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Preface

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematics Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) returned to Boston, Massachusetts for its twenty-fourth annual Conference on Research in Undergraduate Mathematics Education from February 24 - February 26, 2022.

The program included plenary addresses by Dr. Paul Dawkins, and Dr. Karen Keene, and the presentation of 145 contributed, preliminary, and theoretical research reports and 75 posters. The conference was organized around the following themes: results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education.

The proceedings include several types of papers that represent current work in the field of undergraduate mathematics education, each of which underwent a rigorous review by two or more reviewers:

- Contributed Research Reports describe completed research studies
- Preliminary Research Reports describe ongoing research projects in early stages of analysis
- Theoretical Research Reports describe new theoretical perspectives for research
- Posters are 1-page summaries of work that was presented in poster format

The conference was hosted by the Wheelock College at Boston University. Many members of the RUME community volunteered to review submissions before the conference and during the review of the conference papers. We sincerely appreciate all of their hard work by the 152 reviewers. We wish to acknowledge the conference program committee for their substantial contributions to RUME and our institutions. Without their support, the conference would not exist.

This was also the first year the RUME Conference was held in a hybrid format. The conference was simultaneously accessible in person and online using GatherTown. There were 397 registrants in total. Of these, 208 attended in-person and 189 attended virtually.

Finally, we wish to express our deep appreciation for Wheelock College and Boston University. Their support enabled us to have our conference and continually support our community.

Shiv Smith Karunakaran, RUME Organizational Director
Samuel Cook, RUME Conference Local Organizer

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Counterexamples and Refutations in Undergraduate Mathematics

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How do undergraduate mathematics students interpret refutations? We investigated this question by asking participants to 1) decide whether statements are true or false and provide refutations, 2) evaluate counterexamples and 'correct versions' of the statements as valid or invalid refutations, and 3) judge which potential refutations are better, explaining why. We report a study in which 173 undergraduate mathematics students completed this task. Results reveal that participants did largely understand the logic of counterexamples but did not reliably understand the broader logic of refutations.

Keywords: Counterexample, Refutation, Conditional Statement, Logic, Undergraduate

Introduction

Logical reasoning is core to mathematics. It is promoted officially in educational policy and guidance documents (e.g., DfE, 2014; NCTM, 2000), it is addressed explicitly in undergraduate introduction-to-proof courses (e.g., Hammack, 2013), and its development is considered in depth in studies in undergraduate mathematics education (e.g., Hub & Dawkins, 2018; Yopp et al., 2020). Moreover, logic and content knowledge are interlinked. Some teaching sequences rely upon students using reasoning skills to construct content knowledge (e.g., Dawkins & Cook, 2017), and there is evidence that intensive mathematical study at age 16-18 develops abstract logical reasoning skills (Attridge & Inglis, 2013).

But what of situations in which content knowledge interferes with logical reasoning? Understanding logical statements is difficult (Dawkins, 2017; Evans, Handley, Neilens & Over, 2007), and even relatively simple situations can mislead students into using familiar content knowledge without considering all relevant logical possibilities (Dawkins & Cook, 2017; Epp, 2003). The study reported here addresses this issue from a novel angle. Its design is based on observation of real analysis students' responses to questions asking them to state whether statements were true or false and, for each that they declared false, to justify their answer with a counterexample or a brief reason. One such statement, for which results are reported here, was

$\forall\, a, b \in \mathbb{R}, |a+b| < |a| + |b|$.

We might anticipate that some students would incorrectly answer *true* because the statement 'looks like' one that appears in the course, so inattentive students might fail to notice the difference and declare it true without considering possible counterexamples. Or students might search for counterexamples, but fail to find any due to insufficiently developed example spaces (Goldenberg & Mason, 2008) or insufficient checking that a relevant example really has the required properties (Edwards & Alcock, 2010). In the real analysis course, some students did indeed give incorrect *true* responses. However, many also gave correct *false* responses accompanied by invalid refutations. The following was common for the statement above.

FALSE. Reason: It should be $\forall\, a, b \in \mathbb{R}, |a+b| \le |a| + |b|$.

The provided reason can be considered a 'correct version' of the original statement. But it does not refute the original. This illustrates the possibility of interference between content knowledge and logical reasoning, but the mechanism, as discussed below, is not obvious.

Theoretical Background

Perhaps, to explain invalid refutations like that above, accessibility is key. If correct versions come readily to mind for students confronted with similar but false statements, they might write those down as refutations and think no further. This would be consistent with phenomena observed in research around dual-process thinking in mathematical situations (e.g. Evans & Stanovich, 2013). Alternatively, perhaps accessibility is key in that counterexamples do *not* come to mind. Students' *example spaces* take time and effort to develop (Watson & Mason, 2005), and reasoning about concepts is often skewed by early or familiar examples (Zazkis, Liljedahl & Chernoff, 2008). This might make counterexamples relatively inaccessible, especially where concepts are or poorly explored (Sinclair, Watson, Zazkis & Mason, 2011; Zazkis & Chernoff, 2008). If accessibility is key, we would predict that on seeing appropriate counterexamples, some students would recognise them as such and switch answers accordingly.

It might be, however, that invalid refutations arise from explicit value judgements. Perhaps students who give correct-version reasons do think about the logic of their answers—even generating counterexamples—but nevertheless judge correct-version reasons to be better. This might occur if students' mathematical epistemologies are not well developed: some might believe that mathematics is primarily about dutiful recitation of correct answers (Muis, 2004). Alternatively, students might think about the logic of their answers but fail to arrive at valid conclusions. Extensive research has shown that people in general do not reason in normatively correct ways (e.g., Evans, Handley, Neilens & Over, 2007), and that mathematics students typically perform better but far from perfectly (Attridge, Doritou & Inglis, 2015; Attridge & Inglis, 2013). If explicit value judgements are key, we would predict that if presented with appropriate counterexamples, some students would stick with correct-version reasons as better.

The study presented here investigated these possibilities using a three-stage instrument. Undergraduate mathematics students were presented with three items and asked to give initial true/false and counterexample/reason responses. Then, for each item, they were asked to evaluate the validity of both a counterexample and correct-version reason, and subsequently to decide which of the two is the better response, explaining their decision. The outcomes have pedagogical relevance: instructors must understand the origin of this phenomenon in order to intervene in a way that connects with student thinking. They also have theoretical relevance: we know that mathematical and logical reasoning are linked (Inglis & Attridge, 2016), but we know little about how reasoning does or could develop alongside content knowledge in the logically demanding world of early proof-based mathematics (Dawkins & Cook, 2017; Lee, 2017).

Method

Our study used a specially constructed true/false instrument, completed by students in a lecture in a real analysis course that covered sequences, real numbers, and series. Approximately two thirds of the large class (more than 200 students) were first-year students registered for a UK mathematics degree; they spent 75-100% of their study time on mathematics. The remainder were second-year students on joint-degree programmes involving approximately 50% mathematics. All had high but (usually) not elite prior mathematical attainment.

The instrument used the three items:

- $\forall\, a, b \in \mathbb{R}, |a+b| < |a| + |b|$.
- If $x < 3$ then $1/x > 1/3$.
- A sequence (a_n) is increasing if and only if $\forall n \in \mathbb{N}$, $a_{n+2} \geq a_n$.

For the first, *absolute value item*, the obvious correction is to the triangle inequality, which involves correcting the conclusion. Providing a counterexample requires two numbers. The second, *reciprocal item* invites correcting the premise by ruling out cases where $x < 0$. It is simpler than the first in that providing a counterexample requires just one number. The third, *sequence item* invites correction to the definition of *increasing*, which in the course was given as 'A sequence (a_n) is increasing if and only if $\forall n \in \mathbb{N}$, $a_{n+1} \geq a_n$'. Because all sequences that satisfy this definition also satisfy $\forall n \in \mathbb{N}$, $a_{n+2} \geq a_n$, a counterexample must not be increasing and is nontrivial to construct because it cannot be drawn from the sequences that, for most students, would be considered prototypically non-increasing (Alcock & Simpson, 2017).

The true/false instrument contained four pages. On the *initial response* page, participants were presented with all three items and the prompt 'Answer TRUE or FALSE to each question. For those that are FALSE, give a counterexample or a brief reason'. On each remaining page, participants saw one item again, together with a possible counterexample response and a possible correct-version reason. As shown in the sample page in Figure 1, they were asked to evaluate each response by ticking boxes, then state which answer they thought better and why.

Evaluation of Answers to Question 1 1

$\forall a, b \in \mathbb{R}$, $|a+b| < |a| + |b|$.

Below are some examples of typical answers. Tick one circle to show how you would evaluate each answer.

1. FALSE 1.2
   Counterexample: If $a = 1$ and $b = 6$ then $|a+b| = 7$ and $|a| + |b| = 7$.

   ◯ The answer is correct and the counterexample is valid.

   ◯ The answer is correct but the counterexample is not valid.

   ◯ The answer is incorrect.

2. FALSE 1.3
   Reason: It should be '$\forall a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$'.

   ◯ The answer is correct and the reason is valid.

   ◯ The answer is correct but the reason is not valid.

   ◯ The answer is incorrect.

3. Which answer is better and why? 1.4

Figure 1: Evaluation and forced-choice page for the absolute value item (in original layout); the small-font numbers to the right label which item is which to facilitate analysis of the randomised-order versions.

We constructed six versions of instrument with different permutations of the three items. For each evaluation page in each version, multiple copies were created in which the orders of the counterexample and correct-version reason were randomised. The instrument was distributed on individual paper copies. The front page contained informed consent information, explaining the purpose of the study and the way in which the data would be used. Participants were given 15 minutes in week 3 of the course to complete their versions of the instrument; 173 agreed that their data could be used. We report here on the initial, evaluation and forced choice responses.

Results

Initial Responses

The initial responses were classified according to whether participants stated *true* or *false*, and qualitative analysis revealed several categories of counterexample/reason. The categories (and short codes) are shown below, with illustrative responses from the absolute value item.

- CEX: Single correct counterexample.

  'False. $a=5$, $b=6$, $|a+b|=|a|+|b| \rightarrow |5+6|=11$, $|5|+|6|=11$.'

- CEXG: Correctly specified general class of counterexamples.

  'False. If a and b are both positive, $|a+b|=|a|+|b|$.'

- CEXI: Single incorrect/incomplete counterexample.

  (Participant simply wrote the symbol '$\leq$' without numbers.)

- CVR: Correct-version reason.

  'False. $\forall\, a, b \in \mathbb{R}$, $|a+b| \leq |a| + |b|$.'

- CVRI: Incorrect correct-version reason.

  'False as $\forall\, a, b \in \mathbb{R}$, $|a+b| = |a| + |b|$.'

- E: Alternative form of explanation.

  'False. Arguments can be equivalent/equal, not stated.'

- EI: Incorrect alternative form of explanation.

  'False. Counterexample: $(a+b)^2 \nless a^2 + b^2$, $a^2 + 2ab + b^2 < a^2 + b^2$.'

- T: True.
- O: Other (blank, incomplete, or inconsistent responses like 'true' with a counterexample).

Regarding the CEX and CEXG classifications, we note that some authors value general classes over single correct counterexamples due to their power to provide insight into why a claim is false beyond 'exceptions' (Giannakoulias, Mastorides, Potari & Zacharides, 2010) and to their consequent relationship to mathematical theory building (Yopp, 2017; Zeybek, 2016). Here our focus is on simple refutation, but we respect the distinction in these descriptive results.

Table 1 documents the distributions of initial responses. For all items, about one fifth of the 173 participants answered *true* (not the same participants each time: 19 answered *true* twice and none answered *true* three times). For *false* responses, justifications that were incorrect on their own terms (with an 'I' in the code) were uncommon, though more appeared for the sequence item. Beyond this, distributions across the three items differed considerably. The reciprocal item was the 'easiest' to respond to in a normatively correct way: 119 participants provided only counterexamples and a further nine included valid counterexamples. Thus 74% of participants gave a counterexample for this item, and 95% of those who correctly answered *false* included a counterexample. This shows that, of those not tripped into answering 'true', a majority understood the logic of counterexamples well enough to provide these in some circumstances (cf. Stylianides & Al Murani, 2010). For the absolute value and sequence items, however, the data clearly document the observed reasoning problem. For these items, correct-version reasons formed the entire response for 43 (25%) and 72 (42%) of participants respectively.

Table 1: Initial responses summarized by type for the absolute value, reciprocal and sequence items. Responses involving extra explanation are included with the main type except where they occurred separately.

	CEX	CEXG	CEXG+CEX	CEXG+CEXI	CEXI	CEX+CVR	CEXG+CVR	CEX+CEG+CVR	CEXG+CVRI	CVR	CVR+CEXI	CVRI	E	EI	T	O
Abs	50	9	2		1	18	8			43		4	1	1	34	2
Rec	64	29	26		1	3	4	1	1	5					38	1
Seq	23	1		1	8	12				72	3	3	6		35	9

Evaluation Responses

Almost all participants who erroneously said *true* at the initial response stage recognised their mistake during the evaluation stage and accepted at least one suggested *false* answer as correct. The striking feature of the evaluation data was that the majority of those who had correctly answered *false* neither stuck with their initial response type nor switched. Instead, they accepted *both* the counterexample and the correct-version reason as valid. This is captured in Table 2 where, for each sub-table, the shaded cell highlights the number and percentage of normatively correct responses: counterexample valid and correct-version reason invalid. For all items, this combination was much less frequent than accepting both as valid.

Table 2: Counts of validity judgements for the evaluation responses. CV=answer evaluated as correct and counterexample/reason as valid, CI=answer evaluated as correct and counterexample/reason as invalid, I=answer evaluated as incorrect. Blank or unreadable responses mean that one participant is omitted for each question.

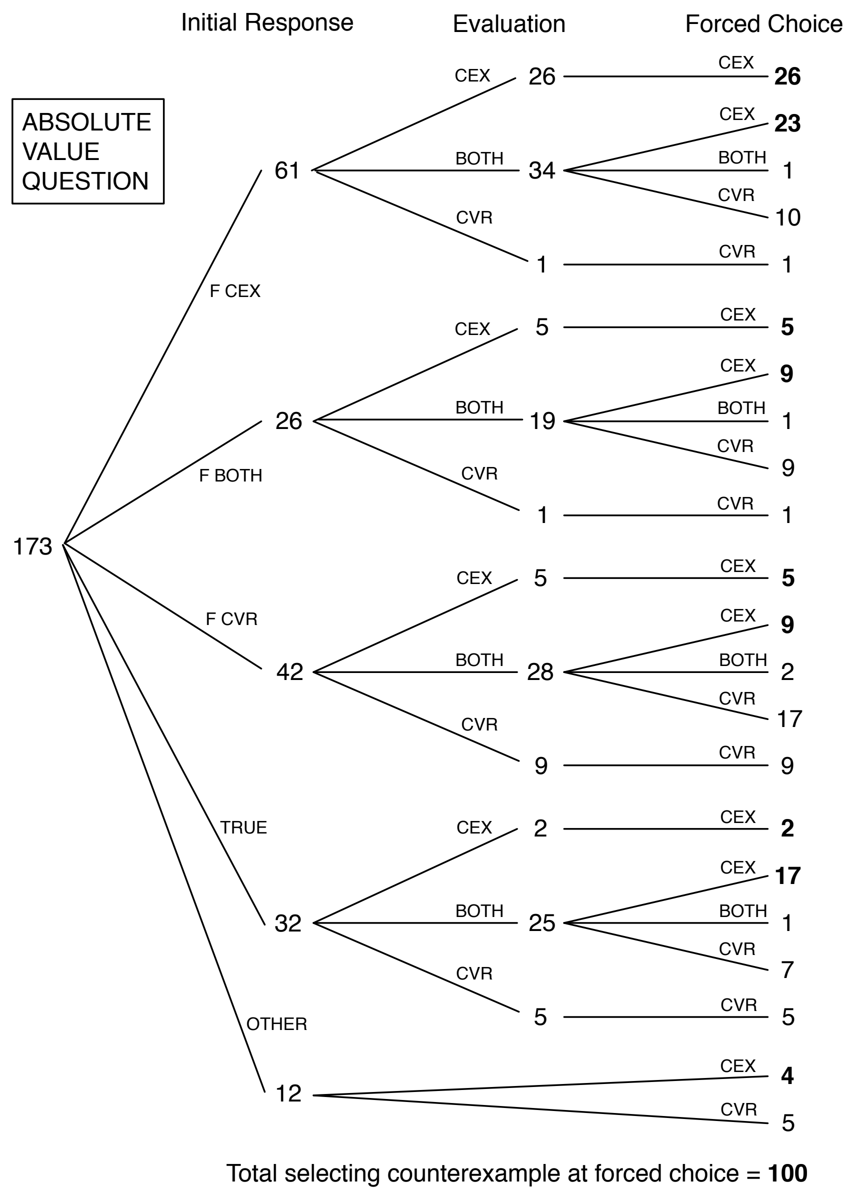
			Counterexample		
			CV	CI	I
Absolute value	Correct-version reason	CV	114 (66%)	12 (7%)	5 (3%)
		CI	36 (21%)	2 (1%)	0 (0%)
		I	2 (1%)	0	1 (1%)
			Counterexample		
			CV	CI	I
Reciprocal	Correct-version reason	CV	119 (69%)	6 (3%)	0 (0%)
		CI	44 (26%)	0 (0%)	0 (0%)
		I	2 (1%)	0 (0%)	1 (1%)
			Counterexample		
			CV	CI	I
Sequence	Correct-version reason	CV	95 (55%)	15 (9%)	13 (8%)
		CI	33 (19%)	2 (1%)	1 (1%)
		I	4 (2%)	5 (3%)	4 (2%)

Thus, only a small minority of participants rejected the counterexamples, indicating that they understood these to be valid refutations. But, for each item, only around one fifth to one quarter of participants rejected the correct-version reason. This means that participants largely did not interpret the broader logic of refutations in a normatively correct way.

Forced Choice Responses

The above result renders the forced-choice responses of particular interest: when asked to choose, which would participants judge better? Responses for the absolute value item are summarised in the tree diagram in Figure 2. To render these readable, initial response categories are combined so that F CEX includes all *false* responses with valid counterexample-only justifications, F BOTH includes all *false* responses with valid counterexamples and correct-version reasons, and F CVR includes all correct-version-reason-only responses.

Figure 2: Tree diagram showing contingent responses for the sequence item. Numbers differ slightly from those in Table 1 because, for example, a participant might have given a counterexample at the initial stage but introduced an inconsistency later, and thus be classed as 'other' here.



Responses for the reciprocal and sequence items were broadly similar and, in theoretical terms, these findings cannot be explained by the accessibility account alone. When offered counterexamples, some participants who initially gave correct-version reasons switched, but more did not. Moreover, some participants switched from counterexamples to correct-version reasons and, for all three items, close to half judged the reason better when forced to choose. This suggests that value judgements account at least partly for students' responses; it does not appear that the result of reflective thought is necessarily a mathematically valid answer.

We conclude these results with a by-participant perspective. Table 3 shows the distributions of participants according to scores out of three for the number of initial counterexample responses (initial score), the number of evaluations only the counterexample judged valid (evaluation score) and the number of counterexamples selected at forced choice (choice score). This confirms a lack of simple interpretations: a majority of participants gave and accepted counterexamples and judged them better some but not all of the time. This indicates that for most, valid reasoning was possible but not reliable (cf. Dawkins & Cook, 2017; Epp, 2003).

Table 3: Distributions of initial, evaluation and choice scores out of three.

	0	1	2	3	Mean (SD)
Initial Score	23	69	62	19	1.45 (0.86)
	13.3%	39.9%	35.8%	11%	48.2%
Evaluation Score	104	35	16	18	0.70 (1.01)
	60.1%	20.2%	9.2%	10.4%	23.3%
Choice Score	33	36	44	60	1.76 (1.13)
	19.1%	20.8%	25.4%	34.7%	58.6%

Discussion

This study investigated reasoning about refutations in undergraduate mathematics. For all three items in our instrument's initial response stage, a large majority of participants correctly answered *false*. However, for the absolute value and sequence items, substantial minorities justified *false* answers only with correct-version reasons. In the evaluation stage, participants who had answered *true* recognised their mistakes, and very few rejected counterexamples as invalid refutations. However, a majority also accepted correct-version reasons as refutations, indicating that they did not understand their logical inadequacy. By-participant analysis confirmed that most participants gave normatively valid responses in some but not all cases.

Methodologically, we acknowledge that a survey provides limited insight into what students 'really think'. Our instrument provided information on both spontaneous reasoning and more considered reflection in response to structured prompts but, if pushed in an interview setting, some students might change their minds or reveal thinking that was not captured on paper (cf. Stylianides & Al Murani, 2010). We also acknowledge that our instrument's instruction that a *false* response should be accompanied by 'a counterexample or a brief reason' implicitly condones justifications of both types, and might have influenced some participants toward or away from providing counterexamples. It would be interesting to compare responses to versions of the instrument that ask specifically for counterexamples and generally for justifications.

In the meantime, our findings shed light on student understanding of refutations as an aspect of logical reasoning, and lend larger-scale support to arguments (e.g. Dawkins & Cook, 2017; Hub & Dawkins, 2018; Ko & Knuth, 2013) that students specialising in mathematics might benefit from interventions designed to help them interpret issues of logic in normatively valid ways and consistently across mathematical content.

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The Synchronous Online Environment as a Mediator of Collective Proving Activity

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Technological tools can transform how students interact not only with each other but also with the mathematics itself. This study investigates how students in a synchronous remote Introduction-to-Proof course operationalize the technological tools available to them to engage in collective proving activity. I identify several utilization schemes students developed for these tools and describe how students coordinated them and their impact on the role of proof.

Keywords: Proof, Computer Mediation, Instrumental Genesis, Collaborative Learning

Technology fundamentally changes the way students interact by acting as a mediator (Vygotsky, 1962), impacting not only interactions between students but also the process of developing knowledge through those interactions (Borba & Villarreal, 2006). Consequently, mathematics courses offered remotely present new challenges as well as opportunities for students to learn collaboratively with one another. This is only magnified in proof-based courses which traditionally do not engage students in collaborative work.

The impact technological tools can have on student learning has been investigated by various scholars. Stahl (2006) found that shared online whiteboards can provide students with a consistent reference to their work and Clark et al. (2007) discussed how they support students during argumentation. When it comes to collaborative writing, Zhou et al. (2012) found that technological tools that like Google Docs can support students in developing a shared meaning, providing a rich learning experience. Lastly, Öner, (2008) noted that dynamic technology can promote a doing perspective of math that focuses on the process rather than the result.

While this literature is very insightful, very little of it is in the context of proof and if it is, it is primarily related to geometrical proofs (e.g., Fukawa-Connelly & Silverman, 2015; Öner, 2008). This study hopes to build on this literature by addressing how technological tools can impact students' collective proving activity in a remote Introduction-to-Proof course.

Theoretical Perspective

Instrumental Genesis

In order to study how the remote learning environment impacts students' collective proving process, this study will draw on Instrumental Genesis Theory (Lonchamp, 2012; Rabardel & Bourmaud, 2003). This theory describes a relationship between artifacts and their users, and the impact of that relationship on cognition.

An *instrument* is differentiated from an artifact through a process called *instrumental genesis.* Béguin and Rabardel (2000) define an instrument as made up of the physical structures of the artifact and what is referred to as a *utilization scheme* which is a cognitive structure that designates a method for using an artifact (Carvalho et al., 2019; Lonchamp, 2012). It is only when an artifact is paired with a utilization schemes that it becomes an instrument (Rabardel & Bourmaud, 2003). The process of instrumental genesis contains two sub-processes. First, *instrumentalization* is oriented toward the artifact and describes the process of exploiting the properties of an artifact either temporarily or permanently (Carvalho et al., 2019; Lonchamp, 2012). Alternatively, *instrumentation* is oriented toward the subject and describes the development and evolution of utilization schemes (Lonchamp, 2012). In classroom settings, instrumentation is specifically focused on the practices students develop to create an instrument in support of knowledge-building activities (Hakkarainen, 2009). This study will be particularly focused on the students' instrumentation process as it focuses on how students learn to use the artifacts available to them to work with their peers on proving activities.

Students in remote environments have access to multiple artifacts which can both support each other and complete similar tasks. Together, these artifacts make up an *instrument system* (Rabardel & Bourmaud, 2003). In courses where students meet for short periods of time (like synchronous online courses) the instructor has an important responsibility in preparing the system (Lonchamp, 2012). When preparing for a course, the instructor develops initial utilization schemes that shape the artifacts for the students, students then develop additional and personal (to themselves and their group) utilization schemes. Development of these schemes with accompanying goal-directed activity, will be the focus of this study.

Proof as Problem-Solving

In this study, the students' proving activity is viewed as a form of problem-solving. Scholars with this view work to "understand the skills, competencies, and dispositions that students need to produce adequate performance on proof related activities" (Styliandes et al., 2017, p. 239). Carlson and Bloom (2005) developed a *Multidimensional Problem-Solving* framework that identifies problem-solving phases and attributes that impact each phase. For this study I draw specially on the problem-solving phases to describe the structure of students' proving process. There are four major phases in Carlson and Bloom's problem-solving cycle. *Orienting* involves taking actions to develop an understanding of the problem situation through behaviors like sense making, organizing, and constructing. *Planning* involves developing a strategy for how to approach the problem. Savic (2015) noted that this can happen at a local level (e.g., identifying warrants for individual lines) or at a global level (e.g., choosing a proof framework). Carlson and Bloom identified a sub-cycle that occurs during this phase, a conjecturing cycle where students: a) *conjecture* an approach to the problem, b) *imagine* how that approach would play out, and c) *evaluate* that approach. This cycle is repeated until a plan has been established. After planning, students *execute* their strategy that was developed in the planning phase. Lastly, *checking* involves verifying the accuracy and adequacy of the executed strategy. Like the planning phase, this can happen at a local or global level. As students work through these phases, they either *cycle forward* when an executed solution is deemed valid, or *cycle back* when an approach needs to be reworked.

This study is focused on understanding how students operationalize the remote learning environment to engage in collective proving and the impact of that operationalization on their collective proving activity. In particular, I will use these theories to help me answer: *How do students operationalize the remote environment to engage in the collective proving process? How does that operationalization impact their engagement with the mathematics itself?*

Methods

Data Collection

The data for this preliminary analysis is from a university Introduction-to-Proofs course that was part of a larger, ongoing NSF-funded project (ASPIRE in Math, DUE #1916490) that is developing Introduction-to-Proof curricula and accompanying instructor support materials. This course was taught remotely with 14 students and met synchronously over Zoom for 10 weeks, three times a week for an hour and five minutes each day. For each day of the class, screen recordings were used to capture the entirety of the students' activity both in breakout rooms and whole class. Moreover, the screen recordings were used to capture simultaneous use of the different modalities each day (e.g., Google Docs and Zoom).

This study focuses on three students' collective work during one day of this course. The, three students Abigail, Alison, and Justin, worked together to collectively write a proof of what the class called the Sudoku Property of group Cayley tables. This refers to the fact that for each row and column of a Cayley table, every group element appears exactly once (i.e., it exists and is unique). The instrument system was made up Google Docs and Zoom. The instructor had prepared the Google Docs before class by including the task with instructions for the students.

Data Analysis

Analysis for this study drew on Powell et al.'s (2003) methodology for studying video data. The first stage of this plan is identifying *critical episodes* that are significant to the research agenda. The episode described above was identified as a *critical episode* since from an initial impression, it seemed that the technological artifacts played an important role in students' ability to work together. A multimodal transcript (Hoffman, 2018) was created using the video data to capture all student interactions as accurately as possible. The transcript was then analyzed using Carlson and Bloom's (2005) problem-solving framework and Instrumental Genesis. Coding of different problem-solving phases involved identifying the problem-solving phase a student's action belonged to relative to the group's collective problem-solving. To capture the students' instrumentation process, each time the students used a technological artifact I answered the questions: 1) What is the goal of using the artifact? 2) How did the student(s) use the artifact?

Once the multimodal transcript was created and coded, I conducted dual thematic analyses on the Instrumental Genesis coding (Braun & Clarke, 2006). First, to identify themes for students' goal directed activities (using my answers to the first question) then, to identify themes for the utilization schemes for each goal directed activity (using my answers to the second question). This dual thematic analysis resulted in the identification of different goal directed activities and associated utilization scheme(s). I then went back to the transcript and video to analyze how the students' instrumentation process impacted their collective proving process by coordinating the students' instrumentation process with the different problem-solving phases. Moreover, identified how the instrumentation processes impacted the kind of proving activity the students engaged in and how the instrumentation processes supported (or hindered) the collective activity. From this, I created descriptions of the role of students' instrumentation processes in their collective proving process.

Results

Students' Different Instrumentation Processes

I identified seven sub-goals that students developed toward the general goal of engaging in collective proving activity, five of which corresponded to a specific problem-solving (P-S) phase (see Table 1). In what follows I discuss how the students coordinated some of these schemes and how this coordination aided in them writing collectively.

Table 1: Identified themes of students use of technology to engage in collective work

P-S Phase	*Goal-directed Activity*	*Artifact*	*Utilization Scheme*
Orienting Phase	Understand Task	Zoom	• Discuss Goals with Peers
		Google Doc	• Read from or Add Information to Worksheet about the Task
Planning Phase	Communicate Idea	Zoom	• Verbally Share Ideas
		Google Doc	• Write Out Proposed Idea • Use Cursor to Reference
	Evaluate Peers' Idea	Zoom	• Listen to Peers' Ideas • Read Peers' Chat
		Google Doc	• Read Peer's Written Proposed Idea • Identify What Peer is Referencing
Execution Phase	Create Team Solution to Task	Zoom	• Verbally Dictate Answer
		Google Doc	• Add on to Existing Text (*repurposing/adapting*) • Write Full Answer
Checking Phase	Check Team Solution to Task	Zoom	• Verbally Share Concern or Acceptance
		Google Doc	• Read Full Answer
Throughout	Update Workspace	Google Doc	• Delete Unnecessary or Irrelevant Text • Update Visual Aids
	Receive Instructor Feedback	Zoom	• Ask Instructor Questions Verbally • Move to Whole Class Discussion
		Google Doc	• Read Instructor Comments

Coordinating between instruments. Several of the students' instrumentation processes involved developing utilization schemes for one artifact that relied on the utilization scheme for the other. Specifically, the students coordinated the textual affordances of the Google Doc with the verbal affordances of Zoom in two ways. First, during the planning phase, the students used the Google Doc to aid in *communicating and evaluating an idea.* Consider the following two students who discussed why a group element must show up at least once in a Cayley table.

Alison: …Once an element shows up once, it can't show up again. So, it has to be filled by a different one. An empty space. Are you following my train of thought? […] Here, I'm gonna write it out and then then read and tell me what you think. (Alison writes in google doc "We already proved that each symmetry can show up at most one time in each row. So, if B appears once, it cannot show up again….")

Abigail: (as Alison is typing) Oh, yeah, that makes sense. Because there's only like, Okay, if it's a six-by-six table, then and you place one. And then you can't place it again. So, you have to pick another one from the five remaining.

Even though Alison wrote down almost exactly what she said out loud, to Alison the Google Doc was an instrument that could better help her communicate her ideas. For Abigail, the writing on the Google Doc acted as a visual mediator, providing additional insights to Alison's approach.

The second way this coordination was observed was by students visually referencing existing text with their cursor during their verbal communication. For instance, when discussing further justifications that their proof needed, two students had the following conversation:

Alison: Should we regroup it too?

Justin: Probably at **that point**. Do we use two steps somewhere?

Alison: Maybe associativity **right here**.

As seen in Figure 1a and Figure 1b, Alison's cursor was visible to the everyone viewing the Google Doc. At this point, the group had established the utilization scheme of using the cursor as an indicator of where in the document the cursor's owner (i.e., the student's) attention was. With just the placement of the cursor, Justin knew where in the proof Alison's attention was when he suggested they regroup "at that point" and Alison knew that her cursor would indicate to Justin what she meant by "right here". Students would also highlight the text to visually reference parts of their document. This coordination allowed the students to use indexical terms (e.g., this one, that one, here) in such a way that avoided ambiguity during conversation.

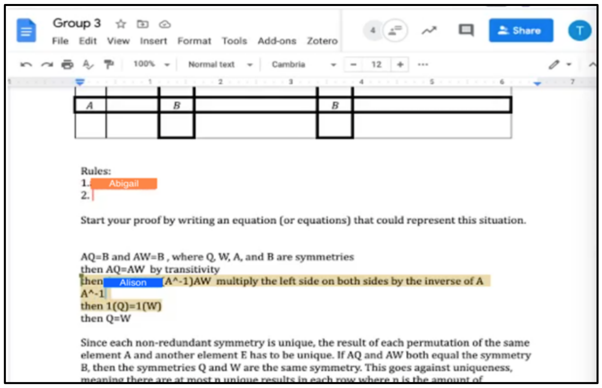

Figure 1a. Alison's cursor identifying "that point"

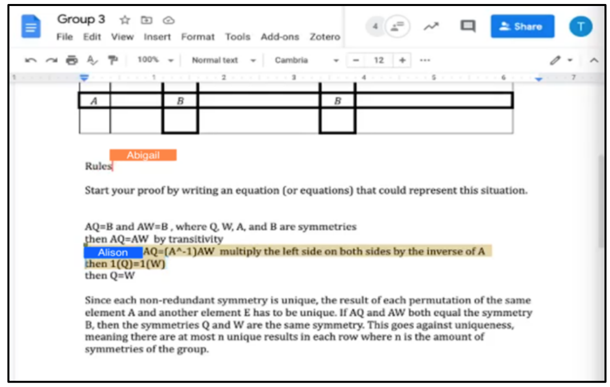

Figure 1b. Alison's cursor identifying "right here"

Writing collectively. The students in this study developed various utilization schemes to achieve the goal of *creating a team solution.* Engaging collectively in the execution phase of the problem-solving process often involved different people executing different parts of the task. For instance, Figure 2 shows the group's completed proof of the uniqueness of group elements in a Cayley table that was written among all members. The different colors represent different student contributions, showing how the proof writing was carried out by all students.

While at times the group engaged in collective writing by writing simultaneously, a particularly interesting use of the instrument system was the students' schemes to repurpose and adapt existing text. As was discussed earlier, students' schemes of communicating ideas involved writing text down in the Google Doc as an aid to their verbal communication. This syntactic communication scheme supported the development of a repurposing scheme in which students repurposed text originally serving a communicative function so that it was now serving a final solution function. This scheme was often utilized without any explicit comments from the students and instead the text was implicitly identified as serving this new function when students moved on to the execution phase. For example, consider the proof in Figure 2, the lines "AQ=B and AW=B, so AQ=AW, then Q=W" were originally written by two students to communicate

AQ=B and AW=B , where Q, W, A, and B are symmetries
then AQ=AW by transitivity
then $A^{-1}AQ=A^{-1}AW$ multiply the left side on both sides by the inverse of A
then $(A^{-1}A)Q=(A^{-1}A)W$ by rule 1
then 1Q=1W by rule 3
then Q=W by rule 2

Since each non-redundant symmetry is unique, the result of each permutation of the same element A and another element E has to be unique. If AQ and AW both equal the symmetry B, then the symmetries Q and W are the same symmetry. This goes against uniqueness, meaning there are at most n unique results in each row where n is the amount of symmetries of the group.

Figure 2. Proof written collectively with different colors representing different student contributions

their ideas about how the proof by contradiction should work during the planning phase. This exact text was then built on and slightly edited during the group's execution phase and makes up an important part of the final solution. By employing the repurposing scheme, students opened up the pool of ideas to include more than just contributions during the execution phase.

The Instrumentation of Proof

Students' instrumentation of the technological artifacts (i.e., Zoom and Google Docs) transformed the role that the proof played in their collective activity. Beyond being a document that represented a complete, formal argument, a proof became an evolving artifact that represented the sharing of ideas and their dynamic argumentation during the proving process. The students developed several utilization schemes that leveraging this feature transforming the proof into an instrument.

Due to the students repurposing and adaptation schemes, their proof was encoded with the group's discussions. Using their referencing scheme, the students developed a utilization scheme to use the proof to *engage in argumentation* with each other. For example, consider the case when a group of students were discussing why their current proof proved that an element appears at most once in each row of a Cayley table:

Justin: Because they have to be the same thing. Because we're- that shows that Q and W equal each other.

Abigail: Yeah, it does the same thing as being, like like R equals R (writes R=R in the Google Doc), we wouldn't even have that because those are the same symmetry. You know what I mean? So, these are like the identical symmetry (highlights R=R and then W=Q). And so, it wouldn't be one of the like, one's on the outside of the table…

By writing *R=R* and then immediately highlighting *W=Q*, Abigail is doing more than just referencing to bring attention to the very last line of the proof. Instead, she is relying on the last line to communicate the argument from the full proof: if an element B shows up twice in a row *A* then there must exist two elements *W* and *Q* in the header row that are equal to each other.

In addition to the proof acting as an instrument to communicate ideas, the students' instrumentation processes transformed the proof into an instrument that represents their evolving argument. Specifically, the students developed a utilization scheme of using the proof as a *record of their current understanding*. The students' proving process involved multiple iterations of

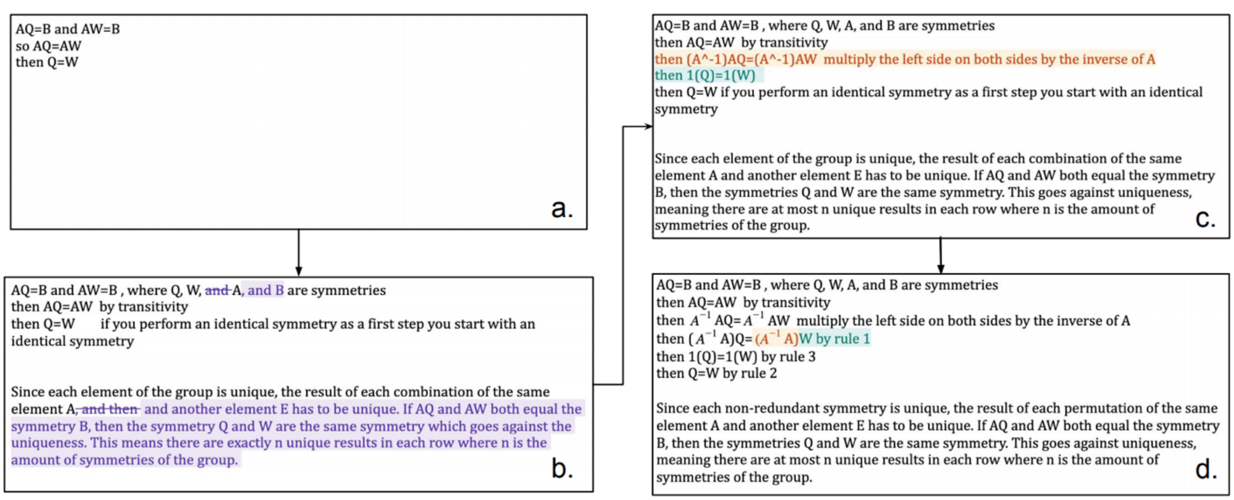

Figure 3. Evolution of the Students' Proof

proposing and evaluating warrants for different parts of the overall argument; often including simultaneous executing and checking cycles. Throughout these cycles the students employed several utilization schemes for the technological artifacts in such a way that their written proof was continuously updated and thus, evolved as their argumentation evolved. To model this evolution, consider Figures 3 which represents a few steps in how the group's proof evolved as they engaged in several problem-solving cycles.

Discussion

These results show that students use and operationalize the instrument system of the remote environment in many ways to engage in collective proving activity. Moreover, through an instrumental genesis lens, we can see how students can successfully transform artifacts into collaborative instruments.

Students used multiple instruments in a complimentary way to help organize their collective activity around a common task (e.g., writing on the Google Doc to supplement a verbal explanation). Not only did this dual scheme enrich students' communications to one another, but it also enriched their abilities to understand and listen to one another. Scholars have noted that due to the many moving parts of the remote environment, successful coordination of instruments can play an important role in developing a collective understanding (e.g., Çakir et al., 2009; Stahl, 2006). The students in this study also developed schemes that leveraged this coordination to create a collective solution. Their repurposing schemes allowed their communication to be embedded into their final work and the editing features of the Google doc allowed for easy and simultaneous editing to occur among the students. Together these schemes supported the creation of a collective proof that reflected the ideas of each individual.

The students' instrumentation of the technological artifacts resulted in the instrumentation of their proof. Due the various utilization schemes applied to the Google Docs and Zoom, the students' proof became an evolving artifact that represented the sharing of ideas and their dynamic argumentation during the proving process. As such, the proof was utilized to communicate with one another and to record their evolving understanding of the argument. CSCL literature on students' use of Dynamic Geometry Systems often emphasizes how students' use of these artifacts results in an emphasis on the process of *doing* math rather than on the final product (Öner, 2008). Results from this study show evidence of an analogous view of math when it comes to proving activity. This evolving feature of proof can support students in seeing proving as a process that involves first, second, etc. attempts, where ideas are proposed, executed, and refined. If proof is viewed as a dynamic object and not as a final document full of formal (and perhaps foreign) language, it might become more accessible and inviting to students.

This study presented a case of how one group of students successfully navigated and leveraged the remote environment to engage in collective proving activity and how that impacted their proving process and their conception of proof overall. Future work is needed to further explore these results further and how students can be supported to leverage technological tools in a productive way.

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Developmental Math Students' Perceptions of (Re)learning Outcomes: The Value of Algebra All Over Again

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We report findings from an exploratory study on developmental mathematics students' perceived experience re-learning content they had already studied in middle- or high-school. Our findings suggest that these experiences may be largely shaped by students' expected and perceived learning outcomes associated with that content. Rather than describing the mathematical concepts learned, students focused on the additional value of the Intermediate Algebra course when describing their expectations and perceptions learning about a topic they had seen before in previous algebra courses. We describe how six of those learning outcomes depended on students' confidence in their previous understanding of the content to be relearned, and how those learning outcomes influence students' modes of engagement during (and emotional reactions to) their relearning experience.

Keywords: developmental mathematics, relearning, confidence

Developmental (or remedial) education courses are commonly offered at U.S. colleges and universities for students that are deemed underprepared for "college-level" work in mathematics, reading, or writing. Traditional developmental courses are often nestled within a sequence, meaning that a student needs to pass multiple courses before enrolling in a credit-bearing course of the same subject. The Conference Board of the Mathematical Sciences Survey (CBMSS) found that in Fall 2015 approximately 41% of all two-year college and 11% of all four-year college and university mathematics and statistics enrollment was in developmental courses. Despite this sizable enrollment, only an estimated 50% of students beginning at public two-year institutions and 58% of students beginning at public 4-year institutions pass or earn some credit for all the developmental mathematics courses (DMC) they attempt to take (Chen, 2016). Reasons for these failure rates have been proposed at various levels, but most research has thus far focused on entry and exit problems with the developmental course sequence such as placement and attrition (e.g. Bahr, 2008a; Bahr 2012a; Bailey, Jeong & Cho, 2010). These studies give us a broad sense of the paths students take to credit-bearing courses, but leave the reasons for rates of attrition, passing, and graduation obscured with a "black box" of teaching and learning in DMC (Grubb, 2001). In order to address the black box problem in developmental mathematics reform, more research needs to be done that characterizes student experiences in developmental mathematics programs and identifies aspects that appear to be critical to the formation of end-of-course outcomes over time. This exploratory study began to fill this gap in the literature by describing student experiences with content through the theoretical lens of *relearning*, or the experience of trying to learn about something one has already tried to learn about before.[1]

[1]A brief report of these results appears in the proceedings for the 43rd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.

Literature Review

Student experiences in DMC involve significant amounts of time learning about content students have seen before, either from a previous K-12 mathematics course or from a previously-attempted DMC (Ngo & Velasquez, 2020; Fay, 2020). Stigler, Givvin and Thompson (2010) summarize student experiences in DMC as being, "presented the same material in the same way yet again" (p. 4). In their study of 306 first year students placed into intermediate algebra at California State University, Benken, Ramirez, Li, and Wetendorf (2015) found that 60% of these students had taken courses beyond Algebra II in high school. Likewise, Ngo (2020) investigated the percentage of college students who take "redundant" math courses, or courses whose content is either at the same or lower level than their highest completed math course in high school (or lower than what they would be predicted to pass given their results on a 12th grade math assessment). He found that while roughly 20 percent of all college students take redundant math courses, that percentage increases to roughly 40 percent when looking at developmental math students specifically. Taken together, these studies indicate that the experience of learning about content one has seen before (possibly even to some level of mastery) occurs more frequently for developmental mathematics students than students enrolled in other math courses in college. For students enrolled in a multi-course sequence or for the large proportion of students who fail DMCs, this repeated interaction with content seen before occurs more often.

This similarity to content seen before across multiple timepoints is a point of concern for developmental math educators due to the high numbers of students who are recommended to take developmental mathematics courses who fail to complete them because they never enroll in the final course in the first place (Bahr, 2008a; Bailey, Jeong & Cho, 2010). Accordingly, the past 10 years has seen an influx in the adoption of new forms of developmental mathematics experiences throughout colleges and universities other than the traditional, sequential, semester-long experiences. These reforms change the amount of time students spend in remedial courses, but nevertheless involve the underlying phenomenon of learning about content seen before. Across traditional and reform-oriented DMC, studies of effectiveness have thus far relied on end-of-semester outcomes such as student passing rates and graduation rates. However, the over-reliance on this method of evaluation combined with the lack of information about student experiences in DMC has made it difficult for researchers and practitioners to understand the reasons for these outcomes and apply them to other contexts. In order to address this issue, researchers have been called to open up the "black box" of instruction in DMC (Grubb, 2001; Sitomer et al., 2012; Mesa, Wladis & Watkins, 2014), particularly when it comes to student experiences with content. Although such research is limited, the research that does exist suggests that students' past experiences with content influence their current experience each time that content is re-encountered (Sitomer, 2014; Zientek et al., 2013; Givvin, Stigler, & Thompson, 2011). However, currently no such studies have been conducted in service of describing what it is like for students to learn about content they have seen before, despite the defining role of this phenomenon in developmental math classes. The purpose of this study is to begin to fill this gap by characterizing student experiences with content they have tried to learn about before, an experience we define below as "relearning".

Theoretical Framework: Relearning

At the most basic level, we define relearning as requiring three things: some (mathematical) content, a "time 1" (T1) representing a past occurrence when an individual has tried to learn about that content, and a "time 2" (T2) representing the most recent time an individual has tried to learn about that same content again. While the content at T1 and T2 need not be identical, it does need to cross a threshold of similarity such that the content learning goals at T2 are essentially the same as those at T1. Although the name *relearning* appears to suggest some degree of mastery of content at T1, we make no such assumption in our conceptualization. As argued previously, although it may not be the case that absolutely every piece of content a student encounters in DMC has been seen before, it is likely that most of the content is familiar. Further, by nature of their design, the learning goals of DMC are identical to those of middle and high school algebra courses considered prerequisites for undergraduate mathematics instruction. Given the scarcity of research on student experiences with this phenomenon in DMC, we turned to what research does exist that in some way considered developmental mathematics student experiences with content in order to begin to structure our investigation. In order to incorporate a focus on the influence of students' past experience with that same content, we additionally anchored our analysis in variables that have proven useful in research from cognitive psychology on the influence of past experiences on perception in interaction (Kleinschmidt & Jaeger, 2015).

First, in two papers on community college developmental mathematics students, Stigler, Givvin, and Thompson (2010) and Givvin, Stigler, and Thompson (2011) suggested that repeated exposure to instruction that associates learning mathematics with rote memorization may explain data indicating that developmental mathematics students' procedural knowledge is disconnected from actual mathematical concepts, resulting in minimal ability to evaluate the validity of their answers. Second, in their research on 169 developmental classes in 20 community colleges, Grubb and Gabriner (2013) provided an account that pushed back on the idea that student experiences would be homogenous. Instead, they proposed that there were five different types of developmental mathematics students according to their "learning needs". The two that focus on student relationships to content are "brush up" students (deemed only to need a quick reminder of content) and students with insufficient basic academic studying skills, which leaves them with an understanding of content that is just good enough to pass, but not good enough to retain concepts beyond a final exam. While the later is reminiscent of Givvin, Stigler, and Thompson's hypothesis, the former presents another possibility. The other categories (students who have been misplaced, students with learning disabilities, and students with mental health issues) are less relevant to our focus on relationships to content, but nevertheless highlight the important role that affective states and attitudes towards the course on student experiences. Finally, Cox and Dougherty (2019) described types of student experiences from their interviews with 25 pre-algebra students across four sections of a developmental math course at a community college. The resulting categories were defined by either the expected or perceived impact of the course on students' relationship to the content they had seen before. In a notable departure from the results of the previous studies, some students characterized their experience as wanting to deeply understand the reasoning behind their use of procedures in a way that they had not done before (although, it was relatively uncommon for students to feel this had been accomplished by the end of the course). Others indicated that the course was useful in "refreshing" their memory, similar to the "brush up" students described previously. However, unlike Grubb and Gabriner, Cox and Dougherty described student perceptions of their own experiences rather than instructor's perceptions of students, and thus were able to identify variation within the "brush up" student category in terms of affective disposition (e.g., while some students were satisfied with the function of the course as a 'refresher', others found 'reviewing' to be a waste of time given their prior experience).

The convergence of these results suggested that the question of what students get out of a DMC in terms of their relationship to the content might be a defining experiential feature because it is unique to this type of mathematics course. Given the previously-noted possible variation in this type of experience in terms of engagement and affective disposition as well as the lack of theory that would allow for a targeted investigation of this concept, we turned to research in cognitive psychology on the influence of past experiences on perception (Kleinschmidt & Jaeger, 2015) for further guidance. In particular, the variables of perceived *similarity* of experience and *relevance* of the prior experience in the new situation were variables that seemed to characterize how individuals perceive and approach scenarios they have been in before. With these variables as well as methods of engagement and affective disposition in mind, we set out to explore how developmental mathematics students experience relearning content they had seen in previous algebra courses. In particular, we focused on the following questions addressing two aspects of that experience: how similar do students think the content of DMCs are to that of their school algebra courses, and how relevant do they perceive their current knowledge of algebra to be in their success in DMCs?

Methods

This was a multiple-case study in which three students enrolled in an Intermediate Algebra course at a four-year public university in the Spring 2020 semester participated in one-hour, semi-structured interviews before and after learning about a topic they indicated they had seen before (Equations of a Line and Polynomials) in a previous algebra class. Each student also participated in one follow up interview in the Fall 2020 semester. Simon and Zena were students of Instructor A, had never previously taken an algebra course in college, and were an 18-year-old first year student and a 19-year-old second year student at the time of the study, respectively. Valeria was a student of Instructor B, had previously taken three algebra courses in college, and was a 20-year-old third year student at the time of the study. Both Instructor A and B were recommended by the head of the department for their quality of instruction, and as many participants as possible were recruited through an online survey. All course meetings in which these topics were covered were conducted in person, prior to the switch to online instruction due to the Covid-19 pandemic. Each of these meetings were observed and recorded with the first author taking fieldnotes on participant behaviors and problems worked on during class. In interview 1, students were asked to describe their history learning about the topic, confidence in their current understanding, and to predict what the experience of learning about the topic again would be like. In interview 2, after the topic had been taught, students were asked to describe what it was like to learn about the topic again, including how predictions aligned with what occurred. Discussions were anchored in problems gathered from field notes or student work as often as possible. A detailed discussion of the instructor interviews and student follow-up interviews is beyond the scope of this report. The results reported here focus on student interviews from the Spring 2020 semester. Interviews were transcribed and analyzed using a modified thematic analysis. While data collection was guided by some theoretical propositions, the novelty of the research called for a more exploratory approach. Thus, coding proceeded in five stages beginning with in vivo coding (Creswell, 2007) and progressing to a modified thematic analysis (Braun & Clarke, 2012) as familiarity with the dataset increased.

Results

Due to space constraints, we report results around one variable central to our findings on student experiences: learning outcomes, or the resultant relationship between a student and material they have seen before at the end of a relearning experience. This is not a grade or an indication of passing/failing but instead an answer the question: what was the value of this learning experience in terms of students' understanding of content this time around? Across Interview 1 and 2, students described six types of learning outcomes (Table 1).

Table 1. Students' expected/perceived learning outcomes when relearning algebra content seen before.

Outcome	Definition	Example
Gain a Deeper Understanding	Students see previously-learned material in a new way and use this shift in perspective to gain a "deeper" understanding than they did before.	Zena, Interview 2: I think [in high school] I definitely just memorized it short term for the exam or class and now I actually understand what I'm solving for and what I'm trying to do with the equations.
Confirm my Understanding	Students already know the content and verify that their existing understanding is adequate.	Simon, Interview 1: We're just learning what I already know. It takes a load off the class cause you just know what's gonna happen and so it's kind of fun to be able to [solve problems] while he's explaining.
Jog my Memory	Students refresh their memory of material they have forgotten due to matters unrelated to inadequate prior understanding such as time. After the class is completed, they do not require further practice.	Valeria, Interview 1: It will be like a refresher. Cause I took it already and like I said, I'm bad with names of things, so I don't really know what it looks like, but once we're doing it in class I'll know what to do.
Reconstruct my Memory with Guidance	Students are guided through a reconstruction of their memory of previously-learned material and avoid remembering things "incorrectly". After the class is completed, they require further practice.	Zena, Interview 1: If I'm trying to remember what I first learned I could get confused easily with what I vaguely remember and what I don't; so I just want a concrete 'this is how I should do it'…and I think that would help me understand faster.
Fix Past Mistakes	Students identify and address inadequate understandings of course material gained from past experiences.	Zena, Interview 2: When I have to make my own linear model, what I have trouble with is corresponding the x and y with the variables they give…I think I've gotten better at it in this class just because he really kind of breaks down why we're doing it.
Accept what I Don't Understand	Students acknowledge that they hold inadequate understanding of course material gained from past experiences, but see themselves as being unable to address it.	Valeria, Interview 1: I just can't get it. [Word problems don't] make sense to me at all, and I try, like I've been watching videos and all that stuff, but I still won't get it at all.

While the codes are not ordered hierarchically, they were each associated with a degree of confidence in the student's perception of their current understanding of the relearned material. *Confirm my Understanding*, for example, was associated with a high degree of confidence in a student's perception of their understanding of the relearned material, thus allowing them to simply confirm that their current understanding is adequate. On the other and, the learning outcome *Reconstruct my Memory with Guidance* was associated with a lower degree of confidence, as students who mentioned this outcome felt that they understood the material enough to partially rely on memory, but not well enough to do so without significant assistance from an instructor. Some of these outcomes, as with *Jog my Memory* are reminiscent of prior conceptualizations of the value of developmental math courses (i.e., Grubb and Gabriner's (2013) brush up students, Cox and Dougherty's (2019) refreshing). Others, such as *Confirm my Understanding* and *Accept what I Don't Understand* are novel. Each of these outcomes were informed by students' pasts, and represent a multitude of frames through which they could interpret their experiences. Importantly, discussions of their expected and perceived learning outcomes were associated with particular methods of engagement with re-learned material and affective dispositions while relearning. Differences in engagement and affect were frequently observed across the same learning outcome. For example, although all three students noted *Confirm my Understanding* as a learning outcome, Simon and Valeria associated this outcome with not paying attention during class, whereas Zena did not describe engaging in this behavior.

This variation in engagement and affect for each learning outcome was even observed for the same student across Interview 1 and 2. For example, Simon described the experience associated with *Confirm my Understanding* as "kind of fun" in Interview 1, and as boring in Interview 2. Due to the exploratory nature of this study, it remains an open and critical question about the extent to which students' degrees of experience with re-learning informed their dispositions towards the learning outcomes as well as the types of learning outcomes they perceived as viable. Student descriptions of their experience with re-learning also shifted both from Interview 1 to Interview 2 when discussing the same topic (from expected to perceived learning outcomes), and within the same interview when discussing different aspects of a topic. Because each of these interviews centered around one unit (3-4 classes), this meant that students were describing different learning outcomes for different problems, subsections, and problem-solving strategies within one unit as well as different learning outcomes from those that they expected going into the lesson. For example, in Interview 2, Valeria described different anticipated learning outcomes for sub-topics within the polynomials unit. With many subtopics in the unit that involved adding or multiplying polynomial expressions, Valeria found herself frustrated with the experience of learning about content that she already understood (confirm my understanding) to the point where she saw little value in paying attention during class.

Valeria: Polynomials is just like, a thing to me like I honestly could care less. So when I know, I know. I'm like, I'm kind of getting lazy with this, cause this is like the third time of taking this class. But usually, I don't know, in class I just, I don't like it cause of the time.

Interviewer: Cause it takes so long, you mean?

Valeria: Well like the class period and cause she explains like every little thing sometimes. Like she's like 'what's 3 times 2?' like we all, we all should know that, so there's no point in asking it. But she does that sometimes.

Interviewer: And when you said like the stuff that you know, you know. Can you say more what you mean by that?

Valeria: Like if, like when she starts doing examples and I do them on my own and get the right answers, like maybe she'll do five and I get the first three right then I'm like I don't really need to pay attention any more. So then I'll just either like daze off or like go on my phone to be honest.

However, for other sub-topics that she found more difficult, Valeria's experience seemed to be dominated by avoidance behaviors and anxiety gained from her past history with algebra. For example, when discussing factoring techniques she remarked, "If [a polynomial] has a number in front of the x^2 one, then I'll usually just sit there and just stare at it, but if it doesn't then I can usually just do it...I think it just scares me or something. It just like scares me." In a follow up interview, Valeria went on to suggest that her difficulties with this type of factoring, both conceptual and emotional, could only be resolved if she had learned about them in a different way the first time they were introduced (*Accept what I Don't Understand*). As Valeria's case illustrates, due to their previous experience with algebra, developmental mathematics students are frequently making judgments about what they expect to get out of their re-learning experience often based only on perceived similarity of content (to what they've seen before) and their confidence in understanding the material at that time. Interestingly, she saw her experience with *Accept what I Don't Understand* as similar to *Confirm my Understanding*, given that both outcomes were motivated by her unwillingness to engage with course content. This suggests that the relationships between these learning outcomes would be a worthwhile pursuit for future investigations in this domain.

Conclusions

Given that some learning outcomes may be more or less desirable than others, the results provided here may be useful to DMC instructors seeking to understand the potential impacts of their pedagogy on student understanding, and how this may vary from other courses in which students are learning new mathematics for the first time. The fact that student perceptions of their learning outcomes shifted not only by topic within an interview, but across interviews as they gained more information casts doubt on literature that attempts to classify students as one particular type, or as having one type of learning need. Given the exploratory nature of this study, we would encourage future inquiry into the types of learning outcomes across various DMC contexts: both topic-by-topic and considering course experiences more holistically. It may be the case that DMC as well as other courses that involve relearning may be designed in ways that encourage or discourage particular learning outcomes from being achieved. A mathematics department looking to evaluate the effectiveness of a DMC could ask: what learning outcomes are desirable for us in these courses? Beyond setting expectations for the content students should be learning in these courses, instructors of courses that contain relearning experiences need to ask: what relationship to this content could a student build in this course given the type of instructional support and content engagement I am providing? Practically, the language of learning outcomes also may provide more specific information about not only what students learned while in the course, but *how* they approached the task of relearning in order to achieve that outcome. Having language that can go beyond 'a review' could facilitate more meaningful conversations with students that result in setting more appropriate expectations for their learning experience. Additionally, the notion of "fixing" one's past mistakes shares similarities to findings from Dibbs (2019) who investigated the experiences of undergraduates retaking calculus courses, another type of relearning experience. The extent to which the findings described here might be relevant to other such contexts is an area for future inquiry.

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Contextual and Mathematical Conceptual Resources for Reasoning about Null Spaces

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Vector spaces are often taught with an axiomatic focus, but this has been shown to rely on knowledge many students have not yet developed. In this paper, we examine two students' conceptual resources for reasoning about null spaces drawing on data from a paired teaching experiment. The task sequence is set in the context of a school with one-directional hallways. Students' informal reasoning about paths that leave the room populations unchanged supported more formal reasoning about null spaces. We found that one student used context-based resources (such as 'loops' in hallway) to reason about null spaces, while the other student drew largely on previously formalized mathematical resources (e.g. free variables, linear dependence). The use of formal resources sometimes required recontextualization, which may function to constrain student sense-making or afford opportunities for broadening students' formal prior knowledge.

Keywords: linear algebra, subspaces, null spaces, conceptual resources

Axiomatic treatments of vector spaces are thought to be often inaccessible to students because they unify and formalize many ideas that many students have not yet developed (Dorier, 2000; Grenier-Boley, 2014). In their genetic decomposition of vector spaces, Parraguez and Oktaç (2010) identified the binary operations of scalar multiplication and vector addition, as well as the closure of sets under linear combinations of these operations, as the critical constructs for consideration in regard to students' learning about vector spaces. We thus argue that subspaces function as a more accessible entry point for supporting students' learning about vector spaces.

Given the increasingly important role of linear algebra in real world applications, more work is needed to support students in developing robust conceptualizations for subspaces, and particularly null spaces, given their applicability to a wide range of closed systems problems. In this paper, we examine data from an instructional sequence on subspaces whose design is informed by the principles of Realistic Mathematics Education (RME; van den Heuvel-Panhuizen & Drijvers, 2020). In particular, we address the research question: What are students' conceptual resources for reasoning about null spaces in the context of an RME-based instructional sequence focused on subspaces?

Literature and Theory

A limited number of studies target students' early reasoning about vector spaces, subspaces, and especially null and/or column spaces (Stewart et al., 2019). Two recent exceptions in the area of subspaces are the work of Wawro et al. (2011) and Caglayan (2019). Wawro et al. (2011) documented 8 undergraduate students' concept images for subspaces in relation to how they reasoned about the concept definition -- identifying geometric and algebraic interpretations, and the critical role of reasoning about a subspace as a part of a whole for making sense of the formal textbook definition presented to students. Caglayan (2019) interviewed 14 undergraduate math majors who had recently taken linear algebra, asking them to use a digital geometry system

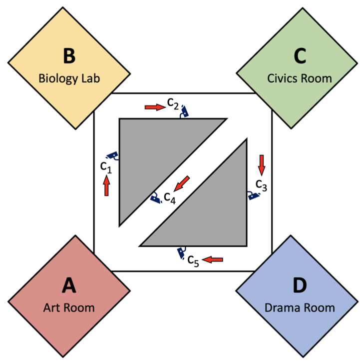
(Matlab) to classify 57 subsets of 7 different vector spaces as subspaces or not -- particularly highlighting ways in which students used the zero vector to show the set was not closed. These studies and others (e.g. Açikyildiz & Kösa, 2021; Dogan, 2018) highlight the potential value of geometry for introducing ideas related to vector spaces. Our study differs as we examine an experientially real context for engaging students in reasoning about ideas related to subspaces and null spaces, but do not leverage the geometry of two and three dimensional space.

We draw on diSessa's (2018) Knowledge in Pieces (KiP) view of learning and cognition, which is influenced heavily by constructivist and cognitivist traditions. We value this as a lens to theorize conceptual resources (Taber, 2008) and adopt anti-deficit views of students and their learning (Adiredja, 2019). KiP emphasizes the complex, continuously shifting, and contextually-bound nature of knowledge, where learning involves reconfiguration of "naive" knowledge into more normative understandings (diSessa, 2018). For example, the notion that 'multiplication makes bigger' is a concept that is considered normatively correct in some contexts (numbers larger than 1) and not others. It is a "small knowledge structure" that can either be cast as an unuseful idea from a deficit perspective or a useful idea that can be more explicitly contextualized and reconfigured to reflect broader and more normatively accepted understandings.

Study Context

This work was conducted as part of a broader research project focused on developing a series of research-based linear algebra materials. These materials were developed in alignment with RME principles of experientially real starting points, guidance from an instructional figure, and the critical role of model-of, model-for shifts in students' mathematical activity (Van den Heuvel-Panhuizen & Drijvers, 2020). In this manuscript, we leverage a KiP lens for examining students' mathematical activity. To our knowledge, KiP frameworks have not been heavily leveraged in the literature to analyze student reasoning in RME-inspired task sequences and we believe coordinating these perspectives may provide a meaningful contribution to the field.

In its current form, the sequence consists of three core tasks. In this manuscript, we focus our analysis on students' reasoning in the context of the second and third tasks. Our team's approach to subspaces was organized around the notion that subspaces are non-empty subsets of vector spaces that are closed under linear combinations. To support students' development of meaning in relation to this notion, we leverage the scenario of one-way hallways during a pandemic (Plaxco et al, 2021). The hallway scenario can be viewed as a coordination between two sets of quantities: the number of people who pass through each of the one-way hallways as observed by cameras and the change in the population of each classroom that occurs during a period of observation (Figure 1a). For example, assuming people do not linger in the hallway, if 3 people pass by Camera 1 and Camera 2 during a class changeover, then the net change of the number of people in the Biology Lab is zero. This induces a mapping of the set of n-tuples encoding of the number of people passing by the camera in each hallway ("*camera vectors*"; in the West Wing, vectors of the form $\langle c_1, c_2, c_3, c_4, c_5 \rangle$) to the set of m-tuples showing the net change in each room's population over the period of hallway observations ("*room change vectors*"; in the West Wing, vectors of the form $\langle \Delta A, \Delta B, \Delta C, \Delta D \rangle = \langle c_4 + c_5 - c_1, c_1 - c_2, c_2 - c_3 - c_4, c_3 - c_5 \rangle$). In Tasks 1 and 2, students are asked to represent journeys of students walking through halls using camera vectors, reason about the relationships between the *camera vectors* and the *room change vectors* within the problem context, and reflect on and make generalizations based on their reasoning. These activities are focused on supporting students to reason about the set of camera vectors that do not change the populations of the rooms (which corresponds to the null space of the matrix that maps camera vectors to room change vectors). This activity culminates in the students representing the problem context using a matrix transformation. During the third task, students are asked to reason about a new wing of the school (the East Wing; Figure 1b) based on a given matrix, rather than a given hallway diagram.



(a)

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{pmatrix}$$

(b)

Figure 1. Information about the West Wing (a) and East Wing (b) of Ida B. Wells High School

Participants, Data Sources, and Methods of Analysis

The study had two white male participants (who we refer to with pseudonyms Drew and Carson) at a predominantly minority public post-secondary institution in the Southeastern U.S. Drew and Carson were the only two volunteers satisfying the study constraints of their age being 18 and older and having taken an inquiry-oriented linear algebra course based on the IOLA curriculum. Both students had earned an "A" in the course. Our data sources for this analysis consist of video-recordings of a four-day paired teaching experiment (Steffe & Thompson, 2000), as well as any inscriptions created by participants or the teacher-researcher.

The paired-teaching experiment (PTE) consisted of four, 90-minute sessions. Each session was conducted and recorded on Microsoft teams and included two research team members. One was the teacher-researcher across all sessions; the other collected field notes and asked clarifying questions. Students were asked to think aloud and respond to questions regarding their work. Student work was captured by having students upload work, hold up their work to the screen, or work on a shared whiteboard. The research team debriefed after each interview session, noting mathematically significant aspects of student reasoning that emerged in the interview.

To identify students' conceptual resources for reasoning about null spaces, our team began by first assigning two team members to review each session of the PTE and note themes in students' mathematical reasoning that related to ideas about topics related to null spaces (including linear combinations of vectors, span, linear transformations, and solutions to homogeneous systems of linear equations as well as their applications in the problem context.) In this analysis, we focus on the final two days of the teaching experiment, when the definition of null spaces was formally introduced to students in relation to their prior work, and they were asked to extend their reasoning to a new context (one-directional hallways in relation to room locations in the East Wing of the high school, with information inscribed in a matrix rather than a map). Based on team members' notes, videos were selectively transcribed, and we identified two broad themes that functioned as conceptual resources for reasoning about null spaces: closed loop reasoning and RREF reasoning.

Findings

Though the data presented in this analysis comes from the final day of the PTE, it is important to note that students developed important ways of reasoning about camera vectors, room change vectors, and relationships between them on the previous days. Namely, the students described the sets of camera vectors corresponding to possible paths a single person or 5 people could take from room A to room C, and from room C to room C. They identified the former sets as not closed under linear combinations, and they identified the latter sets as being closed under linear combinations (with vectors in these latter sets earning the designation of "closed loop" vectors). See Plaxco et al. (2021) for more detail on this. The students agreed that closed loop vectors left room populations unchanged and identified ways of expressing the relationship between camera vectors and room population change vectors. In this section, we highlight how closed loop reasoning emerged as the core conceptual resource for one student across a pair of questions, whereas linear dependence and RREF reasoning emerged as the core conceptual resource for the other student. (Though interviewed as a pair, the students rarely engaged in one another's reasoning so for clarity we discuss their reasoning in ways that are largely separate.)

Closed Loops as a Context-based Conceptual Resource

Early on the fourth day of the PTE, both students were then given the equation in Figure 2 and asked to interpret what it meant "for that four-tuple on the right-hand-side to be all zeros." Almost without hesitation, Drew responded, "there is no net change in the room population after all movement is complete" indicating a fluency between the two contexts of hallway movement and the matrix equation. The students were then asked if the set S of vectors **c** in R^5 such that A**c**=**0** (in correspondence with the matrix equation shown in Figure 2) is a subspace of R^5 (with the reminder that this would mean that if we sum of any two vectors in S the result must also be in S, and if we scale a vector in S the result must also be in S).

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Figure 2. Matrix equation given to students to reason about the null space

Drew initially argued that the sum of two vectors in S would be in S by stating "yes, because they both equal zero." Drew later elaborated:

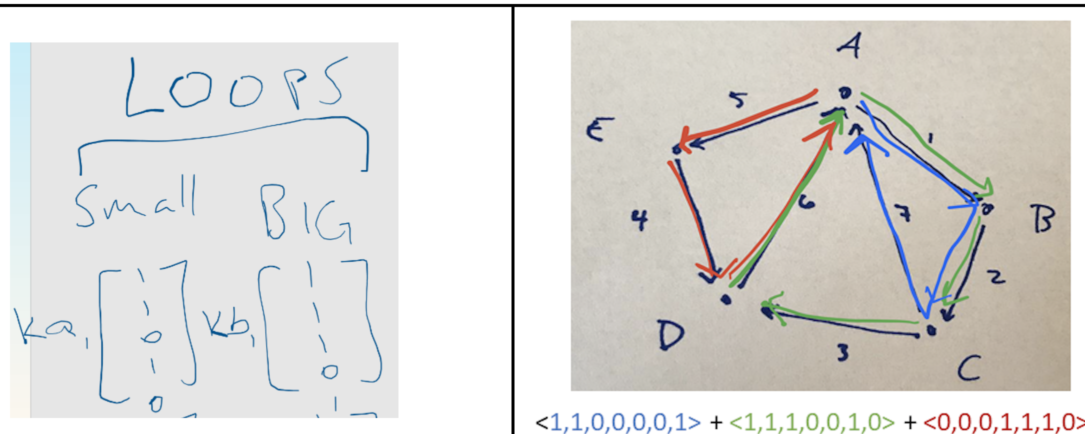
Drew: "Since both vectors are going to be equal to zero, the subspace will cover zero, so zero plus zero will still be in the subspace."

Interviewer: Are you saying that the camera vector has to be zero?

Drew: No, because they're both equal to zero after multiplying by the matrix. That's what we're saying… In order to get back to zero, we need to have a bunch of loops. That's how we get zero. If we keep adding more loops matter what and we just add the vectors of the loops together and we multiply it by that, it would still be zero, would it not? This is the way I'm seeing it. I'm saying 'cause like the way we get zeros is we just add a bunch of loops together. Uh, we just have a bunch of loops there and they're just constantly repeating. So if we had another vector, it's just another, butt ton of loops. We'd just have more loops which will get back to zero. The way I'm seeing it.

Drew later connected to the students' prior work describing paths from C to C:

> … it's like the linear combination that we did a while back, I think it's probably day one where it was like, like the big loops in the small loops and you could have as many of them as you wanted if you just continuously add more and more loops to it. It's still going to get you back to the same point. So, as long as we have loops like that one and the other one. It would, it would basically just be two different scalars… multiplied by those loops. And, that would be the combination of the vectors that we're adding together.



(a) Instructor's inscription of West Wing loops | *(b) Drew's East Wing map and loop vectors*

Figure 3. Inscriptions related to Drew's closed loop reasoning

As Drew spoke, the interviewer made the inscription shown in Figure 3a, as these corresponded to what students had described as the big and small loops on prior days of the PTE. The interviewer prompted, "if you just combine these two vectors together…." and Drew responded that we would "just have more loops." When asked by the interviewer if "more loops is also just a solution to this equation here," Drew again reverted to his hallway reasoning, "yeah, because they're just looping around." Drew similarly extended this line of reasoning when asked about scalar multiplication, saying in reference to figure 3a that, "if you multiply a_1 and b_1 by like ten each, it's just more loops."

When later presented with the matrix M in Figure 1b and asked to describe all the possible hallway flows that leave the number of students in each room unchanged, Drew generated the map shown in Figure 3b -- which is an appropriate translation of the matrix back into a hallway diagram -- and correctly identified three closed loop vectors (which mathematically speaking are a basis for the null space of the matrix M).

Drew's use of closed loops (or closed loop vectors) as a conceptual resource was characterized by four core ideas around which his reasoning was largely organized:

1. Closed loops are paths from a room back to itself
2. Closed loops leave room populations fixed
3. Linear combinations of closed loops are closed loops
4. Closed loop vectors are solutions to homogeneous equations

The first three ideas were developed on prior days of the PTE as indicated by Drew in his comments. However, Drew also connected these ideas to the matrix equation presented in Figure 3 - initially by connecting the zero vector on the right hand side of the equation to unchanging room populations, and then by further connecting these unchanging room populations to "closed loop" camera vectors (which sum and scale to closed loop vectors when scaling by non-negative integers). We argue that in this way, Drew's closed loop reasoning, as characterized and contextualized by this set of ideas, functioned as a robust conceptual resource for reasoning about null spaces in the hallways task sequence.

Multiple Solutions, Linear Dependence, and RREF as Formalized Conceptual Resources

In contrast with Drew, Carson did not leverage closed loops as a central conceptual resource in his reasoning on the tasks presented on the final day of the PTE. Rather, he drew heavily on more formal ideas from the linear algebra class but found that the hallways task setting required contextual reinterpretation of these ideas in ways that were not always straightforward.

When asked to reason about closure of the set S in relation to Figure 2, Carson did not immediately engage in the question set forth by the interviewer (namely, whether any two vectors that satisfy the given homogeneous equation would sum to a vector that also satisfies the homogeneous equation). Rather, he focused on interpreting the matrix equation, inquiring "So we're trying to solve the homogeneous equation, right?" After the interviewer restates his previous prompt (presumably to get Carson to focus on the closure aspect of the question), Carson continued. "There's a free variable, correct? … which means there is more than one solution to the homogeneous equation." Several minutes later, Carson seemed to continue this line of thought, saying "I have a question. If it wasn't linearly dependent, I don't think you'd be able to do it, be able to do loops and stuff. Is that correct?" The interviewer agreed that if the column vectors didn't form a linearly dependent set, then the only solution would be the trivial solution. The following exchange ensued.

Interviewer: Ok. So if those column vectors are linearly independent, then the loops would be what?

Drew: They wouldn't exist. They would be null, they wouldn't actually exist. They would get you a homogeneous answer.

Interviewer: Right. So the only way to get the classrooms to not change?

Drew: Would be to replace the c vector with zeros.

Interviewer: So what does that mean about student movement?

Carson: No one moves.

This episode highlights Carson's ideas about solutions to the homogeneous equations, free variables indicating multiple solutions, and linear dependence being important for the ability to "do loops." He then correctly reinterprets the trivial solution to the homogeneous vector equation to mean that no one moves in the hallway context.

When Carson is asked to find the hallways flows that leave room populations unchanged for the matrix shown in Figure 1b, he suggests "since it's dependent, find something that makes two of them zero. I make it, two of them, zero I guess? Try to find out the homogenous solution cause it wants to be unchanged… You could solve it using reduced row-echelon form is one way." Carson then used Maple to row reduce the matrix M. The conversation then turned to Drew's map (as discussed above) before the interviewer asked how Drew's loop vectors related to Carson's row reduced matrix. Drew argued that if you multiplied his vectors by Carson's matrix (excluding the final column of zeros), it should give you the zero vector, and verified it does. Carson commented that he “thought it would because it gives an absolute answer every single time. That’s why I like matrices. They give exact answers every time.” When asked if Drew’s closed loop vectors might be extracted from Carson RREF, Carson’s initial reaction was:

> It’s basically the idea of, there’s a free variable, right? So you could use any scalar to get to any other location that you wanted to by using scalars and adding them together. This is the part where I kind of get confused. I did all the steps, I just don’t know, I know each of these are column vectors that equal zero, zero, zero.

We interpret this comment to reflect Carson’s effort to brainstorm information he can glean from row-reduced matrices: the presence of free variables, information about span (e.g. places you can “get to”) -- but indicating that the connection between his row-reduced matrix and Drew’s closed loop vectors was not obvious.

We argue that Carson’s use of multiple solutions, linear dependence, and row reduction as conceptual resource was characterized by three core ideas around which his reasoning was largely organized:

1. Free variables indicate multiple solutions
2. Linear dependence is needed for (closed) loops
3. Row reducing matrices gives exact answers

Carson productively leveraged the first two ideas to reason about some aspects of null spaces, but the final p-prim did not provide sufficient detail to extract the desired camera vectors. Potentially complicating this issue is the fact that, in cases with infinitely many solutions, row reduction produces solution sets with free variables that can presumably range through all real numbers -- yet the hallways problem context requires limiting to solutions in which all entries are non-negative integers.

Discussion

The contextually developed conceptual resources organized around the idea of closed loops provided a productive entry point for several aspects of reasoning about null spaces (e.g. specifying elements, reasoning about closure, and identifying a basis), and both students had ideas related to closed loops. Previously formalized mathematical resources also offered useful insights into some aspects of null spaces, but were more useful in connecting to the hallways context at some times (e.g. connecting multiple solutions to linear dependence and closed loops) than at others (trying to extract camera vectors that leave room populations unchanged based on the RREF of a given matrix).

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Investigating the effectiveness of short-duration workshops on uptake of inquiry-based learning

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Short-duration and long-duration professional development workshops on inquiry-based learning (IBL) were designed to increase participants' capacity to teach using IBL methods. This study used a sample of 66 participants in short-duration workshops (one day or less) and 199 from long-duration workshops (four days plus ongoing support) held from 2016-2019 to investigate the role of workshop duration in increasing IBL capacity, and in fostering uptake of IBL teaching practices. After participating in professional development, both short-duration and long-duration workshop participants reported comparable levels of IBL capacity–meaning the beliefs, knowledge and skills that prepare them to use IBL. However, short-duration workshop participants implemented IBL teaching practices less intensively than long-duration participants. These findings support the use of both short-duration and long-duration professional development as a means to increase instructors' capacity to use IBL and their adoption of IBL teaching practices.

Keywords: Inquiry-based learning, Professional development, Workshops, Teaching

Introduction

Research-based instructional strategies (RBIS) have been shown to promote student learning and academic success in US undergraduate STEM education (Freeman et al., 2014; Ruiz-Primo, 2011). In undergraduate math contexts, a form of RBIS, inquiry-based learning (IBL), has been associated with positive student outcomes (Kogan & Laursen, 2014). However, use of RBIS by undergraduate STEM instructors is not common; approximately 20% use RBIS extensively (Stains et al., 2018; Eagan, 2016).

Prior research has shown that teaching-focused professional development (PD) can increase STEM college instructors' use of RBIS, including those of undergraduate mathematics instructors using IBL (Archie et al., 2021; Benabentos et al., 2020; Chasteen & Chattergoon, 2020; Manduca et al., 2017; Bathgate et al., 2019). The focus of these studies were long-duration PD experiences (e.g. multi-day workshops), rather than single instance "one-off" experiences. Duration has been identified as a "critical feature" of professional development (Viskupic et al., 2019), and research on PD in higher education settings has consistently demonstrated that PD is effective when it occurs over extended periods of time (Desimone, 2009; Allen et al., 2011; Ebert-May et al., 2015; Garet et al., 2001; Pelch & McConnell 2016; Postareff et al., 2007; Wilson, 2013). A review of PD in higher education found that PD that takes place over an extended period of time results in more changes in instructor teaching practices than one-time PD activities (Stes et al., 2010). However, this review contained few studies focused on short duration, one-time events.

K-12 contexts also have shown similar findings. For example, Supovitz and Turner (2000) found that 40 hours or more of PD participation was needed to make a detectable impact on teachers' use of inquiry-based teaching practices. Other studies suggest ideas about teaching and teaching practices change over time, rather than from "one-shot" workshops (Loucks-Horsley et al., 2009; Postareff et al., 2007). However, Kennedy's 1999 review of PD indicated that PD content was more strongly related to participant outcomes than duration. Given the costs of professional development for facilitation, participant support, and travel, it is important for providers and funders of PD to know what features of PD generate measurable impact on teaching practice and, by inference, on students' experiences and success in courses where research-based teaching is applied.

While research indicates that long-duration PD is effective in generating change to teacher practice, few studies have shown that short-duration workshops were less effective in doing this, thus more research is necessary to determine the importance of duration, among other variables, in PD. This study addresses this gap by investigating the outcomes of short-duration IBL workshops for undergraduate math instructors. In this study, we compare short workshop findings to those from long workshops that have already been shown to be effective in increasing math instructors' capacity to use IBL and to increase their use of IBL methods (Archie et al.; 2021). Specifically, this study sought to answer the following research questions:

RQ1: Do short-duration workshop participants report the same gains and levels of IBL capacity (attitudes, knowledge, and skill) as long-duration workshop participants?

RQ2: Do short-duration workshop participants implement and use IBL teaching practices to the same degree as long-duration workshop participants?

IBL workshop description

Long-duration workshops are an established, research-supported method of professional development for instructors to learn to use IBL. Participants travelled to workshops that occurred over four consecutive days, utilized consistent content and format, and featured ongoing support from workshop leaders and peers following the workshop (see Hayward & Laursen, 2018; Yoshinobu et al., 2021 for a full description of the long duration workshops). Long-duration workshops have been shown to increase participants' capacity to use IBL methods and subsequently, increase their use of IBL teaching practices (Archie et al., 2021).

Short-duration workshops were designed to complement long-duration workshops. Both models serve as professional development opportunities for mathematics instructors to learn about IBL and increase participants' capacity to use IBL teaching methods. Short workshops were intended to provide an introduction to IBL, while the long-duration workshops provided a deeper, implementation-focused experience. In both long and short workshops, leaders modeled IBL techniques through their facilitation. For example, leaders asked participants to engage in Think-Pair-Share activities to help participants understand how to use this technique in their own classrooms. Short and long-duration workshops were planned and led by the same cadre of trained workshop facilitators, working within a single, coordinated project grounded in PD literature, sharing a broad philosophy of IBL (Laursen & Rasmussen, 2019), and drawing on their training and experiences with long workshops to develop short workshops. Evaluation data show that both long and short workshops received very high ratings of participant satisfaction.

Between 2017 and 2020, the project supported 26 short workshops reaching at least 500 participants. The workshops were between 1-8 hours in length and were held at various institutions across the country, including two- and four-year colleges and sessions held at professional development conferences of the MAA and AMATYC. Short workshop content was non-standardized and varied by audience and facilitator preference, but most workshops provided a general introduction to IBL. A few workshops emphasized enhancing specific skills, such as assessment techniques, facilitating discussions, and generating classroom materials, and one centered on a particular audience, instructors of pre-service elementary teachers. The primary goals of the short workshops were to: increase awareness of IBL methods, enhance component skills of IBL, recruit new audiences of faculty into IBL teaching, reach departments or groups not yet active in IBL, and offer an "on-ramp" to IBL practice.

Methods

Data collection

Long-duration workshop participants completed a pre-workshop survey about one month before their workshop, a post-workshop survey immediately after, and a follow-up survey about 18 months later. Of 293 long-duration workshop participants from 2016-2019, 291 (99%) completed the pre-workshop survey, 275 (94%) completed the post-workshop survey and 199 (68%) completed the follow-up survey.

Short-duration workshop participants completed a brief post-workshop survey immediately following the workshop. In all, 328 post-survey responses were collected from an estimated 470 participants from 24 workshops, for an overall response rate of 70%. A more extensive follow-up survey about IBL implementation was sent in late 2020 to all participants who provided contact information when they attended a short-duration workshop from 2017-2020 (n = 270). Sixty-six follow-up survey responses were collected for a response rate of 25%. While the response to the post-survey was high (70%) and likely representative of the workshop population, the follow-up survey had a lower response rate (25%) and cannot be considered representative. Moreover, the follow-up was sent to all workshop participants in late 2020, so the time between workshop attendance and follow-up varied, ranging up to 3 years. Given this response rate, the high reports of IBL implementation in particular may be skewed due to non-response bias where the follow-up survey respondents may be those who were most interested or enthusiastic about IBL.

Measures

Measures of IBL capacity were used to answer RQ1 and are indicators of short-term outcomes resulting from workshop participation. Measures of IBL capacity include participants' attitude about IBL, their knowledge of IBL, and their skill using IBL. Short- and long-duration workshops used identical measures of IBL capacity attitude, knowledge, and skills on follow-up surveys and on long workshop pre and post surveys. As a measure of IBL attitude we asked "To what extent do you believe inquiry strategies are an effective learning method?" and was measured on a four-point scale (1 = Don't know, 2= Not very effective, 3= Somewhat effective, 4 = Highly effective). IBL knowledge and skill were measured with similar separate measures asking "How would you rank your current level of knowledge/skill in inquiry-based teaching" with both items sharing the same four-point response options (1=None, 2=A little, 3=Some, and 4=A lot). Thus, we can directly compare levels on these items at the follow-up timeframe and can compute gains for long workshop participants by comparing rankings across timepoints.

Since short workshop participants did not take a pre-survey, we instead asked them to rate their perceived gains in IBL attitude, knowledge, and skills on the post-workshop survey using a five-point scale (1= A lot less, 2= Less, 3=About the same, 4= A little more, 5= A lot more).

To answer RQ2, we directly measured short and long workshop participants' self-reported implementation of IBL and, as an indirect measure of implementation, their self-reported teaching practices. On short and long workshop follow-up surveys, we asked respondents to self-report their implementation of IBL methods by asking “Have you implemented an IBL course since the workshop?” with four response options: “No; Not a fully IBL course, but applied some approaches; Yes, one IBL course; Yes, more than one IBL course.”

Separately, we asked long workshop participants on the pre-workshop survey, and both short and long workshop participants on a follow-up survey, to indicate their frequency of use of 11 teaching practices using the following scale: 1= never, 2= once or twice during the term, 3= about once a month, 4= about twice a month, 5= weekly, 6= more than once a week, or 7= every class. As described in Hayward et. al (2016), five of these 11 teaching practices are classified as ‘core IBL’ practices because they characterize all variations of IBL that were emphasized in workshops: decreased use of instructor activities, including lecture and instructor problem-solving on the board, and increased use of student activities, especially student presentations of their own work and student discussion in small groups or as a whole class.

Data analysis

To answer RQ1 we computed frequencies of short workshop participants who reported gains in IBL attitude, knowledge, and skills in the post-workshop survey. For comparative purposes, we computed frequencies of the long-duration workshop participants who reported gains in IBL attitude, knowledge, and skills from pre-workshop and post-workshop surveys. We also calculated means of IBL attitudes, knowledge, and skill of short and long workshop participants from their respective follow-up surveys. We conducted an independent samples t-test to check for differences in follow-up survey IBL capacity measures by workshop duration.

To answer RQ2 we computed IBL intensity scores for each instructor based on their self-reported frequencies of the five core IBL teaching practices as follows: IBL intensity= student group work + student presentation + class discussion – lecture – instructor solving problems. IBL intensity scores were computed at the follow-up time point for both short- and long-duration workshop participants. We conducted an independent samples t-test to check for differences in intensity of use of IBL teaching practices by workshop duration.

Results

To answer RQ1, we compared both the proportion of reported *gains* in IBL capacity measures, and the final *levels* of IBL capacity, by workshop duration. First, as shown in Figure 1, a majority of both short and long duration workshop participants reported gains in all three IBL capacity measures (attitude, knowledge, skill). Since we measured and calculated gains differently across workshop types, it was not appropriate to make direct statistical comparisons of gains in IBL capacity. However, it is evident that the relative gains in capacity (attitude, skill, and knowledge), are the same for both short and long workshops; that is, the greatest gains are in IBL knowledge, followed by less extensive gains in skill and IBL attitude.

Also addressing RQ1, we conducted an independent samples t-test to check for differences in mean IBL capacity reported by workshop participants in respective follow-up surveys. While *gains* measures differ, the measure of final *levels* is the same in surveys sent to both long and short workshop groups. Descriptively, short workshop participants reported lower mean IBL capacity than did long workshop participants (Table 1). However, the only statistically significant difference in the individual indicators that make up IBL capacity was in IBL attitudes, and the effect size indicates that this difference was minimal. We found no statistically significant differences in either IBL knowledge or skill by workshop duration.

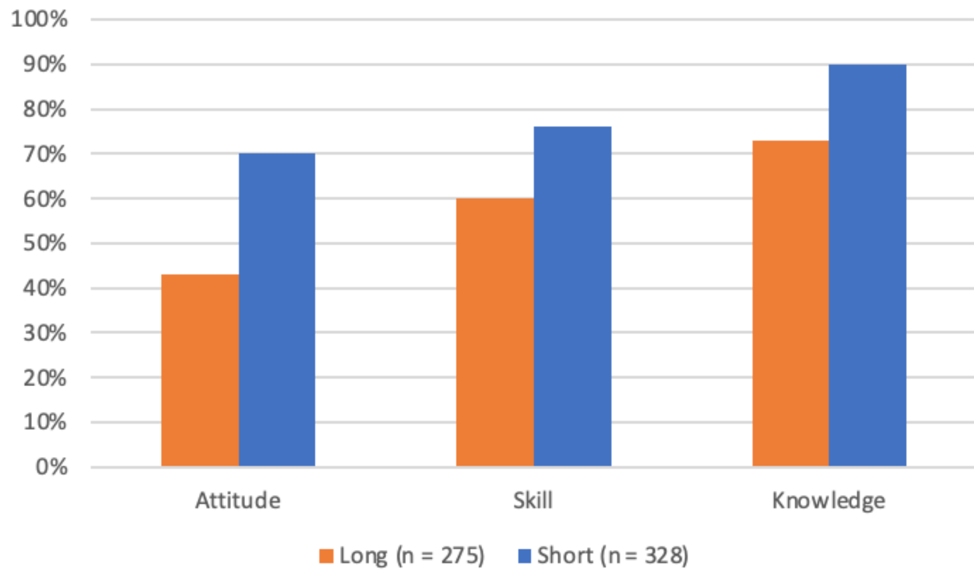

Figure 1. Proportion of workshop participants who reported gains in IBL capacity measures by workshop duration

Table 1.
t-test of mean IBL capacity follow-up measures by workshop duration

	Short workshop (n = 53)		Long workshop (n = 189)				
IBL Capacity	*M*	*SD*	*M*	*SD*	*t* (df = 240)	*p*	*d*
Attitude	**3.59**	0.50	**3.71**	0.51	-2.96	0.003	0.23
Knowledge	**2.96**	0.65	**3.24**	0.61	-1.54	0.126	0.46
Skill	**2.64**	0.74	**2.80**	0.64	-1.49	0.137	0.24

Note: *d* = cohen's *d* and is a measure of effect size.

To answer RQ2, we compared the proportions of participants who implemented IBL methods, by workshop duration. The measures used were identical for both survey groups. As shown in Table 2, we found that a higher proportion of short workshop participants (19%) did not implement IBL than long-duration workshop participants (5%). A greater proportion of long workshop participants (29%) reported implementing IBL in one fully IBL course than did short duration workshop participants (13%).

Table 2.
Proportion of workshop participants who implemented IBL methods by workshop duration

	Workshop duration	
IBL implementation	Short (n = 62)	Long (n = 199)
No	19.%*	5%*
Not a fully IBL course, but have applied some IBL approaches	53%	44%
Yes, one fully IBL course	13%*	29%*
Yes, more than one fully IBL course	15%	22%

* short and long workshop proportions differ at $p \leq 0.05$

RQ2 was also answered by comparing participants' intensity of use of IBL teaching practices after participating in their respective short or long workshops. An independent samples t-test indicated that average intensity scores after long workshops (M = 7.87, SD = 5.44) were significantly higher, (t(226) = 2.71, p = 0.007), than the IBL intensity scores after short workshops (M = 5.43, SD = 4.37. The effect size (C*ohen's d* = 0.46) indicated a moderate difference in the use of IBL teaching practices between long and short workshop participants.

Discussion

Both the long workshops and short workshops were designed to increase participants' capacity to use IBL teaching methods, and our findings suggest that short and long workshops may be effective in doing so. This study provided the opportunity to compare workshops that are driven by a shared philosophy and led by the same facilitators, but with different durations. However, the limited sample of short-workshop follow-up survey respondents (discussed in detail in the final section) provides a sense of the *possible* outcomes from short-duration workshops, while the robust sample of long-duration workshop participants provides a *generalization* of typical outcomes from the long-duration workshops. Therefore, direct comparisons of short and long workshops should be interpreted with caution.

Both short and long workshop participants reported the same pattern of gains in the three individual IBL capacity measures. The lowest gains by both groups were in IBL attitude; participants self-selected to participate and thus many participants already had a positive attitude about IBL, so their gains were small. Both short and long workshop participants reported higher gains in IBL knowledge than in skill. This is a logical finding considering that skills take time and practice to develop. Overall, both workshops seem to be improving participants' capacity or readiness to use IBL methods in a consistent way.

The lack of statistical and/or meaningful differences in follow-up measures of IBL capacity by workshop duration were unexpected and are inconsistent with prior research which has shown that long duration PD is more strongly related to outcomes than short duration PD (Stes et al., 2010). We expected that, in short workshops, the limited time participants had to acquire IBL knowledge and skill would translate into lower gains in IBL capacity than from long workshops. This finding is likely due to the small, and likely biased, sample of short workshop participants who completed a follow-up survey.

Findings related to RQ2 indicated that short duration workshop participants implemented IBL teaching practices to a lesser degree than long workshop participants. About 20% of short workshop participants who responded to the survey implemented no IBL teaching practices, and a greater proportion of long workshop participants implemented one course they considered to be fully IBL. Follow-up reports of IBL intensity also showed that short workshop participants used particular IBL methods less intensively. These findings are consistent with prior research, which has shown that long duration PD is positively associated with changes in teaching practices (Stes et al., 2010). These findings were expected given that the long workshops were implementation-focused: they used a consistent structure designed to meet implementers' needs and allowed participants work time to plan their own IBL course. Long duration workshop participants committed to four days, so they had a prior high level of commitment to implement IBL. Short workshops were less focused on implementation; rather, they sought to create broader awareness of IBL. Short workshops also required less participant commitment to implement IBL in their teaching and did not allow time for participants to plan how to implement IBL in their teaching.

Limitations and conclusion

While some aspects of these findings seem to support the efficacy of short workshops, several limiting factors of this research must be considered in interpreting these findings. The relatively small sample size from participants in short workshops (n = 66) is likely biased. Those short workshop participants who implemented IBL may have been more likely to respond to a follow-up survey than those who did not. Thus, the short workshop findings represent a best-case scenario for outcomes. They suggest that some participants respond to the workshops by implementing IBL methods in their own classrooms, but these findings should not be considered to be representative of all workshop participants.

While we measured pre-workshop and follow-up IBL capacity and teaching practices, we did not collect corresponding measures for short-duration workshop participants (Archie et al., 2021). Since short-duration workshop participants were not required to pre-register, we were unable to administer a pre-workshop survey. This prevented us from knowing participants' initial levels of IBL capacity and their teaching practices before attending a workshop. Without this data, we can't be certain about how much short workshop participants gained and the degree to which their teaching practices changed. Although we can't know the outcomes for all participants of short workshops, these results suggest that for some participants, short workshops may be an effective way to build IBL capacity. Others have suggested that starting with less intense implementations and building over time may lead to more widespread adoption, a process known as "phased inquiry" (Yarnall & Fusco, 2014) or "trialability" (Rogers, 2003); one interpretation of the findings is that short workshop participants are moving through this process. Thus, short workshops may have served in some respects as an effective "on-ramp" for instructors as they work towards full adoption of IBL teaching methods, while the long workshops led to greater measured change in teaching practice. Because duration is a basic PD design parameter that in turn affords or constrains other choices about content and engagement, these findings are important for PD providers and funders to consider as they make decisions about where to invest their effort and funds.

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Comparing Student Strategies in Vector Unknown and the Magic Carpet Ride Task

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We present findings from a study analyzing and comparing the strategies participants deployed in playing the game Vector Unknown and completing the Magic Carpet Ride task. Both the game and task are designed to give students an introduction to basic concepts about vectors needed for success in linear algebra. We found that participants used a diverse array of strategies, tending to favor algebraic approaches to the Magic Carpet Ride task. We also found that participants tended to try the same strategies in both tasks, but did not usually follow through with the same strategy in both contexts. These findings have implications for instructors considering using one or both tasks in their linear algebra class.

Keywords: linear algebra, game-based learning, inquiry-oriented instruction

Game-based learning (GBL) has proven to be a popular approach in STEM education and STEM education research (Klopfer & Thompson, 2020). However, much of the research into GBL in mathematics education has been focused on K-12 and especially K-8 education (Byun & Joung, 2018). One of the few games developed specifically for undergraduate mathematics instruction is Vector Unknown (VU; Mauntel et al., 2021), an adaptation of the Magic Carpet Ride (MCR) task from the Inquiry-Oriented Linear Algebra (IOLA) curriculum (Wawro, Zandieh, et al., 2013). VU, like the MCR task, is designed to give players an introduction to basic concepts about vectors needed for success in linear algebra. While the goals of both tasks overlap, the differences between them may lead to differences in the kinds of thinking students engage in when playing VU or solving the MCR task. This could be important to instructors deciding how they might use either or both in their own instruction. To begin to explore these differences, we present findings from a qualitative interview study with the following research questions:

RQ 1: What strategies do students deploy in solving the Magic Carpet Ride task and in playing levels in the Vector Unknown game?

RQ 2: What patterns are apparent in the use of strategies across tasks?

Context and Background

Literature Review

The idea that games and puzzles are environments where people engage in mathematical thinking is not new. Puzzles like Sudoku and games like Chess and Go have been the objects of study for mathematicians over the years (Silva, 2011). Moreover, the use of video games for the teaching and learning of STEM topics has been a popular application of GBL given their computational nature, their ability to simulate complex situations, and the active engagement they demand (Klopfer & Thompson, 2020). Vector Unknown is one of the few video games that have specifically been developed for undergraduate linear algebra (Mauntel et al., 2021).

Drawing from K-12 literature on GBL in mathematics education, there are clear indications of potential positive outcomes for student learning. In their meta-analysis on GBL research in K-12 math education, Byun and Joung (2018) computed an average effect size of d = 0.37 from quantitative studies, indicating a small-to-moderate-sized positive effect on math learning outcomes. Even so, some quantitative (Beserra et al., 2014) and mixed method studies (Ke, 2008) comparing games to similar non-game active learning opportunities have found that games may not always offer additional advantage over other active learning activities in terms of learning outcomes. While the VU game (Mauntel et al., 2021, Mauntel et al., 2020, Mauntel et al., 2019) and the MCR task (Wawro et al., 2012; Wawro, Rasmussen, et al., 2013) have each been the subject of several publications, no work thus far has compared student thinking in these tasks.

The Tasks

To properly contextualize the remainder of this paper, we present a brief summary of the two tasks being compared in this study below.

Magic Carpet Ride. The MCR task used in this study comes from the IOLA curriculum (Wawro et al., 2012; Wawro, Zandieh, et al., 2013) and is designed as a “day one” task as part of a larger “Magic Carpet Ride” unit that introduces concepts related to vectors, span, and linear (in)dependence. The day one MCR task asks students to determine if they can reach Old Man Gauss’s cabin at the point (107, 64) using two forms of transportation represented by vectors <3, 1> and <1, 2>. The next task in the unit asks students to consider whether there are “some locations that [Old Man Gauss] can hide and you cannot reach him with these two modes of transportation.” This task was used as a follow-up in some of the interviews as time allowed.

Vector Unknown. In VU, each level randomly generates a goal position (represented by a basket) and two pairs of vectors that are scalar multiples of each other (so one possible set of vectors is <-3, 2>, <-9, 6>, <1, 3>, and <2, 6>). Players then use any two of those vectors and integer scalars to get a rabbit from the origin to the goal. In this study, the current first three levels were used for the interviews. All participants played Levels One and Two, which work the same except for the Predictive Path feature. In Level One, as players choose their vectors and scalars, a Predictive Path line shows them where the rabbit will move when they hit “Go;” this feature is absent from Level Two. Level Three includes the Predictive Path and has an added component of a player first needing to get to three keys on the map and *then* go to the goal position. Completing Level Three usually requires the player to move from a location other than the origin after gathering some or all of the keys. See Figure 1 for an illustration of the game.

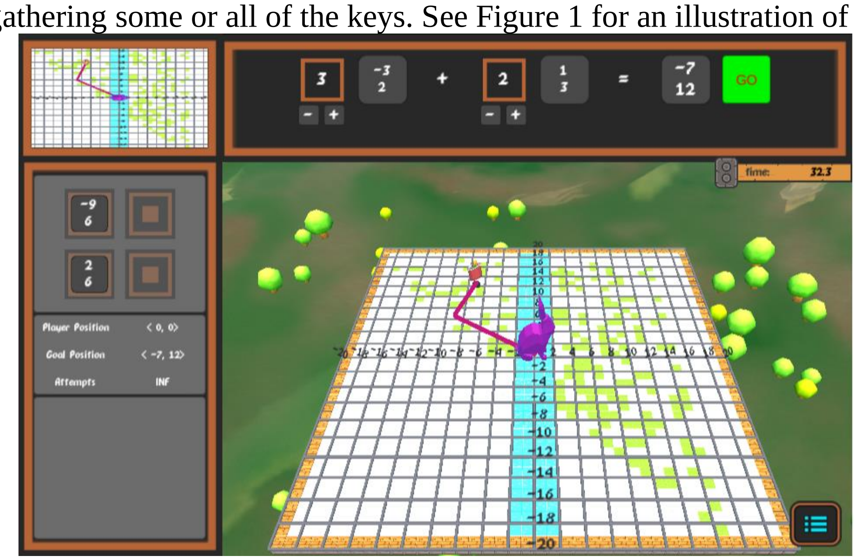

Figure 1. Gameplay of Vector Unknown.

Conceptual Framework

In answering our research questions, “strategy” needed to be operationalized. Because the purpose of this research was to compare two specific tasks, and because both of those tasks have published material to draw from, we used past work to develop a conceptual framework to operationalize strategy. This framework was based primarily on one developed with VU (Mauntel et al., 2021), supplemented by student sample work from the MCR task (Wawro et al., 2012; Wawro, Zandieh, et al., 2013), and further modified during the analysis process. Figures 2, 3, and 4 outline the conceptual framework.

Strategy	Mauntel et al. Description	Adaptations / Notes
Guess and Check	Player presses buttons while attending to how the vector equation changes or to how the geometric Predictive Path changes.	Name changed from “Button Pushing” to “Guess and Check,” to also include trying random scalars in the MCR task.
Quadrant-based Reasoning	Player chooses a vector to match the signs/quadrant of the goal position, references the direction of the Predicted Path or a quadrant on the graph to make sense of the direction of a vector, or understands vectors as slopes.	Slope-based strategies are the most common way this appears in the MCR task, as both the goal and given vectors have the same signs/quadrant.
Focus on one Coordinate	Player reduces the aim of the goal to one coordinate and attempts to reach that one coordinate.	In MCR, this occurs when the participant focuses on the North or East direction, one at a time.
Focus on one Vector	Player focuses on getting as close to the goal as possible with one vector and then utilizes another vector to reach the goal and/or alternates between the two.	In MCR, this occurs when the participant focuses on one of the modes of transport at a time.

Figure 2. Conceptual framework, part 1.

In past work with VU, Mauntel et al. (2021) used an iterative approach which sorted the strategies players used into four categories: Button-Pushing, Quadrant-based Reasoning, Focus on one Coordinate, and Focus on one Vector (see Figure 2). Additionally, each strategy participants used was classified as either Numeric or Geometric (see Figure 3), depending on whether the participant was relying on the numeric data (like the vector equation) or visual data (such as the predictive path) to solve the problem.

Strategy Type	Descriptors for Strategy Type
Numeric	Using arithmetic to solve or check a possible solution Referring to the vector equation (VU) or numeric values of vectors/goal
Algebraic	Setting up a system of equations Creating an equation for a line Creating symbols for unknowns
Geometric	Interpreting graphical information Drawing vectors or lines on a graph Using the Predictive Path (VU)

Figure 3. Conceptual framework, part 2.

Our review of student sample work for the MCR task (from Wawro et al., 2012 and Wawro, Zandieh, et al., 2013) lead to three adjustments to the framework. As a minor change, we renamed “Button-Pushing” to “Guess and Check.” More substantially, we noticed that algebraic solution strategies – strategies involving written equations with unknowns or variables – were more prominent in the MCR student sample work than in the VU research. To address this, we first added a fifth strategy: “System of Equations,” (see Figure 4), to specifically categorize the use of a system of linear equations. Second, Algebraic strategies were separated out from Numeric and Geometric as a third category for any strategy where the participant employed an algebraic equation or expression to attempt to solve the problem (see Figure 3).

Strategy	**Description**	**Basis for Addition**
System of Equations	Player creates a system of equations with two unknowns and then solves it to solve the problem.	Review of MCR student sample work (Wawro, Zandieh, et al., 2013).
Linearly Independent Vector Selection	Player chooses two vectors based on which pairs of vectors are scalar multiples of each other.	Strategy observed during analysis that did not fit into existing conceptual framework.

Figure 4. Conceptual framework, part 3 (new strategies).

Finally, the conceptual framework was further revised during the analysis process. In particular, an additional novel solution strategy for VU of “Linearly Independent Vector Selection” was noted. This strategy will be described and explored in the findings section.

Methods

Data Collection

The participants for this study were five students recruited from a third-semester calculus course at a large public university in the southwestern United States. All five students who indicated interest in the study participated in task-based interviews. For the purposes of this study, no demographic data were collected. As such, all participants will be referred to with the gender-neutral pronoun ‘they’ and pseudonyms generated from a list of gender-neutral names (Van Fleet & Atwater, 1997). Participants were asked if they had ever taken a college-level linear algebra course, and only one participant (Chris) said they had.

Interviews were conducted via Zoom due to the COVID-19 pandemic. The components of these interviews focused on in the analysis presented here consisted of two approximately 30-minute task portions for each of the MCR and VU tasks. In these interviews, the interviewer primarily described the tasks to be completed and did not typically interrupt the participant’s solving process, unless they had not spoken for a long time or were nearing the end of the allotted time. The order of the task portions varied from interview to interview. Figure 5 lists the participants by pseudonym and shows the order they completed the two tasks in.

Participant	Rikaine	Pat	Terry	Auren	Chris
First Task	VU	VU	MCR	MCR	MCR
Second Task	MCR	MCR	VU	VU	VU

Figure 5. Participants and task order.

Data Analysis

The conceptual framework outlined earlier in this paper was applied as a codebook. The two authors developed the conceptual framework over a sequence of meetings, settling on the approach as described above. Then, the lead author coded the transcript by identifying each time a participant applied or attempted to apply one of the strategies outlined in that framework. Further, a student's strategy was stratified based on whether they were applying Numeric, Algebraic, or Geometric versions of that strategy. Once identified, the strategies used by each participant were collected together and reviewed for accuracy and clarity before identifying which strategies best characterized the participant's solving process for each respective task. For MCR, this meant identifying which strategy was ultimately used to arrive at a solution, if any. For VU, where participants completed multiple levels, this meant identifying which strategy was used most frequently to arrive at solutions.

Findings

Our findings are oriented around our two research questions. Given space limitations, the first subsection addresses the first research question by summarizing the strategies students deployed in the MCR task and VU. The second subsection focuses on the novel strategy of Linearly Independent Vector Selection, as previous literature (Mauntel et al., 2021; Wawro et al., 2012) showcases detailed examples of the other strategies. Then, the two subsequent subsections address the second research question, by articulating two notable patterns apparent in the strategies used: the relative prevalence of Algebraic, Geometric, and Numeric strategies and the repetition of strategies across the tasks.

RQ 1 – Overall Distribution of Strategies

To begin to address RQ 1, we use Figure 6 to visually provide a summary of the diverse set of strategies that were used by the sample of participants. Within each box, an A, G, or N represents that the participant used an Algebraic, Geometric, or Numeric instantiation of that strategy, respectively. Bolded letters indicate the strategies that best characterized their performance on that task as defined in the Data Analysis section above.

Participant	Rikaine		Pat		Terry		Auren		Chris	
Task	VU	MCR	VU	MCR	MCR	VU	MCR	VU	MCR	VU
Guess and Check	G	N	G			G				
Quadrant-based Reasoning	G, N	**G, N**			A, N	**G, N**		**N, G**		
Focus on one Coordinate			**G, N**	N			N			N, G
Focus on one Vector	**G, N**	N	N		N					
System of Equations				**A**			**A**		**A**	**A**
Linearly Independent Vector Selection	N		N					N		N

Figure 6. Summary of strategies used by each participant

RQ 1 - Linearly Independent Vector Selection

As mentioned in the conceptual framework, a novel strategy for VU was observed in the analysis process. In four of the interviews, participants chose vectors based on the observation that some of the vectors were scalar multiples of each other. This strategy is perhaps best summarized by Chris's explanation [the vectors referred to are in square brackets]:

Chris: So what I'm thinking is that we can, uh, look for. First, identify any same vectors, any vectors that are linearly dependent, and I can already tell that the top right one [<-1, 5>] here and the bottom left one [<-2, 10>] here are linearly dependent, so they're the same vector, and then the also the other ones, the top left [<1, 1>] and bottom right [<-9, -9>] are linearly dependent so they're the same vector so that means that I can just use these top two as those are the only truly independent vectors available to me.

As Chris was the only participant who had taken linear algebra previously, they were the only participant to describe this strategy in terms of linear independence. Other participants typically only noted that pairs of vectors were scalar multiples of each other, such as in this excerpt from Rikaine:

Rikaine: Um and then, since these are, since these point in the same direction, I'm only really considering this one [<-3, -1>]. Just because I can scale it up to <-9, -3> if I need to.

This strategy is only implementable in VU (notice it is only present in VU columns, bottom row of Figure 6), as MCR has only two choices for modes of transport. Another notable aspect of this strategy was that, in two of the four cases where participants made use of this fact, this only occurred after additional interviewer questioning led them to observe that there were always pairs of vectors which were scalar multiples of each other. Thus, only one player (Rikaine) made an observation and selected vectors in this way without prior prompting or prior linear algebra experience.

RQ 2 - Algebraic, Geometric, and Numeric Strategies

Across participants, Geometric and Numeric instantiations of strategies were both common for VU (see the VU columns on Figure 6). Often, Geometric thinking was more apparent when the level had the Predictive Path feature. Only Chris used an Algebraic solution in VU. After completing the first level using a Focus on One Coordinate Strategy, and toying around with strategies in the second level, Chris reluctantly decided to pursue an Algebraic solution:

Chris: Ehhh, I wanted to avoid the algebra, but I think I'm going to have to use the algebra [laughs]. Going back to the method, okay setting up a two-by-two matrix.

They made this choice after having already solved the MCR task using a system of equations and an augmented matrix and having previously taken linear algebra at the college level.

On the other hand, MCR was mostly solved with Algebraic strategies, with Rikaine solving it primarily Numerically and Terry being unable to complete the task in the time allotted. This is not to say that Geometric thinking did not appear at all in solving the task – all of the participants drew a graph at some point. Some did so to set up their coordinate system and/or to visualize the problem, while a few did so after interviewer prompting to illustrate their solution. In these cases, however, drawing the graph was not directly linked to any of the solution strategies in our conceptual framework, and participants did not use these graphs as tools throughout the problem solving process.

RQ 2 - Repetition of Strategies

Most of the participants would at least attempt the same strategy in both the MCR task and VU. This can be seen by comparing the MCR and VU columns for each participant in Figure 6. For example, in the following excerpts, Pat attempts to apply the Focus on one Coordinate strategy first in VU and then in MCR:

Pat [During VU]: How to get to -7 with my x.…That [mousing over <-1, 3>] will at least get me to negative seven x.

Pat [During MCR]: I wonder if it's better to get to 64 first. So if it's 32 hours by magic carpet, that's the point (32, 64). Which would arrive me at the y, the y, the y value of his house.

In each case, Pat attempts to match one of the coordinates first. However, while they carried this strategy through to completion with VU, they ultimately chose to solve the MCR task with a

System of Equations. The only participant who did not at least attempt the same broad strategy across both environments was Auren. They relatively quickly settled on using a System of Equations on the MCR task, which they completed first. When they played VU, they did not ever consider this strategy, preferring instead to use Quadrant-based Reasoning.

Discussion

The findings above reflect the different affordances for students that VU and the MCR task offer. The fact that participants tended to try the same strategy for both tasks but did not tend to complete both tasks with the same strategy particularly supports this conclusion. This suggests that the two tasks are different enough that students may decide a strategy that worked in VU is not the best strategy for the MCR task, thus engaging them in different ways of thinking.

Two clear ways VU differs from the MCR task are in the offloading of computation and the limited magnitude of goal positions. Because the vector equation at the top of VU automatically updates whenever you change a scalar, it offloads computational work. The addition of the Predictive Path in some levels offloads even more work, as it allows the player to see both the Numeric value of the result and the path it takes on the coordinate plane to get there. In addition, goal positions have their x and y coordinates each somewhere between -20 and 20. This means that the goal position is relatively small in magnitude, particularly when compared to the goal position of the MCR task, (107, 64). These differences in features may be related to the differences in how students use Algebraic, Numeric, and Geometric thinking across the two tasks. Because the MCR task involves working with larger numbers, and because all computation and graphing is left to the student, students may be more inclined to think algebraically to avoid having to do a lot of computations or draw a precise graph. Conversely, because VU handles many computations for the player, asks the player to work with smaller numbers, and shows the player information on an already-made graph, it can be easier to engage in Numeric and Geometric thinking while solving this task. While each individual task may better support different kinds of student thinking, using them in conjunction with one another may support students using all three of these kinds of thinking.

Another difference is that players have a surplus of vectors to choose from in VU, with the four vectors available consisting of two pairs of vectors that are scalar multiples of each other. Thus, students not only have choice in what vectors they use, they also may be able to make observations about linear independence and dependence through playing levels of the game. We saw this with the strategy of Linearly Independent Vector Selection. In comparison, the first task in the MCR unit only includes two linearly independent vectors. However, the rest of the MCR unit does lead students toward considerations of linear independence and dependence. In addition, simply having students play VU may not automatically lead to students having any insight about linear independence. We can see this in the fact that three of the students who used this strategy only did so after additional interviewer questions. This suggests an important caveat: it is not just the design but also the implementation of the tasks that matters. Instructors who subtly or not-so-subtly prompt students to look closer at the vectors that are available to them may be able to scaffold these kinds of observations for students.

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Students' Written Homework Responses using Digital Games in Inquiry-Oriented Linear Algebra

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In this report, we characterize seven of twenty-five students' responses to a single written homework assignment from the Spring 2021 academic semester. The homework was designed to incorporate the Vector Unknown 2D digital game to investigate how students answered questions about span and linear independence after playing various levels of the game. We present our modification of the roles and characteristics framework of Zandieh et al. (2019), our identification of students' grammatical use of game language and math language, as well as the results of analyzing students' homework responses using our framework.

Keywords: Linear Algebra, Inquiry-Oriented Instruction, Game-Based Learning

The teaching and learning of linear algebra are important due to their prevalence in many STEM disciplines. One of the goals of our team was to develop a sequence of game-based written homework assignments that paralleled the course progression of the Inquiry-Oriented Linear Algebra (IOLA) curriculum (Wawro, Zandieh, et al., 2013) for use in any linear algebra classroom. Here we build on past research stemming from the IOLA curriculum that uses Realistic Mathematics Education (RME) constructs to inform the design of curricular materials by iterating research, design, and implementation (Gravemeijer, 1999; Wawro, Rasmussen, et al., 2012).

Figure 1. Two screenshots of the Vector Unknown game for Level 3 (left) and Level 4 (right).

Vector Unknown (VU) introduces novice linear algebra learners to beginning topics such as linear combinations of vectors and span. Players move a bunny through a level to collect baskets and keys by dragging vectors and setting scalar values to create linear combination equations as seen in Figure 1. VU draws heavily on the use of the travel metaphor that is leveraged in the IOLA curriculum during the Magic Carpet Ride (MCR) task sequence (Plaxco & Wawro, 2015; Wawro et al., 2012). We extend the use of VU by augmenting its instructional use with a set of homework assignments. Since this was our first attempt at designing this set of game-based homework, we wanted to understand how our students thought about span and linear (in)dependence after playing several levels of VU. To accomplish our goal, we analyzed students' responses and gameplay screenshots from their second written homework assignment.

Research question: How did students use the Vector Unknown 2D game in their responses to questions about span and linear independence in their written homework?

Background Literature and Theoretical Perspective

Stewart, Andrews-Larsen, and Zandieh (2019) provided a thorough overview of the current state of the research literature on the teaching and learning of Linear Algebra ideas. Stewart and colleagues synthesized 54 research articles across various themes, including span and linear (in)dependence, that are relevant concepts within most Linear Algebra curricula. We use the literature review of Stewart et al. to inform our own.

Span and Linear (In)dependence

Plaxco and Wawro (2015) determined that travel is a consistent metaphor that students use to reason about span in the context of the IOLA curriculum. Adiredja and Zandieh investigated students' everyday examples of basis using an anti-deficit perspective (Adiredja & Zandieh, 2017; 2020). Extending their initial framework for students' example generation, Zandieh et al. (2019) determined that travel was a ubiquitous metaphor for the notion of basis even though the students had never encountered the IOLA curriculum. There have been several studies that analyzed how students think about linear (in)dependence: Ertekin et al. (2010) discussed student thinking about linear independence and dependence in geometric contexts, Hannah et al. (2013; 2016) and Stewart and Thompson (2010) made connections between students' embodied ways of reasoning about linear independence and dependence even when the students were unable to produce a formal definition. Rasmussen et al. (2015) analyzed video data to investigate students' progressive mathematization of examples of linearly (in)dependent vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Most relevant to our work is a similar study by Dogan-Dunlap (2010) who analyzed students' homework responses with an online module to learn more about the various ways students think about linear (in)dependence. When using the geometric web module, students' responses were characterized by Dogan-Dunlap across 17 categories, 11 of which were geometric. It could be the case that using manipulatives and educational video games support students in forming geometric interpretations of span and linear (in)dependence. However, no study has considered how students think about span and linear independence by analyzing students' written homework responses after playing a research-designed video game.

Game-Based Learning

The research team that developed the game VU incorporated Game-based learning (GBL) research as a lens to better understand the instructional affordances and limitations of using video games for teaching mathematical ideas (Coleman & Money, 2020; Gresalfi & Barnes, 2016). There has not been a lot of research leveraging games within higher-level mathematics courses. As such, our research adds to the burgeoning area of GBL within the IOLA curriculum (Mauntel et al., 2021). Digital Game-Based Learning (DGBL) has gained popularity in the K-12 literature over the past 20 years, "...DGBL in mathematics education can be regarded as the use of digital games within the context of learning mathematics," (Byun & Joung, 2018, p. 114). Other researchers within the IOLA research group have investigated the development of students' gameplay strategies using VU (Mauntel, Levine, et al., 2020; Mauntel et al., 2021). We agree that playing a well-designed video game does not imply students are mathematizing anything, "...a young child counting blocks, may...be experiencing mathematical play; whereas a middle school student who is being forced to play a very well-designed mathematics game in the classroom may not," (Williams-Pierce, 2019, p. 592). We view our design of written homework as a structured set of questions intended to support students in forming and testing conjectures about the game to engage in mathematical play while playing VU.

Roles and Characteristics of Basis Vectors Framework

We view the framework of Zandieh, Adiredja, and Knapp (2019) as viable to analyze the spontaneous game examples given by students in their written homework. Zandieh et al. built upon their previous framework of *roles* and *characteristics* codes to understand how students thought about basis through everyday example generation. Zandieh and colleagues used *roles* codes to describe the role that a set of basis vectors has relative to the ambient vector space. Zandieh and colleagues' *characteristics* codes described important conditions of a set of vectors. We did not explicitly ask students to generate examples using VU on their written homework. However, students' spontaneous use of game examples included descriptions of the roles and characteristics of the list of vectors given to the player in VU. We later elaborate on our alteration of the roles and characteristics of basis vectors framework of Zandieh, Adiredja, and Knapp (2019).

Methods

We designed a single written homework assignment to be used in the third author's IOLA class during the Fall 2020 semester. After reviewing student responses and meeting weekly, we refined the single homework assignment and wrote four new assignments during the Spring 2021 semester. Out of the forty-two students enrolled in the third author's IOLA class during Spring 2021, we collected the written homework responses of about twenty-five students (not every student completed each assignment). We analyzed the responses of seven out of the twenty-five students that submitted the second homework assignment due to their use of spontaneous game examples from VU in their response to the first question pictured in Fig. 2: Adrian (chose not to identify), Kieran (chose not to identify), Brooks (White man), Michael (White man), Jing (Asian/Asian American man), Tanner (White man), and Aahan (Asian/Asian American man).

The second homework was written with two mathematical concepts in mind: span and linear (in)dependence. In both the first and second written homework assignment, we explicitly used the word span, though we did not explicitly mention linear (in)dependence in either assignment. Students had encountered the IOLA lessons covering both ideas (Magic Carpet Task and Return to Home Task) by the time they received the second written homework assignment. Students were asked to play Levels 3, 4, and 5 on the hard difficulty of VU multiple times. Level 3 of VU includes four game objects (collect three virtual keys then travel to the location of the lock), Level 4 includes three collinear baskets where the line passes through the origin, and Level 5 includes three collinear baskets where the line of the baskets does not pass through the origin.

5.) (2 points) If you remove one of the vectors from your list in number 3) would the span of this new list be the same as what you marked in number 4)? Explain why or why not. Include in your answer whether it matters which vector you removed.
7.) (Extra Credit) Reflect on the similarities and differences between your answers in the Level 4 column and the Level 5 column. Make connections between as many of the boxes on your answer sheet as you want to receive up to 5 possible extra credit points.

Figure 2. Statement of the problems Q1 and Q2 from the second written homework

We conducted our first pass at data analysis of the second homework by open-coding every student's responses to each question to characterize the nature of the data and identify trends and themes (Strauss & Corbin, 1990). During this initial pass, we noted some students appeared to make connections between their understanding of span, linear independence and the explanations of their gameplay. While there were some selected response questions in the homework, we focused on the free response questions to better characterize students' connections. The framework of Zandieh et al. (2019) proved useful once we determined the subset of seven students who provided explanations using gameplay examples. To focus on how the students were using the game to reason about Linear Algebra concepts, we used the code *game language* to focus on pieces of the data where this might occur. An excerpt was originally coded as *game language* if the excerpt mentioned the student's actual gameplay, mechanics of the game, features of the game, or limitations of the game. For example, Adrian discussed increasing the distance traveled by the bunny in game language as opposed to saying something about increasing the value of the scalar multiple of said vector:

> "Therefore, it would likely only affect the first **movement** by increasing the distance **traveled** by the **bunny** in those two directions," -Adrian.

After determining the seven students that used game language to respond to Level 3 Question 5 (Q1), we used our modified framework of Zandieh et al. (2019) to code the seven students' responses to Q1 and Level 4 and 5 Question 7 (Q2) using our span and independence codes framework pictured in Figure 3. We will now focus our discussion primarily on the responses of the seven students who used game language in their responses to Q1.

Findings

Our findings include (1) students' mixed use of game and math language when responding to Q1 and Q2, (2) our modification of the Zandieh et al. (2019) basis framework, and (3) the results of using our frameworks to analyze students' responses to Q1 and Q2. In Figure 3, we present our span and independence framework. For each code, a portion of a students' response is given as an example to the reader. In Figure 4, we present the results of coding each student's response to Q1 and Q2 using our modified framework. In the column for students' responses to Q2, we separate the codes using semicolons if the student broke their response into paragraphs or sections.

Students' use of Game Language and Math Language After we determined some students responded to the homework using spontaneous examples from their gameplay, we wanted to know how some students used game examples in their open-ended homework responses to Q1 (25 responses) and Q2 (19 responses). We created various lists of words that would fall under *game language* usage (e.g. "bunny", "baskets", "route") and *math language* usage (e.g. "span", "plane", "magnitude"). We added a *mixed language* category for words that could be considered relevant to the game, but weren't always used to refer to one of the elements of the game (e.g. "vector", "line", "scalar"). We coded the second homework assignment itself and every students' responses to Q1 and Q2 at a word-for-word grain of analysis using our lists. We wanted to know if any students used *game language* when responding to Q1. We thought it would be interesting if a student responded primarily in game language to Q1, which was phrased primarily in mixed and math language. While many students' responses to Q1 involved math and mixed language use (69 instances of math language across 23 responses; 86 instances of mixed language across 22 responses), our seven research subjects were the only ones who responded to Q1 using game language. Out of the twenty-five responses to Q1, there were only 18 instances of game language across 7 out of the 25 responses. The data presented here is from the written responses to Q1 and Q2 of these seven students.

The Span and Independence Framework Zandieh et al. (2019) used *roles* codes to describe students' examples relative to how the basis vectors relate to the ambient vector space. In our modification of their framework, we instead use *span* codes to characterize students' descriptions of how vectors or objects relate to the span of said vector set. Further, Zandieh and colleagues used *characteristics* codes to describe students' examples based on the conditions that a set of basis vectors has. We instead use *independence* codes to characterize students' descriptions of the conditions of a set of vectors or objects. We view our use of *span* and *independence* codes as highly overlapping with the use of *roles* and *characteristics* codes. We present a description of the *span* and *independence* codes in Figure 3 with some short examples. Figure 3 is altered from the framework used by Zandieh et al. (2019) with the addition of our own restricting code, modification of three codes, and exclusion of two codes. There are some codes in Figure 3 whose names have been changed to better describe our data (i.e., restricting, redundancy, and sameness). If we did not find evidence of a particular code in our students' written responses, we did not include it in Figure 3 (i.e., systematic and structuring).

Span Codes	Information	Independence	Information
Generating (Gen)	Creating a line or plane through combining vectors or game objects. E.g., The spans being used **form** a plane; you could **form** a plane using the 4 vectors provided	Minimal (Min)	The set being the smallest or having the least number of vectors to fulfill the role in space. E.g., to accomplish the task with **smallest number of vectors**
Covering (Cov)	Filling or encompassing the space. E.g., The span must be a plane because all vector combinations **lie within** a certain range	Redundancy (Redun)	Having extraneous objects in the set. E.g., If one of the two vectors was **removed**, it likely **wouldn't have made much of a difference**
Traveling (Trav)	Moving (themselves or the bunny) through the game space or along a path. E.g., you can **reach** anywhere within R2 with no problem.	Sameness (Same)	Vectors in the set or objects related to the vectors (direction, parallel lines, etc.) are the same. E.g., The vectors <-1,5> and <-3,15> **represent the same line**
Restricting (Restr)	Certain conditions keep the span of a set of vectors from having a higher or lower dimension. E.g., There is also no z point in the vectors allowing 3D traversal, **limiting** it to a plane	Essential (Ess)	The critical importance of including a vector in the set or using a vector while playing the game. E.g., Two [vectors] **were needed** instead of one

Figure 3. Table altered from Zandieh et al. (2019)

Results from Span and Independence Coding We used our modified *span* and *independence* codes as a codebook to produce the set of codes for each student's response to Q1 and Q2 as shown in Figure 4. Looking across columns it can be shown that Q1 and Q2 elicited different sorts of responses from our research subjects. For example, there are far more uses of the essential code for Q2 than Q1. Looking across the rows of Figure 4, it can be seen that students emphasized different aspects of the roles or characteristics of the set of travel vectors provided in Levels 3 through 5 in VU.

Student Name	Q1: Level 3 Question 5 Response	Q2: Level 4/5 Question 7 Response
Adrian	Redun, Same, Trav	Ess, Gen, Ess, Trav; Trav, Ess, Trav; Trav
Kieran	Restr, Trav, Redun	Gen; Restr, Cov; Cov, Ess
Brooks	Cov, Trav, Restr	Trav, Min; Ess, Trav, Redun, Ess, Restr
Michael	Trav, Cov	Ess, Gen; Ess, Trav, Cov
Jing	Redun, Trav, Same	Ess, Cov, Ess, Same; Min, Gen, Ess, Trav; Gen, Cov, Same, Trav; Gen, Ess, Trav, Ess, Gen, Min, Trav, Gen, Same
Tanner	Restr, Trav, Gen	Gen; Gen, Ess
Aahan	Same, Trav, Redun, Trav	No response

Figure 4. The results of coding students' responses to Q1 and Q2

To demonstrate how we distinguished between students' responses using game language, consider the following portion of Adrian's response. For clarity, game language will be *italicized*, math language will be underlined, and mixed language will be *both* italicized and underlined.

> "Therefore, in order to reach all of the *goals*, the *bunny* needed to use a *vector* to get into *position* before using a second *vector* to *follow* the *linear* *path* towards each of the *goals*," -Adrian.

This excerpt was chosen to show a response that primarily used *game language*. Adrian claimed that an additional vector was needed from the perspective of the bunny, which was an indicator of *game language*. Further, to reach the goals (i.e., the baskets), Adrian claimed the bunny needed to first "get into position" to collect the baskets in Level 5. This need to "get into position" was related to the setup for Level 5 of VU (i.e., the line of the baskets did not pass through the origin).

To demonstrate how we used our *span* and *independence* codes, consider the following portion of Jing's response. For clarity, each indication of a *span* or *independence* code will be **bolded**. We coded Jing's response as Trav, Ess, Gen:

> "In level 5, I need to **reach** (Trav) the point of the line and that will **need an extra vector** (Ess) to make it happen, and it will **form** (Gen) a plane," -Jing.

This excerpt was chosen to show a common pairing of codes, *essential* and *traveling* as well as a response given primarily in *math language*. The reader may note the similarity in this portion of Adrian and Jing's responses to Q2 about their Level 5 gameplay in that both excerpts consist of an *essential* and *traveling* pairing of codes.

Discussion and Conclusion

We begin our discussion by first comparing students' span and independence codes across their responses to Q1 and Q2, pictured in Figure 4. We will also discuss trends in pairings of codes that emerged in students' responses similarly to the work of Zandieh et al. (2019).

In Figure 4, we found that we had assigned more students the *generating* code in Q2 than Q1. It may be the case that asking students to compare the span of their vector list in Level 4 and Level 5 led to more examples indicative of creating and forming lines and planes. This makes sense based on the way we asked Q1. Our emphasis was on having students mentally compare the span of their vector list in Level 3 and then comparing that to the span of a new vector list with one fewer vector. Students were assigned the *redundancy* code more times in their responses to Q1 than Q2. Since students were thinking about whether the span of their initial vector set was different than the span of the new vector set (with one fewer vector), this may have contributed to students' responses being characteristic of having extraneous objects in their imagined vector set. We see far fewer *sameness* and *redundancy* codes in general across students' responses to Q2, which also makes some sense. Instead of being asked about the effects of adding or removing a vector from a particular set, students were instead asked about whether the span of the set of vectors they used to win the level was a line (Level 4) or a plane (Level 5).

The *essential* and *traveling* code pairing was the most common pairing in students' responses to the Q2 which asked the students to compare Level 4 and 5. The major difference in these two levels is whether the baskets lie along a line that goes through the origin (Level 4) or not (Level 5). In the *essential/traveling* pairing, students were describing the need for an extra vector to "get to" the line or "detour" to the line containing the baskets in Level 5. The inclusion of the "offset" vector in Level 5 changed the span of the set of vectors needed to win Level 4 from a line to a plane (five out of the seven correctly answered this question). Students' use of gameplay language to compare Levels 4 and 5 may be indicative of the students' explicating an intuitive need for an offset vector, while still working towards more formal mathematization.

It is interesting that many students' intuitive notion about traveling on the offset vector, as indicated by the prevalence of the essential and traveling code pairing, did not appear to translate to more formal mathematization when writing the parametric equation for the line of the baskets in Level 5. It may be that thinking about "traveling to the line of the baskets" is a useful travel metaphor to support students' in transitioning towards more formal ways of reasoning. For example, we view the need to first travel to the line of baskets before collecting them as an informal way of reasoning that foreshadows a transition towards symbolizing the parametric vector equation for the line of the baskets in Level 5. Students were asked to write the vector equation for the line of the baskets on their homework, but only Michael, out of our seven research subjects, produced an equation of the form we had in mind (i.e., $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + a\begin{bmatrix} 1 \\ 2 \end{bmatrix}$).

Our results add to the body of knowledge about student thinking about span and linear (in)dependence in the context of the IOLA curriculum. Specifically, our contribution is through incorporating the use of a research-designed game and written homework assignments to support students in thinking about the notions of span and linear independence while playing the game. The results of this study demonstrate the ability of the VU game, when accompanied with a written homework assignment, to evoke different ways of thinking about span and linear independence in an experientially real context. We see this as an extension of some of the other work using VU in the context of IOLA (Mauntel et al., 2021).

Further, some students adopted the perspective of the bunny when responding to the homework questions. Students adopting the perspective of the bunny seems to be related to Serbin, Wawro, and Storms' notions of *physics-centered* and *physicist-centered* grammatical constructions (Serbin, Wawro, & Storms, 2021). For example, a student may give an explanation that includes a description of a person within the span of a set of vectors (*physicist-centered*). We conjecture that students' use of *game language* may be connected to *physicist-centered* language, and *math language* use may be extremely similar to the notion of *physics-centered* grammar. Future work may investigate the nature of students' use of *game language* in conjunction with students' use of "bunny-centered" language when discussing their responses to the written homework.

Since we collected all five written homework assignments as data from every student in the third author's IOLA classroom during the Spring 2021 semester, we intend to expand our current analysis across all five homework assignments. Further, we intend to use the results of this study to motivate further revision of the game-based written homework assignments in the context of the IOLA curriculum.

Limitations and Acknowledgements

We acknowledge that this data was collected from a single IOLA course at a large University in the United States. This likely impacted the data we collected. The hope would be to eventually scale these homework assignments to more institutions using the IOLA curriculum as motivation to incorporate the Vector Unknown 2D and 3D game into their classroom.

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Students Facing and Handling Challenges in Programming-Based Mathematics Inquiry Projects

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In this paper, we present an empirical study examining challenges undergraduate mathematics students (N=73) encounter when engaging in programming-based mathematics inquiry projects, and how they handle these challenges. Results suggest that a majority of students find the programming or the programming of mathematics the most challenging aspect of their engagement, for different reasons. Also, with more experience engaging in such projects, it seems students may become more able to handle their challenges in an independent manner (e.g., through reviewing concepts, persevering, or planning).

Keywords: Computing; Math Inquiry; Challenges; Student Learning; Instrumental Genesis.

There has been a growing interest in studying and implementing innovative approaches in undergraduate mathematics, such as inquiry-based mathematics education (Artigue & Blomhøj, 2013; Laursen & Rasmussen, 2019), in which students are invited to engage in the practices of professional mathematicians. There is also a recent push to integrate computer programming – or more broadly, computational thinking – in different subject areas and at all levels of education (Guzdial, 2019; Wiebe et al., 2020; Wing, 2014); though the potential of integrating programming in mathematics learning has been known for a long time (diSessa, 2018; Papert, 1980), including at the undergraduate level (e.g., Leron & Dubinsky, 1995; Wilensky, 1995). In particular, programming can support a certain inquiry-based approach, where students engage in computational practices used by some professional mathematicians (Weintrop et al., 2016).

Some undergraduate mathematics curricula have integrated programming in this sense. For example, at Carroll College in the United States, mathematics majors use programming as a problem-solving tool throughout their mandatory coursework (calculus, linear algebra, modelling, abstract algebra, etc.) and may eventually apply their programming skills in senior projects or theses (Cline et al., 2020). Another example, in the Canadian context, is a sequence of three courses called Mathematics Integrated with Computers and Applications (MICA), which have been implemented at Brock University since 2001. Throughout these courses, students engage in a sequence of 14 projects (including end-of-course projects on student selected topics), in which they design, program, and use interactive computer environments to investigate mathematics concepts, conjectures, theorems, or real-world situations (Buteau et al., 2015).

In their recent “Call for Research that Explores Relationships between Computing and Mathematical Thinking and Activity,” to the international RUME community, Lockwood and Mørken (2021) argue that “serious consideration of machine-based computing [including programming] is largely absent from much of our research in undergraduate mathematics education” (p. 2). They suggest that much more needs to be investigated, “including identifying and exploring potential benefits, affordances, challenges, and problematic issues” (ibid.). They also point out that the various approaches to integrating computing in university mathematics classrooms should provide opportunities for research. In our 5-year research study, we address the above gap, using the opportunity provided by the natural MICA environment. In particular, we examine the teaching and learning of using programming for engaging in pure or applied mathematical inquiry. As part of our study, we seek to better understand the challenges students face during their learning, which we see as potentially providing insights for implementation. In this paper, we present some initial findings addressing the following research questions: What challenges do undergraduate mathematics students encounter when engaging in programming-based mathematics inquiry projects? How do students explain their challenges? How do they handle their challenges? Are there any differences among different demographics of students?

Theoretical Framework

In our work, we frame students' engagement in programming-based mathematics inquiry projects using a development-process model (dp-model, Figure 1) proposed by Buteau and Muller (2010). According to the model, students' engagement involves different steps, which arise in a dynamic, non-linear fashion. For example, at Step 3, students design and program an "object" (or interactive environment) they will use for their inquiry, and this may occur in a cyclic manner with Step 4 (verification and validation of the programmed math). This model was developed through an analysis of MICA projects, a literature review (Marshall & Buteau, 2014), and analyses of student data (Buteau, Gueudet, et al. 2019). It has also been argued to align with the programming-based practices mathematicians use to conduct research – i.e., inquiry (Buteau, Gueudet, et al. 2019; based on Broley, 2015). Balt and Buteau (2020) provide a 5-minute video illustrating the model in the context of two selected pure and applied student inquiry projects.

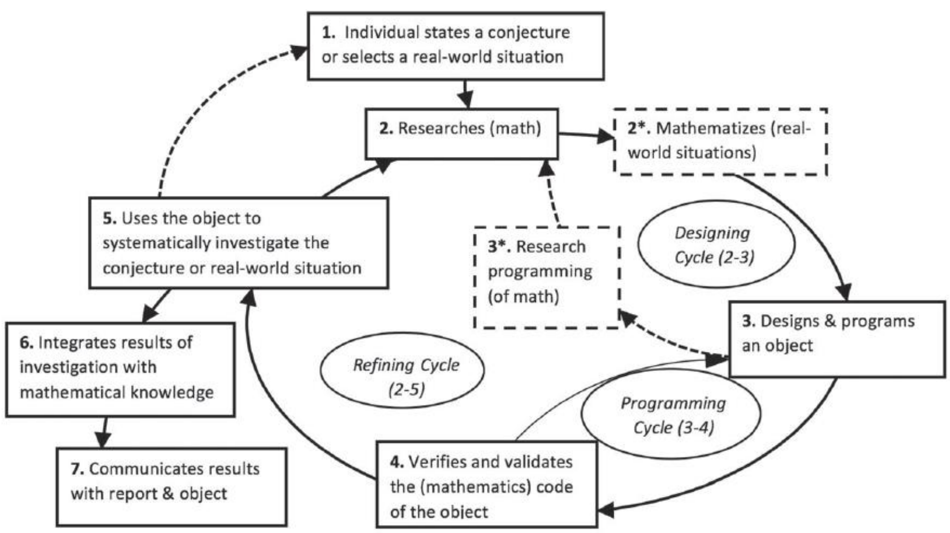


Figure 1. A model of students' engagement in mathematics inquiry projects (Buteau, Gueudet, et al., 2019, p. 6).

We further frame students' learning in the above model using the instrumental approach (Rabardel, 1995, 2002), whereby programming is an artefact (a human product) that may be transformed into a meaningful instrument (e.g., for conducting mathematical inquiry). This transformation – called instrumental genesis – involves the development of schemes (Vergnaud, 1998), including stable strategies (and principles underlying strategies) that enable the student to effectively achieve a goal (e.g., articulating a mathematical process in a programming language, as part of step 3 in the dp-model; Buteau, Gueudet, et al., 2019). In our case, a student's instrumental genesis involves the development of a web of schemes (Buteau et al., 2020), i.e., interrelated schemes found at different steps of the dp-model.

Instrumental genesis is a complex process that can be challenging for students (Laborde, 2002). In our research, we have been documenting individual students' instrumental geneses through the sequence of inquiry projects offered in the MICA courses mentioned above, including mentioning some challenges that arose for those few individuals, in certain projects (Buteau, Gueudet, et al., 2019; Buteau, Muller, et al., 2019; Gueudet et al., 2020). In this paper, we move away from this in-depth approach and move to a wider student population to examine, more systematically, challenges (i.e., aspects of students' engagement that appear to cause the most issues, in terms of schemes, steps from the dp-model, or more general elements), and how students mostly handle these challenges (i.e., to move their mathematical inquiry forward).

Methods

As mentioned above, we work in the context of a sequence of three programming-based mathematics courses, MICA I-II-III, offered at Brock University, which engage mathematics majors and future mathematics teachers in 14 inquiry mathematics projects. Our study is part of a larger five-year naturalistic (i.e., not design-based) research aiming at understanding how students learn to use programming for authentic mathematical investigations, if and how their use is sustained over time, and how instructors support that learning.

As part of Years 2-4 of this research, all MICA students were invited to respond to an online questionnaire (~12-15 minutes long) at the end of their course (MICA I, II, or III). Participation was voluntary. Table 1 shows some relevant demographic information for the 73 participants. The questionnaire contained different sections: demographics; confidence in programming (for mathematics investigations); usefulness of programming; etc. In this paper, we focus on the questions asking 'what' students found most challenging in MICA projects, 'why', and 'how' they mostly handled it. Participants provided short written responses to each question.

Table 1. Information about participants that will be used in this paper.

Demographics				
Gender	MICA I	MICA II	MICA III	**Total**
Female	25	14	9	**48**
Male	14	7	4	**25**
Total	**39**	**21**	**13**	**73**

Each 'what', 'why', and 'how' response was coded, first individually (among 2-5 coders), and then through consolidation (which led to lists of 'codes'). Codes for the 'what' question were first grouped into 'themes'. Since each participant's responses to the 'why' and 'how' questions elaborated on their 'what' response, these codes were sorted first into groups by the 'what' themes, and then codes in each group were looked at and grouped into themes. This thematic regrouping was consolidated by 2 coders. Finally, we created different graphical representations of the frequency distribution of the results, including in relation to different demographic groups. We then interpreted our results using the theoretical frame elaborated in the previous section.

Since participation was voluntary, we cannot claim that our sample is representative of all MICA students. This is a limitation of our study.

Results

We present selected results in four sections aligning with our four research questions.

What students find most challenging

Our analysis led to eight themes, with distribution across participants given in Figure 2.

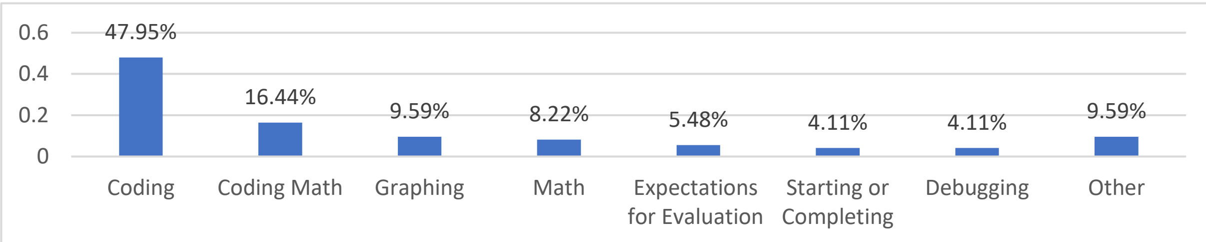


Figure 2. What participants (N=73) found most challenging about the MICA inquiry projects.

The part of the programming-based mathematics inquiry that most participants found most challenging was *coding* (e.g., "anything that requires a loop, an array, or long segments of code"), followed by *coding mathematical processes, including translating mathematics to code* (e.g., "making the computer do the math"). Both of these relate to students' (engagement with the) activity in steps 3-4 of the dp-model (i.e., the programming cycle). The other most challenging parts were mentioned in smaller proportion (at most 10%). Some are related to the programming cycle, such as *graphing and graphics*[1] (e.g., "getting graphs and picture boxes to do things") and *debugging* (e.g., "overcoming a roadblock or mistake in the coding"), making the issues at the programming cycle the most dominant ones (78% in total). All other challenges appear to relate to more general aspects of students' engagement (i.e., not necessarily related to a specific step in the dp-model): the *mathematics* (e.g., "understanding the math components"), *starting or completing the inquiry projects* (e.g., "at the beginning and setting up the program"), *expectations for evaluation* (e.g., "understanding what the professor expects"), and *other responses* such as "the imaginative process."

When reviewing participants' responses to both the 'what' and 'why' questions, we find that some of the latter challenges may be related to several steps in the dp-model. For example, the student mentioning the imaginative process explained: "I've always struggled with feeling confident in my own ideas ... coming up with my own conjecture …[was] the hardest for me," which we associate to step 1. Another student in the "other" theme specified: "I had a hard time understanding the content of the course in terms of how to apply my program to the questions asked of me," which we associate to step 5. And a student included in "math" said: "I found the written reports to be the most challenging … it just seemed to require more in depth thinking, rather than just knowing the math and programming it," which we associate to steps 6-7.

Why students find 'coding' and 'coding of mathematical processes' to be most challenging

We now consider the reasons behind the two greatest challenges reported by the highest proportions of participants. Our analysis of 46 participants' responses led to 11 themes, which can be categorized according to 4 different aspects of instrumental genesis.

The most dominant reason (32.6%) was *struggling with knowing what to do while coding, or starting or getting stuck with coding* (e.g., "I find it difficult to start as sometimes I don't know how to approach the math in a programming sense"). We infer that these students were

[1] Programming graphs in VB.Net requires coding a change of coordinate system.

struggling to mobilize or develop certain schemes (e.g., for articulating a math process in the programming language). Another reason, *struggling with making efficient code* (6.5%), also points to struggling to develop a certain scheme (e.g., "It's one thing to get the code to do what you want, it's another to get it to do that as efficiently as possible"). Thus, 39.1% of participants seem to suggest the development of certain schemes as causing the 'coding (math)' challenge.

Some reasons (with proportions totaling 36.9%) relate more to the nature of instrumental genesis: a process that takes time, effort, and confidence. This includes *coding (math) is new* (15.2%, the second most common reason), *needing more guidance* (10.9%, e.g., "I am expected to know commands and what to write with only being taught the basics"), *not being confident in coding* (6.5%, e.g., "I am not totally confident as a programmer"), and *needing more time* (4.3%, e.g., "It's a hard concept to understand in a short amount of time").

Two reasons (totaling 19.5%) point to the nature of the artefact (in this case, programming). This includes the third most common reason, *coding is finicky or picky* (13%, e.g., "one small thing can throw the whole code off") and *struggling to understand how a program works* (6.5%, e.g., "I really struggle to visualize how the program handles these codes and tend not to understand … how it is getting to the answer").

Finally, two reasons (totaling 8.6%) concern the nature of the mathematics: *finding it difficult to think computationally* (4.3%, e.g., "It generally takes a difficult path to go through the math and make it work") and *struggling with coding complicated or brand-new math* (4.3%, e.g., "some of the math is complicated enough on paper let alone telling a computer how to do it").

How students mostly handle what they find to be most challenging

Looking at how all 73 participants said they handled their most challenging part of their inquiry projects, 9 themes were found (summarized in Figure 3, left).

53.4% of participants indicated *seeking help from the instructor, teaching assistants (TAs), and/or peers* (e.g., "Asking as many questions, and getting as much help from the TA/Instructor as possible"; "I speak with peers and the professor in order to get on the right track").

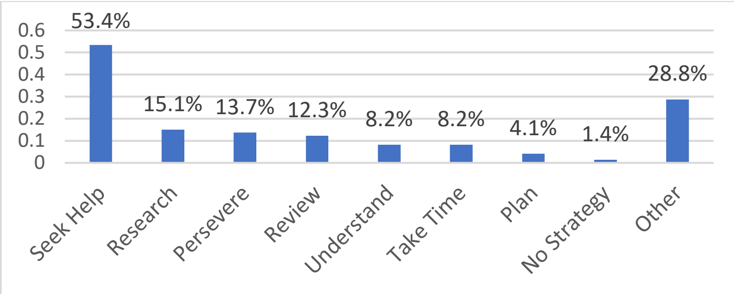
Some other strategies (with proportions totaling 35.6%) include ways of independently gaining the knowledge required to address the challenge: *researching beyond my lecture material* (e.g., "Using the textbook and online sources to self-teach"), *reviewing my (previous) lecture material* (e.g., "sit down with my lecture notes and review them"), and *seeking a deeper understanding* (e.g., "just trying to understand the topics more deeply").

Two other strategies (totaling 21.9%) are associated with taking the time needed to adapt or develop one's schemes for the given situation: *persevering* (e.g., "Just testing new ideas until one works") and *taking my time* (e.g., "I took things one step at a time").

One strategy is specific to 'coding (math)' (4.1%): *planning before coding* (e.g., "trying to structure out some sort of design before writing programs"), which we infer as students developing part of a scheme for articulating a (math) process in the programming language.

A very small proportion (1.4%) indicated having *no strategy* (e.g., "I haven't [handled it]").

There were also 28.8% of *other strategies* mentioned, such as "I've been handling it by staying on top of my academics." Many of these could be connected through a larger category of *working through it on my own*, as could most of the themes in Figure 3, left (excluding *seeking help* and *no strategy*). We thus regrouped students according to their ability to work independently on programming-based mathematics inquiry, as summarized in Figure 3, right: 32.9% of students mostly *just seek help from others*, 38.4% mostly *just work through it on their own*, 23.3% *do both*, and 5.5% mostly *just struggle and/or have no strategy*. We interpret students using a strategy including working through it on their own (61.7%) as suggesting that they may be "further along" in their instrumental genesis or more able to facilitate it themselves.



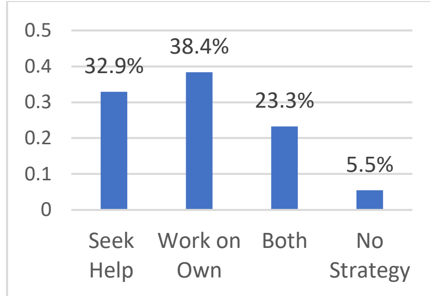


Figure 3. How participants (N=73) mostly handled their most challenging part of the MICA inquiry projects.

Some demographic comparisons in the 'what' and the 'how'

Finally, we present some results considering different demographic categories: year of study (MICA I vs. 'upper MICA,' i.e., MICA II or III) and gender (female vs. male).

When comparing MICA I and upper MICA participants, we found no clear difference in what was found to be the most challenging part of the inquiry projects. As for the way they mostly handled challenges, a greater proportion of MICA I participants indicated solely seeking help (38.5% vs. 26.5%), while a greater proportion of upper MICA participants indicated solely working through it on their own (47.1% vs. 30.8%); and a similar proportion of MICA I and upper MICA participants indicated utilizing both strategies. This may suggest that upper MICA students are more comfortable working independently in programming-based mathematics inquiry projects and they may be further along in their instrumental genesis.

There also seems to be no clear difference between what females and males find most challenging about MICA inquiry projects, except maybe for a slightly larger proportion of females pointing to the coding of mathematics (21% vs. 8%). Female and male participants also reported similar strategies for handling their most challenging part, in terms of solely working through it on their own or seeking help, though female students appear to do both slightly more (27% vs. 16%) and male students may be more likely to have no strategy (12% vs 2%).

Discussion

With this paper, we seek to answer Lockwood and Mørken's (2021) call for research by starting to examine the challenges students face when computing (including programming) is integrated in undergraduate math education, specifically in the context of mathematics inquiry.

In our study, student participants indicated in greatest proportion (~3/4) that the most challenging part of their inquiry is related to what we call "the programming cycle," which involves the design and creation of a program for the purposes of the inquiry, including the intertwined processes of translating mathematical processes into algorithms and validating the programmed mathematics. This aligns with work in computing education that has documented the many difficulties and misconceptions that novice programmers may have (Qian & Lehman, 2017): e.g., "their lack of well-established strategies and patterns often leads to various challenges in planning, writing, and debugging programs" (p. 6). The participants in our study pointed to different factors that may be contributing to their "coding (math)" challenge: not only the inadequacy of their current patterns of doing (which we framed in terms of schemes), but also the nature of the process required to develop such patterns (e.g., it takes time; Laborde, 2002). Participants also pointed to the nature of programming as a tool and the mathematics involved in their programming-based mathematics inquiry. It is possible that the "newness" of this experience is underlying participants' responses, in which case we may expect different results in the future if students start learning programming in STEM subjects from a young age (cf. Wiebe et al., 2020). We note, nevertheless, that students gaining fluency in programming need not imply their fluency in using programming to conduct mathematical inquiry (Buteau et al., 2019). Moreover, the latter will always be inherently challenging: "the inquiry process develops as interplay between known and unknown in situations where some individual or group of individuals is *faced with a challenge*" (Artigue & Blomhøj, 2013, p. 798-799, our emphasis).

Work in computing education emphasizes that challenges may inhibit students' ability to learn and make progress (Qian & Lehman, 2017). In contrast, our study found that just because a student finds something "most challenging" does not mean that they struggle: participants described several strategies for handling their challenges, with only a few suggesting that they had no strategy. The high proportion of students indicating that they "seek help" aligns with principles of inquiry-based mathematics education. Laursen and Rasmussen (2019) identify four pillars of such an approach, which highlight the importance of students collaboratively processing mathematical ideas and instructors facilitating students' thinking in equitable manners (e.g., encouraging students to explain their ideas and ask for others' explanations and help). Importantly, this also aligns with how most professional mathematicians do their work: i.e., in cooperation, collaboration, or consultation with one another (e.g., Bass, 2011; Burton, 2004). This said, our study also found that students seem to develop more independent strategies as they progress in the MICA courses. We conjecture that students may be learning when to collaborate and when to work on their own. One limitation of the current study is that we did not distinguish between "seeking help" and "collaboration." Moreover, we only considered how students *mostly* face their *most* challenging part of their inquiry. Looking more into students' interactions with peers, instructors, and other resources could be an interesting direction for future work.

In their call for research, Lockwood and Mørken (2021) mention equity issues as one of four key research foci for the RUME community investigating computing, concluding that a potential downside to integrating computing in undergraduate mathematics is that it could perpetuate inequities in certain populations. Indeed, research in computing education has highlighted the greater challenges typically faced by female undergraduates (e.g., Margolis & Fisher, 2003). The small differences we observed among female and male participants seem to be promising and is consistent with some of our past work (Buteau et al., 2014). It is possible that this is connected to the inquiry approach taken in MICA courses. For instance, Laursen and Rasmussen (2019) indicate that "current studies show that inquiry classrooms can level the playing field for women … and argue why this may occur … but also show that this is not automatic" (p. 138). Looking further into which aspects of the MICA environment may support equitable outcomes could be another pertinent direction for future work.

Overall, our study seems to suggest that the teaching approach in the MICA courses may positively support students in learning to engage in programming-based mathematics inquiry. To think about implications for teaching, we need to examine more closely the teaching that the students actually receive. In particular, there is the question of what instructors can do to best support their students in handling their challenges.

Acknowledgments

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Effectiveness of a Project-Based Approach to Integrating Computing in Mathematics

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In this naturalistic study, we examine students' learning as they engage in programming for mathematics investigations through a project-based approach. We focus on undergraduate mathematics students' (N=41) engagement in a sequence of 14 programming-based math investigation projects, with data primarily collected through online questionnaires. Results suggest that students learn the most when they engage in projects they are passionate about. Results also provide empirical evidence supporting the effectiveness of a project-based approach by demonstrating the potential richness of students' learning in projects (e.g., learning general ways of doing and succeeding, in addition to specific mathematical knowledge).

Keywords: project-based learning, computing, student learning, mathematics investigation, operational and predicative knowledge

There are different approaches to integrating programming – or more broadly, computing – in undergraduate mathematics education, including: as a required skill (e.g., a computer science course requirement), within specific courses (e.g., modeling, numerical analysis), or through a more integrated approach (e.g., throughout a program or in a sequence of specially-designed courses). For instance, a survey of 46 mathematics departments in the U.K. found that 89% of undergraduate mathematics programs teach programming to all students, most commonly in numerical analysis or statistics (Sangwin & O'Toole, 2017). In comparison, at Manchester Metropolitan University in the U.K. (Lynch, 2020), University of Oslo in Norway (Malthe-Sørenssen et al., 2015), and Carroll College in the United States (Cline et al., 2020), programming is integrated across the undergraduate mathematics curriculum as a learning and/or problem-solving tool. In the Canadian context, Brock University's Department of Mathematics and Statistics has integrated programming since 2001 in a sequence of three specially designed project-based courses called Mathematics Integrated with Computers and Applications (MICA) I, II, and III (Buteau, Muller, & Ralph, 2015).

In a recent "Call for Research that Explores Relationships between Computing and Mathematical Thinking and Activity in RUME," Lockwood and Mørken (2021) suggest that different approaches provide "opportunities for systematically studying different ways for [the] integration [of computing] to occur" (p. 6). They point out, in particular: "there are an increasing number of examples of meaningfully-integrated programs across the world, and *the RUME community can explore what kinds of programs are effective and why*" (ibid., our emphasis). Four potential research foci are identified by the researchers, one of which is teaching. On page 7 of the call, potential research questions are proposed, including (our emphasis):

1. How should computing be introduced and taught in postsecondary mathematics classrooms and how might we design *effective tasks and curricular materials* to integrate computing into postsecondary mathematics classrooms?
2. What are *effective (or ineffective) program, department, and institution-level models* for integrating computing into mathematics classrooms?

The study we present is connected to the above call for research and potential questions.

We focus on the effectiveness of a particular kind of task and program model: those that are situated within or based on projects. There are some studies reporting on the effectiveness of project-based learning (PBL) in post-secondary STEM courses (Ralph, 2015; e.g., in chemistry courses that incorporate the use of computerized models, Barak & Dori, 2005). Nevertheless, a literature review on PBL in K-20 mathematics education (Jacques, 2017) highlights a lack of research and mixed results on the topic, concluding that “we cannot say at this time if PBL is or is not an effective approach” (p. 431).

In our 5-year research, we are interested in examining the effectiveness of teaching and learning environments such as in the MICA courses mentioned above. These courses use a PBL approach, engaging math majors and future math teachers in a sequence of 14 programming-based mathematics investigation projects (see Table 1 in Buteau, Muller, & Ralph, 2015 for examples of the 14 projects). Previous work has reported on the potential effectiveness of this sequence, based on task analyses and reflections from a few students (cf. Buteau et al., 2016). In this paper, we further investigate the (comparative) effectiveness of the MICA projects, as reported by a larger group (N=41) of students, based on their responses to a questionnaire.

Theoretical Framework

The origins of PBL can be traced to first century philosophers such as Aristotle, who believed that humans mainly learn by doing. More recently, theorists connected to the establishment of PBL (e.g., John Dewey) were inspired by an apparent gap between what students learn in school and the skills and attitudes they need to succeed in a constantly changing world. Building on Dewey’s work, Kilpatrick (1921) defined a “project” as any unit of experience dominated by a purpose, which guides its process and drives its attainment. Other researchers have since provided a more elaborated definition, suggesting that projects should: (a) include complex tasks based on challenging questions or problems that involve students in design, problem-solving, decision making, or investigative activities; (b) give students the opportunity to work relatively autonomously over extended periods of time; and (c) culminate in realistic products or presentations (Jones, et al., 1997; Thomas, et al.,1999). Several other defining features of PBL can be found in the literature: e.g., (d) authentic content and assessment; and (e) teacher facilitation but not direction (Moursund, 1999).

The constructionist paradigm (Papert & Harel, 1991) embodies a particular kind of PBL approach in which students consciously and actively engage in constructing (e.g., through programming) tangible and shareable objects. Resnick’s (2014) “4 P’s” – projects, peers, passion, and play – describe some of the key instructional elements that can support an effective constructionist approach. He argues, for instance, that “when people work on projects they care about [i.e., that they are passionate about], they work longer and harder, persist in the face of challenges, and learn more in the process” (ibid., p. 1).

Though not explicitly intended, the MICA courses have been found to be intrinsically constructionist (Buteau, Muller, & Marshall, 2015). Also, the courses align with the features of PBL described above: MICA “projects” involve students in using programming for authentic pure and applied mathematics investigations ((a)/(d)); which are worked on autonomously, that is, facilitated rather than directed by instructors ((b)/(e)); and which culminate in useful computer environments and realistic project reports (c). Recall that the sequence of 14 MICA projects occurs over 3 courses: MICA I (Projects 1-4), MICA II (Projects 5-9), and MICA III (Projects 10-14). It is important to note that the last project in each course differs from the rest in that students can select a topic that interests them (all other projects concern topics that are specified by the instructor of the course). For instance, mathematics majors may decide to construct a program for investigating a problem or an area of research they find interesting; in a similar vein, mathematics teacher candidates may decide to program a “learning object” (Muller et al., 2009), i.e., a step-by-step guided learning of a school math concept, which may be relevant to their future profession (see Brock University, n.d., for some example projects). Inspired by Resnick (2014), we call these end-of-term projects “passion projects.”

In this paper, we look at the “effectiveness” of projects in a PBL approach in terms of “student learning.” Since our work focuses on a certain kind of PBL – where students are learning to use a particular digital tool (a programming language) – we further frame “student learning” using the notion of instrumental genesis (Guin et al., 2005; as described in Buteau et al., 2019). In mathematics education, instrumental genesis has been conceptualized as a complex process involving the intertwinement of learning techniques for using a digital tool and learning specific mathematics concepts (Artigue, 2002). For instance, the second MICA project in the sequence prompts students to implement an RSA algorithm to encode and decode messages. Students typically learn how to use functions and modules in order to program the algorithm, and they also learn the specific mathematics concepts underlying RSA encryption (e.g., modular arithmetic, Euclid’s algorithm for finding the greatest common divisor, finding powers and inverses in Z_n, …). To elaborate further on these two kinds of learning, Vergnaud (2009) offers a conceptualization of knowledge that distinguishes between operational knowledge (which provides means to do and succeed) and predicative knowledge (which consists of means to express ideas in words or symbols). Although both types of knowledge may be involved in PBL, early proponents of the approach seemed to be trying to shift from the focus of math education on predicative knowledge, to also include an appropriate emphasis on operational knowledge.

In light of the above theoretical framework, we pose two research questions:

1. What kind of programming-based math investigation projects are most effective?
2. In what ways are they effective (e.g., what kind of knowledge is learned)?

In this paper, we address these questions from the student’s point of view.

Methodology

The study we present is part of a larger 5-year (2017-22) iterative design research focusing on the learning and teaching of programming for authentic pure and applied math investigations. This is a naturalistic research contextualized in the MICA courses, which are semester-long mathematics courses including 2-hour lectures and 2-hour labs each week. Part of the research closely follows some MICA students over the MICA I, II, and III courses, using individual semi-structured interviews related to their engagement in each of the 14 programming-based mathematics projects (P1, …, P14), as well as their lab reflections and project reports. To complement this in-depth study of individual students’ learning, in each MICA course students are invited to voluntarily participate in pre and post anonymous[1] questionnaires.

The questionnaire data has been collected so far in Years 2-4 of the research. Questions in the pre-questionnaire differ from those in the post- since the latter invites students to reflect on their learning during the course. The questionnaires feature several sections, such as participants’ demographics and participants’ perceptions of: the importance/usefulness of digital technology (including programming), their own knowledge/confidence level in programming, what it means to learn/do math, and the course (the assignments, the teaching, and their learning).

[1] Students who participate in the in-depth study are not anonymous.

This paper focuses on a post-questionnaire question that contains two parts. In the first part, participants are asked to indicate which project they learned the most from, from all the projects in the MICA courses they had taken (for instance, a MICA II participant could select one project from either MICA I or II, i.e., from among P1 to P9). In the second part, participants are then asked to select a reason for the project they chose. Options are provided and participants can choose more than one option if they wish. The possible options are: (a) I learned a lot of new math; (b) I learned a real-world math application that really speaks to me; (c) I completed it all by myself without help; (d) It was challenging but I finally understood it; (e) I had to use new programming concepts or skills; (f) I discovered something I did not expect; and (g) Other.

41 students responded to the above question (24 from MICA I, 5 from MICA II, 12 from MICA III), which is not necessarily representative of the entire MICA student population (a limitation of our study). Responses were analysed using comparative bar charts and frequency tables. The first part of the questionnaire question was key to answering our first research question: When a participant indicated the project they learned the most from, we interpreted this as the "most effective" project from that participant's point of view, and we reflected on the kinds of projects that were selected. This reflection was further supported by the second part of the questionnaire question (students' reasons for their selected project), which was also key to answering our second research question (concerning the ways in which the projects are effective). As part of our analysis, we regrouped the above reasons into two categories based on whether they were indicative of learning predicative or operational knowledge (as defined in our Theoretical Framework). In the predicative knowledge category, we include (a), (b), and (f), which we see as reflecting project-specific learning of math concepts. In the operational knowledge category, we include (c), (d), and (e), which we see as reflecting learning of more general ways of doing and succeeding in using programming for math investigation. We note that each of the "Other" responses were reviewed to determine if they could be categorized as any of (a)-(f) (in which case they were recategorized and frequencies were adjusted accordingly).

Results and Discussion

We organize our results and discussion according to our two research questions.

The Most Effective Projects

Table 1 shows which programming-based mathematics investigation projects from among the sequence of 14 that participants selected as the ones they learned the most from. Note that we distinguish what we called "passion projects" with the notation "PP."

Table 1. The projects participants said they learned the most from.

Course	P1	P2	P3	PP4	P5	P6	P7	P8	PP9	P10	P11	P12	P13	PP14
MICA I	1	10	5	8	-	-	-	-	-	-	-	-	-	-
MICA II	0	0	0	1	0	0	0	1	3	-	-	-	-	-
MICA III	0	1	0	2	0	0	0	1	2	0	1	0	1	4
Total	**1**	**11**	**5**	**11**	**0**	**0**	**0**	**2**	**5**	**0**	**1**	**0**	**1**	**4**

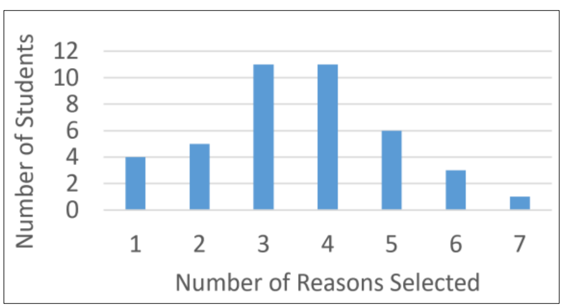
We observe that students selected a variety of projects. Some concern pure mathematics (P1: conjectures about primes or hailstone sequences, P3: a discrete dynamical system, or P11: simulations related to Bertrand's paradox), while others concern applied mathematics (P2: RSA encryption, P8: battle simulations, or P13: randomness of DNA sequences). Some have a higher ceiling in terms of potential for discovery and investigation (e.g., P1, where students are invited to choose or formulate a mathematical conjecture to explore, vs. P2, where students apply the RSA method to encrypt and decrypt messages). Some projects also differ in the degree to which they are scaffolded by instructors (e.g., P2, where students are guided through some steps to construct their program, vs. PPs, where students must determine the design and use of their programs). Finally, the projects chosen by participants show up at different moments in the sequence (e.g., a MICA III participant selected P2, which occurs towards the beginning of the sequence; another selected P13, which occurs towards the end).

Among the projects that were selected, nearly half (20/41) were PPs, in which students choose the topic of their investigations. This provides empirical evidence in support of certain constructionist claims: e.g., that students learn best when they are working on projects that are meaningful to them (Papert, 1980), i.e., on topics that they are passionate about (Resnick, 2014). We also note that about half of the participants who selected PPs indicated their passion project from MICA I, evidencing the low-floor-high-ceiling affordance of programming for mathematics learning (Gadanidis, 2017): already in their first year, with minimal programming background, these students said that they engaged in meaningful learning in such a project.

When looking at MICA I participants only, P2 was the most selected (41.7%), even over the MICA I final project (PP4; 33.3%). At first, this seems to be a rather surprising result, for example, due to P2's lower ceiling in terms of potential for discovery and investigation (Buteau et al., 2018), especially when compared to PPs. We discuss this result more in the next section.

The Ways in Which the Projects are Effective

In selecting the reasons for the project they learned the most from, participants had the option to select more than one reason. Figure 1 shows the number of reasons selected by participants (left) and the percentage of participants who selected the different possible reasons (right).



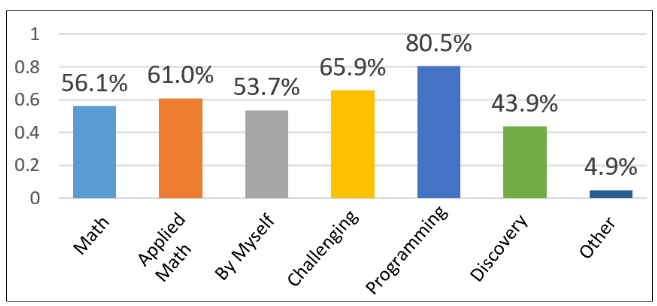


Figure 1. Number of reasons selected (left) and percentages of participants (N=41) selecting each reason (right).

A high percentage of participants (78%) selected at least 3 reasons. This highlights the richness of the students' learning experiences in their chosen projects.

"I had to use new programming concepts or skills" was selected by the highest percentage of students (80.5%). This shows that as students engage in these projects, they typically encounter avenues where in addition to applying their prior knowledge in programming, they also have to use new knowledge to create their program and complete their projects. Further, as seen in Figure 1 (right), at least half of participants selected that they persevered in the face of challenges (65.9%), learned a real-world math application that spoke to them (61.0%), learned lots of new math (56.1%), or completed the projects by themselves without help (53.7%). Also, close to half (43.9%) said they discovered something they did not expect. We argue that the prevalence of these different reasons, as well as the selection of multiple reasons mentioned above, provides evidence in support of the effectiveness of a PBL approach.

Only 2 students (4.9%) gave "Other" reasons for why they benefitted the most from the projects they chose (both of which were PPs). One student said, in addition to (a), (b), and (d): "I was paired with a cool partner and we worked well together." The other, who had designed and programmed a learning object, said, in addition to (c) and (e): "How to create a lesson plan which incorporates programming to teach mathematics to grade 9 students." These responses are linked to two characteristics of effective constructionist learning, as highlighted by Resnick (2014): the former to "peers" and the latter to "passion." It is notable that these two participants felt the need to specify these additional reasons for the effectiveness of their chosen projects.

As mentioned above, PPs and P2 were the most commonly selected projects by the participants. Participants' reasons for selecting these are depicted in Figure 2.

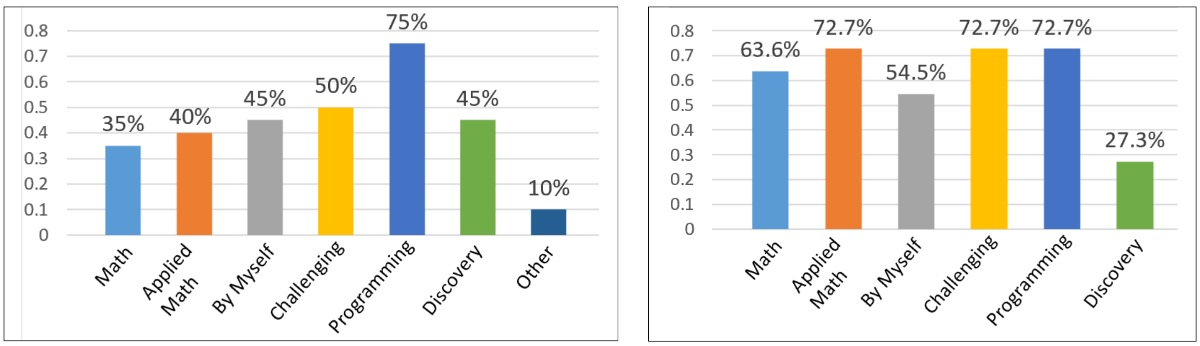


Figure 2. Percentages of participants who selected each reason for PPs (left; N=20) and P2 (right; N=11).

When just looking at PPs (Figure 2, left), we note that only two responses were selected by at least 50% of participants: "I had to use new programming concepts and skills" (75%) and "It was challenging but I finally understood it" (50%). In the case of PPs, students select their mathematics topic, which can naturally lead to new demanding needs in terms of the programming (as opposed to just using the programming learned during prior MICA projects). These results suggest that students' learning experiences in PPs may be more pointed.

In comparison, 5 reasons were selected by at least 50% of participants who chose P2, suggesting that P2 provides an opportunity for students to experience multiple facets of learning. This is likely linked to the position of P2 in the sequence of 14 MICA projects. For most students, P2 is the first project where they must program a more elaborate mathematical process, in addition to using new programming concepts and skills. This represents a steep learning curve for students: overcoming it, possibly almost all by themselves, would constitute a major accomplishment. The fact that students indicate learning a real-world application that speaks to them is not surprising: P2 concerns RSA encryption, a topic that relates to the digital world in which the students live. Interestingly, the other reason ("I discovered something I did not expect") was selected by less than 30% of participants, which seems to indicate that the low ceiling for discovery in P2 (as speculated in Buteau et al., 2018) is reflected in students' perceptions. We also point out that students coming from high school may not be used to investigating conjectures: therefore, after a more unsettling experience with P1, students may feel more comfortable with a more directed project like P2, where they can follow instructions and easily see the progress of their engagement. Anecdotally speaking, we have heard many other students state that P2 was a particularly memorable project.

Operational and predicative knowledge. In this subsection, we report on the kind of knowledge students indicated they learned in their chosen projects, based on our regrouping of reasons discussed in the Methodology. Table 2 summarizes the results.

Table 2. The kind of knowledge students learned in their chosen projects.

Projects	Operational & Predicative	Only Operational	Only Predicative
All	30	8	3
PP	13	5	2
P2	9	2	0

It appears that over 70% of participants learned both operational and predicative knowledge in the project they learned the most from. This reflects the different kinds of knowledge required in using programming for mathematics like mathematicians do (Broley, 2015). Also, over 90% of participants indicated learning operational knowledge. This aligns with proponents of PBL (e.g., Kilpatrick, 1921), who talk about the approach as an attempt to engage students in learning the general processes and attitudes involved in (mathematical) problem solving, in addition to learning specific (math) content. One could wonder about the 7.3% of participants who indicated learning solely predicative knowledge in the project they learned the most from. We note that these three participants were future math teachers. Two selected PPs, which could have been a learning object connected to their future career. If they did it in pairs, it is possible that the participant did not lead the coding of the object, and thus did not feel as though they gained operational knowledge.

Results for P2 in Table 2 align with the above results. In comparison, Table 2 offers a different perspective on the richness of learning experiences in PPs. Despite only a few reasons being identified by high percentages of participants (Figure 1), Table 2 suggests that many (65%) gained both operational and predicative knowledge.

Conclusion

In this paper, we answered Lockwood and Mørken's (2021) call for research by exploring the effectiveness of one approach to integrating computing in undergraduate mathematics: namely, PBL, as defined by general work in education (e.g., Jones, et al., 1997; Moursund, 1999; Thomas, et al.,1999) and the constructionist paradigm (e.g., Papert, 1980; Resnick, 2014). One main contribution of our study is that it addresses the gap of empirical evidence related to the effectiveness of such an approach (e.g., reporting on the different kinds of learning that may occur in programming-based mathematics investigation projects).

Our study was based on student (N=41) responses to an online questionnaire and was exploratory in nature. The theoretical framework and results inspire us to refine our questionnaire: in particular, to rework the reasons that students can choose to explain the project they learned the most from. There seem to be some reasons that could be revised (e.g., 'I learned a real-world math application that really spoke to me' could be divided into 'I learned a real-world math application' and 'I learned something that really spoke to me'). There also seems to be other reasons that are important to include (e.g., 'I worked with others'). As part of our larger 5-year research, we also have interview data, which could support us in digging deeper into the effectiveness of these projects and the types of learning that are involved (e.g., the more pointed yet significant learning that may be happening in passion projects, the rich learning in P2, or the combined learning of operational and predicative knowledge across the project sequence).

Acknowledgments

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Formative Assessment in a Gateway Quantitative Reasoning Course

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Quantitative reasoning is an individual's ability to understand quantitative information in context, represent and model such information, solve real-world problems using mathematical and statistical knowledge, and communicate ideas using quantitative arguments. Using formative assessment practices in a QR course allows students to develop their QR competencies and demonstrate their learning. In this study, we explored whether and to what extent Quantitative Reasoning instructors implement formative assessment and examined their experiences and perceptions of using formative assessment. Herein we report on the formative assessment practices of six instructors who taught Quantitative Reasoning courses at public universities and public community colleges in Ohio during spring 2020. We analyzed the instructor's course syllabus and conducted a semistructured interview with each instructor. The results show that the instructors exhibited various formative assessment practices with a wide range of weights.

Keywords: Quantitative reasoning, formative assessment, projects, student presentations, feedback.

Quantitative reasoning (QR) is an individual's ability to understand and use numbers in context. It is associated with a person's critical understanding of quantitative information and sense-making in context (Elrod, 2014; Meyer & Dwyer, 2005; Wilder, 2012). Central to QR is the ability to think deeply and critically about real-world phenomena and link them to relevant mathematics and statistics, as suggested by Figure 1 (Foley & Wachira, 2021).

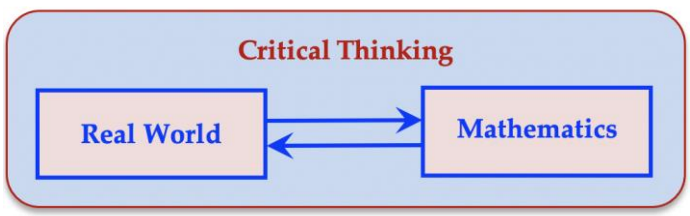

Figure 1. A model of student engagement in Quantitative Reasoning.

An introductory postsecondary (i.e., gateway) QR course aims to prepare students to solve problems in everyday and professional contexts. It includes some nontraditional mathematical goals, like thinking critically about quantitative information and contexts, collaborating with peers to solve real-world problems, and communicating orally and in writing using quantitative arguments (Mathematical Association of America [MAA], 1996; Stump, 2017). Policy documents and professional opinions suggest that instructors should use teaching and assessment approaches that support students to build QR competencies such as interpretation, representation, calculation, analysis/synthesis, assumptions, and communication (American Association of Colleges and Universities [AAC&U], 2009; Boersma et al., 2011).

Assessment in a mathematics course has no value if it does not measure appropriate goals (Steen, 1997). Lutsky (2008) and the MAA (1996) recommend that QR assessments include projects involving real-world tasks and student presentations; these assessments provide opportunities for students to solve problems and construct associated knowledge and skills. Learning via such assessments can provide constructivist and collaborative opportunities for students, with which they can lead themselves and their peers toward academic and non-academic success (Harwood, 2018; Thomas, 2000; Virtue & Hinnant-Crawford, 2019). Unlike traditional assessments—such as quizzes, tests, and exams—these recommended assessment strategies for QR may provide valid measures of the desired QR student learning outcomes of interpretation, representation, calculation, analysis/synthesis, assumptions, and communication.

However, a problem exists in that not all instructors of gateway mathematics are familiar with the sort of recommended assessments that use authentic situations and tasks to measure students' QR skills. Many instructors are still unclear about the most appropriate practices for assessing students' learning in a QR course (Wallace et al., 2009). Most of the assessments used to measure students' QR skills are traditional summative examinations or even multiple-choice tests administered on a computer (Boersma et al., 2011; Roohr et al., 2014). However, such practices do not match the suggestions for periodic and continuous assessments and feedback practices aligned to the desired student learning outcomes, suggested by the National Research Council (NRC, 2001) and Tunstall (2019). Thus, a major challenge in implementing gateway QR courses is aligning learning goals and assessments (Bae et al., 2019; Shavelson, 2008).

Compared to course development and instruction, fewer research studies and policy documents discuss the assessment of students' learning in a QR course (Madison, 2014; Sikorskii et al., 2011; Wright & Howard, 2015). Existing research publications that discuss assessment practices in such a course mainly focus on ensuring the reliability and validity of tests (Boersma et al., 2011; Taylor, 2009). Ward et al. (2011) found that (a) QR assessment practices vary widely across the institutions, (b) the majority of QR assessment initiatives were conducted at small universities and colleges, and (c) midsize and large universities generally lacked such empirical research on assessment of gateway QR courses.

Formative Assessment

Formative assessment is the diagnostic use of assessment during ongoing instruction (Boston, 2002). A key aspect of such assessment is timely feedback to instructors and students about their current practices to make immediate changes to improve student learning. Formative assessment is something instructors do *with* and *for* students to advance student learning (Heritage, 2010). Moreover, it *informs* instructors about the effectiveness of their current practices to use this information as needed to support student learning (Black et al., 2003; Ogange et al., 2018). Spector et al. (2016) explained that a formative assessment emphasizes *forming* judgments about students' progress, affecting the subsequent flow of instruction and learning. The NRC (2001) described a subsequent modification of teaching to promote student learning as the purpose for any formative assessment. Therefore, improved instructor and student practices are the chief aims of a formative assessment (Spiller, 2009).

Formative assessment is a lever for a paradigm shift in pedagogical practices and the effective delivery of instruction (Ogange et al., 2018). In addition to promoting student learning, it promotes democratic practices by valuing student thinking, putting forward their ideas, and challenging others' reasoning. Therefore, it is one of the effective methods of encouraging student engagement and achievement (Spector et al., 2016).

After their extensive research project, Black et al. (2003) reported four central formative assessment practices: *questioning, feedback, formative use of summative tests,* and *peer- and assessment by students*. Questioning is a process in which instructors pose questions to an individual student or a group of students, listen to their responses, and continue by either providing appropriate feedback or probing follow-up questions. Feedback consists of information about students' current levels of achievement and suggestions for improvement.

Formative use of summative tests can take several forms, including the use of student responses on tests to change instruction in the subsequent unit. Peer- and self-assessment are the practices in which students assess their peers and their own learning to determine whether the learning aligns with the course goals and develop reflective understanding; these practices give students the power to evaluate self and others' knowledge, make appropriate decisions, and enhance individual and interpersonal learning (Noonan & Duncan, 2005).

Formative Assessment and QR Learning

It is difficult to assess the QR competencies of interpretation, representation, calculation, analysis/synthesis, assumptions, and communication using only traditional summative quizzes, tests, and exams. Moreover, these high-level skills are new to most college freshmen, and they need support and feedback to develop these QR competencies. Hence, formative assessment is a natural fit for a gateway QR course. Sundre (2003) stresses linking and implementing QR assessment and instruction simultaneously, explaining that such practices enhance QR teaching quality. Moreover, Shavelson (2008), Spector et al. (2016), and Wilder (2012) suggest using formative approaches such as regular and periodic assessment accompanied by detailed feedback to enhance student engagement in QR activities and their consequent achievement. In addition, such practices improve the instructor's choice of learning activities. Reflecting on the information obtained through such assessments, QR instructors can determine students' needs and attitudes in time to make the necessary adjustments to provide learning experiences for students that address course goals (MAA, 1996).

Students' collaboration and communication are common characteristics of QR learning and formative assessment. Consistent with QR student learning outcomes, several formative assessment practices, like peer-and self-assessment, require students to work together to solve real-world tasks and develop abilities to work together as they need in their everyday life. In addition, such formative practices even involve students sharing their thinking and ideas in oral and written forms, arguing with evaluative arguments, and providing appropriate feedback to improve their mutual learning. Also, a formative assessment constitutes frequent dialogues between instructors and students and among students, which can help teachers and students articulate the course goals, address issues related to measuring students' communication skills, and reduce other challenges they face during QR assessment (Boersma et al., 2011). Moreover, formative assessment practices influence students' beliefs and motivations toward understanding and building their explaining, interpreting, and reasoning (MAA, 2018; Spector et al., 2016).

The purpose of this study was to explore instructors' preparedness and practices of using formative assessment in undergraduate-level Quantitative Reasoning courses. We examined the instructors' preparedness before they started teaching QR for the first time and their preparation before each class. Similarly, we explored their formative assessment practices by collecting and analyzing data about the nature, types, and frequencies of assessments and how they use such assignments to improve student learning. The findings of this study will be significant to novice and experienced instructors in planning, contextualizing, and implementing formative assessment in their QR teaching and enriching their student learning. Also, these results contribute to narrowing the research gap in the field with the ability to be transferable to other mathematics courses of equivalent level and in similar settings.

Data Collection

We employed a multiple case study design in this investigation. We defined our cases as the instructors who taught at least a QR course in the public community colleges and universities and their regional campuses in Ohio during the Spring of 2020. Then we employed purposeful selection to recruit 6 QR instructors (3 male and 3 female) teaching at least one section of QR at two public universities and three public community colleges in Ohio (2 participants taught at the same community college). Among the 6 instructors, 5 instructors taught the Ohio Department of Higher Education (ODHE) (2015) approved QR course that their QR course was transferable across other institutions in Ohio. The instructors had a wide range of experience teaching undergraduate-level mathematics between 9 and 30 years. Their experiences in QR ranged between one semester and three years. Also, one instructor worked as a QR coordinator at her institution during data collection, while another instructor had experience developing a QR course in his past institution.

The first author (Budhathoki) collected data by analyzing QR course syllabi and conducting semistructured qualitative virtual interviews with the instructors. In advance of each interview, he studied the instructor's course syllabus to explore their plans to implement the project(s) during the semester, know the weights provided to such project(s) to determine student grades, and prepare a guide for the interview. He used the findings of the syllabus analysis to supplement as well as to triangulate the assessment practices shared by the instructors. The interviews with the instructors focused on investigating the instructors' rationales for choosing their assessments and the ways they implement the assessments. Budhathoki even focused on confirming whether and to what extent the instructor implemented assessments differently than stated in the syllabus.

We employed the cross-case analysis to examine themes to explore commonalities and differences in the instructors' actions, activities, and processes assessment implementation (Cruzes et al., 2015; Stake, 2006). For this, we first analyzed each instructor's project practices separately and then organized their assessment practices into six categories: (a) student projects and oral presentations; (b) attendance, participation, and student collaboration; (c) homework and other written assignments; (d) quizzes and in-class assignments; (e) periodic tests and exams; and (f) final exam. Among these six categories, we considered the first two categories of assessments—projects and oral presentations; and attendance, participation, and collaboration—to be innovative practices as these assessments are not typical to an undergraduate-level mathematics course. For this, we relied on the fact that instructors mostly use quizzes, tests, and exams in other similar-level mathematics courses. We grouped the last four categories—homework and written assignments, quizzes and in-class assignments, periodic tests, and final exams—to be traditional approaches to assessing student learning. These assessments have been typical in undergraduate mathematics courses in the United States for many decades. Then, we categorized the instructors into two groups: (a) Group A consisted of the instructors who provided more than 40% weights to innovative assessments, and (b) Group B consisted of the instructors who provided lesser weights to innovative assessments. Then, we sought *literal replication* by comparing formative assessment implementation of the instructors of the same group and *theoretical replication* by comparing that of instructors across the groups (Yin, 2018).

Findings and Discussion

We discovered a great deal of variation across the instructors' approaches to assessing student learning and assigning grades. On average, the instructors gave 32.3% of the student grades through these two innovative types of assessments and set 67.6% of the student course grade the traditional these types of evaluations. Table 1 shows the frequency and average weights of different categories of assessments used by the instructors.

Table 1. Assessment Frequency & Weights Used by All 6 Instructors

Assessment Categories	Number of Instructors	Number per Category	Weight Range	Average Weight
Projects & oral presentations	6	1–3	4–45%	19.8%
Attendance, participation, & collaboration	6	N/A	5–20%	12.5%
Homework & written assignments	6	4–10	10–20%	17.3%
Quizzes & in-class assignments	4	2–10	7–30%	10.3%
Periodic tests	4	2–3	30–60%	26.7%
Final exam	4	1	10–25%	13.3%

Similarly, Table 2 shows each instructor's weights to each assessment category. Four (3males, 1 female) among the 6 instructors relied on traditional approaches to determine student learning and assign grades. These 4 instructors assigned 70–85% of student grades, with an average of 77.75%, through traditional assessments, mainly periodic tests, and final exams. On average, they assigned more than half of the student grades, 56.2%, through these assessment categories. Despite the prominent callouts for enhancing and assessing QR students' learning by allowing them to work on authentic projects to solve real-world problems using their mathematics and statistics knowledge, the four instructors slightly emphasized projects and students' oral presentations. Each instructor just employed one project during the semester, and on average, they assigned only 8.5% student grades through projects and oral presentations. It seems like these 4 instructors included projects as their assessment to meet the ODHE (2015) requirements. These instructors argued that students do not have enough confidence in math to do projects and make presentations. Only 1 of these 4 instructors had student presentations as part of their projects. They gave reasons for that, including time consumption and students' readiness. Instructor 1 explained his reason for not having student presentations by saying, "my students' confidence level in terms of their perception of their level and ability in mathematics is very low. And then, you compound public speaking on top of it, which is again something they don't feel very confident about."

Table 2. Weights to Various Assessment Categories Provided by All 6 Instructors

Assessment Categories	Instructors					
	1	2	3	4	5	6
Projects & oral presentations	10%	10%	10%	4%	45%	40%
Attendance, participation, & collaboration	15%	20%	5%	15%	15%	5%
Homework & written assignments	20%	20%	18%	16%	20%	10%
Quizzes & in-class assignments	5%	0%	7%	0%	20%	30%
Periodic tests	30%	30%	60%	40%	0%	0%
Final exam	20%	20%	0%	25%	0%	15%

On the contrary, the other 2 instructors (both female) substantially employed innovative assessments to measure student learning and assign grades. In Table 2, we have represented these 2 instructors as Instructor 5 and Instructor 6, who allocated 52.5% of student grades through innovative assessments. Both the instructors emphasized their projects and oral presentations among their assessment categories. Both of these 2 instructors employed three projects during the spring of 2020 to assign 40–45% student grades. After projects and oral presentations, both of these instructors gave more weight to quizzes and in-class assignments, which they employed almost every weekend to collect information about student learning. These instructors used the information from their quizzes and in-class assignments to make instructional decisions about whether they needed to discuss the concept further, clarify students' misconceptions, and rediscuss the significant ideas. For example, Instructor 6 explained the purpose of her chapter quizzes, "see how they understand the material in general." She also further explained that it is her chapter quizzes from where she gets information about student knowledge and provides feedback after then. Likewise, Instructor 5 mentioned that she used her quizzes and in-class assignments to check if the students had understood the content and, if not, rediscuss the material. She said, "If there are lots of questions or people having trouble with a certain idea, stopping and saying, let's clarify what the issue is and how can we fix it." She also mentioned reteaching and reevaluating if found. On average, they assigned 25% of student grades through quizzes and in-class assignments, which contributed to 25% of students' grades on average.

There was also a great deal of variation among the instructors' approaches to assess student learning and assign grades. Specifically, there were variations between the assessments employed by the instructors of the two different groups. The six instructors had three out of six assessment categories in common. However, for the discussion and analysis, we collapsed the last three assessment categories used above—(d) quizzes and in-class assignments; (e) periodic tests; and (f) the final exam—into the single category (d) quizzes, tests, and final exam in Figure 2. That resulted in two categories in both the traditional group and the innovative group of assessments. Figure 2 shows the comparison of average weights that the instructors from the two groups assigned to different categories.

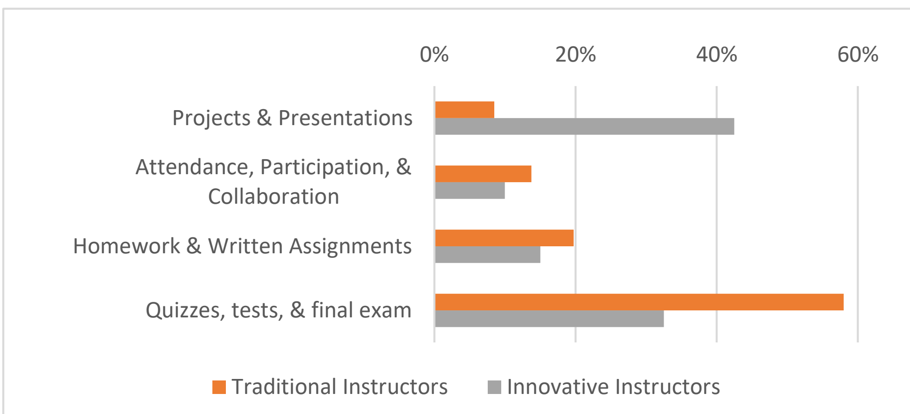

Figure 2. Comparison of Average Weights Assigned by Traditional and Innovative Instructors to Different Categories of Assessments'

The 6 instructors had varying practices of and perceptions about giving feedback to students. Though all 6 instructors claimed to provide feedback on student work, they primarily relied on written feedback indicating mistakes in student work, suggesting alternative ideas, and sometimes writing positive remarks for outstanding jobs. Even some instructors in the traditional group claimed that they provided feedback by posting solutions of homework and tests in their learning management system. All 6 instructors did not systematically practice giving oral feedback on student work. Nevertheless, the 2 instructors in the innovative give mentioned giving oral comments after student presentations but admitted that they still need to work much on their oral feedback practices. Also, only the 2 instructors in the innovative group and 1 instructor in the traditional group had some basic peer-assessment activities. All 6 instructors undermined the learning opportunities that students can get by evaluating their and peers' work. In addition, only the instructors in the innovative group let students redo their work by incorporating instructors' feedback. Likewise, only two instructors in the innovative group believed that providing feedback and allowing students to improve their work after receiving feedback helped students improve their performance and enhance their understanding. However, one of the instructors from the traditional group argued that feedback was helpful only for some students and only during some early weeks of the semester. Similarly, the instructors did not have similar perceptions about giving feedback.

The instructors had variations within the categories that were common to them. In Table 2, it is evident that they had only three common categories: (a) projects and oral presentations and (b) attendance, participation, and collaboration. They varied mainly in their frequency of a particular category, nature of student engagement, and afterward activities. For example, despite the project being common, some instructors required students to collect data and solve real-world situations. In contrast, some other instructors made students solve the worksheet's problems. Such variations remained both across and within the traditional assessment group and innovative assessment group. Variations in the implementation provided different learning opportunities for their students.

Conclusion

This study concluded that the instructors had a lot of variation in their categories of assessments and the ways they implemented their assessments. Though all six instructors had two innovative assessments in common—(a) projects and presentations and (b) attendance, participation, and collaboration—the variations in their implementations provided varying learning opportunities to students. Seeing altogether, the instructors primarily relied on traditional assessments such as homework, quizzes, and exams, with 67.6% of the weight going to these assessments to determine student grades.

Not all instructors implement their assessments to identify student learning gaps, improve their instructional practices, and provide support to enhance student learning. Despite loud callouts for student collaboration and communication in a QR course, the instructors did not orchestrate adequate opportunities for students to learn by working together and sharing ideas. Likewise, lacking communicative options limited students to develop evaluative arguments using quantitative information critical thinking. Providing students with oral feedback is valuable and powerful (Michael-Chrysanthou & Gagatsis, 2014). However, not all instructors had positive perceptions about giving feedback and giving opportunities for students to redo their work by incorporating instructor feedback.

Instructors' practices were imbalanced regarding student attendance, participation, and collaboration assessment. All six instructors used this category, but surprisingly the instructors in the traditional group provided more weight to this category than their counterparts in the innovative group. However, none of the six instructors had clear rubrics to assess these student learning attributes, and they primarily relied on their subjective evaluation.

Suggestions for Future Research

This study was conducted over a short period, including a reasonably small number of instructors. Plus, the instructor represented only a few higher education institutions. Thus, we strongly emphasize some future studies in this issue, including more instructors representing other institutions, and collecting data through other means, such as class observations. Reports from such a study may help policymakers and novice and experienced QR instructors successfully implement projects to develop students' QR competencies and support students for their academic and non-academic successes.

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Comparing STEM Majors, Practicing and Prospective Secondary Teachers' Feedback on Mathematical Arguments: Towards Validating MKT-Proof

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Mathematical Knowledge for Teaching Proof (MKT-P) has been recognized as an important component of fostering student engagement with mathematical reasoning and proof. This study is one component of a larger study aimed at exploring the nature of MKT-P. The present study examines qualitative differences in feedback given by STEM majors, in-service and pre-service secondary mathematics teachers on hypothetical students' arguments. The results explicate key distinctions in the feedback provided by these groups, indicating that this is a learnable skill. Feedback is cast as a component of MKT-P, making the results of this study significant empirical support for the construct of MKT-P as a type of knowledge that is unique to teachers.

Keywords: Secondary Teachers, Mathematical Knowledge for Teaching Proof, Feedback

Calls for students to experience deeper engagement with mathematical proof and reasoning across a variety of levels have been long standing (Nardi & Knuth, 2017; NGA & CCSSO, 2010; NCTM, 2009, 2014; Stylianides, Stylianides, & Weber, 2017). In order for this aim to be realized, mathematics teachers must be equipped with a substantial knowledge, both subject matter and pedagogical. The present study is part of a larger, NSF-funded research project, which studied how Mathematical Knowledge for Teaching Proof (MKT-P) of prospective secondary teachers (PSTs) evolves as a result of participating in a specially designed capstone course *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020). The course design and the assessment instruments draw on the MKT-P theoretical framework proposed by Buchbinder & McCrone (2020), which we describe below. At the same time, the project seeks to understand the nature of (MKT-P) empirically; specifically, whether this knowledge is unique to teachers and if it is learnable. To explore these questions, we compared the performance of STEM majors, in-service teachers, and prospective secondary teachers on an instrument designed to assess MKT-P (Buchbinder & McCrone, 2021). This examination of groups with distinct mathematical and pedagogical backgrounds aims to draw out the ways in which MKT-P is unique to teachers. Distinctions in performance of the practicing teachers with respect to individuals with little teaching experience or extensive theoretical knowledge of teaching, but strong knowledge of proof-specific subject matter, could reveal the components of MKT-P, which are unique to teachers (Krauss, Baumert, & Blum, 2008).

The study reported herein analyzes a particular aspect of the MKT-P - the ability to provide instructional feedback on student mathematical work. A set of items on the MKT-P instrument asked participants to provide hypothetical students with feedback on their mathematical arguments. This feedback was quantitively assessed in the larger study through a coding scheme concerned with correctness and mathematical richness of responses (Buchbinder, McCrone & Capozzoli, Butler, in preparation). As we coded these responses, we noticed significant qualitative differences between the feedback provided by the three groups, which could not be captured by the existing scheme. The current study provides in-depth exploration of the differences that caught our attention in the larger study. The research questions are:

1) How does the feedback to a hypothetical student on mathematical arguments compare across three groups of participants: practicing teachers, STEM majors, and PSTs.

2) How does the feedback of PSTs change before and after participation in the capstone course?

Background and Literature Review

In this study, providing feedback on students' arguments is taken as one practice specific to Mathematical Knowledge for Teaching Proof (MKT-P). Inspired by research on the notion of MKT-P (e.g., Corleis, Schwarz, Kaiser, & Leung, 2008; Lessig, 2016; Lin et al., 2011; Steele & Rogers 2012; Stylianides 2011), Buchbinder and McCrone (2020) proposed an MKT-P framework, which consist of three domains: knowledge of the logical aspects of proof (KLAP), knowledge of content and students (KCS-P), and knowledge of content and teaching (KCT-P). KLAP encompasses subject matter knowledge related to the mathematical content of proof, such as knowledge of definitions, proof types, and proof validity. KCS-P describes knowledge about student perceptions of various proof related concepts including knowledge of common difficulties or misconceptions. KCT-P is knowledge of pedagogical praxis specific to facilitating student engagement with reasoning and proof and encompasses such skills as design of classroom activities, the making of instructional moves, and the ability to provide students with productive feedback on their proof related activities.

Feedback is commonly described as "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding" (Hattie & Timperley, 2007, p.81). Extensive research across populations and disciplines has explored the ways in which various characteristics of feedback are related to student achievement (e.g., length, timing, positive or negative nature, and format) (Shute, 2008). Recent work in mathematics education builds on these established relationships between feedback and student learning by focusing on feedback as a teaching practice.

Kastberg et al. (2016) found that mathematics PSTs tend to center their feedback on the performance of the task, processes involved in completing the task, and personal evaluations with more attention to correct than incorrect responses. This finding is echoed by Crespo (2002) who found that PSTs tended to focus their feedback on the process and correctness of student responses, with PSTs supplying students with correct solutions in the case of incorrect responses and issuing praise in the case of correct responses. Over the course of a mathematics methods course, Crespo (2002) found that PSTs shifted the focus of their praise toward the process students used to obtain their answers and shifted the focus of their critiques toward probing student processes via questions and offering guidance toward correct solutions. These shifts are also evidenced by Santos & Pinto (2010) who tracked the changes in written feedback given to secondary students by a single teacher over the first two years of her teaching. They found that the focus of her feedback shifted away from the task and toward the student, errors were corrected less often, symbolic feedback was given less often, the nature shifted from stating facts toward leading clues, and length of feedback increased. Bleiler et. al. (2014) analyzed feedback given by PSTs on sample student proof strategies, finding that PSTs tend to critique student work for use of specific cases over general mathematical representations. Overall, the literature indicates that feedback is a practice which teachers develop over time with this development centered on shifts from product to process and from telling to guiding. This supports our assumption that providing feedback on student mathematical arguments is a particular task of teaching specific to classroom situations involving reasoning and proof; in other words, an element of MKT-P. Comparing feedback across groups with comparable mathematical knowledge but different pedagogical background can provide empirical support for this assumption and illuminate the nature of MKT-P.

Methods

Participants and the Study Setting

Three groups of participants were involved in this study; 17 in-service secondary mathematics teachers, 22 university STEM students, and nine PSTs involved in the capstone course *Mathematical Reasoning and Proof for Secondary Teachers*. The in-service mathematics teachers (11 female and 6 male) had teaching experience ranging between 2 to 25 years ($\bar{x}$ = 12.18, *SD* = 8.26). The teachers were recruited via presentations given at departmental meetings and were compensated for their participation with a $35 ex gratia payment.

The STEM majors and PSTs came from the same 4-year university. The STEM students, mostly computer science and mathematics, second year majors, were enrolled in a mandatory mathematical proof course. The course instructors reviewed the MKT-P questionnaire and confirmed that the students should have all the relevant proof knowledge included in it. The students who volunteered to participate in the study received a small extra credit in the course.

All but one of the PSTs enrolled in the capstone course were in their final year of their secondary mathematics education program, so they had completed the majority of their mathematical coursework, including the same Mathematical Proof course as the STEM majors, as well as two courses specific to mathematics pedagogy. While these latter courses did require some classroom observations, the participants had no personal experience with teaching.

The capstone course (taught by the second author of this paper) intended to enhance PSTs' MKT-P by helping the PSTs to connect their university-level knowledge of mathematical reasoning and proof with secondary school teaching (Buchbinder, McCrone, 2018; 2020). The course was comprised of four modules, each devoted to a particular proof theme: (1) direct proof and argument evaluation, (2) conditional statements, (3) role of examples in proving and (4) indirect reasoning. In each module, the PSTs refreshed their subject matter knowledge of the proof theme and subsequently examined common student (mis)conceptions of the topic. Next, they applied this knowledge by planning and teaching a lesson on that proof theme at a local school. They recorded their lessons using 360 video cameras and wrote a reflection report on their teaching (Buchbinder, Brisard, Butler, McCrone, in press).

Instruments

The MKT-P questionnaire contained 29 items designed to evoke participants' knowledge in each of the three domains of MKT-P. Ten questions were devoted to measuring KLAP, eleven to KCS-P, and eight to KCT-P. The content of the questions spanned the four proof-themes of the capstone course (direct proof, conditional statements, role of examples in proving, and indirect reasoning) framed across a range of topics in the secondary curriculum: number and operations, geometry, algebra and functions. The KCT-P items, which are the focus of this study, consisted of two parts: identifying errors in a sample student's argument and providing feedback to the student, with attention to both the strong and weak points of their work (Figure 1). This type of question requires participants to translate their proof-specific mathematical knowledge into the pedagogical practice of communicating with a student in a way that is productive for their learning. Hence these questions are appropriate for assessing KCT-P. By examining KCT-P items qualitatively, we hoped to detail the differences among the groups of participants, which we noticed in the larger study.

13. Mr. Briggs asked his students to prove the following statement: ***The sum of any two rational numbers is a rational number.***

a) **Paula's solution**:
Suppose r and s are rational numbers. Since r is a rational number, it can be written as a fraction. Similarly, s is a rational number, so it can be written as a fraction by definition of rational number. But the sum of two fractions is a fraction, and a rational number is a fraction, it follows that $r + s$ is a rational number. ■

i) Identify errors (if any) in the student's argument. If none, write "no errors".
ii) Provide feedback to the student, highlighting strengths and weaknesses of their argument.

Figure 1: Sample KCT-P item

Data Collection and Analytic Techniques

The same MKT-P instrument was administered to all three groups. The in-service teachers were given two weeks to complete the questionnaire in an on-line format. The STEM majors completed the paper-and-pencil questionnaire near the end of the course. The questionnaire took 1.5 – 2 hours to complete. The PSTs completed the MKT-P questionnaire *twice*; at the beginning of the course (as an ungraded assignment) and at the end of the course as a final exam.

The feedback provided by participants on KCT-P items was first analyzed for mathematical correctness and pedagogical depth, e.g., whether a student receiving this feedback would be able to improve their mathematical work. Through this analysis, detailed in Buchbinder et. al. (in preparation), we noticed stark differences in the three groups' feedback, which our coding scheme was not capturing. Consequently, we re-coded all data using a newly devised a coding scheme, along four dimensions. First, we counted the sheer *amount of words* in the feedback. Second, we analyzed the *perspective* - to whom the participants addressed their responses. Addressing a student directly (e.g., "you are on the right track") was coded as 1, using the collective "we" (e.g., "think what we are trying to show") was coded as 2, and referring to the student in the third person, was coded as 3. The third dimension was the presence of *questions* in the feedback. Lastly, since participants were asked to highlight both positive and negative aspects of the student's work, we analyzed the content of the feedback for complimentary and critical feedback. We used open coding and the constant comparative method (Strauss & Corbin, 1994) to identify recurring themes (Table 1). The unit of analysis was the appearance of a code in a given response, so occasionally multiple codes were assigned to a given response. Two researchers coded the data individually with initial Kappa agreement of 0.67 (moderate) for compliments and 0.53 (weak) for critiques. All disagreements were discussed and reconciled.

Table 1. Coding scheme for the content of the feedback

Code and Description	Examples
Empty Providing feedback which is devoid of substance beyond recognition of correctness	*Compliment*: Good proof *Critique*: Not proven enough
General Mathematical Comment A general statement about the mathematics of the student's work or describing work in a neutral manner	*Compliment*: Strength: showed multiple examples showing its true *Critique*: An example does not prove so a counterexample is needed.
Student Mathematical Work An explicit connection to the student's work is made in the feedback	*Compliment*: Molly had a great idea to show how the fractions add together to make a new fraction and used the closure property to argue details about the integers.

	Critique: Nicole only uses one example to argue that the statement is true.
Student Understanding Feedback gauges what the student knows or does not know	*Compliment*: You show your strong understanding of what a rational number is and how to use variables to generalize a situation. *Critique*: Calvin lacks an understanding in conjecture which is good to know for the teacher.
Mathematical Value Centered on some mathematical value (brevity/clarity), mathematical writing, or argument structure	*Compliment*: I like that the student attempted to use algebra in a general case to prove it *Critique*: Try to write it more in the 3rd person
Directing Solution The participant tells the student how to correct their work, either through suggestion or direct explanation	*Compliment*: N/A *Critique*: Nicole needs to use variables to show that the statement holds true for all rational numbers

Results

Although the *number of words* in the feedback does not give insight to its substance, it gives a broad sense of the participant's willingness to take seriously the hypothetical student and the mathematics of the problem. The groups showed notable differences in the mean number of words, with teachers averaging 37.43 words per response and STEM majors averaging 27.18 words per response. The PSTs'-Pre[1] feedback was initially closer to the length of the STEM majors', averaging 29.33 words per response; however, PSTs-Post surpassed the teachers averaging 48.69 words per response. This gain could be attributed to the high-stakes nature of the post-questionnaire, but it is nevertheless a valuable indication of engagement.

The groups also differed with respect to *perspective* in which they framed their responses. The teachers had an average perspective rating of 1.87 while the STEM majors had an average rating of 2.29. This means that teachers more often framed their feedback as an address to the student while STEM majors tended to talk about the student or to the collective. The PSTs-Pre aligned with the STEM majors, with an average rating of 2.54, which slightly lowered upon their second attempt to an average of 2.17 (for PSTs-Post). This shift indicates that PSTs began to address students directly in their feedback more often, but still less frequently as the teachers.

Participant's responses were also analyzed for the presence of questions, although these only appeared in about 25% of the data: across all groups, only 74 responses contained some form of question. Of these, 13 (17.6%) were generated by STEM majors and 37 (50%) by teachers, meaning that teachers tended to pose questions to the hypothetical students more frequently than the STEM majors did. The PSTs-Pre posed 9 questions as a group, and 15 questions on the Post, representing 12.2% and 20.2% of all questions in the data set respectively. Yet again, the PSTs-Pre performance on the questionnaire resembles that of the STEM majors, while the PSTs-Post performance shifted toward the teachers' performance.

Whenever possible, the content of the feedback was analyzed for the presence of compliments and /or critiques on student arguments. Complimentary feedback appeared in 62% of STEM majors' responses, 64% of teachers' responses, 73% of the PSTs'-Pre, and 91% of the PSTs'-Post responses. Figure 2 shows the frequency of each *compliment* type (detailed in Table 1) as a proportion of the complimentary feedback from each group to facilitate cross-group comparison.

[1] For ease of communication, we refer to the pre-course data as "PSTs-Pre" and the post-course data as "PSTs-Post"

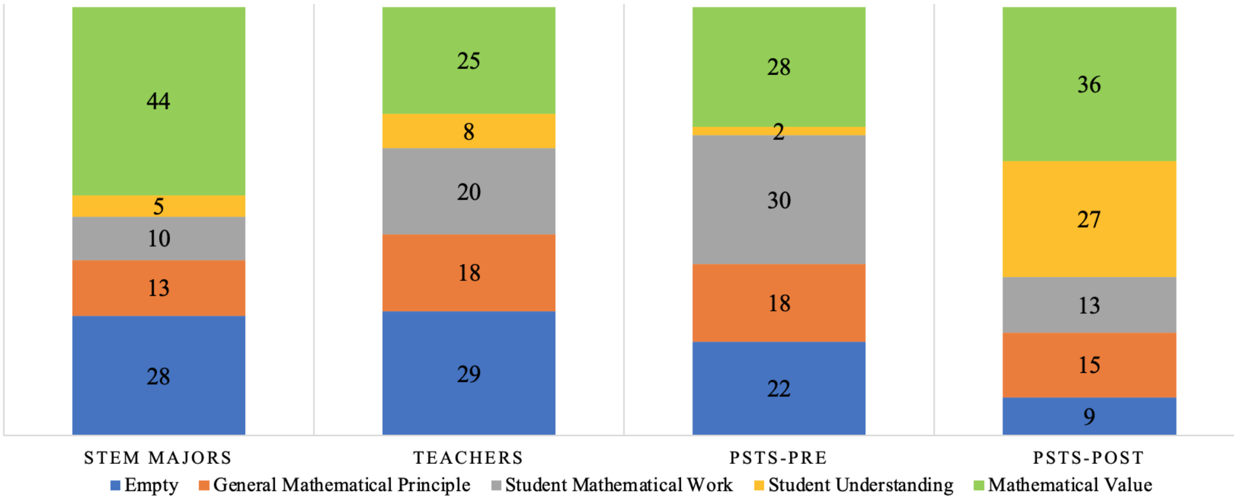

Figure 2: Distribution of compliment types by group (in percent of total complimentary feedback in each group)

Figure 2 shows that the only category for which all groups displayed similar proportions (within 5% of one another) is that of general mathematical principles. Among the notable differences between the groups is the drop in empty compliments of PSTs-Pre to PSTs-Post. While all other groups had similar proportions of empty praises (within 7%), PSTs-Post gave such feedback less frequently, with only 9% of their compliments considered empty. PSTs-Post also showed a large drop (17%) in complimenting student mathematical work, while dramatically increasing their complements of student understanding (2% to 27%). Both teachers and STEM majors showed low rates of such feedback (5% and 8% respectively) making this category a distinguishing feature of the PSTs'-Post complimentary feedback. PSTs also slightly increased their compliments of mathematical values from 28% to 36%. While PSTs-Pre were closer to the teachers on this category (25%), the PSTs-Post fell between the teachers and the STEM majors (44%).

Feedback of a *critical nature* was more frequent than complimentary feedback for both STEM majors and teachers, appearing in 86% of STEM majors' responses, 84% of teacher responses. 68% of PSTs-Pre and 89% of PSTs-Post feedback contained some sort of critique.

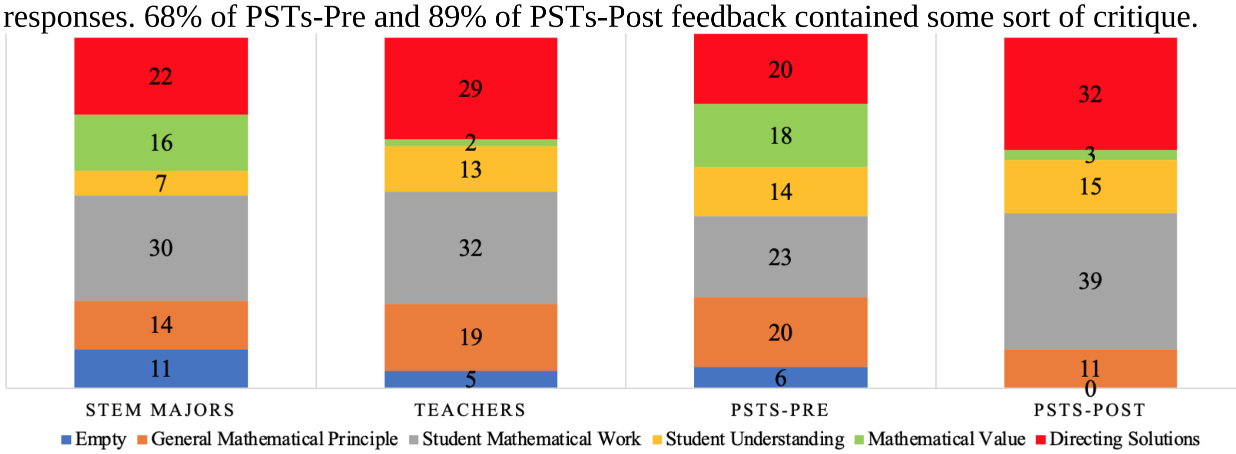

Figure 3: Distribution of critique types by group (in percent out of total critical feedback in each group)

Figure 3 shows the distribution of critical feedback categories for each group in percent of total critical feedback in each group. Compared to Figure 2, each group provided empty critiques less often than empty compliments, with such critiques being entirely absent for PSTs-Post. This indicates that the critiques provided by all groups are substantive and contain more information regarding the nuances of student's arguments. The types of critiques vary by group. Notably, PSTs increased their critiques of student mathematical work and directing solutions. PSTs-Post decreased their focus on general mathematical principles and mathematical values compared to PSTs-Pre. This indicates shifts toward specific aspects of existing student work and revisionary work rather than making general comments. The increase in directing solutions and decrease in mathematical values mark a shift toward the critique profile of the teachers while the decrease in general mathematical principles marks a shift toward the critique profile of the STEM majors.

Discussion

The two research questions of this study asked about qualitative differences in the feedback given by the groups and changes in PSTs feedback before and after the capstone course. Significant differences were found through these comparisons. Teachers tended to provide longer feedback, address the student directly, and frequently posed questions. STEM majors tended to write shorter responses in the third person. Content wise, teachers' feedback tended to focus on the particularities of student work while STEM majors focused on the mathematical conventions and qualities of mathematical writing. Considered holistically, the feedback provided by the PSTs-Pre is qualitatively close to that of the STEM majors, while PSTs-Post feedback is qualitatively close to that of the teachers. This is primarily evident in the shift toward longer feedback addressed to the student and inclusion of questions. It is also evident in the shift of critical feedback toward *directing solutions* and away from critiquing *mathematical values*.

These shifts in the PSTs feedback are consistent with the existing literature exploring PST feedback. PSTs in this study tended to provide complimentary feedback frequently, concurring with Kastberg et al. (2016) and Crespo (2002) that PSTs tend to focus on the correctness of responses. This study also echoes the trend in which PSTs write longer feedback focused on processes as they gain experience teaching (Santos & Pinto, 2010). The literature is furthered by this study as it compares the work of PSTs to both experienced teachers and STEM majors, revealing aspects of feedback specific to teachers and providing insight to their development.

While limited by small sample sized and the high-stakes nature of the PSTs post assessment, the results of this study elucidate MKT-P as a construct; namely, these qualitative distinctions contribute to the empirical validation of this theoretical construct. By drawing out the differences in feedback provided by groups with varying mathematical and pedagogical backgrounds, this study illustrates that there is an underlying difference in the MKT-P of these groups, thereby lending credibility to the construct as something which is both unique to teachers and learnable.

This connection of feedback to MKT-P also carries implications for teacher education. While the course that the PSTs of this study were enrolled in was not specifically aimed at cultivating specific feedback practices, changes in PST feedback were evidenced after taking this course. These changes along with the substantial differences in teachers' and STEM majors' feedback indicate that giving feedback is a learnable practice. By consciously attending to the development of MKT-P in PSTs, teacher preparation programs can better equip their participants to evaluate student work regarding proof and communicate their thoughts about this work back to the student in meaningful ways.

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Shifts in External Authority and Resources for Sense-making in the Transition to Proof-Intensive Mathematics: The Case of Amelia

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This paper explores how one mathematics major, Amelia, perceived and used a diverse set of resources to address specific challenges and learn mathematics through three semesters of proof-based course work. Drawing on data from interviews, homework logs, and classroom observations, we trace how shifts in her available resources shaped the actions she took to maintain personally meaningful mathematical reasoning and learning. Specifically, we highlight the ways that available resources shaped her mathematical agency and autonomy, as we have defined these constructs. These results have implications for further investigation of the resources available to support students' transition to competent proof work.

Keywords: transition to proof, resource use, student agency, student autonomy, authority

In the U.S. context, the transition in collegiate mathematics from computationally-focused introductory coursework to proof-intensive upper-level coursework substantially changes what is expected of students and their mathematical work—both their reasoning and written products Moore, 1994; Holton, 2001; Winsløw et al, 2018). The shift from being a consumer of mathematical algorithms and arguments to a producer or author of mathematical proofs is a nontrivial one, both in how one sees oneself and how one understands the nature of the discipline of mathematics (Selden, 2012). Such an epistemological shift has implications for what students believe about the role of instruction, the kinds of experiences and resources that mathematics class should provide them with, and the ways that they should organize their learning activity, both in class and out of class (Brousseau, 1997). For students who were previously successful in computational mathematics course, upper-division proof-based courses can be quite jarring as they grapple with how to adapt their prior ways of effectively studying and learning mathematics to the new requirements of proof-intensive mathematics courses. The broad aims of our research project are to better understand how undergraduate students construe the opportunities to engage in mathematical reasoning, both in and outside of their mathematics courses, and how these opportunities support (or not) their development as capable sense-makers and proof writers.

In this paper, we focus on a particular issue that is intertwined in the experience of undergraduate students: the role of external resources in students' developing reasoning processes. One way to see students' productive entry into proof work is that in the face of challenges, they consult with perceived mathematical authorities (teachers, texts, internet resources, peers) as appropriate and needed, while retaining their own central role in developing arguments and writing proofs. This paper explores how one mathematics major, Amelia, perceived and used external resources (social, material, and digital) to learn mathematics through three semesters of proof-based course work. We trace how shifts in her available resources shaped the actions she took relative to her mathematical reasoning and learning in her courses. Building upon prior work, we characterize the interplay between opportunities for sense-making and students' actions in terms of agency, autonomy, and authority.

Conceptual Framing

We understand agency as the self-defined space of action that students perceive in any given situation—a space where actors see some actions as possible/permitted and others as risky, costly, or even impossible. In conceptualizing agency, we follow Ahearn (2001) in linguistic anthropology, Bourdieu and Wacquant (1992) in sociology, Calabrese-Barton & Tan (2010) in education in not separating students' actions from the larger social and institutional structures that afford and constrain them. That is, we understand agency, not in terms of students' actions per se, but as their felt capacity to take action in the contexts where they do mathematical work, in or out of the classroom. Conceptualizing agency in this way is consistent with our orientation to understand how students experience the transition to proof, by centering their judgments and sense of what is possible.

> **Mathematical Agency:** The felt capacity to take action in the face of challenges related to mathematical work. This capacity is jointly shaped by contextual norms and resources in specific courses and students' narratives about their views of mathematics, their future in it, their learning processes, their selves relative to authorities.

Students see mathematical authority as residing in their instructors and the solutions they provide, some of their peers, textbooks, and internet sites.

In analyzing the actions that students take, we have been informed by Piaget's notion of intellectual autonomy (Piaget & Inhelder, 1969) where an important facet of an action is whether it is deemed sensible and owned by the actor (intellectually autonomous action) or whether she is dependent on some external authority to validate the action. We depart from Piaget's perspective in our focus on the more or less autonomous character of actions, not individual students.

> **Mathematical Autonomy:** A quality of actions, that reflects the student's active resistance to endorsing, following, or replicating the reasoning of mathematical authorities (e.g., texts, internet sources, instructors, peers), without engaging in sufficient sense-making to make that reasoning their own.

Specifically, we see students' acting more autonomously when they use resources judiciously and locally to resolve specific snags or obstacles in reasoning. That is, our view of more autonomous mathematical actions is not that they preclude students' engagement of resources, but that their use is bounded and subject to the overall goal of making personal sense of mathematics. Acting autonomously precludes the abdication of control for one's reasoning to another.

Grounded in these definitions we pursue the following research question: How do students exercise mathematical agency and display mathematical autonomy in the transition to mathematical proof and subsequent proof-based mathematics courses?

Methods

We observed classroom activity for two semesters of a multi-section introduction to proof course (ITP) and recruited student participants from multiple sections. During that semester, participants completed a baseline interview early in the semester, weekly homework logs, and a task-based interview later in the semester. The baseline interview focused on understanding their experience in mathematics prior to the ITP course and in its early weeks. One of the foci of the later task-based interview was to understand their view of the course, its challenges, and their learning in it. After completing the ITP course, we conducted interviews at the end of each of their subsequent proof-intensive mathematics courses.

We applied our conceptualization of agency and autonomy to these data sources for a small number of our participants, by coding for: challenges, actions taken, and felt capacity, while attending to the opportunities for action provided by courses and the resources that students engaged. During this process, one student's data, Amelia, appeared to shed light on how students' agency can change and in turn how that agency, as felt capacity to act, affects the autonomy of their actions. We analyzed Amelia's experience across her mathematics courses with respect to her work both in and out of class. Memos were constructed of patterns that occurred across time or distinct changes in her agency and autonomy. This work led to a pattern of resources being a key part in her narrative with respect to autonomy and agency both within and outside of the classroom. These were then used to begin to understand how STEM majors exercise agency and display autonomy during mathematical reasoning processes in their ITP and subsequent courses.

Amelia's Transition to Proof Work

Amelia was a junior mathematics major who ultimately planned to become a high school mathematics teacher. We began working with Amelia during her ITP course. She came to her major late after switching from another field and was working through her mathematics degree requirements to finish in four years, so that her fifth year would focus on student teaching. Amelia took four proof-based courses after her ITP course and in that time, her university shifted from in-person to online instruction in response to the COVID pandemic, as depicted in Figure 1. Our analysis focuses on two distinct portions of her transition to proof work: finding and using out of class resources and searching for resources to compensate for lectures she found unproductive.

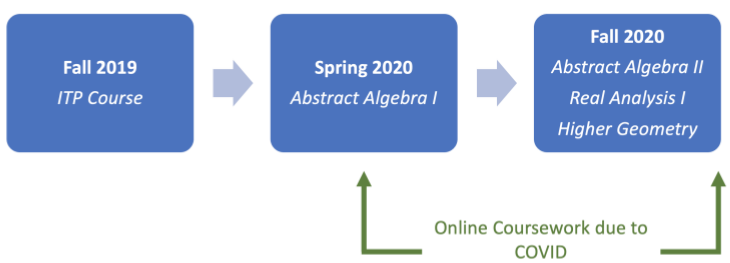

Figure 1. The progression of Amelia's mathematics proof-based courses from Fall 2019 to Fall 2020. The figure denotes when classes switched online (mid-semester), an important contextual shift, via the green arrows.

In her data corpus, we located four main types of resources: people, notes, solution keys, and the university's mathematics learning center (MLC), that she used across the observed semesters. We highlight that notes appear within both parts of our analysis; hers were initially produced in class (as most students do), but Amelia also adapted note-taking as an out of class activity. In each of the following sections, we summarize the narrative she relayed to us relative to opportunities and resources, and then discuss how these informed our analysis of students' agency and autonomy in challenging proof work. In Amelia's transition to proof-based mathematics, shifts in her sense and use of resources were especially apparent.

Finding and Using Out of Class Resources

Amelia's father, an engineer, was a central resource in her mathematical experience prior to her entry intro proof work. During her calculus courses Amelia would call him for help such as writing out the mathematics or explaining different topics. This dynamic began to shift in her ITP course. Initially he hedged his aid by stating he could not "help [her] with any of the fancy words, but this is how you do it" and expressed that he would not have succeeded in the ITP course since he generally cared more about the answer than the process that produced it. As the class progressed, she still reached out and he helped form negations or contrapositives, but ultimately, he remarked that he had 'overestimated' his abilities. Her father could no longer be her 'trump card' in mathematics. Despite the ITP experience, she reached out during abstract algebra I, but to no avail, as she had surpassed him in mathematics.

Once Amelia recognized that she had exceeded her father's expertise in mathematics, she began to explore other helpful out of class resources. She visited the MLC as one potential source; she logged in her (minimal) required hours, but overall found the resource of limited value relative to her homework practice.

Amelia: Sometimes it was just because I was like, busy and the only time I could go is on Wednesdays. And then like other times, it was I was understanding it. I didn't need to go. And then sometimes I just was too lazy to go.

Interviewer: Was it also too early because you, like you said you started homework Tuesday, Wednesday. So you sometimes started homework later than that?

Amelia: Yeah. So sometimes I like hadn't gotten around [to going] because I didn't want to go if I didn't have questions that I was going in with. Like I wanted to have like started the homework and found a spot where I got stuck or like a problem that wasn't in the notes or something like that.

Further, Amelia stated she was more likely to seek out help when she felt confident; when she was not confident in her approach proof tasks, she would put away her work and did not share it. But her central resource was her notes. When she worked on homework, her approach was to, "go through my notes. Because usually it's something that's in the notes recently. And I use my notes on my homework. I never do homework without… I don't think I'd get through it if I did it without the notes." Amelia did not describe her note-taking process, so we don't know her own sense of that process or how close to the instructor's written presentation they were.

Once Amelia completed her ITP course, the MLC was no longer available as it was not staffed for more advanced courses. Due to this change and others, she expressed frustration over lack of resources to her father:

> I was talking to my dad about this that it makes me feel like now I'm like alone on an island of like just, because the only other person I can ask about like, but it's just, she's not in my [abstract algebra I] class. She's not taking that right now. So it's like that class I feel like yes, I have gotten my contact information of other people, but there's no like MLC to go to or like, I don't really want to go to his office hours…That does not help me.

Amelia also reached out to a peer from her ITP course in a different section, but this possible resource was not fruitful as the two sections covered different topics. Ultimately, she was able to get the number of another student in her section, and she would text him sparingly if she was stuck on a part of a problem. During this course, the instructor provided solutions for homework and practice exams, and she used them to compare against her work and for exam review.

She started her abstract algebra II course not knowing anyone, but by the end of the semester she had made a friend. They were taking in multiple courses together and "would kind of compare notes on homework, and if [they] were struggling on a problem or something”. When they collaborated, their typical questions were, ‘Did you do this homework problem? I'm really confused on it. Or something like that, and then either we both be confused… or, Yeah, I did that one. Here's how I did it. Or like, I used this theorem, or whatever.’ Despite establishing this productive relationship with her peer, Amelia was frustrated by the lack of solutions for tasks on practice exams and review sheets. The instructor never uploaded solutions, which deviated from her prior courses, and she remarked that “If I am going to practice this, how do I know that I am getting it right? Because I don’t want to practice it wrong.” She appreciated prior instructors who uploaded them so she could compare her work, but she had now lost access to this out of class resource.

Interpretation Relative to Agency and Autonomy

Amelia’s transition to proof-based mathematics was oriented by the fact that she worked through her high school and early collegiate courses with a strong mathematical authority: her father. But during her ITP course and after, he could no longer serve as a mathematical authority and social resource; Amelia faced the fact that she surpassed him in mathematics. This was dual loss: he was both someone whom she could comfortably interact with about mathematics and was also able to declare whether she is right or wrong. This loss of a resource resulted in a decrement to her agency, as she no longer has the same felt capacity. In her previous courses, she did not always call her father but knew that she *could* call him if she was stuck or was uncertain. In response to this loss, she worked to find resources where she could comfortably discuss mathematics, but also indicated that she felt more comfortable when an authority (e.g., an instructor’s solution key) validated her reasoning and written work.

With respect to solution keys, she remained engaged in her sense making process and completed problem sets by herself, a pattern of actions we see as substantially autonomous. Although her deference to the solution keys for judging whether her solutions were ‘right’ or not suggested a pattern of less autonomous action. We see a similar pattern in her views and use of the MLC. One reason why she found it less beneficial than she hoped was that she wanted to work through problems and hit “stuck points” before turning to the MLC—paralleling her work with her father. This response to the resource of the MLC shows her resistance to replicating the thought processes of others. Instead, she wanted to engage in her own sense making and use resources judicially to resolve stuck-points in her own reasoning—an indication of substantial autonomy in her actions. Amelia sought out others to discuss the mathematics throughout her transition, and one of her highest points of frustration was when she felt that she was “alone on an island.” She reached out to peers in her sections, some of whom she knew were other sections. Although some of these attempts to secure social resources were not fruitful, she was successful with some, particularly in abstract algebra II. Overall, she tried to make decisions and take actions that maintained her level of agency by locating and using a variety of resources. Some actions were more autonomous than others, but overall, she retained, rather than abdicated, her control over her reasoning in her proof-based courses.

Responding to the Increasing Barrenness of Lecture

Amelia attended and actively engaged in her ITP course by asking questions of her peers during group work and of her instructor during segments of lecture. During one observed class session, the class worked on a problem where the TA gave a hint, but after working on the problem Amelia called the TA over because she did not understand a part of the resulting proof. As they worked together, they realized the hint’s algebra was incorrect. Amelia explained the problem to the class so her peers could also fix their algebra. Overall, she engaged in lectures with questions, discussed possible solutions with her peers, and remained focused on making her own sense during class.

Reflecting on abstract algebra I course, Amelia relayed she no longer asked questions in class, partially because the instructor "never left a space for questions to be asked." She added,

> I feel like maybe it's because I just didn't feel as comfortable [asking questions], because when people did ask questions, they would ask a question I would have too, but I wouldn't feel satisfied by the response. Like it didn't help much. So then it kind of was like, Why should I ask my own questions?

Her dissatisfaction with the instructor's responses to questions was one reason why she became more broadly unsatisfied by the lectures. Lectures were almost "word for word from the notes," so she "didn't really get much more from [the instructor]" than reading his notes, other than having someone audibly read them. Dissatisfaction and loss of motivation led her to decide to not synchronously attend lectures (after the shift to online instruction), as she judged this was not a productive use of her time. She continued this pattern of work for the duration of the semester, taking her own notes rather than attending lecture, although she reported that she fell behind towards the end.

In her proof-based geometry class, she listened during lecture rather than copying everything down. Her instructor posted her notes online, allowing Amelia to take notes on them at a later time and then use them as a resource for her homework. She did not take notes during class in her abstract algebra II course.

> But I wouldn't really take many notes during lecture, because a lot of them were very, very similar, to the same examples to what were in the book, is what I started to notice kind of like a couple of weeks in… I would more listen and then copy the notes from the textbook and then use those to help me on the homework So, it kind of felt like... it didn't feel like I was on my own and I was teaching myself, but I was doing those things outside of class, instead of having group discussion, or doing things in groups.

Amelia did attend the synchronous lectures in that course, even when they were not very helpful, because they kept her on track. She noted that if she treated it like an asynchronous course and viewed the lecture later, she would fall behind. In both the geometry and algebra classes she did not ask questions, feeling that the instructor wanted to just move ahead with the content. This dissuaded her from asking, even though upon reflection she felt she should have.

Interpretation Relative to Agency and Autonomy

In Amelia's responses to her courses, we see that she adapted to the interactive emptiness of lecture, in a manner roughly paralleling her adjustment to the loss of her father as a mathematical authority. Specifically, her change in question asking in classes subsequent to her ITP course highlights a key portion of the relationship between agency and autonomy. In her ITP course, she felt that questions were not only encouraged but were ultimately helpful for her mathematical understanding. She reported positive experiences in how her questions were received and answered, both with regards to instructors and peers, allowing her to make effective sense during class. Her comfort with question asking was highlighted in her narrative of not taking a hint at face value even though it was provided by the TA, a mathematical authority. She had the felt capacity to engage in the classroom, and more specifically engage in her personal sensemaking since she had opportunities that aided her in this process. In later classes, her felt dissatisfaction with answers to her questions indicated her continuing focus on personal sensemaking. If she had simply written out the lecture notes during class without processing the meaning or challenging the validity, this practice would indicate less autonomy. Instead, she interacted with the material enough to realize that she needed a different explanation, as she did not have a personal understanding of the topic, leading her to look for other resources.

When reflecting on question-asking in her abstract algebra II, she noted that she should have asked questions. It was not that she did not have questions, rather she felt the space during lecture did not permit this action. Her felt capacity thereby constrained the potential autonomy of her actions. She looked for alternate pathways and support but found even the lectures themselves unhelpful. She exercised her agency by producing and using her notes on her own time, and these became a reliable resource for completing homework and studying—while also seeking productive interactions outside of class. For Amelia, this process of notetaking felt more as if she were having a mathematical discussion, paralleling her search for resources other than her father, with whom she could have these discussions. Her engagement with the textbook and the process of creating her notes substituted in part for the opportunities provided by her prior ITP groupwork. Even though Amelia had not necessarily started with the same felt capacity that she was accustomed to, or was critical for her mathematics learning process, she worked to find other substitutions for her in-class experience thereby increasing her agency.

Conclusion and Future Work

We have used this case of an undergraduate major's transition to proof-based mathematics to illustrate the importance of longitudinal inquiry to understand such transitions and the central role of activity outside of the classroom in them. Conceptually, we have shown how analytic attention to course opportunities (or their absence) and different types of resources can be used to characterize changes in students' mathematical agency and autonomy. Our focus on agency, autonomy, and authority allows insight into how students experience the transition to proof-intensive mathematics courses and why their experiences play out the way they do, something that tracking grades, subsequent course taking, and major/minor completion cannot explain.

The particularities of the case are evident. Many students' transition to proof will not be oriented by the presence and retreat of a mathematical authority like Amelia's father. Further, Amelia's notetaking became a productive practice and resulted in a resource for her where it might not for others. But even in its particularity the case suggests ways to support students in addressing similar challenges in their proof-based coursework at many universities. In identifying these possibilities, we draw on the experience of other project participants, majors and minors whose transitions to proof-based mathematics have shown similar patterns to Amelia's experience as well as clear differences. One design option simply takes seriously the felt barrenness of lecture and to experiment with openings for interaction. Dissatisfaction with lecture as unproductive for learning is not isolated to Amelia as literature questions the modality of lecture both with respect to students' learning and their experiences (Freeman et al., 2014; Laursen et al., 2014, Lew et al., 2016, Prince, 2004). Another design consideration not only encourages collaboration but allots some classroom time for students to find peers for productive work outside the classroom. More generally, it may well be productive to devote class time to a discussion of the range of resources and how to use them—to support rather than undermine students' mathematically autonomous action.

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Challenges in Mentoring Mathematical Biology Model Construction: Quantification and Context

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The researcher observed a group of undergraduate students and faculty mentors collaborating in the development of a model for a student chosen topic in epidemiology. Results suggest that both students and mentors struggled with key understandings necessary to develop the model. Students struggled with conceiving of their compartment model as relating quantities, and mentors struggled with tracking and attending to the biological constraints of the problem the students chose.

Keywords: Ordinary differential equations, mathematical modeling, model construction, student-led projects, quantification

Choosing quantities and their relationships is a critical part of successfully approaching a mathematical model. Equations and graphs in word problem and modeling contexts represent relationships between quantities and their rates of change. However, the notion of quantity receives little focus in the teaching school mathematics (Smith & Thompson, 2007; Thompson, 2008, 2011). Thompson describes quantification as the process of conceptualizing of an attribute of an object as having a measure. Understanding how students and mathematicians imagine and interpret quantities is critical to understanding the process of model development.

Research in mathematical modeling and quantification typically focuses on students working pre-chosen tasks (Bliss et. al., 2006; Gravemijer, 1994; Goldin, 1997; Lesh & Doerr, 2003; Thompson, 2011; Steffe and Thompson 2000). Sometimes these tasks are quite open, and students go through cycles of model development; however, in assigning a task to students, there are constraints placed on students as part of intentionally guiding the students' conceptual development (ibid). It has been argued in the past that these constraints limit students' experiences in developing their own research questions (Castillo-Garsow, 2014; Castillo-Garsow & Castillo-Chavez, 2015). Furthermore, pre-chosen tasks place teachers or teacher-researchers in a dominant role where the teacher has an opportunity to learn and know everything there is to know about the problem. So these pre-chosen tasks rarely highlight difficulties in modeling or mentoring modeling that a teacher would experience.

For example, Camacho et al. (2003) found that choosing one's own project and research question creates situations in which students take the lead in researching topics far outside a mentors' area of expertise, essentially reducing the mentor to a role of consultant rather than leader. Mentors provide mathematical expertise, guiding students by suggesting appropriate tools and techniques. However, students take the lead in these projects, providing the background that defines the problem from subject area research and personal experience (Castillo-Garsow & Castillo-Chavez, 2015). These role reversals in which students are the experts are important sources of motivation and self-efficacy. Choosing a topic that mentors know less about allow students to participate and contribute in ways other than mathematical performance (Rubel, 2017), which is an area in which students cannot compete with mentors (Camacho et. al 2003; Castillo-Garsow & Castillo-Chavez, 2015).

This perspective creates a dichotomy of two ways in which difficulties with quantification can occur: understanding the background context but having difficulty mathematizing, or having an expert understanding of the mathematical tools for modeling, but having difficulty understanding and connecting those tools to the background context due to inexperience with the context itself. Quantification research typically focuses on the former, but rarely focuses on the latter. This study focuses on both. As students and mentors interacted with each other, this paper identifies challenges that arose in quantification for both students and mentors as they collaborated to construct a model in a student-led project.

Methods

This study occurred in the fifth week of an eight-week summer REU in mathematical biology. Prior to this study, the students had taken a three-and-a-half-week course consisting of lecture, computer lab work, and textbook exercises in dynamical systems. Following this course work, students self-recruited into groups of three to five, and chose a topic of interest. During the fifth week, students made daily presentations on their topic to a panel of faculty and graduate mentors who provided feedback. In the final three weeks of the program, students completed the analysis of their model and wrote a technical report on their project. Four groups of students chose to participate in the study, and this a case study from one of those groups. Analysis of the other groups can be found elsewhere (Castillo-Garsow, 2021, 2022).

The group of students in this study was formed of five undergraduate students who chose to construct a model for controlling a disease that can be treated but not cured. The students working on this project eventually developed their work into a published journal article, the citation for which is omitted for privacy. The students made six presentations over six days to a panel of faculty and graduate mentors who provided feedback to the students. Each proposal presentation was video and audio recorded, and the audio recordings were transcribed. Transcripts were open coded (Corbin & Strauss, 2014), and from that coding, themes emerged that identified and explained the primary areas of conflict between mentors and students. The results here are a case study of those transcripts, focusing on creating a narrative of those conflicts (Flyvbjerg, 2006). The purpose of this case study is to identify challenges than mentors and students may encounter while collaborating on a student-led applied mathematics project, both to inform mentors and to inspire future research.

Results

The groups' research question was focusing on the cost effectiveness of treating individuals with mild symptoms of a disease, compared to the current practice of only treating patients in the severe symptom stage. These mild symptoms occurred in many different diseases, meaning that treating individuals with mild symptoms would result in treating many individuals who did not have the disease of interest with medication for the wrong disease. The students proposed studying this question with a system of ordinary differential equations (ODE model).

Student Challenge: Quantity vs. Category

In the students' first attempt at constructing a disease model (Figure 1), the students classified individuals only by their symptoms. The category I_1 therefore contained both individuals who had the disease of study, and individuals who had the same symptoms of a difference disease. Students imagined that individuals who did not have the disease of study would return to S, while individuals who did have the disease of study would progress to I_2 or L. Describing this model, a student said: "After presenting mild symptoms, those mild symptoms go away, then they go back to the susceptible class. Only those who have [the disease] proceed to a progression to the severe symptoms, which are only for [the disease]."

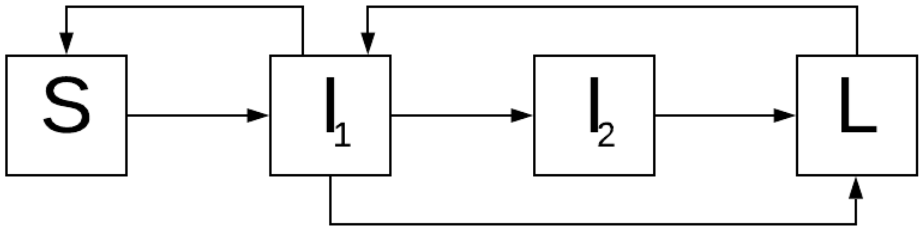

Figure 1: A simplified flow diagram of the disease group's first model. Box S represents susceptibles, I_1 represents individuals with mild disease symptoms from several diseases.. I_2 represents individuals with severe symptoms unique to the disease of study. L represents asymptomatic individuals.

However, in an ODE model, these variables only track a number of individuals. I_1 would necessarily be a numerical quantity of a number of individuals with mild symptoms, meaning that information about who does or does not really have the disease of study could not be stored in this information structure. Students may have made this mistake because they were imagining tracking individuals moving through the categories S, I_1, I_2, and L; rather than imagining S, I_1, I_2, and L as simple numbers of people. In other words, the students were not conceiving of S, I_1, I_2, and L as the values of quantities that could be measured (Thompson, 2011).

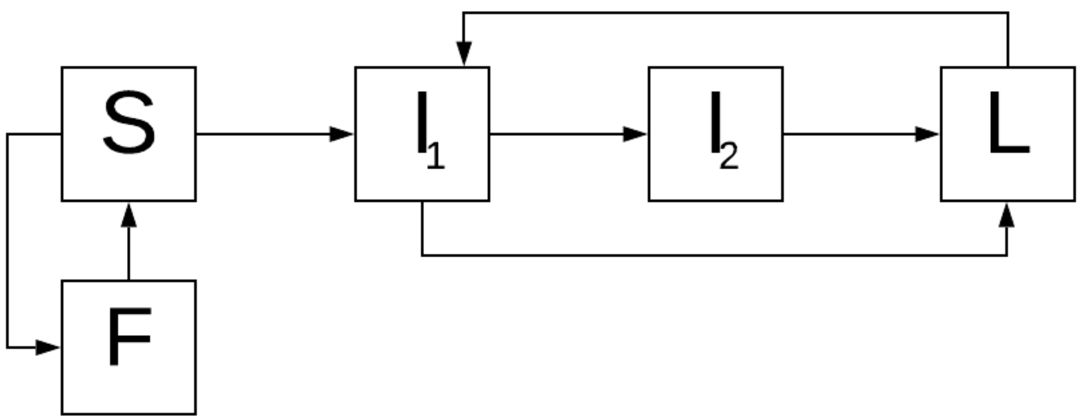

Figure 2. Students' second model, showing an F compartment for individuals falsely diagnosed with the disease of study. The students' fourth and fifth models also had a similar compartment forming a closed loop with S.

Mentors provided feedback on this model informing them that they needed a separate compartment for individuals who did not have the disease "I think you're going to need a separate class for those people" (day 3) and "There's no way to do this without a separate compartment" (day 3). But at this time, mentors did not explain that tracking individuals was not possible in an ODE, or that S, I_1, I_2, and L were numbers. Students responded by creating a compartment F for individuals falsely diagnosed with the disease of study (Figure 2), but this was changed again in the third version (Figure 3).

In the students' third version of the model (Figure 3), they repeated their categorization mistake with a new compartment. Here students imagined that all individuals exhibiting symptoms would receive the same treatment, so all treated individuals were placed in a single compartment. Again, the students imagined that from this compartment, individuals who did not have the disease of study would return to S, and individuals who did have the disease would advance to L, and that the model would somehow keep track of which individuals were which. This resulted in a model with a path from susceptible to asymptomatically infected passing only through treatment – implying that it was possible for treatment itself to cause infection. This time, mentors addressed problems with tracking individuals. As one mentor put it:

> You can't do that because if a person who doesn't have the illness and a person who does have the illness, and they go to the same thing. What you're doing after that is you're saying, both people who don't have it and do have it can now become [asymptomatic]. (Mentor, day 5)

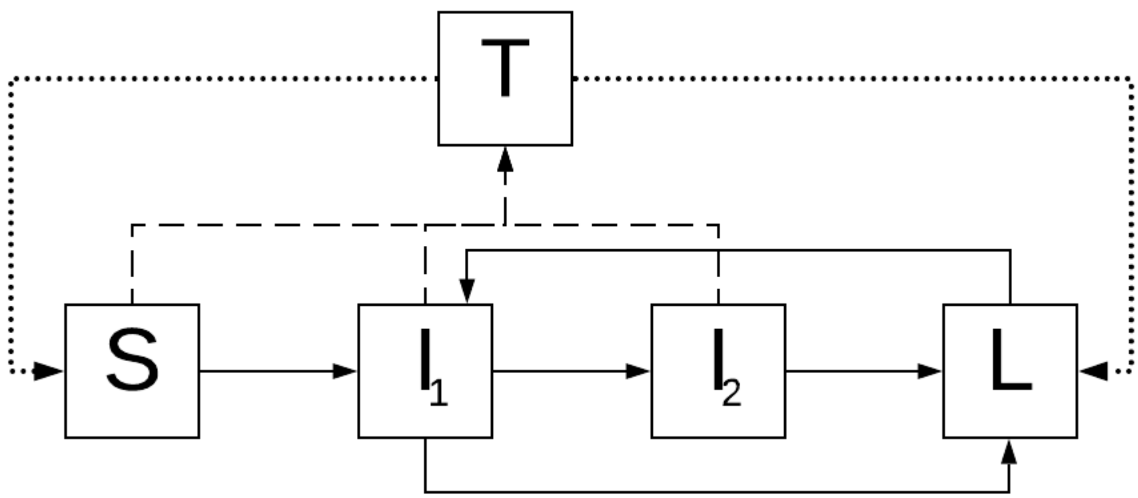

Figure 3. A simplified flow diagram of the disease group's third model, with an added T compartment for treated individuals. Dashed lines represent individuals moving into treatment. Dotted lines represent individuals moving out of treatment.

Another perspective here is that students may have been adopting the point of view of a physician-observer, rather than the point of view of the disease itself. Students wanted to lump all the mild symptoms into a single category because all mild symptoms look alike. Similarly, they wanted to lump all treated individuals together because they all were receiving the same treatment. However, from the point of view of the disease, individuals in these categories had very different diseases, and therefore needed to be tracked separately. In either case, I_1 and T did not represent numerical quantities.

Mentoring Challenge: Dynamics over the research question.

Because students were tracking the cost-effectiveness of treatments that had a risk of being wasted on individuals with another disease, the students needed a way to track the number of falsely diagnosed individuals. These falsely diagnosed individuals would add to the cost without controlling the disease. The students included a compartment, F, for this in their second, fourth, and fifth models (Figure 2).

This F compartment was isolated from the rest of the model in that the variable F did not appear in equations for I_1, I_2, or L. Because this compartment did not affect disease dynamics, mentors objected to its inclusion as unnecessary, and frequently forgot that the compartment was needed to answer the research question of cost. Examples include: "If you only consider dynamics, F compartment does nothing" (day 3) and "You asked the question how to treat early treatment for [the disease]. That part [F] has nothing to do with [the disease]. Why do you have to include this here?" (day 6).

Mentoring Challenge: Testing vs. diagnosis

In the disease model, a key component of the cost was the risk of treating other diseases. This risk was increased because no available test that could distinguish mild symptoms of this disease from mild symptoms of other diseases. The tests that did exist could only be used during severe symptoms, when testing was unnecessary because the symptoms were characteristic. The students frequently stated that there was no test, or that testing was only possible in the I_2 severe stage. However, mentors frequently assumed that there was testing or screening occurring in I_1. See the following excerpt from day 3:

Student: The current test that we have now, there is no way to test if you have [the] disease. There's absolutely no way. The only way you test it if you go here [I_2] and you have a lesion here to take samples

Mentor: The cost of this testing, and the patient, the I_1. The same test?

Student: There's no testing for I_1 in that.

Confusion about testing continued through day 5, where mentors continued to ask questions and suggest changes to the model that involved "testing" or "screening" individuals in I_1. For example, on day 5, suggesting changes to the third model (Figure 3) by incorporating screening: "So then you screen that [S] and once you screen that you put it here [I_1], [you] do not go here [T]." At least some of the confusion arose from students frequently referring to falsely diagnosed individuals as a "false positive," suggesting the presence of a test.

Discussion

Previous research from the project showed that mentoring had the most impact on students' decisions when the mentors focused on asking questions about the biological background, and making suggestions about the mathematics (Castillo-Garsow, 2021). That result is consistent with perspectives found in literature on these student-led projects, which describe students as having topic context expertise, while mentors have mathematical expertise (Camacho et al, 2013; Castillo-Garsow & Castillo-Chavez, 2015). Effective mentoring of a student-led project involves respecting the respective expertise of both students and mentors. This project shows an alternate perspective on the same phenomenon. Here, the challenges in model development arose from students struggling to adopt a mathematical perspective on the problem, while mentors struggled to understand the biological context.

The students in this project had a strong understanding of the biology, but had difficulty communicating that understanding to the mentors. The mentors had difficulty with setting aside their preconceived ideas of what the biology of this disease would be and imagined that testing and treatment occurred in ways that they were more familiar with. Mentors also struggled to attend to contextual concerns – such as cost – over mathematical concerns, such as the dynamical behavior of the model itself.

The students also struggled with quantification. They imagined the modeling process as the story of individuals moving through categories, and/or the perspectives of observers of those individuals. However, writing a mathematical model requires imagining not just individuals, but also numbers of individuals. The students' repeated difficulties in making the transition from category to quantity resulted in errors in the base structure of the model and the corresponding mathematical equations. Mentors initially responded by only correcting the surface level mistake. It was only after the mistake was repeated in a new way that mentors addressed the foundations of ODE model construction with students, specifically the principle that individuals did not have histories that could be tracked through compartments. Here the necessary mentoring expertise was in mathematics in understanding the assumptions and limitations of an ODE model, but also in pedagogical content knowledge by forming a model of the students' interpretation of the model and addressing individual tracking.

Conclusion

The results here add to the literature on student-led projects in mathematical biology modeling. The results of this paper suggest that the reversal of roles described in previous literature (Camacho et al. 2013; Castillo-Garsow & Castillo-Chavez, 2015) is not only sufficient for a successful project (Castillo-Garsow, 2021), but also necessary. In this example, difficulties in model construction came from participants operating outside of their areas of expertise. Students struggled with mathematical concepts, and mentors struggled with biological concepts. However, this struggle outside of ones area of expertise should not be avoided. Rather it was mutual teaching between students and mentors that enabled the participants to collaborate and develop a successful model that was eventually published.

The results of this research also suggest that more research is needed in the ways that mathematicians come to understand or struggle to understand scientific concepts as part of mathematical quantification. Quantification research cannot only address the mathematization of well understood scientific contexts, but must also explore how a developing understanding of the context influences the conceptualization of quantities and the development of quantitative relationships. In particular, further study of mathematics experts developing models of unfamiliar scientific problems would greatly add to our understanding of quantification.

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Challenges in Mentoring Mathematical Biology Model Construction: Mechanisms and Listening

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The researcher observed the project proposal week of an undergraduate research program in applied mathematics, where students took the lead in choosing a topic and developing their own research question. Results from the stories of two of the groups suggest that students struggled with identifying mechanisms that would inform the development of their research question and model, while mentors struggled with allowing students to direct the project and presentations.

Keywords: Mathematical biology, mathematical modeling, mechanism, research question, student-led project

Mathematics as it is taught in secondary and post-secondary classes differs greatly from mathematics as it is practiced by professionals (Lewis & Powell, 2017). Mathematics as it is taught in undergraduate classes is also divorced from the immediate needs of partner disciplines in the sciences, humanities, and social sciences (Ganter & Barker, 2004; Ganter & Haver, 2011). Reports from industry and professional societies repeatedly emphasize the importance of professional skills in communication, collaboration, problem solving, mathematical modeling, and creativity on top of a solid foundation of procedural skills and coherent mathematical understandings (Bliss et. al., 2016; Casner-Lotto & Barrington, 2006; Ganter & Barker, 2004; Ganter & Haver, 2011).

The skill of interest to this report is a skill for STEM and non-STEM described undergraduate mathematics classes described by Bliss et. al. (2016) as "Distilling a large ill-defined problem into a tractable question," which I will simply call "developing a research question." Smith, Haarer, and Confrey (1997) found that graduate students in applied mathematics struggle with developing research questions and mathematical models that are too applied and mathematically intractable, or are too mathematical with little application.

Camacho, Kribs-Zaleta, and Wirkus (2003) found that choosing one's own project and research question creates situations in which students' take the lead in researching topics far outside a mentors' area of expertise, essentially reducing the mentor to a role of consultant rather than leader. And these unique situations create opportunities to identify key components of soft-skills such as developing a research question. Encouraging students to follow their interests and positioning students as experts is also an equity directed practice that engages students with high level questions where students can participate and succeed in more ways than just performance, and students are engaged in contexts that connect with and respect their own experiences (Aguirre, 2013; Castillo-Garsow & Castillo-Chavez 2020; Goffney et al, 2018; Rubel, 2017).

This study follows two groups of undergraduate students who developed a topic of their own interest into a research question and accompanying mathematical model. Because the students in this project work in close collaboration with both graduate and undergraduate mentors, we have an opportunity to see how mathematicians at different stages of their career view the task of developing a research question and accompanying model. The purpose of this paper is to identify challenges than mentors and students may encounter while collaborating on a student led applied mathematics project, both to inform mentors and to inspire future research.

These two groups were selected for discussion due to a common theme in the results. A recurring idea that students encountered in constructing their projects was mechanisms. Although the meaning of “mechanism” remains debated in the scientific community (Craver & Taber, 2015; Hunt et al., 2018), the term mechanism comes from the idea of describing the function of a machine that would replicate a hypothesized scientific process (Craver & Taber, 2015). For our purposes, a mechanism is an interaction between biological parts that could explain an observed behavior, and is precise enough to suggest a mathematical relationship.

Methods

This study occurred in the fifth week of an eight-week summer research experience for undergraduates (REU) in mathematical biology. Prior to this study, the students had taken a three-and-a-half-week course consisting of lecture, computer lab work, and textbook exercises in dynamical systems. Following this course work, students self-recruited into groups of three to five, and chose a topic of interest. During the fifth week, students made daily presentations on their topic to a panel of faculty and graduate mentors who provided feedback. In the final three weeks of the program, students completed the analysis of their model and wrote a technical report on their project. Four groups of students chose to participate in the study, and this a case study from two of those groups. Analysis of the other groups can be found elsewhere (Castillo-Garsow, 2021, 2022).

The two groups in this study were formed of four undergraduate students each. One group wanted to study the relationship between eating disorders and the menstrual cycle (menstrual group). The other group was interested in developing a spatial simulation of tumor growth (tumor group). Both groups completed a technical report that was reviewed by the mentors. Neither group has published their research in a peer-reviewed journal by the time this paper was written. Both groups presented over six days to a panel of faculty and graduate mentors who provided feedback to the students. The menstrual group presented once per day. The tumor group gave seven presentations over the six days, with two presentations on day five. Each presentation was video and audio recorded, and the recordings were transcribed. Transcripts were open coded (Corbin & Strauss, 2014), and themes emerged that identified and explained the primary areas of conflict in goals between mentors and students. The results here are a case study of those transcripts, focusing on creating a narrative of those conflicts (Flyvbjerg, 2006).

Results

Menstrual Cycle Model

The menstrual group was interested in studying the interactions between eating disorders and disruptions to the menstrual cycle. They identified a mediating hormone: leptin, that affected both appetite and the secretion of hormones related to the menstrual cycle (Figure 1). However, the group’s final model did not include any leptin or eating disorders, focusing only on modeling the menstrual cycle.

Mentoring Challenge: Following the students’ lead. Students in this group were extremely consistent about their interests and research question. Their research question changed very little over the six day period, from “Tracking eating disorders through hormonal irregularities in the menstrual cycle” (day 1) to “what is leptin's effect on the induction of the menstrual cycle?” (day 6). However, mentors expressed concern that this research question would lead to a model complex enough to have interesting dynamics: “You cut from [leptin] because that only can impact to the forward. You do not have anything feedback.” (day 2). Eventually, mentors simply forbade students to work on leptin and appetite. “This is your model, only focus on this one, no leptin yet… then, if there is time, you go adding the leptin.” (day 6).

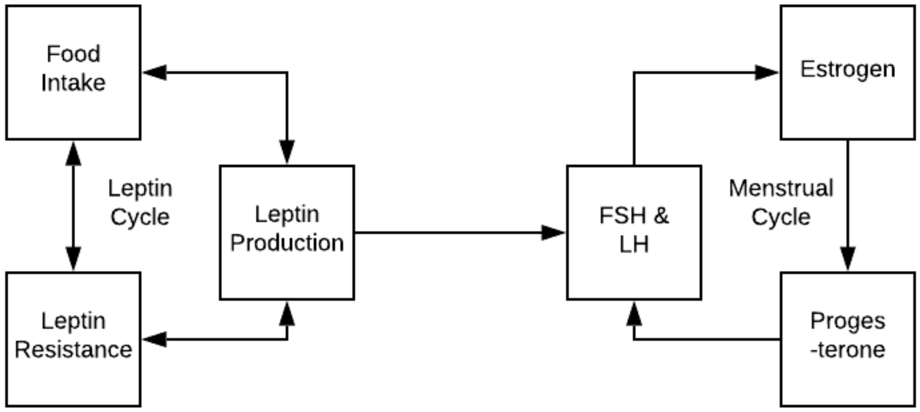

Figure 1. Leptin influences both eating habits and the release of hormones involved in the menstrual cycle.

This problem occurred because the students struggled to explain the mechanisms and relationships that they were interested in when describing their research question. Fundamentally their biological context had two feedback cycles: a cycle in which eating habits and leptin influence each other, and the cycle of hormones that regulate menstruation (Figure 1). What students were interested in studying was really the dynamics of eating disorders (Figure 2).

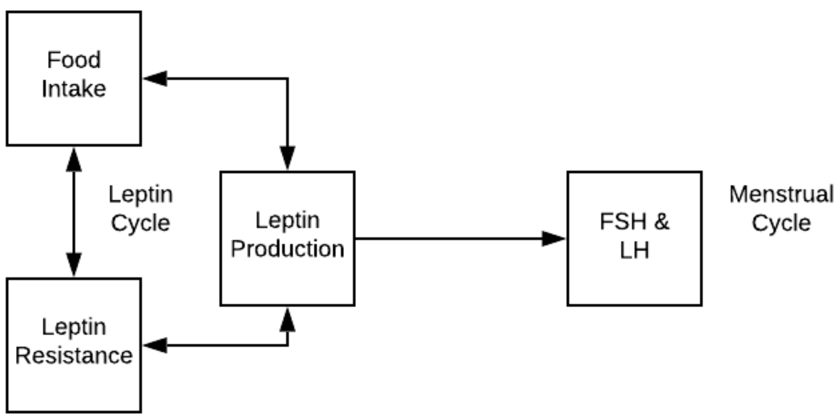

Figure 2. Representations of the part of the problem that held the students' interest.

The mentors had three objections. First was that there were other regulatory processes that influenced the hormones FSH and LH (Figure 1), and that made isolating these hormones from the entire menstrual cycle difficult. Second is that students focused early on leptin early on, and simplified their message to "We're studying the effect of leptin on the induction of the menstrual cycle." This created in the mentors an understanding that students were interested in studying a problem more like Figure 3. This led to the last objection: In both Figures 1 and 3, the effect of Leptin is only feed-forward. There was no obvious impact of the menstrual cycle on leptin production, meaning that leptin could be reduced to a simple input function in a menstrual cycle model, and didn't need to be studied dynamically.

Additionally, with the students found and presented prior modeling research on the menstrual cycle, and mentors, encouraged the students to dive deeper into these models, giving the project a greater and greater appearance of being about the menstrual cycle over time. The mentors did also make recommendations for prior modeling research on leptin, but as the mentors focused more and more on menstruation, opportunities to present leptin research didn't occur.

The students could likely have completed a successful project studying only the leptin cycle that would have been more in line with their interests. However, due to the problems in communication, the mentors steered the group towards a successful project studying only the menstrual cycle.

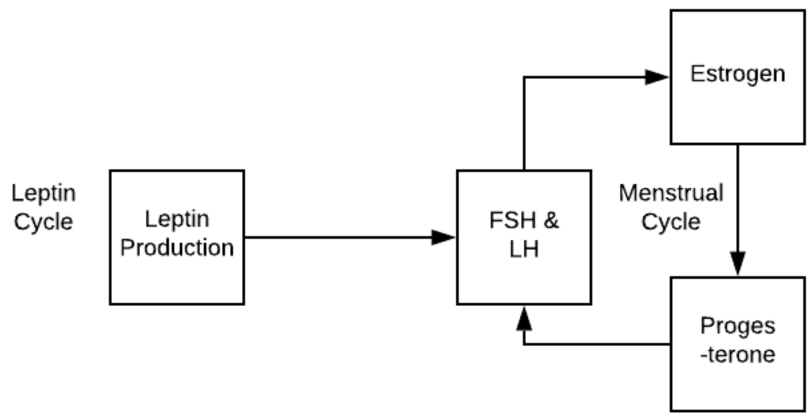

Figure 3. A depiction of a mentor's understanding of the menstrual cycle group's problem.

Student Challenge: Behavior vs. Mechanism. The menstrual group's communication difficulties were exacerbated by their confusion over biological behavior and mechanism. Students would answer questions about mechanisms with answers about biological behaviors. See this excerpt from day 2:

Mentor The levels of FHS and LH, how do they change?

Student: Here's a phase here called the follicular phase and that's where there's follicles that the FSH-- when the secretion of FSH happens, they create follicles that create granules that create the LH. Once the LH is there, these two hormones are able to mature the whole egg for release into the fallopian tube.

In this excerpt, the mentor is asking about mechanisms: asking the students to explain interactions that cause concentrations of FSH and LH to go up and down, or impact the rates at which these concentrations go up and down. The student responds not with an explanation of how FSH and LH change (a mechanism), but instead with an explanation of what FSH and LH do, how they impact the body (a behavior).

The idea of a mechanisms as a biological interaction that defines mathematical expressions is a relatively nuanced idea, and at least one mentor found it unsurprising that students struggled with this idea:

*Student***:** Can I ask a question? You're saying we're not understanding what is the mechanism. Honestly, I don't know what exactly that means in terms of mathematics I'm not sure what that means. Can you explain that to me?

Mentor: This is a very beautiful question that you're asking actually many of us are understanding in a different way also. That's why it will help you when you will draw that flowchart. We will precisely say, "Oh this is what we mean by saying mechanics." [The mentors] are using words like feedback which is almost the same thing as mechanism. Some people use different words for the same concept.

With feedback, students improved at describing mechanisms, “Estrogen at low concentrations has a decreasing effect on LH. The higher concentrations of estrogen, LH seems to have this peak at higher concentrations of estrogen” (day 3); however, they continued to struggle with this distinction and described biological behaviors when mechanisms were asked for: “FSH is entered through the ovaries through the capillaries.” (day 3).

The Tumor Model

Student Challenge: Finding a research question. The tumor group was interested in simulating the growth of a tumor in the lungs using a spatial model. However, with this focus on methodology, the students in the group struggled to find a research question. Over seven presentations and five days, the group proposed several different research questions, including studying the effect of a treatment strategy, studying attacks between cells, studying competition between cells for nutrients or space, or studying the internal processes of a cancerous cell in the lung. Finally with the encouragement of mentors who provide prior modeling research, the students settled on studying a tumor’s chemical defense against the immune system. What remained largely consistent for the group was the choice of topic: tumor growth in the lungs, and the choice of methodology: a two dimensional spatial simulation. The students were a group with a chosen topic and methodology in search of a problem.

Student Challenge: Mechanism vs. model. Part of the reason for the students’ difficulty in identifying a research question was that they struggled to understand the biology in terms of mechanisms of interaction. In contrast with the menstrual cycle group, which described biological behaviors instead of mechanisms, the tumor model frequently described abstract modeling concepts instead of mechanisms.

The core of the tumor model’s struggle was defining the concept of “fitness” in terms of mechanisms. The group wanted to explore a hypothesis that cells could sense the fitness of neighboring cells and induce apoptosis in neighboring cells that were less fit. However, the group struggled with explaining mechanisms that would determine which cells would win in an interaction cells. The could not identify what process a cell was sensing in a neighboring cell, how that process was being sensed, or what determined whether an attempt to induce apoptosis would be successful.

Definitions of fitness given by the students included “an ability of the cell to thrive in a given environment” (day 1), the growth rate of a cell (day 2), a change in the equilibrium of the tissue (day 2), “how normal the cell is” (day 2), “the length of a cell’s cycle” (day 2), “the rate of protein synthesis” (day 2), and the number of cells that a cell divided into (day 2). Several of these ideas, such as “ability of a cell to thrive” or “how normal a cell” is were based on abstract concepts that a modeler might define a measure for, but that a cell would not be able to sense. Others, such as the number of cells a cell divided into, required access to information that the cell could not react to for the fitness sensing hypothesis, because the replication would have already completed. Some of these ideas, such as the rate of protein synthesis, provided potential explanations for what a neighboring cell might be sensing, but students did not follow up on these ideas. Students also answered questions about this fitness sensing process with modeling rules and assumptions, such as: “We wanted to do a spatial model using a grid… If the neighbor cells will have more fitness, it would be able to occupy their space.” (day 2).

One complete mechanistic explanation that the students provided was that a cell would sense a nearby cell’s preparation to divide and release a toxin, and that tumor cells were more fit because they were resistant to the toxin. However, this explanation was not consistent with the hypothesis that neighboring cells sensed “fitness,” as the fitness mechanism occurred after the cell released its toxin, instead of the toxin being released in response to what the cell sensed.

Ultimately students abandoned the fitness sensing hypothesis, and instead built their research question on a previously published model for the interaction between the immune system an tumor growth, focusing on the ability of tumor cells to inhibit the maturation of immune cells by releasing a chemical. The students retained the interest in spatial models and cancer modeling by focusing on the local effects of the chemical release on interactions between tumor cell and immune cell pairs.

Mentor Challenge: Letting students explain at their own pace. Once the students settled on a research question, they developed a spatial continuous time Markov-chain model to simulate the tumor growth. However, the students were not familiar with this terminology, and instead explained the model as a cellular automaton. “We're going to model this with a stochastic cellular automata where our events are going to be discreet and the time of the transmission between the pair of events is going to be continuous.” (day 5 presentation 1). The students had an outline in mind, where they would introduce the idea of a stochastic cellular automata, then random time between events, the list of possible events, examples of these events, and then finally introduce equations for how the probabilities of events were calculated. However, the students were quickly thrown off script.

One mentor focused on the idea of cellular automaton and imagined a simulation driven by simple rules, similar to Conway’s Game of Life. This mentor expected a slide of rules that would be explained all at once. The students felt their simulation was much more complex, and wanted to explain rules and interactions one at a time. This created in mentors the impression that students did not understand the rules that governed their simulation: “Please go learn about Conway's Game of Life and then this will make it much clearer” (day 5 presentation 1), “If T encounters this condition it does this. If T encounters this condition, you do this. You have to work all this out otherwise you're not going to be able to do it.” (day 5 presentation 1). The students were called back for a second presentation on the same day.

In the second presentation, the students maintained a similar structure. Introducing the random time between events before moving to a list of events. “We consider continuous-time Poisson process, the events on the wheel and the waiting time between events are continuous, distributed as exponential” (day 5, presentation 2). The main difference in the presentation was that as one student discussed each event on their slides, another student wrote the equation for the probability of the event on the whiteboard. Mentors were much more receptive in this second presentation, offering direct feedback on the equations and identifying the methodology as a “standard continuous time Markov chain” (day 5, presentation 2). In total, the first presentation lasted almost 42 minutes, while the second presentation was completed in under 10 minutes.

One factor that may have contributed to the mentors’ responses to the first presentation is concern about time. Mentors had previously expressed concern about the short time frame left to complete the project “Remember you have three weeks” (day 4), and this presentation on day 5 was supposed to be the second to last presentation, meaning the mentors had only one more opportunity to review the students’ model as a whole group. The time pressure may have contributed to the mentors’ anxiety about students not having a model or not presenting the model efficiently, leading mentors to take more control of the presentation than was helpful.

Discussion

Both groups of students struggled to understand the modeling concept of a mechanism. The menstrual cycle group identified biological behaviors – such as the maturation of a follicle, or the release of a hormone – rather than identifying causes of those behaviors. They also struggled with identifying mathematical quantities such as the concentration of a hormone, describing instead qualitative behaviors. The tumor group also struggled with identifying mechanisms, although rather than focus on biological behaviors, the tumor group identified abstract constructs such as fitness, and struggled with connecting those modeling concepts to biological behaviors such as sensing the concentration of a particular protein.

While the students struggled with identifying mechanisms, the mentors struggled with letting students take charge. In the case of the menstrual group, concerns about the potential for the project to generate interesting dynamics caused mentors to encourage students to move away from their interest in eating disorders to focus more on the menstrual cycle. In the case of the tumor group, taking charge of the research question was necessary. The students’ focus was primarily on methodology (a spatial simulation of cancer), and they struggled with identifying a problem that would match that methodology while providing innovative insights. Where mentors struggled with the tumor group was letting students take control of the presentation and present the tumor model in a way that made sense to the students. This may have been due to anxiety about the remaining time, or difficulty trusting in the students’ ability to construct a model.

These anxieties are not unique to faculty. For example, Smith et. al. (2021) found that pre-service teachers of mathematics also struggled with anxiety related to trusting students with open ended projects: concerns that students would be capable of completing the project, and anxiety about letting go of control of the classroom. Similarly, the mentors felt anxiety about the tumor group’s ability to construct a model in the one day remaining and took control of the presentation on day 5. In the menstrual group, anxiety about student’s ability to construct an interesting model also led mentors to take some control away from the group, but in a more subtle way.

Although neither group completed the project that the students initially envisioned, both groups successfully completed a project related to their initial idea. And the students in both groups expressed satisfaction and pride in their projects when they were complete.

Conclusion

The students in this study had stated interests in topics, and the mentors were concerned with ensuring that students could complete a scientifically significant project within the time limit. In both cases, students struggled to explain the mechanisms of their projects to mentors, and this led mentors to take more control over the direction of the projects. Understanding and identifying mechanisms is a key challenge in mathematical biology (Reed, 2004), and scientists continue to debate the precise meaning of the term (Craver & Taber, 2015; Hunt et al., 2018). More research is needed in the area of how scientists, mathematicians, engineers, and students understand mechanisms differently.

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How do students think about inverses across contexts? Theory-building via a standalone literature review

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The topic of inverse is one of the most ubiquitous and important in the K-16 mathematics curriculum. We observe that the literature base on inverses is substantial yet context-specific and therefore compartmentalized. In other words, though much research examines students' reasoning with inverses within algebraic contexts, it is currently unclear what might be involved in productively reasoning with inverses across algebraic contexts. To address this issue, we employed a theory-building methodology for systematic, standalone literature reviews to develop a conceptual analysis for inverses across contexts. Our findings include the explicit description and illustration of three cross-context ways of thinking about inverse: inverse as an undoing, inverse as a manipulation of the original element, and inverse as a coordination of the binary operation, identity, and set.

Keywords: inverses, ways of thinking, conceptual analysis, standalone literature review

Introduction

Few topics are as ubiquitous throughout the K-16 mathematics curriculum – and, indeed, in mathematics itself – as inverses. Inverses are initially treated at the elementary level in the context of basic arithmetic operations (in the form of subtraction/additive inverses and division/multiplicative inverses) and eventually manifest in a variety of forms at the secondary level as students engage with the real numbers, functions, and matrices. Undergraduate mathematics majors encounter inverse across a broad spectrum of additional contexts, such as modular arithmetic and symmetries of planar polygons. Students' activity with inverses has generally been well-documented, yet much of what is known concerns only to how students reason *within* specific contexts and thus in its current form provides limited insight into how students might reason with inverses *across* algebraic contexts. Thus, in this study we seek answers to the following research question: what ways of thinking are necessary for reasoning productively with inverses across contexts? To answer this question, in this study we bring coherence to the extensive (yet largely context-specific and, hence, disconnected) literature base on inverses by employing a theory-building methodology for systematic, standalone literature reviews (Wolfswinkel et al., 2011). The theory we develop takes the form of a *conceptual analysis* (Thompson, 2008) in which we explicitly describe and illustrate three cross-context ways of thinking about inverse: *inverse as an undoing*, *inverse as a manipulation of the original element*, and *inverse as a coordination of the binary operation, identity, and set*.

Background

Fink (2014) characterized a standalone literature review as a "systematic, explicit, and reproducible method for identifying, evaluating, and synthesizing the existing body of completed and recorded work produced by researchers, scholars, and practitioners" (p. 3). The word *standalone* is used to distinguish such reviews from those found in typical research articles, which usually do not explicate their inclusion criteria or offer a substantial enough contribution without being paired with the study for which they provide context. Standalone literature reviews, on the other hand, provide explicit search parameters and inclusion criteria, clear descriptions of the analytical techniques, and produce a substantial contribution to the field without collecting any new primary data. The nature of these contributions can vary: while such reviews are often conducted to establish a certain threshold of evidence in favor of a particular policy, strategy, or intervention, they can also serve as a means by which to build theory – that is, to develop "a framework for positioning [future] research endeavors" (Okoli, 2015, p. 3). Wolfswinkel and colleagues (2011) argued that such an approach can help researchers to "extract the full theoretical value out of a well-chosen set of published articles" (p. 2).

Before proceeding, we note that the kinds of research questions we are asking are often associated with methodologies that involve direct interactions with students, such as task-based clinical interviews (e.g. Goldin, 2000) and teaching experiments (e.g. Steffe & Thompson, 2000). This raises a natural question: why answer these questions by conducting a systematic, standalone literature review instead of these more conventional methodologies? First, we propose that the full theoretical value of the inverses literature is currently not being realized: inverses are ubiquitous in the literature, yet nearly all of this literature focuses on students' reasoning *within* particular algebraic contexts and thus there is no coherent, overarching theoretical perspective for analyzing students' reasoning with inverses *across* algebraic contexts. Conducting a standalone literature review therefore offers a rich opportunity to "extract the full theoretical value" from the context-specific inverses literature to develop a coherent cross-context theory. Second, the kind of theory we aim to develop in this study (e.g. a cross-context conceptual analysis of inverse) is an essential tool that informs the conduct of teaching experiments and task-based interviews, yet, as we have noted, no such theory currently exists. Thus, we see the development of a conceptual analysis using a standalone literature review methodology as a precursor to – and not a replacement of – more conventional methodologies involving direct interactions with students.

Theoretical Perspective

Conceptual Analysis

The theory that we develop will take the form of a *conceptual analysis*. A conceptual analysis is an explicit description of "what students might understand when they know a particular idea in various ways" (Thompson, 2008, p. 57). The conceptual analysis we develop in this paper explicitly describes ways of thinking "that might be propitious for students' mathematical learning" (Thompson, 2008, p. 60). Specifically, we are interested in identifying and explicating ways of thinking about inverse that can support productive reasoning with inverse across algebraic contexts. A conceptual analysis can also serve as an integral component of what diSessa and Cobb (2004) call a *domain-specific instructional theory*, informing the selection of productive ways of thinking targeted in learning trajectories and also providing a mechanism with which to analyze and build upon students' thinking along that trajectory (e.g. Simon & Tzur, 2004).

Ways of Understanding and Ways of Thinking

We view students' reasoning in terms of Harel's (2008) *ways of understanding* and *ways of thinking*. These constructs afford a way to distinguish between and characterize students' mathematical thinking at various levels. Ways of understanding are the cognitive products of a mental act – that is, the explanations, descriptions, and procedures that one utilizes in response to a specific scenario. Ways of thinking, on the other hand, are broader perspectives and habits of mind that can be utilized to reason about a variety of scenarios and tasks. These can be inferred by identifying "common cognitive characteristics among [a] multitude of products of each act" (Harel, 2008, p. 497). That is, researchers infer ways of thinking by examining the common characteristics that manifest amongst students' ways of understanding. Though Harel did not set a specific number of times in which one must observe such common characteristics, we infer from his use of 'multitude of products' that such characteristics should be observed across at least two ways of understanding in at least two different contexts.

These constructs provide a clear, tractable way to characterize and illuminate key elements of students' reasoning about inverses. As ways of understanding are the explanations, descriptions, and procedures that one uses in *specific* scenarios, we therefore consider the explanations, descriptions, and procedures that a person associates with inverse within a *specific* algebraic context to comprise that person's ways of understanding inverse. A way of thinking about inverse is therefore inherently more general (and not specific to a single context) because it involves common elements observed in students' ways of understanding in a multitude of algebraic contexts. In this way, ways of understanding characterize elements of students' reasoning about inverses *within* an algebraic contexts while ways of thinking provide the desired insight into what is involved in reasoning productively with inverses *across* algebraic contexts.

The current study involves analyzing the descriptions, explanations, and procedures related to inverse exhibited in published research articles across the K-16 mathematics curriculum. Typically, ways of thinking and ways of understanding are constructs used to explain the mathematical activity of an individual – a way of thinking is characterized by patterns in that individual's reasoning. The current study, however, involves identifying commonalities exhibited across the ways of understanding of multiple (perhaps many) individuals. To resolve this issue, we employ Thompson's (2002) notion of an *epistemic student*, which is a mental depiction of a hypothetical student that a researcher develops in order to explain her mathematical activity. Thompson (2002) notes that "images of this type allow us to propose ways of thinking that are not specific to any one person" (p. 195). Thus, in order to thematically organize and identify common characteristics amongst the ways of understanding inverse that appear in the literature, we do so by imagining that these ways of understanding were exhibited by a particular epistemic student.

Methods

We follow Wolfswinkel and colleagues' (2011) 5-stage process for conducting standalone, theory-building literature reviews. Our procedures are summarized in Table 1.

Table 1. Summary of data collection and analysis procedures and results

Stage		General descriptions outlined by Wolfswinkel and colleagues (2011)	Summary of procedures
Data collection	1	*Define* the search parameters and outlets	Google scholar search for "inverse" in 7 high-quality mathematics education journals
	2	*Search* for relevant articles using the procedures outlined in Stage 1	Executing the search parameters yielded 895 potentially relevant articles

	3	*Select* relevant papers that resulted from the search in Stage 2	We *included* articles in which "inverse" was mentioned (a) on at least 30% of the pages in the article, and (b) at least 10 times total. We *excluded* articles in which "inverse" was used primarily in the context of logic, functional variation, and probability. This resulted in 53 relevant articles.
Data analysis	4	*Analyze* the papers that are selected in Stage 3 in order to develop a theory	Conducted a *thematic analysis* (Braun & Clarke, 2006) of all 53 papers in the sample to identify, describe, and illustrate ways of thinking about inverse
	5	*Present* the categories of the theory that emerged in Stage 4	Related the cross-context ways of thinking and associated themes to research questions and selected vivid excerpts to illustrate these ideas in the paper

In Stage 1 (Define), we used Google Scholar's "advanced search" feature to search for the word "inverse" in each of the 7 journals that were consistently rated as being of "high" or "very high" quality by both of the most recent assessments of journal quality (Toerner & Arzarello, 2012; Williams & Leatham, 2017): *Journal for Research in Mathematics Education*, *Educational Studies in Mathematics*, *ZDM*, *Journal of Mathematical Behavior*, *Journal of Mathematics Teacher Education*, *For the Learning of Mathematics*, and *Mathematical Thinking and Learning*. Stage 2 (*Search*) produced a list of 895 potentially relevant articles. In Stage 3 (select), we selected papers in which (a) the percentage of pages in the article containing the word "inverse" is at least 30%, and (b) the word "inverse" is mentioned at least 10 times. These criteria narrowed the list to 61 articles. We then recorded the algebraic context(s) in which inverses were discussed for each, removing in the process 8 articles invoking a meaning of inverse that we deemed irrelevant for our purposes (e.g. the inverse of a conditional statement in logic, inverse variation of quantities, and inverse probabilities). Our final sample of 53 articles is listed in a special section ('Bibliography of the 53 Articles in Our Sample') after the references.

For Stage 4 (Analyze) and Stage 5 (Present), we used *thematic analysis* (Braun & Clarke, 2006) to analyze of all 53 papers in the sample. Thematic analysis aims to develop theory by "systematically identifying, organizing, and offering insight into patterns of meaning (themes) across a data set" (Braun & Clarke, 2006, p. 57). This aligned with the overarching goal of our literature review: to develop a coherent, unifying theory that shines light on themes related to students' reasoning with inverses that are present but perhaps obscured due to the context-specific and fragmented nature of the inverses literature. It also aligned with our theoretical perspective: common themes observed across ways of understanding within particular contexts would emerge as the definitive characteristics of the desired cross-context ways of thinking about inverse.

Results

Inverse as an Undoing

Inverse as an undoing is a way of thinking about inverses in which "inverse is associated with an operation that cancels the previous operation and 'returns to the starting point'" (Zazkis & Kontorovich, 2016, p. 107). We highlight two definitive characteristics:

- *Characteristic U1*: inverse is viewed as an operation (or a sequence of operations)
- *Characteristic U2*: the operation (or sequence of operations) is applied to undo or cancel the effects of the original operation

The following excerpts illustrate various ways of understanding exhibited by students that we associate with *inverse as an undoing* within particular contexts (specified via the set and relevant binary operation):

- *Real numbers*: "The unknown was multiplied by 4 and then 17 was added. I can undo it by first subtracting 17 and then dividing by 4, thus the unknown is 15" (Selter et al., 2012, p. 403).
- *Functions*: "If I were to put 5 into $f(x)$ I would get 8, I would then show them that if I put 8 in $f^{-1}(x)$ then I would get the number I started with originally" (Weber et al., 2020, p. 14).
- *Linear transformations*: "You're pretty much transforming [the vector] into something else, and the inverse really just transforming to, or transforming it back to what it originally was" (Bagley, Rasmussen, & Zandieh, 2015, p. 40).
- *Symmetries of regular polygons*: "**Jessica:** Because any time you go clockwise and then counterclockwise, you're just going back to your spot. The same spot that you started with. **Sandra:** That's a good observation. … you're going to end up with nothing. So they're all just canceling, these four moves are just basically canceling each other out" (Larsen, 2009, p. 124).

Inverse as an undoing is compatible with the notion of *inversion*, a construct from the K-8 literature that involves "recognizing that adding and then subtracting the same number (or vice versa) leaves any initial number unchanged" (Baroody & Lai, 2007, p. 131). For example, "a child might induce that adding 1 to any (small) collection can be undone by taking 1 away" (p. 132). Notice that this description focuses on the relationship between the *operations* of addition and subtraction (Characteristic U1) and, specifically, how addition "can be undone" by subtraction (Characteristic U2).

Inverse as a Manipulation of the Original Element

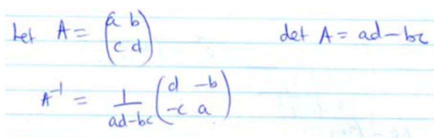
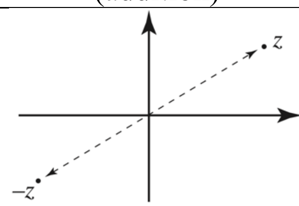
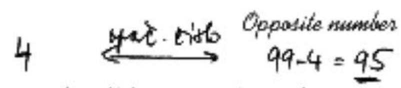
Inverse as a manipulation of the original element is a way of thinking in which inverse is associated with a procedure by which a given element can be manipulated into its inverse element. This way of thinking is consistent with the observations of researchers that some students might understand inverse exclusively in terms of the application of inverse *procedures* (e.g. Paoletti et al., 2018; Wasserman, 2017). We identified two definitive characteristics:

- *Characteristic M1:* inverse is viewed as an *element*;
- *Characteristic M2:* inverse is associated with a procedure by which a given element is manipulated into its inverse element.

Examples of ways of understanding that we associated with this way of thinking are included below; selected illustrations that we associated with this procedure are shown in Table 2.

- *Real numbers*: Additive inverses can be found via "a unary function requiring a single input (the 'opposite of x' denoted by -x)" (McGowen & Tall, 2013, p. 529) – this function involves multiplying the original element by -1.
- *Real numbers*: "'flipping' the numerator and denominator of a fraction creates an inverse element to an original fraction" (Zazkis & Mamolo, 2011, p. 12).
- *Functions*: "We rewrite the […] relation which represents the inverse function of the given function, as $y = (x-3)/2$, by which we mean switching the variable x with the variable y in the equation which was found, after solving it in terms of *y*" (Pinto & Schubring, 2018, p. 901).

Table 2: Illustrations of procedures associated with *inverse as a manipulation*

Matrices (multiplication)	Complex numbers (addition)	Modular arithmetic (addition mod 99)
 (Kazunga & Bansilal, 2020, p. 349)	 (McGowen & Tall, 2013, p. 530)	 (Simpson & Stehlikova, 2006, p. 359)

Inverse as a Coordination of the Binary Operation, Identity, and Set

Inverse as a coordination of the binary operation, identity, and set is a way of thinking in which inverse is viewed as a relationship between a pair of elements of the set in question, the relevant binary operation, and the relevant identity element – specifically, that the combination of an element and its inverse element via the relevant binary operation yields the relevant identity element. We define it via the following three characteristics:

- *Characteristic C1* (binary operation): inverse is viewed in terms of as relationship between elements and their image with respect to the relevant binary operation
- *Characteristic C2* (identity): involves an explicit awareness that the two elements that combine (with the relevant binary operation) to yield the identity element are inverses if and only if their combination yields the relevant identity element
- *Characteristic C3* (set): attends to the fact that both an element and its inverse element are elements of the set in question

The following excerpts are representative of ways of understanding exhibited by students that we associate with *inverse as a coordination* (see also Table 3, which includes images of students' inscriptions that illustrate similar ways of understanding):

- *Real numbers (addition)*: "So we need to actually add 8 in order to do that. […] when you have a number and its additive inverse. When you add those together we get the additive identity zero. So we actually created a zero pair here" (Clay, Silverman, Fisher, 2012, p. 769).
- *Functions (composition)*: "If you're composing functions and you compose the function with its inverse, […] it produces the identity function. So you plug in '*a*' to that

composition, then you get out '*a*'. So we know that functions are a group (pause) and the identity function serves as the identity for that group, composition being the binary operation" (Wasserman, 2017, p. 189).

Table 3: Illustrations of student inscriptions associated with *inverse as a coordination*

Matrices	Symmetries of regular polygons	Modular arithmetic (modulo 99)
$\begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (Bagley et al., 2015, p. 42)	~~If~~ For every move A, if ~~A·B=B·A then~~ A·B=B·A=I, then B is the inverse of A written A^{-1}. Every move has an inverse. (Larsen, 2009, p. 125)	$4 \oplus 95 = 99$ (Simpson & Stehlikova, 2006, p. 359)

Consider, for example, the excerpt above concerning the compositional inverse of a function. Characteristic C1 is readily observable, as the student explicitly mentions composition several times (e.g. "composing functions", "composing the function with its inverse," "composition being the binary operation"). The same is true for Characteristic C2, as the student explicitly identifies that a function composed with its inverse function "produces the identity function." Characteristic C3 is observable but perhaps less explicit: the student refers to "composing *functions*," suggesting that, when he refers to composing "the function with its inverse," he is aware that the inverse is also a function (and therefore a member of the same *set*).

Discussion

The primary contribution of this study is the conceptual analysis of inverse, which provides detailed descriptions of three ways of thinking about inverse: *inverse as an undoing*, *inverse as a manipulation of the original element*, and *inverse as a coordination of the binary operation, identity, and set*. The fact that it can be used to describe the nature of students' reasoning about inverses across contexts makes it – to our knowledge – the only one of its kind in the literature. In addition to venturing ways of thinking that can support productive reasoning with inverses, we note that this framework takes a key step towards bringing coherence to the inverses literature. Though space constraints have prevented us from exploring these ways of thinking in greater detail, here we shall simply note our primary claim: all three of these ways of thinking play a key role in supporting productive reasoning with inverses across algebraic contexts. Though this claim (which, we acknowledge, is not directly addressed in this short report) is supported by the literature, the theoretical nature of its derivation leaves two possibilities for empirical refinement and extension. The first is simply testing (and perhaps refining) this claim using methodologies that involve direct interactions with students. The second involves investigating how students might be supported in developing these three ways of thinking using, for example, the teaching experiment methodology.

We would also like to call attention to the standalone literature review methodology and its value. In the current study, this methodology enabled us to unearth valuable insights about the nature of reasoning about inverses that were indeed already documented in the literature (in some cases very well so) but were far from being widely recognized. More generally, we suspect that the inverses literature is not unique in being fractured and compartmentalized. We therefore see the methodology we employed in this study as a mechanism that researchers can use to combat this issue, particularly in cognitive research as a means of developing an initial conceptual analysis prior to conducting task-based interview or teaching experiments.

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Observing Intellectual Need in Online Instructional Tasks

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The idea of intellectual need, which proposes that learning is the result of students wrestling with a problem that is unsolvable by their current knowledge, has been used in instructional design for many years. However, prior research has not described a way to empirically determine whether, and to what extent, students' experience intellectual need. In this paper, we present a methodology to identify students' intellectual need and also report the results of a study that investigated students' reactions to intellectual need-provoking tasks in first-semester calculus classes.

Keywords: Intellectual Need, Instructional Videos, Flipped Pedagogy

Problem solving has long been viewed as both an essential source and product of mathematical learning. However, Fuller, Rabin, and Harel (2011) characterize much of students' engagement with mathematics as "problem-free" in the sense that the mathematical "problems" students encounter can ordinarily be completed by applying skills and understandings previously developed, and thus are better characterized as "exercises." In contrast, "problem-laden" activity originates in and is sustained by students' construction of a problem in such a way that they (1) recognize their current knowledge structures as insufficient to solve the problem, and (2) construct an image of the understandings that would enable them to progress towards a solution. To address this issue, Harel (1998) proposed the *necessity principle*: "For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to intellectual need" (p. 501). Although the construct of intellectual need has been widely applied in instructional design (e.g., Harel, 2013b; Koichu, 2012; Caglayan, 2015; Foster & de Villers, 2015) and analysis (e.g., Rabin, Fuller, & Harel, 2013; Zazkis & Kontorovich, 2016), prior research has not developed methods for empirically identifying students' experiences of intellectual need.

The goal of this paper is to explore the possibility of explicitly identifying students' self-reported experiences of intellectual need and to examine related factors that might be associated with these experiences. In addition, we seek to explore the relationships between students' experiences of intellectual need and their learning from instructional videos that were designed to present solutions to intellectual need-provoking [IN-P] tasks and to explicate the understandings and ways of reasoning required to construct these solutions.

Theoretical Framework

The concept of intellectual need is situated within an elaborate theoretical framework called *DNR-based instruction in mathematics* (Harel, 2008a) and is informed by two key theoretical premises: the *Knowing Premise* and the *Knowing-Knowledge Linkage Premise*. The Knowing Premise states, "Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium" (Harel, 2008b, p. 894). Relatedly, the Knowing-Knowledge Linkage Premise states, "Any piece of knowledge humans know is an outcome of their resolution of a problematic situation" (Harel, 2008b, p. 894). The Knowing and Knowing-Knowledge premises derive from Piaget's (1971) genetic epistemology and von Glasersfeld's (1995) radical constructivism.

Informed by the Knowing and Knowing-Knowledge Linkage Premises, Harel (2013b) describes intellectual need as the perceived need to resolve “a perturbational state resulting from an individual’s encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, his or her current knowledge” (p. 122). This perturbation is rooted in the individual’s experience within the discipline and is based on “the learner’s discernment of how and why a particular piece of knowledge came to be” (Harel, 2013a, p. 8).

Intellectual need is distinct from *psychological need*, which is the motivation a student experiences to initially engage in the process of solving a problem (Harel, 2008). Harel (2013b) suggests that psychological need is often linked to students’ perceived obligation to participate in school, to increase social or economic status, or to advance societal goals. In particular, a student’s perception of how interesting or enjoyable they find the context could influence their motivation for engaging and persevering in solving the problem.

There is little discussion in the research literature of what might constitute evidence for students’ experiences of intellectual need. In most research, claims of students experiencing intellectual need have been associated with students’ expressions or activity that indicates puzzlement or curiosity. However, the data-collection protocols used in these studies did not appear to explicitly interrogate students’ experiences of these psychological states or the assimilations that occasioned them.

For this study, we operationalize intellectual need by using the colloquial ideas of puzzlement and curiosity. That is, we can ask students whether they felt curious or were left wondering about something as they engaged in a task or problem context. We also distinguish these feelings vis-a-vis the intellectual content of the task from the student’s interest in the underlying context—that is, an aspect of their psychological need for engaging in the task.

Research Questions

The goal of our study is to identify instances in which students experience intellectual need and some factors that are related to this experience, as well as their learning from the associated instructional videos. Thus, our research questions are:

1. How much variation of students’ intellectual need is there between video sets (i.e., collections of instructional videos, pre/post-video questions, and related material for topics in first-semester calculus)?
2. Are different instructors associated with different rates of students’ intellectual need?
3. Does trying an intellectual need-provoking [IN-P] task and/or watching a student problem-solving video lead to a higher rate of intellectual need?
4. Does a student’s mathematical background knowledge predict their experience of intellectual need?
5. Is there a relationship between psychological need and intellectual need?
6. Is there a relationship between students’ intellectual need and their learning from the associated collection of instructional videos?

Methodology

Our methodology addresses three issues: First, we needed a way to potentially provoke students’ intellectual need. Second, we needed a way to adapt these provocations to an online environment. Third, we needed a way to identify students’ experiences of intellectual need.

Provoking Intellectual Need: Task Construction

We needed to engage students in tasks that had the potential to provoke intellectual need. For each mathematical topic we investigated we examined the epistemology of the underlying concepts and procedures, the curriculum in which the concepts were embedded, and the research literature about the likely background knowledge of the students who would be enrolled in the class. Then, we collaboratively designed a problem for which the target concept was required to arrive at a solution and that provided an opportunity for a student to experience a perturbational state while working on the task or watching the associated student problem-solving video (described below). We also endeavored to situate the problem within a context that, we felt, might be interesting to the student population.

Adapting to an Online Environment: Student Problem-Solving Videos

We hypothesized that simply viewing and trying to solve a task might not lead a student to experience perturbation. In particular, we thought it might be possible for students to not realize that their initial way of thinking about a problem might be inadequate. Thus, we sought a way to help students identify shortcomings in their solution methods or reasoning about the tasks. To do this, we designed a "student problem-solving video" to accompany each IN-P task. In each video, a pair of actors posed as calculus students and attempted to solve the task. These videos were loosely scripted so that the actors demonstrated a variety of compelling ways of thinking about the task and concept that incorporated common student (mis)conceptions about the concept. The videos were presented to the students after they had attempted the IN-P task.

Identifying Intellectual and Psychological Need

After interrogating the concept of intellectual need, we felt that the terms "curiosity" and "wonder" were closely related descriptors. After attempting the IN-P task and watching the student problem-solving video, the students were asked the following two questions:

1. The task you just worked on dealt with the context of [context—e.g., "the speed of a baseball"]. In your honest opinion, how interesting/enjoyable was this context?
2. When you were working on this task, were there any parts where you genuinely were curious or were left wondering about something? If so, please state them in the box below; if not, please leave the box empty.

The first question was designed to enable students to self-identify an experience of psychological need; the second question was designed to enable students to self-identify an experience of intellectual need. Throughout the results, when we refer to a student experiencing an intellectual or psychological need, we mean that they responded "yes" to the corresponding question above.

Materials, Participants & Methods

Materials

We designed a set of 1-3 instructional videos for each of 30 target concepts in introductory calculus; each set of videos included a solution to one of the IN-P tasks and some additional explanation of the underlying concept. We also created a collection of multiple-choice and computational problems to be solved prior to and after watching the instructional videos, an IN-P task, and a student problem-solving video. For each video set, each student was randomly assigned to either try the IN-P task or not and to see the student problem-solving video or not.

Participants

The participants in the study were 2,733 students who were enrolled in first-semester calculus classes at one of 18 universities during the fall, 2018 and spring, 2019 semesters. The universities included both public and private institutions, ranging in size from just over 3,000 to over 35,000 students, from all regions of the United States and one institution in Indonesia. The institutions included small, private colleges through large, research-focused universities.

One member of the research team was the calculus course coordinator at their institution (a large, public university in the South-Central United States; we will refer to this as the Coordinated Institution) and incorporated a subset of the video sets into the curriculum during both the fall, 2018 and spring, 2019 semesters, for a total of 15 instructors coordinated sections. The other 18 instructors (33 instructors total) were voluntarily participating in the project research, and each selected a subset of the video sets to incorporate into their curriculum.

Statistical Design

We coded students as experiencing intellectual need if they responded "yes" to the question described in the methodology and experiencing psychological need if they indicated they found the context "somewhat" or "very" interesting. We measured students' background knowledge by the percent of pre-video questions they answered correctly, and coded students as having learned from the instructional videos if their score on the post-video questions was higher than on the pre-video questions.

Due to the nested and cross-nested nature of our data, we used Hierarchical Linear Models (Raudenbush & Bryk, 2002) to perform our statistical analysis. We use two models to answer our research questions: one with intellectual need as the outcome, and one indicating growth from pre-video questions to post-video questions as an outcome. Since both of our outcome variables were measured as a 1 or 0, the regression at the lowest level is a GLM model using logistic regression. The predictor variables in our models included whether or not the student saw the IN-P task; whether the IN-P task was computationally focused or not; the student's score on the pre-video questions, the number of the video set in the semester, whether a student watch the student problem-solving video; whether or not the student was at the large, coordinated institution; and, for the learning model, the student's response to the intellectual need question. Due to space constraints, we report results for only a subset of these variables. The models were run with about 26,000 online lessons, 1550 students, 25 instructors, 14 institutions, on 30 topics. This data set was smaller than the original dataset due to missing data, or small number of students in some classes/institutions.

Results

Unconditional Model for Intellectual Need

We ran unconditional mixed models to understand the variation at the student, instructor, and video set levels. We found that a typical student working on a typical video set from a typical instructor would experience an intellectual need in 4.5% of the video sets. However, different students have different rates at which they report experiencing an intellectual need: students 1 standard deviation less than the mean only report experiencing an intellectual need on 1.3% of the video sets and students 1 SD above the mean report it at 14.5%. There is similarly large variation between instructors (1.4% to 13.5%), and this increases to (0.4% to 35%) for two standard deviations. This means that some teachers rarely have average students (in a typical lesson) who report experiencing an intellectual need, while at the other end, some teachers have average students reporting an intellectual need on about a third of the video sets.

Because most institutions in our data set are represented by a single instructor, it is difficult to tease apart variation due to instructor and variation due to institution, curriculum, or pedagogy. To address this, we ran an unconditional model using only the data of the coordinated institution and found that the standard deviation at the instructor level was only 10% less than at the other institutions. The variation across video sets is less than between teachers or between students in a class, but the two standard-deviation range is 2% to 9.6%. This shows that some lessons are nearly 5 times more likely to generate an intellectual need than others. At the coordinated institution, the standard deviation across video sets was 20% lower.

Conditional Model for Intellectual Need

The results of the conditional mixed model with intellectual need as an outcome are displayed in Table 1. The asterisks indicate the level of statistical significance (i.e., * designates the result was significant at the p=0.05 level; ** indicates significance at the p=0.01 level; and *** at the p=0.001 level). The coefficients are given in log-odds. The percentages for the coefficients are marginal percentages given a unit increase in the variable from the model intercept, with all other variables equal to zero. The percentages are not additive like in a linear regression model, and the effect of a variable could be larger or smaller than the listed percentage depending on the values of other variables.

Variable	IN Outcome Coefficient	Marginal Percentages
Intercept	-2.12***	10.7%
IN-P Task (*IT*)	-0.294***	-2.5%
Problem-Solving Video (*V*)	-0.553***	-4.2%
*IT***V*	0.532***	6.3%
Psychological Need	0.382**	4.3%
Pre-Test Score	0.768**	9.9% per SD
Lesson Order	-0.044***	-0.4% per lesson
Coordinated Institution	-1.35***	-7.7%

Table 1. Results of the conditional mixed model for intellectual need.

We summarize these results below:

- Both the effects of trying the IN-P task and watching the video are, individually, negative. However, the effects are not additive, and there is a significant interaction term between the two predictors. Thus, students who both saw the IN-P task and the student problem-solving video were more likely to report experiencing an intellectual need than students that only saw the video.
- Typical students are about 40% more likely to report a self-reported IN if they report experiencing a psychological need.
- Students with above average pre-video achievement scores (1 SD above average), are almost twice as likely to report having an intellectual need.
- The rate at which students reported having an intellectual need decreased, on average, across the semester. Lessons near the beginning of the semester had a rate of 16.9%, while lessons near the end had an average rate of 6.9%, for students with zeros on all other variables.

Models for Learning

The unconditional Learning model shows that a typical student would improve from their pre-video to post-video score on 42.4% of the typical video sets. The conditional model for Learning shows that there is a significant positive association between experiencing intellectual need and learning. Typical students show evidence of learning on about 7% more lessons if they experience an intellectual need, an increase of about 23% (from the average/intercept of 29.9% up to 36.8% of lessons) over students that do not experience an intellectual need.

Discussion

Methodological Contributions

One significant contribution of this study is the methods and methodology to empirically identify students' experiences of intellectual need. This included developing tasks to provoke intellectual need, videos related to each task to help students recognize the need, and survey questions—administered at the point where we thought students might be experiencing disequilibrium—to enable students to report feelings of psychological and intellectual need.

There are several potential shortcomings of the methodology. We don't know whether "curiosity" and "wonderment" are the most appropriate terms to identify intellectual need, and students' self-identification might be inaccurate. Students might have been reluctant to respond affirmatively to the intellectual need question because doing so would require them to write additional information, this is supported by the significance of the Lesson Order predictor. Finally, we don't know the extent to which intellectual need can be provoked by a single task, even when the task is accompanied with a student problem-solving video. Instructor interaction and intervention might be essential to moving students into a state of disequilibrium, and our methods might have been insufficient to actually provoke genuine intellectual need.

Factors that Affect Intellectual Need

Overall, there was a relatively low rate of students reporting an experience of intellectual need. However, instructors—and, implicitly, the ways they incorporate the video sets into their instruction—are associated with different rates of intellectual need. Taken together, the results suggest that the instructor variation in our model is due mainly to differences in instructors, rather than other institutional factors. Thus, there is a complex interaction between pedagogy, curriculum, and students' interaction with the out-of-class learning materials, and this interaction needs to be studied in more detail.

Students were much more likely to experience intellectual need in response to some video sets than others. This means that some mathematical topics, tasks, or problem-solving videos were more effective at helping students experience and identify a state of disequilibrium. The relationship between video set content and intellectual need warrants further investigation.

There was a significant relationship between students' experiencing intellectual and psychological need. One explanation for this result is that there is a significant cognitive or emotional overlap between the two types of need, and that it is important to consider problem context when constructing an IN-P task. Alternatively, it could be that there is an overlap between our operationalizations of the two concepts, making it difficult for students to accurately distinguish between them.

Students who had more extensive background knowledge for a task were more likely to experience an intellectual need than other students. One explanation for this result is that students need a certain level of knowledge about the background mathematical concepts to engage in the IN-P task in the intended way. Alternatively, students might need the background knowledge to *identify* their experience as one of intellectual need. Both explanations suggest that IN-P tasks need to be carefully tailored to particular student knowledge and characteristics in order to provoke intellectual need.

Beyond the students' own background knowledge and other characteristics, it appears that the ways in which we structure the video-watching process can impact the students' experience of intellectual need. Students who (only) tried the IN-P task or (only) watched the student problem-solving video were less likely to experience intellectual need. However, for students who watched the problem-solving video, those who also tried IN-P task were more likely to experience intellectual need. This result suggests that merely provoking intellectual need is not a straightforward process, and that it would be useful for educators to have a framework to support the design and implementation of IN-P tasks.

Relationship between Intellectual Need and Learning

There is an association between a student experiencing intellectual need and demonstrating learning from the instructional videos. This result aligns well with the theory of intellectual need, which posits this relationship between need and learning. However, our measures of learning were relatively unsophisticated, and it is possible that we didn't accurately assess the depth or sophistication of students' learning. Furthermore, learning is often intended to take place over an extended period of time, rather than across a handful of short instructional videos, so we might not have adequately measured the intended constructs.

Conclusion

This study makes a significant methodological contribution to the design and evaluation of learning environments and materials. Our methodology and methods provide a first step into empirically identifying students' experiences of intellectual need and connecting those experiences to their learning. Our results also shed light on some of the factors that might impact students' experiences of intellectual need and how these factors influence learning. The relationship between intellectual need, learning, structuring the students' experience of the video sets, the students' background knowledge, and the instructor's pedagogy is complex. Taken together, these results highlight the importance of continuing to study intellectual need and to create a framework for helping instructors design and implement intellectual need-provoking tasks.

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Goals for Student Learning among Mathematics Graduate Student Instructors (MGSIs)

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This study focuses on novice Mathematics Graduate Student Instructors (MGSIs), defined as graduate students who are full instructors of record for the first time. In this semester-long qualitative multiple-case study, researchers examined three MGSIs' goals for student learning with respect to their teaching decisions and planning. Findings indicated four dominant goals that were expressed in their first semesters within the classroom. This study found that novice MGSIs are capable of reflecting on and discussing their goals for student learning. Understanding how MGSIs build on these goals can provide useful avenues for professional development of early-career instructors.

Keywords: Teaching Assistants, Graduate Student Instructors, Goals for Student Learning, Professional Development, Teacher Education

Graduate students are often involved in teaching undergraduates through a variety of roles including grading, leading labs, facilitating discussion groups, recitation sessions, or serving as instructors of record. Although graduate students can have many teaching-related responsibilities, 35% teach their own classes as instructor of record (Belnap & Allred, 2009). This can equate to 17% to 21% of mathematics courses at doctoral institutions (Blair, Kirkman, & Maxwell, 2013), which influences a significant number of undergraduate students nationally. Many graduate students will continue teaching as they aspire to careers in higher education and accept positions in educational settings (Golbeck, Barr, & Rose, 2016). This study focuses on mathematics graduate student instructors (MGSIs). A MGSI is a mathematics graduate student who is serving as full instructor of record for an undergraduate mathematics course, meaning they are responsible for presenting the material, assessing student learning, and assigning final course grades. MGSIs are commonly assigned to teach mathematics service courses (non-mathematics major dominated courses), despite being students themselves, having limited teaching experiences, and limited access to professional development (Speer, Gutmann & Murphy, 2005; Deshler, Hauk, & Speer, 2015; Ellis, 2014). They face challenges when learning to teach that include balancing teaching with their own coursework or research, awareness of low status of teaching at a research university, difficulties getting and interpreting feedback, working with undergraduates who bring negative mathematical experiences to their classrooms, and anxiety in dealing with these challenges (Hauk et al, 2009). There is little research *from the MGSI perspective* to assist with the design of supports to help MGSIs develop as instructors, and little is known about how MGSIs experience learning to teach or implement teaching methods (Belnap & Allred, 2009).

MGSIs' teaching practices are important as calls for changing instruction in higher education settings have been growing (CBMS, 2016; PCAST, 2012). Saxe & Braddy (2015) summarized the recommendations as calls to "move away from the use of traditional lecture as the sole instructional delivery method in undergraduate mathematics courses" and to "seek to more actively engage students than we have in the past" (p.19). These calls for improving instruction have been grounded in research on improving student learning, which suggests teaching methods (such as active learning) can improve undergraduate STEM students' learning opportunities and help decrease failure rates (Freeman et al., 2014). However, studies related to collegiate instructors teaching practices are limited (Speer, Smith & Horvath, 2010).

To address this research gap, a larger semester-long study explored how MGSIs understand their teaching and their efforts to incorporate active learning methods when teaching an undergraduate course for the first time, by examining reasons they gave for making pedagogical decisions and identifying their perceived challenges[1]. It aimed to understand MGSIs' planning, conceptualized as a "set of basic psychological processes in which a person visualizes the future, inventories means and ends, and constructs a framework to guide his or her future actions" (Clark & Peterson, 1986, p.260). The larger study sought to address the research question: During their first semester teaching a new undergraduate mathematics course, how do MGSIs plan (design and reflect) and implement their plans, with a focus on their goals for student learning? To address this broad question, four sub-questions were created. Due to space limitations, this paper focuses on findings to only the first sub-question: *What goals do MGSIs aim to achieve for student learning?*

Theoretical Framework

Schoenfeld's goal-oriented decision-making theory informed the design of this study. The theory's main claim is that "people's decision making in well-practiced, knowledge-intensive domains can be fully characterized as a function of their orientations, resources, and goals" (Schoenfeld, 2011a, p.182). According to this theory, an individual's orientations (beliefs, dispositions, tastes, and preferences) help shape their goals, which are "the things that people consciously or unconsciously set out to achieve" (Schoenfeld, 2011b, p.459). Resources, which include teacher's knowledge, skills and physical materials, are used to achieve goals. Teachers often have multiple, different levels of goals operating at the same time. Goals can be broad overarching content or social goals, or they can be specific to a lesson or problem. When goals are overarching, they are a key aspect of curriculum design (Wiggins & McTighe, 2005) and are described in various standards (NCTM, 2000; NCTM, 2014; Blair, 2006). In this study, goals were defined as "a complex mixture of academic aims: factual, conceptual, procedural, dispositional, and expert-performance-based" (Wiggins & McTighe, 2005, p.58) as this best aligned with how MGSIs discussed their aims when designing and reflecting on their curriculum and instruction.

Methods

Methodological Approach

This study used a qualitative multiple-case study methodology, defined as "an empirical method that investigates a contemporary phenomenon (the "case") in depth and within its real-world context, especially when the boundaries between phenomenon and context may not be clearly evident" (Yin, 2018, p.15). Case study was appropriate for these research questions because the study looked at understanding MGSIs' thinking and decision making about teaching in the context of actual classroom experiences and could provide a more complete picture of MGSIs' planning and interactive teaching throughout a full semester. The research design provided ample reflection time and space for MGSIs to describe their goals and decisions. The site, course, timeframe, and participants themselves defined the bounds of this case (Yin, 2018). This case is comprised of MGSIs teaching precalculus, an undergraduate service course in the

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mathematics department, as part of their teaching assistantship during fall 2019 at a large research university in the southeastern part of the United States.

Participants

All MGSIs teaching precalculus in fall 2019 were invited to participate in this study. Three male MGSIs in their early twenties who were in their second year of graduate school volunteered to participate and were given pseudonyms. Chen was an international graduate student and the other two, Willie and Patrick, were domestic graduate students. All were enrolled in a one-credit, year-long pedagogy course and were supported by peer mentors. Focusing on novice or less experienced instructors is important as universities continually use new MGSIs to provide instruction to undergraduates.

Data Collection

This study followed three MGSIs through their first semester of teaching precalculus. MGSIs were interviewed at the beginning of the semester to understand their prior teaching experiences and beliefs about teaching and student learning. They were asked to select three lessons throughout the semester, roughly one each month, where they planned to seek student engagement. These lessons were observed and video-recorded, lesson plans were collected, and the lessons were followed by a semi-structured interview (50-100 minutes) that explored the MGSIs' goals for student learning, planning and lesson design, and perceived challenges. Other forms of data included weekly journal entries (responding to 3-6 prompts), participant observations from a methods course for MGSIs, their course assignments, and interviews with their peer mentors. At the end of the semester, MGSIs participated in a focus group seeking to gain insight into their overall perspective on teaching the course for student learning.

Data Analysis

Dramaturgical coding which "attunes the researcher to the qualities, perspectives, and drives of the participant" (Saldaña, 2016, p.146) and is useful for exploring underlying psychological constructs was chosen as the coding scheme. Saldaña (2016) describes six types of codes that can be used for dramaturgical coding: objectives, conflicts, tactics, attitudes, emotions, and subtexts. Analysis of a pilot study suggested the emotion and subtext codes were not useful for answering the research questions, thus they were omitted from later analysis. Dramaturgical coding identified objectives defined as motives in the form of action verbs. In this study, objectives often referred to what the MGSI wanted students to know, be or do, although instructor objectives were also identified. To answer the research question, goals of MGSIs were identified through thematic coding of the objective codes. Conflicts or obstacles confronted by the participant-actor that prevent them from achieving the objective were coded to allow for analysis of the obstacles or challenges MGSIs perceived in their planning or teaching. Tactics or strategies used by the participant-actor to deal with conflicts and obstacles and to achieve objectives provided insight into elements of lesson design and classroom practices MGSIs selected and utilized in their classrooms. Finally, attitudes of the participant-actor towards setting, others, and the conflict included much of the MGSIs orientations about teaching and learning. A fifth category called influences was added to capture references to external influences on MGSIs planning.

Memos were written during data collection and throughout data analysis (Saldaña, 2016) to document the data collection and analysis process, reflect on interviews, note connections and questions, contemplate next steps in the research process, and summarize discussions between authors. All interviews were transcribed using Temi, a transcription software program, and cleaned by the first author. Seven data sources were coded for each MGSI: their semester-long journal, three lesson interviews, the background interview, and two written assignments from a pedagogy course. All coding was done in Dedoose and coding proceeded one MGSI at a time so that each instructor could be studied individually before making comparisons across cases. Codes were hierarchical; each code was placed under the dramaturgical categories of objectives, tactics, conflicts, attitudes, or influences (level 1). Objective codes included categories for student learning, students, and teachers (level 2). These goals were also coded at more detail (level 3) and sometimes used in vivo codes (Saldaña, 2016), such as "recognize and use appropriate tools". Code mapping (Saldaña, 2016), a technique to organize data and improve trustworthiness of findings during analysis, was employed after coding the first piece of data (Chen's journal), and again after coding the data from each MGSI. Quotes from categories at levels 2 and 3 were reviewed to confirm quotes described the same concept. The codes were listed, compared, and organized into categories by rearranging or combining similar codes. The codes were then condensed into themes. Generating themes was not a linear process; rather it involved sorting and comparing codes and considering how categories related to one another. As MGSIs expressed many common goals for student learning, findings are reported only through cross-case analysis. The first researcher coded and discussed codes and themes with the second researcher to reach agreement on all codes and themes.

Findings

Although MGSIs work towards many types of goals over a semester, the themes presented below were the most salient goals that MGSIs discussed while preparing for and reflecting on teaching precalculus. These goals related to what MGSIs aim to help students learn and were shared among all MGSIs in this study. MGSIs shared goals for student learning centered on student reasoning, understanding and sense making, developing productive dispositions as learners of mathematics (NRC, 2001), and procedural skills. However, what Saldaña (2016) termed the super objective, "the overall or ultimate goal of the participant" (p.148), focused on preparing students for their future. As each goal was expressed by all MGSIs, their data is presented together to illustrate the pervasiveness of the goal throughout the data set and among MGSIs. Analysis identified four goals for student learning as the most prominent goals in three novice MGSIs' planning. Table 1 summarizes the findings of this study.

Table 1: Summary of Goals and Aspects of the Goals

Goals	Aspects of Goals
Goal 1: Prepare Students for their Future	(1) Coursework; (2) STEM Careers; (3) Personal Study Skills
Goal 2: Develop Reasoning, Sense Making and Understanding	(1) Understanding Why: Deriving or Proving; (2) Sense Making and Developing Intuition; (3) Applying or Connecting Concepts, Facts or Procedures
Goal 3: Develop Productive Dispositions	(1) Enjoy Mathematics and/or Precalculus Class; (2) Build Self-efficacy; (3) View Mathematics as Useful; (4) Willing to Persevere
Goal 4: Develop Procedural Skills	(1) Content Skills to Solve Standard Problems; (2) Recognize and Use Appropriate Tools; (3) Know or Memorize Facts

Goal 1: Prepare Students for their Future

MGSIs frequently spoke about preparing their students for the future, which primarily related to future coursework or career-related skills. All journals contained multiple references and similar phrases to Patrick's journal, where he stated, "my main goal is to prepare these students for calculus," and all MGSIs directly identified this goal in their background interviews. This goal was continuously shared throughout the semester as the clear overarching goal MGSIs held relative to their students. In addition, MGSIs also kept in mind their students' STEM career aspirations. MGSIs' desires to prepare their students included both seeking to help students achieve their career goals and be competent workers in their STEM fields. As Willie remarked during the focus group, "we are trying to prepare them for their life," and knowing precalculus material will be important for students to obtain and keep their jobs. MGSIs noted the importance of their students correctly using mathematics in their future careers for projects that can have real life consequences. As Willie stated, "These kind of kids [STEM majors], they need to know it…[because] in the future, they are going to be using this" in their careers.

Another way in which MGSIs spoke of preparing students was in developing personal work habits and study skills. Less frequently mentioned than the career focused types of preparation, this category included learning how to learn, taking ownership of the learning process, and feeling personal satisfaction or pride in hard work, which are transferable to learning in areas outside of mathematics. Chen shared in the focus group how students' experiences of studying precalculus were important for undergraduates in ways beyond the mathematical content. He explained,

> It's not just like they have to know the material for calculus, but they still have to know *how to learn* the material… It's not just like math is math. Like you also need to learn the skill for learning like new knowledge.

Some of the focus on preparing students may be due to the course they were assigned to teach. For these MGSIs, the precalculus course was the prerequisite for calculus. Teaching precalculus thus became about student preparation and MGSIs' desire for their students to be prepared and successful in their future coursework.

Goal 2: Help Students Develop Mathematical Reasoning, Sense Making and Understanding

The most frequently coded objective for student learning related to students' mathematical reasoning abilities. MGSIs described this goal in many ways, including wanting students to conceptually understand or make sense of mathematics, to apply and connect concepts or ideas, to derive, prove, generalize or conjecture about formulas or theorems, and develop intuition. MGSIs discussed these ideas in a connected, overlapping manner. MGSIs frequently discussed wanting students to understand why theorems or formulas were true. Willie had designed a discovery jigsaw activity about quadratic functions. He explained in the interview,

> They all knew the quadratic formula. So my goal was to not explicitly say, you're finding the quadratic formula. It was for them to derive and say, oh look, it's the quadratic formula! [And have] the epiphany, the whoa! Like what you do when you get a proof and you spent forever doing and then you finally get there.

He reiterated during another interview, "Figuring out where it comes from, I think is pretty important because it gives you that more conceptual knowledge. You found it! Now you know how to do it again instead of me telling you how to find it." MGSIs were aiming to provide opportunities for students to develop mathematical understanding and reasoning skills.

MGSIs also aimed for students to understand and make meaning of mathematical definitions and terms. Chen reflected on his lesson about asymptotes, recalling "I hope they can relate that [where the function is undefined] to the vertical asymptote and the graph and where the problem happened." MGSIs often expressed a desire for their students to be able to apply previous material to the current lesson, to apply current concepts in the future, and to connect mathematical concepts. In another interview, Chen described how he asked students to make connections between the unit circle and the graphs of sine and cosine functions. "If they have to produce this graph, hopefully they can try to recall, to use this unit circle to graph. Again, it's a connection." Connecting the unit circle to the graph of a trigonometric function reflects Chen's aim to help students build connections between mathematics concepts.

Part of encouraging students to make sense of ideas seemed to relate to helping develop what MGSIs sometimes called students' intuition or their ability to think mathematically or creatively. In an interview, Patrick's reflected on what he wanted students to learn from working together to verify trigonometric identities. He discussed how students need to develop "that intuition of where to go or what to do next" and hoped that allowing students to work together in groups would help them build intuition. These quotes illustrate various ways in which MGSIs shared goals for students to develop understanding of mathematical ideas, make sense of mathematics and reason mathematically.

Goal 3: Help Students Develop Productive Dispositions

All MGSIs expressed affective goals for their students related to how students viewed themselves as learners of mathematics and how they understand the field of mathematics. This category included MGSIs' goals for students to enjoy and take interest in mathematics or their precalculus course, to build students self-efficacy, for students to appreciate the usefulness of mathematics, and develop a willingness to work on problems and persevere in solving them.

MGSIs wanted their students to hold positive views towards mathematics. As Willie stated in the focus group, "That's something I want to hear. Someone was like, yeah, I think I enjoy math now." In an interview, Patrick also explained he incorporated a jigsaw activity because "math is not just supposed to be something that you have to do. Math is not supposed to be a chore. So maybe these types of activities will make them more interested in or motivated [to learn mathematics]". Building students' self-efficacy was also a consistent goal for Patrick who stated in the background interview,

> One goal I might have is to sort of change that opinion of some students. You're not bad at math. Math is just a skill. As you practice it, like any other skill, you're going to get better at it.

Through their teaching, MGSIs hoped to improve students' attitudes towards mathematics and build their confidence in their ability to learn and do mathematics.

Additionally, MGSIs wanted their students to recognize ways mathematics contributes to daily life and be willing to persevere when solving problems. Chen wrote in his journal, "I hope to tell my students why they need to learn math...I hope my students appreciate the use of mathematics more." MGSIs also understood that for students to solve mathematical problems, they must be willing to try different ideas and not become discouraged or give up when the first idea is not successful. During the focus group, Chen discussed helping students develop perseverance. He explained, "I want to show them that it's okay to be wrong, like it's fine. Just use another ratio or another number, because the idea of trying is really helpful…especially when they haven't seen the pattern yet." The manner in which MGSIs discussed their goals to build students' confidence and desires for students to enjoy mathematics, hold positive views of mathematics, and be willing to persevere, aligns well with the notion of developing productive dispositions meaning "the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (NRC, 2001, p.131).

Goal 4: Help Students Develop Procedural Skills

Finally, MGSIs also aimed to develop their students' ability to execute various mathematical procedures. Common for mathematics courses at this level, MGSIs stated much of the course includes objectives for students to solve, compute, manipulate or simplify, which often amount to replicating or following procedures. As Chen stated in an interview,

> It's like by the end of the day, the goal for the lecture is for them to be able to solve [a triangle] using the law of sine. They don't have to actually know where it comes from. It would help with their intuition, but it doesn't like align with the goal that I want to reach [that students can solve triangles].

All MGSIs at some point described procedural skills as tools and stated in interviews their goal for students to "recognize and use appropriate tools." Willie explained in an interview that he wanted students to see completing the square as much as possible "because it's a good tool. You'll see it all the time and they definitely see it in calculus." MGSIs did not only want students to be able to execute procedures but also apply them appropriately.

Discussion

Goals were central to the theory, data collection and analysis in this study. A pilot study analysis suggested MGSIs thought about student learning as well as their own hopes and aims to facilitate learning. Schoenfeld (2011a) suggests teachers pursue multiple goals at different levels or sizes and those goals can be conscious or unconscious. Some goals, such as the super objective of preparing students for their future, were clear and conscious in all MGSIs' minds from the start to the end of the study. Other goals emerged or were discussed and identified more clearly as the semester progressed. MGSIs' awareness of and ability to speak about some goals increased throughout the semester. Findings described overarching cognitive or affective goals as they best explained MGSIs' planning on a larger scale and related to reasons MGSIs gave for planning or teaching decisions. All goals described in this paper became conscious goals at some point during the study and were confirmed by participants who reviewed a draft of the study's findings. These goals were remarkably consistent across the multiple cases of MGSIs, thus may provide a fertile starting point for professional development to support new MGSIs (Ellis, 2014).

This study illustrates new avenues of filling in the aforementioned research gap around MGSIs' goals for student learning. Although Schoenfeld's theoretical framework (2011a) asserts goals are shaped by orientations, it does not discuss the formation of goals. This study's use of dramaturgical analysis found that some MGSIs' goals for student learning were molded and shaped by their early impressions of teaching. For example, knowing their students needed to complete the calculus sequence, put added pressure on MGSIs as they recognized the importance of the course for their students' future success (Goal 1). This weaving of goals for student learning and lesson designs calls for the novice MGSI's voice when generating thoughtful, meaningful, and MGSI-centered professional development.

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Analysis of a Mathematical Modeling Assessment

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In this paper we use multiple statistical approaches, including classical test theory (CTT), item response theory (IRT), and principal component analysis (PCA) to understand item behaviours, scale properties, and dimensionality of a developing multiple choice assessment of mathematical modeling competencies designed for tertiary STEM majors. We share analyses and inferences, making recommendations for the field in pursuing such assessments.

Keywords: mathematical modeling, assessments, differential equations

Mathematical modeling is an important skill for students at all levels of mathematics, in part because it can be a vehicle for learning further mathematics and partly because of the capacity of real world problems to motivate students to pursue and persist in STEM fields. As scholarly and pedagogical interest increases in teaching with (or through) mathematical modeling, so increase stakeholder interests in assessing growth in modeling skills. Many assessments for mathematical modeling knowledge and skills have been independently created, however the majority serve a local need and are based on ad hoc constructions or small-scale studies of student work (Frejd, 2013). At the same time, there have been increasing calls for instruments in mathematics education research to undergo evidence-based validity assessments (see, for example Melhuish & Hicks, 2019). The rationale is that documenting properties of tests and test-takers can aid the field in synthesizing, and thus building upon an abundance of research results. Within this context, we have been developing an assessment for evaluating interventions that aim to improve tertiary students' modeling skills. To facilitate scholarship in this area, the instrument is intended to support research designs based in a pre/post measurement paradigm. Thus, the project goal has been to develop two parallel forms of an assessment appropriate to targeting modeling skills of tertiary students. The purpose of the present study was to learn about the psychometric properties of the items and the scales. We ask and answer the research question: What are the psychometric properties of the instrument and do the items behave as intended?

Conceptual and Assessment Frameworks

We adopt a synthesis of instrument development frameworks (2014) as a set of validity criteria advanced as part of the *Validity Evidence for Measurement in Mathematics Education* project. Melhuish and Hicks (2019) recently applied the standards to concept inventories in mathematics education, demonstrating the approach to be viable for research assessments. They make a case that validity of an instrument should reflect empirical evidence to support its (1) content validity, (2) response process validity, (3) relations to other measures, and (4) internal structure. Content validity is established through expert evaluations and literature-informed item development. Seeking evidence of response process validity ensures that both items and distractors tap into students' reasoning patterns. Typically, evidence for response process validity is sought through direct student feedback on the items. Checking relations to other measures can mean calibrating the instrument against other assessments of the same content or instruments assessing distinct constructs. Finally, internal structure validity involves checks on dimensionality, internal consistency, and other psychometric properties for the items and scale. These evidence-based validity criteria guide our instrument development process. In this paper, we briefly summarize development and validity efforts for criteria (1)-(3) which are published elsewhere (Czocher et al., 2020; Czocher et al., 2021) and report on (4).

Integral to establishing content validity, operationalized here as construct validity, we adopt a cognitive view of mathematical modeling as a process consisting in multiple inter-related, idiosyncratic phases (Blum & Leiß, 2007; Kaiser, 2017). We define modeling as the process of rendering a non-mathematical problem about a real-world phenomenon of interest as a well-posed mathematical problem to be solved. The process involves modeling competencies like *understanding* (specifying a problem), *structuring* (identifying relevant and irrelevant factors), *mathematizing* (articulating mathematical relations among quantities), *working mathematically* (solving), *interpreting* and *validating* (checking the model represents the situation and addresses the real-world problem). We developed a pool of 118 multiple choice questions (MCQs) belonging to 9 real-world scenarios based in research reports on students' thinking during modeling and pedagogical materials. We sought scenarios that treated issues prevalent in society, involved situations in the sciences where differential equations might be used, or were suggested by informal interviews with STEM professors. Mathematics content ranged from arithmetic to algebra, to calculus, and to systems of ordinary differential equations. We chose a multiple-choice format to facilitate creating two parallel forms that could support pre/post-test research designs intending to measure gains in competencies. Across multiple rounds of field testing, we solicited feedback from an expert panel of mathematicians and mathematics educators as to the accuracy of the mathematical content and the extent to which items targeted intended competencies. We implemented revisions, culling items that failed to be correct or sensible.

To establish response process validity (Czocher et al., 2020; Czocher et al., 2021), we carried out three rounds of field testing: feasibility, difficulty and distractors, and discrimination. In the feasibility round, we solicited feedback on the items from a group of 12 STEM undergraduates, asking them to evaluate the items, scenarios, and response options for authenticity, sensibility, and rationale for selecting distractors. In the difficulty round, we administered the 63 most promising items on two forms to 35 and 43 STEM undergraduates, respectively. We kept items with difficulty $0.20 < p < 0.70$ and restructured or culled items that did not perform well. We also analysed distractor efficacy, ensuring that each distractor was selected by at least 5% of respondents. In the discrimination round, 30 remaining items were sorted onto two forms and administered to a sample of 25 secondary and 289 post-secondary students participating in an international modeling challenge focusing on applying differential equations (see below for details). The mean item difficulties were $p = 0.39$ and $p = 0.41$ for the two forms. An independent sample t-test ($t = -0.811, df = 144, p = 0.419$) confirmed no significant difference in mean score across the forms and Levene's test ($F = 0.412, p = 0.522$) confirmed equal variances. A set of point-biserial correlations were computed to identify high- and low-discriminating items. We estimated reliability of the two forms using Revelle's Omega Total (ω_T) as a measure of internal consistency, which is appropriate where multiple dimensions (e.g., mathematics content, modeling context, target competence, reading comprehension, item format, native English proficiency) contribute to predicting the construct of interest and when individual items measure the latent construct with differing degrees of precision (Raykov, 1997; Revelle & Zinbarg, 2008). The two forms had $\omega_T = 0.67$ and $\omega_T = 0.63$, respectively, approaching acceptable estimate of reliability 0.7.

To study relations to other measures, we investigated correlations between the instruments measure of modeling competencies and a related instrument's measure of self-efficacy to carry out those competencies (Czocher et al., 2021). In the spirit of Hackett and Betz (1989) and (Bandura, 2006), the Modeling Self-Efficacy (MSE) instrument uses a 0-100 rating scale to measure an "individual's perceived capacity to carry out the interrelated activities that make up mathematical modeling" ((Czocher et al., 2019, p. 13). Of the 314 students who took the MCQ in the discrimination round, 144 also took the MSE. For these students, we found that modeling self-efficacy was a highly significant predictor of modeling competency, as measured by these two instruments. We interpret the strong positive association between modeling competency and modeling self-efficacy as evidence of relation validity; since we expect a positive correlation between mathematics self-efficacy and performance.

We report on efforts to equate two parallel forms and investigate internal structure validity. To this end, we conducted a round of testing two forms using Rasch analysis. We selected Rasch analysis because it produces item difficulty estimates and person ability scores, allowing examination of the scale along both item and ability variables. Non-identical forms can then be equated by anchoring the difficulties of common items to the same values on a common scale. Rasch analysis produces Wright maps which can be interpreted to explain participants' test scores score in the context of the questions that appear on our instrument (Boone, 2016)

Methods

Based on results from previous rounds of testing, we constructed two forms, Ruby and Sapphire, each with 20 items and with 11 of those items common to the two forms. Construction used Classical Test Theory to balance the total (anticipated) difficulty of the forms and content coverage in terms of competencies targeted. In total, each form had 3 understanding, 5 structuring, 5 mathematizing, 2 interpreting, and 5 validating items. The expected mean item difficulties for Ruby and Sapphire were $p = 0.472$ and Sapphire $p = 0.475$, respectively. We administered the forms to a sample of secondary and post-secondary STEM students participating in an international mathematical modeling challenge focusing on using differential equations to solve real-world problems. The challenge took place remotely during the COVID-19 pandemic at the end of Autumn 2020 semester, depressing participation in the data collection. In total, 89 students responded to the items (see Table 1 for demographics), and some response sets were incomplete (detailed below). Additionally, >90% reported typically earning B's or higher in both their mathematics and major classes.

Table 1 Participant demographics

Male	64%	Science	21.3%	Diff Eq	81.8%
Female	33.7%	Mathematics	46.1%	No Diff Eq	18.2%
Non-binary	2.3%	Engineering	25.8%		
		Other	6.7%		

We first performed a classical item analysis to check that all items had positive item-total correlations. Item difficulties were calibrated for each form using a Rasch model. Item and person fit statistics and residuals were calculated to check model-fit assumptions and conduct dimensionality analysis, respectively. We identified items with fit statistics outside of the 0.5 to 1.5 range or exhibiting appreciable item-pair residual correlations (Yen's $Q_3 > 0.2$). Item-pairs with higher residual correlations were examined for patterns in their relationships. We excluded mis-fitting items and persons and re-calibrated the remaining dataset to the Rasch model. Due to small subgroup size (e.g., type of school, major) a Differential Item Functioning (DIF) analysis to assess potential bias was not feasible.

Given all of the assumptions below, which equation **best** models growth for a population?

1. The birth rate is proportional to the population.
2. There are sufficient resources for the population to thrive.
3. Members die of unnatural causes, like murders.
4. Unnatural deaths are proportional to the number of two-party interactions.
5. k_1 and k_2 are proportionality constants.

a. $\frac{dP}{dt} = k_1 P - k_2 \frac{P^2}{2}$

b. $\frac{\mathrm{dP}}{\mathrm{dt}} = k_1 P + k_2 \frac{P(P-1)}{2}$

c. $\frac{dP}{dt} = k_1 P - k_2 \frac{P}{2}$

d. $\frac{dP}{dt} = k_1 P - k_2 \frac{P(P-1)}{2}$ *

e. $\frac{dP}{dt} = k_1 P - k_2 \frac{(P-1)}{2}$

Figure 1 Sample multiple-choice item for mathematizing (choosing a representation)

Results and Interpretation of Rasch Analysis

On Ruby, between 29 to 38 students responded to each item. Ten students were flagged as having response patterns that did not fit model expectations – they had too many missing responses. One item had a large outfit and distractor options with positive item-total correlations. Given the messy response pattern, we excluded the item from further analysis. Ruby was then re-calibrated. Two further items exhibited large outfit values (close to 3) but were kept because the test length would be short for the number of respondents. The Rasch item reliability score was 0.88 (Adj. SD = 0.63) and the person reliability score was 0.63 (Adj. SD = 1.39). We examined the pairwise correlation of item residuals (Yen's Q_3) and a Principal Component Analysis (PCA) of standardized residuals. Both suggested underlying multidimensionality. Multiple item pairs had residual correlations > 0.20 and the first factor had eigenvalue $\lambda = 3.05$.

On Sapphire, between 32 and 41 students responded to each item. We scored two items using a partial credit scale because these items had stems phrased to select an optimal choice. However, most students chose a mathematically correct choice that was not optimal for modeling (Haines et al., 2000). We discuss implications for future test forms below. Most items showed satisfactory fit statistics. One item (Sapphire Item 7, validating) had an outfit value of 1.74 and another (Sapphire Item 5, simplifying) was underfitting (outfit 0.66, infit 0.73). Because the test is short, underfitting items were kept. After recalibration, one common item (Ruby Item 15, simplifying) was removed from just the Sapphire test. Although it performed well on Ruby form, it had high outfit on Sapphire and its residuals were highly correlated with another item's (Sapphire Item 6, simplifying).

Excluding the two problematic items, a recalibration of the Sapphire test produced a Rasch item reliability score 0.88 (Adjusted SD = 1.22) and the person reliability estimate of 0.64 (Adjusted SD = 0.89). Since Sapphire had a larger person reliability estimate, we used it as the anchor form during equating. The Sapphire item-pair correlations and PCA of residuals indicated less concern about multidimensionality than on Ruby form. The two forms were recalibrated from the 10 remaining common items (excluding Ruby Item 15) and anchored to Sapphire's final calibration values. Thus, all Ruby and Sapphire item difficulties could be placed on a common scale. As is customary in mathematics education, we report the Cronbach's alpha as well, for Ruby, $\alpha = 0.91$ and for Sapphire $\alpha = 0.85$.

Table 2 contains the breakdown of each item on the final version of both tests with the competency they target, the context, and the difficulty. The Wright Maps (item-person map) shown in Figure 1 order the ability of the students who took the tests on the left side and the difficulty of the questions on the right side. The left side of the Wright map shows the distribution density of the test takers compared to the item difficulty on the right (high scoring test takers are distributed as positive logits and low scoring as negative). Outside of items 9, 10, and 11, the distribution of the difficulty of the items is good. Items Sapphire 9, Ruby 9, Sapphire 10, and Ruby 10 show high difficulty relative to the ability distribution. Both questions focus on the competency of interpreting in two different contexts (disease and recycling). Item 11 on Ruby and Sapphire which focused on understanding with the context of a wastewater tank, had the lowest difficulty across both tests. Understanding tasks varied in difficulty across the different problems, with Ruby Item 1 being difficulty and Ruby and Sapphire Item 12 having a small negative difficulty (-0.75). Items about mathematizing (excluding Ruby Item 2), simplifying, and validating had difficulty levels clustered around an ability level of 0.

Table 2 Rasch model difficulties (δ) for Ruby (R) and Sapphire (S) forms.

Item #, R	Item #, S	δ, R	δ, S	Competency	Context
1		1.81		understanding	Decay
2		-2.17		mathematizing	Disease
3		-0.31		mathematizing	Population
4		0.32		simplifying	Recycling
5		-0.06		simplifying	Wastewater Tank
6		1.09		simplifying	Wastewater Tank
7		-2.36		validating	Carrying Capacity
8		-0.94		validating	Wastewater Tank
9	9	2.57	2.57	interpreting	Disease
10	10	2.06	0.98	interpreting	Recycling
11	11	-3.35	-3.35	understanding	Wastewater Tank
12	12	-0.75	-0.75	understanding	Disease
13	13	-0.36	-0.36	mathematizing	Wastewater Tank
14	14	0.87	0.87	mathematizing	Recycling
15		0.67		simplifying	Recycling
16	15	-0.98	-0.98	simplifying	Haines & Crouch*
17	16	-0.21	-0.21	validating	Wastewater Tank
18	17	0.53	0.53	validating	Wastewater Tank
19	18	0.67	0.67	validating	Wastewater Tank
	1		0.32	understanding	Recycling
	2		1.08	mathematizing	Wastewater Tank
	3		-1.56	mathematizing	Carrying Capacity
	4		-1.13	simplifying	Recycling
	5		0.24	simplifying	Wastewater Tank
	6		0.32	simplifying	Recycling
	7		-0.2	validating	Carrying Capacity
	8		-0.11	validating	Disease

**Item Ruby 16 is from (Haines et al., 2000) and asks students select variables needed to model the time for passengers to safely exit an aircraft.*

We conducted a Principal Component Analysis (PCA) on the responses for the final forms to investigate any empirically evident dimensionality. The first Ruby component (19.3% of explained variance) contained items 2, 5, 6, 8, 11, 13, 17, 18, 19. These are all but one from the Wastewater Tank context and as a set possess a wide range of difficulties. The items comprise four of the five competencies (4 validating, 2 mathematizing, 2 simplifying, 1 understanding). The second Ruby component (16.8% of explained variance) contained items 9, 10, 12, 13, 15, two of which (9, 10) were the difficult interpreting items. The first Sapphire component (15.8% of the explained variance) includes items 3, 4, 5, 9, 12, 16, from many contexts (2 Disease, 2 Wastewater Tank, 1 Recycling, 1 Carrying Capacity) and competencies (1 each validating, understanding, interpreting, and mathematizing, 2 simplifying). The second Sapphire component (15.6% of the explained variance) included items 2, 14, 17 – 2 mathematizing and 1 validating. Two were Wastewater Tank context and one was from Recycling

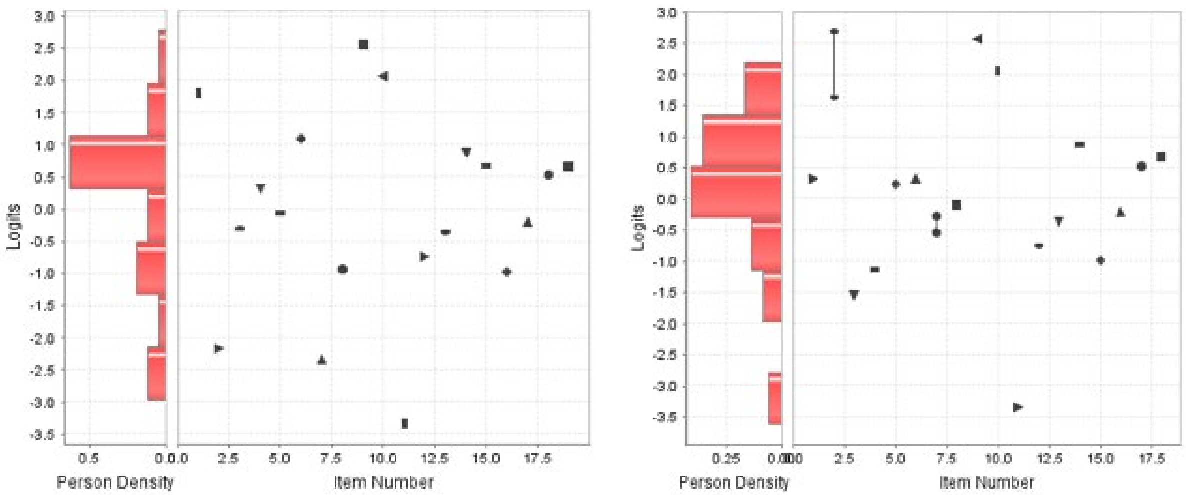

Figure 1 Final anchored Wright (item-person) maps for Ruby (left) and Sapphire (right), ordered by item number.

Discussion and Conclusions

Based on our psychometric analyses, we have created two parallel forms targeting modeling competencies appropriate for tertiary students of advanced mathematics. On the calibrated scale, most items had difficulties $|\delta| < 1$, and the Wright map showed a good difficulty distribution, suggesting that the scales are balanced. The easiest items ($\delta < -1.5$) had a clearly correct answer and could be addressed using test taking strategies to rule out distractors. Items 9 and 10, the most difficult ($\delta > 1.5$), were both interpreting items and had lengthy response options requiring comparison among the options. We hypothesize that a higher cognitive load may contribute to their difficulty, besides the modeling competence.

The low person-ability reliability likely reflects a small sample of test takers, with diverse characteristics, and a relatively short test. It may also show that the test items were not well "targeted" to the ability-levels of students in the sample. We are not pessimistic about this interpretation since we administered the forms as a pre-test to an intervention where students would have the opportunity to practice these skills. Thus, it is sensible that their ability levels would be low, as measured by this instrument. Though administering the forms online permitted us to reach a larger sample from a small population, we note the instrument may be challenging to respond to via mobile platforms and that, because of COVID-19, completing yet another assignment may have been laborious.

Empirically, the forms did not reveal dimensionality according to *a priori* constructs like mathematical content, modeling context, or modeling competency. It is possible that competencies or contexts may form dimensions, but the sample of students was so diverse in terms of their personal characteristics and prior knowledge that the instrument could not detect it. Instead, given the relatively low person reliability (due to large standard error), it is possible that dimensions may include aspects of guessing, English comprehension, test fatigue, or that items may require judgment rather than offering a clear correct answer. One notable exception was that the PCA on responses revealed all the Wastewater Tank problems loading to Ruby Component 1 and it also was strongly present on Sapphire Component 2. We also suspect that other constructs, such as facility with quantitative reasoning, may play a role. In any case, since the breadth of competencies was well-represented on the extracted components, we infer that no one item type is responsible for all the variance. Instead, variance is pleasantly distributed among item types.

Our approach to designing the MCQs is atomistic (Blomhøj & Jensen, 2003), targeting each item toward a competency. This approach facilitated the multiple-choice format but carries limitations in its capacity to assess modeling as a composition of those competencies. More testing is necessary to explore suitability as a measure of individual students' modeling capacity. Future rounds can also examine patterns in item difficulty according to competency or reliance on other latent constructs like quantitative reasoning. We have provided an evidence-based validity evaluation of the internal structure of the parallel forms to evaluate pedagogical innovations. This effort is integral in moving towards a valid and reliable instrument for measuring growth in modeling skills of tertiary students, and more broadly towards establishing a shared empirical basis for interpreting results of studies of student's modelling across education levels.

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Advanced Students' Actions for Operationalizing Quantification in Analysis

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This study identifies seven categories of student meanings for quantification evoked in the course of reasoning about and proving a theorem in functional analysis. By observing how two graduate students flexibly foregrounded and backgrounded the variation in different quantities within a proving task, we identify ways that quantification operates in proof-oriented reasoning. We call these quantification operationalizations *or* quantops*. While some of these quantops mirror findings from prior studies of reasoning about quantified statements, our framework introduces new aspects of reasoning about quantification. Also, we observe that each quantop has two forms: one that backgrounds quantification and one that foregrounds it. Backgrounding appeared highly important for managing the myriad of quantities constituent to the task.*

Keywords: quantification, analysis, proof

Learning advanced undergraduate mathematics requires fluency in the use of quantifiers, often multiple nested quantifiers, in the same definition or theorem. Like many matters of logic, prior research shows why these ways of reasoning pose challenges to student success in proof-based courses (e.g., Epp, 2003). Unfortunately, we have comparatively fewer insights about successful reasoning and learning. Some recent studies have worked to remedy this limitation regarding research on quantification. Sellers, Roh, and Parr (2021) characterized undergraduates' quantification actions for quantified variables. Vroom (2020) documented how students developed and distinguished different multiply-quantified relationships to express a concept that they sought to define. This investigation builds on productive aspects of those studies by 1) documenting students' quantification actions for multiply quantified relationships as those students 2) sought to express, relate, and justify highly complex relationships.

In particular, we analyze a series of task-based interviews with graduate students taking a functional analysis course to identify their meanings for multiply quantified relationships and their habits for notating and proving about such relationships. These students had years of experience learning proof-based mathematics and working with multiply quantified relationships, so we seek to learn from their expertise. We document the actions they took during problem solving and proving to reason about quantification relationships in the Arzela-Ascoli Theorem and its proof. We call these *quantification operationalizations* or *quantops*. We use the term *operationalization* in part because we observe that the definitions and proofs featured in these interviews involve an impressive number of quantified objects[1] such that reasoners must find ways to manage working memory to imagine the definition of the concepts and the lines of inference for proving implications between them. We extend the insights of prior studies (e.g.,

[1] The Arzela-Ascoli Theorem can be stated "A subset S of $C([a,b],\mathbb{R})(\sup)$ is equicontinuous and bounded if and only if it is totally bounded," but if we substitute the definitions of those terms, which one must coordinate to understand the theorem or write a proof, it would appear as the following: "$\forall S \subseteq C([a,b],\mathbb{R})(\sup), [\forall f \in S, \forall \varepsilon > 0, \exists \delta > 0 \ni |x-t| < \delta \Rightarrow |f(x)-f(t)| < \varepsilon] \wedge [\exists M > 0 \ni \forall f \in S, |f(x)| < M] \Leftrightarrow [\forall A \subseteq S, \forall \varepsilon > 0, \exists \{a_1, a_2 \dots a_n\} \subseteq A \ni \forall f \in A, \exists i \in \{1,2,\dots n\} \ni \forall x \in [a,b], |f(x) - a_i(x)| < \varepsilon]$." We present the second rendering to emphasize that proving this theorem is a rich opportunity to understand how students conceptualize and operationalize quantification.

Durand-Guerrier & Arsac, 2005; Roh & Lee, 2011) especially by documenting how students make meaning of quantification beyond syntactic structure (the order of quantifiers in a claim).

Prior literature on reasoning about quantification

In this section we review mathematics education literature on reasoning about quantified variables and multiply quantified statements.

To which and how many values do variables refer?

The key function of a quantifier in mathematics is to specify for which and how many values of a variable does a statement need to hold to be a true statement. The truth value of a statement like "x is even" or "$x = 2 \cdot k$" depends on the value(s) of the variable(s). If we want to claim such a statement is true for all relevant values of the variables, we use *universal quantfication* (often expressed "for all"). If we only require at least one value to make the statement true, we use *existential quantification* (often expressed "there exists"). These two types of quantification are the most common in mathematics and in prior research. As Sellers et al. (2021) and Vroom (2020) point out, students may also reason about what is conventionally called *unique existence* in which a condition must be true for one value of the variable and false for all others.

Sellers et al. (2021) developed a framework of five meanings undergraduate students exhibited for quantified variables when interpreting statements and deciding their truth values. The first three meanings correspond to the way mathematicians use existential quantification, unique existential quantification, and universal quantification. The fourth meaning describes when students take terms like "any" to mean they can choose one or a few values, which entails variability but not universality. The fifth meaning describes when students showed no evidence of perceiving the variable as varying, typically by picking a single value. This framework provides a rich starting point for understanding quantification as a mental construction that students enact while reasoning about mathematical statements or problems.

Using quantification to interpret multiply quantified statements

Many studies begin by presenting students with quantified statements to interpret and determine their truth value (e.g., Dubinsky & Yiparaki, 2000; Dawkins & Roh, 2020; Roh & Lee, 2011, Sellers et al., 2021). Most such studies use multiply quantified statements, which introduce the complexity of understanding how the two variables take on values in tandem and how they vary together or independently. Such studies tend to focus on the relationship between syntax (the grammatical form of the statement) and semantics (how the statement refers to objects and construes meaning). Durand-Guerrier and Arsac (2005) explored the *dependence rule* that a variable quantified second (in the syntax) generally depends upon the variable quantified before it. Roh and Lee (2011) emphasized the *independence rule* that the variable quantified first must remain independent of the variable(s) quantified thereafter. The order of quantifiers is thus accorded great significance in how mathematicians use such statements.

Using quantification to express and reason about mathematical concepts

Students draw upon quantification to construct and express meaning, in addition to simply reading quantified statements (Vroom, 2020). Apart from mere syntax, students gave meaning to multiple quantification by constructing relationships among various quantities sequentially in order to express a known concept. Dubinsky, Elterman, and Gong (1988) similarly contrasted negating the entire meaning of quantified statements (e.g., failing to be "continuous") from negating layers of quantification.

Methods and Theoretical Framing

The student data presented in this report was taken from a larger study that explored students' understandings of metric space results in function space contexts. The goals of the study were to learn about how students connected function spaces and general metric spaces and how students drew upon prior understandings of analysis concepts in proof construction. We report on data from two students' paired, task-based clinical interviews (Hunting, 1997). The sessions were video recorded, with both students working together on a whiteboard to carry out the interview tasks and responding to prompts from the researchers present.

We recruited these two students, whom we call Adisa and Fenfang, based on initial interviews. The primary purpose of these interviews was to observe students' mathematical activity to make inferences about their in-the-moment understandings and their mental resources for engaging in proof (Clement, 2000; Thompson, 2008). Accordingly, we held no other criteria for selection or paring beyond our anticipation that we could observe and interpret sufficient mathematical activity and utterances from the pairs to make inferences about their thinking from interpretive and conceptual analyses (Clement, 2000; Thompson, 2008).

We report on episodes taken from the third and fourth 90-minute interviews with Adisa and Fenfang. In these interviews, we asked the students to attempt proofs of both directions of the Arzelá-Ascoli theorem: "A subset of $C^0([a, b],\mathbf{R})$ is bounded and equicontinuous if and only if it is totally bounded in the sup metric." The students had attempted the proofs of both directions on a take-home exam prior to the interviews. Fitting for a take-home exam, this is a highly challenging and complex proving task. Unlike many task-based interviews, students had worked on it before the study. Our focus in this analysis was not on Adisa and Fenfang's success or lack thereof in producing complete proofs, but rather to understand their ways of reasoning with and about quantification in the process of proof construction. Their exams show that neither had produced a complete proof, suggesting there was still rich proof production to capture in the interviews, which took place some time later. As we noted before, this theorem relates concepts rich in quantification. Our analysis focused both on their reasoning about quantification and that reasoning's interaction with their problem-solving process more generally. Further, since they had already attempted to prove this version of the Arzelá-Ascoli theorem, we were confident that their familiarity with the equicontinuity, boundedness, and total boundedness definitions would allow them to begin the task of proving without onerous development of each concept.

We approached our study through a Radical Constructivist (von Glasersfeld, 1995) lens, which in this case meant we interpreted students' meanings as schemes to which they assimilate features of goal-oriented activity. We sought to elaborate the specific schemes supporting their reasoning involving quantification. The interviewers frequently asked students to explain and justify arguments they made, describe how they understood the ideas they were utilizing, and how they understood any visual representations they made. In accordance with Thompson (2008), the initial goal of our data analysis was to build second-order models of the students' understandings and their ways and means of operating. Interviewers posed questions to test hypotheses about student thinking, not to foster particular understandings. Once initial models were built of the students' reasoning, we observed commonality across the ways that the students were operationalizing variation and pair-wise relationships. We then returned to the data to engage in interpretive analysis (Clement, 2000) with a particular focus on generating new theoretical models of students' reasoning with multiply-quantified statements.

Our initial models described ways that Adisa and Fenfang reasoned about (or backgrounded) the variation in and interrelationships among the quantities in the proofs. As noted above, we call these actions *quantification operationalizations* because they simultaneously give meaning to the quantification and provide ways of using the quantification for proof construction. We attended to the ways the quantops helped Adisa and Fenfang reason about these highly complex proofs, helped us explain their lines of reasoning, and often entailed actions and counteractions by which quantification could be backgrounded and foregrounded as needed. By iteratively analyzing the students' quantification actions and developing our analytic descriptions of each quantop, we refined a list of quantops 1) that could account for the range of student reasoning across the data and 2) in which each quantop occurred multiple times across the interviews.

Results

We present our results in two sections. First, we present and define the seven quantops. Then, we present illustrative data from the students' work on the Arzelá-Ascoli theorem.

Defining the quantops

We identified seven categories of quantops, each with a foregrounding and backgrounding aspect. These two aspects were important for managing the complexity of the statements and relationships being reasoned about. Table 1 presents all seven of the quantops. There are two forms of each quantop: one that backgrounds the quantification to allow the prover to reason about other matters and another that foregrounds the quantification as the focus of reasoning.

Table 1. The foregrounding and backgrounding forms of the seven quantops.

Quantop	Foregrounding	Backgrounding
Variation quantop	Imagining variation in a quantity	Fixing an otherwise variable quantity
Unitizing quantop	Unpacking variation within some unit	Encapsulating variation in some way
Dependence quantop	Constructing relationships of dependence or invariance	Quantifying two variables with no identified relationship between their variation
Cardinality quantop	Making inferences about a variable by explicitly referencing the cardinality of its possible values	Making inferences about a variable based on implicit imagery or acknowledging possible alternatives in the cardinality of variation
Notational quantop	Explicitly linking notation to some aspect of an object's quantification status	Making inferences about a variable based on implicit assumptions about notation or an image of suppressed notation
Existential quantop	Warranting existence by stipulation or construction	Assuming existence to allow further proof activity
Inference quantop	Tracking a line of inference as quantities vary	Constructing lines of inference within other quantification backgrounding actions

Variation Quantop: Fixing a quantity that varies/ imagining variation in a quantity. Universal quantification is frequently operationalized by reasoning about a fixed, but arbitrary element of the set (often expressed by the word "let"). This means imagining fixing a quantity with the awareness that it could otherwise vary. The complement action to fixing a quantity is intentionally imagining variation in the quantity. It is worth noting that imagining variation can happen in different ways such as imagining continuous variation of a number, imagining iterative variation in some countable set, or imagining variation through elliptical reasoning about a few representatives. The intentional variation may need to be coordinated with any reasoning previously performed on the fixed quantity, which often involves some of the other quantops.

Unitizing Quantop: Unitizing an object that involves variation. This quantification action occurs when the reasoner has some means by which they can encapsulate variation via some representation or property that allows them to reason about a range of values at once. For instance, the graph of a function can be understood to represent the range of values of the input and output. A neighborhood stands for all of the values of the variable bounded within the neighborhood. Graphical representations in particular provide tools for unitizing variation. This allows an association of numerical bounds ($|f(x) - f(t)| < \varepsilon$) with spatial regions (e.g., inside ε-tubes on a graph). The use of neighborhoods to encapsulate a range of values is essential to the interface between topological and metrical interpretations of analysis.

Dependence Quantop: Dependent covariation and independent invariance. In an each-to-some pairwise relationship, one quantity may be understood to depend on the other. Accordingly, varying one quantity results in the dependent quantity varying with it, hence the two co-vary. In a one-to-every pairwise relationship, variation in one quantity does not require variation in the other. The other quantity remains invariant, and hence is considered independent. Constructing relationships of independence/dependence or co-variation/invariance are an essential quantop for reasoning about pairwise relationships, and other quantops are often employed toward this end.

Cardinality Quantop: Attending to the cardinality of a quantity's range of values. This quantop is important for imagining variation and reasoning about dependence/independence. An invariant quantity takes on exactly one value. A covariant quantity takes on the same cardinality of values as the quantity with which it varies (or at least no larger a cardinality). A quantity that varies finitely affords certain actions such as taking a maximum. A quantity that varies countably may be reasoned and proved about iteratively. A quantity that varies uncountably must be reasoned about without taking maximums or assuming a "next" choice.

Notational Quantop: Notating features of quantification. A number of notational tools and conventions are used to maintain a record of the quantops performed to aide reasoning about the range of relevant objects and their quantificational status. When two quantities co-vary, they are often notated with the same subscript or one serves as a subscript for the other (e.g., f_i and ε_i, δ_ε). When a quantity is independent (cardinality of 1), it may be notated using *, lack of a subscript (implying lack of variation or dependence), or some other marker of uniqueness. If a quantity varies countably (including finitely), they are often indexed by the natural numbers to express the cardinality of variation. Mathematical concepts are sometimes named to indicate a particular quantification structure; the adjective "uniform" in analysis often means that a relationship holds independent of the variation in another quantity. However, neither dependence nor variation are always notated. When a proof operates fully by universal generalization, an arbitrary value stands for all possible values without further notation (such as "Let $\varepsilon > 0$").

Existential Quantop: Warranting existential quantification. Proofs of existentially quantified claims require that provers warrant the existence of an object meeting the stated requirements. The two primary ways of warranting existence in a direct proof are by construction or by stipulation. One warrants existence by construction when you show how the quantity could be produced from other quantities (e.g., $\varepsilon = \frac{\delta}{M}$ or $M^* = \max\{M_i\}_{i=1}^{n}$). One warrants existence by stipulation when a definition guarantees that something exists. For instance, if we know a function is bounded we can stipulate that M exists such that $\forall x, |f(x)| \leq M$ (this is sometimes called *existential instantiation*). The complement action to warranting existence is to assume existence. We observed several times when students assumed that some quantity existed to elaborate a line of inference before later returning to warrant that claim. This backgrounds the need to warrant existence to allow progress on the proof.

Inference Quantop: Imagining a line of inference under variation. Proofs entail inferences and constructions performed on quantities within some imagined quantification structure. These lines of inference become nested within other quantops such as fixing or varying certain quantities, which often means the whole line of inference must be analyzed under variation. If a quantity is constructed from another that is temporarily fixed (e.g., $\delta = \min\left\{\frac{\varepsilon}{3}, 1\right\}$), then varying the first quantity may result in varying the newly constructed one. This induces a relationship of dependence. If one wants to justify a relationship of independence, the reasoner must attend to why the construction of one quantity remains invariant while other quantities are allowed to vary (e.g., the maximum or supremum of a set does not vary as the values in the set vary).

Examples from student data: Proving totally bounded implies bounded

We limit our data presentation to Adisa and Fenfang's initial orientation to the concepts since it highlights many of the quantops and is easier to describe succinctly as compared to the extended proof production that followed. They began their work with the statement of the Arzela-Ascoli theorem written on the board as "A subset S of $C([a, b], \mathbb{R})$ (sup) is equicontinuous and bounded iff it is totally bounded." Since we aim to link the imagery of the various quantops to students' goal-oriented activity, we organize this description by the apparent goals the students were pursuing throughout the task-based interviews.

Goal 1: To recall and write the definition of equicontinuous. Fenfang began writing a definition of equicontinuity as "$\forall \varepsilon > 0, \forall n \in \mathbb{N}, \exists \delta > 0,$ s.t. if $|x - t| < \delta$, $|f_n(x) - f_n(t)| < \varepsilon$." Adisa asked if this was "uniform" or "pointwise." Fenfang explained why this was "uniform" in two ways. First, she said that a pointwise meaning would have "$\forall x$" at the beginning. Then, she explained that for pointwise "we fix one x [tracing a vertical line with her hands] and then we have… epsilon and delta." They agreed this expressed "uniform equicontinuous" while Adisa wrote the definition of "point-wise equicontinuous" as "$\forall x, \forall \varepsilon > 0, \exists \delta > 0\ |f_n(x) - f_n(t)| < \varepsilon, |x - t| < \delta. \forall n$." Adisa explained the difference saying, "For every x, there is a δ that you pick such that this will cause [points to the inequality on f values]."

We observe two quantops in this quick interchange. Fenfang fluidly shifted from moving the quantification expression "$\forall x$" to the front of the definition and imagining fixing that quantity before introducing the others (variation quantop). They recognized that this meant the δ that followed depended on the choice of x (dependence quantop). While they expressed this in writing using syntactic order, they did not completely rely on order to express quantification relationships. In the process of writing their definitions, both wrote the final predicate "$|f_n(x) - f_n(t)| < \varepsilon$" before they wrote other quantification expressions. Adisa reversed the common syntax by writing the conditions on x, t, and n after the predicate inequality. Fenfang used conditional "if, then" structure to quantify x and t, which is common for the nested universals in analysis definitions (a backgrounding variation quantop). We thus observe that their ways of constructing and expressing pairwise relations were more robust and varied than mere syntactic order, which remained true throughout the interviews.

Goal 2: To recall and write the definition of bounded. Adisa then questioned whether "bounded" meant "one M for all of them or one M for each one." The interviewer asked the students to explain the difference. Adisa initially said it "boils down to the same" suggesting the two were equivalent. To explain this, Adisa began exploring whether he could prove one from the other. He wrote the definition, "$\exists M > 0$ s.t. $|f(x)| < M\ \forall f \in \mathcal{E}$." He then wondered aloud about the cardinality of $\mathcal{E}$. He explained that if this set were finite and "I could always take the max of all of them and use that as the M for every one of them." He later expressed this writing, "$M = \max\{M_1, M_2, \dots, M_n\}$." Alternatively, he explained that when $\mathcal{E}$ is infinite, this would not be possible since you would have to take the supremum of the bounds (which may not be finite). He thus decided that the definition must mean "one M that bounds all the f's in the subset."

We observe that Adisa used implication between the two possible meanings of bounded to decide what was the correct definition. First, he used dependence and cardinality quantops to interpret how the conditions were different. It seems he implicitly assumed that having one M implied having a bound for each (one M is a stronger condition) and thus focused on proving the converse implication. Within the condition "one M for each," he imagined M varying as f varied so that finitely many functions meant finitely many bounds (variation and dependence quantops). He expressed the maximum of these bounds by constructing the universal bound with no subscript as the maximum of the set of varying bounds with natural number subscripts (notational quantop). He then considered the available inferences on the set of bounds (inference quantop). Since an infinite set of bounds would not always be bounded above, he decided that having many M's did not imply having one M and that the definition was the stronger condition.

Goal 3: To recall and write the definition of totally bounded. Adisa had written on the board "$\forall \varepsilon > 0\ A \subset M\ \exists$ finite covering by ε-ngbhds" as a definition for a totally bounded set M. He elaborated using a diagram of a square, which represented the set A. He said if you fix $\varepsilon = 1$, then there would be a finite set of neighborhoods that cover A. He represented this by drawing circles on the square that covered most of its area. He then elaborated his written definition with the set $\{a_1, \dots, a_k\}$ explaining "every member of A will be in one of these ε-neighborhoods, centered at one of those." As Fenfang explained her interpretation of the diagram by putting a_i at the center of each circle, Adisa also wrote "Let $\varepsilon = \varepsilon_k$. $\{M_\varepsilon(a_i)\}_{i=1}^{k}$." Both students were working to elaborate that the a_i were centers of epsilon neighborhoods with radius ε.

In this discussion, the students constructed a complex each-to-some relationship in which each of the (infinitely many) functions was in some one of the (finitely many) neighborhoods, which involves the cardinality and dependence quantops. However, the specific functions in the neighborhoods were nowhere represented on the board except as locations in the square. This reflects the idea that the cover means that neighborhoods stand for the range of values within them (unitizing quantop). The neighborhoods themselves represented covariation of three entities: center, radius, and set. Adisa began discussing only the radius and centers before he noted that each center represented a whole neighborhood, which he then notated using $M_\varepsilon(a_i)$ (dependence quantop). Both students felt the need to recognize the various moving parts even while they clearly linked them as compound units that covaried. This helps explain why Adisa, after initially fixing ε=1 (variation quantop) later wrote "$\varepsilon = \varepsilon_k$" to show that the radius did not vary with the neighborhood (dependence and notation quantops).

Conclusion

In this paper, we set forth seven categories of quantops that capture the complex ways two graduate students managed the high number of quantities involved in the Arzelá-Ascoli theorem and the relationships between them. These categories contribute to the literature by identifying new aspects of how quantification is enacted in reasoning about analysis proof tasks. Furthermore, we note the interplay between foregrounding and backgrounding quantification to manage the cognitive load of the tasks. Due to space concerns our data presentation was limited to a largely illustrative function. However, we anticipate that future reports and studies can continue this work to characterize how quantification is enacted in reasoning. A natural future step is to explore how learners can develop these highly complex skills for proof production.

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A Symbolizing Activity for Constructing Personal Expressions and its Impact on a Student's Understanding of the Sequence of Partial Sums

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This paper reports the results from a set of exploratory teaching interviews designed to engage students in reasoning about partial sums and constructing individualized algebraic expressions (called personal expressions) to describe their meanings for partial sums. Our analysis focused on one particular student, Emily, who constructed two sets of personal expressions to describe her conceptions for partials sums as (a) a process and (b) the result of an additive process, one novel and one based on her image of summation notation. We conjecture that investigating individual students' purposes for creating personal expressions and how these expressions evolve will provide greater insight into how students use mathematical notations and their meanings for particular mathematical topics such as infinite series.

Keywords: Infinite series, symbolization, exploratory teaching interview, calculus, summation symbol

Introduction and Literature Review

Mathematicians and students use semiotic representations such as graphs, algebraic notation, formal statements, and diagrams ubiquitously to convey their images for mathematical concepts. Algebraic notations are often privileged in teaching calculus topics such as infinite series (González-Martín et al., 2011) and constitute a focal point of students' reading of mathematical proof (Shepherd & van de Sande, 2014). Algebraic expressions such as summation notation and limit notation can be difficult for students to interpret because expressions often convey dual meanings, such as a dynamic process (i.e., additive process, limit process) or the result of this process (i.e., sum, limit value), and students must determine which meaning is appropriate in a given context (Gray & Tall, 1994; Güçler, 2013; Kidron, 2002; Martínez-Planell et al., 2012). Gray and Tall (1994) claimed that professional mathematicians' ability to reason about mathematical concepts successfully comes from flexibly using notations to describe either a process or the result of the process. Although cognitive theories related to students' construction of stable concepts from processes abound in the literature (e.g., Dubinsky, 1991; Glasersfeld, 1995; Sfard, 1991), there has been little research on the algebraic expressions that students create in correspondence with this cognitive transition (e.g., Tillema, 2010).

This paper constitutes the results from one portion of a more extensive study designed to investigate undergraduate students' symbolizing activities related to infinite series. In mathematics, the symbolic expression $\sum_{n=1}^{\infty} a_n$ for an infinite series indicates (1) the process of summing consecutive terms of an infinite sequence (a_n) and (2) the metaphorical "result" of this process. The method to establish whether an infinite series converges and the value to which it converges is to determine the limit of the corresponding sequence of partial sums, typically denoted using the expression $\sum_{i=1}^{n} a_i$. To construct the sequence of partial sums, the student must abstract the process of computing individual partial sums into a cognitive entity that can be coordinated with an index to order the partial sums into a sequence.

The conventional algebraic notation for denoting the general term of the sequence of partial sums is summation notation, i.e., $\sum_{i=1}^{n} a_i$. Some researchers have reported the potentially problematic nature of the indices of the notation (i.e., i's in a_i; Katz, 1986) and how students' meanings for the indices can affect their ability to interpret or expand summation notation (Strand et al., 2012; Strand & Larsen, 2013). However, there is scant research on how students impute their various meanings for partial sums to algebraic expressions or the ways in which these expressions change as the student's conceptions of partial sums evolve. Consequently, this paper will describe the evolution of one students' algebraic expressions for individual partial sums as she transitioned from considering the process of computing individual partial sums to envisioning arbitrary partial sums as a concept. The research questions addressed by this study are: (1) *what purposes do students have for constructing and utilizing symbolic expressions for partial sums?* and (2) *how do students' constructed expressions evolve as a student moves from considering partial sums as a process to a result?*

Theoretical Perspective

In this paper, we adopt a radical constructivist ontological perspective on algebraic expressions. Glasersfeld (1995), who initiated radical constructivism, proposed the following definition for symbol: "a word will be considered a symbol, only when it brings forth in the user an abstracted re-presentation" (p.99). In this context, a *re-presentation* is an abstracted component of an individual's previous experience that she posits in her mind's eye to metaphorically re-experience the situation. From this standpoint, an algebraic expression itself becomes a symbol (to a student) only when she imputes a component of her previous experience (e.g., relationship, process, concept) to the expression.

We will use the term *personal expression* to denote an algebraic expression that a student has created and to which, in the researcher's mind, the student has imputed a re-presentation. We will call the basic unit of an *expression* an *inscription*, which we define as a written mark utilized by an author to succinctly describe a property, action, or relationship that the author has envisioned. A student may adopt a conventional notation to construct a personal expression or create a unique and novel expression. However, we will only call an expression a personal expression if it appears, to the researcher, that the student has "something to say through [it]" (Thompson & Sfard, 1994, p. 6).

There are several "things" that students say through their personal expressions. For example, Gray and Tall (1994) identified three meanings that a student can impute to an expression: (a) a process or mathematical operation such as adding consecutive integers, (b) a concept such as the sum of the first ten positive integers, or (c) a dualistic meaning encompassing both the process and the concept such that either may be called upon at will. Gray and Tall (1994) used the term *procept* to identify the third meaning, a personal expression that a student employs fluidly to re-present either a process or a concept (according to her needs). In the results section of this paper, we report a student's construction of a set of personal expressions that she initially utilizes to re-present a process (meaning "a") and her subsequent construction of a new personal expression to re-present a proceptual meaning for partial sums (meaning "c").

Methodology

The data for this study consisted of four 90-minute individual exploratory teaching interviews (Castillo-Garsow, 2010; Moore, 2010; Sellers, 2020) conducted during the Summer 2021 session at a large public university in the Southwestern United States. This paper focuses on the first two interviews, which entailed tasks related to the sequence of partial sums. The student participants were enrolled in a second-semester calculus course and completed the unit on sequences and series between the two interviews. Due to the pandemic, the interview was conducted remotely, and student work was recorded on a collaborative whiteboard application. The first author served as the teacher-researcher, and the second author served as the witness.

The task sequence was based loosely on a three-phase approach described by Radford (2000). In the first phase, the students described how they might determine individual partial sums from a set of five infinite series presented in expanded form (e.g., $1 + \frac{1}{2} + \frac{1}{4} + \cdots$). In the second phase, the teacher-researcher prompted the students to reflect on their actions during Phase I and construct a written rule for determining an arbitrary partial sum. In the final phase, the teacher-researcher introduced the concept of a personal expression, asked the students to construct personal expressions to denote an arbitrary partial sum, and describe their meaning of partial sums, the sequence of partial sums, and infinite series through their expressions.

Results

This section focuses on one student, Emily, and her journey to construct a personal expression that she could use to re-present her image for partial sums. Emily was a chemical engineering student who had just finished her freshman year and took second-semester calculus during the Summer 2021 session. We focused our analysis of Emily's interviews on (a) describing the purposes that Emily had for constructing her written rule and personal expressions to describe partial sums and (b) examining the evolution of her personal expressions throughout the interviews. In the following subsections, we discuss Emily's actions and personal expressions during each of the three phases of tasks, focusing mainly on her construction of expressions to describe a process for computing a partial sum and adopting a holistic view on partial sums.

Phase I and Phase II: Reasoning about Partial Sums and Developing a Written Rule

During the Phase I task, Emily developed a four-step process to determine the value of an arbitrary partial sum. These steps included (a) examining the initial terms in the series to develop a recursive or explicit pattern to generate additional summands, (b) test the potential pattern on all known values in the series, (c) use the pattern to generate all terms in the series until the desired term (e.g., 37^{th} summand), and (d) add the terms together to compute the partial sum. In the Phase II task, Emily constructed a written note to reflect these four steps (see Figure 1). Her purpose for constructing this note was to describe the process of computing a partial sum in clear, distinct steps, as evidenced in the transcript below:

Interviewer: Go ahead and explain to me your rule, your written note.

Emily: Ok, so first, um. Well, first, before, when I was thinking about how, like, what steps I would do [*to determine a partial sum*], I could not describe a way that I would tell someone it, because. I'm a visual learner, and I would have to show, I would have to have a sequence (*Emily used sequence and series interchangeably*) in front of me to show them like, "Ok, this is what you do." So, I was like, Ok, so I'm going to change it and I'm going to put it like an instruction manual. Um, so, in that tense, the first step is to…

In this episode, Emily was unsure how to describe her thinking in written form until she decided to create an ordered set of instructions delineating her method to determine an arbitrary partial sum. After Emily decided that the purpose of her written rule was (to her) to depict a process, she quickly and efficiently wrote the four-step written note shown in Figure 1. In this regard, we conjectured that Emily could leverage her *describe-a-process purpose* for developing her written rule to construct a personal expression through which she could re-present to herself this process.

Phase III, Purpose 1: Developing a Personal Expression to Describe a Process

During the Phase III task in the first interview, we presented Emily with a video clip describing the meaning of personal expressions and providing examples of constructing personal expressions (the video clip may be viewed at https://youtu.be/PdKkhZVPulA). After the video clip, we asked her to create a personal expression for an arbitrary partial sum and record any inscriptions that she created in a glossary. Emily's written rule, personal expressions, and glossary are contained in Figure 1.

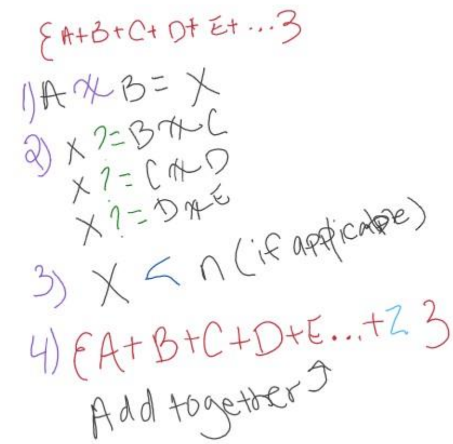
Emily's written note:

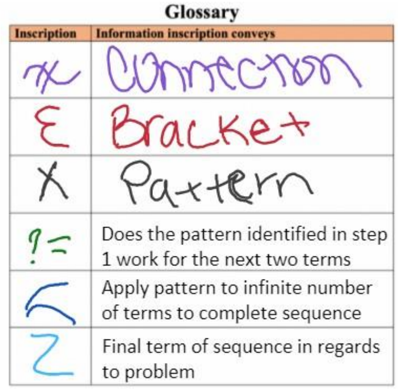
In order to find the sum of any number of terms in a series, follow these steps:

1) Compare the first and second terms in the series. Meaning look at these first two terms and think about how they connect to each other. Ex: 1 > 2 (Explain how you can get from 1 to 2 and usually you will find the pattern to the whole sequence quickly.
2) Once you think that you have figured out the pattern. Use it on the rest of the sequence. If it works out with all of your terms then you have found the pattern correctly. If not return to step 1 and repeat until correct.
3) After you have found the correct pattern for the sequence. All you have to do is apply it until you reach the term you wish to calculate.
4) Now you have calculated the correct number of terms needed. Lastly, you will take all of your solutions and add them up to find the sum of your number of terms.

Then CONGRATs you have found the sum of the number of terms.

Emily's Personal Expressions





Glossary

Inscription	Information inscription conveys
ϰ	Connection
{	Bracket
X	Pattern
?=	Does the pattern identified in step 1 work for the next two terms
≺	Apply pattern to infinite number of terms to complete sequence
Z	Final term of sequence in regards to problem

Figure 1. A comparison of Emily's written rule, personal expressions, and glossary

In Emily's personal expression, the order of the summands of the series is designated by capital English letters. Emily used her "connection" inscription to indicate a relationship between consecutive summands in the series and her quantification of this "pattern" with the letter x. Emily's use of these two inscriptions closely corresponded with Step 1 of her written rule, such that Emily could re-present the first step in her written note through line (1) of her personal expressions. Emily's introduction of her fourth inscription in the glossary, $?=$, functioned as a testing operator through which Emily could re-present the second step of her written note—testing subsequent pairs of summands to determine whether pattern x holds for all summands. Emily utilized her fifth inscription, similar in appearance to $\prec$, to re-present the process of generating the first n summands of the series by repeatedly applying the pattern x to consecutive terms. Finally, Emily introduced a sixth inscription, Z, through which she could designate the value of the nth summand to re-present the partial sum generated by computing the sum of the terms. Although Emily's personal expressions more closely resemble a form of shorthand than a conventional mathematical expression, her personal expressions fulfilled the purpose that Emily proposed when she initially constructed her written note: to serve as a symbolic medium through which she could re-present her process for computing individual partial sums. After Emily successfully created the personal expressions to re-present her instruction manual for determining individual partial sums in the first interview, we decided to challenge Emily's thinking in the second interview by asking her to describe individual partial sums and the infinite series itself with her expressions in the second interview.

Phase III, Purpose 2: Developing a Personal Expression to Describe a Procept

In the second interview, Emily utilized her personal expressions from the first interview to describe the sum for any number of terms in a series. When the teacher-researcher asked Emily to use her personal expression to describe the "entire series," she immediately stated that she would need to create a new inscription. After creating and rejecting a novel inscription, Emily proposed using a variation of conventional summation notation, Σx, to convey a "holistic view" of a partial sum, as evidenced in the following transcript:

Interviewer: Why did you pick this inscription [Σx] for representing the entire series?

Emily: I used it because, so, I know it is used in, like, you know, like, official math symbols, and it's used quite a lot, at least in calculus it's used a lot. And, I do not think I 100% really understand the importance, or maybe the, um, symbolism that this little thing (*indicates sigma*, Σ) does. But I do know that it's used to represent a holistic view of something. And so, that's the closest symbol I can think of. All my symbols were going to look something like that, the ones I was thinking of before I made it. So, I do like the holistic view of what it means no matter, whenever it is used, it always means, like, all of something. So that's why I choose it.

At this moment, we claim that Emily's purpose for constructing a personal expression changed from a *describe-a-process purpose* to an *adopt-a-holistic-view purpose* that resulted in constructing a personal expression to re-present both the additive process of computing a partial sum and the result of the process. We further claim that Emily's "holistic" personal expression functioned as a *procept* through which she could fluidly consider either the process of generating summands and adding them together or the resultant sum. The transcript below indicates this fluidity in re-presentations that Emily imputed to her "holistic" expression, which she employed to describe specific partial sums for individual series (see Figure 2).

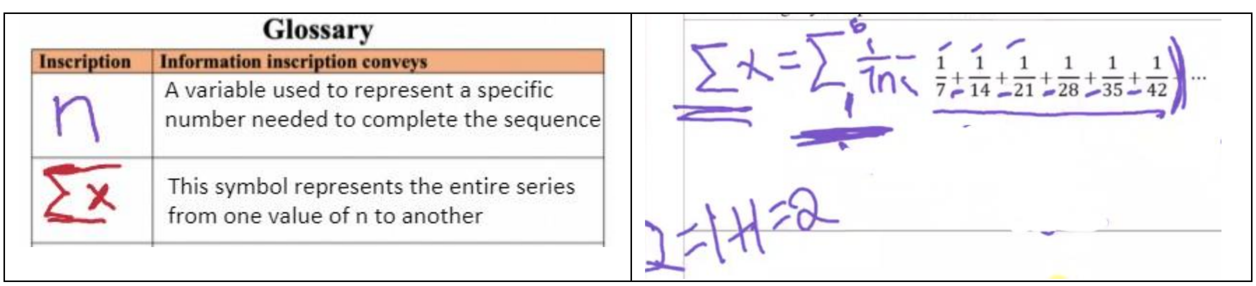


Figure 2: Emily's personal expression for a "holistic" view of a partial sum

Emily: So $1+1=2$, right? So, if you have this $(1+1)$ equals to Q (*writes* $Q=1+1=2$), then Q equals not only $1+1$, but it always equals 2. Does that make sense? So, yes, in the inscription ($\sum_1^6 \frac{1}{7n}$) when you say what it equals, it would equal all of these written out (*indicates* $\frac{1}{7}+\frac{1}{14}+\frac{1}{21}+\frac{1}{28}+\frac{1}{35}+\frac{1}{42}$). But when you add them together, it also equals that (*indicates* $\sum_1^6 \frac{1}{7n}$) .

Emily's new personal expression through which she re-presented partial sums became a powerful tool for her to describe partial sums, infinite series, and components of infinite series throughout the remainder of the second interview. However, Emily also continued to use her original shorthand personal expressions (see Figure 1) to describe computing individual partial sums in the series task following the introduction of her "holistic" personal expression. For example, after Emily constructed instantiations of both types of personal expressions to describe (a) the process of computing any partial sum (using expressions in Figure 1) and (b) the partial sum itself (using expressions in Figure 2), the interviewer asked Emily whether her personal expression for an arbitrary partial sum, $\sum_1^n x$ (part "b"), could be used to compute any partial sum (part "a"). The transcript below details Emily's response, which indicates that she was beginning to consider that her "holistic" personal expression might subsume her original personal expressions that functioned as shorthand for her written rule:

Emily: Yeah, you could because, like I said before, as long as this top n does not equal infinity, x, or you know whatever, this (*indicates* $\sum_1^n x$) equals this (*indicates* $\frac{1}{3}+\frac{1}{8}+\frac{1}{13}+\frac{1}{7}+\frac{1}{35}+\cdots$)… But, um, so this, the "holistic" inscription that I made equals the $\frac{1}{3}+\frac{1}{8}+\frac{1}{13}+\frac{1}{7}$. So, yes, as long as n is not infinity in the way that I've written it, um, this whole thing (*indicates* $\sum_1^n x$) can answer number 1 [part "a"] and number 2 [part "b"].

For the remainder of the interview, Emily utilized only her "holistic" personal expression, $\Sigma_1^n x$, to describe partial sums and series. However, the remainder of the tasks were related to describing infinite series or partial sums as results, so we do not have clear evidence that Emily would not have utilized her original shorthand personal expressions to describe computing individual partial sums. This experience indicates that even though Emily created an expression through which she could re-present a proceptual view of partial sums, the "holistic" expression might not have entirely subsumed her original shorthand expressions. Instead, Emily's two types of personal expressions likely evolved to fulfill two separate (to her) purposes: re-presenting the process of computing a partial sum and re-presenting the result of this process.

We acknowledge that because Emily participated in an accelerated sequences and series unit in the week between the first and second interview, it is likely that summation notation was more readily available to her in the second interview than the first. However, we make two points about Emily's use of summation notation to clarify that her exposure to summation notation in her course did not unduly affect her construction of personal expressions. First, Emily persisted in using her original personal expressions in both interviews to re-present the process of computing a partial sum, including after the portion of the interview where she created her "holistic" personal expression, $\Sigma_1^n x$. It appeared that only when Emily's purpose for constructing a personal expression changed from a *describe-a-process purpose* to *adopt-a-holistic-view purpose* did the need for summation notation arise. Second, Emily initially stated that she did not want to use summation notation but settled on this convention after realizing that her image of what summation notation conveyed (i.e., a holistic entity) aligned with her purpose for constructing a personal expression.

Conclusion and Discussion

The research questions for this paper addressed (a) the purposes for which students construct personal expressions and (b) how these personal expressions evolve as students' thinking changes about partial sums. We have shown the example of one student, Emily, who created two distinct sets of personal expressions—one entirely novel, the second borrowed from convention—to re-present to herself the process of computing partial sums and the resulting partial sums. We have further indicated that neither of Emily's personal expressions seemed to wholly subsume, even as her purpose for creating the expressions changed from re-presenting a process to re-presenting a result. At one point near the end of the Phase III tasks, Emily claimed that (to her) her personal expression resembling summation notation could be used to answer the questions that she had previously utilized her process-oriented personal expressions to answer. However, there is insufficient data to conclude whether she believed her "holistic" personal expression could wholly replace her initial instruction-based shorthand personal inscriptions.

It is worthwhile to note that conventional mathematical symbols all began as personal expressions for individual mathematicians. For example, suppose that a theoretical mathematician spends several months considering a hitherto unreported topic within a particular branch of abstract mathematics. During this research, the mathematician identifies a novel process or concept that she would like to report to the mathematical community formally. In order to both name this mathematical idea and distinguish it from others, the mathematician proposes a new mathematical symbol—or a variation of an existing symbol—to indicate this idea. In other words, the mathematician constructs a personal expression to re-present to herself the mathematical idea that she has constructed. After the publication of the mathematician's work, other scholars might adopt and utilize the newly-constructed symbol in their own work. Over time, the symbol might evolve into a convention with a "normative meaning" assigned by the mathematics community. Mathematicians and instructors might then utilize the conventional expression for decades, or even centuries, to convey this "normative meaning" in their practice. However, the conventional expression would not exist without the work of the original mathematician, who considered an idea that was (to them) unique and constructed a personal expression to which they could re-present their meaning for their novel idea.

We acknowledge that research on communication and symbolization between individuals is essential and helpful to further understanding social dynamics and education at the group level. In this paper, we step even further and move the discussion on symbolization to the realm of individual student cognition. Exploring students' creation of personal expressions is a novel approach that does not focus on the power of semiotic representations to foster communication between individuals and groups. Instead, the focus is on an individual's designation of a semiotic device as an instrument through which she can re-present her meanings to herself (a prerequisite to effective communication with others). In this regard, personal expressions almost always make sense to students, both from a semantic and syntactic perspective, because students create or adapt them to re-present their specific meanings. Research investigating students' personal expressions, students' purposes in creating personal expressions, and how students' personal expressions evolve and interrelate can better identify the various meanings that students impute to their expressions. Individuals construct their own meaning, and by studying students' construction and utilization of personal expressions, we can potentially gain deeper insight into how students use symbolic notations and their meanings for mathematical topics such as infinite series.

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Confronting Abstraction: An Analysis of Mathematicians' Concept Images and Definitions

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Mathematicians frequently engage in mathematics involving highly formal and abstract concepts. These concepts are often challenging or impossible to accurately visualize. However, mathematicians are still successful at answering new, difficult mathematical problems, despite confronting this obstacle of mathematical abstraction. This study attempts to understand how mathematicians think about and understand abstract mathematical concepts. Tall and Vinner's (1981) notion of concept image and definition are used to model and analyze the understanding of these mathematical concepts. One research mathematician participated in an individual Zoom interview, where both interview questions were answered, and a mathematical task was completed. Analysis provided evidence that for abstract mathematical concepts, a large portion of a mathematician's evoked concept image consists of informal and visual examples. These examples are supplemented with important formal definitions and properties from one's concept image. Implications for the teaching of undergraduate and graduate mathematics are discussed.

Keywords: Abstract algebra, Concept image, Mathematical abstraction, Research mathematician

Introduction

In academic settings, mathematicians frequently engage in solving difficult mathematical problems. For these mathematicians, and in particular those who study pure mathematics, these problems often involve highly formal and abstract concepts. With the introduction of more recent mathematical theories, such as category theory, much of the newest, most cutting-edge mathematical research is teeming with exceedingly abstract terminology, definitions, theorems, and proofs. One research area in which the extent of the abstraction can be clearly acknowledged is in the broad field of abstract algebra, which includes some subfields such as algebraic geometry and commutative algebra. A major challenge in understanding abstract, formal mathematics is that as the abstraction grows, the ability to accurately visualize the mathematics frequently becomes more challenging. In contrast, much of primary and secondary school mathematics consists of concepts that can be, at least somewhat, visualized. Simple algebraic concepts like functions can be visualized in the Cartesian plane, geometric objects are easily sketched on paper, and core calculus concepts, such as differentiation and integration, can be visualized through rate of change and area under a curve. This is not to question the difficulty of learning these mathematical concepts, but it illuminates a new obstacle that mathematicians encounter when studying mathematics. However, despite the abstraction of formal mathematics, mathematicians continue to be successful at solving problems and publishing new results. Thus, it is worth investigating how mathematicians think about abstract mathematical concepts.

In mathematics education, the research of mathematicians' mathematical practices is largely uncharted territory. Moreover, there is even further scarcity in the literature on how mathematicians think about and understand mathematics. One mathematical practice that has been closely studied is how mathematicians interact with proofs, such as their reading, writing, and evaluation of them (e.g., Mejía-Ramos & Weber, 2014; Weber, 2008; Weber & Mejía-Ramos, 2011, 2013). Leone Burton (1999) conducted an interview-based study of seventy research mathematicians about how they "come to know" mathematics. Though this study is insightful, her emphasis was on their mathematical practices and how they engaged in mathematical communities of practice, not on modeling the mathematicians' thinking.

Furthermore, mathematicians are valuable members of the broad mathematical community. They instruct and mentor undergraduate students, and they are responsible for the presentation of much of the advanced mathematics that these students will encounter. In addition, some of these students will go on to teach mathematics in primary and secondary schools. Thus, understanding how mathematicians think about and understand mathematics could provide valuable insight into mathematicians' teaching practices and why they are successful or not. Moreover, this could then also have far-reaching benefits for the undergraduate and graduate students who are taught by these mathematicians. Lastly, mathematicians are excellent at understanding highly abstract concepts. Working to understand and model their mathematical thinking may be advantageous in learning how to help and instruct students to develop this same level of understanding.

Literature Review

The term "abstract" in mathematics is often used to describe formal mathematics because formal, or pure, mathematics is centered on the use of generalized definitions, concepts, and symbolism. I will define *abstraction* to be the process of generalizing a concept, phenomenon, or observation by extracting the underlying properties or structures, without reference to the particular instances that they came from. In formal mathematics, abstraction serves a pivotal role because it allows mathematicians to solve the general case of a problem, rather than just a specific scenario. As an individual continues to learn new mathematics in a specific field, it is likely that the mathematical concepts may become more abstract. Dienes (1961) describes this notion as follows: "When we say 'more abstract', we mean that more, seemingly rather different, situations may be described at the same time… Getting more abstract then means extending the field of applicability" (p. 293). Moreover, Hiebert and Lefevre (1986) use the term "abstract" to refer to the degree to which a unit of knowledge is tied to specific contexts (pp. 4-5).

To be successful, mathematicians need to be proficient at understanding and utilizing abstract concepts, and potentially also, constructing new abstract concepts. This raises the question of how are mathematicians able to successfully work with abstraction in mathematics? More specifically, how do mathematicians understand and think about mathematics that they cannot accurately visualize? One thread of research that begins to address this question is on how mathematicians leverage the use of examples to explore and prove conjectures (Lockwood, Ellis, & Lynch 2016; Lynch & Lockwood 2019). Constructing examples for an abstract concept allows for a way to connect the unfamiliar mathematical concept to a familiar concept that is already known or understood. Lynch & Lockwood (2019), when comparing the use of example between mathematicians and students, observed that "mathematicians were much more explicitly attuned to the logical structure of conjectures and how examples fit into those structures" (p. 337). Though this observation is in the specific context of mathematical proof, it demonstrates that for mathematicians, examples and the formal concepts they exemplify are deeply intertwined. Vinner and Dreyfus (1989) also state that, "The student's image [of a mathematical concept] is a result of his or her experience with examples and nonexamples of the concept" (p. 356). It could then be assumed that mathematicians' understanding of a concept is, at least partially, structured by, and reliant on, insightful examples that illustrate the properties of that concept.

Theoretical Framework

The theoretical constructs of *concept image* and *concept definition* (Tall & Vinner, 1981) have been used in a variety of contexts. Tall and Vinner define the concept definition to be "a form of words used to specify that concept" (p. 152). This concept definition can vary over time, and different definitions can be used in varying contexts. Furthermore, this definition can differ from the formal concept definition that is generally accepted by the mathematical community at large. The more complex construct is that of the concept image, which Tall and Vinner define as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). In particular, the concept image not only consists of visual imagery. Vinner (1991) states that an individual's concept image "can be a visual representation of the concept in the case the concept has visual representations; it also can be a collection of impressions or experiences" (p. 68). A person's concept image will likely change over time, and at any given moment, a portion of a person's concept image may be used and become activated, which Tall and Vinner (1981) define as the *evoked concept image*. Moreover, two different components of one's concept image or definition may be in conflict with one another, which can lead to any number of negative responses. Tall and Vinner refer to this notion as a *potential conflict factor* (p. 153). Research related to concept image and definition frequently focuses on content from high-school or undergraduate mathematics courses, such as calculus, linear algebra, or abstract algebra (e.g., Melhuish et al, 2020; Rösken & Rolka, 2007; Wawro, Sweeney, & Rabin, 2011). I am unaware of there being any research utilizing concept image and definition to study mathematical content beyond the undergraduate level.

Methods

Research questions

On the basis of the aforementioned background and framework, I examined mathematicians' concept images and definitions for various graduate-level abstract algebra concepts. These concept images and definitions are then used to analyze how mathematicians think about and understand abstract mathematics. The following two research questions are central to this study:

1. How much of a mathematician's concept image for an abstract mathematical concept is composed of visual components and images?
2. What other components of mathematicians' concept images of abstract mathematical concepts are evoked when engaging in mathematical thought and activity?

Design of the study

The participant in this study was one male mathematics professor at a large public university in the southeastern United States. The participant (Luke) is an assistant professor with research interests in various topics in abstract algebra. He was contacted and asked to participate based off of his personal background and experience in mathematical research relating to abstract algebra. The data for the study was collected through an individual interview, where interview questions were answered, and Luke engaged in a mathematical task.

The participant first participated in one 15-minute Zoom meeting where he was asked about his mathematical background and understanding of certain algebraic concepts. The answers provided during this meeting informed which mathematical task was given during the interview. No data was collected during this preliminary meeting. Luke later participated in one 45-minute individual, semi-structured Zoom interview, which was recorded for more-detailed analysis. The interview started with questions pertaining to how he thinks about modules, tensor products, and direct limits, which are abstract concepts commonly used in algebra. These interview questions were designed to help begin delineating his concept images and definitions for these three mathematical concepts. Motivation for the phrasing of the questions was taken from how

Melhuish et al. (2020) articulated their questions about concept definitions of homomorphism and function. For example, two of the interview questions were:

1. How would you define what a module is?
2. What other mathematical concepts, objects, or ideas do modules make you think about?

Interview questions mentioning the word "example" were purposely excluded from the interview, in order to ensure that he was not influenced into sharing examples, but any examples he shared would be an unbiased part of his evoked concept image. Afterward, he was given the following mathematical task, asking him to compute the direct limit of a particular direct system of modules. The task was chosen because it was a problem that he had likely never encountered.

Consider the directed set on $\mathbb{N}$ given by the partial ordering $m \leq n$ if and only if m divides n. Define the directed system of $\mathbb{Z}$-modules $\{M_i\}_{i\in\mathbb{N}}$ where $M_i = \mathbb{Z}$ for all $i \in \mathbb{N}$ and $f_{mn} : M_m \to M_n$ is multiplication by $\frac{n}{m}$. Compute $\varinjlim M_i$.

Figure 1. Mathematical task from the interview, asking Luke to compute a specific direct limit of modules.

The participant utilized tablet screen-sharing to display his work while completing the task. The participant was encouraged to also communicate verbally what he was doing and thinking. The task took about 20 minutes, and all written work was collected after the interview to assist in the data analysis. After the task was completed, interview questions pertaining to how the participant thinks about and approaches mathematical abstraction were asked. This provided an opportunity to collect data tailored more directly to the broad research questions, while also allowing him the opportunity to reflect on the task he had just completed. Following the interview, the interview data was transcribed and analyzed. The analysis process focused on identifying key components of the participant's concept images and definitions, as well as identifying how he approached and understood mathematically abstract concepts. Recurring components were noted and used for more guided analysis.

Results

From the analysis, I found that Luke's evoked concept images for modules, tensor products, and direct limits primarily consisted of examples and visual images, as well as some relevant formal mathematical properties. Pertinent results from the interview are discussed below. Furthermore, results are also presented on how he spoke about mathematical abstraction and how he confronted it. Lastly, Luke was unable to complete the mathematical task during the allotted time of the interview, and some possible reasons for this are discussed below. It should be noted that not every part of the interview, relevant to the research study, is mentioned and discussed.

Examples and visual images

When asked about modules, Luke commonly spoke using simple, visual examples. For example, he often referenced vector spaces when discussing modules. In particular, consider the following interaction:

Interviewer: How would you define what a module is?

Luke: It would probably depend on who I'm talking to, but to a student who has already done linear algebra, I would say that a module is just like a vector space, except the scalars come from a ring rather than a field.

Even though a vector space is a well-behaved, special case of a module, the core properties of this structure can be visualized through images evoked by the concept of vector spaces. Vector spaces also resurfaced in the interview when asked to define what a tensor product is. After Luke discussed the process of finding a basis for a tensor product of free modules, he states:

> If you're just doing vector spaces, every module is free, and you can work with it this way. But it's not always good to pick a basis. We like to work independent of basis, so there is a very abstract way to think about tensor products.

Interestingly, despite the fact that the tensor product of vector spaces provides a way for Luke to better comprehend tensor products, he acknowledges that there are limitations to this restricted understanding. This acknowledgement of an imprecise, but helpful, correspondence between simplified, visual descriptions and formal mathematical properties reoccurred throughout the interview. For instance, he described tensor products as "in a sense like multiplying the two modules."

Later in the interview, when asked to define what a direct limit is, Luke immediately stated, "The simplest possible case of a direct limit is when you have an increasing union." Luke never addressed the formal definition in his discussion of direct limits but chose to primarily provide examples to describe the structure. As we will see later in the discussion, it is likely that this was because he was not certain about his formal concept definition of the direct limit.

Formal mathematical properties

There were multiple instances where the participant called on formal mathematical properties or definitions when answering questions about these mathematical concepts. Some of the examples he provided in the discussion were highly abstract and connected with his specific area of research. However, this was significantly less prevalent than his providing of simple, visual examples and informal descriptions. The first instance of referencing formal mathematical properties is when Luke referred to how he would define what a module is to a class. He mentioned, "If I was teaching a class, I would say that a module is an abelian group together with a map from $R \times M \rightarrow M$ where you have the distributive property and the associative property. So, I would just write all those nuts and bolts out."

Predominantly, Luke mentioned formal mathematical concepts and properties when asked to reference other related mathematical concepts and ideas. One example of this came during the discussion of direct limits:

> *Interviewer*: What other mathematical concepts, objects, or ideas do direct limits make you think about?
>
> *Luke*: Definitely one of the biggest ones is what's called the stalk of a sheaf. And another instance of this is when you talk about, it's the same concept actually, germs of holomorphic functions…Let's just talk complex analysis.

Immediately following this, Luke proceeded to elaborate on many of the details related to the concept of "stalk of a sheaf", using a variety of formal terminology from complex analysis and category theory, such as "contravariance." Another observation is that as the major concepts (module, tensor product, direct limit) became more abstract or complex, Luke tended to answer questions with more informal language and examples. This is interesting because as mathematics becomes more abstract, the formal definitions and properties become that much more important. However, it is possible that he relied on simpler, more visual components of his concept images to effectively understand and communicate his thoughts about abstract mathematics.

Confronting mathematical abstraction

After completing the task, Luke answered more general questions pertaining to abstraction in mathematics. When asked what he does when encountering new mathematical concepts or problems he does not know how to solve, without hesitation he said that one has to do examples. Additionally, he made comments about using strategies such as relating it to something known, understanding the object's properties, and trying to simplify the object. This is evidence that one method of confronting abstraction is to relate the new concept to simpler components of other related concept images. When asked about concepts he cannot accurately visualize he stated:

Interviewer: How do you understand and work with mathematics that you cannot accurately visualize?

Luke: You have to have a way to write it down and work with the objects, and if I can't visualize it, I have to focus symbolically on how can I represent this object, and what manipulations can I do to it.

Luke stated that when he cannot visualize the concept, he has to resort to symbolic manipulation. Even though a visual understanding may not exist for the concept, a formal, symbolic understanding can exist and still be successful in working with the concept. Finally, Luke ended his interview with sharing an informative comment about mathematical abstraction. He asserted, "I think abstraction is useful, extremely useful for, as a lens to view things through and to see general properties. But it always has to be balanced with examples… I don't see much value in abstraction just for abstraction's sake. There has to be a rich class of examples that it applies to." These "rich classes of examples" provide a way to support the formal and abstract components of one's concept image.

Potential conflict factors and mathematical problem solving

While working on the mathematical task, Luke did not complete the problem in the allotted time. The primary reason for this appeared to be that he conflated properties of direct limits and inverse (or projective) limits. Prior to the task, Luke remarked, "For the dual notion, the projective limit, I actually know more." Even though he demonstrated that his concept image for the direct limit was full of coherent examples and informal properties, it is likely that he is more familiar with using inverse limits in his own mathematical work. He later repeated this comment while working on the task. When starting the task, Luke began writing down some of the transition maps between modules in the directed system. Luke stated that he wanted to "use the universal mapping property", but he applied the universal mapping property of the inverse limit instead of the direct limit. He expressed concern that he may have the property backwards.

Here we can see that two different components of his concept image for direct limits were at conflict, which led to a conflict factor. Since his formal concept definition is at odds with another part of his concept image, this led to cognitive conflict when working through the problem. Luke eventually made a guess for the answer, despite using an incorrect definition. However, after working to verify his prediction, he concludes that his prediction cannot be correct. In particular, Luke constructs a homomorphism to help check his prediction, but he realizes that the map is not compatible with the original transition maps he wrote. Nonetheless, after the interview, Luke sent me an email later that day with the correct answer to the problem, providing further evidence that the formal definition was a conflict factor in this situation.

Discussion

The notion of concept image and definition have been utilized to a great extent to try to model mathematical thinking and understanding for secondary school students and undergraduates (e.g., Melhuish et al, 2020; Rösken & Rolka, 2007; Wawro, Sweeney, & Rabin, 2011). However, I was unable to find any instances where this framework was used to study graduate students or mathematicians. I utilized concept image and definition to try to model their thinking in the context of understanding how they confront abstraction. The primary goal of this study was to determine, for a specific abstract mathematical concept, how much of the concept image consists of visual components, and what other aspects are prevalent in that concept image. The analysis shows that mathematicians' concept images may be heavily comprised of visual components, examples, and informal descriptions. However, these visual and informal components are supplemented by relevant formal mathematical properties and definitions. This agrees with Lynch and Lockwood (2019), who state that mathematicians are well-accustomed to how examples fit into logical structures. I note that this study consisted of only one male mathematician with research focusing on abstract algebra, and the mathematical concepts that were analyzed were all algebraic structures. In order to further support these conclusions, it would be valuable to conduct similar studies interviewing more mathematicians, also with a more mathematically diverse group of individuals.

The analysis also demonstrates that Luke's main response to confronting mathematical abstraction and new mathematical problems is to work through examples. For example, when discussing direct limits, Luke mentions:

> Some people are just really good at seeing an abstract definition and getting it. I really think the examples have to be right there to sort of get a feeling for it… I would lean on examples to understand what the concept is about.

Thus, in order for Luke to develop a coherent concept image for an abstract mathematical concept, the concept image needs to also consist of relevant, supportive examples. This supports the claim by Vinner and Dreyfus (1989) that, "The student's image is a result of his or her experience with examples and nonexamples of the concept" (p. 356). A major implication of this analysis is that it highlights the value of integrating supportive examples and visual mathematics into one's concept image for an abstract mathematical concept. More specifically, it could be beneficial to guide students into this process of integrating examples and visual images when learning new mathematical concepts. Moreover, teachers should provide opportunities for students to not only see relevant examples, but also successfully integrate them into their concept images by understanding their value and relationship to the concept. Lynch and Lockwood (2019) came to a similar conclusion about example-related activity in proof, "An implication of this is that it may be beneficial to provide students with experiences of seeing and constructing powerful examples that give insight into a conjecture or a proof" (p. 337).

The task in this study was specifically chosen to represent a concept that was familiar to the participant, but he was uncomfortable working with it. This would provide opportunities to analyze how mathematicians think and work in contexts where they must actively struggle with the mathematical problem. Though he did not complete the task during the interview, the analysis illuminates the value of the formal concept definition in the overall concept image for an abstract mathematical concept. The non-task interview analysis demonstrated the importance of having well-chosen, supportive examples and visual images in one's concept image, but the task showed that the formal properties can be equally valuable in doing mathematics. Luke was able to provide well-chosen examples of direct limits in the interview, but this was not enough for him to overcome the conflict factor with the formal concept definition. In future studies, it could be valuable to see what mathematicians' evoked concept images are in the case where the task was completely solved. Furthermore, analyzing mathematicians' thinking through different mathematical tasks could provide further support for these conclusions.

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The Teaching of Proof-Based Mathematics Courses

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We survey and synthesize the literature from 2000 to the present about the teaching of proof-based undergraduate mathematics courses and find that the field has learned a great deal about what instruction looks like and why mathematicians teach the way that they do. We identified 112 papers that explored teaching via surveys, interviews, or observations. We report the findings in three broad categories; (1) what occurs in class, (2) the beliefs, knowledge, and rationales that shape instruction, and (3) the relationship between teaching and learning.

Keywords: Teaching, Lecture, Non-Lecture, Literature Review

In undergraduate mathematics education, there are a number of compelling reasons that researchers should explore mathematicians' beliefs, goals, and practices around teaching and learning. First, if we, as a research community, want to better explain the ways that student' mathematical behaviors, understandings, and beliefs, then it is useful to examine how they were taught. An understanding of mathematical instruction can provide us with a lens to analyze students' mathematical knowledge and behavior and provide explanatory accounts for how those behaviors arose. Second, mathematicians are successful learners of mathematics and have a large body of practical teaching experience and expertise, especially in terms of advanced undergraduate proof-based mathematics. Hence, studying their insights and practices can provide a rich source of insight into both students' reasoning and behavior in proof-based mathematics (Alcock, 2010). Third, as Alcock (2010) and Larsen (2017) noted, mathematicians have amassed a large body of expertise on many aspects of pedagogy, including how students reason mathematically and how students can be movtivated.

Based on their literature review published in 2010, Speer, Smith, and Horvarth observed that there was a noticeable gap in the undergraduate mathematics education literature. Although there was a large and growing body of literature on undergraduate students' understanding of mathematical concepts, there was little research on how they were taught. Without knowing how undergraduate mathematics were taught, we do not know why students develop the (often problematic) understandings that they do. Further, without knowing how mathematicians typically teach, mathematics educators may find it challenging to identify aspects of their instruction that are problematic or to suggest areas for improvement. In the last decade, the situation certainly has changed in the context of advanced mathematics courses. We have undertaken a literature review as a sequel to Speer, Smith, and Horvarth's (2010) influential review of research on university mathematicians' teaching practices. In their review, Speer et al. (2010) accurately documented the dearth of empirical research into mathematicians' teaching practices. Since the publication of this review, mathematics educators have answered Speer et al.'s call, so much so, that the field needs a clear picture of what we know and what questions continue to need additional research. Since the Speer et al. (2010) review, there has been substantial work exploring mathematicians' beliefs, goals, and practices in a variety of areas, especially in advanced undergraduate mathematics courses. This growth in research necessitates a sharper focus to the literature review. In the following study, we set the goal of synthesizing the different papers in the area of proof-based mathematics and summarizing what we, the field, know and outline future directions for research. That is, the purpose of this article is to synthesize the growing research literature on mathematicians' pedagogy when teaching advanced mathematics courses.

Methods

Corpus Search and Selection

We did an initial search of Ebsco Education, Ebsco ERIC, and Scopus for papers published since 2000 in 16 journals that focus on mathematics (and science) education. We also added papers (including conference proceedings) found in citations and books if we knew of them. We acknowledge that there might have been papers published in more general journals that we might have missed as a result. We searched on language in both the abstract and title using Boolean 'or' and 'and' to attempt to find all possible relevant papers. This initial search resulted in 774 potential papers. At that point, a member of the research team read the title, abstract, and, as needed, the methods section of the paper to determine whether papers are 'relevant' to our goals of describing the teaching of proof-based undergraduate mathematics courses. Meaning, from the pool of papers, we excluded those that were about undergraduate mathematics teaching, but not proof-based courses, were not about mathematics, or not related to what happens inside the classroom.. If a paper included exploration of both proof-based and other undergraduate courses we kept it in the corpus. This parsing left 112 studies that explore issues of teaching of proof-based undergraduate mathematics in the corpus.

Analysis Scheme

For each of the 112 papers in the remaining corpus, two members of the research team read each. We first parsed them as either empirical or not. For the empirical papers, we then noted where the study was carried out, the course content. We also coded a study an intervention if the authors of the paper attempted to directly influence the way that the class was taught, either by having a member of the research team teach the course or by working with the instructor to change his or her instruction. A study was coded as observational if the research team merely observed some aspect of the instruction or the beliefs of the instructor, without trying to change his or her teaching. We also drew on the study's characterization of the classrooms as either lecture or 'not-lecture.' For interview studies where this was not made explicit, we assumed lecture based on the relative frequency of lecture as found in large-scale surveys (Johnson, et al, 2019). We described the means by which the data was collected (e.g., interview, observation, survey) and number of participants. Where both professors and students were participants, we noted each and further distinguished between professors and Graduate Teaching Assistants. Finally, we described the main claims of each article and how those claims were warranted. Because we wanted to focus on instruction, we developed three broad categories of reporting for each of lecture and non-lecture pedagogies and synthesized results within those categories. Those categories are (1) what occurs in class, (2) the beliefs, knowledge, and rationales that shape instruction, and (3) the relationship between instructor actions and student learning.

Results

Lecture

Most advanced mathematics courses continue to be taught by lecture and practices share many commonalities. The evidence for this claim comes from two sources. First, Johnson, Keller, and Fukawa-Connelly (2018) and Johnson, Keller, Peterson, and Fukawa-Connelly (2019) administered surveys to 219 algebraists from the United States who had recently taught an abstract algebra course, where the participants were solicited from a range of universities, including Ph.D granting institutions to institutions only offering a bachelors degree. Among those at research universities, Johnson et al (2018) found that 85 percent of their respondents claimed that they taught via lecture. When considering all universities, they found that 83% of the respondents reported using lecture at least one quarter of the time. These findings are largely consistent with mathematics educators who have observed mathematics teaching. Artemeva and Fox (2011) observed 50 mathematicians teaching both calculus courses and advanced mathematics courses in seven different countries. Fukawa-Connelly, Weber, and Mejía-Ramos (2017) summarized the teaching practices of 11 mathematicians teaching a variety of advanced mathematics courses in the United States.

Commonalities in lecture practice. There are significant commonalities across countries and universities within the United States. Artemeva and Fox (2011) observed 50 mathematics teachers teaching university mathematics courses in seven countries, at the calculus and advanced mathematics level. What struck these scholars were the commonalities across all the lessons they observed. Artameva and Fox found that these teachers collectively engaged in a genre of teaching that they called “chalk talk”. Chalk talk consists of lecturing where (a) the lecturer writes mathematical results as inscriptions on the blackboard; (b) the lecturer would concurrently provides a running commentary in which she verbalizes what was being written and her thought processes as she was doing this mathematics (i.e., modeling the “doing” of the mathematics that was written); and (c) the lecturer occasionally stops to offer meta-commentary, reflecting more globally on what was written. They also claimed that questioning is a very common practice among lecturers.

By analyzing 11 individual lecturers in different mathematical classes, Fukawa-Connelly et al. (2017) corroborated Artameva and Fox’s claim that mathematics courses were taught in the genre of “chalk talk” lectures. Fukawa-Connelly et al. further documented that mathematicians made seven mathematical contributions. Fukawa-Connelly et al found the following contributions: definitions, theorems, and proofs (which the authors labeled as “formal mathematics”) and methods/heuristics, informal representations, modeling mathematical behavior, and examples (which the authors called “informal mathematics”). Their main findings were the following: There were many informal mathematical contributions in the mathematics lectures. The formal mathematical contributions were nearly always written on the blackboard. With the exception of examples, the informal mathematics was only said orally and rarely written down.

Why do mathematicians continue to lecture? It is natural then to wonder why mathematicians continue to lecture, especially considering mathematics educators’ frequent contention that lecturing in advanced mathematics leads to poor learning outcomes (e.g. Leron & Dubinsky, 1995). The answer to this question is surely multifaceted (Fukawa-Connelly et al., 2016), but the simplest answer is that mathematicians teach via lecture because they believe lecture is the best way to teach. This finding is warranted by the writings of professional mathematicians (Krantz, 2015; Pritchard, 2010, 2016; Wu, 1999), surveys with mathematicians (Fukawa-Connelly et al., 2016), and interviews with mathematicians (Woods & Weber, 2020).

The practice of lecture varies and the variation can generally be explained by differences in professors' beliefs, goals, and rationales. Pinto (2019) described two mathematicians, Yoav and Amit, implementing the same lesson plan in a real analysis course. The formal content of the lessons was the same: both mathematicians introduced the formal definition of the derivative, stated a theorem about derivatives (differentiability implies continuity), and showed several examples and applications. However, Pinto argued that their lectures were markedly different. For instance, in elaborating on the formal definition of derivative, Yoav looked at each term in the definition of the derivative and then offered a precise definition of each term. In contrast, Amit used visual and metaphorical imagery to give an informal account of what the derivative meant, describing the derivative in terms of "cones", "swings", and "traps". Yoav and Amit further presented different narratives of how students should make sense of definitions when they encounter them.

Weber (2004) described similar variance in real analysis lectures. Weber presented a case study of a single mathematician, Dr. T, and illustrated how Dr. T's lecturing style changed over the course of the semester. A theme across both studies is that mathematicians may emphasize different types of "informal content" (in the sense of Fukawa-Connelly et al., 2017) in their lectures, which lead to different lecturing styles, even when the formal content remains the same (Pinto, 2019). The key point that we wish to draw from Pinto's and Weber's studies is that mathematicians' different beliefs about how mathematics is done and what students need to do, and their related pedagogical goals, can exert a strong influence on their pedagogical actions. In short, Mathematicians' beliefs and goals can explain variance in their pedagogy. This depth of thought about teaching is also illustrated and warranted by numerous interview studies.

Researchers have also observed substantial variation in how mathematicians engaged students in mathematical activity during their lectures. In their survey, Johnson et al. (2019) found that many mathematicians who lectured frequently professed to use some non-lecture-based student-centered activities in their classes some of the time, such as having students work collaboratively to solve a problem. Dawkins (2012, 2014) and Fukawa-Connelly (2012a) presented case studies of highly interactive classrooms along with analyses of the classroom norms that encouraged student participation. Alcock (2018) described techniques that she used in her own lectures to engage students, such as asking them to explain the meanings of concepts to a peer. The key point here is that while most lectures in advanced mathematics are largely teacher-centered where the students are passive, there are some lectures that seem to be at least somewhat interactive. The existence of lectures in which students are somewhat active is important. Alcock (2018), Braun et al. (2017), and Dawkins and Weber (in press) argued that mathematicians need not radically overhaul their teaching to improve it. Finding ways to *improve* lectures rather than *replace* them is an avenue for future research in undergraduate mathematics education.

There is little research on how lecturers' actions influences students' learning. The extant research illustrates one mechanism by which students may fail to learn from lectures. Students may fail to recognize the purpose of a lecture, which can lead them to ignore aspects of a lecture that are considered most important (e.g., Lew et al., 2016) or to misinterpret what the professor is asserting (Krupnik et al. 2018). However, while the preceding research illustrates how this may occur, the fact that there were only two studies with a limited number of participants naturally leads us to question the generalizability and robustness of these findings.

Non-Lecture

In some ways, the literature on the teaching of more student-centered or inquiry classes has quite a different focus with the research questions, methods, and claims about instruction in lecture classes. Notably, this literature base is dominated by intervention studies and also includes a few small scale case studies of exceptional teaching. This focus makes sense as lecture is the primary mode of instruction and thus more active classrooms can be seen as counter to the norm and often occurring via intentional change. Out of 39 papers that reported on interventions to teaching, 23 described any of the work of teaching rather than focusing on task sequences or student outcomes. Those 23 papers studied 125 total classrooms (some lecture for comparison) or professors with 46 classrooms actually observed. Eleven of those papers studied a single classroom and only four studies included more than 10 classes or professors. We also note there were studies (e.g., Laursen, et al., 2014) covered a broad array of undergraduate mathematics classes (including, but not limited, to proof-based mathematics), but did not distinguish between the courses and thus are not included in these counts. In this section, we broadly attend to the complement of the prior section: non-lecture courses broadly defined those that include students being actively involved in generating disciplinary ideas.

The Literature on Active Learning is Dominated by the "Inquiry" Paradigms. As recently discussed in Laursen and Rasmussen (2019), much of the reform in advanced undergraduate courses stem from either Inquiry Based Learning or Inquiry Oriented Instruction. In Laursen and Rasmussen's theoretical paper they sought to operationalize such instruction as hinging on common underlying principles related to "student engagement in meaningful mathematics, student collaboration for sensemaking, instructor inquiry into student thinking, and equitable instructional practice to include all in rigorous mathematical learning and mathematical identity-building" (p. 129). Further, Kuster et al. (2019) and Hayward et al. (2016) have developed and use observational measures that incorporate specific rubrics that operationalize and serve to measure inquiry-oriented and inquiry-based teaching practice, respectively.

In contrast to the lecture literature base, we do not know much about how non-lecture unfolds in classrooms that are not associated with a researcher intervention. In the literature, there are only a handful of case studies about such instruction (e.g., Dawkins, et al., 2019; Fukawa-Connelly, 2012b; Ticknor, 2012). While the Johnson, et al. (2018) survey discussed above points to 85% of abstract algebra instructors self-report lecturing, that leaves 15% of instructors teaching in more active manner. We argue that there is still a substantial amount to learn about non-lecture instruction observationally. The Johnson, et al. survey found that self-identified non-lecturers were not using the researcher-designed materials that have been the primary focus of research in this area point to a mismatch between actual practice most research.

Non-lecture instructional practice is varied, even by different teachers using the same curriculum. This variation can generally be explained by differences in beliefs, goals, and knowledge. Much like the case of lecture above, student-centered instruction is similarly not a monolith, and instructors' beliefs and knowledge shape their instruction. Two large-scale studies point to this Kuster et al.'s (2019) study of instructors implementing inquiry-oriented curricula and Laursen et al.'s (2014) study on inquiry-based instruction. In one sense, we can see commonalities across classes including more time that is student-centered, less instructor authority, and more formative feedback than in lecture courses. On the other end, the reported standard-deviations and range of rubric scores point to differences. For example, in Kuster et al.'s (2019) analysis of 13 instructors implementing the same inquiry-oriented abstract algebra lesson, the degree to which teachers engage students in each other's reasoning which ranged from a score of 1.5 to a score of 4.5 (on a 5-point) scale. Thus, even within a relatively constrained setting, teaching practice can have substantial range.

Differences in practice has been tied to several sources including: tensions in managing coverage concerns; beliefs and values about mathematical practice; and challenges involved in incorporating students' thinking in genuine and productive ways. In some ways, teaching practice becomes more complex in classrooms where students are generating ideas -- ideas that may be unexpected. For example, Johnson and Larsen. (2013) illustrated ways that a mathematician implementing inquiry-oriented abstract algebra can be supported by common content knowledge, but may be constrained by a lack of pedagogical content when listening to interpret students and progress the lesson based on their ideas. Andrews-Larson et al. (2019) found that inquiry oriented linear algebra instructors increased their pedagogical knowledge and are better able use student ideas with support and repeated use of the curriculum.

Mesa et al. (2019) and Johnson et al. (2013) provided insight into instructors' views and beliefs when implementing mandated inquiry-based learning linear algebra and a voluntary inquiry-oriented abstract algebra, respectively. In both cases, there was discussion of the tension between coverage and active learning. This is an idea echoed in surveys of why instructors do not lecture (Johnson, et al., 2017) and amongst those who design interventions such as Nwabueze (2004) who suggested an interactive approach to abstract algebra would not allow for covering the full material. Johnson et al.'s study of three mathematicians implementing inquiry-oriented curriculum similarly pointed to general pacing tensions, but like in Mesa's study the instructors' individual beliefs shaped their concern and management of such a tension. One mathematician valued content goals as the primary outcome of the course and therefore was most concerned with coverage; one mathematician saw disciplinary activity as the primary goal of the course and was not concerned about coverage; while the third occupied a middle ground emphasizing the importance of "student opportunity to interact with math on their own terms" (p. 751) in order for students to learn. The hierarchy of goals and values is consistent with other studies of instructors who select an inquiry approach, such as in Dawkins et al. (2019) where the instructor stated their goals for an inquiry-based real analysis course was focused on student creation of mathematics and overcoming challenges. In general, beliefs and goals mediate instruction; however, these studies are few and are situated in particular intervention settings.

There are cautiously positive results related to student learning and non-lecture interventions. Unlike the general results about lecture, non-lecture interventions have generally pointed towards positive (or in some cases mixed) results linking teacher practice and student learning. There are two large-scale studies that warrant this result: Laursen et al. (2014) and Johnson, et al. (2021). Laursen et al. found that students in inquiry-based learning classes (some proof-based) reported greater learning gains than those in lecture. They also found that IBL courses seemed to have a positive impact on womens' confidence and interest. Johnson, et al. provided a more mixed result finding that students in inquiry-oriented abstract algebra courses scored just as well on a content assessment, but it was only the men in these classes that saw estimated test scores higher than lecture sections. Smaller intervention studies have found positive links such as between modeling reform teaching and pre-service teachers' view on the nature of discourse in mathematics classes (Blanton, 2002), and questioning and discussion practices supporting students in more authentic activity norms and taking responsibility for knowledge (Fukawa-Connelly, 2005). Further, a great deal of literature illustrates productive ways that students engage in proof activity within intervention contexts; although the details of instructor practice is often left underspecified (e.g., Larsen & Zandieh, 2008; Yee, et al., 2018).

Discussion

Based on the 24 published papers, we have a strong understanding of what goes on in lectures in advanced mathematics. With regard to lecturing, we have good evidence that most mathematicians teach by lecture and they do so because they believe that it is the best way to teach. We also know that most lectures consist of the mathematician writing the official mathematics on the blackboard, while offering a running commentary about the informal meaning of the mathematics and metacommentary about the nature of doing mathematics orally. While mathematics lectures show some uniformity, there is also variance in what informal mathematics that the mathematicians choose to emphasize and how they use student-centered activities (if they use them at all). Finally, we know that mathematicians can offer sophisticated rationales for their instructional choices. However, much of the research examining other phenomena tends to involve conducting case studies or exploratory analysis, with the output being "frameworks." Unfortunately, these frameworks have not been applied in future research. One consequence of research into how mathematicians lecture is that the most basic suggestions for how to improve instruction—such as not being overly formal, offering explanatory proofs, and explaining the reasoning and motivation behind definitions, theorems, and proofs—is not likely to be helpful. Mathematicians already do these things. What we know little about is how the instructional actions the professors' take influence student learning and why the apparently sophisticated and nuanced thought that goes into lecturing has not produced more promising learning outcomes. Indeed, we do not yet have good evidence that changes to lecturing influences learning at all. More research in this area is needed.

The literature on non-lecture includes 39 papers, 23 of which included any study of classroom activities while 16 focused on task-sequences and/or student outcomes. Most of the papers about non-lecture report on mathematics-educator developed curricula in the RME or IO traditions. This is, even though the data suggests that the majority of non-lecture classes rely on instructor-developed curricula. With interventionalist studies, we have obained a rich understanding of how task sequences and classroom environments affects and improves students' practices and understandings. We are still developing our understanding of productive teaching practices in these environments, as well as the challenges that instructors face when using student-centered curricula. More research in this area is needed.

An understudied area is what mathematicians who do not lecture choose to do in their classrooms. There are many studies of how mathematicians or mathematics education researchers behave when teaching with inquiry-oriented curricula developed by mathematics educators. However, the mathematicians who use such material is scant (cite your survey showing that this is so). There are far more mathematicians who do not use lecture, but also are not influenced by mathematics education research (cite your survey showing this). This is a sizeable population that has received little attention, but could produce interesting findings about how mathematicians change their practice and what they are willing to do.

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Characterizing Undergraduate students' Problem Posing Products and Processes in an Introductory Business Mathematics Course

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This study investigated the nature of students' problem posing products and processes and the potential for using these products and processes to assess students' level of learning. First, the kinds of modifications made to existing problems are examined followed by the impact of these modifications on the students' solution strategies. I argue that by examining the students' solution strategies alongside the strategies that they encounter in class, we can gain insights into the students' understanding of the concepts learned. Furthermore, posed problems that are unsolvable but consistent and coherent can be used as a launchpad into new topics and ideas in mathematics.

Keywords: Problem Posing, near transfer, far transfer, Coherence, Consistence

Although many researchers have consistently documented the benefits of engaging students in mathematical problem posing (PP), many students, especially at college level, experience PP less often than they do problem solving (Lavy, 2015, Silver et al., 1996). This is despite some researchers (e.g., Mestre, 2002) asserting that PP could be cognitively more engaging than problem solving (PS). Much research into why PP has not been adopted as widely as PS agree that the main reason behind this is the limited knowledge about students' cognitive processes when they are engaged in PP (Cai et al., 2015). Specifically, there is need for the field to understand how undergraduate students pose mathematical problems and if PP is a skill that can be taught and learned (Crespo & Sinclair, 2008). Furthermore, the question of how PP can be used to assess students' learning in mathematics is still less explored. In calling for more research into PP, Cai and Huang (2013) noted that:

> Little is known about the cognitive processes involved when students generate their own problems and therefore about the ways problem posing can be used as an assessment tool. Furthermore, little research has been done to identify instructional strategies that can effectively promote productive problem posing or even to determine whether engaging students in problem-posing activities is an effective pedagogical strategy. (p. 58).

To understand the cognitive processes involved when students engage in PP, it is necessary to first understand the nature of the student-posed problems (SPPs) in relation to the problems that they meet as part of their learning (class problems). This consideration is important since much of PP literature (e.g., Cai & Cifarelli, 2005; English, 1998) suggests that problem modification is one of the ways through which students pose problems. Thus, an examination of the aspects of class problems (CPs) that students modify and how they do so is important in understanding the students' cognitive processes in PP. Furthermore, to be able to use PP in assessment of learning, an important consideration would be whether or not students recognize the consequences of their modifications on the solution processes. To address these issues, this study sought to answer the following questions:

1. What kinds of modifications do undergraduate students make on class problems in order to generate their own problems?

2. How can these modifications and their consequences on solutions be used to gain insights into the students' level of learning?

Background and Theoretical Framework

Role of Problem Posing in students' mathematics education

There is consensus among most studies that PP is an important skill that should be included in the mathematics curriculum at all levels of schooling (Bonotto, 2013). Some of the benefits of PP that these studies have documented include better problem solving skills (Cai & Hwang, 2003), higher creativity in mathematics (Voica & Singer, 2013), and increased motivation towards learning mathematics (Akay & Boz, 2010). Despite these benefits, however, PP has not been made a prominent part in mathematics curricular (Lu & Wang, 2006).

Defining Problem Posing

Problem posing has been conceptualized in various ways in the mathematics education literature. For example, Silver (1997) pointed out that PP and PS are naturally intertwined, and that PP occurs at various phases when someone is engaged in PS. For example, *before-solution* PP happens when one rephrases or breaks down a problem in order to understand what is required while *during-solution* PP serves the purpose of checking one's progress in PS as a way to monitor their PS process. On the other hand, *post-solution* PP happens after the PS process is complete and serves the purpose of reflecting on one's solution process to check if the final answer(s) makes sense. Silver's (1997) view of PP implies that PP happens spontaneously as long one is involved in PS.

Stoyanova and Ellerton (1996) on the other hand view PP in terms of the conditions under which it occurs. They offer three categories of PP including *free*, *structured*, or *semi structured*. In *free* PP someone is asked to create a problem without being given any specific conditions or a starting problem. One can write any problem drawn from any content in mathematics. In *structured* PP, one is given a starting problem and is asked to make modifications to come up with a new problem. *Semi-structured* PP, on the other hand, provides a limited set of conditions such as a topic or context that must be adhered to.

Brown and Walter (1990b) suggested a PP strategy known as the "What-If-Not?" strategy. In this strategy, one comes with a new problem by altering the givens of an existing problem. This strategy is similar to Stoyanova and Ellerton's (1996) structured PP but focuses mainly on the constraints in a problem. I argue that there are other modifications besides changing the constraints of a given problem that students may use. For example, a student may add new parameters to a problem that could create a need for a change of other aspects of the problem. Thus, the current study is based on Ellerton and Stoyanova's (1996) notion of semi structured PP and an expanded form of Brown and Walter's "what if not" strategy.

Analytic Framework

Voica and Silver (2013) developed a framework for analyzing mathematical problems in terms of their textual elements and the relationships between the elements. According to the framework, a mathematical problem includes some or all of the following elements: a *background theme, parameters,* data, *operating schemes,* and *constraints*. As an illustration of these elements, consider the following problem posed by one of the participants called Zive (all names are pseudonyms).

Marco can get a $ 50, 000 loan from one of two banks. The first charges 7%(12) and the second charges 7.05%(2). If he thinks he will pay off the loan in five years, which bank offers a better deal?

The *background theme* of a problem is what the problem is about or the context in which the problem is based. Zive's problem above is about an individual (Marco) making financial decisions. To provide more detail to the background theme, a mathematical problem needs parameters. In the case of Zive's problem, the *parameters* are bank interest rates, loan periods, and the loan limit. For the problem to make sense, these parameters need to be associated with *data*. The data in the problem include 7.05% (2), 7%(12), and $ 50, 000. *Operating schemes* are the actions suggested in the problem such as a comparison of the rates for the two banks. Finally, *constraints* are ways to preserve certain relationships in the problem structure by imposing restrictions on the other elements. An example of a constraint in the above problem is that Marco can get the loan form only one of two banks. The interest rates and the associates compounding periods can also be interpreted as constraints on the data. Furthermore, the loan amount cannot exceed $50,000. If someone modifies the data by changing the $50 000 value to $55 000, the resulting problem is considered a new problem even though the overall structure of the solution could remain unaltered. Various modifications to existing problems have various impacts. For the purposes of this study, modification is not limited to changing the data, rather it can include extending the problem by adding data or even adding more elements not in the starting problem. If one adds a third bank, for example, that charges simple interest, that is an extension of the problem. A problem that provides its elements in a correlated manner with no ambiguities is said to be *coherent* whether it is solvable or not. On the other hand, a problem for which a certain mathematical model can be assumed, has at least one solution (or a proof that there is none) and for which the data are not contradictory is said to be *consistent*. Zive's problem above is an example of a *coherent* and *consistent* problem. It is possible for a problem to be coherent but inconsistent and vice versa.

In this study, I used Voica and Silver's framework to characterize the SPPs and compared them with CPs to determine the kinds of modifications that students made. If the modifications made to a problem significantly impacted the structure of the starting CPs, then, I called such problems *far transfer* problems. Similarly, modifications that resulted in little impact on the overall structure of the problem (e.g., changing the data values only) were labelled *near transfer* problems (Pelca & Voica, 2011). To gain deeper insights into students' understanding of the learned material, I compared the solutions that the students had encountered in class with the solutions that they presented in their SPPs. If a student was able to adapt a class solution in such a way that they were able to use it on a far transfer problem, then, that served as an indicator of deeper learning. I made a similar determination whenever a student used a novel strategy (not used in class) on any *near transfer* problems.

Method

Participants and Setting

I gathered data from 16 undergraduate students enrolled in a mathematics course on financial mathematics during the fall semester of 2020. There were three chapters covered in the course namely, simple interest, compound interest, and annuities. The students completed a comprehensive problem posing assignment (PPA) towards the end of the semester that required them to pose at least one problem from each of the chapters covered. In this PPA, students were asked to state the starting CP that they used to pose their own problems, explain their posing processes and then solve the problems. Students' written artifacts were the main data sources for this study. There were also one on one meetings with some of the students via zoom during which they clarified their work and explained their solutions.

The main data sources for the study included written artifacts in form of students' posed problems alongside explanations and solutions. The PPA was graded based on three-point criteria involving creativity, solvability, and complexity. This rubric was made clear to the students. They were free to use any CPs as starting points as long as they indicated so as part of their PP process explanation. Problems that greatly resembled class problems scored low on creativity than those that did not. Similarly, problems that were linguistically well articulated and provided clear context scored high on complexity. Problems that required use of more operations or concepts from multiple sections also scored high on complexity. One received maximum credit on solvability if their problem had at least one solution and an attempt at finding such solutions was made.

Analytic Procedures

The data analysis was conducted in three phases. In the first phase, my colleagues and I independently coded the SPPs using codes derived from Voica and Singer's (2013) framework and determined the relationships between the elements. Following this, we classified each SPP into one of four categories namely category A (coherent and consistent), category B (coherent but inconsistent), category C (consistent but incoherent), and category D (incoherent and inconsistent). We later met to compare our coding/classification and the inter-rater agreement for this part turned out to be 96%.

In the second phase of analysis, we compared the SPPs with the corresponding CPs to determine if they were *far* or *near transfer* problems. To do this, we examined the modifications made to various elements of the CPs and the impact of such modifications on the solutions to the CPs. Modifications to a CP were regarded significant if they called for a different a solution strategy or a modification of the solutions presented in the CP. Significant modifications resulted in far transfer problems while insignificant ones resulted in near transfer problems. In some cases, students used completely different strategies from the ones presented in CPs. In such cases, the problems were considered far transfer even if they only reflected minor modifications on CPs. This is because the students viewed them in different perspectives.

Results

First, I present the findings related to the first research question, addressing the kinds of modifications that students made on CPs during their PP processes. To provide context on how the findings were generated, we provide an illustration of elements in one CP (Figure 1) and how a student made modifications on the CP to generate their own SPP (Figure 2).

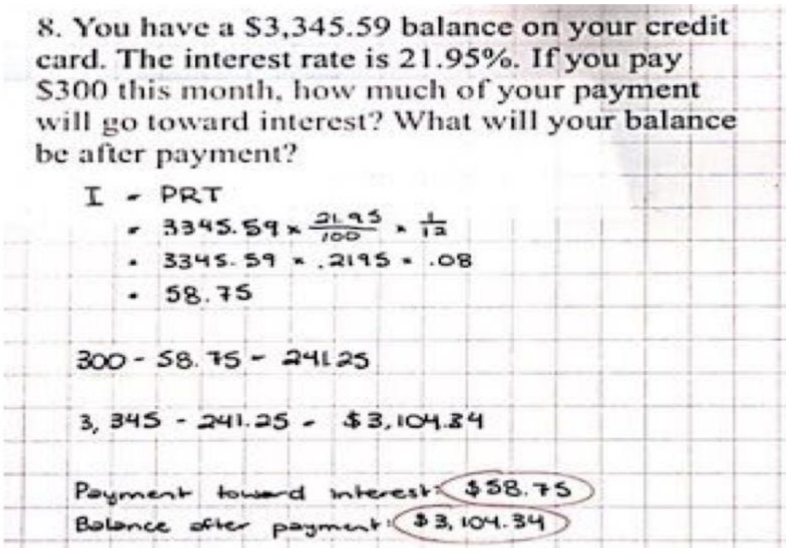

Figure 1. Sample CP from chapter 1

This CP is situated in the context of credit cards (background theme) with two main parameters namely credit card balance and interest rate. The corresponding data for these parameters are $3,345.59 and 21.95% respectively. The $300 repayment amount is a constraint that sets up the problem question (the new balance). Based on this CP, Cegra posed and solved the problem in Figure 2.

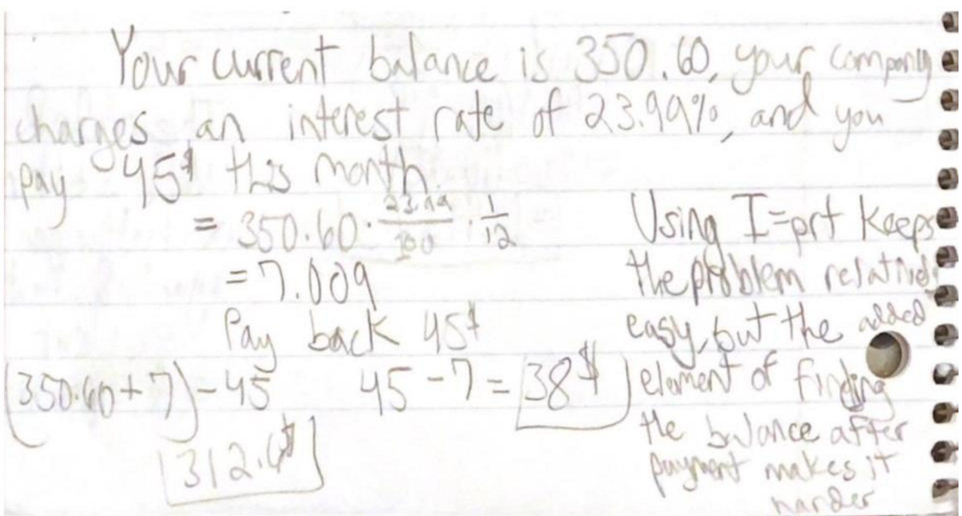

Figure 2. Cegra's problem from chapter 1

An examination of Cegra's problem shows that it keeps most elements and the overall structure of the CP (Figure 1) on which it was based. The background theme, the parameters, and the operating schemes (finding the new balance) are all unchanged. Her modifications only targeted the data (i.e., she changed $3,345.59 to $ 350.6 and 21.95% to 23.99%). Although Cegra's problem is a lot similar to the CP, we still found it to be both consistent and coherent (category A) since all elements are well correlated and the problem has a solution. All SPPs were classified in an analogous manner. The results are presented in Table 1.

Table 1. Classification of SPPs by Modified Elements and whether or not they were Category A

Modified element in a CP	# of problems (n=46)	# of Category A problems (n=30)
Background theme only	2	3
Data only	15	13
Constraints only	8	6
Operating schemes only	3	1
Multiple elements	18	7

Table 1 shows that most students (39.1%) made modifications to multiple elements in the CPs. Of the 18 SPPs that had multi-element modifications, however, only 7 (38.9%) were category A problems. The next most represented group were problems that had modifications on the data only (32.6%). These problems involved changing certain numbers but keeping everything else about the problem. Interestingly, most of these problems (86.7%) turned out to be category A problems.

Regarding the second research question, we found that a majority of category A SPPs (80%) were near transfer problems. The accompanying solutions to these near transfer problems were also strikingly similar to the CP solutions that the students had encountered. Of the 7 category A *far transfer* problems, 5 (71.4%) came from problems that had multi-element modifications. Table 2 provides a summary of the problem transfer alongside the modifications that the students made on CPs.

Table 2. Problem Transfer vs Modified Element

	Modified Element				
Problem Transfer	Background theme only	Data Only	Constraints only	Operating Schemes only	Multiple elements
Far	0	1	1	0	5
Near	3	12	6	1	2

Although we expected far transfer problems to have solutions that were different from ones in CPs, we found a couple of cases where students attempted to use the strategies used in CPs without any modifications. We concluded that such students did not have a deeper understanding of the concepts learned. Nevertheless, there were a few students who posed far transfer problems and modified the CP strategies in a manner that reflected a deeper understanding of the concepts learned. Rano's problem (see Figure 3), for example, is based on the earlier CP (see Figure 1). Her modifications involved expanding the background theme to include two loan scenarios and adding more parameters/data. These modifications also made it necessary that she change the operating schemes in the problems. An examination of her solution strategy to this problem shows awareness of the impact of such modifications. This awareness resulted in relevant modification on the solution strategy presented in the CP. For example, she used algebraic manipulations involving an unknown value (A) representing the amount owed on the car and used it to solve the problem. It should be noted that there was no CP of this kind or that required the solution strategy implemented by Rano. Therefore, Rano's problem can be considered a *far transfer* problem and her solution as evidence of deeper learning.

You owe a total credit of 13,900 on your car and on your boat from different companies charging you 14% pa and 11% respectively. If the total amount of interest charged this month is \$146.16, how much do you owe on your boat?

Let A = amount owed on car

$\Rightarrow$ 13900 − A = amount owed on boat.

$$\text{Int on car} = A \times \frac{14}{100} \times \frac{1}{12} = 0.01166\,A$$

$$\text{Int. on boat} = (13900 - A) \times \frac{11}{100} \times \frac{1}{12} = 127.42 - 0.0092A$$

$$\text{Total interest} = 0.01166A + 127.42 - 0.092A = 146.16$$

$$0.00246A = 146.16 - 127.42$$

$$\frac{0.00246A}{0.00246} = \frac{18.74}{0.00246}$$

$$\boxed{A = \$7{,}712}$$

$$\text{Amount owed on boat} = 13900 - 7{,}712 = \boxed{\$6188}$$

Figure 3. Rano's Problem from Chapter 1

Discussion

The findings of this study show that when students are asked to pose their own problems by modifying existing ones, they tend to focus more on multiple elements as opposed to single elements. When doing this, however, they tend to end up with inconsistent and/or incoherent problems. Nevertheless, these problems provide students with opportunities to use existing strategies in new ways and hence a possibility for new knowledge. If the goal is simply to pose more consistent and coherent problems, the findings of this study suggest that modifying single elements (especially data) is the most successful strategy. It should be noted that doing this also results in solutions that are highly related to the already encountered solutions and hence less room for creativity. The findings of this study also suggest that if we are to understand the depth of student learning in a certain topic, far transfer problems are more superior. This is because they challenge the students to modify already learned strategies so as to use them in novel settings.

Finally, I note that even when students are unable to solve their far transfer problems, they still gain from attempting such solutions. Instructors can leverage the difficulties arising from such situations to help students delve into new topics if the posed problems require so.

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Connecting STEM Retention to Student Affect in Pathways Mathematics Courses

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We investigate how particular dimensions of student affect (e.g., sense of community, mathematical identity) relate to student retention in STEM in the context of active learning college algebra and precalculus classrooms. We developed and implemented a survey with Likert-style questions informed by four known dimensions of student affect and used exploratory factor analysis to identify five distinct factors actually being measured in the survey. These include student perceptions of their instructor's actions, mathematical identity, confidence, and feelings of belonging. We report on statistically and practically significant relationships between student affect and two indicators of continuation in STEM: (a) their expected grade in the course; and (b) their status as a STEM major. These results contribute to our growing understanding of how student affect may impact STEM retention.

Keywords: active learning, student affect, STEM retention

Student retention has long been a focus of STEM education research, particularly within the context of introductory mathematics courses at universities in the United States. Much attention has been focused on single-variable calculus, but courses which prepare students to take calculus (e.g., college algebra, precalculus) are also high enrollment gateway courses which many students must pass through on their way to a STEM major. However, students starting in precalculus at the university level are less likely to complete a STEM major than students who start in calculus (Bowen et al., 2019; Krause et al., 2015; Wilkins et al., 2021; Van Dyken & Benson, 2019). Some proposed reasons for precalculus courses reducing student retention in STEM include poor quality of instruction (Olson & Riordan, 2012; Seymour & Hewitt, 1997) and departments not allowing students to take introductory engineering courses without precalculus (Van Dyken et al., 2015). We posit that another reason is student affect in their introductory mathematics courses.

In this paper, we define student affect in the general psychological sense, as any expression of the student's feelings or emotions. We use exploratory factor analysis to examine dimensions of student affect relative to students' ability to complete a STEM degree (i.e., they have to pass STEM classes and be a STEM major) to answer the following question: How does student affect (e.g., sense of community, mathematical identity) relate to students' major and expected grade in online college algebra and precalculus courses taught using active learning principles?

Literature Review and Framing

Our framework is comprised of the four dimensions of student affect we intended to measure when we initially designed our survey. We define each of the four dimensions to convey what we had in mind when we decided which items aligned with a particular dimension.

Four Affective Dimensions

The research team chose four affective dimensions - mathematical caring, sense of community, sense of inclusion, and mathematical identity - to investigate student affect in relation to students' major and expected grade in active learning Pathways college algebra and precalculus courses. Note that these items are not wholly independent of one another. For instance, students' sense of inclusion in the classroom may impact their perception of the mathematical caring relationship between themselves and their instructor, or the development of their mathematical identity.

Mathematical caring. Hackenberg (2010) defined mathematical caring relations (MCR) as "a quality of interaction between a student and a teacher that conjoins affective and cognitive realms in the process of aiming for mathematical learning" (p. 237). Hackenberg argued that if students are continually asked questions they are not prepared to answer, they may leave the classroom with negative perceptions of their experiences which may ultimately impede the learning process. As such, MCR provide an important connection between cognitive and affective impacts on students' perception of their instructor's actions within the classroom.

Sense of community. We operationalize students' sense of community using Strike's (2004) four C's of community: coherence, cohesion, care, and contact. Strike argued that schools must have a shared vision, care about initiating students into this vision, and have structural features that facilitate this care in order to establish a sense of community. A key aspect of community to LeChausseur (2014) is social space, which includes "people and their relationships; assets and liabilities; rules, regulatory bodies, and punishments; opportunity structures; the diffusion of power; and connections to other communities" (p. 307). Not only must school administrators and teachers construct a shared vision and give students the opportunity to align with that vision, but the students must *choose* to align with these structures while forming relationships with one another. Taken together, Strike and LeChasseur's definitions of community account for both the structures and the interpersonal relations needed to develop a sense of community.

Sense of inclusion. Roos (2019) conducted an extensive literature review to investigate how the notion of inclusion was leveraged in mathematics education studies on both a societal level and a classroom level. Within the classroom level, some studies use inclusion to refer to participation in the classroom and some refer to inclusion in relation to exclusion. We use inclusion to refer to students' sense of being included in the classroom as it is related to their participation: participating in the classroom discourse and contributing to group work.

Mathematical identity. Cribbs et al. (2015) defined mathematical identity as "how students see themselves in relation to mathematics based on their perceptions and navigations of everyday experiences with mathematics" (p. 1049). In this paper, Cribbs et al. found that students' perceptions of themselves (in terms of interest in mathematics, recognition of mathematical ability, and competence/performance with mathematics) had both direct and indirect impacts on students' mathematical identity. The link between students' mathematical identity and their self-perceived performance in their math class informs our investigation into mathematical identity.

Methodology

Curricular Context

The students in our study were enrolled in courses taught using the *Pathways* precalculus curriculum. This research-based and conceptually-driven curriculum has been implemented and refined over the past few decades at multiple colleges and universities. The pedagogical goals of Pathways have been to support students' meaning making (Clark et al., 2008) and construction of conceptual understandings for constant rate of change, exponential growth, and trigonometric functions (Carlson et al., 2015; Kuper & Carlson, 2020; Moore, 2014). The conceptual emphasis of the course is highlighted by structured task sequences designed for group work. At the university where these data were collected, instructors are encouraged to attend a weekly teaching seminar focused on the mathematical and pedagogical goals of Pathways. Since student thinking is the emphasis of the pedagogical approach of Pathways, most Pathways courses tend to de-emphasize lectures. Many of the modules and investigations are accompanied with dynamic applets and instructional videos to support students' repeated reasoning about the relevant course topics. In such a class, student-student interaction and discussions of their own mathematical thinking may impact student affect differently than in a didactic lecture setting.

Participants and Data Collection

The survey was conducted at a large research university in the U.S. toward the end of the Fall 2020 semester. The participants of the surveys were enrolled in College Algebra (CA) and Precalculus (PC) classes. Both courses were taught using the research-based Pathways Precalculus curriculum, though CA introduces fewer topics. The survey was distributed and completed electronically using the online survey platform Qualtrics (2021). Instructors distributed the survey to their students, and many instructors offered extra credit to their students for taking the survey, in accordance with approved procedures from the university's institutional review board. A total of 193 students agreed to participate in the study and responded to the survey questions. Data was collected from two sections of CA taught by the same instructor (n=106), and seven sections of PC taught by seven different instructors (n=87). These data were collected during the COVID-19 pandemic, and all the classes used either online or hybrid (online and in-person) instruction. The courses were taught in a synchronous virtual format using an online video conferencing software.

Survey Instrument

The survey was designed to understand student experiences inside and outside of the mathematics classroom relative to four primary constructs: mathematical caring, sense of community, sense of inclusion, and mathematical identity. The survey questions and design were informed by both literature on each construct and former published surveys (Apkarian et al., 2019; Carlson et al. , 1999; Code et al., 2016; Walter et al., 2016). The survey consists of 55 Likert items asking students to indicate the extent to which they agree with given statements on a four-point scale: 1 (Disagree), 2 (Slightly Disagree), 3 (Slightly Agree), or 4 (Agree). The survey also consisted of demographic and contextual questions, including asking for student major (STEM, not STEM, or unsure), type of course (PC or CA), and expected grade in course (A(±), B(±), C(±), D, E, or other).

New Likert items were written for our survey when previously validated instruments did not adequately or directly attend to these constructs. All items were reviewed by the project team to ensure face validity, and items were organized into groups based on the focal aspect of the question (i.e., instructor, classmates, self) as opposed to being grouped by theoretical constructs. This was intended to reduce cognitive fatigue on the part of participants. A web-based version was then created in Qualtrics (2021) for distribution.

Analysis

Analysis was conducted on de-identified survey data (aliases were used for instructors), and free response answers (which sometimes named students, GTAs, or instructors) were kept separately from these multiple-choice data. The underlying constructs driving students' responses to the Likert items were first identified using exploratory factor analysis (EFA), and the resulting factors were used to create composite scales; these scale scores were then tested against other student responses using analysis of variance (ANOVA) testing, and main effects were investigated using Tukey's HSD post hoc test. EFA is a multivariate statistical method for identifying relationships between measured variables (here, responses to each individual Likert item). Instead of depending on the four a priori constructs we intended to measure, we used EFA on student responses to detect underlying constructs which were salient to students (these factors are described in the results section). Composite scale scores were calculated as the average of students' responses to each item in the detected factor. As response options ranged from 1 (disagree) to 4 (agree), a composite score of 2.5 can be interpreted as "neutral" with respect to the factor; scores below 2.5 indicate disagreement while those above indicate agreement. We tested for relationships between composite factor scores and (a) major (three levels); and (b) expected grade (six levels). This was done using ANOVA, while Tukey's HSD post hoc test was used to interpret the results of the ANOVA. Analyses were performed in R (R Core Team, 2021). These tests were performed in the aggregate and disaggregated by enrolled course.

Results

Factor Analysis Results

Comparison of multiple models using EFA established that a seven-factor model was satisfactory (RMSEA = 0.043; Tucker Lewis = 0.871). Two of the resulting factors consisted of fewer than four items and did not have face validity; these were not included in the subsequent analysis. Descriptions of the five retained factors and sample items from the clusters are presented in Table 1. While not entirely aligned with the initially targeted dimensions, these five factors do capture aspects of student affect.

Table 1. Descriptions of and examples for the five factors determined through EFA.

Factor	Description	Item Examples
Factor 1	Items about the instructor (their role, actions, orientation); related to mathematical caring.	"My instructor tries hard to help students understand." "My instructor cares about my learning."
Factor 2	Items about students' perception of their mathematical identity and abilities.	"I am confident in my ability to verify the correctness of my answer." "Thinking about mathematics makes me anxious."
Factor 3	Items related to participation and belonging in the classroom community.	"I am a valued member in this mathematics class." "My classmates motivate me to try my best in class."
Factor 4	Items related to students' orientation towards a sensemaking approach when doing math, and their instructor's encouragement, expectation, or facilitation of such approach.	"When making sense of a mathematics word problem, I represent the situation with a drawing." "My instructor expects us to explain the thinking we used to determine our answer."
Factor 5	Items about students' insecurity or discomfort with their own mathematical ability.	"The course moves so fast that I don't have time to understand ideas well." "I do not like to share my thinking during class."

We used composite scores for each participant in each factor and investigated distributions of data based on these composite scores. Specifically, we used ANOVA and Tukey's HSD post hoc test to look for relationships between factor scores and the variables (a) students' expected grade in the class (GradeExpected, possible responses: A(±), B(±), C(±), D, E, Other), and (b) students' enrollment in a STEM major (MajorSTEM, possible responses: Yes, No, Unsure). Based on the results of a chi-square test for independence ($\chi = 11.98$, $p > 0.10$), the variables GradeExpected and MajorSTEM were determined to be independent.

The average factor scores for each response of GradeExpected and MajorSTEM in the full data set is provided in Table 2. No students reported expecting an E or "other" grade in the course, so those items have been omitted from the table.

Table 2. Average factor scores based on GradeExpected or MajorSTEM items in full data set.

Variable	Item	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5
GradeExpected	A(±)	3.72	3.12	2.67	3.47	1.65
	B(±)	3.61	2.91	2.55	3.28	1.88
	C(±)	3.55	2.32	2.41	3.16	2.16
	D	3.03	1.60	1.92	2.63	2.44
MajorSTEM	Yes	3.61	3.01	2.64	3.40	1.87
	No	3.63	2.77	2.56	3.25	1.79
	Unsure	3.67	2.72	2.36	3.33	1.99

ANOVA and Tukey's HSD Post Hoc Test Results

A collection of ANOVA models was used to determine the effects of ExpectedGrade and MajorSTEM on each factor. The full data was examined, as well as data for the CA and PC courses individually. GradeExpected was determined to be a main effect ($p<0.05$) on (a) the full data for Factors 1, 2, 4, and 5; (b) the CA data for Factors 2 and 4; and (c) the PC data for Factors 1, 2, 4, and 5. MajorSTEM was determined to be a main effect on (a) the full data for Factor 2; (b) the CA data for Factor 3; and (c) the PC data for Factor 2.

We then used Tukey's HSD post hoc test to look at the differences between particular MajorSTEM and ExpectedGrade responses for the relationships that appeared significant in the initial ANOVA. We omit discussion of expected grade comparisons involving those expecting to receive a D, as there were only 5 such students. A difference of 0.5 corresponds to a difference of half a scale-point on the agree/disagree response scale; we further omit discussion of these results as their practical significance cannot be established. Thus, we restrict further discussion to two main findings: (a) differences in Factor 2 scores between students expecting an A-grade and those expecting a C-grade; (b) differences in Factor 3 scores between students intending to major in STEM and those who reported being unsure if they would major in STEM or not (Table 3).

Table 3. Noteworthy differences in factor responses based on ExpectedGrade or MajorSTEM.

Factor	ExpectedGrade Difference	MajorSTEM Difference
Factor 2	Full Data, C(±) - A(±), diff = -0.80 ***	N/A
Factor 3	N/A	CA, Yes - Unsure, diff = 0.58 *

Note. Three pieces of information are given for each difference: where the data comes from (Full Data, CA = College Algebra Data, PC = Precalculus Data), which items are being compared, and the statistical significance of the difference (* = p<0.05, ** = p<0.01, *** = p<0.001).

Discussion

We briefly discuss the results of the exploratory factor analysis in comparison to the affective dimensions initially targeted. We then discuss the two findings above, difference in average Factor 2 scores based on ExpectedGrade and difference in average Factor 3 scores based on MajorSTEM. These two findings support the notion that student affect is related to a students' ability to remain in STEM.

Factor Analysis

The survey was constructed with four dimensions of student affect in mind: mathematical caring, sense of community, sense of inclusion, and mathematical identity; exploratory factor analysis instead identified five robust factors. While not identical to the original design, these factors are not unrelated to the original affective dimensions. In particular, Factors 1, 2, and 3 primarily aligned with the constructs of mathematics caring, mathematical identity, and sense of community, respectively.

Firstly, Factor 1 closely aligned with the construct of mathematical caring. Many of the items in Factor 1 were similar to the example items provide in Table 1, in that they asked about students' perceptions of their instructor's actions and motivations. These affective student perceptions form the basis for Hackenberger's (2010) mathematical caring relationships. Secondly, Factor 2 closely aligned with the construct of students' mathematical identity. The examples in Table 1 show items related to students' recognition of their mathematical ability and their interest in math, two factors associated with mathematical identity (Cribbs et al., 2015). Other items in Factor 2 were related to Cribbs et al.'s constructs of recognition, interest, and competence/performance. Lastly, Factor 3 closely aligned with the construct of sense of community. As shown in Table 1, the items in Factor 3 primarily focused on students' relationships to other in their class, an important aspect of LeChasseur's (2015) social space. The remaining factors incorporated some items from these three constructs, as well as the construct of sense of inclusion.

Factor 2 Discussion

We interpret Factor 2 as a measure of students' mathematical identity. The Tukey's HSD post hoc test showed a difference of -0.80 in average Factor 2 scores between students expecting a C(±) in the course (M = 2.32) and students expecting an A(±) in the course (M = 3.12), for all students (CA and PC combined). This practical significance, along with the statistical significance of this result ($p < 0.001$), implies that the students that responded to this survey who expect a C(±) in their Pathways College Algebra or Precalculus course on average report a weaker mathematical identity than students who expect an A(±). In particular, C-expecting students have an average score below 2.5, implying they generally disagree with statements that align with positive mathematical identity, while A-expecting students have an average score above 2.5.

We acknowledge that expected grade is not a comprehensive measure of a students' self-perception in their classroom, but it does provide some insight into how students either (a) view their ability to do math in their classroom, or (b) view their teachers' perception of their mathematical ability. In either case, this finding weakly supports the finding by Cribbs et al. (2015) that there is a connection between mathematical identity and self-perception. We conjecture that having a more robust understanding of how a student connects their expected grade in the course to their self-perceived mathematical ability may provide more insight into how a student develops their mathematical identity.

Factor 3 Discussion

We interpret Factor 3 as a measure of students' sense of community. The Tukey's HSD post hoc test showed a difference of 0.58 in average Factor 3 scores between students who were definitively STEM majors ($M = 2.79$) and students who were unsure if they were or were going to be STEM majors ($M = 2.21$), for students enrolled in the College Algebra course. This is a small, but still meaningful, practical difference which is also statistically significant ($p<0.05$). That is, student respondents who are sure of their status as STEM majors on average report a greater sense of community in their classroom than students who are unsure about their status as STEM majors.

We note that there was not a statistically significant difference in Factor 3 scores between students who identified as STEM majors and those who identified as non-STEM majors. This suggests that the uncertainty of students who responded "Unsure" is an important factor in this difference. This result complicates the finding by Rainey et al. (2018) that students who report a greater sense of belonging are more likely to persist in STEM. Perhaps the students who are unsure of their status as STEM majors might consider sense of belonging as a bigger factor than others in deciding to stay in STEM.

Conclusion

We have shown that in the context of a research-based undergraduate active learning mathematics curriculum, there is a connection between (a) students' mathematical identity and their expected grade in the course, and (b) students' sense of community and their major. More generally, our data support the notion that students' affective needs are related to their potential to complete a STEM degree. Our study shows a correlation though, not a causation. Future work needs to be done in examining the complexities of these relationships and leads to further questions: How are these factors related to one another? Do students who report higher affect according to our survey graduate with STEM degrees at higher rates? We believe that by answering these questions, we will be better equipped to attract, support, and retain STEM majors.

Further directions for this work include a redesign of the survey items to better align with the factors that were discovered. In particular, the survey items designed to elucidate students' sense of inclusion in the classroom should be reexamined and rewritten. Further, to better understand the transferability and validity of our EFA model, more data from other institutions implementing the Pathways innovation should be gathered. This is especially important given that these data were recorded during the COVID-19 pandemic, where instructional practices were constantly changing and adapting.

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The Relationship Between an Instructor's Mathematical Meanings for Teaching Sine Function and Her Conception of and Intended Use of an Applet: The Story of Kendra

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Digital technologies including graphing software and applets are frequently included in mathematics curricula as resources for supporting students' learning and performance (Swidan, 2019; Thompson, Byerley, & Hatfield, 2013). Research has shown that teachers' conception of didactic objects is largely influenced by their mathematical meanings (Thompson, 2002; 2008; Guy, 2021). As such it seems reasonable that there is a relationship between teachers' mathematical meanings for teaching (MMT) and their conception of and intended use of an applet. This report presents results that demonstrate how a graduate student instructor's (GSIs) MMT influence her conception of and intended use of an applet. Moreover, we illustrate how a teacher's use of an applet can lead to advances in her MMT.

Keywords: Applets, Didactic Objects, Quantitative Reasoning, Mathematical Meanings

As technology advances, new and emerging technology is being regularly adopted to support both teachers and students in the teaching and learning of mathematics (Pope, 2013). Current research has documented that teachers play a critical role in the integration process of technology in curricula (e.g., Artigue, Drijvers, Lagrange, Mariotti & Ruthven, 2009; Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010; Daher, 2009). As one example, teachers are using applets – interactive computer-based objects – to support students in visualizing attributes of specific mathematical ideas (Gadanidis, Gadanidis, & Schindler, 2003). The dynamic abilities of an applet can promote discussions that focus on quantities and their relationships within a problem context as they vary in tandem (Moore, 2009). Cobb, Boufi, McClain, and Whitenack (1997) referred to these discussions as reflective discourse. A teacher strives to engage in reflective mathematical discourse when implementing an applet as a *didactic object*.

Thompson defines didactic object as "a thing to talk about' that is designed with the intention of supporting reflective mathematical discourse" (Thompson, 2002, p. 198). An object is not considered didactic until the teacher conceptualizes an image of a conversation designed to support students in constructing coherent mathematical meanings. A teacher's conceptualization of a didactic object largely depends on the instructor's *didactic model*. A didactic model is "a scheme of meanings, actions, and interpretations that constitute the instructor's or instructional designer's image of all that needs to be understood for someone to make sense of the didactic object in the way he or she intends" (ibid, p. 212). A teacher's didactic model is thus largely dependent on her MMT, and therefore may be different from the instructional designers of the applet.

Thompson (2016) described a teacher's mathematical meanings for teaching (MMT) as her images of the mathematics she teaches and intends students to have. As such a teacher's MMT encompass her meanings for the idea together with the teacher's (1) image of epistemic students, (2) image of how to support students in developing similar meanings, and (3) an image of activities to support students' construction of these meanings (Silverman & Thompson, 2008). Although researchers have used applets as didactic objects to support students' learning of mathematics (Moore, 2009; Guy, 2020), few have investigated the relationship between teachers'

MMT and their use of applets. This report addresses the following question: *In what ways does teachers' MMT influence their conception of and intended use of an applet while teaching?*

Theoretical Perspective

Prior research has identified *quantitative reasoning* (Thompson, 1990, 2011) as a foundational way of thinking for supporting students' understandings and teachers' instruction of angle measure and trigonometric functions (Moore, 2010, 2014; Hertel & Cullen, 2011; Tallman, 2015; Tallman & Frank, 2018; Rocha & Carlson, 2020; Rocha, 2021). An individual engages in *quantitative reasoning* when she conceptualizes a situation in terms of *quantities* and *quantitative relationships* (Thompson, 1990). A *quantity* is an attribute of an object that one imagines measuring. When someone has conceived of three quantities, two of which determines the third, they have conceived of a *quantitative relationship.*

A productive meaning for sine function involves quantitative (Thompson, 2011) and covariational reasoning (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Covariational reasoning entails the coordination of changes in two quantities' values while attending to those changes as the quantities vary simultaneously. Specifically, sine function relates the measure of an angle swept out from the 3 o'clock position to the vertical length of the point on the terminal ray of the angle (*terminal point)* measured from the center of the circle. Moore (2012, 2014) found that is productive to measure an angle's "openness" (i.e., the length of arc from the initial ray to the terminal ray) by determining the fraction of the circle's circumference that is subtended by the angle. Meaning "for the measure of the angle to be independent of the size of this circle, the subtended arc must be measured in a unit whose magnitude is proportional to the magnitude of the subtended arc and the circumference of the circle that contains it" (Tallman & Frank, 2018, p. 5).

The value of sine can be determined by leveraging relative size reasoning in comparing the length of the terminal point's vertical distance from the center of the circle to length of the circle's radius. One of the benefits of conceptualizing sine function in terms of the radius's length is that "the values convey numerical measures for every circle all at once because regardless of the circle, one obtains equivalent numerical values when the quantities are measured relative to that circle's radius" (Moore, 2014, p. 13). Similarly, other trigonometric functions require one to formalize a relationship between the covariation of angle measure and a ratio of lengths (see Figure 1).

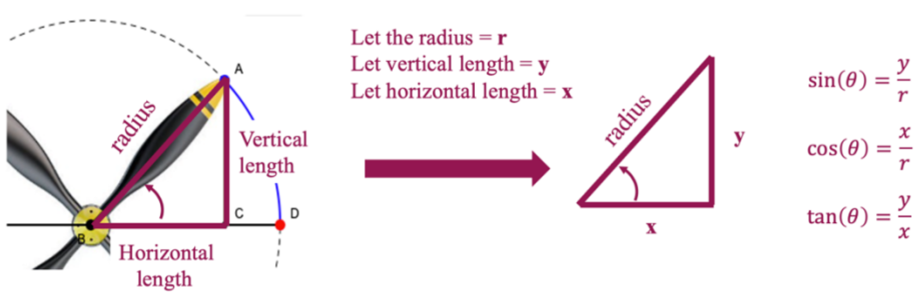


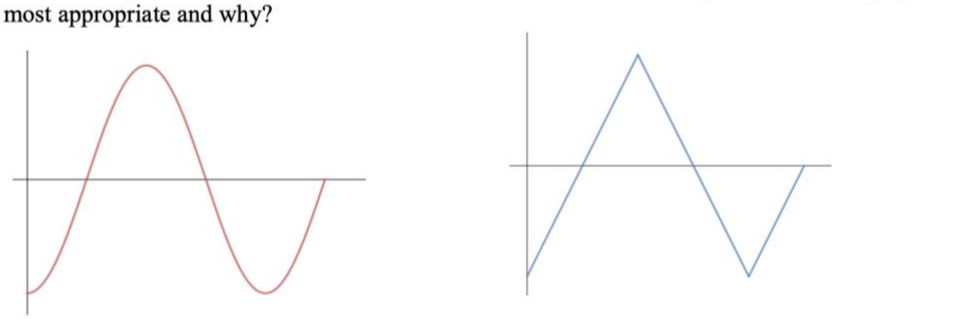
Figure 1*: A connection between unit circle trigonometry and right triangle trigonometry.*

The Fan Blade (FB)-Applet (Figure 2) was designed to promote quantitative meaning for sine if engaged in an instructional conversation that focuses on the directed vertical distance for a given angle measure for various radius lengths. For instance, for an angle measure of 0.79 radians (or 45 degrees), one may increase and decrease the radius (point D) of the circle which produces various arc-lengths and vertical distance measured in feet. But, in radius lengths (radii), the measure of the directed vertical distance measured in radii is the same numerical value as the vertical distance of the terminal point from the center of any circle rotated at 0.79 radians is constantly proportional to the length of the radius. This meaning for sine function is especially productive for students who view the unit circle as a basis for any circle context, in which they rotate 0.79 radians, view the vertical length of the terminal point to be $\sin(0.79) \approx 0.71$ radius lengths, then multiply the radius of the circle in the given context (e.g. a propeller blade with a length of 3 feet) to the value of $\sin(0.79)$ to produce the accurate y-coordinate of the fan blade's vertical position from the center of the circle (i.e. $3 \cdot \sin(0.79) \approx 3 \cdot 0.71 \approx 2.12$ feet).

Methodology

The purpose of this study is twofold (1) to investigate teachers' meanings for sine function and (2) to investigate how teachers' meanings for sine function influence their conception of and intended use of an applet. To accomplish this goal, we interviewed an instructor who was in her second-year teaching precalculus with a research-based Pathways curriculum (Carlson, Oehrtman, & Moore, 2018) at a large public university in the southwest United States. During the interview, the instructor was asked to respond to two tasks (shown in Table 1) designed to reveal her meanings for sine function. The GSI was also presented with an applet designed to support instructors' teaching of sine function. The instructor was then asked to think aloud as she interacted with the applet and its features. The first author then posed questions to the instructor to gain insight into her perception of the applet including the mathematical ideas and understandings the applet could support and if/how the instructor intended to use the applet while teaching. The first author analyzed the clinical interview using Simon's (2019) three phases of analysis to form a second-order model (Steffe & Thompson, 2000) of the GSI's thinking and meanings for sine function.

Table 1. Tasks posed to instructors during the clinical interview.

Task 1	Jamie is a pilot for Beta Airlines. He noticed that there was a piece of gum stuck at the tip of one of the propeller's fan blade. The length of each fan blade from the center of the propeller is 3 feet. Jamie rotates the fan blade with the gum attached in a counterclockwise (CCW) direction from the 6'oclock position. Two of your fellow teachers constructed the following graphs that represents the gum's vertical length (position) from the center of the propeller with respect to how far the gum at the tip of the fan blade travelled from the 6'oclock position. Which graph is most appropriate and why? 
Task 2	Let's say you were lesson planning and prepping to present this problem (below) in class to your students. *Jamie is a pilot for Beta Airlines. He noticed that there was a piece of gum stuck at the tip of one of the propeller's fan blade. The length of each fan blade from the center of the propeller is 3 feet. Jamie rotates the fan blade with the gum attached in a counterclockwise (CCW) direction from the 6 o'clock position.* Create a graph that shows how the gum's vertical length from the horizontal diameter varies with the angle swept out by the gum's fan blade as it rotates CCW from the 6 o'clock position.

Results

We share the following results from a GSI's interview that entail her responses and interactions with Task 1, Task 2, and the FB-Applet included in a research-based Precalculus curriculum (Carlson, Oehrtman, & Moore, 2018). We first highlight the GSI's meanings for sine function. We follow this with a description of the GSI's goal(s) for supporting students in developing coherent meanings for sine function. This includes a description of the GSI's image of a productive meaning for sine function and the questions she intends to pose to students while using an applet in class. We conclude with a description of the GSI's interaction with applet and the meanings for sine function she expressed while interacting with the applet.

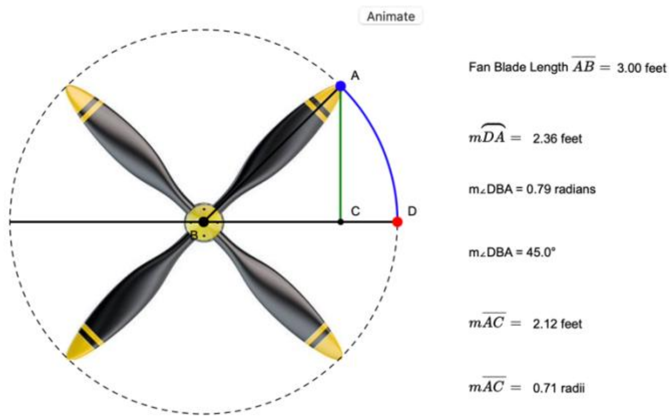


Figure 2: *Fan Blade (FB)-Applet*

Kendra's Meaning for Sine Function

When responding to task 1, Kendra immediately identified the red graph (see Table 1) as the correct graph of sine function. She then expressed that the phrase "how far the gum traveled" (included in task prompt) is a description of a rotation from the 6 o'clock position. Following this she labeled the red graph's horizontal axis as angle measure and the vertical axis as the vertical distance from the center of the fan blade (see Figure 3). When Kendra was asked how she intended to support students in identifying the correct graph in the first task, she expressed that she wanted students to compare changes in the vertical distance of the terminal point over equal-sized intervals of the domain. Specifically, Kendra expressed that in the fourth quadrant (see Figure 3), she wanted her students to focus on the amount of change in the vertical distance of the gum for "equal changes of radians" in the counterclockwise direction (CCW). Although Kendra expressed that she wanted her students to coordinate changes in the vertical distances of the gum with changes of radians, she did not make explicit how one could measure the vertical distance of the gum from the horizontal diameter.

Figure 3: *Applet 1 - Angle Measure*

When Kendra was asked to describe the meanings for sine that she wanted students to have she expressed that she wanted students to think of the sine as a function machine that relates the terminal point's vertical length from the horizontal diameter (the output) to an angle of rotation from the 3 o'clock position (the input). Kendra further expressed that she could determine the vertical distance of the terminal point above the horizontal diameter by comparing the relative size of the vertical length of the gum's position to the length of the radius (Excerpt 1). As one example, Kendra explained that if the gum's terminal point was along a circle's circumference (indicated by blue point in Figure 2), then one can estimate the vertical length (solid vertical blue line) to be 0.9 or 90% of the radius's length (the dashed vertical blue line on vertical diameter).

Excerpt 1

Kendra: So if my terminal point is here [draws the blue point], I'm going to be measuring this length [draws the blue vertical line from the blue terminal point to the horizontal diameter] in comparison to my entire radius [draws dashed line to represent the radius] so that [the solid blue line] will never be bigger than my one radius length so we'll be taking basically a proportion of that radius length. So, how many times as large that vertical length is compared to the length of my radius. Maybe, this one [references the solid vertical line] is point nine. So, your output for the sine function without adjusting it will be between negative one and one every time because it can't go beyond one because you'd be outside the circle."

Kendra explained that by measuring the vertical length in units of the radius, she hoped students would recognize that the output of sine function must range between -1 and 1 units of the radius. She further explained that no matter the size of the circle, the vertical length of a terminal point along a circle's circumference whose rays are centered at the center of the circle will be proportional to the size of the circle's radius.

Excerpt 2

Kendra: To find the vertical distance of that terminal point above the horizontal diameter, in terms of radius's length [i.e. from units of radius lengths to feet]. Then we can multiply by the radius's length [i.e. 3 feet] to get it in terms of the units that the radius length is measured in. Because if I have an output of one and I multiply by 3 …I'm going to get 3, so my radius length of 3 feet that value three is 3 times as large as the radius length itself being one. And if I know that my vertical distance is 0.9 times as large as the radius length, then, if I multiply that by 3, that's going to tell me how many feet it would measure.

Excerpts 1 and 2 show Kendra's conceptualization of both radius lengths and feet as an appropriate unit of measurement for the gum's vertical distance from the ray's terminal point to the center of the circle. Moreover, it appears as though Kendra's meaning for sine function is grounded in quantitative reasoning as she identified an attribute to measure (vertical distance of the terminal point above the center of the circle), a unit of measure (radius length or units of the radius (i.e. feet)), and a process to measure the vertical distance (determining the relative size of the vertical distance to the radius length) (Tallman & Frank, 2018).

Kendra's Expressed Meaning for Sine through Fan Blade Applet

The start of Kendra's interaction with the FB Applet entailed animating the applet, searching for what points on the circle she could move, and the meaning of the information displayed on the right side of the panel (Figure 2). The first thing Kendra discussed was her interest in varying the length of the radius (i.e. dragging point D horizontally to the right which varied the size of the circle). As she did this, she began to describe the affordances of being able to vary the length of the radius (see Excerpt 3).

Excerpt 3

Kendra: The affordance of this is that I can change the size which can adjust my output when it's in terms of feet, but I like that [*moves her mouse to circle in the bottom right corner of the FB applet*]…I like this *m* times AC is always going to stay 0.93 matter what size right because it doesn't matter what size the circle is if we're talking about in terms of radius length.

Following this, Kendra expressed her appreciation for the designer in choosing to not include the term "sine function" on the applet as she imagined having an initial conversation with her students about the quantitative relationships between the arc-length, the radius, and the vertical position of the gum as they vary in tandem prior to defining sine function explicitly in her lesson. Kendra then described the affordance of the applet's dynamic capabilities – the ability to physically rotate the gum's position along the circumference of the circle could support a conversation with students about the quantities involved in the problem and how they vary together. This was important as this aligned with her initial goal of supporting students' imagery of two quantities covarying. While interacting with the applet, Kendra also expressed that she felt it was important to have a fruitful discussion with her students about the applet's capabilities and to observe their reactions. Moreover, Kendra anticipated that students would have difficulties constructing the meaning she intends to convey from the applet in the absence of her (the teacher's) support based on prior experiences from teaching this idea.

Once Kendra seemed comfortable using the applet and all its features, we asked her to describe if and how she would use the applet in class to support students in constructing the meaning for sine that she had described previously. Kendra explained that she would first animate the applet and ask students "what all is changing or staying the same". Kendra expressed that she wanted to animate the applet to allow students to "silently internalize [the] animation and develop their own understanding for what's there or maybe start thinking about what they don't understand." Kendra then expressed that her students might notice that "there are multiple things changing at the same time", which would then provide an opportunity for a conversation on the "need for breaking down" each mathematical notation (see Figure 2) and how it relates to the image shown in the applet. She stated the following:

> "When I vary the size of this propeller, which values are changing, and which values are staying the same. That's definitely something I want to point out, because you can clearly see the arc length is changing the length of the radius is changing, vertical length is changing in feet, but it's staying the same with radii when in terms of the radius, that kind of re-emphasizes that if we measure something in radius length it doesn't matter, the size of the circle, so when we start multiplying it [vertical length in radii] by a number to get an output in feet, that is really dependent upon on a circle's size and a specific context. Whereas the regular sine function is just talking about in general, any circle. And we can relate that vertical length to the radius's size and that's always going to remain proportional, no matter how big or small that circle is. That's definitely something I'd want to emphasize with this one."

It is important to note that while expressing her image for how she intends to use the applet to support an instructional conversation, Kendra continuously dragged point D and varied the length of the radius. She later stated that without the applet, she did not think she would be able to support students in recognizing the proportional relationship between the vertical distance of the terminal point above the horizontal diameter and the radius of the circle as one could demonstrate with the applet. To further explain, Kendra described that the feature at the bottom of the applet that states $m\overline{AC}$, which is the length of the vertical distance measured in units of the radius, continuously staying the same as she varies the radius length, and the size of the circle is a form imagery that she believes displays a productive way of seeing the proportionality of the vertical length to the radius length. Lastly, Kendra also expressed that the applet could be used to support conversations about (1) other applied trigonometric problems and (2) the construction of the sine graph.

Discussion and Conclusion

The results of this study illustrate a teacher's MMT sine function and how her MMT sine function influenced her conception of and intended use of an applet while teaching. During the clinical interview, Kendra expressed a meaning for sine function grounded in quantitative reasoning. Namely, Kendra, described sine as a function "machine" that relates the vertical distance of a terminal point above the center of a circle and the measure of an angle swept out as they vary in tandem. Kendra also expressed that the value of sine represents how many times as large the vertical distance of the terminal point is to the radius of the circle, no matter the size of the circle. Her meaning for sine function supported her conception of the applet's features including the varying radius, angle measure, and vertical distance within a problem context of a fan blade. It is important to note that Kendra's image for supporting her students in conceptualizing how various quantities within the problem context vary in tandem through an instructional conversation (see Thompson 2002 for a more detailed description) was prevalent as she conceived of ways in which the applet could be implemented during those discussions. Specifically, Kendra expressed that she wanted to support students in developing an understanding of sine that encompassed a proportional relationship between the terminal ray's vertical position from the center of the circle to the length radius. She stated that the use of the applet's ability to vary the circle's radius as the vertical length of the terminal point from the center measured in radius lengths stayed constant aligned with her goal for in supporting students in conceptualizing this proportional relationship.

From our findings we hope to interview more instructors to further investigate in what ways teachers' MMT influence their conception of and intended use of an applet while teaching. Future research should investigate teachers' use of applets and how it may support their development of images of teaching practices to advance students' learning. More concretely, we hypothesize that a teacher's MMT influences their conception of an applet and its features, and conversely, a teacher's use of an applet enriches their MMT to include new images of effective teaching practices (e.g., ways to use an applet's features to support students' covariational reasoning). It is important for researchers and professional development leaders to be aware of the relationship between teachers' MMT, their conception of and intended use of an applet, and how they implement the applet within the classroom to advance students' mathematical understandings. In additional studies, we hope to investigate the interactions of four graduate student instructors' MMT for sine function and their conception of and intended use of an applet(s) over multiple sessions while exploring teachers' experiences of using the applet(s) in a classroom.

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Making Mathematics Meaningful for All Students: An Exploration of Self-Efficacy in Teaching Mathematics

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Making math meaningful for all students is a consistent expectation held by students and a fundamental requirement of teaching for equity. Yet little is known about what makes instructors self-efficacious in helping students find meaning in the mathematics they study. This paper presents results from interviews with four college math instructors during the 2019 spring academic year. The instructors taught several different courses and described having significantly different senses of self efficacy for making math meaningful. A characteristic of those instructors who had strong senses of self-efficacy was notable background experiences with marginalization. Through analysis of these interviews we present evidence showing how these background experiences supported self efficacy in addressing meaningfulness questions in classroom instruction.

Keywords: Meaningfulness of Mathematics, Teacher Self-efficacy, Teaching Self-efficacy, Marginalization

Students in mathematics courses have often voiced complaints to the effect of "when is this ever going to be useful?" (Gough, 1998). We interpret this question and others like it, such as 'why are we learning this?', 'why should we care?' as students asking for the purpose, usefulness, relevance, and/or meaningfulness of what they are learning. Under the assumption that these terms are closely related, if not synonymous in an instructor's and students' eyes, we group these questions under an umbrella term we call the 'meaningfulness question(s).' Such questions put an onus on instructors to make mathematics meaningful to their students, especially marginalized students who are documented as receiving mixed messages about the importance of mathematics in their lives (Martin, 2000).

Some instructors, however, may not feel confident in their ability to meet this challenge. Although little is known about mathematics teaching self-efficacy in higher education, K-12 mathematics teachers differ in their feelings of self-efficacy for teaching mathematics. Furthermore, these differences impact their practice in domains such as their use of high-quality instructional practices (Depaepe & König, 2018) and their willingness to experiment with new approaches (Evers, Brouwers, & Tomic, 2002; Ghaith & Yaghi, 1997). It is, thus, worth exploring what differences might exist in instructor self-efficacy for the more specific domain of making mathematics meaningful.

This kind of work has not been a priority in the field thus far. Research on K-12 teacher self-efficacy, such as the studies cited in the previous paragraph, focus on the broader construct of mathematics teacher self-efficacy (Xenofontos and Andrews, 2020). There has been little explorational work of other, more fine-grained aspects of teacher self-efficacy (Xenofontos and Andrews 2020). Thus, there is a need for qualitative work that investigates more specific components of teacher self-efficacy in mathematics such as self-efficacy in making mathematics meaningful for students. Given the lack of research on mathematics teacher self-efficacy in higher education, work in higher education is especially important. In this study we explore this issue by investigating the research question, *What makes undergraduate instructors' self-efficacious with respect to making mathematics meaningful for all students?*

Theoretical Framework

How then can instructors think of the terms 'meaning' and 'meaningful'? Drawing on Brownell (1947) and social theory, Chowdhury (2021) describes four types of responses to students' meaningfulness questions which derive from different ways to construe the term 'meaning' in mathematics. Brownell (1947) initially talks about the *meaning of mathematics* as referring to the understandings required to grasp a concept mathematically, e.g., the *meaning of* arithmetic requires understanding whole numbers, fractions, operations. By contrast, the *meaning for mathematics* entails understanding mathematics' significance for non-mathematical purposes, e.g., applications to life beyond the classroom or even skills for test performance.

These two conceptions focus on how an individual understands meaning. An individual makes sense of concepts mathematically by understanding requisite concepts (*of*) or values mathematics for its everyday applicability (*for*). However, some may focus on neither, instead focusing on preparing to be a functioning member of the mathematics community or society more broadly (e.g., critical citizenship). For these reasons, Chowdhury (2021) expands the prior by drawing on social theory, specifically aspects of Wenger's (1998) framework. Wenger holds "Practice is about *meaning as an experience of everyday life*" (p. 52, emphasis added). Synthesizing these and Wenger (1998) more generally, Chowdhury interprets Wenger as saying that meaning is how one engages in the everyday experiences of a specific community. Thus, individuals can become competent members of the mathematics community by learning how to engage in the everyday practices of mathematicians or individuals can learn how to engage in practices that enable them to function in non-mathematics communities. Chowdhury proposes that instructors' goals could be oriented towards having students learn practices of either kind, but also mathematical understanding or personal relevance as Brownell originally outlined, as shown in Figure 1 below.

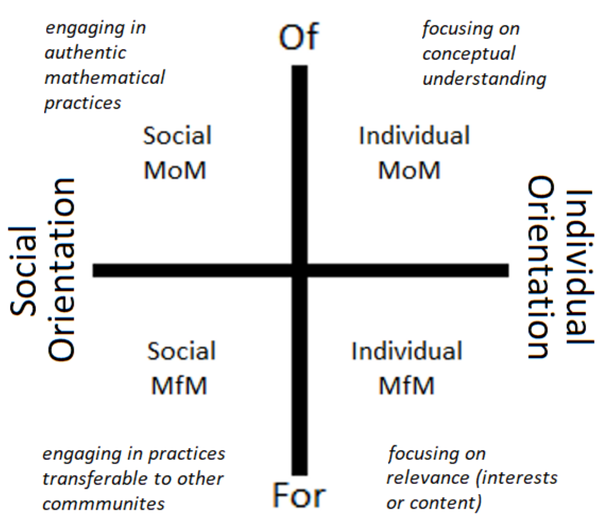

Figure 1. Four Orientations of Mathematical Meaning as a Model of Instructors' Goals.

Thus, Chowdhury (2021) outlines four categories, or *orientations*, for responses instructors could have to students' meaningfulness questions. The *individual meaning of mathematics* (iMoM) refers to meaning as conceptual understanding of mathematical ideas while the *social meaning of mathematics* (sMoM) refers to engaging in the practices of the mathematics community. The *individual meaning for mathematics* (iMfM) refers to the relevance of mathematics to non-mathematical things in an individual's life while the *social meaning for mathematics* (sMoM) refers to engaging in practices transferable to non-mathematics communities. Chowdhury uses the four resulting orientations on meaning to categorize instructors' goals, where some of those goals could be instructors' responses to students' meaningfulness questions.

Methods

The data in this study is from a larger multiple case study of four instructors teaching during the 2019 spring term at a Western U.S. university (Chowdhury, 2021). Each of the instructors taught a different course. Archy (he/him) taught quantitative literacy for non-STEM majors. Jordan (she/her) taught introductory statistics for health science and business majors. Benjamin (he/him) taught calculus III for STEM majors. Natalie (she/her) taught mathematics for teachers to preservice teachers. Jordan, Benjamin and Natalie all had substantial prior experience teaching their courses. Archy was teaching his course for the first time.

The data used in this study were interviews, class observations, collected homework, notes, and email correspondence. Chowdhury interviewed each instructor three times throughout the term. The first interview focused on an instructor's background and course goals. The second and third interviews followed Speer's (2005) video clip interviews: Chowdhury brought clips he suspected may have implications for an instructor's goals and asked the instructor what they did, their goals in the moment and how those goals related to the goals the instructor voiced in the first interview. Interviews were transcribed and analyzed following Braun and Clarke's (2006) approach to thematic analysis. For the analysis, Chowdhury assigned initial codes regarding goals, beliefs, instructor background, and concerns. He then looked over the codes to find broader themes and create thematic maps to see how coded experiences, beliefs, and meaning orientations were organized.

Findings

In the following sections, we will first pose representative examples of instructional goals as evidence of the different meaning orientations as answers to students' potential meaningfulness questions. We will then highlight one way that this distinction in meaning orientations for goals can matter: instructors can have differing senses of teaching self-efficacy to meet goals with different orientations. We will then posit a possible explanation for why this difference in teaching self-efficacy exists (experiences of marginalization).

Instructor's Goals

Instructors had goals spanning all four meaning orientations that they posed in response to students' meaningfulness questions. In terms of goals we classified as iMoM- and sMoM-oriented, instructors had goals around building conceptual understanding and doing mathematics that they posed in response to the meaningfulness question. Natalie had students engage in reconstructing mathematical knowledge to convey that "there aren't all these different individual concepts to learn. that there's bigger ideas." Natalie was having preservice teachers reconstruct mathematical knowledge to see that there are bigger connected ideas (this reasoning being the justification for the iMoM-orientation). Archy stated, “you can just enjoy it [math] the way you enjoy playing basketball. It's just like, basketball's meaningless too, but it's fun.” In trying to convey this fun then, Archy devoted a section of his class to covering sequences and series. There, he wanted students to go through and see how there was a “sweet connection” between sequences of partial sums to the sum of the first n integers. The aspect of showing “sweet connections” focused on drawing connections (iMoM-oriented) while the focus on having students *do* mathematics the way one plays basketball focused on engaging in the practices of mathematicians (sMoM-oriented), and Archy connected these to being meaningful in their own right. Benjamin’s main goals consisted of conveying two overarching “storylines” (sequences and series and preparing for calculus in 3D), and he talked about how he fell back on these goals in response to meaningfulness questions. He stated, “what I end up focusing on more, is simply that this subject [mathematics] is fascinating, … it doesn't have to have anything to do with the people around you, that it can just be its own safe haven.” In all the prior examples, a focus on having students understand mathematical concepts and how they connect to other mathematical concepts (iMoM-oriented) or on doing mathematics (sMoM-oriented) were posed in response to students’ meaningfulness questions.

In terms of goals we classified as iMfM- and sMfM-oriented, instructors had goals around building community engagement and personal relevance that they posed in response to the meaningfulness question. For example, Benjamin had personal relevance goals of building students’ algebra skills by picking algebraically complicated problems because “maybe they'll have spent this much extra time being forced to do hard algebra that that will be a skill that they can carry on and actually utilize in their lives.” Similarly, Archy devoted a section of his class to personal finance. Generally, these personal relevance goals instructors connected to students’ meaningfulness questions focused on promoting competencies for students’ individual use outside any commitment to the discipline of mathematics (i.e., iMfM-oriented goals). On community engagement, Jordan talked about bringing in articles on how statistics were used because “they may not use this specific statistic in their life going forward, but… we hope that they become more educated consumers and not get fooled by statistics… understand that that statistic was manipulated by a politician.” Similarly, Archy had a section on data collection so that his students could critically evaluate claims propagated throughout society. Here, Jordan and Archy are focused on getting students to learn how to function as critically informed citizens in a democratic society. Natalie talked about how “the whole point is to teach them [preservice teachers] useful math…‘Why are we doing it this way? Why aren't you just telling us [how] to do it?’ One of my answers would be, you know, this is how people learn.” The point here was that having the preservice teachers understand math in this way will enable the future teachers to teach their students math effectively (which is sMfM-oriented). In all of these, instructors focused on preparing students to function in (non-mathematics) professional communities or society more broadly and positioned this work as meaningful. The focus on community engagement signals an sMfM-orientation.

Why Might Orientations Matter?

So why do the orientations matter? We found while no instructors described self efficacy concerns relating to i/sMoM-oriented goals, there was a stark contrast in self efficacy in meeting goals with i/sMfM-orientations that broke down along background (childhood experiences, gendered experiences in mathematics, etc.). We note that the absence of voiced concerns about i/sMoM-oriented goals does not establish that all the instructors had high levels of self-efficacy when attending to those goals, but we find it significant that no concerns were stated relating to i/sMoM-oriented goals while some concerns were stated relating to i/sMfM-oriented goals.

Benjamin and Archy expressed discomfort attending to some i/sMfM-oriented goals. For instance, Benjamin commented, "how can we get more people of African American heritage to take math classes and be successful in math classes, and how can we make math relevant to them? I don't have an answer for that question." Benjamin noted the importance of i/sMfM-oriented goals because of racial inequalities. We took Benjamin's view of math as "relevant" as either iMfM- or sMfM-compatible since he went on to talk about cultural relevance, and it was not clear if this meant relevant to an individual (iMfM-oriented) or relevant to functioning in the African American community (sMfM-oriented). As a result of his discomfort in the last quote he noted, "but what I end up focusing on more, is simply that this subject [mathematics] is fascinating, … it doesn't have to have anything to do with the people around you, that it can just be its own safe haven." Since Benjamin did not feel he had the resources to attend to i/sMfM-oriented goals that could be important for students from marginalized populations, he focused on i/sMoM goals. Archy described similar difficulties when he stated, "my concern is that like I don't have any material for this course and that like I'm just making it all up." He later went on to say "I've never done anything useful...if I had examples on point, then I would be able to definitely hit the, the more personal side of how is this gonna effect you, like either via society or directly." Like Benjamin, Archy noted preparation struggles to meet i/sMfM-oriented goals.

By contrast, the Natalie and Jordan felt comfortable implementing i/sMfM-oriented goals. Natalie highlighted how mathematics connects to other topics so that the preservice teachers can more effectively do their job. She stated, "I care about them [preservice teachers] engaging our students and that's [connecting to other topics] a way to engage." Here she demonstrated personal relevance and community engagement goals, which we previously discussed as generally being i/sMfM-oriented. Jordan highlighted how statistics play into other fields and broader culture. She stated, "I always, on the last day, emphasize to them that they may not use this specific statistic in their life going forward but … we hope that they become more educated consumers and not get fooled by statistics." Here she demonstrated the community engagement goal, which we previously discussed as generally being sMfM-oriented.

Possible Explanation of Difference in Self-Efficacy

One possible explanation of this contrast in self efficacy is a difference in the instructors' historical mathematical identities, specifically experiences of marginalization amongst the women and lack thereof amongst the men. Benjamin always had a curiosity around mathematics and had his interests supported. He noted that in the fifth grade, "my teacher saw that I just had a knack for mathematics and … allowed me to do both the 5th and 6th grade math together. … and after that I was always a year ahead." Archy stated, "some component of my identity is… you know how people are. They're like, 'that's a math person.' And I'm like, well I guess I am." Those around both men recognized and supported their mathematical interest. By contrast, Jordan grew up struggling to learn mathematics in her childhood and later in a patriarchal college environment devaluing women. For example, she noted, "I took an engineering course, ... and then I sit down and he [the professor] announced to the whole class, 'yeah we actually have a girl in here. Don't worry, she won't be here by the end.'" She further stated, "it [patriarchal college environment] pissed me off in a lot of ways and I wanted to change those views … I still am up against sexism constantly in my committees and in the work I do." While Natalie was supported in her success in mathematics, she struggled to integrate that success into her identity. She recalled, “my mom remembers me being horrified to have to go up and get this math award….I was definitely not excited to be recognized.” Natalie had further experience with mathematical marginalization when she taught historically marginalized students as part of a teacher certification program centered around issues of access and equity.

Thus, all of the instructors described a need to focus on i/sMfM-oriented goals of personal relevance and/or civic community engagement, sometimes because of how societal inequalities have kept underserved minorities and women from achieving in STEM. However, there was a noteworthy demarcation in translating these realizations into practice. The men with historical identities rooted in mathematics expressed discomfort attending to these i/sMfM-oriented goals, while the women, who had more contentious historical identities associated with mathematics, felt confident attending to these goals.

How might we explain this contrast in self-efficacy? While we don’t have evidence of a mechanism for iMfM-oriented goals, we suspect that, with respect to sMfM-oriented goals, the experiences of marginalization amongst the women may have been a motivational resource. The women went out of their way to find how mathematics is meaningful for life in various professions or society more broadly. Jordan was observed pulling up and discussing various research articles using different statistics. When asked why she did this, she stated,

> I think the point of this is, just this course in general is just to give them a taste of what they may see in a research article or in the media or maybe they're setting up their own surveys … I have about six different articles I'm going to show them that kind of use the things that we've been using throughout the quarter, like where they might see a p-value in a research article. There's one on a replication of a study. So it was a study that was never able to be replicated. And then there's another story I cite where the data was faked.

In this example, it’s notable that Jordan goes out of her way to find ways in which statistics is applied to various fields, but more specifically how to be critical of studies (which related to her community engagement, sMfM-oriented goals). As was mentioned before, Natalie similarly showed how mathematics plays into other fields (e.g., geology) because it would help preservice teachers draw in their students, thus enabling the preservice teachers to function in their profession (sMfM-oriented). In contrast, Benjamin demonstrated less motivation to improve his ability to address sMfM-oriented goals,

> We don't have to have applications for students to appreciate what it [math] is ... really, the breadth of where this could be used is well beyond my understanding and my knowledge. People come up and show me things that-, where math is used, all the time, that I've never seen before and I'm always fascinated [by]. I'm like, 'that's so cool. I wish I had the time and energy to go and like study that specific thing more.'

Benjamin laments the personal knowledge gaps that inhibit his self efficacy, but does not make filling these gaps a priority.

There is a clear contrast between Benjamin who admits to not having the expansive knowledge to attend to some sMfM-oriented goals, and the women who go out of their way to attend to such goals. By having experiences of marginalization, the woman perhaps saw a greater need to push themselves to become knowledgeable of how to draw these connections to other professions. This need could have been motivated by their own learning needs, which was in fact the case with Jordan, or the learning needs of others, as Natalie may have realized as a result of teaching in underserved communities. The example above only applies to sMfM-oriented goals of community engagement; we do not know if a similar dynamic holds for iMfM-oriented goals.

Conclusions and Implications

We found that, for this group of instructors, attending to different meaning orientations (Chowdhury, 2021) provided a useful lens and possible explanation for understanding differences in self-efficacy in making mathematics meaningful for students. None of the instructors expressed self-efficacy concerns with respect to i/sMoM-oriented goals, while two of the four instructors expressed concerns with respect to i/sMfM-oriented goals. The instructors who were more comfortable addressing students' potential i/sMfM-oriented questions were those with contentious historical mathematical identities and experience with marginalization, while those with self-efficacy concerns in this area had positive identity-shaping experiences in mathematics during their childhood. For the case of sMfM-oriented goals, our analysis suggests that these experiences with marginalization acted as a motivational resource.

Thus, a characteristic that would traditionally be thought of as a deficit, marginalization in the instructors' experience, could be a resource for instructors to meet the needs of marginalized students. This finding builds on previous work demonstrating the effectiveness of K-12 teachers of color in working with students of color (McKinney de Royston et. al., 2021; Milner, 2006). In addition to shared racialized experiences, shared experiences of marginalization could support instructor success in making mathematics meaningful to students. This finding also builds on previous work suggesting that normative identities and mathematical abilities may alienate marginalized students from teachers who emphasize i/sMoM-oriented goals (Mayes-Tang, 2019, Martin, 2003). Students with meaningfulness questions may make stronger emotional and intellectual connections with instructors who have and demonstrate a deeper understanding of the marginalization that informs their queries.

One question in our work is the role of gender. We posited that background experiences with marginalization in mathematics could explain the difference in self efficacy to meet i/sMfM-oriented goals, but it is worth noting that this difference in experience also fell along gender lines. It may be that gendered experiences, instead of solely experiences with marginalization in mathematics, could explain the difference in self-efficacy. Or even that both gendered experiences and marginalization intersect to impact instructors' sense of self efficacy to answer the meaningfulness question in various ways. Another limitation to our work is Archy's lack of experience teaching math for liberal arts; he might develop more examples as he teaches his class, and build his sense of teaching self-efficacy in attending to various meanings, regardless of his lack of experience with marginalization.

The surprising finding, that a historically consistent and strong mathematical identity could be a liability in instructor self-efficacy with addressing students' meaningfulness questions, is worth further exploration. If it is indeed a broader phenomenon, it has important implications for instructors and teacher educators. For instructors with mathematical backgrounds similar to Archy and Benjamin, this finding suggests value in consulting with their peers who may have a stronger understanding of how to bring meaning to the mathematics they teach. For teacher educators, this finding suggests that understanding pre-service teachers' historical mathematical identities is a valuable undertaking that could have important implications for individual pre-service teacher needs in teacher preparation. If the field of math education can address these issues and, thus, better attend to instructor's feelings of self-efficacy in making mathematics meaningful for all students, a future where all students see meaning of and for mathematics in their classes could be within reach.

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Differential Impacts on NSTEM Graduation: Exploring a Multi-Institutional Database

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Calculus is a quintessential "gatekeeper" course to degrees in the natural sciences, technology, engineering, and mathematics (NSTEM). However, first-year courses in chemistry or physics, for example, can also impact an institution's ability to retain high-quality students in the NSTEM pipeline. For women and underrepresented students, these courses may be particularly impactful. Using the Multiple-Institution Database for Investigating Engineering Longitudinal Development (MIDFIELD), we investigated the role of sex and race/ethnicity on graduating with a NSTEM degree while controlling for prior academic experience and students' first experiences with NSTEM courses. This project highlights the utility of and need for databases such as MIDFIELD. We estimate the probabilities of students from various sexes and races/ethnicities of obtaining a NSTEM degree as they intended when starting college.

Keywords: Differential Outcomes, Race/Ethnicity, Gender/Sex, STEM, Graduation

The American job market is poised for an 8% increase in science, technology, engineering, and mathematics (STEM) careers during the 2019-2029 decade, over twice the growth expected for all careers (Zilberman & Ice, 2021). While this is a fantastic opportunity, there are serious hurdles to filling those positions. For instance, we struggle to retain students in the STEM pipeline (Pell, 1996). Indeed, only about half of students who begin college seeking a degree in STEM will attain such a degree within six years (Ehrenberg, 2010). Ellis, et al. (2016) reported that after taking calculus the odds of a woman switching out of a STEM major are 1.5 times those of a comparable man. This hemorrhaging is not exclusive to women as race matters, too. Asai (2020) reported that students classified as science PEERS—persons excluded because of their ethnicity or race—are over-represented at the start of pursing STEM degrees but leave at a much higher rate than non-PEERs.

While numerous studies have explored factors contributing to differential outcomes for women and other under-represented students, they tend to coalesce around topics such as academic preparedness and course grades (Ehrenberg, 2010), and student persistence/confidence (Ellis et al., 2016; Harris et al., 2020; Riegle-Crumb et al., 2019). Academic performance in gateway courses, such as calculus, plays a central role in such studies.

Within this landscape, we sought to address two research questions. First, we explore differential outcomes by sex and race on graduating with a degree in the Natural Sciences, Technology, Engineering, or Mathematics/Statistics (NSTEM), accounting for their academic preparedness as well as their performance *in initial NSTEM courses*. Second, we look at the probability of a student attaining such a degree.

An important aspect of our work is a narrowing of the National Science Foundation's definition of STEM (National Science Foundation & National Center for Science and Engineering Statistics, 2013). Specifically, we restrict to the natural sciences (e.g., physics, chemistry, biology, etc.), instead of all sciences. Thus, the primary difference between NSTEM and STEM is that we have excluded the social sciences (e.g., psychology and sociology) as well as the health sciences (e.g., kinesiology, nursing).

Methods

To explore our research questions, we drew upon the Multiple-Institutions Database for Investigating Engineering Longitudinal Development (MIDFIELD; Ohland & Long, 2016). Currently consisting of 20 partner institutions with engineering programs from across the United States, the database consists of all student records (not just engineering students) who attended each institution as reported by their respective registrars. The oldest records stretch from 1988 all the way to 2018. While there are more than 1.7 million student records in MIDFIELD, we have restricted our attention to students who began college between 2005 and 2012, inclusive, following them to 2018 ($N = 225{,}944$). This time window allowed us to look through the six years after each student started to see if they graduated and with what degrees as well as having consistent ACT and SAT scores (avoiding significant changes in how these scores get calculated). Additionally, this time frame situates the data after Chen (2013) while overlapping with MAA's national calculus student (see Ellis et al., 2016) and many of the studies reported in *Talking about Leaving Revisited* (Seymour & Hunter, 2019).

Given the size of our restricted sample, we used a 70%-30% train/test split for our logistic regression model fitting, stratifying along sex and race/ethnicity. We wish to emphasize that there are important distinctions between sex and gender. While the database uses "sex", there is ambiguity as to whether this attribute refers to sex or gender. In our models we will use the term sex for reproducibility purposes. We controlled for past academic preparation through students' high school GPA (Galla et al., 2019) as well as standardized tests (ACT and SAT). We used ACT composite scores and a concordance table (ACT & College Board, 2009) to convert SAT scores to ACT composite when ACT was missing. In addition to variables for sex and race, we also included variables for students' intent to get a NSTEM degree as well as the number of Ds, Fs, and withdrawals ("W"s) students received in NSTEM courses in their first term (semester or quarter).

To account for the multiple/simultaneous inference problem, we have two inference families. The first are the regression coefficients for the multiple logistic model, which we have controlled using a False Coverage-Statement Rate method for selective interval construction (Benjamini & Yekutieli, 2005) at 5%. This method strikes a balance between frequentist and Bayesian approaches while directly dealing with the selection of important terms in fitting a regression model. Our second family consists of the probability estimates for student profiles using Šidák's method to control the familywise Type I error rate at 5%.

Population

There are 12 participating institutions who provided student records for our selected time frame. However, we dropped one institution when that institution's reported student demographics substantially departed from their publicly available demographics. This left 11 institutions varying from a small, private liberal arts college to public R1 and R2 universities.

Of the 225,944 students (with complete information) in our sample, approximately 13.4% ($n = 30{,}368$) graduated within six years of beginning at their institution with at least one degree in a NSTEM field. Table 1 provides the breakdown of the sample by race and sex. Given the racial demographics in U.S. higher education as a whole, it is unsurprising that the sample is majority white.

Table 1. Student Demographics—Race and Sex.

Race	White	Asian	Black	Hispanic/Latinx	Native American	Total
Male	101,076 (44.73%)	6028 (2.67%)	5,427 (2.4%)	8,160 (3.61%)	1,259 (0.56%)	121,950 (53.97%)
Female	85,871 (38.01%)	4,180 (1.85%)	5,291 (2.34%)	7,328 (3.24%)	1,324 (0.59%)	103,994 (46.03%)
Total	186,947 (82.74%)	10,208 (4.52%)	10,718 (4.74%)	15,488 (6.85%)	2,583 (1.14%)	225,944 (100%)

Students' ACT composite scores varied from a 3 to 36, with a *sample arithmetic mean* (*SAM*) value of 24.46 (*standard deviation* [*SD*] value of 4.02) and a *sample median* value of 24 (*median absolute deviation* [*MAD*] value of 4.45). Their high school GPAs varied from 1.0 to 5.0, values for the *SAM* and *sample median* of 3.36 and 3.4, respectively. The values for the *SD* and *MAD* are 0.57 and 0.58, respectively. We calculated the number of Ds, Fs, and Ws each student received in their first term for core NSTEM courses (mathematics/statistics, chemistry, technology, biology, and physics). The number of DFWs went from a low of 0 to a high of 7 courses. Approximately 83% of the students had no DFW's in NSTEM courses in their first term, another ~13% had only one DFW, while the remaining 4% had two or more DFWs.

Of the students in our sample, approximately 60% (n = 135,387) did not intend to get a NSTEM degree at the onset of their college experience while 40% (n = 90,557) did. Nearly 6% of the total students (n = 13,384) switched into NSTEM programs and 14% of students (n = 31,708) switched out of NSTEM programs.

Results

We present our results in two parts. First, we present information about the multiple logistic regression model and then share results of student profiles. The first part allows us to examine the general impacts of various factors on students attaining a NSTEM degree while the second provides a more interpretable look at the model through profiles of students. In both cases, the response attribute of our model is a student's attainment of a NSTEM degree within six years of starting at their MIDFIELD participating institution.

Model

Our multiple logistic regression model consists of two covariates (ACT composite score, high school GPA), three factors (sex, race, and no NSTEM intent), as well as a potential mediator, the number of NSTEM DFWs in the student's first term. We included several interaction terms (sex and race, NSTEM DFW count by race, NSTEM DFW count by sex) in the model as well. The estimates provided below come from the 70% split of the data. To check the model's fit, we used a receiver operating characteristic (ROC) plot and the area under the curve (AUC; Bowers & Zhou, 2019) shown in Figure 1. The ROC curve and the AUC of 0.841 taken together indicate that our model explains the data well.

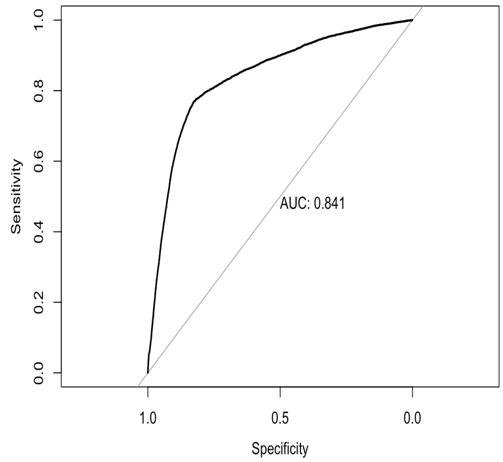

Figure 1. ROC Curve of Fitted Model on Testing Data Set

Table 2. Odds Ratios from Logistic Regression Model

Term	Estimate (Std. Error)	95% Adjusted Conf. Interval
Odds of White, Male, NSTEM Intending, Student with 0 ACT, 0 GPA, and 0 DFWs (Intercept)	0.046 (0.078)	(0.040, 0.53)
High School GPA	2.173 (0.017)	(2.105, 2.44)
No NSTEM Intent	0.070 (0.019)	(0.068, 0.73)
NSTEM DFW Count	0.589 (0.024)	(0.563, 0.616)
Female	0.751 (0.020)	(0.724, 0.780)
Hispanic/Latinx	0.720 (0.051)	(0.655, 0.792)
Black	0.652 (0.074)	(0.568, 0.749)
ACT Composite Score	1.008 (0.002)	(1.003, 1.012)
Black and NSTEM DFW Count	0.769 (0.102)	(0.636, 0.930)
Native American	0.741 (0.131)	(0.581, 0.945)
Female and Hispanic/Latinx	1.180 (0.075)	(1.027, 1.356)
Female and Asian	1.184 (0.077)	(1.027, 1.366)
Female and Black	1.160 (0.102)	Non-significant
Asian	0.933 (0.048)	Non-significant
Asian and NSTEM DFW Count	1.109 (0.074)	Non-significant
Native American and NSTEM DFW Count	0.905 (0.186)	Non-significant
Female and Native American	0.902 (0.193)	Non-significant
Female and NSTEM DFW Count	1.013 (0.039)	Non-significant
Hispanic/Latinx and NSTEM DFW Count	1.004 (0.076)	Non-significant

Note: Based on n = 158,156 observations (~70% split).

Table 2 presents the point estimates from our resulting logistic regression model. We've transformed the regression coefficients from log odds to odds ratios except for the intercept. This term's estimate represents the odds of the reference class attaining a NSTEM degree vs not; that is, a white, male student who intends to get a NSTEM degree with a high school GPA of 0, an ACT Composite score of 0, and no NSTEM DFWs in their first term. All remaining estimates are odds ratios. We've reported adjusted confidence intervals for those terms which are selected as significant following the Benjamini-Hochberg procedure in conjunction with the Benjamini-Yekutieli procedure for a false coverage statement of 5% (i.e., the probability of constructing an interval which does not cover the true value is no more than 5%; Benjamini & Yekutieli, 2005).

From Table 2, we notice several things. First, consistent with Galla, et al. (2019) a student's high school GPA is a much stronger predictor of attaining a NSTEM degree than their ACT composite score. Second, holding all other factors constant, a woman student's odds of attaining a NSTEM degree are only 0.751 times as large as a similar man student's. This is consistent with Ellis, et al.'s (2016) finding that women students were 1.5 times as likely to switch as comparable men students. Our estimate is equivalent to an estimate of 1.33 in Ellis et al.'s framing. Third, while the number of DFWs in NSTEM courses matter in attaining a NSTEM degree, race and sex are still conditionally associated with NSTEM degree attainment above and beyond DFW counts.

Student Profiles

While the logistic model can offer some insights, odds ratios are not very intuitive, especially when you wish to vary multiple factors and covariates simultaneously. However, we can use the model to estimate the probability of various students attaining a NSTEM degree within six years. For our profiles of NSTEM degree intending students, we used a fixed high school GPA of 3.5 and ACT composite score of 26 as these are consistent with the values of the *SAM* and *sample median* of NSTEM intending students in our sample. The resulting profiles (varying race, sex, and number of NSTEM DFWs in the first term) appear in Table 3.

Table 3. Predicted Probabilities of Attaining NSTEM Degree

		Number of NSTEM DFWs		
Race	Sex	0	1	2
White	Female	0.388 (0.375, 0.401)	0.274 (0.254, 0.295)	0.184 (0.154, 0.214)
Asian	Female	0.412 (0.366, 0.458)	0.317 (0.259, 0.374)	0.234 (0.151, 0.318)
Black	Female	0.324 (0.273, 0.376)	0.180 (0.131, 0.230)	0.092 (0.041, 0.142)
Hispanic/Latinx	Female	0.350 (0.310, 0.390)	0.244 (0.196, 0.292)	0.162 (0.099, 0.225)
Native American	Female	0.298 (0.199, 0.396)	0.186 (0.088, 0.284)	0.110 (0, 0.221)
White	Male	0.458 (0.447, 0.469)	0.332 (0.315, 0.349)	0.226 (0.201, 0.252)
Asian	Male	0.441 (0.404, 0.477)	0.340 (0.286, 0.393)	0.251 (0.169, 0.334)
Black	Male	0.355 (0.302, 0.408)	0.200 (0.144, 0.255)	0.101 (0.044, 0.159)
Hispanic/Latinx	Male	0.378 (0.341, 0.415)	0.264 (0.216, 0.312)	0.175 (0.110, 0.241)
Native American	Male	0.385 (0.288, 0.482)	0.250 (0.134, 0.366)	0.151 (0.006, 0.296)

Table values are the predicted probabilities with 95% Šidák corrected confidence intervals.

The predicted probabilities in Table 3 paint a stark picture: not even the advantaged white males who intended to get a NSTEM degree when they started college reach the status of a fair coin flip for attaining a NSTEM degree. Across the board, women students are less likely than men students of the same race to attain an NSTEM degree (odds ratios: White students—0.75, Asian—0.89, Black—0.87, Hispanic/Latinx—0.89, Native American—0.68).

The number of DFWs in NSTEM courses do not affect students equally. White and Asian men students who have at least one DFW in NSTEM courses in the first term have a higher probability of attaining a NSTEM degree than Black and Native American women students who don't have any DFWs. Black and Native American women and Black men with 2 DFWs in their first term will obtain a NSTEM degree no more than 11% of the time. However, for White and Asian men as well as Asian women with the same number of DFWs will get NSTEM degree more than 20% of the time.

Discussion

This study leveraged a large, multi-institutional database (i.e., big data) to explore factors contributing to differential outcomes in NSTEM degree attainment. However, there are several important limitations to this work.

Limitations

An important limitation is that our models are not causal in nature. Our work might best be viewed through the lens of exploratory data analysis (Behrens, 1997) and is observational in nature. Further, our sample does not constitute a random sample of students or institutions. Institutions self-select to participate in the MIDFIELD program and students self-select universities. The sheer size of our sample brings benefits that small, random samples do not. As comparisons, Chen (2013) used a sample of 13,400 students, Ellis, et al. (2016) used 2,266 students, and Riegle-Crumb, et al. (2019) used a high of 4,828 students for some of their models. The size of our 70% training set (n = 158,156) is nearly 12, 70, and 33 times as large as the aforementioned sample sizes, respectively. While large sample sizes are not a replacement for random sampling (e.g., there is still the existence of bias in big data), they do come with the risk of making Type I errors based off the sample size and not any actual effect from the factors/predictors. We have guarded against this by using a training/test split and adjustments for multiple comparisons. While the previous work is important and contributes to our collective understandings of (N)STEM disparities, the prior three studies did not account for multiple comparisons (or, if they did, made no mention of it); a weakness we sought to overcome.

This large sample size would not have been possible without the MIDFIELD data. However, such a database comes with its own limitations. Participating institutions use their own idiosyncratic methods for coding student data. While the data stewards do what they can, using such a large-scale database does require considerable time in cleaning/processing the data. For example, within the race attribute of students we encountered two categories which we omitted from our analysis: Other/Unknown and International. These two categories are too vague for interpretable use of even though there might be important effects related to underrepresented minorities within them. (A similar Other/Unknown category appeared for sex.) Second, high school GPAs exist on at least two scales: 4.0 and 5.0. It is unclear whether institutions converted all GPAs to the same scale before sending their data to MIDFIED. After much internal discussion, we opted to work with GPAs as reported.

A final limitation we wish to mention relates to both the database and our response attribute (attaining a NSTEM degree within six years from the institution they started at). While using six years is a ubiquitous practice, perhaps future work could re-examine such a limit. More importantly though is that the MIDFIELD database is constrained by what *registrars* can report in anonymized ways. Other longitudinal studies make use of *student self-report* data collection methods which allow for knowing whether the student did graduate (up to study attrition). For our data, once a student leaves a MIDFIELD participating institution, they effectively vanish as registrars are not going/able to track such a student outside of that institution. A NSTEM intending student could start at a participating institution but then transfer to a non-participating one and still graduate with a NSTEM degree. In terms of a generalizable model, such a student would be ultimately misclassified as not attaining a NSTEM degree. However, from the institutional level perspective, they are not misclassified. This is an important limitation of the current work. We see such an issue as an opportunity for the development of databases such as MIDFIELD as well as push for better data collection/standardization and sharing.

Key Points

Our present work highlights the utility of large, multi-institutional databases for modeling issues in education, especially as related to issues of diversity, equity, and inclusion. When working at a single institutional level, the number of students in a demographic category might be 0 at worst and small number at best. This often leads researchers to collapse many categories together and/or forgo intersectional analyses of overlapping traits (e.g., race and sex). For example, at one of the institutions included in this study, there were only two black female students who began college between 2005 and 2012. Using the MIDFIELD database allowed us to overcome these issues.

Our resulting model, while observational, is in alignment with the results of several other studies (e.g., Chen, 2013; Ellis et al., 2016; Riegle-Crumb et al., 2019; Seymour & Hunter, 2019). Our work also expands the prior work not only by using MIDFIELD data, but also moving away from the typical notion of gateway/gatekeeper courses (e.g., calculus). Rather than exclusively looking at students' grades in just Calculus I, we looked at students' grades in all the NSTEM courses they took in their first term. Not only does this bring in the potential of gatekeepers directly in computer science, engineering, physics, chemistry, and biology, this expands the mathematics/statistics side to include courses such as college algebra and pre-calculus. This expansion allows us to better capture the impact of students' first experiences in college/university NSTEM courses, especially given that higher proportions of under-represented minority students often begin in developmental/remedial math courses (Chen, 2016). Our results suggest that we can't just focus single mindedly on addressing issues in Calculus I; we need to take a broader view of the experiences of our NSTEM intending students.

Our model also highlights the effects of sex and race which are not attributable to course grades. We take this to signal that those efforts focusing on academic support for women and under-represented minority students might only see marginal gains. We believe that for successful transformations, institutions must consider students holistically as well as critically examining and addressing their own systemic biases. The probabilities of attaining a NSTEM degree for various student profiles provides a useful tool for institutions. Using their own data, institutions can calculate the proportions of various student profiles that attain a NSTEM degree and then compare those proportions to the values in Table 3. While a blunt instrument, such comparisons can be eye-opening to institutional leaders/administrators, bringing the issue of differential impacts of NSTEM degree attainment home.

A final note we wish to make is that our results appear to be in line with Galla, et al. (2019) for the usefulness of standardized test scores in predicting the attainment of a NSTEM degree. While not a focus of this work, our results raise new questions for exploration and continued calls to question whether the benefits of using standardized tests such as ACT/SAT outweigh their (historical) systemic, racial problems (Rosales & Walker, 2021).

Researchers have documented differential outcomes within the (N)STEM pipeline for many years. Our work affirms and expands the previous research in this important arena by using a large, multi-institutional database. However, this work is but a first step. We need to take the knowledge from studies such as this and begin working towards meaningful changes at multiple levels (class, department, institutional, societal) to fully tap into the creativity and talent which leave the (N)STEM pipeline.

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The Practice of Naming and its Role in the Collective Productive Struggle of an Undergraduate Summer Research Community

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We explore productive struggle as a disciplinary practice in the context of undergraduate summer research. In contrast to the classroom, students and professors collectively engage in productive struggle, providing a rich context to study how professors model and support students in appropriating the nuances of this practice. Results elevate a particular sub-practice of productive struggle, that of naming, which the summer research group used to jointly clarify and negotiate their differing understandings. We identify the different components of naming and discuss how they support collective productive struggle (CPS). While naming is a mathematical practice familiar to most mathematicians, taking the lens of CPS to analyze this process elevated the important role that discourse and other social mechanisms have in perseverance. Comparing different instances of naming, we document how such a process supported the group in grappling with both conceptual and symbolic ideas and ultimately establishing a unified understanding.

Productive struggle has been identified as a key component of learning mathematics with understanding (Hiebert & Grouws, 2007). To make sense of the mathematical structure of problems presented as well as the underlying mathematical concepts, students must be given opportunities to explore and navigate situations where the path forward is not immediately apparent (NCTM, 2014). Such authentic experiences promote deeper, more flexible reasoning and problem solving skills. Students develop the ability to draw on their own resources to resolve problematic situations rather than resorting to the authority of others. Moreover, engaging in productive struggle fosters in students a positive attitude about learning and a confidence in their own self-efficacy as learners (Jackson & Lambert, 2010; Hassi & Laursen, 2015). This results in an increase in student empowerment, as students are able to see their own role in generating mathematical insight instead of viewing mathematics as a discipline that only involves the application of procedures and the ideas of others.

To date, most studies have focused on the instructional methods and environments that teachers use to foster and support productive struggle. Researchers have explored and documented how teachers navigate the natural tension that exists between maintaining cognitive demand and simultaneously encouraging students to engage in productive struggle. For example, several studies have highlighted the type of questioning that supports productive struggle (e.g. Freeburn & Arbaugh, 2017). Warshauer (2014), offering a more comprehensive framework, characterized different types of struggle and offered up a continuum of possible teacher responses that vary in the level of cognitive demand they support. This detailed study illustrated the situatedness of productive struggle and how it differs depending on the students, the task, and the norms of the classroom. Notably, when these situational factors overwhelmed students, teachers responded with direct guidance, often telling students what to do and lowering the complexity of the mathematical task.

Productive Struggle in the Context of Undergraduate Research

In contrast to previous work, we chose to explore productive struggle in the context of undergraduate research. Such engaged learning programs have become increasingly popular in undergraduate institutions. However, these programs create an experience of productive struggle for students that differs from the typical instructional setting. In a classroom, the teacher is already familiar with the solution method and must judiciously decide when and how to offer support to students (NCTM, 2014). In a mathematics research setting, the path forward and even the ultimate destination are often unclear to both the students and the professors. As such, instead of being tasked to create an environment that supports productive struggle for the class, the professors themselves experience struggle along with the students. Consequently, in an undergraduate research setting, productive struggle arises naturally and manifests itself as a quintessential disciplinary practice which professors engage in firsthand, modeling how they react and overcome the challenges presented. This type of authentic participation in productive struggle highlights how summer research takes place in a community of practice (Wenger, 1998). In contrast to a classroom, understanding of productive struggle in a research setting occurs as the group works together towards a common goal. Students appropriate the norms and strategies associated with productive struggle as they interact with the professors, grappling together with how to productively move forward.

Research Question

Leveraging this context, we endeavored to understand how such a community collectively engages in the disciplinary practice of productive struggle. Specifically, we explored the research question: How does the practice of collective productive struggle manifest itself in an undergraduate mathematics research setting?

Theoretical Framework

An individual's willingness to engage in productive struggle is usually viewed as a personal disposition, denoted by a tendency to embrace struggle or an appreciation that struggle is a key component of learning (e.g. NCTM, 2014). For example, DiNapoli (2018) offered a detailed characterization of productive struggle, distinguishing it from other character traits such as grit and persistence. He maintained that productive struggle, like perseverance, is characterized by the flexibility to switch tactics and improvise to accommodate in-the-moment challenges. Moreover, he analyzed productive struggle through the lens of the individual, choosing not to incorporate the contextual features that shape how individuals respond to adversity.

No doubt the community and its norms provide resources from which we draw on to persevere. As Warshauer (2014) noted, the nature of student struggles seemed related to the sociomathematical norms that were in place in the different classrooms she studied. This finding suggests that the sociocultural practices established in the classroom, which impact and mediate students' learning behaviors (Yackel & Cobb, 1996), should be expanded to include productive struggle.

Collective Productive Struggle

In contrast to the individual perspective, Sengupta-Irving and Agarwal (2017) conceptualized productive struggle as a collective enterprise in a middle school classroom setting. They documented the interplay between group members, illustrating how the collective group supported each other in engaging in productive struggle. Students problem-solved independently, but repeatedly reconvened, to share ideas and draw upon their collective thinking to move forward together. Taking such a collective lens, they documented the teacher's impact on groups' collective engagement in struggle. Expanding on Vygotsky's notion of Zone of Proximal Development (ZPD), they leveraged the "collaborative ZPD" (Goos, Galbraith, & Renshaw, 2002) to emphasize that benefits of interaction are bi-directional between all members of a community. A collaborative community does not simply expand the problem-solving ability of the novices, but that of the entire group.

As student research takes place in a community of practice, we adopt a collective view of productive struggle. By taking a broader view of productive struggle, we aimed to explore the environmental and social resources that summer research groups draw upon and use to support each other in productive struggle.

Naming

Our exploration within a collective research context elevated one particular sub-practice, which we refer to as naming. Naming involves group members initiating a conversation to negotiate and define a mathematical idea that has collectively emerged as significant, but that has not been explicitly discussed. As Hewitt (2007) wrote in describing his theory of symbolic interactionism, names are linguistic symbols that are social in nature. They draw their meaning from a community adopting them and using them in a consistent way. Symbols transform abstract ideas into concrete entities, giving rise to a new form of social interaction and enabling the transference of ideas (Hewitt, 2007). As such, in the context of mathematical research, the act of naming not only supports the group in communicating their divergent ideas with each other, it serves to establish a new mathematical idea, embedding the dense often divergent thinking into a single symbol or name. Moreover, names serve as a repository of those ideas that have proven important for a particular community (Hewitt, 2007). The process of naming signals to a research community a productive idea, focusing their attention and creating a newly created tool to use moving forward.

Methods

This study takes place during a summer research program in a mathematics department at a small liberal arts university. This program supports a wide range of projects in pure and applied math and the groups involved vary in size, experience, and community norms. After observing multiple small research groups, we focused our analysis on one group that was attempting to determine the minimal number of time steps required to move a fixed number of objects along the edges of a directed graph. This group consisted of two students, a female rising junior, Allison, and a male rising senior, Bobby, as well as a female and male professor, Dr. Alister and Dr. Brown, respectively. We focused our analysis on this group because the students and professors were highly collaborative in their approach. Not only did they discuss their work aloud, they used a large white board to record their collective ideas and support their interactions. Such public communication, both verbal and written, allowed us to observe their collective thinking. Moreover, from the very first observations, it was clear the group had established strong norms, where they listened and built off each other's ideas, providing a rich context to explore their mutual engagement in productive struggle.

Data was collected over the last three weeks of an eight-week, daily summer research project. To understand how the group collectively engaged in productive struggle, we observed and filmed the research group working together each day, two hours with all four group members and two hours with the students only. To understand how the group collectively engaged in productive struggle, we identified specific moments when the group worked together to overcome a challenge that emerged.

Analysis

One significant moment emerged through this process in which the two students and two faculty members collectively grappled with naming and defining a particular concept that had emerged. Three members of our research team reviewed this instance and met to discuss observations and to identify salient moments within the instance that seemed to support the group in productively moving forward. We then returned to the data to look for other cases of naming. Taking this lens, two other instances were identified. To understand the different components of naming and their apparent purposes for the group, we compared these three instances looking for commonalities and differences. Contrasting these moments enabled us to identify several significant markers in the collective naming process (see Table 1), as well as two distinct classes of naming instances (conceptual naming and symbolic naming). This analysis resulted in a framework for classifying critical stages in the naming process that highlights the manner in which naming--a common practice in mathematical research--supports collective productive struggle. We describe this framework in detail in the results section that follows.

Table 1. Stages in the naming process and their presence in naming exemplars

Stages	LMC	W	C-1-2
1. Elevation of the concept	✓	✓	✓
2. Negotiation			
• Establishment of collective definition	✓	✓	
• Discussion of importance	✓		✓
• Discussion and choice of name	✓	✓	✓
3. Adoption of name	✓		

Results

The first interaction consisted of the research group naming what they called a “leftmost minimal cut” (LMC). This was an example of conceptual naming and was the most robust exchange of the three instances. Notably, it was the only interaction in which group members engaged in all stages of the framework (see Table 1) and the only one in which all group members participated. The second interaction focused on the concept which the researchers referred to as “waits” (W). This interaction involved only the two students and illustrated their appropriation of the practice. While similar to LMC, the W interaction exhibits evidence of only parts of the naming framework. The final interaction, which we call C-1-2, is the only example of what we identified as symbolic naming. While this example differs from the other two instances in both representation and substance, it is similar in that it involves the group attaching a single name to a fundamental notion or in this case a reoccurring algebraic expression.

In all of these instances, the group possessed what we call a preliminary collective understanding. One researcher’s understanding of a notion that is central to the mathematical exploration of the group may not always completely coincide with the understanding of another researcher within the same group. We thus view the preliminary collective understanding as the intersection of the individual conceptualizations of the idea being named.

The first stage of the naming process is marked by an assertion and shared understanding that the idea being named is central to the mathematical exploration; evidence of this stage often includes repeated references to the concept during conversation or problem solving. In the LMC interaction, we can see the idea first introduced when Dr. Alister interjected, “You know, I wonder if there’s something to that.” Shortly afterwards, Dr. Brown elevated the same idea, stating, “The beauty of the minimal cut, though, is…” and “[the] important thing [is], whether or not [a minimal cut] occurs, you know?” In all of these examples, various group members interjected in an attempt to focus the attention of the group on a concept they saw as central to the mathematical investigation with which they were engaged. At this point in the process, group members possessed different conceptualizations and ways of reasoning about the idea at hand. While the repeated inclusion of the idea elevated the concept, the individual group members’ understanding of the concept remained mainly implicit. In fact, Stage 1 results in the (often diverging) individual conceptualizations being made public. Notably, all three instances of naming we analyzed contain evidence of this stage. As such, this initial stage appears essential to the naming process; without it, there is no motivation for the group to engage further in naming the concept, as the concept is not sufficiently elevated to warrant a name.

Stage 2 is characterized by a collective negotiation of the concept. We identified several components that support this process: establishment of collective definition, discussion of importance, and discussion and choice of name. This negotiation process can be quite disjointed as the meandering discussion in the LMC interaction exemplifies. At the beginning of this stage the different members of the group attempted to come up with an initial collective definition. Dr. Brown revoiced his understanding of Dr. Alister’s interpretation, saying “Just so I understand what you’re saying [...],” and Dr. Alister’s reply with the question, “What’s the definition of minimal?” These different interjections illustrate the group’s attempt to arrive at consensus. The negotiation ultimately arrived at the following exchange, where it is clear that the group had come to an initial agreement about the concept being defined:

Dr. Alister: I am changing the definition[...]
Dr. Brown: No don’t change the definition, come up with a new definition
Dr. Alister: This is a new definition that I don’t have the terminology for yet
Bobby: Can we call it a minimal ideal cut, maybe?
Dr. Alister: Why is it ideal?
Bobby: Because it is as far back as we can do it and that’s how you really define it.
Dr. Alister: Ok [...] be careful because ideal means something else in algebra
Bobby: Ah ok nevermind
Dr. Brown: So let’s just be clear, so minimal [...]
Dr. Alister: What about a quasi-min? Or pseudo-min?
Dr. Brown: Let me ask you this question though, if you have a minimal cut, why could that ever be bad?

This interaction also included a second part of the negotiation stage, discussion of importance. During this component of the negotiation phase, participants discussed what makes the concept central to the mathematical exploration and in doing so, identified the key aspects of the concept that they believed should be reflected in the name. Dr. Brown’s questioning of the importance of this concept at the end of this exchange illustrates this reflection.

The exchange above also illustrates a third part of the negotiation stage: discussion and choice of name in which members of the group propose various names and collectively consider the affordances, drawbacks, and implications of each name. As the exchange above illustrates, a critical part of this stage is the group intentionally selecting a name which they believe conveys important information about the mathematical idea they are discussing and avoids a name that could potentially be misleading. After the negotiation above, the group finally arrived at the name of "leftmost minimal cut" as we see in the following, which occurred shortly after the previous exchange.

Allison: So are we just saying `find one and …'

Dr. Alister: Well [Bobby] is suggesting that because of the special properties of these graphs, that maybe if you find one, you can adapt it [...] so in some sense you achieve a ...

Dr. Brown: ...Leftmost cut...

Dr. Alister: ...A leftmost minimal cut

We highlight that the negotiation stage is inherently fragmented as the discussion can meander among various tangential ideas with different components occurring simultaneously or out of order. For example, at the end of the exchange above involving the choice of name, Dr. Brown brought the group back to a discussion about the importance of the concept by asking them to think about whether this particular type of cut is really important to the exploration. This comment encouraged the other researchers in the group to question the significance of the particular type of minimal cut they were attempting to define, even after they had begun proposing names.

Finally, Stage 3 is an ongoing stage marked by the adoption of the chosen name by all group members. Following the LMC interaction, the name "leftmost minimal cut" was employed by all group members whenever the notion was being discussed. However, as the following exchange later the same morning illustrates, the group was still working to internalize the name and concept.

Dr. Brown: If you had minimal cuts and you had a notion of a…

Dr. Alister: ...Leftmost minimal?

Dr. Brown: ...Leftmost or topmost, does that solve the problem?

Notice that Dr. Alister used the name established earlier, leftmost minimal cut, despite the fact that the group was examining a graph that was oriented differently than their previously examined graphs, and thus the notion of "leftmost" was in fact "topmost." However, all group members tacitly accepted the name "leftmost," even though it did not reflect the more general case they were discussing. It seemed that since all group members were fully involved in the earlier LMC interaction, the name "leftmost minimal cut" did not lead to any discrepancies or confusion despite the name failing to accurately represent the notion being considered. Such acceptance suggests that the earlier discussion around the meaning and importance had established a strong, collective understanding of the concept.

It is also noteworthy that a group can engage in the naming process without engaging in all of the stages. For example, in the W interaction, Allison and Bobby did not discuss the importance of the name (Stage 3) nor collectively adopt the name (Stage 5). In the exchange below we see in Allison's first comment that the group was engaging in Stage 1 and began the negotiation. Bobby also participated in the negotiation process, by suggesting names.

Allison: Should we say, like, stretch or, like, add... how can we explain adding those loops?

Bobby: I've been thinking of them as w-a-i-t, waits but you could also think of them...

Allison: I think it's so funny because we have so many different meanings for so many different words

Bobby: yeah add stalling, I guess, or wait

However, the establishment of a collective definition was strikingly less intentional in this interaction than in the LMC interaction. The two students discussed the name, but while the potential for misunderstanding was acknowledged ("I think it's so funny because we have so many different meanings for so many different words"), it was never resolved. This interaction also did not reach the "adoption of name" stage: Bobby continued to use the name he proposed (waits), despite Allison's apparent confusion about the name he was using. Ultimately, both referred to the same concept using different names (Bobby using the word waits while Allison used the word pauses), even within the same presentation to researchers outside of the group. Though the negotiation stage was not as robust in this instance (missing the discussion of importance) and the group failed to engage in the adoption of a name, the interaction still supported productive struggle by indicating Bobby's understanding of the concept given his name suggestions, and displaying a potential misunderstanding.

Similarly, the C-1-2 interaction is missing some stages, but still supports productive struggle. Specifically, the C-1-2 interaction did not include an explicit discussion of the importance of the concept being named, nor did it include an adoption of the name by group members; the conclusion by the participants in the interaction was that the concept was not fundamental to the mathematical exploration and this symbolic naming was only motivated by a need to save space on the board (Bobby says, "we can name it, sure, but we're gonna get rid of it in just a second," to which Allison replies, "well we're just running out of space.") The discussion and choice of name in the C-1-2 interaction is also less substantial than in the LMC interaction, possibly precisely because it is an example of symbolic naming, which seems to require less discussion of implications associated with possible names.

Discussion

Naming is a mathematical practice familiar to most mathematicians. As demonstrated by the various exchanges of the research group, the act of naming signals the perceived importance of a concept, brings to light possible divergent understandings among group member, and results in an expansion of their collective understanding. This work sheds light on the process of naming by decomposing the practice and identifying constituent parts. Such analysis highlights the critical role that discourse and other social mechanisms play in collective problem solving and in supporting perseverance.

While it is notable that the students found value in this naming practice and adopted it to support their problem solving in the one case we analyzed in which they worked independently, our investigation revealed they omitted stages that were present in the interactions involving the professors. We suggest that the disciplinary practice of naming and the understanding of its importance and effectiveness was not fully established in the mathematical problem-solving toolkits of the students. Notably, the student researchers each continued to refer to the notion using their own preferred nomenclature, even using their own preferred terms within the same presentation to other researchers in the program outside of their research group. We propose that the missing stages help to establish a common understanding of the concept. This supports our claim that these stages are critical in supporting group progress and we encourage other research groups to use this framework to guide their work in engaging in collective productive struggle. We suggest that professors endeavor to demonstrate and instill the practice of naming in their students, conscious of the need for a robust discussion for the naming to be an effective tool.

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How Students Learn Math Best: Tutors' Beliefs about Themselves Versus Their Tutees

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Undergraduate and graduate tutors play an important role for students who utilize math tutoring centers. Here we draw on our previous work on our modified version of the Teacher Beliefs Interview (TBI) protocol by Luft and Roehrig (2007) to assess mathematics tutor beliefs (Pilgrim et al., 2020). While validating item wording through cognitive interviews, we noted tutors had different beliefs for how their students learn math best compared to how the tutors themselves learn math best. This paper investigates this difference and describes the process, coding, and results of this qualitative data as well as implications for tutor professional development programs.

Keywords: Tutors, Belief, Practice, Professional Development

Of 105 higher education institutions surveyed in 2015, 102 indicated that they had a tutoring center—of these, 87% had undergraduate tutors and 33% had graduate tutors (Johnson & Hanson, 2015). In another study specifically on higher education mathematics centers, 96% of the 75 centers utilized undergraduate tutors and 65% used graduate tutors, some of whom also had instructional responsibilities as graduate teaching assistants (Mills et al., 2020). Both studies show that undergraduate and graduate tutors are commonly involved in providing academic support to undergraduate students. Their work in tutoring centers can impact academic outcomes such as tutees' final course grades, persistence, retention, understanding, and academic confidence (Colver & Fry, 2016; Kostecki & Bers, 2008; Rasmussen et al., 2014). However, tutors' personal views and biases influence how they perceive their tutees, which can then affect their approaches to tutoring including session goals and instructional decisions.

In one study by Derry and Potts (1998), tutors were shown to use adaptive modifications to tutoring sessions in terms of content, pedagogy, and complexity based on their categorizations of tutees. This study's tutoring context was different from the current study in that three of the five tutors were assisting tutees with a computer-based instructional system, and only four out of the five tutors were tutoring math. Derry and Potts also utilized Kelly's (1955) personal construct theory to identify constructs that the individual tutors used to think about tutoring sessions. In this context, *constructs* can be thought of as "major components or elements" that tutors use to distinguish tutees (p. 70). Overall, the two main constructs or types of classification had to do with motivation and intellectual ability.

Having an adaptive tutoring style can be desirable, but it could be also problematic if a tutor is incorrectly categorizing a tutee or making poor tutoring decisions. Because of this, understanding tutor beliefs and designing tutor professional development are important for the success of a tutoring center. In particular, tutors' reflection on and awareness of their own beliefs and biases can initiate reform to their pedagogical practices (Nardi et al., 2005).

To improve tutor professional development, we first need a way to measure beliefs. Our tutor beliefs research group has modified the Teacher Beliefs Interview (TBI) protocol (Luft & Roehrig, 2007), a semi-structured interview protocol designed to "explore the beliefs of beginning secondary science teachers" (p. 38), in order to create an open ended survey instrument that assesses the beliefs of undergraduate and graduate tutors. Our goal with the modified TBI is to assess changes in tutor beliefs over time as a result of experience and professional development. In our pilot study, we utilized seven TBI open-response items, and we coded the data in a way similar to Luft and Roehrig but adapted for the tutoring context (Pilgrim et al., 2020). Responses were coded on a spectrum (Figure 1) that ranged from *instructive* (i.e., more traditional approaches to instruction) to *adaptive* (i.e., more closely aligned with reform methods of instruction). During this process, we found that some tutor responses contained excerpts that spanned across the codes including ones that are not adjacent on the spectrum. This highlights the complexity of responses for general tutor beliefs. We will provide detailed examples in the next two sections.

Figure 1. Codes for the qualitative responses.

Before deploying our modified TBI questions to tutors at higher education institutions across the United States, we wanted to assess the validity of our items through cognitive interviews (Desimone & Carlson Le Floch, 2004). For this article, we focus on the question of how students (including tutors) learn math best, which is a component of our larger project's research question: what are the mathematical beliefs and practices of graduate and undergraduate tutors? In the next sections, we describe the cognitive interview process, details of coding, findings, and implications about tutor professional development.

Methods

We conducted two rounds of three cognitive interviews (i.e. six interviews total) to test the validity of the TBI item wording. This allowed us to ensure that participants were interpreting questions with consistency and were considering similar contexts while responding to the items (Desimone & Carlson Le Floch, 2004). The first three participants who voluntarily responded to a recruitment email were part of the first round of interviews and the next three individuals who responded were part of the second round of interviews. All interviews were conducted over Zoom. They started with participants reading the TBI items and then verbalizing their streams of consciousness while interpreting the questions, recalling relevant experiences, and determining their answers. Before the interviews, we predicted where item wordings may be problematic, and we preemptively created follow up questions addressing these concerns. However, spontaneous follow up questions were additionally asked when the interviewer wanted clarification or elaboration on a previous response (Betty, 2004). These interviews were recorded and transcribed for later analysis.

After the first round of interviews, we made a modification when the transcripts indicated participants were not interpreting and answering the questions as intended or with similar contexts in mind. The modified item wordings were then used in the second round of cognitive interviews. All participants attended the same urban, public research institution. Tutors 2, 3 and 4 were undergraduate tutors who regularly worked in the campus math tutoring center and who led precalculus review sessions. Tutors 1, 5 and 6 were graduate students who worked in the campus math tutoring center in addition to holding office hours for college algebra or precalculus. For simplicity, we refer to both graduate and undergraduate tutors when we say tutors.

One of the items on our modified TBI was, "How do students learn math best?" (Our modification of the original TBI merely replaced "science" with "math"). We were initially concerned that some respondents may interpret the group "students" to include themselves since they were current undergraduate or graduate students. We thought that some might also restrict "students" to the group of students they tutor, thereby excluding themselves. Because these two interpretations could influence the way respondents were answering the question, it could influence the validity of claims based on responses to this TBI item. Therefore, in the first round of cognitive interviews, we asked the follow up question, "Did you include yourself in the group 'students' when originally answering this question?" In this case, all three participants answered, "no," so we further asked if their answer would change if they did include themselves. When all three of these participants did change their response, we decided to modify the protocol. In the second round of cognitive interviews, we first asked, "How do students learn math best?" and immediately followed with, "How do you learn math best?" In both rounds of interviews, there was a qualitative difference between the tutors' responses to the two questions, so we investigated this further using the coding scheme previously developed in Pilgrim et al. (2020).

In addition to the aforementioned spectrum, we further describe the details of the codes here. The *instructive* (red solid pattern) code was applied to responses that primarily focused on doing practice problems and mimicking instruction. In contrast, the *adaptive* (blue polka dot pattern) code was applied to responses that involved transferring knowledge to new problems and making conceptual connections. *Transitional* (yellow zig zag pattern) was in between, with such responses being characterized by engaging in mathematics in various ways (e.g., in groups) and going beyond pattern matching or following a recipe to solve problems. Example responses for each of instructive, transitional, and adaptive can be found in Pilgrim et al. (2020). Some responses could not be strictly coded as instructive, transitional, or adaptive. At times, responses constituted a blend (orange horizontal stripe pattern or green vertical stripe pattern) of these categories. Here is one example: "Student[s] learn math best by doing. Some examples of this include group work and homework." This was coded as a blend of instructive and transitional (orange horizontal line pattern) because while there is an aspect of working problems in the response, there is also an awareness that students learn through collaborative group work.

However, there were challenges in applying single codes to some responses, especially lengthy ones. At times, parts of responses encompassed both *instructive* and *adaptive* codes, so averaging to *transitional* did not seem appropriate for the entire excerpt. Thus, for this analysis, we applied multiple codes to responses by pinpointing specific portions of the answers. This was more appropriate for the cognitive interview data given the longer descriptions and because respondents changed their answers when asked about different contexts.

Findings

In both rounds of the cognitive interviews, we noted a difference in tutor descriptions of their tutees' learning in comparison to their own.

Differences in Belief Spectrums

In some cases, the baseline methods for learning mathematics were the same, but the tutors described additional methods for learning when talking about themselves (see Table 1). For instance, Tutor 6 believed that tutees learn math best "by seeing examples and then repeating it themselves," an idea that Tutor 5 also expressed. When describing their own learning, Tutor 5 agreed that "do[ing] a bunch of problems" was still helpful, but they also added that "working collaboratively with others" or asking questions "to fill in the gaps" was helpful too. Tutors 1 and 4 had a more instructive response for the tutees but a wide range of instructional, transitional, and adaptive descriptions for themselves. Tutor 4, for example, gave a broad range of ways to learn for themselves by suggesting that the best ways to learn are to work on practice problems, to check answers, to "understand why it's happening," and to utilize multiple problem-solving methods. In another case, the descriptions of the tutee and tutor learning methods were on completely different ends of the spectrum. For instance, Tutor 2 had an instructive description for the tutees and an adaptive description for the tutors. When asked to describe the learning of tutees, Tutor 2 brought up how "students who don't like math or aren't confident [in] their mathematical abilities do best when they do a lot of practice." Yet, when describing their own learning, Tutor 2 preferred to reflect about the problem, "break the problem into pieces," and "try a different way" as needed. Tutor 3 also had very different responses for tutees versus tutors, but in their case, they primarily described how they were "anti-homework" for their tutees since "bad practice" only leads to students "not understanding it." Since Tutor 3 answered in the negative, we coded their response as not-red/solid pattern and not-orange/horizontal-stripes, but this did not necessarily mean that they had transitional or responsive answers.

Table 1. Summary of cognitive interview codes.

Tutor	"How do students learn math best?"	"How do you learn math best?"
1		
2		
3	Not- Not-	
4		
5		
6		

Tutors Categorize Mathematics Learners into Various Groups

We noted tutors making various, explicit comparisons of two different groups during the six cognitive interviews.

Mathematics Course Level. Sometimes the tutors categorized students into lower level versus upper level courses or undergraduate versus graduate courses. Tutor 1 even described the changes in their own learning when comparing their personal experiences in high school, undergraduate work, and graduate studies. In their case, they found that study groups were useful in graduate school "because everyone has a different perspective." Tutor 1 did not explicitly generalize to all high school and undergraduate students, but they implied that their own experience with groups was not truly collaborative prior to graduate school, so they did not think it was a particularly helpful resource for their tutees. On the other hand, "working independently" on problems is a reliable method of learning math for Tutor 1 across all categorizations. Interestingly, Tutor 1 also referred to how they think "the whole entire curriculum is based on tests" for the tutees. Because they perceived their tutees' classes to be problem- and test-oriented, then it made sense to Tutor 1 that doing problems was the best way to learn in those courses. Tutor 4 did not refer to the curriculum but also echoed Tutor 1's thoughts by saying that doing practice problems "doesn't necessarily give you the same ability to differentiate and use a different method, but, at the very least you can still get the result that you're looking for." This implies that Tutor 4 also thinks the answer is more important than the process for the tutees' coursework.

Types of Learners. Some of the tutors also categorized students into types of learners. For example, both Tutors 1 and 3 mentioned "visual" or "auditory" learners without any prompting from the interviewer. Tutor 3 used these types of learners to talk about their tutoring decisions. They sought to be "super flexible" tutors—if a tutee did not appear to be one type of learner (e.g. "visual learner"), then they would "switch to something else." Tutor 1 also mentioned these categorizations of learners in the context of types of courses. Tutor 1 believed that many people were visual learners, but that the subject of abstract algebra was "pretty auditory," so "being able to articulate math in a high level" is another level of understanding as it is "more like telling a story and how all the pieces come together." In addition, Tutors 2 and 3 also discussed "creative" versus "logical" thinkers. Tutor 2 described themselves as always having been "a logical thinker versus a creative thinker." They associated their logical thinking with their mathematical thinking, and for them, it was their logical thinking that differentiated them from their tutees. In fact, Tutor 2 even said that math learning is "a lot more different for [the tutor] and [their] precalc students." Tutor 3 also mentioned creativity but in a very different way than that of Tutor 2. Tutor 3 said that many of their tutees "feel bad about themselves" when facing math courses and that they "can't be creative" when they are "terrified of failing." Rather than contrasting creativity with logic, Tutor 3 seemed to value creativity in math problem-solving but believed that affective factors held their tutees back from learning math in the best way possible.

Discussion

Although their responses varied on the coding spectrum, the cognitive interview process indicated that our questions about tutor beliefs were consistently interpreted across tutors. In the case of the questions about how students learn math best, tutors had a range of responses from adaptive to instructive, and their answers were different for themselves versus tutees. For multiple tutors, categorizing tutees by math course or by perception on type of learner was important for how the tutors adapted their tutoring sessions.

Our findings are similar to those of Derry and Potts (1998), whose study indicated that experienced tutors largely classified students by motivation level and intellectual ability for the purpose of adapting their tutoring sessions. These categories are different from the ones we observed with our tutors, but there are overlaps in the tutors' decision-making. For instance, Tutor JS in Derry and Potts' study would give more "direct verbal instructions" with "weaker students," which would be coded as instructive for us. On the other hand, Tutor JS would have "discussions of theory" with "stronger students" and would allow "higher ability students" to lead the tutor sessions tasks, which is a more transitional (or possibly a blend of transitional and adaptive) response in our coding scheme. This seems to parallel our Tutor 1's remarks about higher levels of understanding for students in upper level math classes. Additionally, Derry and Potts' Tutor PM "emphasized the goals of independence, understanding, self-evaluation, and enjoyment for all students [...] little direct instruction was given by Tutor PM. Rather, content was conveyed by discussions that employed the conceptual language of mathematics" (p. 83). We would code this excerpt as a more transitional or even adaptive response. Lastly, for three out of the five tutors in the Derry and Potts study, "motivational/affective considerations were reported as more important than judgments of cognitive ability" (p. 95). Although the reasons behind this were not further explored, this is related to our cognitive interview comments from Tutors 1 and 4 about problem- and test-oriented courses. In our case, these two tutors implicitly suggested that tutees' motivations were affected by an instructive course structure. Because of this, the tutors also chose instructive methods in their tutoring sessions to match the course's instructive style.

For some of our tutors, authentic mathematical practices (such as understanding connections between concepts, being able to utilize a variety of methods to solve problems, and working collaboratively in a group) only became necessary for students in upper division or graduate courses. Tutor 1 explicitly indicated that they formed this distinction by examining how undergraduate courses are structured and because of the types of mathematical activities necessary to succeed on assessments. Tutor 1 expressed awareness that many lower division courses merely require students to memorize equations and procedures for solving problems, and that this type of "curriculum is kind of controlling how you teach it." This distinction between lower and upper division undergraduate mathematics courses is not unfounded, as seen in studies on assessments. Multiple studies have found that lower level assessments largely focus on applying procedures and utilizing algorithmic reasoning, while upper level assessments had more emphasis on higher order skills such as evaluating or creative reasoning (Mac an Bhaird et al., 2017; Maciejewski & Merchant, 2016).

It also seems likely that tutees themselves are aware of the requirements for success in lower division undergraduate mathematics courses. This may explain why Tutor 6 said, "I feel like very few students actually care about the why." Maciejewski and Merchant (2016) found that non-expert conceptions of mathematics correlated with superficial, procedural approaches to studying. Furthermore, they found that those conceptions and procedures could result in obtaining a desired course grade in lower level courses but not in higher level courses. On the other hand, expert conceptions of mathematics correlated with deep approaches to studying, which they found to increase the chances to succeed in upper level courses. Additionally, Code and colleagues (2016) found that students moved away from expert-like orientations about mathematics after taking a semester or year of an introductory undergraduate mathematics course, but they reported more expert-like orientations after taking a specialized mathematics course. The implication here is that students who are successful using superficial approaches in lower level courses will likely underperform in upper level courses if they are not supported in changing their studying approaches or their beliefs about mathematics.

There is a tension that exists between what tutors experience as students, what they perceive to be important, and what they may be taught during training and professional development activities. While student-centered approaches to tutoring are recommended by mathematics education research and emphasized in tutor training and professional development, institutional practices can often still reflect teacher-centered instructional practices (Carlson & Rasmussen, 2008; Stains et al., 2018). It may be that these practices are pressuring tutors to engage with students in ways counter to the recommendations from mathematics education literature. Alternatively, it could be that instructors' reformed practices have not been clearly communicated to the tutors, or that tutors' roles haven't been updated to match these practices.

In any case, tutors will likely have varied beliefs and practices reflected in their tutoring, and we advocate for training activities to meet tutors where they are, allowing for differentiation in their professional development process. If a tutor's initial beliefs are aligned with more *instructive* approaches, then activities to support their growth to a more *transitional* approach may be more appropriate than *adaptive* activities, as such tutors may not be ready for this shift. Additionally, tutors may think that some adaptive approaches are only fit for certain tutees, so unpacking these complex beliefs and potential biases can better train tutors to foster more adaptive approaches for all students. Furthermore, we recommend that those running tutor professional development explicitly examine their own tutors' beliefs. It may be that tutors have a limited view of the curriculum, so their beliefs are misaligned with those of the instructors. In this case, professional development activities to align tutor and instructor beliefs may be appropriate. It may also be that tutor beliefs and practices are appropriately reflecting departmental differences between lower division and upper division courses. In this case, it may be appropriate for a department to reevaluate the course structures and assessments to ensure that best practices are available to all students.

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Oral Assessment in University Mathematics: The Role and Variations of Follow-up Questions

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We investigate how four assessors conducted a virtual oral assessment for an Introduction to Proof course when minimal direction was given. Our qualitative analysis of assessors' behavior showed that most, but not all students were asked follow-up questions during their oral assessment. When the assessor engaged students in follow-up questions, we identified four purposes of these questions: to seek details of the thought process, to fill gaps in a student's presentation, to provide the student access to the problem, and to foster deeper learning. Although the assessors had been given minimal guidance on how to follow-up with students during the assessment, there were several instances of assessors asking similar questions. Our findings culminate in suggestions for others wanting to use oral assessments with multiple assessors.

Keywords: Oral assessment, online education, reasoning and proof, technology in assessment

Creating assessments of student knowledge is a critical part of course design. Romagnano (2001) argued that designing an objective mathematical assessment is an impossible task. However, instructors can include a variety of assessments to provide multiple perspectives of their students' knowledge and capabilities (Iannone & Simpson, 2012). By having multiple sources of evidence, instructors are more likely to produce a more accurate picture of students' knowledge and understanding (Romagnano, 2001). While written modes of mathematics assessment are dominant in the English-speaking world, it is well-documented that varied assessment practices add valuable and arguably necessary insights into students' understanding (Iannone & Simpson, 2012). Implementing less-familiar assessments in your classroom comes with advantages and disadvantages to both students and the teacher. In this paper, we will evaluate our implementation of a less-familiar assessment in U.S. higher education mathematics classrooms: oral assessment.

Background

Oral assessment allows an instructor to have a constructive dialog with students. In this alternative to written exams, students can express their knowledge in a variety of ways, and the assessor can ask clarifying questions when needed. Studies have shown several positive aspects of oral assessment, such as helping students develop communication skills (Iannone & Simpson, 2015), being more authentic (Iannone & Simpson, 2015), and being more inclusive (Huxham et al., 2012). For instructors, oral assessment allows them to rephrase a question so that a student can demonstrate more knowledge than may have been evident from a written response (Huxham et al., 2012). For students, providing a verbal explanation may generate deeper understandings of material (Iannone & Simpson, 2015). Additionally, cheating on an oral assessment is much harder than on a written assessment (Huxham et al., 2012; Joughin, 1998). This last benefit is particularly pertinent given the widespread use of virtual instruction and assessment due to the global pandemic. As is demonstrated by the data we present, oral assessments are highly adaptable to the virtual environment with minimal disruption in the implementation and integrity of the assessment.

There are potential drawbacks to oral assessment with overwhelming workload, fairness, and anxiety among them. Oral assessment requires more time to proceed, especially when the class is large. The workload issue could be minimized by conducting a group oral assessment or using multiple assessors. The issue of fairness can arise from the absence of anonymization, variations in students' experiences, and difficulties with post hoc moderation by assessors not present in real time. These issues around fairness were addressed by Iannone and Simpson (2015). Students in their study were concerned about fairness because different assessors would ask different follow-up questions when working on the same problem. Lastly, students' stress and anxiety were more likely to be high because of unfamiliarity with oral assessment (Akimov & Malin, 2020). The data presented here were from a practice quiz that was one of several low-stakes opportunities for students to experience oral assessment throughout the semester.

This exploratory investigation considers the practicalities of this implementation in terms of grading and variation in students' experiences when a virtual setting was used, and limited direction was provided to each assessor. This led us to the following research questions:

1. How consistent are assessment ratings when using a minimally directed oral assessment protocol in an introduction to proof course?
2. How consistent is a post hoc assessment of the same student population?
3. What are the purposes of follow-up questions that students are asked in an oral assessment? What are the differences and similarities in these follow-up questions?

Methods

Participants

Twenty-five of 29 students enrolled in an introduction to proof class at a large, research focused university in the United States agreed to participate. One student was excluded from data analysis because she misunderstood the instructions and prepared for the wrong assessment. The four assessors were all experts in mathematics education with graduate-level backgrounds in pure mathematics: an associate professor who was also the course instructor, a post-doctoral scholar, and two doctoral students. Depending on their mathematics course history, students' may have been familiar with any subset of the assessors as all were instructors of mathematics courses (including the calculus sequence prerequisites) at the institution. All data were anonymized using pseudonyms prior to data analysis.

Procedure

This oral assessment under consideration here is a 'practice quiz' conducted in week 4, one week before their first low-stakes oral assessment that counted toward their course grade. One week ahead of this quiz, students were provided with a set of problems that they would be asked to talk about and to which they could prepare solutions in advance. Each problem on the quiz had multiple parts, and assessors had the freedom to ask students to present any or all parts. This study focuses on the second question of the quiz, presented in Figure 1. This item was chosen as it had the smallest number of parts and was the only item that all assessors used all parts for their assessment with every student.

Before assessing students individually, all assessors met to discuss some general rules such as ethical considerations and options for which problems to ask during the assessment. For example, the first problem on the assessment had four parts covering the set definitions for subset, powerset, union, and intersection. It was discussed that in the interest of time, students need not be required to answer all four parts. Rather, the assessor could ask them to explain their reasoning for part one that focused on unions, and then ask them how the answer changes if the union were changed to an intersection. It was discussed that if a student was stumped, then it was ok to restate the problem or give hints to help the student make progress. If substantial hints were needed, then the student should receive a score of growing or not yet (rather than success) with encouragement and suggestions for future studying. This discussion included how to start the assessment (brief introductions) and how to handle incorrect responses.

2. Concept Check 2. I can state the definition of a Cartesian product and use this definition to describe the elements of the Cartesian product of two given sets.

   (a) List all the elements of the set: $\{x \in \mathbb{R} : x^2 = 2\} \times \{x \in \mathbb{R} : |x| = 2\}$

   (b) Sketch the set $X = \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 9\}$ on the plane $\mathbb{R}^2$. On the second set of axes, shade in the set $\overline{X}$.

Figure 1: Quiz problem 2

The practice quiz was conducted through one-on-one video meetings that lasted approximately 30 minutes. Students shared their prepared responses using the website scratchwork.io; a whiteboard application that allows collaboration with others in real-time. This platform let students display pictures or pdf files, which was important for this study because the students were allowed to prepare their answers and could display and present their responses from those files if they wished.

The course instructor provided all assessors a three-point rubric: *success*, *growing*, and *not yet* that was used to give students immediate feedback on their responses. This was the rubric that would be used to assess their responses on all quizzes throughout the semester. Raw, anonymized grading data and rubrics are available at [redacted].

Analysis

Each member of the research team conducted about one-fourth of the oral assessments. After all data was collected, each member of the research team watched the recordings of the other assessments and scored them independently. An inter-rater reliability analysis revealed that reliability between assessors was very high; there was disagreement between assessors on only two items. This high level of agreement suggests that students' course grades would not be substantially different regardless of who conducted their oral assessment. Numeric or categorical ratings are a small part of the oral assessment picture. To better understand how students' experiences may differ in this setting, we conducted a qualitative analysis of the recordings.

We used an inductive thematic analysis approach in which the codes and themes were driven by the data itself without any pre-existing codes (Braun & Clarke, 2012). The qualitative data came from the 24 video recordings of each one-on-one oral assessment meeting. After watching all videos and choosing to focus on question 2, the relevant parts of the videos were transcribed using online resources (e.g., zoom transcript) and edited as needed. Transcript excerpts presented here have been lightly edited for readability. Each member of the team wrote notes on what they observed during the initial video viewing, such as who assessed which questions and characteristics of follow-up questions. We observed that different assessors asked different follow-up questions and sometimes no follow-up questions. We then proceeded with a more detailed analysis of the transcripts, focusing on how each assessor used followed up questions. This analysis led to the identification of two broad categories and three sub-categories of follow-up questions, and the roles they play in oral assessment.

Results

Purposes of Follow-up Questions

It is natural that assessors would ask different follow-up questions during an oral assessment, especially given the minimal guidance provided. In Table 1, we summarize the types of follow-up questions asked by each assessor. We noticed that some question types were asked by multiple assessors despite no guidance on this. For example, both Paige and Sandy asked questions about how the definition of Cartesian product was applied in the given context, but Nancy and Barb did not. There were follow-up questions that were only asked by one assessor, such as Barb asking about the size of a Cartesian product in general terms.

We first distinguished two broad categories for follow-up questions: directly related to the problem and not directly related to the problem. We found that questions that were directly related to the problem had three main purposes: probing details, filling gaps, and bridging access for students in need. Meanwhile, only Barb asked questions that were not directly related to the problem, such as 'what is the size of $A \times B$ in a general sense?' or 'is $A \times B = B \times A$?' We observed that these types of follow-up questions played a role in fostering learning.

Table 1. Follow-up questions identified among assessors

	Part (a)	Part (b)
Paige	- No follow-up - How Cartesian product works	- Did not assess
Nancy	- How to get elements of each set - Elicitation of missing element(s)	- The length of radii of the circles - Inclusion/exclusion boundary
Sandy	- How elements of sets were attained - How Cartesian product works	- Inclusion/exclusion boundary - How they got the circular shape
Barb	- Clarification of terminology - Elicitation of missing element(s) - Size of $A \times B$ in general sense - Whether $A \times B = B \times A$	- Inclusion/exclusion boundary - How they got the circular shape. - Identify length of radii of the circles - Slight modification of the problem to a new problem

Probing details

Oral assessment provides an opportunity for an interaction between assessor and student in which the assessor may ask questions to probe the student's knowledge and reasoning. We found that some assessors sought details in students' reasoning even after students presented a complete answer to the problem.

When following up on part (a), some assessors asked for details on students' thought processes for obtaining the Cartesian product, such as "how did you get this set as your Cartesian product?" (Sandy to D2) or "So, how did you end up getting those, those ordered pairs? What was your thinking there?" (Paige to A6). In another instance, Nancy asked C6 how she obtained the elements for one of the sets in the Cartesian product, "how did you figure that out?"

For part (b), some students explained the why the boundary would be solid or dashed without being prompted. When the student did not address why they had drawn the boundary as they did, we noticed that all assessors asked further questions about the boundary. The questions were often, "why was the boundary included/excluded/solid/dashed?"

Follow-up questions that seek additional details are likely asked to ensure that students can speak and reason knowledgeably about what they are presenting. This was especially important because the students could use any resources to prepare their answer ahead of time. Furthermore, we found that the questioning provided opportunities for students to recognize errors and misconceptions. In the following excerpt, we observe how the probing question from Sandy let D1 recognize her misconception.

D1: All right, so since it's between one and nine, I just kind of saw that it was, if you took one and one, that would be one. So, I know it’s at least here, here. And then the largest part is nine, so three squared plus three squared is nine.

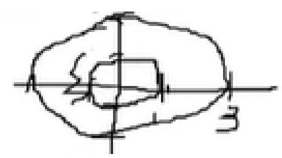

Figure 2: Diagram drawn by D1

Sandy: Hold on, you said three squared plus three squared equals nine?

D1: I did not mean to say [mumbling]. I think I screwed up somewhere in my thoughts. [long pause]. All right. Hold on.

Sandy: Take your time.

D1: I feel like it still is a washer with a radius with the inner circle being a radius of one. And I think that at the end that it's a radius of three still. I just didn't understand exactly how…

Based on her drawing (Figure 2), we can infer that D1 knew what the shape of the graph looks like, but her explanation for how she obtained that did not follow. Although the correct result was presented, it may have been obtained through memorization, help from someone else, or the use of graphing software. If this were a written assessment, this issue might go undetected by both student and teacher while the teacher thought the student had mastery of the problem.

Filling gaps

We identified filling gaps in a student’s presentation as another role of follow-up questions. For instance, two students missed the element $-\sqrt{2}$ when listing the elements of the factor set for problem 2(a). Barb and Nancy asked questions in hopes of eliciting the missing element. The initial questions were broad and moved towards being more explicit when the student continued to struggle. In the transcript excerpt below between Barb and B2, the student quickly realized that something in their solution needed to be addressed when Barb expressed a qualm about the presentation.

Barb: So, I agree with everything you said. Except we have one small qualm. It is about the first set. So, you’ve told me that we’re looking for all real numbers x such that x squared equals two.

B2: Oh, wait…

Barb: Can you think of any other numbers that might be in that set?

B2: So, this … [writes negative root two on the scratchwork board]

From this excerpt, we observed that a brief prompt from Barb caused B2 to realize that she was missing something. However, filling the gap was not always easy. Another case shows that the student needed more explicit direction.

Nancy: So, the square root of 2 definitely makes that true. Is there any other real number that also makes that true?

C2: Something square root…I don't think so…
Nancy: What about negative square root of two?
C2: Yeah… (smiling)

In both situations, we observed that through follow-up questions, the assessors led the students to think and help them realize that something had been missed.

Bridging Access to Knowledge

Students who cannot access problems, such as those who do not know a certain definition or those who do not understand what is being asked, might become discouraged and would most likely receive a low score on a written assessment. However, we found that oral assessment could help those students by providing some "help" without losing the integrity of an assessment. Let us look at the following excerpt in which Nancy was providing a bridge for C6 to access problem 2a.

C6: I wasn't like entirely sure when we have um... like for example when it says x squared equals to 2 and then the Cartesian product of absolute value of x equals 2. I don't understand how you would do that.
Nancy: So that's telling you the two sets of your Cartesian product. So, that first one, that means x is a real number such that x squared is equal to 2. So, what value of x makes x squared equal to 2? that are real numbers?
C6: Um… negative square root of 2 and square root of 2.

This reiteration of the problem seemed to be a big help to C6. She was able to proceed with the rest of this problem on her own. Note that Nancy did not provide a hint to the problem, but she just restated the set notation in language that was more accessible to the student.

Fostering Learning

Across our data set, Barb was the only assessor to ask follow-up questions that were not directly related to the problem. These questions were mostly in the form of a modification of the actual problem or a generalization of some aspect of the problem. It appears these questions were asked to encourage deeper student reflection and learning. For instance, after students presented their answers for part 2(a) correctly, Barb asked about the order of $A \times B$ in general. In the transcript excerpt below, Barb asks about the size of a generic Cartesian product of sets.

Barb: So, if I told you that the size of a set, some random set A was four, and the size of a different set B was five, and I asked you how big was A cross B? or the Cartesian product of A and B, without knowing what the elements were, would you be able to know how big that set is?

When assessing part b, follow-up questions included modifications of the problem, such as changing the inequality into a strict inequality. For instance, Barb followed B1's explanation for 2(b) with, "say we took away the less than or equal to, and we made that just less than. What does the picture look like now?" On another occasion when following-up on part b, Barb modified the universal set to $\mathbb{R}^3$.

Discussion

Using multiple assessors resulted in different experiences among students. During our quizzes, some students experienced no follow-up questions while other students received follow-up questions that were beyond the problem they were presenting. Although the students had different experiences, scoring was highly consistent among assessors. Hence, we conclude that students' grades would not be negatively affected regardless of who their assessor was. The differences in student experience were related to the nature of the follow-up questions they received. Each of these contribute to answering our research questions.

In answer to our first two research questions, we found that the assessors in our study were surprisingly consistent in their ratings of student performance, even when assessing student knowledge after the fact. Although our sample was small, we have evidence supporting Iannone and Simpson's (2015) claim regarding the feasibility and consistency of post hoc moderation of oral assessments in undergraduate mathematics.

In answer to our third research question, we identified four purposes for asking follow-up questions: seeking details, filling gaps, building a bridge for students to access problems, and fostering learning. There were often similarities to the questions asked by assessors. As noted, all four assessors asked follow-up questions about how the boundaries of the circles were decided to be solid or dashed. Three of the assessors asked questions related to the equation of the circle and how the radius of the circle was determined. The questions that were similar across assessors were the ones that sought details and filled gaps.

Questions that sought details or asked for additional information to fill gaps allowed the student to demonstrate the depth of their knowledge. When students found themselves unable to proceed, questions that created bridges to knowledge allowed the students to extend their learning by making connections between their ideas and their representations. Follow-up questions that fostered deeper learning asked a student to make a generalization or modification of some aspect of the problem. Each type of follow-up question provided a path for students to reflect and learn more, which is the ultimate goal of education.

We conclude that in preparing to administer an oral assessment, many follow-up questions could be prepared in advance. This aligns with Joughin's (2012) advice to have prepared probing questions to help provide uniformity and consistency in both grading and students' experiences. To prepare these questions for a mathematics setting, we recommend writing questions for each problem that: 1) probe for detail and fill gaps by asking for explanations on key ideas/concepts, 2) bridge access to knowledge by anticipating trouble spots and scaffolding hints, and 3) foster reflection and deeper learning by extending ideas beyond what was initially asked. We acknowledge that it is impossible to prepare for all possible scenarios in team preparation, but an extensive list of possibilities should improve the experience and consistency. It is encouraging that even with the minimal guidance our assessors had, the scoring of students was highly consistent. This consistency was likely due, in part, to using an assessment scheme that was quite coarse.

We believe that our implementation of a virtual oral assessment was a successful endeavor considering its novelty in undergraduate mathematics classrooms in the United States, and we have learned much from this experience. Student evaluations of the course were very positive, with one student writing, "I was pleasantly surprised by how much I preferred the oral quizzes to normal quizzes. It allowed me to really think about difficult questions beforehand, and I often had breakthroughs in the middle of the quizzes." The most promising benefits to students when incorporating oral assessments into assessment practices are the opportunities to demonstrate their knowledge using modalities beyond writing and to deepen their learning by engaging in authentic mathematical conversations. When used as part of varied assessment practices, the focus of classroom activities will shift away from merely trying to accumulate enough points for a given grade to one in which students are working toward richer mathematical knowledge.

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Conceptions of active learning held by undergraduate mathematics instructors participating in a statewide faculty development project

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We explore the perceptions about "active learning" among college and university mathematics faculty involved in early stages of the Mathematical Inquiry Project (MIP), which supports long-term collaboration across mathematics departments at the 27 public institutions of higher education in the state of Oklahoma. Our analysis indicates that faculty beliefs about active learning varied widely across individuals and significantly differed from the MIP characterization, even though participants believed their conceptions to be aligned. We document changes in participants' beliefs as a result of participation in the MIP that faculty attributed to engagement in rich mathematical tasks, conversations with other participants, small group discussions of research literature, and conversations with project team members. Participants also reported enacting their conceptions of active learning in their classrooms more often as a result of their involvement in the professional development.

Keywords: Community of Practice, Inquiry-Based Learning, Entry-Level Math, Active Learning, Professional Development

Introduction and Background

The Mathematical Inquiry Project (MIP) is a statewide collaboration among mathematics departments at the 27 public institutions of higher education in Oklahoma to foster sustainable, large-scale reforms to improve instruction in entry-level mathematics courses. To promote awareness of and attention to the mathematical, epistemological, and affective considerations in instructional design, the MIP is guided by definition of *mathematical learning through inquiry* that entails three interdependent components: (a) engaging students in *active* learning, (2) incorporating *meaningful applications*, and (3) supporting students' development of broader *academic success skills*. These components are defined as follows:

Students engage in active learning when they work to solve a problem whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.

Applications are meaningfully incorporated in a mathematics class to the extent that they support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure.

Academic success skills foster students' construction of their identity as learners in ways that enable productive engagement in their education and the associated academic community.

The MIP aims to foster instructors' professional growth by fostering a *community of practice* (Wenger, 1998) in which participants engage in a joint enterprise to design, disseminate, and implement instructional resources, as well as develop as leaders within the community's emerging view of expertise. In this report, we focus on faculty perceptions of *active learning*.

In the summers of 2019-2021, the MIP led five multi-day initiation workshops during which Oklahoma mathematics faculty identified key priorities for courses in the four state pathways—

functions and modeling, college algebra/precalculus, quantitative reasoning, and Calculus I—as well as on academic success skills across all courses. Participants engaged in readings, presentations, and small- and whole-group discussions about the three components of inquiry, their dependence on conceptual analyses (Thompson, 2008), and their implications for constructing hypothetical learning trajectories (Simon & Tzur, 2004). The purpose of the initiation workshops was threefold: (a) to initiate a statewide community of practice, (b) to build the capacity for faculty to design and implement instructional materials that support learning through inquiry, and (c) to identify the key conceptual threads in these entry-level courses for future instructional design collaborations. Following the workshops, the MIP participants were encouraged to join Collaborative Research and Development Teams (CoRDs) comprised of groups of 2-5 faculty tasked with developing, testing, and refining an instructional module related to one of the conceptual threads identified in an initiation workshop. Later stages of the MIP will involve broadening the community of practice through disseminating resources via the MIP website, regional workshops, and peer mentoring.

Research on faculty professional development highlights that change strategies should seek to alter individual's beliefs as opposed to enacting top-down policy to impact teaching or disseminating "effective" curricular resources (Henderson et al., 2011). This demonstrates the importance of characterizing faculty's conceptions of mathematical learning through inquiry and describing how particular professional development experiences contributed to their evolution. Furthermore, by adopting communities of practice as a model of faculty change, the project forgoes attempting to directly impact participants' conceptions of inquiry-oriented mathematics instruction, leaving that to emerge as part of the community's shared priorities. As such, we sought to evaluate participants' conceptions of active learning after having participated in one or more of the MIP initiation workshops. Specifically, we were interested in the extent to which participants' conceptions of active learning aligned with the MIP's definition and what project activities facilitated any changes in these conceptions. We consider the following research questions:

1. What are participants' conceptions of active learning, and to what extent do they align with the MIP's definition of active learning?
2. What aspects of the MIP influenced participants' conceptions of active learning?

We expect our exploration of these research questions to inform how we might operationalize the general mechanisms of individual learning through social engagement articulated by Wenger (1998) to influence the conceptions of active learning held by mathematics faculty at different stages of participation in a statewide professional development project.

Theoretical Framing

The MIP seeks to effect changes in the cultural practices of mathematics instructors by cultivating a community of practice that enables professional growth through individual participation. A community of practice is a social entity in which individuals negotiate meaning through their mutual engagement in a joint enterprise around a shared repertoire of reified artifacts (Wenger, 1998). Each stage of the MIP seeks to engage mathematics faculty in experiences that require their negotiation of meaning around the MIP's three components of inquiry.

The purpose of this research study is to explore how individuals experience their involvement in the MIP community of practice as they negotiate meaning through their engagement with other members and through their interaction with the community's established set of reified artifacts. Through an individual's interactions with other members of the community and its reified artifacts, they become increasingly cognizant of the knowledge base and skillset required to participate competently within the community. A central goal of the MIP is to cultivate a community of mathematics faculty that, through their participation in the community's activities, negotiate a notion of competence reflecting the three components of the MIP's definition of mathematical learning through inquiry. Characterizing participants' initial and developing conceptions of the three components of inquiry is essential to this effort as these characterizations can inform the MIP Team's participation in the community and allow for the strategic introduction of reified artifacts into the community's activities.

Methods

We conducted semi-structured interviews over Zoom with 15 MIP participants in spring 2021. The interviews were audio recorded and transcribed for use in analysis. The interview questions included the following:

1. Please describe your image of active learning in entry-level college math courses.
   a. Why is this important for entry-level math courses?
   b. Can you describe a specific example of active learning in an entry-level math course, yours or someone else's?
   c. What made this example effective? What could have been better?
   d. Has your participation in the MIP activities changed your thinking about active learning? How?
2. Here is the MIP's definition of active learning. [Participants were presented with the definition].
   a. Are there parts of this that you think are important but haven't discussed yet?
   b. Do you particularly agree or disagree with emphasizing any aspect of the MIP definition for improving instruction in entry-level college mathematics?

We employed the constant comparative method (Strauss & Corbin, 1994) to identify themes in the data. One author read all the transcripts, highlighting words and phrases that characterized participants' images of active learning. When a new word or phrase was added to the list, the author reread all previous transcripts seeking instances of that word or phrase. This generated a list of 30 items. The author then grouped similar items into themes and described them using the words/phrases from the list (Table 1), resulting in a list of open codes. The author then re-coded the transcripts using the working descriptions of the codes, and then refined these descriptions until they captured all highlighted words from the first reading. We note that that these codes are not mutually exclusive; for instance, if a participant discussed motivating students with real-world examples, that was coded as both 'real world examples' and 'affective.' Similarly, a participant suggesting enacting active learning with a class discussion was coded both a 'format in which learning would take place' because it described the plan for the lesson, and as 'nature of student engagement' because students would be interacting with other people.

Results

Our analysis suggests that participants' conceptions of active learning focused on three major themes: the class setup, the mathematical content of a lesson, and the affective facets and benefits of active learning.

Table 1. Emergent themes

Theme	Sub-theme(s)	Description
Class setup	Nature of student engagement	Participant describes a way students might be involved, such as answering questions, interacting with other students, a class discussion, exploring, group work, students giving presentations
	Format in which the learning will take place	The participant describes something that could be thought of as lesson planning, like group work, class discussion, guided work, scaffolding information, using manipulatives, using Desmos or a graphing calculator, doing projects
Content	Problem solving	Participant mentions “problems” or “problem solving” without explicitly identifying that it is a novel task
	Problematic situations	Participant mentions engaging students in a productive struggle and/or in situations that are problems (not exercises), or describes students selecting, applying, and evaluating tools or actions
	Understanding	Participant describes that active learning should help students understand or know rather than memorize
	Real-world examples	Participant mentions that active learning should entail real world examples
Affective	n/a	Participant describes active learning in terms of its implications for, or dependence on, students’ interests, motivation, perseverance, mindset, anxiety, etc.

Regarding class setup, all participants mentioned that they associated active learning with particular instructional formats, such as collaborative group work or a class discussion. This demonstrated participants’ attention to ways students might interact with each other or with mathematical tools (e.g., graphing calculator or dynamic visualization software) and the class structure (e.g., scaffolding information instead of lecturing, making a class like a lab or workshop). Jack, who attended one workshop and was on one CoRD, described his class:

> I incorporate a lot of collaborative project learning.... I like to use a lot of manipulatives. I have a limit of how much I want to actually speak to them in a formal setting and having

> them actually do things while I'm there as a mentor is a lot more helpful ... the more I can make my class like a lab, the better I am at really letting active learning [happen].

Most participants provided both examples of active learning that foregrounded the structure of students' mathematical activity and its relation to their conceptual development and an example that foregrounded students' participation without attention to how the activity might support students' construction of particular mathematical meanings. For example, Reagan, who had attended two workshops and was on a CoRD, said in response to interview Question 1b,

> In the college algebra class when we talk about the function rate of change and like a main concept at the beginning.... I designed the pre class activity for students to work on [review] problems.... I also let a student to come up with examples. So I give problems, I give applications and let them come up with what kind of additional related example in the real life they can come up with.... so like in the rate of change when we talk about constant rate, normally we start with talking about the distance and the time.... some of students will come up with... go to the grocery store and you buy the grocery and it to sell by the pound, what is the unit price? That is also a constant rate.

Reagan did not connect her example of students working on review to a specific conception of rate of change that she intends students to construct. We consider this portion of Reagan's statement to be more aligned with a colloquial meaning of active learning because, although she referenced a mathematical topic (rate of change), she did not specifically attend to how the problems in question might elicit actions that reflect the multiplicative structure of a function's constant (if linear) or average rate of change. Instead, Reagan considered the example she proposed to be an instance of active learning because students were working on problems (instead of observing her work on these problems in lecture). On the other hand, she related students production of an example of average rate of change to the underlying structure of corresponding changes in quantities' measures such as changes in "distance and time" and the proportional relationship between a grocery item and its weight and cost. Her attention to aspects of the specific conceptual structure of rate of change in this activity indicates the potential for the students to select actions whose structures are equivalent to the concept to be learned.

All participants acknowledged the affective requirements and affordances of engaging in active learning. Actively engaging in meaningful applications of mathematics both requires and fosters academic success skills (e.g. mathematics self-efficacy, growth mindsets, persistence in problem solving). Adam and Eden's comments are representative of those coded under this theme. Adam had worked on a CoRD, and Eden had attended three workshops and was participating on a CoRD at the time of the interview.

> *Adam*: I think [active learning] is important [because] students who engage with math tend to have better perception of it. ... I think it allows the students to gain a sense of autonomy and, um, confidence in math that they may not be accustomed to.
>
> *Eden*: it's self-efficacy, the whole thing and that goes into the academic success skills, but I mean it's building, especially for students that are going to need to take math past that entry level, it's, you're, you're, you're creating if you like your own machine, you're, you're starting at the entry level and getting them used to this idea. So, as they progress through their math courses, they will be more successful.

Both Adam and Eden discuss the affective benefits of active learning: Adam's response focused on students' developing a sense of autonomy, potentially increasing their mathematical confidence, and Eden's remarks highlight the importance of mathematical persistence.

Affective affordances and requirements of active learning were not part of the MIP's definition of active learning, but do appear in the MIP definition of academic success skills. The MIP three components of inquiry are interdependent, and we agree that active learning can both require particular affective states (e.g., a willingness to engage in productive struggle) and afford particular affective states (e.g., foster increased willingness to engage in productive struggle). The MIP had explicitly stated affective affordances and requirements only in the definition of academic success skills. That participant talked about affective affordances and requirements in active learning indicates negotiating meaning in the community of practice. Specifically, participants seemed to favor a definition of active learning that incorporates the affective affordances and requirements of active learning explicitly.

The participants also reported that various aspects of their engagement in the MIP influenced their conceptions of active learning. Generally, participants cited specific examples of rich tasks, conversations with other participants, small group discussions of research literature, and conversations with project team members. For example, Ellison, who attended one workshop and was on a CoRD, felt the problem-solving literature she read for her CoRD had influenced her thinking that an important part of active learning was not to give the answer too soon. Gemma, who attended one workshop, said

> I really liked those types of problems that we did as samples... something that gets you to think outside of the box on math and not have to be like... sitting there doing x's and y's and whatever. Thinking about real life and how can I connect this and then you know problem solving

The sample problem Gemma referred to were generated by the MIP team. The goal of the problems was to model a conceptual analysis, hypothetical learning trajectory, and how those informed the design of tasks that operationalize the three MIP components of mathematical inquiry. We take Gemma's statement about the problems making students to "think outside of the box" and "problem solving" as possibly indicative that she attended to the tasks as problematic situations.

Discussion

We developed the MIP definition of active learning to foreground the implications of the nature of students' activity for their construction of particular mathematical meanings and to serve as a guide for the project design of engaging faculty in a community of practice. While we expect to see some change of participant's goals, values, and beliefs that might make aspects of the MIP definition more meaningful to them, we equally expect the community to develop its own priorities and standards. We present our characterization of participants' goals, values, and beliefs about active learning to inform subsequent project activities in ways that will better support participants to (a) understand the nature of their conceptual learning goals; (b) act in ways that foster those goals in their instruction; and (c) reinforce the development of the community of practice toward similar sensitivities.

While participants universally described general formats in which students might be participate in class (e.g., group work, class discussions, and projects), they often did not attend to the nature of students' engagement with carefully designed mathematical tasks that deliberately support abstraction of underlying mathematical structure. We note that many of these perceptions were internally consistent, based on individualized implicit learning theories and prior experience, and thus highly stable. However, it is important to the PD project to know that participants' definitions of active learning did not explicitly attend to students' selecting, performing, and evaluating specific mental actions, because we view the ‘active’ in ‘active learning’ as *mental activity*, and hence propose that focusing on the nature of the mathematical tasks (as opposed to whether they will be done in groups, or in back-and-forth question-and-answers between the class and instructor) is critical in effective instruction. In short, there is some misalignment with participants’ definitions and the MIPs in regard to how each views what ‘active’ means in active learning.

The organic evolution of a normative conception of competence within a community of practice—which reflects the current and developing conceptions of those participating in it—restricts the range of possible interventions that seek to influence how participants conceptualize both the practice in which they engage and the nature of competence this engagement requires. Participation in the community is the mechanism of individual identity transformation (i.e., learning), and is directed towards the normative conception of competence implicitly negotiated by the community through its pursuit of a joint enterprise. A principal affordance of becoming aware of one’s own conceptions of learning is that it positions an instructor to *purposefully* develop and implement instructional sequences that are consistent with it (Tallman, 2021). We view a central priority for the project to be fostering the community of practice to (a) make these goals, values, and beliefs explicit; (b) create the intellectual need for critical reflection on them; and (c) provide opportunities to develop, implement, evaluate, and refine new strategies based on the MIP characterization of inquiry. Based on our analysis, we recommend the following forms of interventions that might support the refinement of participants’ conceptions of active learning through their engagement in the MIP community of practice:

- *opportunities to engage in instructional design with community-recognized experts to foster the MIP components of inquiry*
- *opportunities to critically evaluate curricular artifacts that reflect explicit operationalization of the MIP components of inquiry*
- *feedback from peers that suggest concrete ways to modify their proposed instructional materials to support more effective implementation of the MIP components of inquiry*
- *guided reflection on results of pilot lessons and refinement to improve implementation of the MIP components of inquiry*

Future research

One direction for our future research is to include additional data sources. For example, recordings of participants teaching or artifacts from their class materials could lend additional insight into participants’ conceptions of active learning and the extent to which they align with the MIP’s definition. An analysis of such data would provide more robust findings by allowing us to describe how participants enact their conceptions of active learning.

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A Characterization of an Undergraduate Mathematics Instructor's Conception of Learning Through Mathematical Inquiry

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This research report focuses on documenting the conceptions of and relationships between three interrelated components of mathematical learning through inquiry (active learning, meaningful applications, and academic success skills) demonstrated by an experienced mathematician throughout a series of semi-structured clinical interviews concurrent with the participant's engagement in a large-scale professional development initiative. Our analysis revealed that the participant demonstrated stable and coherent conceptions of the three components of mathematical learning through inquiry that were generally resistant to intervention. We reflect on the implications of our analysis for influencing the conceptions of inquiry-oriented instruction held by undergraduate mathematics faculty.

Keywords: Active Learning, Mathematical Inquiry, Professional Development, Teacher Knowledge, Case Study

Introduction

Researchers, professional organizations, and policymakers have emphasized a need for instructors to attend to meaningful student engagement in mathematics classes (CBMS, 2016; Freeman et al., 2014; Kober, 2015; NRC, 2012; NSTC, CSE, 2013; PCAST, 2012; Saxe & Braddy, 2015). Instruction with higher levels of student interaction enhances problem-solving skills (Prince 2004), demonstrates greater conceptual gains (Freeman et al., 2014; Hake, 1998; Svinicki, 2011), and improves retention of information (Dirks, 2011; Prince, 2004; Sokoloff & Thornton, 1997). Although a preponderance of evidence indicates the benefits of active learning, many mathematics instructors struggle to effectively engage students in active learning experiences. Active learning strategies realize their potential when an instructor operates with an explicit understanding of *how* the actions in which students engage might engender the cognitive mechanisms necessary to construct productive meanings for targeted mathematical ideas (Tallman, 2021). Unfortunately, undergraduate mathematics instructors are rarely afforded the opportunity to construct this essential component of pedagogical content knowledge.

Perceptions that mathematics lacks relevance to one's interests and goals is a common deterrent to engaging in active learning. Students with this perspective tend not to appreciate the broader skills they can acquire by studying mathematics. Modeling real-world problems with mathematical tools can provide students with greater opportunities to understand course content and its relevance (Frykholm & Glasson, 2005; Jacobs, 1989; Koirala & Bowman, 2003; Pyke & Lynch, 2005). However, due to a need to cover a broad range of topics, the techniques in entry-level undergraduate mathematics courses are rarely taught in the context of real-world situations. Additionally, even when applied contexts are incorporated into instruction, students often struggle to notice similarities in mathematical structure that would allow them to apply content learned in other contexts (Gick & Holyoak, 1983; Lobato & Siebert, 2002). These observations establish a need for instructors to incorporate relevant applications into mathematics courses for the purpose of both promoting students' positive affect and engaging them in active learning.

Students' conceptions about the origin of mathematical intelligence and the skillset they attribute to being proficient at mathematics have additional implications for their capacity to engage in active learning. Fortunately, students' mindsets are not immutable psychological traits, and specific supports can enable them to reconceptualize what it means to become proficient at mathematics (Middleton, Tallman, Hatfield, & Davis, 2015). Tallman and Uscanga (2020) specified the types of support instructors might provide to foster the positive affect necessary for students to meaningfully engage in active learning. These scholars also cautioned against the ineffectiveness of common efforts to foster students' productive affective engagement in mathematics by reducing these efforts to a list of pedagogical prescriptions without consideration for the mathematical conceptions they are supposed to support.

Taken together, the literature summarized above demonstrates that designing instruction to effectively engage students in active learning is a complicated task for which undergraduate mathematics instructors often receive insufficient preparation, and are thus generally underprepared. To address this need, we designed the Mathematical Inquiry Project (MIP) to support mathematics faculty at all 27 public institutions of higher education in Oklahoma to engage students in learning through inquiry in entry-level mathematics courses. More broadly, the MIP complements principles of organizational change with social learning theory to foster effective, scalable, and sustainable cultural shifts in mathematical learning through inquiry defined in terms of *active learning*, *meaningful applications*, and *academic success skills*.

This research report focuses on documenting one MIP participant's conception of these three interrelated components of mathematical learning through inquiry. In particular, we explored the following research questions: (1) *What conceptions of the three elements of mathematical inquiry and their mutual influence are held by an MIP participant?* (2) *To what extent do these conceptions align with the definitions proposed by the MIP?*

Theoretical Background

Active Learning

At the level of student engagement, the MIP synthesizes constructivist, neo-constructivist, and social-constructivist perspectives, in which conceptual structure is consistently characterized as abstracted from reflection on the structure of one's *actions* to resolve a problem (diSessa, 1982; Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Hickman, 1990; von Glasersfeld, 1995; Piaget, 1970, 1980). The MIP operationalizes this perspective through emphasizing processes that focus students' attention to the nature of a problem, selection of appropriate mathematical tools, application of those tools, and attention to the reciprocal influences of the tool both applied to, and evaluated against, the problem. The MIP definition of active learning is:

> *Students engage in active learning when they work to resolve a problematic situation whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.*

Designing this level of student engagement first requires an instructor to become cognizant of the targeted concept's conceptual structure (Tallman & Frank, 2020), and then to purposefully inquire into students' mathematical thinking by listening, interpreting, and respond to the interplay between intuitive and informal ways of reasoning students express.

Meaningful Applications

The MIP leverages educational theory and empirical research that offers insight into how relevant applications might effectively support active learning. One of the most immediate features of our emphasis on applications is that mathematical representations refer to real-world objects and quantities that can be described and imagined to support intuitive reasoning, which is often not elicited when students only encounter abstract representations. This concrete reasoning enables students to engage in mental constructions that can subsequently be represented abstractly by variables, expressions, diagrams, and graphs. The MIP definition of meaningful incorporation of applications is:

> *Applications are meaningfully incorporated in a mathematics class to the extent that they support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure.*

Ultimately, for applications to be used effectively, they must foster students' engagement in the mental activity on which they will later reflect to construct targeted understandings of particular mathematical concepts. Additionally, the phrase "generalizing across contexts" should be interpreted within the domain of mathematics and not across other academic disciplines.

Academic Success Skills

Students' active learning is initiated and sustained by components of their affect. The MIP leverages research about growth versus fixed mindsets, the nature of memory and expertise, the integration of academic and social communities, academic identity, stereotype threat, and study skills to equip instructors with the tools to support students active learning of mathematics. We express this amalgam of affective qualities, dispositions, and states as *academic success skills*, which the MIP defines as follows:

> *Academic success skills foster students' construction of their identities as learners in ways that enable productive engagement in their education and the associated academic community.*

Methods

The lead author conducted 11 semi-structured clinical interviews (Clement, 2000; Hunting, 1997) with an experienced professor of mathematics (Robert) from a small, liberal arts university in the Southern United States. The focus of these clinical interviews was to elicit products of Robert's conception of the three components of mathematical inquiry defined by the MIP and to explore their relation to Robert's values, beliefs, and commitments, and instructional goals.

Our data analysis was guided by grounded theory procedures (Corbin & Strauss, 1990; Strauss & Corbin, 2007). The lead author first identified segments of interviews during which Roberts' remarks captured the essence of his conception of one of the three components of mathematical inquiry, or the relationships between them. This initial analysis enabled us to become sensitive to not only *what* Robert said but also *how* he expressed it. From these segments of data, the lead author abstracted Robert's expressed ideas into larger codes and categories (an application of axial coding) and later combined and related these codes and categories into more general clusters, often through several iterations, resulting in a stable set of final themes.

Results

Active Learning

Robert participated in four interviews during which he responded to a series of questions specifically related to active learning (Interviews 1, 2, 6, and 10). During the first interview, Robert described active learning as synonymous with genuine mathematical engagement, which occurs when students solve non-routine problems for which they cannot uncritically apply a rehearsed procedure. At the beginning of the second interview, Robert proposed the following definition of active learning:

> *Students are engaging in active learning when they are asked to engage with a problem themselves (as opposed to passively observing an instructor solve the problem).*

As this definition suggests, Robert's conception of active learning foregrounds the idea of student *engagement*. This definition does not explicitly address the relationship between the mental actions in which students engage and the cognitive structures they might be expected to construct for the targeted concept of a lesson. Towards the end of Interview 2, Robert requested to revise his original definition. His only modification was to replace the word "engage" with "struggle and wrestle." This modification, as well as Robert's comments throughout Interview 2, revealed his perspective that an essential feature of active learning is that it entails students making strategic decisions in the service of overcoming some sort of struggle. Robert considered such strategic and impasse-driven decision-making to be an essential feature of genuine mathematical activity, and thus active learning.

Robert's emphasis on decision-making as a feature of active learning was apparent during a discussion of what might distinguish a student who is engaged in active learning from one who is not. Addressing this question, Robert explained that students "have to be asked to think critically about the problem. They have to make some decisions. They have to make some determinations. So, decision making would be- for me would be the, the, the line." These decisions, Robert clarified, need to be "strong" decisions, and he provided an example from calculus of students struggling to determine whether to apply the method of *u*-substitution instead of integration-by-parts. A few minutes later, Robert concluded: "As long as they're actively, they're, they're trying hard, they're struggling with it, if there's maybe, maybe the defining line should be there has to be a, a level of struggle with a problem." This comment reinforces the claim that Robert considered students' experiencing struggle to be the essential criterion of active learning. There was no evidence in Robert's responses to suggest that he conceived students' struggle as a precondition for engaging in the cognitive activity necessary to construct specific *meanings* for targeted concepts. Generally, Robert gave no indication throughout the series of interviews that his conception of active learning related to an instructor's goals for students' mathematical learning. Instead, Robert's characterizations of the features and requirements of active learning were dominated by his expectation that students' activity should reflect a mathematician's experiences while engaged with a challenging problem. Indeed, Robert's introspection on his own mathematical activity consistently appeared to be the source of the various features of active learning he proposed.

Meaningful Applications

The lead author conducted four interviews with Robert during which he responded to questions specifically related to meaningful applications (Interviews 1, 3, 7, and 9). Robert proposed the following definition of meaningful applications at the beginning of Interview 3:

Applications are meaningfully incorporated in a mathematics class when problems are presented that piques student interest and highlights a key concept (or some key concepts) of the lesson.

There are two fundamental components of Robert's definition: stimulating student interest and highlighting a key concept of the lesson.

When Robert was asked to interpret the MIP definition for meaningful applications in Interview 1, he recognized that this definition was useful for instructors, but "from a student view, meaningful application would be something that, just a problem that they're, that engages them, intrigues them, has to be some kind, spark some kind of interest." One way that a problem might intrigue students is if it relates to them personally or is an applied, real-world problem that enables students to appreciate the applicability of the mathematics they are learning. When asked about his interpretation of the word "meaningful" in the third interview, Robert stated that meaningful is something that "interests me" and has a "hook" to promote interest.

Robert's interpretation of the word meaningful arose from a student perspective (e.g., piques students' interest) as well as an instructor's perspective (highlights a key concept). Robert stated that another interpretation for the word "highlights" is demonstrates: "Demonstrates is (*pause*), it brings to (*pause*), it shows the, shows the usefulness of these concepts. It demonstrates why, why we're doing what we're doing." Robert offered an illustration for how an instructor might highlight or demonstrate the usefulness of a concept:

> If I can come with an application, a problem that forces you, really encourages you to use one over the other, that's a meaningful application, right. If I, if I try to teach you a shell method, right, usually a shell after the washer method, right, but if I give you a problem where I, where students can just as, just as easily solve it using the washer method, then what's the point, right?

Robert echoed a similar sentiment later in the interview, stating that if he were teaching the shell method he would "steer away" from problems that can be solved equally easy using either the shell method or the washer method (unless he wanted to highlight that sometimes either approach is appropriate).

Academic Success Skills

The lead author conducted three interviews with Robert during which he responded to questions specifically related to academic success skills (Interviews 1, 4, and 8). At the beginning of Interview 4, Robert constructed his definition:

Academic Success Skills are behaviors/actions that help people/students succeed academically (i.e., in their studies/research). Examples include: detailed note taking, a sense of curiosity, the grit/determination to tackle/solve a problem—from several approaches if necessary, and to think critically.

Robert's conception of academic success skills centered around thinking critically and exercising grit and tenacity to persist in satisfying curiosity. According to Robert, *thinking critically* entails making "deliberate decisions" and operating *intentionally* when solving problems as opposed to using a particular technique because peers used it, the book encouraged it, or it proved useful in another context. Critical thinking is a key facet of the academic success skills Robert values most:

> For me, the most important academic success skill is being engaged, and being, uh, curious, and being, and just really getting, getting down and really exploring the concept,

> right, having the grit, uh, and tenacity to, to, work on the problem. I, that to me, that, for me, hands down, that's the most important.

Robert's latter statement offers insight into how he might develop tasks to support students' academic success skills. He expects that the affective qualities he values in students can be reinforced by engaging them in tasks that elicit productive struggle, and encouraging them to reflect on their attempts to solve the problem.

While students may be able to enhance their determination from engaging in tasks that facilitate productive struggle, the interviewer asked Robert if the ways in which an instructor teaches a *specific mathematical topic* can promote students' curiosity, tenacity, or grit. Robert's response emphasized how instructors' actions can be *problematic* for promoting these affective characteristics, stating that making math "mechanical" makes it become artificial and tedious.

Throughout the series of interviews, Robert frequently described active learning as dependent upon students' curiosity and tenacity. The interview data consistently indicated that Robert's conception of active learning was closely connected to his image of mathematical problem solving and the curiosity and tenacity that initiates and sustains it. Robert reasoned that curiosity and tenacity are essential to active learning since the former is both the origin of genuine mathematical problems and the affective state that compels a student to want to solve a problem. Additionally, Robert expressed that tenacity is required to persevere through the struggle inevitably encountered when reasoning about a novel problem. He explained that curiosity and tenacity is "what made me successful in my schooling," again revealing Robert's reflection on his mathematical activity to infer essential features of active learning and their relation to students' affect.

Robert clarified his image of the relationship between active learning and students' academic success skills during Interview 2:

> My style of teaching is for students to struggle with a problem, understand why that problem is hard and why that problem is interesting, then I will come back around and show them a way that is easier. Then they can learn to appreciate the new knowledge that I am trying to teach them.

This excerpt suggests that Robert considered active learning in terms of students' engagement in the precise activity that results in their experiencing an intellectual need for the mathematical content that he will eventually present to them. Robert's comments indicate that he valued this type of activity because it can ultimately position students to more intently and purposefully absorb the mathematical meanings, skills, and strategies he communicates. That is, active learning makes students more receptive to *perceiving* the mathematics that Robert conveys and/or demonstrates. This conception of the relationship between active learning and students' affect stands in contrast to the MIP definition of active learning, which views students' activity as the experiential basis of their knowledge construction through abstraction. The excerpt above reveals Robert's expectation that the insight, method, or strategy required to solve a problem should ultimately be provided by the instructor, rather than constructed by the students through their mathematical activity.

Summarizing Robert's conception of the relationship between meaningful applications and active learning, he described meaningful applications as providing contexts that can stimulate students' curiosity and then require them to be tenacious and perseverant to solve a contextual problem. There was no evidence in Robert's remarks to suggest that he interpreted "meaningful" as a reference to the meanings an instructor expects their students to construct.

Discussion and Implications

Robert' conception of active learning centered around student engagement in thinking critically, making decisions, and wrestling with problems. According to Robert, active learning can occur in many forms, and the distinction between students who are engaged in active learning and those who are not depends on students making decisions and struggling to solve non-routine problems. A primary feature of Robert's conception of meaningful applications includes a context that *cultivates students' interest*. Robert also explained that a meaningful application should also *highlight the key concept* of the lesson, which could involve demonstrating the usefulness of a particular technique (e.g., disc vs. shell method in calculus). Robert's conception of academic success skills focused on students acting on their curiosity with grit, determination, tenacity, and critical thinking skills to solve a novel problem.

While Robert claimed that his conceptions of the three components of inquiry had been influenced through his involvement in MIP activities, his comments throughout the series of clinical interviews demonstrated that he underemphasized and perhaps undervalued the importance of designing tasks that promote students' construction of particular mathematical meanings. Additionally, Robert's remarks revealed his inattention to the nature and development of students' conceptions.

A fundamental distinction between Robert's conception of active learning compared to the MIP definition is that Robert considered struggle, effort, and strategic decision-making as sufficient criteria for engaging in active learning, independent of the *meanings* he expected students to construct. Similarly, Robert conceptualized the incorporation of meaningful applications as having primary implications for students' affective engagement, specifically motivation and interest. In his own practice, Robert explained that he motivates students by providing *interesting* contexts or by demonstrating the *usefulness* of a particular problem-solving approach.

Notably, Robert's comments throughout the series of interviews highlight a fundamental epistemological distinction: that he underemphasized, devalued, or was inattentive to the instructor's role in designing tasks informed by conducting a *conceptual analysis* (Thompson, 2008) for the purpose of enabling students to progress through a *hypothetical learning trajectory* (Simon & Tzur, 2004). While promoting students' affective qualities is important, particularly related to academic success skills, the MIP definition is centered around supporting students' construction of operative mathematical schemes by identifying and clarifying the nature of the mental actions and conceptual operations required to construct an understanding of a mathematical idea in a particular way.

Importantly, this discussion is neither intended to diminish nor devalue features of Robert's instructional design or pedagogical practices, nor to criticize his interpretation of three elements of inquiry proposed by the MIP. Rather, we offer these distinctions to highlight the potential for Robert's conceptions to be extended and refined to include attention to epistemological considerations of students' mathematical learning. Our results suggest that intensive measures are required to disrupt and modify Robert's established, stable, and coherent conceptions of the three components of mathematical inquiry.

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Undergraduate Math and Science Instructor's Attitudes, Beliefs, and Views on Diversity, Inclusion, and Equity

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In Fall 2020 we collected 1,064 responses to a survey asking instructors of first-year math and science courses questions about their attitudes, beliefs, and views of diversity, inclusion, and equity. Here we provide descriptive analyses on three aspects of the survey: items aimed at instructors' views of certain factors that might contribute to disproportionate representation of different race-gender groups in STEM; items that asked participants about their views on strategies for addressing race-gender disparities in STEM; and items about respondent's personal experiences with discriminatory behavior. About two-thirds of respondents recognize that systematic advantages are afforded to white men in STEM and an overwhelming majority agree that they have a personal responsibility to address inequities. However, we also found stronger support among participants for passive actions than more active ones and a sizable proportion of these instructors also believe that different race-gender groups have different interest levels in science and mathematics.

Keywords: postsecondary STEM, instructor beliefs, DEI attitudes

It is well known that the demographic landscape of science, technology, engineering, and mathematics (STEM) professionals in the USA does not reflect that of the population. This inequity is the product of individual actions as well as systemic barriers. The road to a STEM degree starts in first-year mathematics, chemistry, and physics courses. These courses frequently serve to "weed out" potential STEM majors and disproportionately affects students from marginalized populations (Seymour & Hewitt, 1997; Seymour & Hunter, 2019). Black and Latinx students who begin in STEM are most likely to switch majors or leave college altogether, a phenomenon not seen in other majors (Riegle-Crumb et al., 2019); Black women, among all women entering STEM majors, are the most likely to choose physics, engineering, math, and computer science fields, yet attrition rates are also the highest among Black women (Ma & Liu, 2017). These disparate attrition rates are not due to innate characteristics of the students, but rather to features of the societal, institutional, and classroom environments they must navigate.

The lack of diversity, equity, and inclusion (DEI) in STEM is a function of systemic issues, which require systemic change to address. Individual actions by members of the system are required to make these changes. Postsecondary STEM instructors have access to several levels of higher education and disciplinary systems, including: hiring and promotion, research, mentoring, campus and society DEI initiatives, and, of course, teaching. Their beliefs and attitudes in relation to DEI issues and initiatives shape their ability and willingness to agitate for change. It is difficult to create effective change towards DEI without understanding these beliefs and attitudes. To that end, we administered a survey in Fall 2020 and received 1,064 responses describing respondents' professional identity and context, assumptions about the experiences of eight race-gender groups in STEM, their beliefs about (and engagement with) efforts to increase diversity in their fields, and their departmental and personal experiences with discrimination.

Guiding Theories & Relevant Literature

In line with our personal beliefs and the tenets of critical theories, we hold that STEM is not a neutral field with respect to personal identities or circumstance; higher education trades on a myth of meritocracy in which access and success are controlled by greater social structural inequalities that are reflected in postsecondary institutions (Liu, 2011). Thus, our goals of improving postsecondary STEM education to better engage, support, and serve students must include efforts to remove barriers to access *and* to shift the wider culture toward equity and inclusion. Institutional culture determines the implementation and enactment of policies, and so we must consider various stakeholders in these institutions and their potential role in upholding or dismantling the status quo (Reinholz & Apkarian, 2018; Schein, 2010). Instructors are key members of the STEM higher education ecosystem, as they exist at the intersection of their institution, department, discipline, and classrooms; they are broadly the most important actors in the creation, evolution, and enforcement of “the culture” (Kezar, 2014; Reinholz & Apkarian, 2018). Shifting culture (practices, policies, and attitudes) toward a more desirable state requires documenting the current state and identifying appropriate levers and fulcra for various types of change (Henderson & White, 2019). It is in this avenue which our work is relevant – we have documented instructor beliefs about DEI issues and initiatives in STEM (and postsecondary STEM education), and are engaged in an ongoing analysis of how opinions may have shifted during the year 2020, which saw interruptions from the COVID-19 pandemic as well as highly publicized demonstrations for racial justice.

An intersectional lens is critical when considering issues related to DEI. For example, Asian people are overrepresented and women are underrepresented in STEM compared to the US population, which obscures the situation of Asian women (Castro & Collins, 2021). While Asian men are well-represented in academic STEM environments, they face discrimination and stigma (Shah, 2019). Studies of women students’ interactions in STEM courses reveal stark differences in white and non-white women’s perceptions of belonging and instructor care, as well as decisions to continue in STEM (Rainey et al., 2018, 2019). Elementary and high school students of Mexican descent are often tracked toward vocations, as opposed to college, and are pushed out of the educational pipeline at high rates which vary by gender, citizenship status, and family income (Covarrubias, 2011). It is clear that many people experience barriers in STEM education and careers based on their gender and race. These barriers must be addressed as a matter of moral justice, and to fill the “missing millions” who are needed in the science and engineering workforce (National Science Board, 2020).

Data and Methods

The data collected for this paper consists of survey responses from a national sample of postsecondary STEM instructors who had already participated in a larger survey related to STEM instructional practices. The larger survey targeted postsecondary instructors teaching introductory STEM courses at two-year colleges, four-year colleges, and universities in the United States. Data collection for that project was conducted in Spring 2019, and the final sample consisted of 3,769 respondents who were primary instructors of a general chemistry, single-variable calculus, or introductory quantitative physics course in the 2017-18 or 2018-19 academic year. At the end of the initial survey, participants were asked if we could contact them for a follow-up survey focused on diversity, equity, and inclusion in STEM. Of the 3,769 who completed the primary survey, 2,229 indicated a willingness to be contacted for this follow-up. In Fall 2020, all 2,229 were sent invitations for this follow-up, and we received 1,064 usable responses. Of these 1,064 participants, 387 taught first-year (General) Chemistry, 305 taught first-year Mathematics (Calculus), and 372 taught first-year Physics.

Here we focus on three aspects of the DEI survey: items aimed at instructors' views of certain factors that might contribute to disproportionate representation of different race-gender groups in STEM, which provide insights into how faculty might explain the relative over- and under-representation of different identity groups; items that asked participants about their views on strategies for addressing race-gender disparities in STEM, such as affirmative action style hiring policies and diversity statements; and items about respondent's personal experiences with discriminatory behavior. Here we provide descriptive analysis of the survey responses.

This sample is not random and is not representative. However, we have enough information to conjecture about the direction in which this bias might skew our results. First, self-selection bias was introduced through the voluntary nature of participation. Our sample includes people willing to participate in education research, and who were willing to receive a survey about diversity, equity, and inclusion issues in STEM, and who then completed that survey. This suggests that our participants are more likely to be aware of, and possibly supportive of, ongoing DEI work in STEM than a truly random sample of STEM faculty. This group may, through their increased awareness, be even more susceptible to social desirability bias. This second form of bias refers to a tendency for survey-takers to provide answers which they think will be viewed positively by others. Broadly speaking, this means that our results likely portray STEM instructors as more aware of DEI issues and engaged with DEI initiatives than the true population.

Results

Perceptions of Reasons for Disparities

Four survey questions captured instructors' views of certain factors that might contribute to disproportionate representation of different race-gender groups in STEM, suggesting how faculty might explain the relative over- and under-representation of different identity groups. The first, and broadest, of these asked whether STEM culture affords an overall advantage or disadvantage to members of eight race-gender groups. Responses to this item are displayed in Figure 1.

White men were the only group that a majority of respondents identified as having a systematic advantage in STEM; Black women, Hispanic/Latinx women, Black men, and Hispanic/Latinx men were perceived by the majority of respondents as having an overall disadvantage in STEM; the majority of respondents perceive that Asian men, White women, and Asian women have neither an advantage or disadvantage. These perceptions have some relation to the actual representation of these race-gender groups in STEM: white men receive the largest share of chemistry and physical science BA/BS degrees, PhD's, and faculty positions; Asian men are the most over-represented compared to their share of the US population at all levels of STEM; Black and Hispanic/Latinx men and women *combined* make up less than 10% of STEM faculty while accounting for 30% of the US population. We note that these responses suggest that participants view Black and Hispanic/Latinx men and women's disadvantagement similarly, but distinguish between men and women among white and Asian groups; this reflects the broader understanding of how racism and sexism intersect in STEM (Castro & Collins, 2021).

The next questions were prefaced with the statement that "more than half of the PhDs awarded in 2017 in mathematics and the physical sciences were received by white men." Again, we asked participants to consider three possible factors contributing to disparities in representation (aptitude, interest, and opportunities) for different race-gender groups; this time the question specifically asked about phenomena *relative* to the dominant group. Participants' responses are summarized in Figure 2.

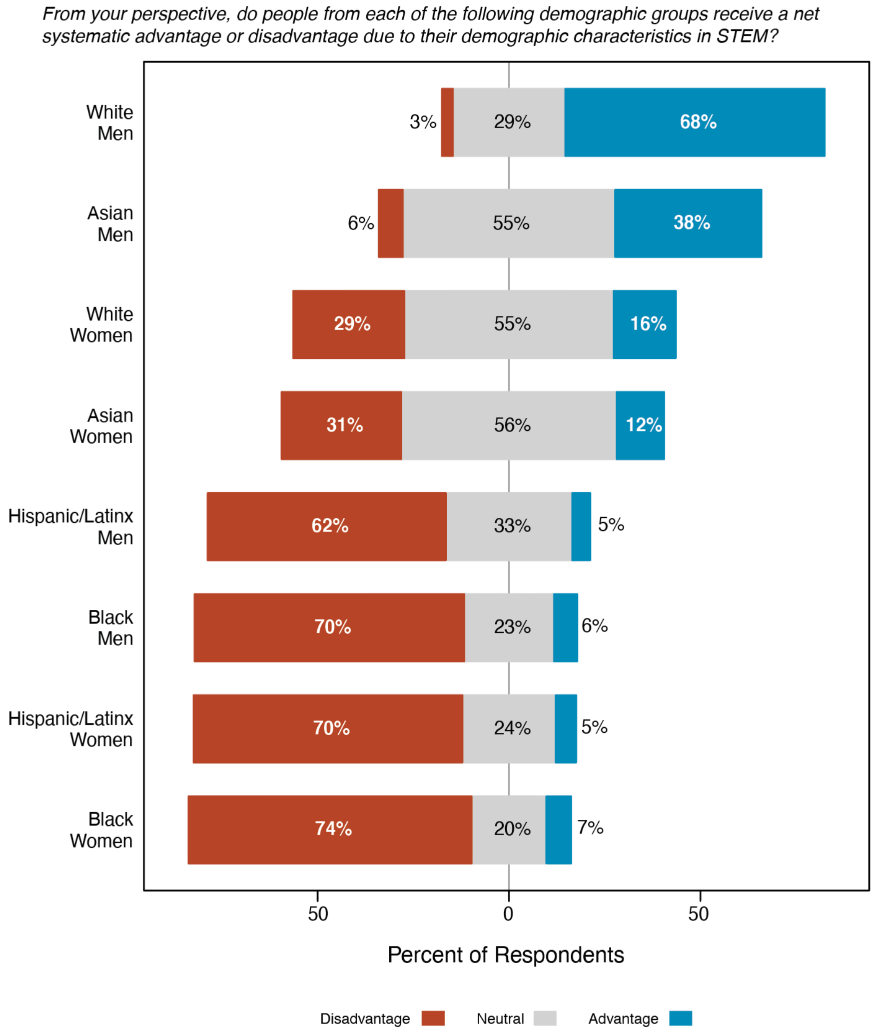

Fig 1. Percentage of respondents reporting that each race-gender group receives a net disadvantage (red, left), neutral (gray, center), or net advantage (blue, right). N=1015

Historically, claims of differential aptitude for mathematics, science, and learning have been used to defend discriminatory policies including segregation and eugenics (Tucker, 2007). These claims have, of course, been refuted, and over 90% of our respondents indicated that members of each race-gender group have equal aptitude for science and mathematics as compared to white men. The other 6-8%, however, are not distributed evenly across groups. The higher ratings of Asian men and women's aptitude for STEM conforms to damaging stereotypes (Shah, 2019; Trytten et al., 2012); the lower ratings for Black and Hispanic/Latinx men and women conform to unfounded racist and sexist tropes. The presence of these beliefs, even among a few, may hamper efforts to diversify our fields; self-selection and social desirability bias likely mean that there are more in the general instructor population who think aptitude is inherently related to race and/or gender than our respondents reported on this survey.

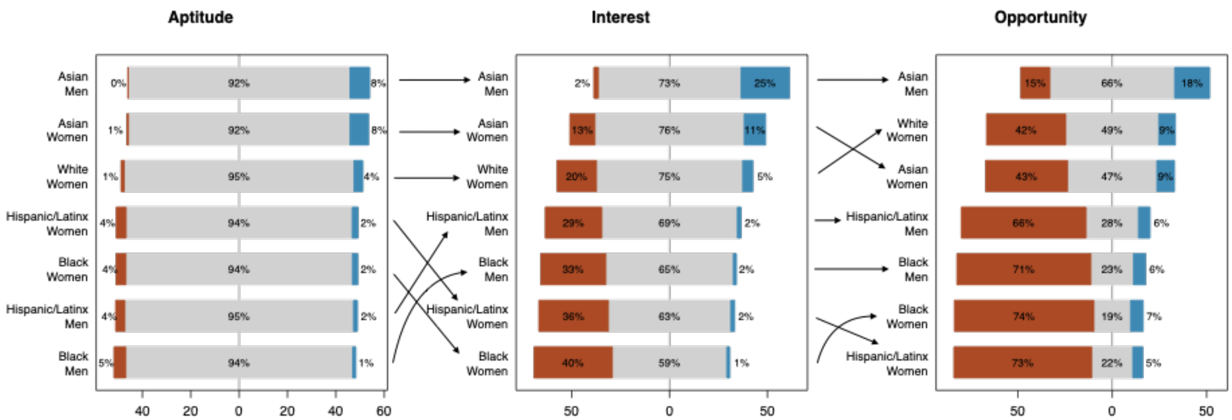

Figure 2. Percent of respondents reporting that, as compared to white men, each race-gender group have less (red/left), equal (gray/center), or more (blue/right) (A) aptitude for; (B) interest in; (C) opportunities to do [your field]. Groups are ordered based on survey responses; arrows indicate changes in group order from one graph to the next. N=1015.

Interest in the field is another justification for unequal representation in STEM (Estrada et al., 2016). We see evidence that many STEM faculty believe that interest in science and mathematics varies across race-gender groups. Many instructors (40%) of introductory courses believe that Black women have less interest in their STEM field than white men. Regardless of where this belief comes from, it is indicative of a significant bias that may play out in a variety of ways that impact students. For example, who is more likely to be seen as a "good fit" for a lab opportunity; who receives what kind of mentoring. Believing that certain types of students are "simply not interested" in STEM shifts the locus of responsibility onto the students, and away from inequitable practices and policies.

Conversations about diversity and equity often also include the topics of access, or opportunities to experience and participate. A large majority (66-74%) of first-year instructors report a perception that Black and Latinx women and Black and Latinx men have less opportunity than White men; just over 40% report the same for Asian and White women. A substantial majority (66%) report that Asian men have equal opportunity as compared to White men. This apparent recognition of systemic disparities in opportunity is heartening as it relates to unpacking the myths of meritocracy which pervade higher education (Liu, 2011). However, additional investigation is needed to understand how this impacts instructors' interactions with their students and their willingness to support or engage with initiatives aimed at redressing disparities. It may contribute to perceptions that individuals from underrepresented groups are less prepared to succeed; or that outreach to certain populations is a priority; or that fault lies with pre-college experiences; or that affirmative action-style policies should be implemented.

Mechanisms for Addressing Disparities

With a general consensus that some race-gender groups are systemically (dis)advantaged, and that opportunities and interest - but not aptitude - are unevenly distributed, we asked participants about strategies for addressing race-gender disparities in STEM. A heartening 86% report efforts to diversify STEM are beneficial for their field (12% had no opinion; 2% said such efforts are detrimental); this is likely inflated by self-selection and social desirability biases. When asked about specific types of initiatives (Figure 3), we find a similarly high proportion of instructors support "diversity, equity, and inclusion statements" by their professional societies, institutions, and departments; a smaller majority support affirmative action-style policies for faculty hiring and student admissions. These results are a positive indicator of support for efforts to diversify the field, but the lower levels of support for more direct actions need further investigation. DEI statements are passive acts, which may not have a direct impact on instructors or DEI; policies related to student admissions and hiring are more active and have direct impacts on the communities. The latter has also been shown to have a concrete and lasting impact on diversity (Bowen & Bok, 1998; Murrell & Jones, 1996), though equity and inclusion do not automatically follow from presence. Understanding what impacts instructors' support, or hesitancy, will contribute to building STEM faculty buy-in for evidence-based strategies for change.

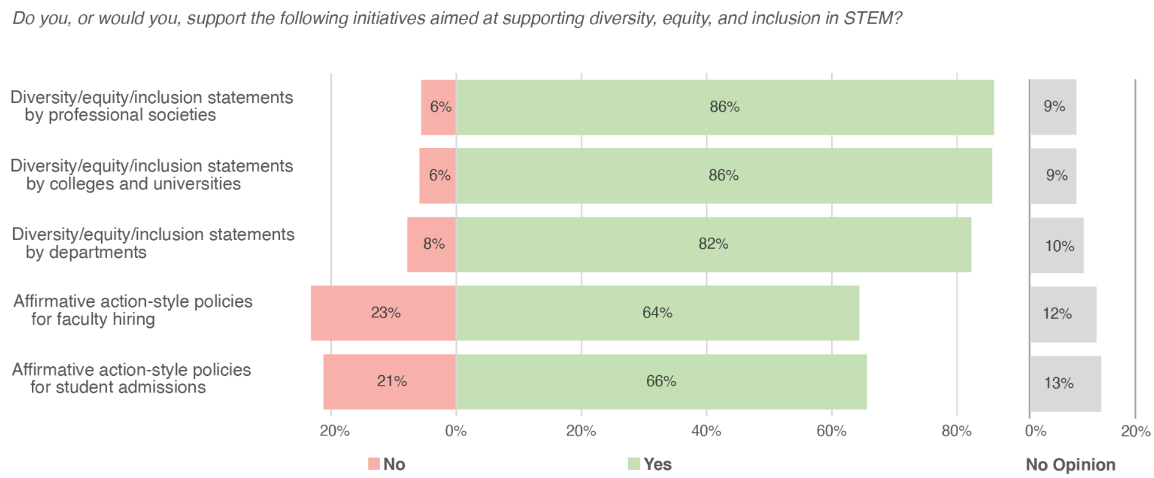

Figure 3. Percentage of respondents indicating that they do not support (red, left), do (or would) support (green, center), or have no opinion (gray, far right) five existing models of DEI initiatives. N=1055, 1054, 1058, 1058, 1057

Personal Experiences with Discriminatory Behaviors

Lastly, our survey asked respondents about their personal experiences with discriminatory behavior. In general, the majority of our respondents reported that they believe their own departments are pretty welcoming, with nearly 70% reporting that they believe their department is either "very welcoming" or "welcoming" to people of different identities, and only about 6% reporting that they believe their department is "hostile" or "very hostile" (with the remaining respondents reporting "neither hostile nor welcoming").

However, a significant proportion of respondents also reported witnessing discriminatory behaviors in a professional setting in their field in the last 3 years: 53% of chemists, 39% of mathematicians, and 45% of physicists. Of those who reported witnessing such behaviors, 66% of chemists, 68% of mathematicians, and 72% of physicists reported intervening and/or speaking up, in the moment or after the fact. Taken together, nearly half of the respondents recently witnessed discriminatory behaviors in a professional setting and, of those, about a third abstained from intervening. We anticipate that responses to this question are particularly likely to be impacted by social desirability bias, suggesting that the true number of people in this sample who have intervened or spoken up about discrimination they have witnessed is lower than reported; self-selection bias suggests that recognizing and responding to discrimination is likely less frequent among the general population. We also note that intervening or speaking up about discrimination carries different risks for different people, so that refraining from speaking out may in some cases be an act of self-preservation.

The question about discriminatory behavior “in a professional setting” could refer to contexts outside a participant’s own departmental context (e.g., research conference), but raises questions about what threshold these instructors might use to consider their department “hostile.” Nearly half of our participants report witnessing discriminatory behavior, and yet 94% report that their department is not hostile toward people of different identities. Perhaps the presence of individuals who commit discriminatory acts is not considered sufficient to categorize the department as a whole as hostile; alternatively, these behaviors are only occurring (or being perceived as occurring) outside their home department.

Conclusion

Undoing systems of marginalization (e.g., racism, sexism, ableism, elitism) requires systemic change through cultural transformation. In the distributed systems of STEM higher education, change can come about best when a critical mass of constituents recognizes existing inequities, believe change is necessary, and act in concert to change systemic structures. Our results provide room for cautious optimism about the potential for future progress. There are at least 688 instructors in American postsecondary institutions who recognize that systematic advantages are afforded to white men in STEM. Furthermore, an overwhelming majority of our respondents (92% of chemists; 89.7% of mathematicians; 90.8% of physicists) agree that they have a personal responsibility to address inequities. While these 1064 survey respondents represent a small fraction of the profession, and may be unrepresentative of the professoriate as a whole, this is still a significant number of first-year math and science instructors who indicate a personal responsibility to address inequity. While we see this number as cause to be hopeful, many questions remain that need to be investigated as new DEI initiatives are designed and implemented.

In our survey, we found that support is stronger among our participants for passive actions (statements) than more active ones (admissions/hiring practices) but whether this support is based on practical convenience and/or beliefs about the impacts of these practices is unknown. Instructors believe that different race-gender groups have different interest levels in science and mathematics - but we don’t yet know if this is viewed as a cause or a result of disproportionate representation. It is also unknown how this perception impacts instructional practice, mentoring relationships, or hiring decisions. Based on our findings here, and questions raised but not answers, thus we argue that future research is needed to document these DEI beliefs more widely and more specifically, including (a) ways in which they impact teaching, hiring, and mentoring; (b) variation across ranks and status; (c) variation by identity group; (d) where they come from; (e) how to change these views in individuals. Answering these questions, among many others, can inform research-based interventions aimed at engaging STEM faculty in more equitable practices and structures.

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The Impact of the COVID Pandemic on Departmental Efforts to Implement Active Learning in Precalculus through Calculus 2 Courses

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For mathematics departments across the United States, everything changed in March 2020 as institutions rapidly moved their instructional delivery model to a model based on remote instruction. This study investigates how four institutions leveraged organizational structures and initiatives to cope with these changes. The findings suggest that the four institutions studied had solid foundations for implementing active learning in their various entry level mathematics classes. Their plans involved different pedagogical, coordination, and communication practices. In light of the strain that the pandemic placed on these efforts, some were found to be more robust and able to leverage digital resources to continue to make progress to their goals for active learning implementation than others. Departments would do well to consider their range of preparedness and the robustness of their efforts to directly impact instruction in Precalculus through Calculus 2 (P2C2) courses and the related coordination and communication.

Keywords: Precalculus through calculus 2 (P2C2); Organizational structures; Active learning; challenges; COVID-19 pandemic

Amongst the challenges present in STEM education, one of the most significant may be the percentage of students receiving grades of D, F, or Withdraw (W) in mathematics courses. Nationally, between 20% and 30% of students receive a D, F, or W in courses ranging from Precalculus through Calculus 2 (P2C2; Apkarian & Kirin, 2017; Bressoud, 2015). Several large-scale efforts have aimed to better understand this problem and work toward reducing DFW rates (Bressoud, 2015; Rasmussen, Apkarian, Hagman, Johnson, Larsen, Bressoud & Progress through Calculus team, 2019; Ström, Webb, Voigt, & Funk, 2021). These studies have found and studied institutions that have had great success reforming courses and reducing DFW rates using a variety of tools (Bressoud & Rasmussen, 2015; Smith, Voigt, Ström, Webb, & Martin, 2021). Most studies have focused on changes that were made at a course, department, or institutional level where individuals or groups of individuals were working to reform instructional methods, student placement, leadership, or department culture. However, in these studies the external environment remained relatively static.

Everything changed in March of 2020 when institutions across the United States shifted their instructional model in response to a rapidly changing environment as the world responded to a pandemic (Sahu, 2020). The recent pandemic provided a unique opportunity to consider the impact of significant alterations in external conditions on P2C2 instruction. This study aims to better understand how departments responded to a radical environmental change and the perceived effectiveness of those responses. Specifically, this study aims to answer the following research questions: *How did the COVID pandemic impact mathematics departments working to include active learning in their P2C2 classes? What organizational structures and initiatives were leveraged in order to make progress toward more active P2C2 practices and to cope with the challenges of the global pandemic?*

Theoretical Framework

The theory that was used as a lens for the study came from Weisbord's Six-Box Model (1976), which has been used in previous case study research for qualitative analysis of P2C2 courses offered by math departments (Moore-Russo, Kornelson, Savic & Andrews, 2021). To better understand how math departments responded to a rapidly shifting environment, we found it useful to analyze how they operated as organizations. Each of the six boxes describes a dimension that should be considered when studying how organizations function. In the first box, *purpose,* are the goals, objectives, missions, and perspectives of the organization. The second box, *structure*, refers to how work is facilitated or divided in an organization. The third is *relationships*, which deals with how organizational members are connecting and their interpersonal interactions. It includes how members behave, communicate with, and regard one another. The fourth box is *rewards*, which includes both formal and informal incentives that motivate (or demotivate) individuals. The fifth box is *mechanisms*, which are the "procedures, policies, meetings, systems, committees, …that facilitate concerted efforts" (Weisbord, 1976, p. 443). It is important to note that in this box the systems, resources, and policies that are available or used to help people plan, budget, and carry out their work as well as coordinate with others may "actually help or hinder people in accomplishing their work" (Yousefi, & Sajadie, 2017). The sixth, and final box is *leadership*, which includes those who have power or influence over other organizational members or over the allocation of resources. The *environment* involves all external demands and pressures that impact (and are impacted) by the organization.

Methods

The four mathematics departments examined in this paper are drawn from a research project involving two dozen mathematics departments engaged in the implementing and sustaining active learning practices in the undergraduate calculus sequence. We loosely define active learning practices to be methods of instruction that acknowledge students should be engaged in the learning process through meaningful mathematical tasks, with peer-to-peer and student-to-instructor collaborative processing of mathematical ideas, and instructor use of student thinking while fostering equity and inclusion (Laursen & Rasmussen, 2019; Strom, Webb, Voight & Funk, 2021). Each mathematics department completed a proposal that described their rationale for increasing the use of active learning, the strategies they were going to use to support changes in practice, and how improvement would be documented. Each mathematics department utilized a localized strategy that considered the degree of instructor buy-in, mathematics instruction, indicators of student success, and ways they might contribute to a Networked Improvement Community (NIC; Bryk et al., 2015) of mathematics faculty and benefit from participating in the NIC.

Mathematics departments participated in the NIC in two groups: a first group received funding in fall 2018 to support the work outlined in their proposal; a second group was invited in spring 2020 to participate in the NIC but without funding to enact their proposed work. Any funding received by the second group was provided by state or campus sources, and not the research project. Both groups participated in and contributed to the NIC, but participation was staggered by about 18 months. Pertinent to this paper: the second group, from which the four cases in this study are drawn, joined in spring 2020 right as the COVID pandemic caused nearly all university instruction to move to remote and online instruction. For this paper, we selected four mathematics departments from the second NIC group after a preliminary analysis using the six-box model. The four departments were selected because they varied in size and geographic locations, and represented a broad range of experiences in the shift to remote instruction: from pre-pandemic changes to a flipped instructions with robust administrative support and departmental buy-in facilitating the transition to remote instruction to significant disruption and retrenchment of previous change efforts. In a P2C2 context, these departments' experiences reflect larger trends surveyed in undergraduate mathematics programs (Kirkman, Blair & Barr, 2022).

For the four mathematics departments analyzed and discussed in this paper, we interviewed mathematics tenure-track faculty, teaching faculty, graduate student instructors, and campus and department administrators between summer and fall 2020. This group of interviewees also included those who co-authored the local proposal for active learning. Each of these interviews was conducted using a common protocol and included questions pertinent to the role of the interviewee in the proposed work. Since the data collection for these four cases occurred after the onset of the pandemic, we also asked questions about how the pandemic influenced classroom practices and department policies. These specific questions from the interview protocol included:

- *In what ways have active learning efforts been impacted by the COVID disruptions?*
- *How successfully has the mathematics department handled the shift to remote instruction for precalculus and calculus classes?*
- *What factors have been important in helping make this transition?*
- *What has the department learned from online instruction that may be used for precalculus and calculus instruction in the future?*

Interviews were conducted virtually using video conference software. These virtual meetings were recorded and transcribed. After the transcriptions were cleaned for accuracy, teams of researchers collaborated in the analysis of interviews for themes that emerged from the interviews. The original proposal from each institution was also used as a data source that documented both the rationale for the proposed work and expectations for progress over the short term. The qualitative analysis of these interviews contributed to summary reports that were sent to the project lead for each mathematics department to confirm and correct how the research team described goals, progress and recommendations from the data sources which included interviews with mathematics instructors, graduate students, and department and campus administrators. In this paper, using the six-box model we used these data sources and summary reports as the basis for our analysis of how each of four mathematics departments responded to rapid changes in the environment.

Weisbord's (1976) model was used to construct cases for four mathematics departments using each of the six constructs. Similarities and differences in these organizational constructs were identified and related to descriptions of departmental responses to the pandemic. Thick descriptions for each case were also developed that included narrative descriptions of topics such as: use of course coordinators, professional development, conceptions of active learning, levers that were utilized, and challenges encountered. Themes and relationships between organizational constructs and responses to the pandemic were then summarized and reanalyzed for lessons that could be learned from departmental approaches and responses to the COVID pandemic.

Findings

The data set came from four different mathematics departments. All four departments are situated in public institutions with a Carnegie classification of a Doctoral University: High

Research Activity. Enrollments and institutional locations varied between institutions, as are described below.

Mid-Size City Campus (MCC)

MCC is a public university located in an urban location with a student population of approximately 15,000. At the outset of the study, the mathematics department at MCC undertook the goal of reforming instruction in P2C2 by including training for P2C2 instructors, focusing particularly on part-time instructors, the development of more shared materials for P2C2 courses, and the hiring and training of both graduate and undergraduate learning assistants for P2C2 courses. Historically, tenure line faculty at MCC do not teach courses below calculus; instruction in these courses is relegated to adjunct faculty. The proposed mechanism for change was a bottom-up approach in which individual instructors make changes with the goal of reaching a critical mass that can eventually influence the faculty at large. In 2019, a new senior vice chancellor noted that math courses were a bottleneck for graduation and committed resources to the math department, particularly, the hiring of a Director of Quantitative Reasoning. The institution has established relationships with another institution that has an established active learning in mathematics (ALM) program; MCC leveraged that relationship to borrow and adapt materials that could be used in many of the P2C2 courses. Active learning is not an expectation for promotion/tenure. However, using innovative teaching methods is grounds for receiving a distinguished rating on teaching.

The pandemic forced MCC to rethink a lot of things ranging from what ALM involved to departmental priorities. One instructor stated, "*We just can't do things, you know, engagement and active learning look very, very different. It's almost non-existent to be honest, you know. You can't let them be talking to each other close to each other.*" However, another instructor argued that the move to remote classes forced instructors to reconsider lecture and the inability to "*[cram] stuff down there in a lecture.*" At MCC, it appears that some instructors felt like they were able to transition into ALM remotely while others felt unable to adapt ALM to remote instruction. The chair at MCC noted that the pandemic interrupted the department's work, "*We missed the last two full department meetings because there was just too much COVID stuff to deal with*." The chair continued to discuss how prior to COVID there were departmental level discussions of how Calculus could be changed, but "*they got blown up because of COVID*." Nevertheless, the chair remarked that they had learned a lot through COVID and the experience would certainly change the teaching in the department.

Large City Campus (LCC)

LCC is a public, urban university of ~25,000 students. Prior to participation in the NIC, LCC engaged in an effort to improve STEM education and increase the flow of students to engineering majors. LCC's upper administration made a key hire to oversee the courses leading to calculus; this director was given authority over hiring, firing, and the budget of all courses before Calculus. The director oversaw a cohort of five instructors who met regularly and were given release time to work on curriculum. Courses were redesigned to include evidence-based practices including active learning. The director created relationships with feedback loops with other "client" disciplines (e.g., Schools of Engineering and Business) as well as the Math Department Chair. He used data to study DFW rates in P2C2 courses as well as in other STEM courses that had math prerequisites. As a result of these efforts, DFW rates in courses leading to calculus dropped significantly and the number of students moving on to calculus grew substantially.

COVID forced classes to change to an online platform. The curricular innovations that had been adopted were mostly abandoned as instructors moved online, both in synchronous and asynchronous environments. Even when classes were allowed to return to in-person instruction, all students faced forward with six feet of separation. The tendency was to return to lecturing, and instructors who tried to engage students struggled. As one instructor said, "*Remote [instruction] is very different from when we are in class.... Previously, the instructor circulated, students might go to board, but instructors weren't really the person who drove things. The groups were. Remotely, I have not found a way to not be the leader of the activity. You can do the pregnant pause all you want, but there isn't the communication between them [the students] that was so integral to the activities we were doing. There has to be a better way, but I haven't found it, and I don't think others in my area have either*."

The feedback loops also faltered due to COVID. Rather than in-person meetings with rich discussion, most communication during this period was via email. Almost all communication flowed through the director in an overwhelming fashion. Communication between the director and the Math Department Chair and heads of the client disciplines fell away. The Chair reported that, "*There is a lot of frustration. Everyone is tired.*" The fatigue and frustration came out in the director's interview. The director questioned the sustainability of what they had accomplished pre-COVID stating, "*COVID has stalled us. COVID has set us backward….I've gotta figure out how to create an active learning environment online, and I don't know where to begin on that.*"

Large Research Campus (LRC)

LRC serves over 30,000 students. Their proposal to expand active learning focused specifically on the move from "*large, coordinated lectures in Calculus I to a system of flipped classroom delivery of this vital course*." Faculty involved in the proposal argued that student success in the large lecture version of the course was mixed and that quality of instruction in the course needed attention. The vision was to move the course into the 21st century using active learning practices.

The leadership for this effort was significant. A well-respected member of the mathematics faculty with an interest in student learning organized the move to flipped classrooms to create more opportunities for student discussions in classrooms (instead of lecturing). Campus leadership noticed these curricular and instructional developments in the mathematics department and described how infrastructure funds were made available to support the shift from large lecture stadium seating to classroom seating that supported group activities and peer interaction. Funding for the hiring of additional tenure line and teaching faculty was also allocated to the mathematics department, in response to these curricular and structural changes in the design and teaching of calculus. As new faculty were brought in, they took over as course coordinators which gave them authority to make structural changes in how communication between instructors was organized. Improved coordination allowed for better uniformity of topics, instructional materials, and assessment. More regular meetings to support this coordination improved communication between calculus instructors.

The move to flipped classrooms motivated an investment in technology-enhanced instruction with video recordings that students could watch before attending in-person sessions. Coincidentally, this all occurred prior to the pandemic. In March 2020, when the transition to remote learning occurred, the move was relatively smooth according to most interviewees as materials were already online and instructors were more comfortable with using technology to support teaching and learning. The motivation of a key leader in the math department to craft a proposal for instructional innovation, the work of calculus instructors to convert course activities to support a flipped classroom model, and the campus support for classrooms that support active learning all converged to support progress toward the goal to expand and sustain active learning. These factors allowed for synergistic processes to emerge amidst the pandemic.

College Town Campus (CTC)

CTC is a STEM-focused university with a student enrollment of ~6,000. Similar to the other institutions, CTC has both tenure-stream ladder-rank faculty and VITAL (i.e., visiting, instructional, TAs, adjunct, and lecturing) faculty. Prior to CTC's participation in the NIC, its stated goal was to improve coordination within and between courses in their calculus sequence for a more uniform experience for both students and instructors. Every lower-division large enrollment course (with one exception) is coordinated with coordinators mainly drawn from the teaching faculty. Two of the common concerns among the teaching faculty were the lack of cohesion among P2C2 faculty and the lack of a unified vision on the role of coordination.

Prior to COVID, the teaching faculty self-organized a departmental-based professional learning community (PLC) open to all faculty and offered during the fall and spring semesters. Participation was voluntary but incentivized with lunches funded by the department. The PLC served as a meeting ground to discuss teaching, active learning techniques, and readings from math education and scholarship in teaching and learning literature (e.g., MAA Instructional Practices Guide). However, the COVID pandemic and the switch to remote instruction put an abrupt halt to these meetings (at least during the fall semester) and funded lunches. One interviewee noted, "*We haven't tried it [having the PLC without lunches] yet. I don't think we've had the bandwidth to do it [the PLC] this fall."* The lack of the PLC meeting was especially missed in creating a community amongst the VITAL faculty. One interviewee reported, "*Some people might be missing, just the community piece and talking to someone."* The interviewees were split between waiting to bring back the PLC during the 2021-2022 academic year or during the 2020-2021 spring semester. In contrast, the department's GTA Teaching Seminar, their PLC aimed at graduate student instructors offered as a regularly scheduled class, continued throughout fall 2020. In response to COVID, a teaching session on using Zoom features of breakout rooms and polling was added to the GTA Teaching Seminar.

With the shift to remote and hybrid instruction, Calculus 1 at CTC was delivered in a flipped model with pre-recorded lectures students viewed before class and in-class activities. There was ambivalence expressed by one of the coordinators on continuing with the flipped model: "*Oh, I mean it [the flipped model] certainly could [continue]. I think it could work but I don't know that I want to do it that way.*" However the coordinator liked the activities created for the flipped model and expressed interest in using these activities in a class model with more in-class lecture-based delivery interspersed with activity-based days.

Discussion

In our case analysis several themes emerged with respect to how mathematics departments responded to a common change in *environment* imposed by COVID. Prior to the pandemic, each mathematics department proposed a plan to make progress toward a common *purpose*, namely increasing the use of active learning in P2C2 courses. As such, *leadership* articulated potential *mechanisms* for enacting change for the support, use, and expansion of active learning. Differences in the emerging development of *structures* and *relationships*, however, created a range of preparedness for managing a crisis that directly impacted instruction in P2C2 courses and the related coordination and communication of norms and practices.

Prior to the shift toward online and remote instruction, LRC had developed their coordination *mechanisms* using resources to change large lecture courses to flipped classroom models through a combination of lecture videos and revised instructional materials to support student discussions in small groups. We acknowledge that the timing of LRC's campus initiatives (*rewards*) to hire coordinators and support curriculum development for flipped classrooms was serendipitous; nevertheless, regular meetings focused on course redesign helped LRC develop communication *structures* and *relationships* while providing resources for an unforeseen crisis. The pandemic motivated faculty at CTC to redesign courses using a flipped model, but urgency to develop and use that model under challenging conditions may have stigmatized use of flipped instruction when CTC returns to in person instruction.

In the cases of MCC, LCC and CTC, communication was disrupted during the pandemic; professional development meetings were discontinued; and instructional resources that could be used in technology-intensive environments were underdeveloped. Many of the existing instructional and communication *mechanisms* faltered in online settings. *Structures* for promoting student engagement and creating a supportive instructional community that had worked while in-person often did not transfer to online platforms. Certainly, there was no way to predict the advantage of developing videos or tech-friendly tasks to support active learning prior to March 2020; however, *the importance of developing robust communication norms and practices as mechanisms to maintain and advance structures and relationships when a department encounters challenges is noteworthy*. When mathematics faculty are working toward a common purpose, there is a need for communication. When workload intensifies, the *relationships* that are supported by communication can create opportunities to share the load through the coordination and delegation of instructional resources, assessments, and ways to implement active learning in virtual environments.

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Boosting Student Preparation in Flipped Multivariable Calculus with Low-Stakes, Daily Quizzes

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Motivating students to watch pre-lesson videos to prepare for class is among the most significant challenges cited with flipped learning. We demonstrate boosting student preparation by up to 29% by implementing start-of-lesson quizzes. This is shown through comparison of a flipped control group without the quizzes to flipped experimental groups including the quizzes. Results indicate daily, low-stakes quizzes can also reduce failure/withdrawal rates and help maintain student motivation to prepare for class. A marginal increase in performance was also noted for students in the experimental group in spite of their lower pre-test scores. Lastly, statistically significant correlations were found between student perceptions of the class and their video-viewing habits, homework completion and graded event scores.

Keywords: Flipped Learning, Low Stakes Quizzes, Daily Quizzes, Student Preparation, Multivariable Calculus

Flipped learning has been implemented in numerous undergraduate mathematics courses with varying instructional activities leading to varying degrees of success. The instructional activity showing the strongest positive effect on student learning appears to be conducting formative assessments during each lesson (Chung, Khe, & C., 2017). This study follows with research questions asking:

- How do low-stakes, daily quizzes affect students':
  - Preparation?
  - Retention of motivation to prepare throughout the semester?
  - Failure/withdrawal rates?
  - Exam performance?
- Does it matter whether the daily quizzes are at the start or end of the lesson?
- How do student perceptions correlate with study habits and performance?

Background

The United States Air Force Academy (USAFA) has a goal of graduating over 50% STEM majors to better satisfy degree requirements of various career fields (NRC, 2010). Multivariate calculus, or Math 243, is a required math class for most USAFA STEM majors, but has a historically high withdrawal/failure rate approaching 20%. Failure rates for multivariate calculus at other universities are high as well, often in the 20-40% range (Caerols-Palmaa & Vogt-Geisse, 2019; Groen, Coupland, Langtry, & Memar, 2015). We therefore seek to reduce failure and withdrawal rates in Math 243 while continuing to uphold high academic standards.

Benefits and Challenges of Flipped Learning

USAFA recently changed Math 243 to a flipped modality, which we define for the purposes of this study as "a technology-enhanced pedagogy that delivers parts of the course materials through video resources before class, followed by the integrated use of assessments, mini-lectures, individual problem solving, and small-group learning activities inside the classroom" (Chung, Khe, & C., 2017). This modality has been shown in other studies to decrease failure rates by 8-30% due to increased active learning and opportunities for targeted feedback and instruction (Clintondale High School, 2014; Cronhjort, Filipsson, & Weurlander, 2018; Petrillo, 2016; Schroeder, McGivney-Burelle, & Xue, 2015).

Aside from decreasing failure rates, flipped undergraduate mathematics courses have also been shown to increase student performance. Seven recent studies of undergraduate integral and multivariate calculus classes examined the performance of students in flipped vs. traditional groups (Adams, 2016; Braun, Ritter, & Vasko, 2014; Kennedy, Beaudrie, Ernst, & St. Laurent, 2015; Maciejewski, 2016; Petrillo, 2016; Schroeder, McGivney-Burelle, & Xue, 2015; Scott, Green, & Etheridge, 2016). On average, the studies showed a performance improvement of 0.14 standard deviations.

Primary challenges were captured in a meta-analysis of 61 studies on flipped mathematics courses. The challenge of student unpreparadness was found to be second only to the challenge of student unfamiliarity with flipped learning (Chung, Khe, & C., 2017). The challenge of unpreparedness is where the benefits of start-of-lesson quizzes comes in.

Benefits of Start-of-Lesson Quizzes

Despite an average increase in performance shown for classes incorporating a flipped pedagogy, study results differ based on differing instructional activities implemented. For example, Figure 1 shows effect sizes for seven studies of flipped learning in undergraduate calculus, three of which incorporated start-of-lesson quizzes (shown in red), and four that did not (shown in blue). Notice that those without start-of-lesson quizzes do not see the same level of benefit as those that did. In fact, a meta-analysis including 22 flipped mathematics courses showed that including quizzes at the start of class sessions led to an average performance improvement of 0.57 standard deviations as compared to only 0.20 for those not including them (Chung, Khe, & C., 2017). Incorporating start-of-lesson quizzes thus appears to be a distinguishing component towards increasing the effectiveness of flipped instruction.

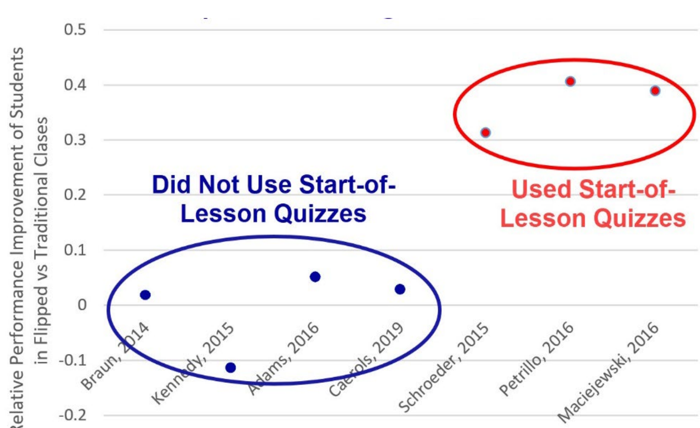

Figure 1. Effect sizes of flipped calculus studies yielding an average mean improvement of 0.14 standard deviations. Red/blue markers indicate studies that did/did-not incorporate start-of-lesson quizzes, respectively.

Theoretical Perspective

The theoretical benefits of low-stakes, daily quizzes include the five primary components listed below:

a) Motivation to prepare: Assigning course points provides extrinsic motivation to come to class prepared, and is considered a best practice (Kennedy, Beaudrie, Ernst, & St. Laurent, 2015).
b) Retrieval Practice: Even no-stakes quizzes have been shown to increase learning from videos (Schacter & Szpunar, 2015) as the required recall process strengthens neural pathways to the required information.
c) Targeted Feedback: The quizzes provide the instructor with helpful feedback on where their students require additional instruction, allowing instructors to address misconceptions before they become entrenched. This type of feedback has been identified as among the most powerful influences on learning and achievement (Hattie, 2007), and is a critical component of frequent low-stakes quizzing (Agarwal P. K., 2021) (Butler & Roediger, 2008).
d) Activation of Prior Knowledge: Quizzes at the start of class activate student knowledge, laying a solid foundation on which students can build their learning during class (Merill, 2002).
e) Attendance: Quizzes at the start of class have also been shown to improve attendance (McBride, 2015). However, this can be separated out of this study since attendance is mandatory at USAFA.

Student perceptions

A complete picture of any intervention requires consideration of student perceptions as well as performance. Studies on flipped instruction often assess both. A recent literature review of flipped mathematics courses showed that of fifteen undergraduate calculus studies, ten reported generally positive and two reported generally negative student perceptions of the flipped vs. traditional methods (Kooistra, 2018). The generally positive perceptions indicate students are typically amenable to the method.

Novelty of Study

Most of studies reviewed above compare the performance of an experimental, flipped group, to a control, traditional group. This study is different in that both experimental and control groups are flipped. This tighter control setting allows investigation into the effects of daily quizzes, including any differences between start-of-lesson and end-of-lesson quizzes. Another important contribution of this study are the correlations examined between students' perceptions, study habits and performance, which is largely absent in the literature on flipped learning.

Methodology

The difference in class structure between the groups is shown in Figure 2. Before class, all groups have the opportunity to earn extra credit by answering a question based on the first video for each lesson. This is recommended to motivate video watching and allow for student practice and self-checking (Chung, Khe, & C., 2017).

Unlike the control group, experimental groups 1 and 2 would take a low-stakes quiz during class. Experimental group 1 would take the quiz towards the end of class, while experimental group 2 would take the quiz at the start of class. Both control and experimental groups would incorporate active learning exercises during class. Experimental group 2 was unique in offering some class time for students to work on homework with peer and instructor assistance during class time. The quizzes consisted of 3-5 short response and/or multiple choice questions based on the associated pre-lesson videos. The quizzes were proctored within 5 minutes through

Microsoft Forms within Microsoft Teams. This facilitated ease of viewing results to provide feedback, ease of grading, and allowed students the ability to review their performance and score on each problem. Open-notes were allowed to encourage use of the notetaking documents provided to align with the pre-lesson videos.

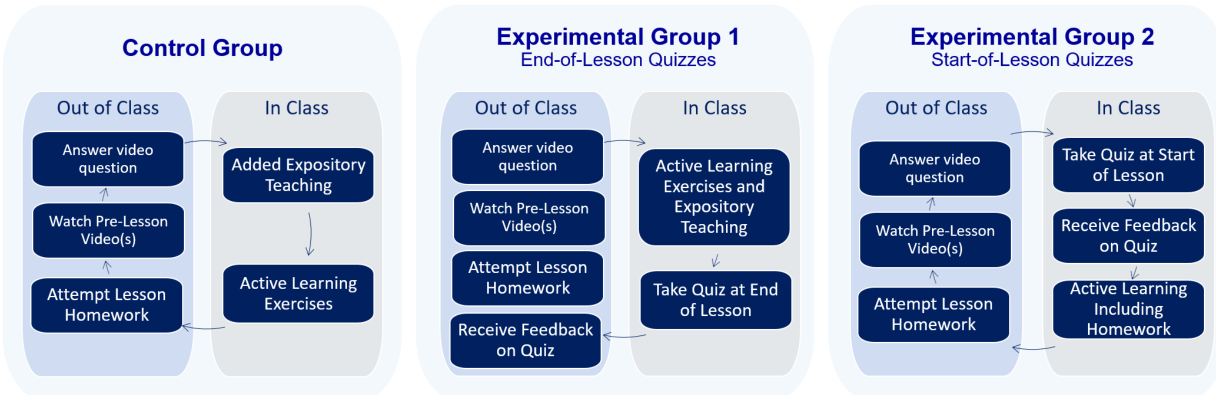

Figure 2. Typical in and out-of-class processes for control and experimental sections.

The control and experimental groups were tightly standardized with the same syllabus, course calendar, learning objectives, homework, midterm exams and final exam. The only significant differences were with 75 out of 1000 course points and the in-class structure. These 75 points were allocated to two medium-stakes quizzes in the control group and to 30 start-of-lesson quizzes in the experimental groups. The two lowest quiz grades were automatically dropped for each student in the experimental group, as recommended to relieve student pressure (Agarwal & Bain, 2019).

The pre-lesson videos were mostly created in prior semesters by various Math 243 instructors to align with course objectives and better establish the student-instructor connection. Videos were typically broken up into 5-10 minute segments as recommended to enhance student engagement (Guo, Kim, & Rubin, 2014). The videos were hosted on Echo360, which enabled tracking of video-watching habits for each student.

Instructor equivalence was controlled for with two instructors teaching the experimental groups and five instructors teaching the control sections. There were a total of 49 students in the experimental groups and 162 in the control group who finished the class. Our primary findings are limited to 1st-year students who took integral calculus the previous semester, which eliminated 7/29 students from the experimental/control groups, respectively. This reduced the sample size but allowed the students' integral calculus final exam scores, taken just 3 weeks before the start of the semester, to be used as a 'pre-test' to control for student equivalence.

Student perceptions were investigated through feedback collected using mid and end-of-semester questionnaires. Student names were associated with their responses in order to draw correlations between perceptions, habits and performance. The questionnaires and analysis were handled in such a way as to guarantee the confidentiality of student responses in order to ensure transparency and reliability of feedback.

Results

Video Viewing

The percentage of students in the control and experimental groups who viewed each of 121 videos in the course can be seen in Figure 3. Each column represents a video with the green, red and grey bars representing the percentage of students who watched the video before the associated lesson, after the lesson, or did not watch it at all, respectively.

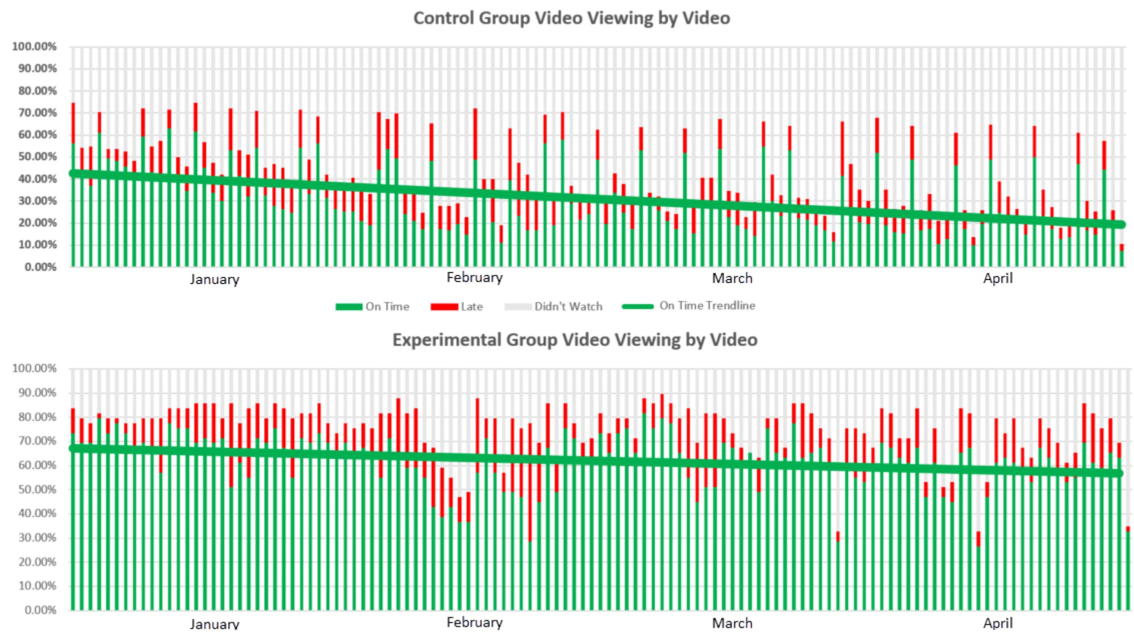

Figure 3. Video viewing percentages of students in the control and experimental groups. Each column represents one of 121 videos in chronological order through the semester.

The green trend lines in Figure 3 show the general decline of video viewing throughout the semester. This is consistent with another multivariate calculus study on flipped learning showing a 30% decline in video viewing throughout the semester (Caerols-Palmaa & Vogt-Geisse, 2019). It is noteworthy, however, that viewership in the experimental group only declined by 10% as compared to a decline of greater than 20% in the control group. It thus appears the start-of-lesson quizzes motivate both increased and sustained pre-lesson preparation.

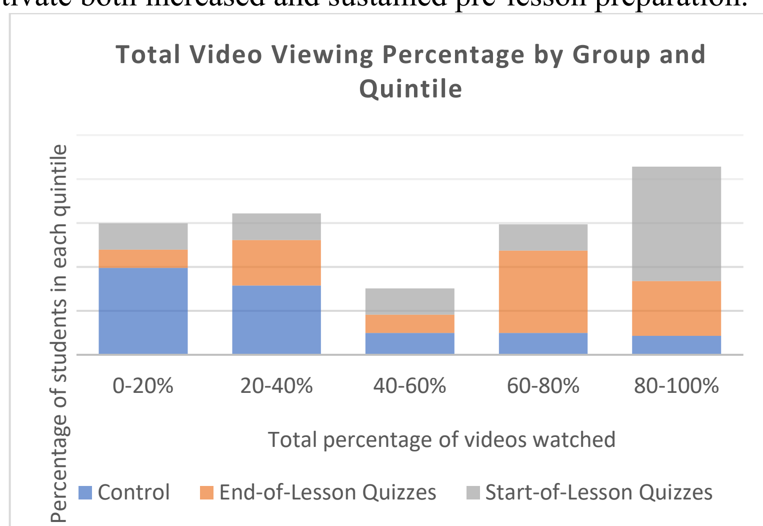

Figure 4. Total video viewing percentage (before or after associated lesson) by group and quintile, showing the lowest/highest percentage of video viewing amongst students in control/start-of-lesson-quiz groups, respectively.

There are noteworthy, periodic spikes in the control group data corresponding to the first video of each lesson. These viewership spikes are due to both the control group instructors emphasizing the importance of watching the first video, and the motivation from extra credit for students who responded to the pre-lesson questions, which were based exclusively on content in the first video of each lesson.

Students in the control, end-of-lesson quiz and start-of-lesson quiz groups watched a total of 49.0%, 61.5% and 78.2% of the 121 videos, respectively, revealing a 29% boost from start-of-lesson quizzes. This can be seen from a different perspective in Figure 4. It shows students with the start-of-lesson quizzes were most diligent in watching the videos before class, while the control group students were the least diligent. Having the quiz at the start vs. the end of class likely led to greater motivation to prepare since having the quiz at the end of class might have led students to trust they could learn the pertinent quiz material during, instead of before, class.

On the mid and end-of-semester feedback, only 9% and 0% disagreed with the statement "The daily quizzes increased my motivation to watch the pre-lesson videos". This confirms, along with the data above, the impact quizzes can have towards motivating student preparation.

Student Performance

Students in the experimental group slightly underperformed by 0.9% on the pre-test and overperformed by 0.3 and 0.5%, respectively, on the midterm exams and final exam. Further detail can be seen in the whisker plot shown in Figure 5.

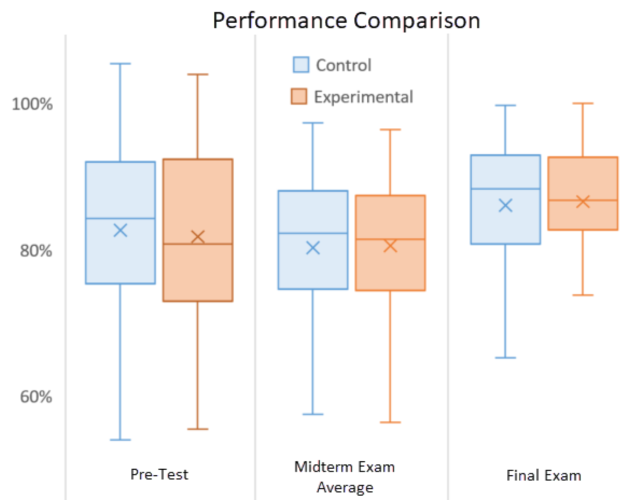

Figure 5. Performance comparison between control and experimental groups on their pre-test, midterm exams and final exam. The mean is shown by the X, the median by the horizontal line, the lower and upper quartiles by the boxes, and the minimum and maximum by the whiskers.

This slight performance improvement from including daily quizzes is perhaps not as pronounced as might be expected from Figure 1. It should be noted, however, the control group in this study also had a flipped modality and implemented many excellent learning activities, such as hands-on and computer-aided 3D visualization techniques, computer-aided calculation tools, active learning exercises, field trips, and pre-lesson video questions. The control group scores also got a slight boost from the late withdrawal of the seven students mentioned earlier.

A performance comparison between the two experimental groups shows students with end-of-lesson quizzes came in underperforming by 2.9% on the pre-test and slightly under and over performed by 0.5% and 0.5%, respectively, on their midterm exams and finals. The end-of-lesson quizzes may have better motivated student attention during class, and the in-class homework done with the start-of-lesson quiz group may have been counterproductive to the benefit they might have gained from completing their homework on their own outside of class.

Withdrawal/Failure Rates

Out of 235 students who started the class, 24 did not pass. Two of these 24 students were in the experimental group and withdrew on the first day of class. Four students in the control group failed the class, seven of them withdrew at some point after the first three weeks of class, and eleven withdrew within the first three weeks of class. Given the expected withdrawal/failure rate of 13% in the control group, the single variable chi squared p value for a correlation between the experimental group having only two students who did not pass the class is 0.08, indicating a good likelihood the daily quizzes increased the odds of students successfully passing the class.

In the mid and end-of-semester feedback, 88% and 93% of students, respectively, agreed with the statement "The daily quizzes helped keep me on pace in the course." Keeping students on pace may very well have contributed to their likelihood of successfully passing the course.

Correlations between student perceptions, effort and performance

Statistically significant correlations were found between student perceptions of the class and their video-viewing habits, homework completion and graded event scores, as shown in Table 1. Students' opinion of the course was determined by their choice of six Likert scale responses to the question "Overall, this course is:"

Table 1. Correlation between first and second items found using the Spearman rank correlation method.

First Correlation Item	Second Correlation Item	ρ value	p value
overall course opinion	video viewing percentage	0.183	0.038
overall course opinion	online homework completion	0.144	0.104
overall course opinion	average performance on midterm exams	0.392	$4.29 \cdot 10^{-6}$

These correlations suggest students who buy into the flipped modality are more likely to adequately prepare for class, complete their assigned work, and perform better on exams.

Conclusion

The vast majority of prior studies on flipped learning compare the performance of an experimental, flipped group, to a control, traditional group. This study is different in that both experimental and control groups are flipped, allowing us to zoom in on the effects of daily quizzes. We find that proctoring low-stakes quizzes at the start of each lesson leads to a significant, 29% increase in students' pre-class video viewing. This confirms the theoretical benefit of daily quizzes that they increase student motivation to prepare. The proposition that daily quizzes also reduce withdrawal/failure rates is supported with a p value of 0.08. This can likely be attributed to the theoretical benefit of targeted feedback from daily quizzes, helping students keep pace in the class. Lastly, while correlation analysis is largely absent in studies of flipped math courses, this study showed that positive perceptions of the course correlate with increased pre-class preparation and better exam performance.

Acknowledgments

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Transitioning Active Learning STEM Recitations Online

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As part of a more extensive study on transitioning active learning STEM recitations to the online setting, twenty mathematics instructors and course coordinators took part in a survey on transitioning their traditional in-person recitations to an online setting. Participants were asked to reflect on what elements from in-person recitations were retained and lost in the online setting, along with challenges and considerations for both students and instructors. The instructors taught coordinated recitation courses in the Pre-Calculus to Calculus II track for engineering and non-engineering students. Key themes that emerged from the surveys included sustaining collaboration, maintaining student engagement, and administering activities to keep students accountable. This article summarizes our findings and provides recommendations for future online recitation implementations.

Keywords: Active learning, gateway courses, online recitations, course transformation

In the broadest sense, active learning refers to classroom strategies that move away from instructor-centered lectures toward a learner-centered teaching model that focuses on learners' problem-solving and knowledge creation skills (Theobald et al., 2020). As instructors continue to explore active learning strategies in their fully online and hybrid courses, knowing which techniques and tools best enable active learning is necessary for determining the feasibility of implementing such tools in an online setting. This paper examines the rapid transition to online mathematics instruction due to the global impact of COVID-19. Specifically, we look at the transition from active learning face-to-face mathematics recitations to active learning online recitations from a faculty perspective. Our online implementations were done in response to "emergency remote teaching" (Hodges et al., 2020) with the anticipation that many sections would return to in-person sections over time. For the sections that will remain online in the future, we tried to extract the key findings that will support an online implementation of an active learning classroom environment.

Review of the Literature

Research in mathematics education demonstrates the effectiveness of active learning in student success (Hsu et al., 2008). In the "active learning" setting, the instructor takes on more of a facilitator role in helping students master targeted learning outcomes (Mayer, 2004). The benefits of active learning have been widely studied in many contexts and have a demonstrated history of students' improved critical thinking and writing skills (Baepler et al., 2014; Bonwell & Eison, 1991; Freeman et al., 2014), stronger learning outcomes on posttests (Freeman et al., 2014), lower overall course failure rates (Baepler et al., 2014; Reinholz, 2015), and stronger conceptual understanding of targeted concepts (Hake, 1998). Active learning instructional strategies in STEM also have been advocated to promote better exam performance (Freeman et al., 2014), to focus on recruiting and retaining racially minoritized students (Dirks, 2006), to reduce the achievement gap (Haak et al., 2011; Seymour et al., 2019), to elevate racially minoritized student persistence (Estrada et al., 2017), to increase the success of female students (Laursen et al., 2014), and to enhance performance across multiple outcomes such as grades and persistence (Kuh, 2008). Some of the most convincing support for active learning came from the Freeman et al. (2014) meta-analysis of 225 studies on active learning in the STEM fields. They concluded that in comparing active learning to traditional learning, students achieved an increase of 6% on examinations in active learning settings. In the following subsections, we relate active learning and recitation research to teaching active learning-based online recitations.

Online Active Learning

With the rapid shift to emergency online instruction due to the COVID-19 pandemic, instructors accustomed to implementing active learning activities in the face-to-face setting were forced to implement active learning in the online environment. Examples of active learning methods for the online setting used include activity-based frameworks, role-playing, problem-based learning, and the jigsaw method (Amador & Mederer, 2013; Anderson & Tredway, 2009; Baker & Watson, 2014; Lebaron & Miller, 2005). More recent studies in online active learning involve students creating their own content under the instructor's guidance. Student-created videos in undergraduate STEM courses (Campbell et al., 2020) and "viral videos" in marketing classes (Purinton & Burke, 2020) are two such more recent attempts at building online active learning opportunities that have led to students' increased understanding of the course content.

Collaborative learning through video tutorials and flipping the classroom also fall under the active learning umbrella. Flipping the classroom has been done for many years and has primarily been used in blended settings where students watch videos or read before coming to class to apply their knowledge to more complex problems once they arrive at the face-to-face class (Bergman & Sams, 2012). Many studies have touted increased learning gains, improved transfer of information, and positive student perceptions in flipped classrooms, particularly in undergraduate mathematics (Adams & Dove, 2018; Albalawi, 2019; Sun et al., 2018; Wasserman et al., 2017). Outside of specific instructional approaches, another study found that successful implementations of online active learning require strong interpersonal interactions and frequent and effective student-instructor interactions to foster a climate of active learning and academic achievement (Jaggars & Xu, 2016).

STEM Recitations

Recitation courses have a long-standing history of supplementing the lectures' content by using smaller class sizes where active learning can be more easily implemented (Laursen et al., 2019). Recitations sometimes include hands-on components to encourage engagement and conceptual understanding (Etkina & Van Heuvelen, 2007). Some rely on computer simulations to provide near-life experiences with illustrated phenomena. Incorporating active learning and collaborative activities in recitation is rooted in the theory that collaboration involves the metacognitive activities of discussing goals and ideas that improve learning and retention (Bruffee, 1984; Hillocks, 1986). Indeed, studies have shown that setting up collaborative groups in recitations has led to students' increased success in mathematics (Bonsangue, 1994; Herzig & Kung, 2003; Springer et al., 1999), physics (McDermott et al., 1994), and chemistry (Gosser & Roth, 1998).

In one such large-scale recitation study in Calculus 1 for STEM majors, Watt et al. (2014) implemented three different types of recitation sessions over six years, each of which had some variation of an active learning component. The most significant improvement in students' learning outcomes was when students had to actively co-construct knowledge on Verbal, Graphical or Geometric, Numeric, and Algebraic (VGNA) concept activities in recitation. The VGNA activities allowed students to explore calculus problems together from various vantage points to develop a shared understanding of key calculus concepts. This form of active learning used in an introductory undergraduate mathematics course improved students' post-test results and increased retention and decreased DFW rates across the course. This study suggests that active learning opportunities in recitations lead to stronger learning outcomes for students.

The aforementioned literature suggests that for active learning to occur in a recitation, the instructor must design purposeful activities and opportunities for students to engage more deeply with the content, with the instructor, and with one another. Achieving this in the online setting is a bigger challenge since novel ways to engage with the content, instructor, and peers must be developed and promoted. Our research project aims to bridge the literature on STEM recitations and active learning by analyzing the recitation course transformation process for Mathematics, the efforts to include active learning in the transformed courses, and the takeaways from the experience.

Theoretical Framework

Constructivist learning theory emphasizes that individuals learn through connecting new ideas and experiences to existing knowledge and experiences to form new or enhanced understanding (Bransford et al., 1999). Instructional strategies that include active learning foster the cognitive work recognized as vital for learning by constructivist learning theory. Active learning methods also habitually involve cooperative learning groups and peer interactions, similar to the recitation instructors' breakout rooms in their online classrooms. As we aim to investigate the components of active learning preserved and modified in the process of transitioning recitations, we utilized the Teacher-Centered Systemic Reform (TCSR) Model (Woodbury & Gess-Newsome, 2002). The TCSR is an instructional model used in secondary and post-secondary contexts that views instructional reform at a systems level. It looks closely at how personal factors, teacher's thinking, and contextual factors influence an individual's instructional practices. Since many instructors came into the Fall 2020 semester having previously used active learning techniques in the in-person setting, it was important to us to see how their practices stayed the same or changed based on the new online context.

The TCSR Model explains "personal factors" as the collection of past teaching experiences, past and present pedagogical training, and demographic profile. Of particular importance in this study is the extent of the instructors' preparation to teach recitations in the online setting and their efforts to continue learning how to incorporate active learning into online instruction. The TCSR Model explains "teacher's thinking" as the collection of the instructor's knowledge and beliefs about learning, teaching, students, and content, sense of individual dissatisfaction with the teaching process, and teacher self-efficacy for implementing instructional strategies. Our study examines some of these aspects of teacher thinking, specifically in the context of active learning in the online setting. Last, the TCSR Model explains "contextual factors" as the broader picture of how the instructor is embedded within the classroom, department, school, and professional organization. The TCSR hypothesizes that the teacher's thinking and contextual and personal factors impact instructional reform efforts. In our study, transitioning in-person active learning recitations to online active learning recitations can be viewed as a reform effort across gateway STEM courses, with instructional decisions impacted by teachers' thinking and personal factors.

Method

This paper addresses the following research questions (RQ):

RQ1: What elements of the recitations were retained in the transition to online teaching?
RQ2: What elements of the recitations were lost as a result of the transition to online teaching?
RQ3: What were the challenges faced and lessons learned during the abrupt transition to

online teaching?

Study Participants and Data Collection

In order to include all variants of the faculty observations that are relevant for our study (Yanow, 2000) and to ensure information-richness (Fossey et al., 2002), the mathematics faculty included four lecturers teaching the large lectures, 15 recitation instructors, and nine course coordinators of gateway courses as survey participants. Some faculty served in more than one capacity during the Fall 2020 semester. These faculty all taught or oversaw recitations in either Pre-Calculus, Calculus 1 (for the Engineering and for Life & Social Sciences tracks), and/or Calculus 2 (for the Engineering track). The small-class recitations were offered for 80 minutes once a week, which met in addition to two full 80-minute large-class lectures twice a week. All but three respondents used Zoom as their recitation platform, whereas the remaining respondents used a comparable platform, Canvas's BigBlueButton Conferences tool. An 18-question Qualtrics survey including the research questions was sent to faculty members during the last three weeks of the Fall 2020 semester. The open-ended and multiple-choice survey questions allowed participants to describe the setup of their online recitations' setup and elaborate on class preparation, group work, learning activities, and assessments. Participants were asked to describe how they collaborated with their colleagues to execute the recitations. The survey concluded with a series of questions on elements of the recitations that were retained, lost, and modified due to the transition to online recitations.

Analysis and Results

All responses were cleaned and coded using a semi-open coding scheme (Corbin & Strauss, 2015) to look for themes across question responses. After several passes of semi-open coding, a frequency count was determined for all open codes. Codes were collapsed into larger categories, such as "accommodations," "in-class activities," "collaboration," and "interpersonal interactions," and "participation." The following sections elaborate on the themes that emerged during our analysis process.

Elements Retained Online

Respondents enthusiastically agreed that they retained many traditional elements of an in-person recitation once it was transitioned online: quizzes, group problems, homework review by the instructor, example problems. One respondent commented that the online format seemed to allow more students "at the board" to work out problems, whereas three respondents indicated that it was challenging to ensure students completed the assigned tasks. Respondents were asked if they continued to grade work completed during online recitations. 80% of respondents showed that they retained grading work as part of the recitation. Of the remaining 20% who did not require submitted work, one respondent shifted from assigning graded work to not assigning it online by citing the challenge of not being able to observe breakout rooms concurrently.

Elements Lost Online

Respondents interestingly shifted their responses from tangible items (worksheets, quizzes, etc.) to behaviors when describing components lost in the online transition. Sixty percent of the respondents described losing interpersonal interaction, coaching opportunities, student interactions, and before/after class opportunities to ask the instructor a casual question. Twenty percent of respondents commented on losing group work once recitations moved online. The remaining respondents took their face-to-face recitations and fully transitioned all elements to the online setting.

Collaboration

When asked about offering collaborative group work opportunities in the online setting, six respondents reported administering no collaborative group work, four of whom previously promoted collaborative group work in person as a form of active learning. Their reasons for the switch included lack of student participation in breakout rooms and the resulting independent work with unresponsive classmates. Some of the fourteen respondents who had online collaborative group work echoed the same difficulty of continuing collaborative learning in the online classroom. One instructor noted that the average student seemed to participate at the same level, whether in person or online, but the less involved students' lack of participation was exacerbated in collaborative groups. Two instructors cited effective methods of ensuring collaboration, including self/peer evaluations and allowing students an opportunity to first enter their responses in the chatbox before working out the problem in a breakout room.

Recitation Setup

Almost all respondents described online recitations as being divided into three phases: a teacher-led "review" phase, followed by a student active learning phase, then a closure phase. The recitation instructor devotes about 10-30 minutes to summarize the week's content, review sample problems, and answer questions in the review phase. In the active learning phase for about 30-50 minutes, instructors create learning opportunities such as online poll questions and breakout groups where students work on new problems. Finally, most respondents reported spending the last 20 minutes on an online recitation quiz to assess what students learned from the previous week's content. Two respondents stated that to have a focused recitation introduction, they had students submit the homework review problems ahead of time.

Challenges Faced

Respondents indicated a wide variety of both controllable and uncontrollable challenges. The stability of internet connection, availability of power, and availability of pens and tablets were cited as challenges students faced when accessing and engaging in online recitations. Three respondents cited the authenticity of students' work on assessments as proctoring methods and availability of answers on the Internet can influence students' academic integrity. Another five respondents cited getting students motivated to participate as a significant challenge. One respondent noted learning various online technologies as a challenge faced by faculty.

Lessons Learned

The final question asked participants what they learned from the experience of transitioning their recitations online and what they think other recitation instructors should learn from their experiences. Two respondents discussed being accommodating. One respondent felt that being graceful with students and submission due dates helped foster stronger relationships, whereas the other respondent had a different perspective stating that, "lenient or inconsistent policies about tech issues and late assignments lead to students trying to abuse the system in the end of the semester." Another instructor suggested holding office hours on the same day as recitation, as office hour attendance typically seemed high when it was on the same day as recitation. One instructor discussed accessibility and building a structured learning management site containing recorded lectures. The remaining respondents shared ways to engage and involve students, such as mandating the microphone over the chatbox, allowing an anonymous live discussion stream during class, having interactive slides that every student gets a copy of and that the instructor can monitor, virtual whiteboards, and using breakout rooms at least one time during every recitation.

Discussion

Although instructors were forced to move their recitations online due to the impact of the COVID-19 pandemic, our survey responses reflected many of the same elements and challenges instructors typically associate with in-person recitations. Some aspects of administering recitations online, including assessment, accessibility, and developing engagement and interpersonal connections, seemed to be amplified in the responses as areas needing more attention as online recitations continue to be developed. Our participants described a typical online recitation structure similar to that of a face-to-face one but expressed concerns about ensuring engagement during the active learning phase. Many of these feelings can be tied back to teachers' thinking and personal factors associated with many years of running in-person recitations in a tried and true, highly structured format (Woodbury & Gess-Newsome, 2002). In the following paragraphs, we discuss recommended changes to instructional practices based on our findings. We consider both online and in-person implementations of active learning recitations, as both formats will be available in the future.

Assessments

Transitioning the mode of in-person assessments from weekly paper/pencil quizzes to online quizzes made the coordinators and the instructors aware of the availability of a new form of assessment. One of the main recitation components instructors hope to retain is to change in-person, paper-based quizzes to online quizzes regardless of the recitation medium. The advantages of this change are three-fold: (1) allowing more dedicated class time for student-student and student-instructor interaction, (2) providing accommodations to students by extending the time frame (about 12 hours to 24 hours) to take the online quizzes instead of the old practice of taking the quizzes in the last 15-20 minutes of class time, and (3) assessing students in the same format as their homework and midterm/final exams are administered. From a personnel management perspective, this also better repurposed full-time and part-time faculty members' time to avoid traveling, printing, and copying assessments for large classes. During the pandemic, it also eased safety concerns of bringing students' papers home and, as a result, saved faculty members time with the removal of some manual grading.

Collaboration

The challenge of managing online group work was mentioned by many of our respondents across multiple questions and was a significant obstacle for many instructors who wanted to replicate the in-person active learning environment online. Setting up group norms and expectations at the start of the semester is one critical step needed to facilitate effective group work regardless of the recitation medium. Instructors also found it essential to develop student-student and student-instructor communications within the group setting. Since students also tend to be more isolated in online classes, the student-student interactions can be very limited in this environment. Keeping group sizes small, coupled with a low stakes graded assignment due shortly after recitation, could be a way to foster more engagement in online groups. Asking questions that cannot be solved by an online program or Google search can improve engagement and authenticity. This can be applied to both online and in-person recitations.

Engagement

Moving into future iterations of online recitations that do not fall under the umbrella of emergency remote teaching, we realize one of the biggest concerns from faculty was student participation and engagement. One possible solution to promote more active participation is to give all students an individual virtual whiteboard or one shared virtual whiteboard that the instructor can monitor. As many respondents noted, assigning a small grade for the classwork can also motivate students to be more active during recitation, provided there is a small window of time to submit the assignment after recitation. Small pre-class assignments and preparation of worksheet problems and solutions before recitation also seemed to help improve engagement in online recitations.

Accessibility

Across all questions, a theme of accessibility also emerged. Contextual factors play the most significant role here, as instructors have many resources available within and outside the department and organization to help students (Woodbury & Gess-Newsome, 2002). We propose having students participate in future online recitations through various means, including microphones, written text in the chatbox, and Google Form submissions. Some students may be more inclined to participate through written text, whereas others may prefer something oral or on video. Being accommodating and demonstrating clarity are also essential components of an accessible course. Students know what is expected of them, find what they need, and connect with their instructors. We recommend consistently providing explicit directions, organized Learning Management Systems (LMSs), and multiple ways of contacting the instructor (Piazza, e-mail, Discord) to accommodate students' learning preferences.

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Secondary-tertiary Transition and Undergraduate Tutoring:
Novice Tutors Make Sense of their Teaching of First-Year Students

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Drawing on the commognitive framework, we construe the secondary-tertiary transition (STT) as a distinctive thread of communication that pervades pedagogical discourses of various communities. Our interest rests with university tutors (or "teaching assistants") in light of the emergent recognition of their impact on undergraduates' mathematics learning. We aim to understand the role of STT-communication in tutors' reflections on incidents that took place in their tutorials. Our participants were undergraduate students at the advanced stages of their mathematics degrees in a large New Zealand university and who enrolled in a mathematics education course. Throughout a semester, the participants led tutorial sessions for groups of first-year students and wrote reflections on noticeable mathematical incidents. In this paper, we focus on how STT-communication woven into tutors' descriptions of classroom incidents and their sense-making of unexpected actions of their students.

Keywords: commognition, pedagogical discourse, secondary-tertiary transition, teaching assistants, undergraduate tutoring

Annie is a soon-to-be mathematics major and a novice tutor (or a "teaching assistant") in a first-semester mathematics service course in a large New Zealand university. In one of the tutorials, her students were given $f(x) = x^2 - 2$ and asked to find $f(2 - x)$. This is how Annie reflected on an incident that she noticed in her classroom:

> During the tutorial, I had more than three students asking me how to solve this question. I tried to explain it by telling them that function is like a factory. $2 - x$ is the input and $x^2 - 2$ is like a machine. This is what my maths teacher used as an example when he taught us the definition of functions in school. But they told me they didn't understand it at all. […] After the tutorial I remembered the first time my maths teacher used the factory example and at that time I also didn't understand it. But things are way more paced in school, I can't imagine how hard it must be for a first-year student to meet this idea for the first time.

Putting the accuracy of Annie's description of the incident aside, notice all the places where she referred to the secondary-tertiary transition (*STT* hereafter). On the face of it, it seems only reasonable for her to bring up STT. Indeed, Annie was aware that this was the first tertiary mathematics course that her students have taken, so why wouldn't she comment on "a first-year student"? And it was not that long ago that Annie was in the same shoes as her students, which explains why she recalled her first encounter with "the factory example". On the other hand, Annie worked with the same group of students throughout a whole semester, and yet, she did not acknowledge STT in all of her reflections on the incidents that occurred in her tutorials. This allows us to propose that weaving STT into her reflection was not an unavoidable necessity derived from how Annie communicates about pedagogy, but rather a deliberate move made as part of her sense-making of her teaching.

In this investigation, we explore how the concept of STT finds its way into the reflection of university tutors who teach undergraduates in transition, the discursive characteristics of their communication on STT, and its roles in tutors' sense-making of their teaching. The literature on the transition from school to university mathematics has been prolific (e.g., Gueudet, 2008; Holton, 2001; Thomas et al., 2015). Yet, to our knowledge, this is the first research attempt to consider students' STT from the tutors' perspective. Our participants are undergraduates themselves. By working with a cohort that was experienced in mathematics learning and was making its first steps in teaching, we were hoping to gain access to vivid stories on STT and observe the impact of these stories on tutors' emerging pedagogical thinking.

Related research

This investigation resides in the intersection of research on undergraduate tutoring and on teachers' perspectives on STT.

Undergraduate tutoring

Over the last two decades, international research has become interested in undergraduate tutors – mostly students of mathematics who are employed by their respective departments to contribute to the instruction of specific undergraduate courses (e.g., Speer et al., 2005). Research indicates that the scope of tutors' responsibilities can vary from one country to another (e.g., Oates et al., 2005; Yee et al., 2020), ranging from marking students' assignments, through leading regular tutorial sessions, to solely teaching a whole course. This diversity can explain the lack of universal terminology for referring to this cohort (e.g., see Speer et al., 2005 for "teaching assistants", Yee et al., 2020 for "graduate student instructors", and Nardi et al., 2005 for "tutors"). Previous research in New Zealand context has used the term "tutors" (e.g., Oates et al., 2005) and we continue this tradition.

In many educational contexts, tutor-student interactions constitute a significant course component, which positions the former as potentially having a considerable impact on the learning of the latter (e.g., Speer et al., 2005). In New Zealand, tutors are often in charge of planning and instruction of weekly sessions that the course students are expected to attend regularly throughout the semester. These tutorials bring together sub-sets of the course students (usually less than 50) and engage them with questions related to mathematics discussed earlier in a whole-course forum (often called "lectures").

Research into undergraduate tutors has explored a range of aspects. Given our interest in tutors' sense-making of their teaching, we mention the Undergraduate Mathematics Teaching Project (e.g., Nardi et al., 2005). Through reflective interviews, the project aimed to unpack the complexities of tutors' epistemologies, pedagogies, and craft knowledge. To avoid discussions on some general and dis-embedded levels, the project engaged its participants in reflections on specific incidents that took place in their tutorials.

University teachers' perspectives on STT

Only a few studies tapped into university teachers' perspectives on STT. Klymchuk et al. (2011) developed a short survey, which was answered by 63 university teachers from 24 countries. One of the survey questions asked teachers to propose reasons for the gap between school and university mathematics. The researchers grouped the responses in categories, the most popular of which was "higher level of thinking at university mathematics". Klymchuk et al. illustrate this category with quotes, in which the respondents criticize school mathematics instruction (e.g., "very mechanical and situational") and emphasize its inadequacies in the context of university mathematics ("we expect more out of the students").

The views surveyed in Klymchuk et al. (2011) seem to go beyond individuals and recent times. Back at the third ICME in 1976, a study group brought together delegates from 15 countries to discuss STT. In their report, STT is described as a problem, the three major aspects of which are: unavailability of "topics supposedly covered in the secondary curriculum […] when needed in later study" (Fey, 1977, p. 406), "[m]any students who […] are unable to see the relations between specific ideas" (ibid), and "students leaving secondary school [with] a narrow and formal approach to mathematics" (ibid, p. 407). Put in less deficient terms, echoing perspectives can be found in the report of the London Mathematics Society (1995).

We notice the connections between perspectives on STT and the actions taken by those in tertiary education. For instance, remedial courses are often presented as a "solution" to the "problem" of school graduates' under-preparedness (e.g., Clark & Lovric, 2008; Fey, 1977; Klymchuk, et al., 2011). On the other hand, endorsing Sfard's (2014) perspective on school and university mathematics as almost distinct disciplines, Pinto (2019) discusses pedagogies that university teachers can implement in their courses to "smoothen" the transition.

Theoretical Framework

We turn to the commognitive framework (Sfard, 2008) for theoretical foundations and analytical tools for our study. Being interested in rather nuanced aspects of tutors' reflections, thinking, and self-making, we were hoping to capitalize on the framework that has been renowned for offering operationalizations for elusive constructs of this sort (e.g., Morgan, 2020). In university mathematics education, commognition has been acknowledged for its capability to account for the complexity of this context (Nardi et al., 2014) and has been used to explore teaching (e.g., Kontorovich, 2021; Pinto, 2019). Furthermore, commognition "has been developed within the field of mathematics education and is designed to address the problems arising in this field" (Morgan, 2020, p. 226). As such, it provided us with a set of theoretically coherent tools to attend to issues of mathematics and its pedagogy.

Commognition assumes that *discourses* underlie all aspects of human activity (both communicational and practical), dividing the society into partially overlapping discourse communities (Sfard, 2008). In the context of teaching, Heyd-Metzuyanim and Shabtay (2019) introduce the notion of *pedagogical discourse* as something that shapes and orients teachers "towards *what* to teach students, *how* to teach them, *why* certain teaching actions are more effective than others and, often not talked about but still very important, *who* can learn (or not learn)" (p. 543, italics in the original). The researchers discuss pedagogical discourses in relation to school teaching, but there seems to be no reason to confine the construct to a particular educational setting. Indeed, Viirman (2014) uses the same term and with a similar meaning to investigate communicational practices of university mathematics lecturers. Given that "the membership in the wider community of discourse is won through participation in communicational activities" (Sfard, 2008, p. 91), we consider university tutors as a community that participates in pedagogical discourse.

When engaging with new discourse, a person is likely to find themselves in unfamiliar situations, in which they consider themselves bound to act. Lavie et al. (2019) explain one's capability to perform in new circumstances with *precedents* – "past situations which she interprets as sufficiently similar to the present one to justify repeating what was done then, whether it was done by herself or by another person" (p. 160). For instance, novice teachers can draw on their learning experiences as precedents and replicate narratives and actions of their former teachers.

In this investigation, we are focused on tutors' sense-making rather than on their actual teaching. Then we build on Sfard's (2020) approach to discourses as a special type of communication that "has been construed along history as a toolbox for constructing potentially useful accounts of different segments of reality" (p. 90). Discourses offer conventional building blocks (specifically, keywords, visual mediators, narratives, and routines) to construct such accounts, but they rarely predestinate individual choices. In other words, the selection of what to bring into a discursive existence and how to do so rests with the individual. This is especially the case when people come to capture in words "segments of reality" that unfold in a classroom. Such *pedagogical accounts* belong to the communicational sphere, which makes their objects *discursive*, i.e. arising through humans' keywords and narratives. For instance, when generating a pedagogical account, a teacher needs to decide how to refer to humans: by name (e.g., “Alex”), their assumed role (e.g., “a student”), gender (e.g., “the girl”), et cetera (cf. Dubbs, 2021).

Deficit approaches have been recognized as a common characteristic of pedagogical accounts. Informed by the Critical Race Theory and sociopolitical perspectives, Adiredja and Zandieh (2020) describe *deficit narratives* as those that “focus on students’ academic and intellectual shortcomings and attribute them to deficiencies located in the students, their values, families, and/or cultures” (p. 240). We direct the readers to the previous section for examples of such narratives and to the work of Adiredja and colleagues for a comprehensive discussion and alternatives ways to construct pedagogical accounts.

The Study

In tune with the commognitive framework, we conceptualize STT as a thread of communication that can feature in pedagogical discourses through distinctive keywords and narratives pointing at the transition from the secondary to tertiary educational context. We use “thread” as a metaphor to highlight that this is only one fiber that occasionally weaves into the fabric of pedagogical discourses. To foreground the discursive nature of this conceptualization, we use the term *STT-communication*. Accordingly, the central question underpinning this investigation is “how does STT-communication manifest in tutors’ reflections on incidents that took place in their tutorials for first-year students?”

Context and participants

Our data comes from an undergraduate course in mathematics education (*MathEd* hereafter) offered in the mathematics department at a large New Zealand university. The course mostly attracts students in the second half of their mathematics majors who are interested in educational issues. In a collaborative and student-centered environment, the students engage with various aspects of university mathematics education (for additional course details see Oates et al., 2005). Mathematics tutoring of first-year students is the central activity of the course.

All students in the MathEd course tutor in *Preparation* or *Service* mathematics courses. The former course can be construed as remedial mathematics and it is intended for students who qualified for university studies but do not have the standards that are necessary to succeed in first-year courses. The service course is a general entry course for non-mathematics majors, usually students of commerce, life sciences, and social sciences.

The novice tutors[1] are adjacent to specific groups for the whole twelve-week semester and expected to co-lead, in pairs, ten 50-min long tutorial sessions. Nearly a week before each tutorial, the lecturers of the Preparation and Service courses publish sets of questions on which the course students are expected to work during the tutorial. In the preparation for each session, the tutors are expected to engage with the question sets, raise issues that could emerge, and develop strategies to address them. Overall, the expected role of a tutor role can be described in the words of Moore (1968, p. 18):

> The tutor is not a teacher in the usual sense: it is not his job to convey information. The student should find for himself the information. The teacher [sic] acts as a constructive critic, helping him to sort it out, to *try* it out sometimes, in the sense of exploring a possible avenue, rejecting one approach in favour of another.

[1] From here onwards, we use “tutors” to refer to students in the MathEd course in relation to their teaching in Preparation and Service courses. Those who attended the tutorial sessions as part of their enrolment in these courses are called “students”.

Data collection, corpus, and analysis

As part of the individual MathEd coursework, the tutors were assigned to submit weekly 500-word reflections on an incident that drew their attention in their tutorial session that week (cf. Nardi et al., 2005). The guidelines directed tutors to distinguish between their incident of choice and its interpretations. To enhance the potential usefulness of the reflection process, tutors were asked to formulate "take-outs" for their further teaching.

Overall, we collected 363 reflections from 42 tutors over four semesters. To construct the data corpus, we scrutinized the reflections in search of STT-words, phrases, comments, and narratives. Eventually, our corpus consisted of 58 reflections generated by 38 tutors. We conceptualized each reflection as a snapshot of the tutor's pedagogical discourse; a snapshot where the tutor was the one to initiative STT-communication.

Our data analysis was informed by the commognitive distinction between "mathematizing" – narrating about mathematical objects and "subjectifying" – narrating about participants of mathematical discourse (Sfard, 2008). Focusing on the objects of STT-instances, we distinguished between narratives about mathematics and its entities, courses the participants tutored, and people. The last category was especially diverse, and then we drew on Heyd-Metzuyanim and Sfard's (2012) three levels of generality of subjectifying narratives. The levels distinguish between somebody's performance of a particular action, a routine performance associated frequently or most of the time with a person, and inherent properties that identify a person in a particular manner. With this classification in mind, we arrived at the corresponding distinction between tutors' descriptions of someone's *actions* (e.g., "I tried to explain it by telling them"), descriptions of *routines* (e.g., "This is how our lecturer taught us"), and *general narratives* identifying people or "things in the world" (e.g., "When coming to uni many students struggle"). The level of detail that the participants provided in their reflections varied, and then we use "routine" in a rather general sense as actions that repeat themselves (see Lavie et al., 2019 for an alternative approach). To interpret the roles of STT-communication in tutors' reflections, we approached each instance with such questions as "what does it do to the reflection?", "what is its added value to the story the tutor tells?", "how will this story change if the instance is deleted?".

Findings

The analysis revealed a range of ways in which STT-communication manifested in tutors' reflections. The manifestations were neither strictly distinct nor exclusive to each reflection. Indeed, a rare reflection contained STT-instances that were ascribed to a single category. Due to space limitations, we focus on two manifestations: describing tutorial incidents and making sense of students' unexpected actions.

Describing tutorial incidents

The reflection guidelines explicitly asked the tutors to separate between their descriptions of an incident and its interpretations. Yet, theorizing, evaluations, and judgement found their ways into many incident descriptions. STT-communication often pervaded tutors' descriptions, turning into a building block in tutors' pedagogical accounts.

Let us consider an excerpt from the reflection of Betty. She wrote:

> After visiting a few groups, it surprised me when I found out that there actually were a lot of [Preparation] students who were struggling with fractions. Most of them had some idea of what fractions are, but they had many misconceptions that they brought from school, they did not know how to multiply or add fractions, or find common

factors. […] Some students with richer background math knowledge did finish all tutorial questions"

In this excerpt, Betty writes about her "Preparation students", whom she identifies as "having some ideas" and "many misconceptions". She points at "school" as a source of both. In turn, those students who completed the tutorial questions are identified through "richer background math knowledge". Betty's association of her students' "math knowledge" with their previous school studies in not obvious. To recall, students come to the tutorial sessions less than a week after relevant mathematics has been discussed in lectures. Then, Preparation course appears as an alternative point of reference for addressing tutorial students' successes and challenges. In other reflections, tutors referred to their students and their learning in relation to the university (e.g., "First-year students" and "students who are new to uni").

In the descriptions of the incidents, students were not the only ones that were identified in relation to STT. Some reflections depicted "mathematics", "a topic", or specific questions around which the incidents revolved with comments affiliating them with school or university. For instance, "to remind them of *this concept from the school math* I said […]", "this tutorial was on *vectors which is the first university math in this course*", "the students used a *notation from high school and not proper math*".

Making sense of students' unexpected actions

Many reflections contained descriptions of incidents, where actions (or lack thereof) of some tutorial students appeared as deviations from what the tutors expected, planned, or hoped for. This made room for tutors to account for this gap, which opened the door for STT-communication. Specifically, two approaches emerged from the data analysis through which the tutors seem to make their peace with what students did: *commonizing students' actions* through general STT-narratives and *associating with the students* through drawing on personal STT-narratives. Due to space limitations we illustrate both approaches with the presented excerpt from Annie's reflection.

Annie's writing suggests that after explaining that "function is like a factory", she was not content with the students telling her that they "didn't understand it all". Probably no teacher would evaluate this development as felicitous, including Annie who was invested in her tutoring. However, students "not understanding" appears differently if considered under the assumption that "a fist-year student […] meet[s] this idea for the first time". Through the lens of this general STT-narrative, what looked special and unexpected becomes a logical derivation of a broader pedagogical "truth". Indeed, if "the factory example" is "hard", especially if encountered in a quickly paced first-year course, there seem to be little surprise in the fact that "more than three students" raise questions and repeatedly declare their "not understanding". We use the term "commonization" to underscore that a tutor accounted for students' actions through drawing on some general STT-narratives, through the lens of which these actions presented as less special and more expected.

Another approach that we associate with Annie making sense of her students' actions is providing a personal STT-narrative. Specifically, she writes that the first time she encountered "the factory example" she "also didn't understand it". Drawing parallels between what Annie presents as a recollection of her experience as a student and the actions of her tutorial students, illuminates the latter in a different light. Now that Annie has access to more than one precedent of what appears as a similar re-action of different people to what she describes as a similar situation, it makes sense to re-evaluate her students' "not understanding". Furthermore, Annie is one of these people, which provides her the opportunity to relate to students on a more personal level. Indeed, we read her sentence "I can't imagine how hard it must be" as an exaggerated version of "I *can* understand how hard it must be because I was in a similar situation".

Summary and Discussion

In this study, we construed STT as a communicational thread that weaves into the pedagogical discourses of various communities and is endowed, by commognition, with the power to shape people's thinking and actions. We focus on tutors' community, whose impact on undergraduate students' learning has been recognized (e.g., Ellis, 2014; Speer et al., 2005). To our knowledge, this is the first study to consider tutors as actors in students' STT.

We explored STT-communication of novice undergraduate tutors in New Zealand as it emerged from their written reflections on incidents that took place in their tutorials. Given the diversity of educational systems worldwide and the specificity of our setting, we abstain from generalizing the obtained findings. Nevertheless, it is hard to ignore the similarity between some general STT-narratives that our tutors generated and existing research. For instance, similarly to the lecturers in Klymchuk et al. (2011), a portion of tutors' narratives was de-evaluative of students' mathematics studies at school. On the other hand, there also were narratives acknowledging the difficulty of tertiary mathematics and transitioning into it.

The two manifestations of STT-communication that we presented in this paper appear to play a role in tutors' sense-making of their teaching. Mason (2002) acknowledges that it takes time and effort to learn to write teaching accounts that are free from theorizing, evaluation, and judgement. The first manifestation demonstrates that maintaining this separation between an incident and its interpretation can be not easy for beginning tutors. Metaphorically speaking, it can be said that STT was "in the air" in some incidents and constituted a segment of the tutor's pedagogical reality; a segment that they could not leave behind when constructing their accounts (cf. Sfard, 2020). Further research is needed to establish whether this manifestation of STT-communication has been specific to our context and tutors or whether this kind of talk is typical to university teachers, working with students in transition.

The second manifestation pertains to tutors accounting for students' actions that deviated from tutors' plans, expectations, and hopes. This is where general STT-narratives became handy as they commonized students' actions by turning them from attention-drawing oddities into instantiations of broader patterns. Unfortunately, this commonization often drew on deficit narratives about students, their knowledge, and abilities. Furthermore, let us recall that these narratives emerged from reflections that the tutors submitted as part of coursework. This may suggest that they did not consider deficiency-based interpretations of students' actions as an issue. In this case, these interpretations may be illustrative of the gap between novices to pedagogical discourse and the growing anti-deficit movement in the research community (e.g., Adiredja & Zandieh, 2020). In the school context, professional development appears as a conventional counter-measure to teachers' deficit approaches (e.g., Anthony et al., 2018). We are not familiar with systematical efforts of this sort in the tertiary context in Australasia. This situation draws attention to colleges and universities in the US, where the development, evaluation, and scrutiny of professional development programs for tutors have been institutionalized (e.g., Speer et al., 2005; Yee et al., 2020).

In their reflections, many tutors harked back to their time as students and shared their experiences through personal STT-narratives. They recalled situations that are similar to the focal incidents of their reflections and, having referred to their own personal struggles and difficulties, wrote that they "relate", "understand", and "empathize" with their students. We note that a sizeable portion of reflections in our data contained tutors' personal STT-narratives. This may result from the accessibility of these narratives to our tutors, who were in their students' shoes not so long ago. Such accessibility may be out of reach to other tutoring cohorts, such as graduate students and research mathematicians, for whom STT is often a fading memory (cf. Speer et al., 2005). We then direct attention to what may be a characteristic attribute of tutors who are undergraduates themselves and invite future research to explore how personal STT-narratives can be leveraged in mentoring undergraduate tutors.

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A Collective Case Study of Three GTAs Participating in the M-DISC Professional Development

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Three graduate teaching assistants (GTAs) took part in a one-semester teaching professional development (T-PD) based on the "Mathematics Discourse in Secondary Classrooms" (M-DISC) (Herbel-Eisenmann et al., 2017) T-PD. This report presents a collective case study highlighting one finding per GTA to understand how their classroom discourse changed (i.e., whether it became more productive) and how the GTAs made sense of these changes.

Keywords: Case Study, Graduate Teaching Assistants, M-DISC, Professional Development

From the *Progress through Calculus* survey (Apkarian & Kirin, 2017) we know that U.S. institutions offering graduate degrees in mathematics do not necessarily offer their graduate teaching assistants (GTAs) any teaching professional development (T-PD). Where a GTA T-PD is offered, it is typically confined to GTAs' first year of teaching as indicated by over 80% of responding institutions. The short duration of such T-PDs seems problematic given how much there is to learn about teaching and given that many GTAs find themselves in a stage of *survival* during their first year of teaching (Beisiegel et al., 2019). GTAs may be more receptive to topics beyond classroom management in later years of their teaching, when they find themselves in the stages of *consolidation*, *renewal*, or *maturity* (Beisiegel et al., 2019; Katz, 1972).

To address the lack of support that mathematics GTAs receive beyond their first year of teaching, I adapted and offered the "Mathematics Discourse in Secondary Classrooms" (M-DISC) (Herbel-Eisenmann et al., 2017) T-PD in spring 2021 to three GTAs who were: (a) teaching undergraduate mathematics, (b) no longer in their first year of teaching, and (c) at an institution offering only a first-year T-PD. The M-DISC T-PD covers a range of topics in service of two overarching foci: cultivating *productive* and *powerful* discourse in the classroom. As the M-DISC developers (Herbel-Eisenmann et al., 2017, p. xxxi) explain:

> We use "productive discourse" to mean "discourse that provides students with opportunities to make meaningful mathematical contributions toward particular mathematical learning goals" (Cirillo et al., 2014, p. 142) and "powerful discourse" to mean "discourse that positions students as people who are capable of making sense of mathematics and supports students' developing identities in terms of status, smartness, and competence in mathematics class" (Cirillo et al., 2014, p. 142).

Beyond the practical goal of offering such a T-PD, I was interested in learning whether taking this T-PD was helpful for the participants and their students. For this report, I focus on how productive discourse was fostered by the participants over the course of the semester that they partook in the T-PD. In particular, the research questions I sought to answer were:

1. How does the participants' classroom discourse (as described by the usage of teacher discourse moves and dimensions of the EQUIP [Reinholz & Shah, 2018]) change over the course of the T-PD?
2. How do participants make sense of these changes in their classroom discourse?

To answer these questions, and given the small-scale implementation of this T-PD, I employ what Stake (1995) termed a *collective case study* approach. In this instance, this approach boils down to three *instrumental case studies* of graduate students participating in the M-DISC T-PD.

Brief Overview of the M-DISC T-PD

The M-DISC seeks to help its participants cultivate productive and powerful discourse in their classrooms. To achieve these ends, the M-DISC materials draw on notions like *discourse* and *register* from systemic functional linguistics (Halliday, 1978; Pimm, 1987) and notions inspired by it (e.g., *communication context*), Chapin et al.'s productive talk moves (2003), and positioning theory (Harré & van Langehove, 1999). Structurally, the M-DISC T-PD consists of an introduction, five core "constellations," and an action research capstone experience. The five core constellations are: (a) engaging students in mathematics classroom discourse, (b) teacher discourse moves and positioning, (c) planning for rich discourse, (d) setting up and gathering evidence of student work, and (e) concluding and contemplating evidence.

Through correspondence with the first author of the M-DISC materials B. Herbel-Eisenmann (personal communication, November 22, 2020), I learned that: (a) the first three constellations are "the most substantial parts," (b) constellations 4 and 5 provide opportunities to explore the ideas from constellations 1–3 in more depth rather than introduce new content, and (c) she has omitted constellations 4 and 5 in some past M-DISC implementations. Thus, when I offered the M-DISC in spring 2021, I implemented an abridged version of the M-DISC T-PD that omitted constellations 4 and 5. Further, as its name suggests, the M-DISC was originally designed with secondary classrooms in mind. Thus, where possible, I modified T-PD tasks to be more relevant to undergraduate mathematics without changing the purpose of a task (e.g., replacing a transcript, working on an abstract algebra problem, working with undergraduate mathematics texts). In the end, my implementation of the M-DISC T-PD consisted of thirteen weekly 2-hour meetings during one semester that were held via Zoom due to the coronavirus pandemic.

Relevant Literature

Since classroom discourse—when not a one-way street—is an exchange between teacher and students or students and students, I chose a two-pronged approach to studying the participants' classroom discourse: one prong focusing more on students, the other more on the instructor.

Discourse (Student Focus)

To learn about the mathematical contributions that students were making in the participants' classes, I drew on the dimensions of the "Equity QUantified In Participation" (EQUIP) tool: *discourse type*, *student talk length*, *student talk type*, *teacher solicitation method*, *wait time*, *teacher solicitation type*, and *explicit evaluation* (Reinholz & Shah, 2018). Although each of these dimensions has been linked to being relevant to issues of equity, each of these is also relevant to understanding how productive classroom discourse is.

One dimension I wish to highlight, given its appearance in the Results section, is *student talk type*, which provides a window into understanding whether students are engaging in reasoning or only limited to supplying memorized facts and doing calculations. Reinholz and Shah (2018) drew on Braaten and Windschitl (2011)'s explanation tool to distinguish between *what, how*, *why*, and *other* student talk type. Braaten and Windschitl (2011)'s distinction, however, stems from differences in explanation in science education. The meaning ascribed to these terms in the context of mathematics discourse by Reinholz and Shah (2018) is not fully clear. Thus, for the purposes of this report, *what* statements are claims made by student without a rationale (e.g., "the answer is 12"), *how* statements focus on the process (e.g., "after adding x on both sides, you then need to divide by y"), and *why* statements are claims made by students with a rationale or providing the rationale to an already made statement.

Discourse (Instructor Focus)

To learn more about the instructor's discourse, and with the T-PD materials in mind, I draw on the teacher discourse moves (TDMs) discussed in the M-DISC materials (Herbel-Eisenmann et al., 2017) and various publications (e.g., Cirillo et al., 2014; Herbel-Eisenmann et al., 2013), which are based on Chapin's productive talk moves (Chapin et al., 2003). The six TDMs are *waiting, inviting student participation*, *revoicing, asking students to revoice*, *probing a student's thinking*, and *creating opportunities to engage with another's reasoning*.

Given their appearance in the Results section, I want to foreground *waiting* and *probing a student's thinking*. As Rowe (1986) demonstrated, increasing the time a teacher waits after speaking (wait time 1) and after a student's turn (wait time 2) can have remarkable effects on student participation, reasoning, confidence, and achievement. By *probing a student's thinking*, the M-DISC developers mean "following up with an individual student's solution, strategy, or question. The goal here is to have the student elaborate on or clarify his/her ideas" (Herbel-Eisenmann et al., 2017, p. lvi).

Method

Participants

Three doctoral students from a large public university in the Midwest participated in my implementation of the M-DISC T-PD: Alice, Finnegan, and Valeria (all pseudonyms). Alice and Finnegan were both mathematics doctoral students, whereas Valeria was a mathematics education doctoral student. I had never interacted with Alice and Finnegan before I contacted them to see if they would be interested in participating; Valeria was a friend. All three had multiple years of teaching experience, be it as teaching assistants at universities (Finnegan and Valeria) or schoolteachers (Alice [*sic*]). None received any compensation.

Data Sources

This study is part of a larger study, and the complete set of data sources consist of audio-recordings of all participants' classes, anonymized chat histories from the classes, three semi-structured 60-minute interviews with each of the participants, video-recordings of all thirteen T-PD meetings, and reflections written by me after every T-PD meeting. For the purposes of this report, I will be drawing on a subset of the classroom audio-recordings and anonymized chat histories, as well as the semi-structured interviews.

Classroom audio-recordings & chat histories. All participants shared audio-recordings of their Zoom classes with me as well as anonymized chat histories that excluded the contents of private messages. Using the chat time stamps, I merged the chats with the classroom transcripts. For Alice and Finnegan, who each taught twice a week for 50–80 minutes, I decided to analyze nine classroom recordings: three from the start of the semester (after the syllabi had been discussed), three from the middle of the semester (after the TDMs had been introduced), and three from the end of the semester (before exam reviews). For Valeria, whose audio-recordings were typically around 60 minutes long and who taught only once a week, I decided to code all twelve classroom recordings. (Valeria taught two sections of the same course, but rather then code parts of both, I decided to code all of one section's classroom recordings.)

Semi-structured interviews. Each participant was interviewed before the start of the T-PD and semester, after 7–8 T-PD sessions in the latter half of the semester (after TDMs had been introduced), and 2–4 weeks after the end of the T-PD (i.e., 1–3 weeks after the end of the semester). The first interview served as an opportunity to get to know each other, whereas the last two served to learn about the participants' reflections on the T-PD and their teaching.

Data Analysis

To answer my first research question, that is, how does the participants' classroom discourse change over the course of the semester, I picked nine classroom recordings from the beginning, middle, and end of semester for Alice and Finnegan and all twelve classroom recordings for one of Valeria's sections. These were then coded along two sets of dimensions.

EQUIP. Each classroom recording was coded along six of the seven EQUIP dimensions (i.e., *discourse type*, *student talk length*, *student talk type*, *teacher solicitation type*, *teacher solicitation method*, *explicit evaluation*). *Wait time* was not coded for but considered in a separate analysis. To align with the TDM of *revoicing*, I also coded whether a turn was revoiced.

For the purposes of the EQUIP, a student's turn is any number of student utterances that are not interrupted by another student (but possibly interrupted by a teacher). Since the EQUIP's focus is on positioning, it makes sense to combine a student's utterances, even when they are neither thematically nor temporally linked for it captures the lack of different speakers. Yet, since productive discourse was foregrounded for this part of my analysis, I modified the unit size to define a student turn as any number of student utterances that are linked topically either by responding to the same question or by elaborating on a previous utterance (even when interrupted by the teacher).

TDMs. Each classroom observation was also coded with respect to five of the six TDMs. Again, waiting was omitted and analyzed for separately. In addition to TDMs, I also coded each instance of an instructor making a mathematical solicitation. Although the EQUIP includes a code for *teacher solicitation type*, it does so only in reference to a student turn. Thus, the EQUIP's *teacher solicitation type* code does not cover no-response teacher solicitations, prompting me to code for teacher solicitations separately.

The combination of coding for the EQUIP dimensions and the TDMs then provided data to answer my first research question regarding how the classroom discourse had changed. Combing carefully through the interviews after the EQUIP/TDM-analysis provided context for how the participants made sense of the discourse in their classrooms and served as triangulation for any impressions gained from looking at the EQUIP/TDM data.

Results

The richness of a case study is at odds with the severe length limitations of a conference proposal, and I am forced to paint with very broad strokes in this report. With each case serving as an instrumental case study, each participant's case serves as an example of what it can mean to be a doctoral student partaking in the M-DISC T-PD and each case brought to light different issues. My goal, given the spatial limitations, is to focus on only one classroom discourse issue per participant and elaborate how the participants made sense of it. Before delving into the cases, I want to note that the participants' teaching needs to be viewed in the context of the coronavirus pandemic; the challenges it brought about cannot be understated.

Finnegan: Probing

Finnegan was nearing the end of his mathematics Ph.D., and he loved teaching. The class he was teaching that semester was an algebra class with over 60 students. He frequently solicited mathematical input from his students, averaging almost 1 solicitation per minute (0.941) during the three observations at the start of the semester. Yet, as we can see from Table 1 under

"Student focus," although students communicated mathematics frequently (note: non-mathematical discourse is excluded from the counts), few student turns were *why*-type discourse at the start of the semester (CR1–CR3).

Looking across the nine recordings, one can see that Finnegan embraced the TDM of *probing a student's thinking* after it was introduced in the T-PD. As can be seen in Table 1 under "Instructor focus," the number of probing interactions in classroom recordings 4–9 appears to have increased in comparison to CR 1–3. (Recall: CR4–CR6 are from the middle of the semester after the introduction of the TDMs and CR7–CR9 are from the end of the semester.) As he described in the mid-semester interview, he used to respond to a student's contribution by "assuming that they got the right answer and then trying to put it in more mathematical terms" but was now trying to "dig deeper." He also felt that students had responded well to probing.

In the post-T-PD interview he provided more detail on how he liked to probe:

> I did actually find a lot of use in probing, when it's like, "Well, why do you think this is true?" or, uh, like my, my prime example is, I was like we're, we're dealing with this, we're dealing with this square root function and, "What does this plus c on the outside does [*sic*]?", and somebody answers and like, "Okay, why do you think it does that?" and they draw the connections. So, I did find a lot of use in probing.

As suggested by his remarks and as suggested in the T-PD, there are different ways of probing. The type that especially resonated with Finnegan is what I call *deepening-probing* in Table 1 and involves following up with a student's answer to go deeper with the content of the original turn (e.g., "Why do you think that?", "How did you get that?"). As suggested by the table, this was the main way in which Finnegan used probing. (The second most common way was asking students to clarify their turn—*clarifying-probing*. For example, "What do you mean by […]?")

Switching to the student perspective, one can see that most of the student turns that were why-talk were in response to probing interactions. (Included in this count are why-responses by students who were not addressed by the probing but chose to answer anyway.)

In summary, the TDM of *probing a student's thinking* was taken up by Finnegan after it was introduced in the T-PD and it helped him create a classroom in which students had and seized more opportunities to reason—perhaps the most central discursive practice in mathematics.

Table 1. Table showing parts of the instructor-focused and the student-focused coding of Finnegan's recordings.

Classroom Recording [# of min. spent together as a class]	CR1 [67]	CR2 [55]	CR3 [82]	CR4 [60]	CR5 [64]	CR6 [49]	CR7 [58]	CR8 [81]	CR9 [82]
<u>Instructor focus</u>									
Probing (total)	1	3	2	6	13	13	11	17	9
Deepening-Probing	0	1	2	3	9	12	8	9	6
Solicitations (total)	59	60	76	60	76	59	59	94	74
Solicitations (why)	0	1	1	3	7	7	6	9	2
<u>Student focus</u>									
Student talk type (total)	126	89	101	58	73	54	52	101	85
Student talk type (why)	0	3	3	6	7	8	6	13	4
Student talk type (why) (post-probing)	0	1	0	3	6	5	5	11	3

Valeria: Waiting 1

Valeria was in the middle of her doctoral career and teaching a mathematics content course for preservice elementary teachers with around 30 students. She wanted teaching to be joyful and expressed at the beginning that she wanted her students to feel supported and comfortable asking questions. A main reason she shared for joining the T-PD was learning about positioning.

Her concerns for positioning appear to have expressed themselves when she reflected on the TDMs in the mid-semester interview:

> I noticed that the things that I grab onto the most are things that I can do, or like, that rely more on what I'm doing than on what my students are doing. So, for example, wait times, I can do that. I can wait a little bit longer, and I can wait after my students, uhm, respond. And then for example, I can also revoice my students' answers, so that I have been trying. I think it's when it involves me asking someone else to do something when I'm like a little bit insecure of trying that out, because of the fear of how it, how they might respond. […] I think I have a very hard time, finding ways, to make people do things, without it sounding like I'm forcing them.

This reflection paired with her observation (made during the T-PD and after listening to her own classroom recording) that she felt she did not wait long enough after asking a question led me to wonder about her implementation of wait time, particularly wait time 1. For the wait time 1 analysis, I rounded the wait time after every teacher solicitation to the nearest second in two classroom recordings from the start and end of semester. As the left of Figure 1 (below) suggests, by the end of the semester, Valeria waited longer for student responses when soliciting questions or asking whether something made sense. (Note that the left-hand graph only includes solicitations that were *not* responded to by students.)

Something the right side of Figure 1 suggests is that Valeria had success in getting students to respond to her questions, even if it meant waiting a very long time (particularly towards the end of the semester). (Note that the right-hand graph excludes wait times if students did *not* respond to the question. After also mapping the wait times after teacher questions that students did *not* respond to, I saw no striking patterns.)

In summary, wait time 1 was a TDM Valeria already used at the start of the semester, but through the T-PD, she waited even longer for responses, particularly after asking for student questions. As aforementioned, analyzing her own classroom recording may have played a large part in that. With an eye towards productive discourse, it is unclear from looking at the coded EQUIP dimensions whether students engaged in more reasoning.

Alice: Identifying Obstacles to Using TDMs

Alice was also in the middle of her doctoral career and taught an algebra class with over 50 students. She wanted her students to know that mathematics was about discovery and creativity. Although she tried out several TDMs and was a proponent of some of them (e.g., revoicing and wait time), I want to focus instead on why she found some TDMs more difficult to incorporate into her teaching as she was very articulate about this matter.

As Beisiegel et al. (2019) had noted, GTAs can return to *survival* mode when teaching a new class. This appears to have been the case for Alice, who had multiple years of experience teaching high school mathematics but found herself both teaching this class for the first time and teaching synchronously online for the first time. She expressed being frustrated with herself because she had not implemented as much of the T-PD as she would have liked because she was feeling such pressure from covering the class' content. She added that, "I feel like I'm focusing so much on just like trying to move the class forward that I'm not spending as much time as I would like to on, uuh, creating like meaningful, group work and like meaningful experiences for students to interact with the material."

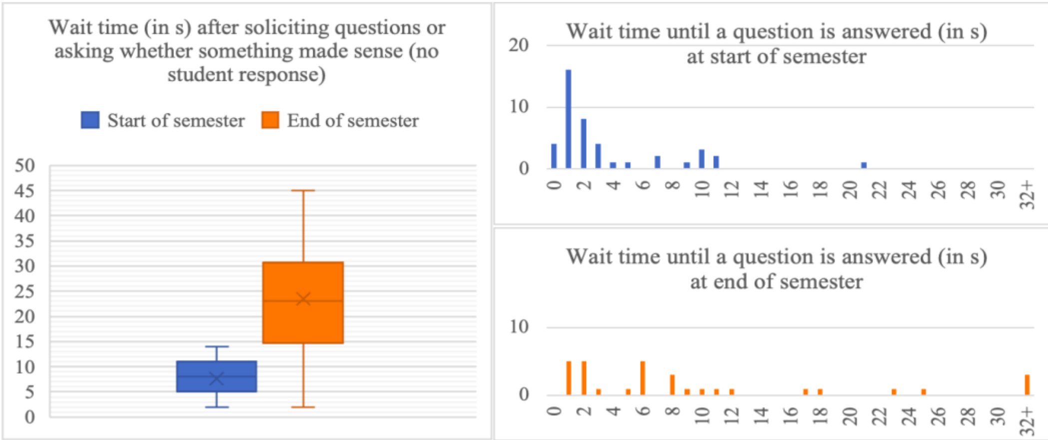

Figure 1. A two-part wait time 1 analysis. The left diagram focuses on wait time after asking students for questions whereas the right two diagrams are focused on wait time after posing questions to students.

In addition to struggling to try out TDMs while teaching a new course in a new mode and suffering from time pressure, Alice noted in the mid-semester interview that some TDMs were just more natural to her: "Like the revoicing and probing and wait time I feel like, uhm, you know, kind of are, they, they kind of naturally happen." Others, like asking students to revoice, was something she noted she did not naturally do. In short, the unnaturalness of some TDMs seemed to present a barrier to Alice to fully committing to them. Some TDMs were more natural but still posed problems: Alice struggled with implementing wait time 2 as she found it difficult to wait after a student's response because she felt the need to reassure them.

Finally, Zoom itself posed some challenges. Although she stated that all TDMs could have been used on Zoom, she felt that Zoom made using them more difficult—a sentiment also echoed by Finnegan who noted the struggles of interacting when many bodily cues are absent.

In summary, Alice encountered and articulated several obstacles to implementing TDMs: feeling too much under time pressure to use novel TDMs, finding some TDMs unnatural, having her teaching style clash with a TDM (i.e., wait time 2), and teaching via Zoom.

Discussion

As seen through the cases of Alice, Finnegan, and Valeria, the M-DISC T-PD encouraged the participants to use at least a subset of the TDMs: (a) Finnegan grew fond of several TDMs, chief among them *probing a student's thinking*, which helped him engage his students in more why-type discourse, (b) Valeria liked wait time (and revoicing) and noted that some of the TDMs were more difficult to use for her because they required asking students to do things, and (c) Alice spoke about using revoicing, probing, and waiting. At the same time, all of them noted struggling to implement certain TDMs, and Alice's case shone light on some of these challenges.

With respect to making discourse more productive in the T-PD participants' classrooms, the M-DISC demonstrated some modest success—although it remains to be seen whether the participants will continue using the TDMs. Yet, the modest success should also be interpreted taking into account that the participants were implementing the T-PD while teaching an ongoing class in which certain discursive norms may have already been established. Further, with the T-PD focusing on productive *and* powerful discourse, the above tells at best half the story.

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A Case Study of Abstract Algebra Learners' Fluency for Quotient Groups towards Efficacy Research

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Abstract: Abstract algebra education researchers continue to document that the quotient concept is difficult for undergraduates (Dubinsky et al., 1994; Melhuish et al., 2020). This proposal explores an undergraduate, a first-year graduate, and second year graduate student's fluency as they work on a "collapsing structure" quotient task in Group Explorer across several registers: Cayley tables, formal-symbolic mappings, and Cayley to Schreier coset digraphs. In addition, it illustrates an avenue for extending a qualitative case study to a mixed efficacy study.

Keywords: representational fluency, semiotics, quotient group, homomorphism

Introduction and Literature Background

A substantial amount of work and funds have been put forth over the past two decades to develop an inquiry-oriented (IO) curriculum for introductory group theory (Fukawa-Connelly, Johnson, Keller, 2016; Larsen, Johnson, & Bartlo, 2013). This curriculum is based on Guided Reinvention and Realistic Mathematics Education principles. Lockwood, Johnson, and Larsen (2013) described the IO curriculum, "Each unit begins with a reinvention phase in which students develop concepts based on their intuitions, informal strategies, and prior knowledge. The end product of the reinvention phase is a formal definition constructed by students and a collection of conjecture" (p. 777). These units are developed through Local Instructional Design Research which entails the creation of in-class activities, anticipation of how learners may interact with the instructor and activities, testing activities by conducting small-scale teaching experiments, data analysis, and refinement. According to a recent large-scale survey study that pooled from 200 institutions across the United States, the IO curriculum materials are not in wide use, in fact there is "almost no uptake" (Fukawa-Connelly, Johnson, and Keller, 2016, p. 280). A possible reason for apprehension or non-incorporation of IO curriculum materials could stem from a lack of sufficient evidence to either support or reject the efficacy of the IO group theory curriculum compared to other approaches.

Recent advances are opening new doors for efficacy studies such as quantitative measures that characterize the instruction environment (IOIM; Kuster, Johnson, Rupnow, and Wilhelm, 2019). However, aside from Johnson et al. (2020), as study that compared IO and non-IO groups with respect to the Group Theory Concept Assessment (GTCA; Melhuish, 2019) and gender, no other large-scale efficacy studies that compared IO to other instruction with quantitative analysis were found.

The purpose of this paper is to propose an avenue towards efficacy research that is restricted to a particular local instructional unit beginning with quotient group concept (Larsen & Lockwood, 2013). Rather than an overall assessment such as the GTCA, this research aims to develop additional local in-depth cognitive assessments for the quotient group concept. This paper illustrates a case study of three learners as they work on a quotient group task and a prototype methodology to assess their conceptual understanding, representational fluency, and flexibility for the quotient group concept. The conceptual understanding dimension includes "conceptual metaphors", language interpretation, and connections between concepts such as equivalence relations, normality, homomorphisms, quotient groups, and FTH (Melhuish et al., 2020; Rupnow, 2021; Lakoff & Nunez, 2000, p. 5). The theoretical framework section covers the representational fluency dimension. Flexibility refers to awareness of multiple solutions or the switching from an invalid to valid solution approach. These dimensions are not completely separable (Heinze, Star, & Verschaffel, 2009).

Theoretical Framework: Representational Fluency

Conversions and Pseudo-Semiotic Representations

A conversion is a maneuver in which a learner generates a representation for a mathematical object in one register and translates it to another (Duval, 2017). The theoretical construct of a pseudo-semiotic representation is proposed in this paper out of a need to describe counter-phenomenon of representational fluency observed in this study. A pseudo-semiotic representation is a sign, interpretation, and signified object triangle in which the signified object assigned to a sign-interpretation pair generated by the learner does not exists mathematically or is not accurately portrayed. The literature was revisited to find existing constructs that resembled the observed pseudo-semiotic phenomenon established by other researchers. A pseudo-semiotic representation could be thought of as a specialized instance that fell under the more general umbrella of pseudo-conceptual behavior proposed by Vinner (1997), hence the added qualifier "pseudo" to the semiotic representation construct proposed by Duval (2006). Vinner (1997) termed "pseudo-conceptual behavior to describe a behavior which might look like conceptual behavior, but which in fact is produced by mental processes which do not characterize conceptual behavior" (p. 100). Vinner admitted that this definition was not satisfactory and provided additional examples of what he meant by pseudo-conceptual processes. He explained that a more novice learner may gravitate towards these processes because they are "simpler, easier, and shorter" (p. 101). Moreover, novices "start looking for ways that will enable them to perform the task. These ways are not necessarily the way thought by the designers of the task when they decided to present it to the students. The task designers probably intended conceptual thought processes; the students came up with pseudo-conceptual processes…formed in a spontaneous way…and not necessarily taught to them by teachers or other agents. Sometimes they are natural cognitive reactions to certain cognitive stimuli. The students use them without going through any reflective procedure, control procedure or analysis of any kind" (Vinner, 1997, p. 101).

The concept image and concept definition distinction proposed by Tall and Vinner (1981) falls under the broad umbrella of pseudo-conceptual behavior in cases where there is conflict between the concept image and definition, however it does not detail differences between various modes of language production (Duval, 2017) or object-specific registers (Lajos, 2021; Ely, 2017). Pseudo-conceptual behaviors that have been described through a semiotic lens includes "gesture and speech mismatch" (Goldin-Meadow, Alibali, & Church, 1993, p. 279). Gesture and speech are two modes of semiotic, language, production (Duval, 2017). This mismatch generalizes to the notion of conflicts between registers (i.e., modes of language production) for a specified concept.

Participants and Settings

Participants were recruited via purposive sampling using the condition that they had exposure to a first-semester undergraduate or graduate-level abstract algebra course. Three students, Max, Jenni, and Alex agreed to participate. They represented cases for three consecutive levels in the same mathematics program at a central R1 level research university in the United States. Max was a fourth-year undergraduate mathematics major in a first-semester introductory abstract algebra course. Jenni was a first-year master's in science (MS) student, she had completed the undergraduate introductory course the previous year. She did not have exposure to the graduate course yet. Alex was a second-year graduate student in the doctoral track program and had completed the graduate course. This sampling is characterized as "maximum variation" which "documents diverse variations of individuals" and "increases the likelihood that the findings will reflect differences or different perspectives" (Creswell & Poth, 2018, p. 159).

Instrumentation and Data Collection

Audio-video data and whiteboard work was collected through semi-structured task-based interviews. The interview task was designed to investigate the registers that students naturally used to support their reasoning about quotient groups, fluency in converting an initial natural response in one register to an alternate register, and flexibility switching strategies. A methodological tactic when Duvalian semiotic theory is used is to design conversion tasks in which the researcher fixes the order of the initial and target registers prior to data collection. For example, McGee and Moore-Russo (2015) phrased a calculus task in a way that directed the participant to go from a numerical to geometric register. Rather than creating tasks with apriori initial and target registers, the present study displayed stimuli for various registers, Cayley digraphs, tables, group presentations, and cycle graphs, and allowed the participant space to choose their initial register(s) and complete their responses before introducing switch prompts that directed them to various targets.

The collapsing structure task prompt asked, "Is the quaternion group of order eight and the dihedral group of order eight isomorphic?" followed by "Is there some way that you can modify both the dihedral group and quaternion group by losing information or collapsing some structure so that the resultant modifications are isomorphic?" The collapsing structure task consisted of four major sub-tasks: determining if the dihedral group and quaternion group of order 8 were isomorphic or not, constructing a homomorphism or quotient from the dihedral group to an image group, repeating the previous sub-task for the quaternion group, and showing an isomorphism between the image groups. The "collapsing metaphor" refers to the identification of elements in a group to form cosets of equal size in which the cosets form a partition of the initial group (Melhuish et al., 2020; Rupnow, 2021). The metaphor of losing information was intended to invoke the fact that a homomorphism loses specificity regarding the orders of elements. The researcher set up the collapsing structure task by arranging *Group Explorer* cards into piles as shown below in Figure 1.

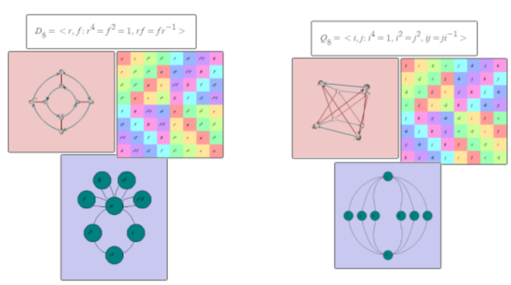


Figure 1. Collapsing Structure Task Display

Participants also interacted directly with groups dynamically through the *Group Explorer* software. They had access to other digraph representations depending on a choice of generators. After these cards were presented, the researcher left it open to the participant to choose an initial register and noted the participant's first natural response. If the participant's natural first response consisted of predominantly visual intuitive responses, such as constructing a quotient map in the digraph register, the researcher followed up with a switch prompt that asked the participant to translate their response using formal-symbolic language. If the participant provided a predominantly formal response that involved writing down cosets or maps using formal notation the researcher applied a switch prompt that asked them to translate what they did using the digraphs.

Data Analysis: Four-Level Coding Framework for Fluency

Notes contained the cues that students noticed or did not notice in certain registers, in what order were registers visited, ways in which representations were used either as a source to stimulate ideas or as a check, and conversions from initial registers entered to prompted registers. These notes were used to develop a chronological summary for how each participant's approach to the task evolved. A comparison highlighted that the three participants invoked many interpretations of what losing information and collapsing structure meant to them within the context of the task. Several strategies that the participants explored were identified in the interview transcripts such as: work towards the Klein-four group, the cyclic group of order two, and the impossible strategy to obtain the cyclic group of order four. Participants also exhibited multiple sign-interpretation pairs for the same sign.

This preparatory round of analysis led to the analytic coding framework in Figure 2. Level 1 of this framework consists of collapsing interpretation themes. Level 2 displays strategies that a learner may enter. Level 3 denotes the registers that learner may use in unimodal sequential order or multiple registers used in parallel. Level 4 focuses on the finest units of analysis, learners' sign-interpretation pairs within each of the registers and conversions between registers.

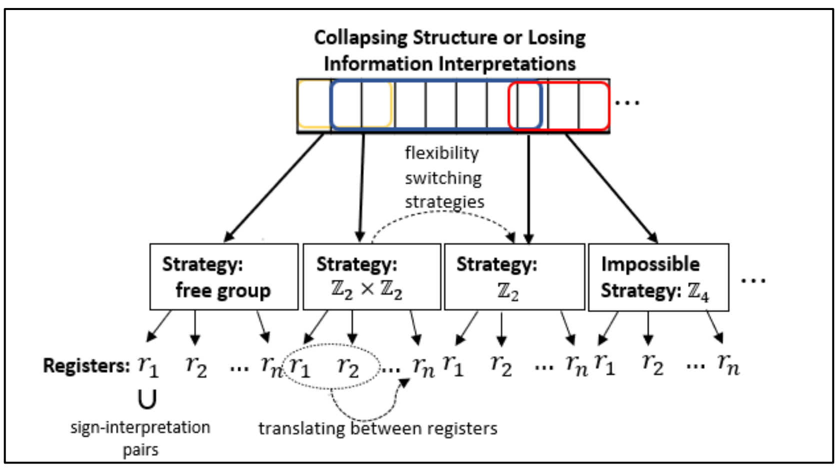


Figure2. Four-level analytic coding framework for quotient fluency

Next, Audio-video files were imported into the Qualitative Data Analysis Software NVivo 12. A hierarchical code book that mirrored the four-level analytic frame was also set up in NVivo as shown in Figure 3. The parent nodes: interpretations, strategies, registers, and sign-interpretation pairs and conversions. Additional selected register codes, Cayley tables, digraphs, group presentations, etc., and open codes were created as sub-nodes under parent nodes. Open coding is the process of abstracting and labeling regularities or perhaps unique instances that are found after immersion in the data (Creswell & Poth, 2018). After coding in NVivo, a summary list of codes across all participants was transferred into a table as depicted in Figure 3.

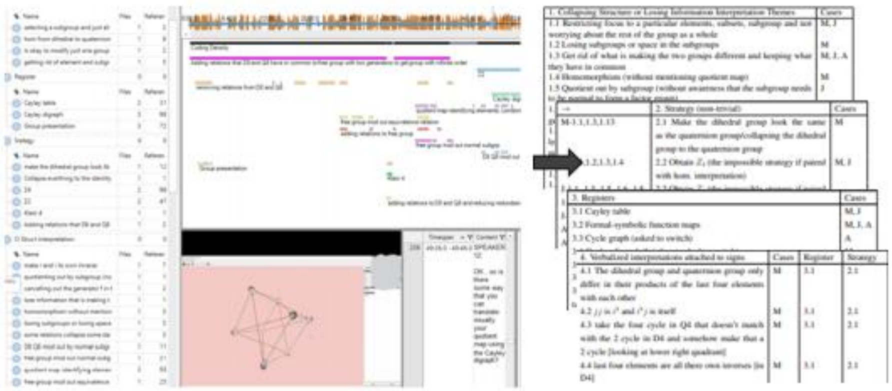


Figure3. Coding in NVivo 12

This table captured aspects of learners' paths in Figure 2. Data was combed through a final time to systematically collect any pseudo-semiotic representations invoked by participants related to their homomorphism or quotient constructions. A pseudo-semiotic representation as a unit of analysis is given by Figure 4.

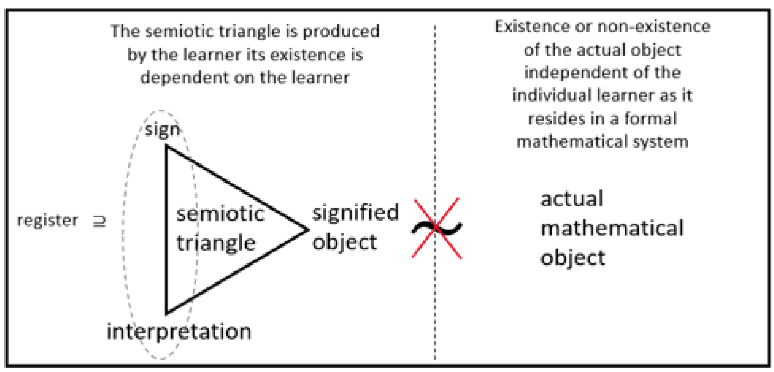


Figure 4. Pseudo-semiotic representation

Results and Discussion

There was a great deal of variability among Max, Jenni, and Alex with respect to their approach to the same task, what collapsing and ignoring information meant to them, their register use, and the sign-interpretation pairs that they produced.

Collapsing, Losing, and Ignoring Structure Interpretation Themes and Strategies

Several of Alex's and Jenni's interpretations of collapsing aligned with the mathematical process of constructing quotients. This included: combining or identifying elements into sets of elements, modding out by an equivalence relation, and partitioning. Max associated the collapsing metaphor with a homomorphism and "losing subgroups or space in the subgroups" but had not assimilated the quotient concept to his conceptions of a homorphism. The initial task prompt was rephrased for Max using the term homomorphism.

In contrast to Alex, Max and Jenni combined interpretation themes in ways that conflicted with the actual mathematical object of a homomorphism. They both mixed the interpretations of getting rid of what was making the two groups different and keeping a subgroup that two groups have in common with a homomorphism. These interpretations manifested into attempts to carry out the impossible strategy, to obtain the cyclic group of order four. The mixing of general every day problem-solving processes, such as looking for perceptual commonalities and differences in the group presentations, with the homomorphism concept led to an implicit assumption that a subgroup of a group would be an image of a homomorphism. However, the image of a homomorphism need not be a subgroup of the domain group. This thinking may have blocked Max and Jenni from entering the Klein-four strategy. During a follow-up interview, Jenni expressed an evolving awareness of this conflict and self-corrected stating that "if I quotient by a subgroup that is not just leaving me with a subgroup" of the starting group.

Quotient Map Conversions for Valid Strategies

Max did not enter a valid strategy to obtain a homomorphisms from the initial dihedral and quaternion groups to the cyclic group of order two or the Klein-four group. Jenni showed flexibility in moving away from the impossible strategy and to the valid strategy of constructing quotient maps from the initial groups to the cyclic group of order two. Alex produced solutions for both valid strategies and never mentioned the impossible strategy. Prior to the researchers switch prompts, Jenni and Alex gave valid solutions in a formal-symbolic register leveraging group presentations. Table 1 summarizes the quotient map conversion that Jenni and Alex made from their initial registers they went to on their own and the prompted digraph register in Table 2. None of the participants performed quotient map constructions in a Cayley table register for valid strategies. Due to time constraints a Cayley table switch prompt was not applied.

Table 1. Quotient map constructions prior to switch prompts

Register/Strategy	M	J	A
Formal-symbolic leveraging group presentation from initial groups to cyclic group of order two	None	$D_4/\sim$ $Q_8/\sim$	D_4/N Q_8/N
Formal-symbolic leveraging group presentation from initial groups to Klein-four group	None	None	D_4/N Q_8/N
Cayley tables/cyclic group of order two	None	None	None
Cayley tables/Klein-four	None	None	None
Digraphs/cyclic group of order two	None	None	None
Digraphs/Klein-four	None	None	None

Table 2. Quotient map constructions following digraph switch prompt

Register/Strategy	J	A
Digraphs/cyclic group order two	$D_4/\sim$ $Q_8/\sim$	D_4/N Q_8/N
Digraphs/Klein-four	None	D_4/N Q_8/N

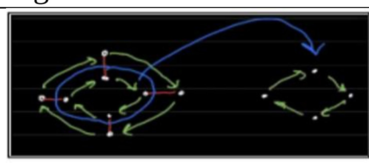
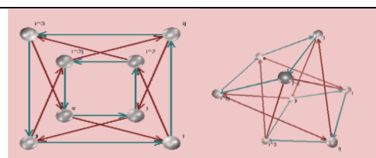
Pseudo-Semiotic Representations

Max could comfortably recall the formal definition of a homomorphism but produced several pseudo-semiotic representations of a homomorphism across multiple registers. The digraph register offered additional insights into Max's difficulty with the homomorphism concept, as illustrated in Table 3. Rather than combining elements into sets of equal size and viewing these sets as elements, Max wanted to erase vertices and edges in the digraph to reduce visual complexity. Max did not mention cosets, partitions, or equivalence relations during the task. Connections between these more fundamental underlying concepts and the homomorphism concept were missing.

Max: Then I guess if I was going to describe what I did, I guess like take this inner piece [circles inner green cycle] and just pull it out and make it hang out by itself and this is a new group. And you don't need to worry about the relationships between these guys or those guys, because those guys don't exist anymore (Table 3, Interpretation 1).

Max: …it's just the cycles in between the inner and outer cube, or squares, are just more complicated. And so, I guess again, I just selected one of the, either top or bottom and took it off on its own. So, if you straighten it out to where the green box is facing down. So, on the top, the top square, there's one four cycle between those elements. And then I just selected those, took them off by themselves. Then I don't have to worry about what they do with the other things on the bottom, because I'm just looking at the top (Table 3, Interpretation 2).

Table 3. Max's pseudo-semiotic representations in a digraph register.

Register	Sign	Interpretation	Signified Object
Digraph		1	Homomorphism from dihedral group of order eight to cyclic group of order four
Digraph		2	Homomorphism from quaternion group to cyclic group of order four

The incorporation of the digraph register made it apparent that Jenni's difficulties and strengths related to a homomorphism were not the same as Max's. Jenni related the term collapsing structure with the quotienting process of modding out by an equivalence relation and viewed sets of elements as elements in the image. She showed a strong understanding of underlying fundamental set-theoretic concepts related to the quotient concept. She consistently partitioned groups into equal size cosets in more formal-symbolic and digraph registers. She got hung up with how to make a partition of a group into a group and did not recall the necessary well definedness/normality condition.

Jenni: My initial instinct when looking at it was that all of these connections [red arrows] were like s's [flips] and so if you just get rid of the s's you collapse these two points into a point and these two points into a point and so you end up with a square and that was along with my process to get to Z four.

Concluding Remarks and Future Directions Towards Efficacy Research

To work towards an efficacy study restricted to the quotient concept, qualitative themes led to a rubric for how qualitative data for the collapsing structure task could be transformed into quantitative data. The quantitative compression revealed that Max had low, Jenni moderate, and Alex had high fluency levels for the quotient concept. This will be discussed further during the talk along with two questions towards future mixed efficacy research. What are educators doing in practice to support undergraduate learners' fluency for the quotient group concept? How can researchers determine if their interventions to provide support are working?

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The World in a State of Panic: Disaster Vulnerability and Digital Technology in Emergency Remote Undergraduate Mathematics Courses

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For undergraduate students across the United States, 2020 was the year of learning alone, together. Each experienced the COVID-19 pandemic and emergency measures in their own way, with interruptions impacting all facets of life: social, political, economic, and educational. Our study catalogues how the pandemic affected students' experience(s) of learning math. We utilized typed free responses - from 461 undergraduate students - to a set of five question prompts. "The world in a state of panic" is the phrase one undergraduate used to describe the backdrop of remote learning. Employing disaster vulnerability theory, a systems theoretic framework borrowed from social work research, we coded for themes of vulnerability and capability within our undergraduate mathematics ecosystem. Our findings indicate that the necessity of learning through digital technology was the greatest source of disruption, with vulnerabilities occurring in internet access, faculty digital literacy, digital content resources, and communication.

Keywords: COVID-19, Social Justice, Mathematics Education Systems, Digital Technology

Introduction

Although Spring 2020 teaching and learning donned the digital trappings of online education, it is vital to not conflate the two. Education practices in Spring 2020 are most accurately characterized as emergency remote learning: a temporary measure deployed during a time of stress to protect students and teachers (Ray, 2020). For some students, the measures taken proved to be beneficial; for other students, the abrupt but necessary changes were detrimental. Thus, the global COVID-19 pandemic provided a unique stressor that allows us to newly assess the strengths and weaknesses of our education systems, with a view towards correcting pre-existing inequity and improving access for all as we build to a new normal. Because of mathematics' unique role in university education as a gatekeeper to degree and career options for individual students (National Mathematics Advisory Panel, 2008), we focus our study on undergraduate mathematics education.

Theoretical Framework

Disaster Vulnerability & Disaster Resiliency Theory (Zakour & Gillespie, 2013) is a subset of system theory; it is primarily used in social work research to understand how and why communities become disrupted and to document the extent of disruptions during a disaster. While this framework is typically deployed in the context of natural disasters such as hurricanes or tsunamis, the theory has been utilized in the context of higher education structure, particularly around inclusive education and supporting students with disabilities (Cedeño, Meza, & Majía, 2018; Moriña, 2017).

Detecting disaster. To justify our use of a disaster vulnerability framework, we need to contextualize the COVID-19 pandemic as a "disaster". Most present-day academic definitions of disaster focus on community-disruption fallout rather than the hazards causing the interruption (Perry, 2018). A *disaster* is a form of collective stress with serious community-wide disruption of social, economic, and environmental conditions (Zakour & Gillespie, 2013). Accordingly, the massive disruption and change in our work and education systems in 2020 constitutes a disaster. A system is *vulnerable* if it has a reduced capacity to adapt in a disaster. Our research lens will identify vulnerabilities in emergency remote undergraduate mathematics education, where some students are resilient and others are injured by our systems.

Liability and capability. Zakour and Gillespie (2013) define a *liability* as a negative characteristic causing reduced capacity. A positive characteristic that allows adaptation is called a *capability*. A model of a disaster can include characteristics of individuals, small sub-communities, or nation-wide policies. Structural factors create vulnerability, which allows hazards to trigger a disaster. The vulnerability existed before and may persist after the hazard and, in fact, disasters exacerbate pre-existing social injustices (Le Masson & Lovell, 2016). We can utilize the process of identifying liabilities and capabilities to characterize our collective disaster experience and to foster community capabilities. From here it is possible to build a more inclusive and resilient normal.

Research Questions

Our study focuses on undergraduate student impressions of emergency remote math instruction at the University of Arizona during the Spring 2020 semester. We considered the following research questions: How did the experience of learning math in the context of a post-secondary mathematics course change as a direct consequence of the pandemic? What aspects of the experience did students view as harmful or beneficial to their learning? In the presence of the novel COVID-19 hazard, what underlying sources of inequity were revealed in our educational systems?

Methods

We conducted a survey in Summer 2020, hosted digitally on Qualtrics. Following IRB approval through the University of Arizona's Human Subject Protection Program, we used a Department of Mathematics email list to invite the 6,761 students who took an undergraduate (100 - 400 level) mathematics course at the University of Arizona main campus in the previous semester, with 461 students responding to five free response questions (see Figure 1).

Q1	After the transition to remote learning, how did aspects of life outside of school affect your experiences in your math class?
Q2	After the transition to remote learning, what aspects of your math class hindered your learning?
Q3	After the transition to remote learning, what aspects of your math class supported your learning?
Q4	Congratulations, you successfully completed the Spring 2020 semester and overcame the emergency transition to remote learning! Imagine that a friend asks for advice in an upcoming remote learning math class. What advice would you give?
Q5	What advice would you give to the instructor of an upcoming remote learning math class?

Figure 1. Free response questions, summer 2020 survey

The qualitative data we analyze in this report is a subset of the data gathered in the surveys, which also included a variety of quantitative measures. Because our free response questions were novel, we presented a draft of the survey to four undergraduate students to assess survey language and structure. Each student participated in a 30 - 45 min individual interview conducted by one-two researchers who asked the students to describe their interpretation of the questions after reading the survey. In compensation for their time, each received a $20 gift card to an online retailer (e.g. Amazon).

We utilized concept coding (Saldaña, 2016) to reveal themes of vulnerability and capability in our data. In our initial coding, we discovered a distinct division between the topics students discussed in relation to their emergency remote learning experience: aspects internal to their math courses and aspects external to their math courses. Aspects inside their math courses included interactions with the instructor or peers, the content, and technology used to create the remote classroom. Outside their math courses, students were affected by the blurring of boundaries between the microsystem, mesosystem, and exosystem (in the vein of (Bronfenbrenner, 1979)). In this paper, we will focus on the themes that emerged internal to the math courses. In future work, we will discuss our findings concerning external aspects that affected their mathematics learning experience.

Results

One of the primary goals of disaster vulnerability theory is to document the structure of a disaster, or how a hazard interacted with the system to cause community-wide disruption. Although we developed our model after careful consideration of the particulars emerging from our data, we present the model first because it informs how we present the underlying vulnerabilities that emerged during coding.

Our Disaster Model

Because technology has become more ubiquitous in our society, especially in educational settings, Kenneth Ruthven (2012) explains an updated version of the classic didactic triangle, see Figure 2. It is important that the didactic triangle be updated to a tetrahedron rather than a quadrilateral, because in an in-person teaching and learning environment, technology is a choice; learning can still occur without passing through the technology node. In other words, there is teaching and learning of mathematics that is not mediated by some form of technology. This updated model allows us to analyze multifaceted complexities, where the vertices, edges, and faces each carry importance and meaning. For example, the leftmost face (digital technology - learner - teacher) addresses "content-independent pedagogical aspects" while the back face (digital technology - learner - mathematical content) concerns "content-specific learning pathways" (Prediger, Roesken-Winter, & Leuders, 2019, p. 411).

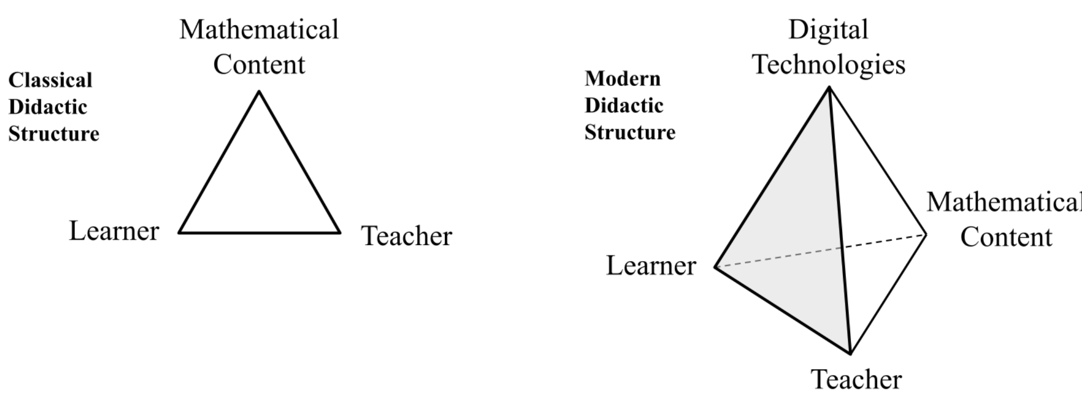

Figure 2. The classic didactic triangle and a modern didactical tetrahedron

When the global pandemic hazard interacted with this educational structure, the classic didactic face was excised from the modern didactic tetrahedron, triggering a disaster, shown in Figure 3. For example, video conferences took the place of office hour conversations between learner and teacher, digital whiteboards replaced chalk-boards, and 2-dimensional digital representations approximated 3-dimensional demonstrations. While technology can be of great value in the classroom, the tragedy was that unexpectedly*, all learning was mediated by technology*. Our emergent themes are inseparable from the digital backdrop in which they occurred.

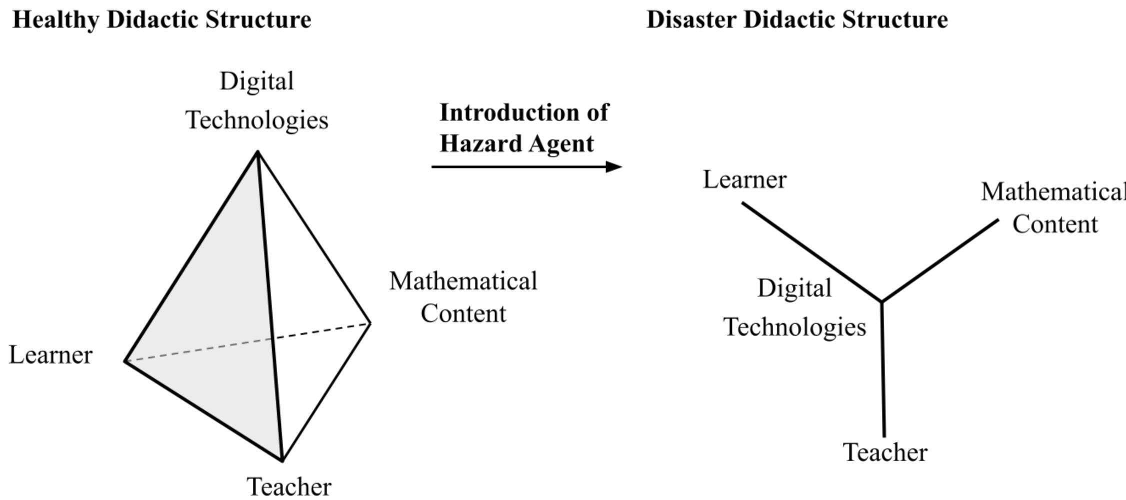

Figure 3. The excised didactic tetrahedron after the introduction of the COVID-19 hazard

A Summary of Vulnerability: The Gap Between Capability and Liability

We will discuss the most common points of capability and liability internal to the emergency remote classroom: internet access & quality, faculty digital literacy, content resources, and communication. Because the gap between capability and liability indicates vulnerabilities in our system, our goal is to understand the broad range of experiences wherein some students did not adapt to emergency remote instruction (see Figure 4). In Table 5, we provide a table with the number of students who experienced each item as a capability and the number of students who experienced it as a liability.

Internet quality & access. Students described inadequate internet as an obstacle to participating in their courses. One student wrote, *"My internet connection is terrible at home, leading to rushes or unfinished assignments."* Some students simply did not have the internet available at their home because of their isolated location (such as on a reservation) or, while the internet was available, it was too costly and they had to wait for COVID relief funds in order to afford it. Students with quality internet celebrated how it improved their access to content and digital resources that assisted in the learning process. One writer described, *"Having the internet and other resources DURING class helped guide learning."*

Faculty digital literacy. Sometimes faculty were not effectively using the available technology to convey course material, i.e. writing on a real-life chalkboard out of view of the webcam. More frequently, students shared how their instructor did not know how to use a program or the video conferencing software. One student wrote, *"My teacher did not know how to use technology and had endless problems that resulted in no lectures some days… "* When an instructor was comfortable with their technology, students appreciated how their remote course felt just like an in-person course. One writes, *"Both of my professors were extremely flexible. I especially appreciated their multiple formats of lecture notes and use of tablet technology to annotate the screen live as they went through content."* In fact, some faculty utilization of digital technology allowed their students to learn better than they would have without these tools.

Digital course documentation. Overwhelmingly, students want video recordings of their lectures to access later. One student expressed, *"being able to record the lectures so i can go back whenever i wanted if i needed help or forgot something was like a miracle happened."* That being said, other students warned that video recordings can hinder their learning process. For example, a student wrote, "*Do live lectures instead of recorded lectures because student will just skip through recorded lectures and there's less incentive to watch them*," while another confided, "*I also lost motivation to attend the Zoom lectures, telling myself that I would watch the recordings, but more times did not*." Other students were frustrated with a lack of documentation of their courses, or they found the quality to be poor. Sometimes a big delay in the posting of lecture recordings left students without the necessary support to participate in their courses.

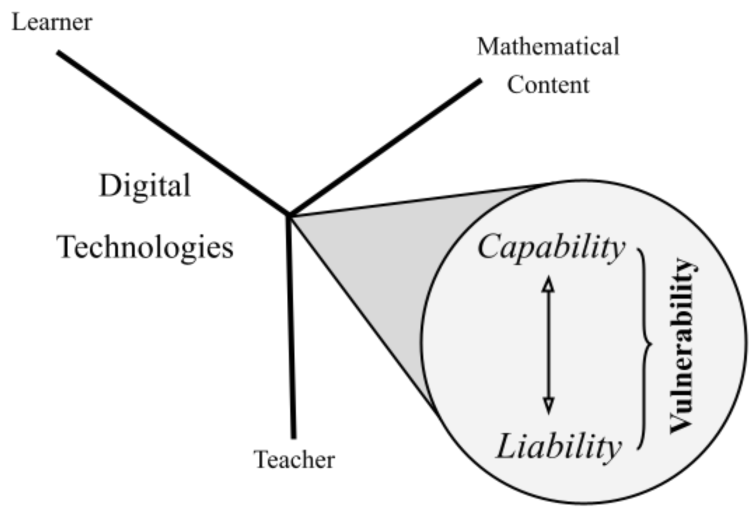

Figure 4. A gap indicates vulnerability in the system

Table 5. Capability and liability frequencies among the 461 participants.

Number of students for which it was a…	**Capability**	**Liability**
Internet Access & Quality	14 (**3%**)	74 (**16%**)
Faculty Digital Literacy	43 (**9%**)	96 (**21%**)
Internal Content Resources	192 (**42%**)	37 (**8%**)
External Content Resources	36 (**8%**)	-
Communication with instructor	160 (**35%**)	120 (**26%**)
Communication with peers	48 (**10%**)	96 (**21%**)

Online communication. Communication with the instructor was extremely important to students. Many expressed frustration that they could not interact in the way they had before the pandemic, for example, *"I think it was harder to ask questions online and explain where you are going wrong because I couldn't actually show my teacher my work."* Other students felt like they could not reach their instructor, with one sharing, *"It became difficult to reach my teachers. They didn't respond to emails and would quite often cancel their office hours."* For some students, online communication is much more comfortable than in person interactions. One participant confided, *" the general anxiety of in-class lectures and having to stop a lecture to ask a question in a giant room of 200+ people was eliminated with zoom questions. It was very easy to communicate mid/post-lecture with my professor and ask questions."*

Roughly two thirds of students who mentioned communication with their peers in the remote classroom discussed how much they missed the in-person experience and the limitations of online engagement. One wrote, "*Breakout rooms for group work don't really work half the time, no one likes to talk over zoom."* For some students, group work online created anxiety: *"It is already stressful enough to work with strangers in person, but that is expected. Trying to work with strangers online is even worse and ends up with no work getting done."* On the other hand, one third of students were satisfied with the online connections that they formed, with one participant explaining, *"I was able to better connect with the people online, as that I don't particularly like in-person speaking, and could more effectively be a better classmate when someone is stuck, and even make better friendships."*

Symptoms of Disaster

We also monitored responses for signs of negative outcomes after a disaster, such as fatalism (Zakour & Gillespie, 2013). Table 6 shows the number of students exhibiting fatalism or erosion of social trust. One student wrote, *"quit school. don't even bother with online learning,"* while another gave the advice, *"Mute your mic if you need to cry."* Some felt abandoned by their instructor: *"My teacher gave up on my class when we went online,"* while others felt abandoned by their country: *"why care about a math test when my country is letting thousands die?"* There were several outpourings of anger at the university, the math department, or the instructor.

Table 6. Fatalism & erosion of social trust frequencies among the 461 participants.

Number of students exhibiting signs of...	
Fatalism	47 (**10%**)
Erosion of Social Trust	28 (**6%**)

On the other hand, we also saw plentiful signs of students adapting to the emergency. A student shared, "*I just want teachers to know they are appreciated for their hard work during this weird time in our lives*," and another wrote "*Stay positive and keep doing what you are doing. We appreciate everything you've done for us!*"

Discussion and Conclusion

Internet infrastructure is a nation-wide level concern. This summer the National Telecommunications and Information Administration (NTIA) released an interactive map that shows the areas of the United States where citizens are not utilizing high-speed broadband internet (McGill, 2021). When triggered by the COVID-19 hazard, this infrastructure liability led to greater harm for certain students. As departments and individual teachers, we are not able to correct this vulnerability, but we can take measures to lessen the harm done to students by adopting a position of grace and understanding.

Prior to 2020, there was a well-documented gap between the digital technologies available and faculty utilization of these resources (Bourrie, Jones-Fermer, & Sankar, 2016). One primary cause is lack of training and support at the university level. TopHat surveyed 808 higher education faculty and support staff in the United States and Canada and found that “40% and 38% of respondents who indicated that learning to use tools to teach synchronously and asynchronously, respectively, has either been a small or non-existent part of the training they received” (TopHat: Higher Ed, 2020). With the increased expectations on faculty to use digital resources in the classroom, we hope to see universities invest more in support and training in these technologies.

Increased documentation of what occurs during class has both the potential to increase access to learning as well as to harm the learning process. From the lens of inclusive education (Moriña, 2017) or becoming a “student-ready-college” (McNair, Albertine, Cooper, McDonald, & Major, 2016), recording class allows students with disabilities or students experiencing a challenge in their personal life to still engage in education. However, increased documentation may lead to students engaging in rote learning, rather than deeper comprehension of the subjects. In fact, Lithner explains that this problem with memorizing over understanding predated the pandemic: “the gravity of the problem, as a main cause behind learning difficulties, is not fully apprehended by students, teachers, textbook writers, syllabus constructers, administrators, politicians, and perhaps also among many researchers” (2008, p. 273). Further, prior to the pandemic, Muir found that online and in-person students utilized online resources differently (2013). With the blending of the online and face-to-face modes, further research is warranted into course documentation that both supports inclusive education while also fostering relational understanding over memorization.

The importance of communication in an effective emergency remote course is becoming well-documented, see (Lanius, Frugé Jones, Kao, Lazarus, & Farrell, 2022), (Pagoto et al, 2021), or (TopHat: COVID-19, 2020). This area is one where individual instructors can make an impact, lessening the gap between capability and liability, removing a vulnerability in our educational systems. It is unreasonable to expect a single instructor to reply to a massive volume of student emails. However, by adopting more novel modes of communication, such as a digital platform which allows students to ask questions in a class forum, instructors can find ways to effectively and efficiently meet this student need.

Research limitations. The primary limitation of our data collection is that we administered it through the internet, which poses a serious obstacle to documenting the experience of students without internet access. We also may have a negative response bias, where students with a negative experience with emergency remote learning might be more likely to respond.

Future directions. We conducted a follow-up survey in Summer 2021. We will complete an analysis to see how our educational systems fared in a prolonged disaster state. In our 2020 data, we see a potential for online education to decrease anxiety for some students. We will further explore what aspects of the online environment can alleviate anxiety and how that might be incorporated into the in-person classroom. Finally, with heightened documentation in classes, we plan to explore how students are utilizing resources and the impact on their understanding.

Acknowledgment

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Student Thinking in an Inquiry-Oriented Approach to Teaching Least Squares

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We present the results of a classroom teaching experiment for a recently designed unit for the Inquiry-Oriented Linear Algebra (IOLA) curriculum. The new unit addresses orthogonality and least squares using Realistic Mathematics Education design principles with the intent to implement the new unit in an IOI (Inquiry-Oriented Instruction)-style classroom. We present an analysis of students' written responses to characterize how they thought about the notion of shortest distance, travel vectors, orthogonality, and dot product in the "Meeting Gauss" context.

Keywords: Linear Algebra, Inquiry-Oriented Instruction, Least Squares Method

Least squares in linear algebra is often introduced as a method for finding the "best possible solution" when solving a system of linear equations, a vector equation, or a matrix equation that has no exact solution. As part of a larger research project involving designing new linear algebra task sequences for classrooms implementing inquiry-oriented instruction (IOI), we designed a task sequence, the "Meeting Gauss" unit, that would facilitate student exploration in $\mathbb{R}^3$ of the notion of "best possible solution" and lead to the reinvention of the approximation equation $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$, least squares solution $\hat{\boldsymbol{x}}$, projection, and least squares error. In the Meeting Gauss task, students were asked if they could reach Gauss' location using three transportation vectors. Students determined they could not reach Gauss and were tasked with finding a location that is closest to Gauss' initial location. Gauss' location is represented by a vector, $\boldsymbol{g}$, in $\mathbb{R}^3$ located off the plane that is the span of the transportation vectors. The shortest distance Gauss could travel corresponds to the magnitude of the orthogonal vector pointing from Gauss' location to the plane. We notate the shortest distance between Gauss and the meeting location as the length of $\boldsymbol{e}$ (error) to be minimized and the vector from the origin as $\boldsymbol{p}$ (Figure 1).

In this paper, we analyze two "snapshots" of student reasoning from the first and second day of a classroom teaching experiment (CTE) (Cobb, 2000) using students' reflection writings as data. Our research question is: *How do students interpret and use notions of shortest distance and travel vectors when learning least squares through an inquiry-oriented task in a linear algebra class?* We investigate various ways students interpreted finding the "shortest distance" (i.e., how students determined which direction Gauss should travel so that his trip is the shortest possible distance). Further, we characterize students' interpretations of the relationships (e.g., orthogonality, projection) that we perceive as important aspects for learning least squares.

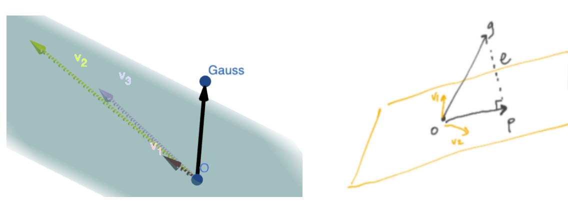

Figure 1. Meeting Gauss task setting in GeoGebra (Left) and Class instruction (Right)

Background Literature and Theoretical Framing

There is little literature on the teaching or learning of least squares or closely related topics within the context of linear algebra. Turgut (2013) created a lesson in which students used Mathematica to solve least squares problems involving finding lines or curves of best fit. He reflected on how students' use of these tasks incorporates Harel's (2000) concreteness, necessity, and generalizability principles. Donevska-Todorova (2015) used Hillel's (2000) three modes of description (arithmetic, geometric, and axiomatic) to explicate three definitions of dot product. Arithmetic referred to multiplying vector components and adding. Geometric referred to the cosine definition and the axiomatic definition was in terms of general properties. She also designed an applet to support students' understanding of dot product of vectors in a dynamic geometry environment (DGE). Cooley et al. (2014) created a module for teaching dot product using the cosine of the angle between two vectors. Their task sequence focused on comparing frequency vectors to determine whether or not the same author wrote two different texts.

Our instructional materials were designed to support students' ideas and support instructors in facilitating conversations around students' ideas. Students make mathematical progress as they participate in class and group discussion, ask questions, and explain their ideas. Instructors guide classroom activity by encouraging students to share their thoughts, asking their thinking about how and why they make decisions, and leveraging their ideas to move forward. Specifically, for developing our instructional sequence, we adopted the instructional design heuristics of Realistic Mathematics Education (RME) informed by Freudenthal (1991). When designing instructional task sequences, we leveraged the notion of *guided reinvention* to support students' transition towards more formal mathematics (Gravemeijer, 1999). From the students' perspective they are not re-inventing anything, our intent with the least squares task sequence is to support students in advancing their mathematical activity through symbolizing, algorithmatizing, and defining (Rasmussen et al, 2005). We designed the "Meeting Gauss" task in the least squares instructional sequence to be an experientially real starting point. This is similar to the Magic Carpet task in the first unit of the IOLA curriculum (Wawro et al., 2012; Wawro, Zandieh, et al., 2013). Students could engage in mathematical activity immediately, and their initial activity should constitute a basis for more formal mathematization.

Rasmussen and Keene (2019) used a river journey metaphor to capture what many in the mathematics education research field know as hypothetical learning trajectories (Simon, 1995). We use the river journey metaphor intentionally to move away from the image of a learning trajectory as a singular path that a researcher hypothesizes as the only way to learn a particular idea. We agree with Rasmussen and Keene that the image of a learning trajectory as the only path to learn a mathematical idea is not representative of the learning process. Further, Rasmussen and Keene utilized the notion of waypoints (Corcoran, Mosher, & Rogat, 2009) from the learning progression literature as "islets" within their river journey metaphor, acknowledging that students may, or may not, visit each of the islets (waypoints) outlined in the river journey (learning progression). To be clear, Rasmussen and Keene's waypoint journey for student reasoning about ODEs is a hypothetical research tool stemming from seven semester-long classroom teaching experiments and years of design-based research. The waypoint journey presented here is more of a "first journey down the river" in the context of a new task being used in an Inquiry-Oriented Linear Algebra (IOLA) classroom for the first time.

Methodology and Research Setting

This study is a part of a larger linear algebra curricular development project funded by the NSF. We conducted a classroom teaching experiment (CTE) to study the first implementation of our instructional sequence and to study how students' reasoning evolves throughout the task sequence (Cobb, 2000). The tasks implemented in the CTE used an *experientially real* situation designed to help students *reinvent* the least squares method. The CTE was conducted with STEM students in two Linear Algebra classes at a large public university of the Southeastern United States. Due to COVID-19, the course was taught synchronously online via Zoom. The course prerequisite a B or better in Calculus I or passing Calculus II. There were 33 students enrolled in one class and 38 students in the other. The CTE lasted four consecutive class days over the course of two weeks towards the end of semester. The fourth author was the course instructor and last author acted as a TA throughout the semester. The breakout group composition was kept as consistent as possible unless student absences necessitated the rearrangement of groups. In total, 22 students consented for their work to be used in research.

Description of the Task

The Meeting Gauss task begins with a callback to a previous IOLA task from earlier in the semester[1]. The vectors $\boldsymbol{v_1}, \boldsymbol{v_2}$ and $\boldsymbol{v_3}$ are presented as three modes of transportation, Gauss lives at a location $\boldsymbol{g}$ in 3D, and students are trying to determine where they can meet Gauss. The Meeting Gauss task is comprised of two parts (Figure 2): (1) investigate if it is possible to reach Gauss at his specific location using the three modes of transportation, (2) explore the trips that you and Gauss should make given the fact that Gauss needs to meet you somewhere you can reach because his house ($\boldsymbol{g}$) is not reachable using the given modes of transportation (i.e., the travel vectors).

Part 1. Remember those modes of transportation you used for 3D travel?

They were $\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}, \boldsymbol{v_3} = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$

Let's say they are a hoverboard, magic carpet, and jetpack, respectively.

Using those modes of transportation, can you get to Gauss if he lives at $\boldsymbol{g} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$

Part 2. Gauss agrees to meet you somewhere you can actually reach. He wants his trip to be the shortest distance possible. So, our big guiding questions are:

a) Where should Gauss meet you?
b) Along what vector would Gauss travel to get to your meeting point?
c) What distance would Gauss's trip be?
d) How would you get to your meeting point using your modes of transportation?

Figure 2. Statement of the Meeting Gauss task

On Day 1 of the CTE, students discussed their ideas related to Part 1 in breakout rooms. At the end of class, students were asked to write a reflection giving their initial thoughts, intuitions, or ideas for questions a)-d) of Part 2. On Day 2, the instructor began class by incorporating many of the students' reflection writings into her mini-lecture. The instructor highlighted students' reflections that incorporated Gauss' shortest trip distance and the span of the travel vectors. In the first breakout session of Day 2, students used two previously created GeoGebra applets to further explore the second part of the Meeting Gauss task. Many of the breakout groups used the applets to obtain estimates for Part 2, questions a)-d). Afterwards, students were called back to the whole class session to share their ideas with other groups and to listen to a set of mini-lectures about lengths of vectors and dot products. In addition to the given transportation vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}$, and the Gauss vector $\boldsymbol{g}$ from Part 1, the instructor defined the vector Gauss travels

[1] In particular, Task 3 from the "Magic Carpet Ride" unit, which introduced linear independence (Wawro et al., 2012).

along as $\boldsymbol{e}$ and the vector from the origin to the meeting point as $\boldsymbol{p}$ (as pictured in Figure 1). In the second breakout session, students were tasked with using the given information and mathematical relationships developed during class (see Figure 3) to provide exact solutions to a)-d). After Day 2, students wrote a reflection summarizing the progress either they or their group made in finding the exact answers for questions a)-d) along with something that was clear and something that they wondered about.

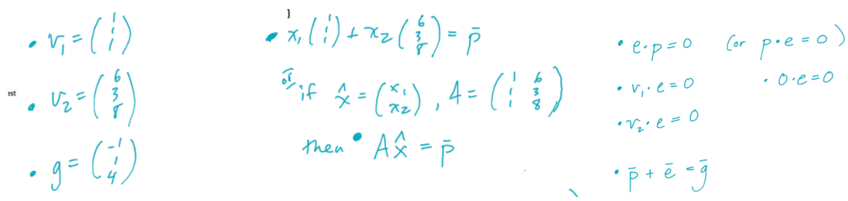

Figure 3. The known information and mathematical relationships developed during class.

Data Sources and Analytic Method

Our data sources include classroom videos, breakout group videos, Jamboard pages, and students' reflections. The whole class discussions and the breakout sessions were recorded via Zoom by the last two authors. The students used Jamboard to communicate and record their work in breakout rooms. Students' reflection writings submitted after Day 1 and Day 2 of the CTE were our main data source. The first two authors watched the whole class discussions and breakout group videos to gain a sense of how class was organized. The first two authors reviewed the Jamboard pages to see how students expressed their mathematical ideas during the breakout discussion. Consenting students were given pseudonyms, and their reflection writings were deidentified and transcribed by the last author. The first two authors engaged in open coding (Strauss & Corbin, 1990) of all four days of student reflections. A code book was created from attending to both (1) student thinking and (2) their descriptions of their methods/symbolization. We used our set of 26 initial codes to analyze students' Day 1 and Day 2 reflections line by line in a spreadsheet. After our initial pass at coding students' Day 1 and Day 2 reflections we identified several categories that emerged within and across our codes. Figure 4 displays our final categories and codes, which were agreed upon by the first three authors. Our process of creating an initial set of codes that identify trends within and across our data is consistent with an emergent coding method (Glaser & Strauss, 2017).

Results

Our findings can be thought of as two snapshots on the river journey. The first snapshot captured students' reflections at the end of Day 1. From their reflections, students were beginning to recognize a number of features of the Meeting Gauss scenario and how those are related. Day 1 reflections are grouped into two themes: (1) shortest distance and (2) the location of the transportation vectors. Students wrote about shortest distance as (1) a point-to-point trip, (2) an orthogonal/perpendicular direction, and as (3) ways to find the distance including the distance formula, the Pythagorean theorem, and trial and error. The second snapshot captures students' reflections at the end of Day 2. From their second reflection, students had moved down the river in various ways. We grouped their writing into (1) a more developed notion of the error vector, $\boldsymbol{e}$, as perpendicular to the plane and transportation vectors, (2) the dot product as providing information about orthogonality and ways to calculate relationships between vectors, and (3) the use of matrix multiplication and systems of linear equations (SLEs) to symbolize the relationships between the perpendicular vectors in ways that allow for finding the error vector.

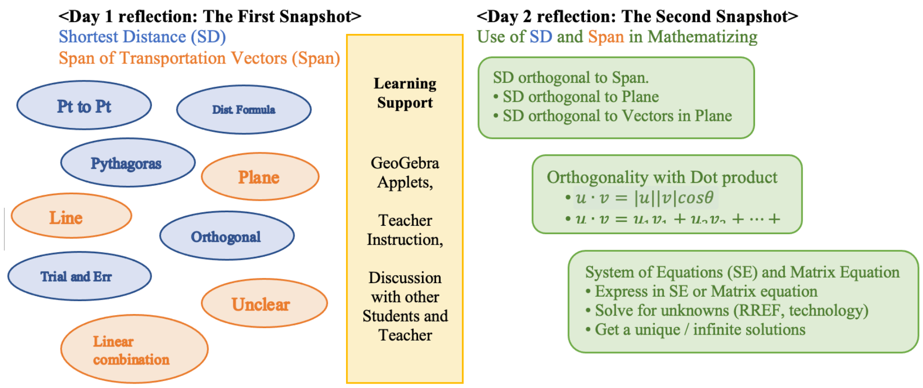

Figure 4. Overview of codes from students' Day 1 and Day 2 of the CTE reflection writing.

First Day of the CTE Snapshot Analysis

On Day 1, most students used row reduction to conclude that Gauss is not reachable using the transportation vectors (see Figure 5). Some students included geometric explanations in their reflections including descriptions of how the span of the transportation vectors was a plane.

Figure 5. A snippet of a group's Jamboard from working on Part 1 of the Meeting Gauss task.

Most students recognized that Gauss can travel around freely to meet the traveler, but the traveler's movement is restricted to use of the given transportation vectors. We will now present examples from students' reflections demonstrating our codes pictured in Figure 4.

Interpretation of shortest distance. Our analysis revealed that students' intuitive notions of Gauss' shortest travel distance included ideas such as *point-to-point distance* and *orthogonality*. Ten students answered that they needed to determine the shortest distance between two points: Gauss' location and the point on the plane closest to Gauss. For example, "We can *minimize the distance* from the *point to a point,*" (Cecil), and "Gauss should take *the shortest distance from his point* (-1, 1, 4) *to the closest point on the plane* our transportation modes cover," (Iliana). Some of the students' thinking of the *point-to-point distance* indicated that they would apply the distance formula or a calculus-based methods to minimize the distance. Five students noted that Gauss's path is perpendicular to the plane. Some of those students also noted that the span of the travel vectors is the plane. For example, "If I had to imagine, Gauss would *travel in a straight line orthogonal to the plane* formed by the three vectors," (Liam) and "…*the vector he travels along should be perpendicular to the final mode of transportation's vector*" (Luka).

Some students mentioned how they would calculate the shortest distance. We view each of the following methods used for measuring the shortest distance as related to *point-to-point distance* thinking: *distance formula*, *Pythagorean theorem*, and *trial and error*. Three students explicitly mentioned that they would apply or did apply the distance formula. For example, "Not sure how to calculate distance, *maybe plot the vectors and then use distance formula,*"(Damien). Tuan did apply the distance formula "Distance: sqrt((-1-(-1))^2+(-1-(-2))^2+(-1-(-5))^2) =

sqrt(0+1+16) = sqrt(17)," There were two students who mentioned they would use the Pythagorean theorem to find the shortest distance. For instance, "We could use *pythagroas theorem to find the distance* of the projection," (Anton).

Interpretation of transportation vectors. Our analysis further revealed that students thought about the transportation vectors in various ways. Some students tended explicitly to the notion of span such as *vectors generate the plane* or *vectors generate a line*. While 13 students mentioned the transportation vectors generate the plane, not every student explicitly mentioned where the plane came from. Other students explicitly used the terms span or linear combination in their descriptions. For example, "The way I am thinking about this is to locate *a spot that is within the area that the 3 vectors span,*" (Ani), "Since Gauss is getting to *the plane that is the span of the three modes of transportation*, a *linear combination* of the three vectors will get us there as well," (Lureyna). Lureyna's response tends specifically to both the idea that the span of the transportation vectors creates the plane, and the traveler must use those modes of transportation to reach the meeting point with Gauss.

Second Day of the CTE Snapshot Analysis

In the following three subsections, we briefly outline the various sections of the river that students went down in the second snapshot from Day 2 of the CTE. Each subsection corresponds to one of three groupings of students' responses outlined in Figure 4.

The 'e' (error) vector is orthogonal to the plane. Our analysis revealed that students' description of orthogonality between the vector $\boldsymbol{e}$ and the plane is multifaceted. We categorized their descriptions into two ways: (1) The $\boldsymbol{e}$ vector is orthogonal to the plane, (2) The $\boldsymbol{e}$ vector is orthogonal to the vectors that constitute the plane. For example, "...We knew that *the line of travel from Gauss to the plane must be orthogonal to the plane*..." (Bianca). From Bianca's response, we see that she made a connection between the plane and the orthogonal path to said plane. On the other hand, some students stated the **e** vector is orthogonal all vectors parallel to the plane. For instance, "I know that *e has to be orthogonal to p and v1, v2,*" (Annalisa).

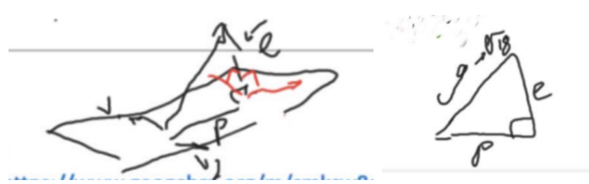

Figure 6. Lucia's pictures from her CTE Day 2 reflection

Dot product and orthogonality. After students returned from the first breakout group session, the instructor gave a set of mini-lectures deriving the lengths of vectors and defining dot product. Students' written descriptions about how they thought about dot product was subtle and sometimes difficult to characterize. For example, Lucia stated (Figure 6, right), "The vectors p and e create vector g, from origin to Gauss, *and the dot product of p and e is zero because they are perpendicular,*". Lucia seemed aware that orthogonality is related to dot product. She also mentioned " We had *tried to combine the statements p·e=0, e·v1=0, and e·v2=0 into one big equation*, and *I think you can eliminate the ||e|| from all sides*, but the issue is that *I do not know how to get the angles between the different vectors* and such," (Lucia). From Lucia's complete response, it was possible to determine she was leveraging the cosine definition of the dot product in her thinking. Further, some students expressed the orthogonal relationship between the error vector ($\boldsymbol{e}$) into a system of dot product equations (Figure 7, left). Other students mentioned that they knew they could use the dot product equations in some manner, but it was unclear if they were thinking about orthogonality. There were some students who seemed to try connecting dot product with the transportation vectors and the $\boldsymbol{e}$ vector but were not comfortable with the transition towards a more formal mathematical expression. For example, "...looked at *plugging in the v1 and v2 vectors into their dot product equations with e* ...using the coefficients we found as a matrix, *whose dot product with e was 0*. This didn't make much sense to me..." (Iliana).

Systems of equations and the matrix equation. Some students appeared to be moving towards formalizing the notion that the $\boldsymbol{e}$ (error) vector was orthogonal to each travel vector to create a matrix equation $\boldsymbol{A^T e} = \boldsymbol{0}$ using their system of dot product equations. For example, "From the dot product of e and v1 and the dot product of e and $v2$, we were able to create a matrix, $[[111], [638]]$, which when multiplied by e would equal 0... if two vectors are orthogonal, their dot product must be zero, which is very helpful..." (Lureyna) and "...we had the matrix A that was $v1$, $v2$ times $e\ (x\ y\ z)\ = 0$…but didn't know where to go from there...I know that e has to be orthogonal to p and $v1, v2$," (Annalisa). Other students, like Cecil (Figure 7, middle), expanded his dot product equations into a system of linear equations (SLE). Cecil used a SLE to write a matrix equation that we recognize as $\boldsymbol{A^T e} = \boldsymbol{0}$. We are unsure how Cecil thought about the result of his row reduction since he stopped after row reducing.

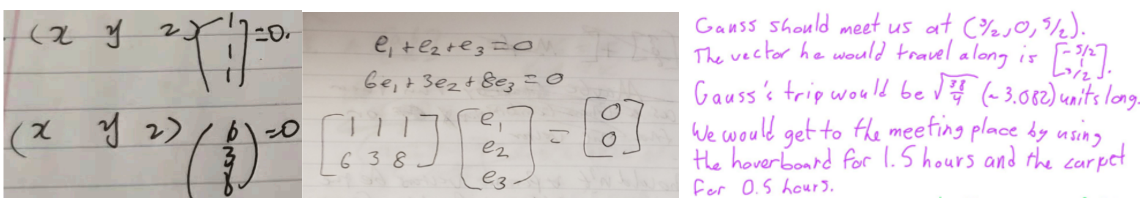

Figure 7. Min's(Left), Cecil's (Middle), Vaki's (Right) written solution.

Discussion

In the first journey down the river of developing the least squares solution method, we see that some of the students leveraged their intuitive notions of shortest distance and the span of the transportation vectors towards a more formal symbolization using the dot product. Students traveled down the river in their own way, spending different amounts of time on each piece of the river journey as seen in the variety of answers within both snapshots. One takeaway is that the least squares river journey may require more time in more areas than others. Vaki was one of the only students to interpret $\hat{\boldsymbol{x}}$ as the amounts to travel on each mode of transportation to meet Gauss. This meaning of $\hat{\boldsymbol{x}}$ is what we as experts recognize as the least squares solution to the Meeting Gauss context. This paper is limited to an analysis of the first two days of the CTE. Future work may investigate students' progress on the final two days in formalizing their thinking towards reinventing the least squares solution $\hat{\boldsymbol{x}} = (A^T A)^{-1} A^T \boldsymbol{b}$. On our next journey down the least squares river, we intend to reflect on where students ended at the end of each day of the CTE to inform future revisions of the Meeting Gauss context within the IOLA curriculum.

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Learning to Teach Reasoning and Proof in an Online Setting: The Case of Nancy

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Preparing prospective secondary teachers (PSTs) to teach mathematics with a focus on reasoning and proving is an important goal for teacher education programs. A capstone course, Mathematical Reasoning and Proving for Secondary Teachers, *was designed to address this goal. One component of the course was a school-based experience in which the PSTs designed and taught four proof-oriented lessons in local schools, video recorded these lessons, and reflected on them. In this paper, we focus on one PST – Nancy, who took the course in Fall 2020 during the pandemic, when the school-based experience moved online. We analyzed how Nancy's Mathematical Knowledge for Teaching Proof (MKT-P) evolved through her attempts to teach proof online and through repeated cycles of reflection.*

Keywords: Reasoning and Proof, Prospective Secondary Teachers, Online Teaching, Reflection.

Supporting prospective teachers developing skills needed to teach mathematical reasoning and proof is an important goal of mathematics teacher preparation (Association of Mathematics Teacher Educators, 2017). However, there is limited theoretical or practical knowledge on how to provide prospective teachers with such support (Stylianides, Stylianides, & Weber, 2017). To address this knowledge gap, Buchbinder and McCrone (2018, 2020) developed a capstone course *Mathematical Reasoning and Proof for Secondary Teachers* and studied the development of PSTs' dispositions towards proof and Mathematical Knowledge for Teaching Proof (MKT-P), with a particular focus on the manifestations of MKT-P in classroom practices. Among other things, this involves an ability to plan and enact proof-oriented lessons.

The breakout of the global pandemic and the schools' pivoting to online teaching in 2020 put additional demands on PSTs who had to learn how to teach proof online "on the spot." We report on a case study of one such PST, Nancy (a pseudonym), who successfully surmounted these challenges. Rather than choosing a "representative" case, by focusing on a successful and articulate PST like Nancy and studying her learning processes and strategies, we hoped to gain insights that could inform the mathematics education community (Seawright & Gerring, 2008).

Theoretical Framing

Mathematical Knowledge for Teaching Proof

It has been suggested that in order to support students' learning of reasoning and proof, teachers need a special type of knowledge: Mathematical Knowledge for Teaching Proof (MKT-P). Several researchers (e.g., Lesseig, 2016; Stylianides, 2011; Steele & Rogers, 2012) proposed frameworks delineating the components of MKT-P and the relationships among them. Akin to Steele and Rogers's (2012) approach, Buchbinder and McCrone (2020) conceptualized MKT-P beyond declarative knowledge captured by written tests to include related classroom practices. Their resulting framework comprised three facets: Knowledge of the Logical Aspects of Proof (KLAP), Knowledge of Content and Students (KCS-P), and Knowledge of Content and Teaching (KCT-P). Each knowledge facet has its corresponding classroom practices. In the context of mathematics classrooms, KLAP corresponds to the use of precise mathematical language and notation adjusted to students' grade level, and the remediation of students' reasoning errors.

KCS-P corresponds to facilitating discussions that address students' common proof-related misconceptions and make proof concepts explicit. KCT-P corresponds to designing and enacting proof-oriented tasks. The three knowledge facets are interrelated, e.g., designing proof-related tasks (KCT-P) requires knowledge of students' proof-related conceptions (KCS-P) and proof-specific subject matter knowledge (KLAP). Distinctions between the MKT-P facets aided the design of learning experiences enhancing PSTs' MKT-P and the design of MKT-P assessments both in written form and through lesson planning and enacting (Buchbinder & McCrone, 2021).

Reflective Noticing

Teachers advance their professional expertise by reflecting on teaching (Seidel et al., 2011), which, in turn, entails teachers noticing elements of classroom environments that are most likely to support student learning (Sherin, Jacobs & Philipp, 2011). Although many definitions of noticing exist, we follow Stockero's (2021) definition of noticing as comprised of attending and interpreting. While noticing is often tacit, reflecting requires conscious engagement and processing. In this paper, we use the term *reflective noticing* to capture both of these processes.

There are several types of reflection according to timing: reflection-*in*-action, which occurs during teaching; reflection-*on*-action, which occurs after teaching (McDuffie, 2004; Schön, 1987), and reflection-*for*-action, which connects particular events to future actions (Jay & Johnson, 2002). Since merely descriptive, anecdotal, or non-critical accounts of teaching have little benefits for learning, researchers suggested that *productive* reflection entails attending to multiple aspects of classroom environments; interpreting, analyzing, and integrating them, and connecting them to theoretical principles, future actions, and past experiences (Moore-Russo & Wilsey, 2014). We take the last part of this definition - connecting to the past experiences –as yet another type of reflection: reflection-*back*.

Although PSTs often do not have access to classrooms, whenever possible, PSTs should be encouraged to reflect on their teaching (Jacobs et al., 2010; van Es, 2011). Learning to notice and analyze classroom instruction has been shown to benefit PSTs' professional development (Stockero, 2021). Similarly, Buchbinder et al. (2021) identified a variety of learning opportunities afforded by PSTs reflecting on their own teaching using 360° video technology.

In this study, we examine an overarching question: "How did Nancy's MKT-P evolve as a result of her planning, enacting, and reflecting on four proof-oriented lessons? We operationalize this question by examining:

1. How did Nancy integrate reasoning and proof in her planned and enacted lessons?
2. What did Nancy notice in the video recording of her lessons; how did she reflect on her lessons, and what learning was afforded by this?

Methods

This study is a part of the larger project that designed and studied the capstone course *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020). The course includes four modules, each focusing on one proof theme: (1) direct proof and argument evaluation (DP); (2) conditional statements (CS), (3) quantification and the role of examples in proving (RE), and (4) indirect reasoning (IR). Each module includes activities to help PSTs *crystalize* their subject matter knowledge of a particular proof theme, *connect* that knowledge to students' proof-related (mis)conceptions and to secondary curriculum, and *apply* that knowledge through a structured, school-based teaching experience. In this experience, the PSTs *plan* a 50-minute lesson on a particular proof theme, *enact* the lesson with a group of students from local schools, record the lesson using 360° cameras that capture both students and the PST, and *reflect* on the lesson. This process repeats four times during a semester, once for each proof theme. In Fall 2020, due to the pandemic, the PSTs taught their lessons online, via *Zoom*. Otherwise, the structure of the course remained the same.

The Case of Nancy

Nancy was a senior mathematics education major in a high school certification track. Prior to taking the capstone course, Nancy completed a prerequisite course on Mathematical Proof, a proof-writing-intensive Geometry course, and one Mathematics Education course. Nancy was a straight-A student in both mathematical and educational coursework. She regularly tutored undergraduate students taking entry-level calculus courses at the university tutoring center but had no classroom teaching experience. Like other PSTs in the course, Nancy completed the pre- and post- MKT-P questionnaire and Dispositions towards Proof survey (Buchbinder & McCrone, 2021). Her high scores on the pre- instruments indicated strong mathematical knowledge and positive dispositions towards proof. In addition, Nancy was very articulate and active during the class discussions, making her a strong case to study (Seawright & Gerring, 2008).

Nancy was placed in Ms. Meyer's high school geometry classroom. Due to the pandemic, a yearly geometry course was condensed to one semester, and Ms. Mayer relied on Nancy to plan lessons closely aligned with her curriculum. Ms. Meyer taught the whole class via *Zoom* for about 30 minutes and then divided students into two groups: one group remained with her, while another group (6 – 7 students) learned with Nancy. Ms. Meyer determined the geometric topic of the lesson, while the capstone course schedule dictated the proof theme.

Data Sources and Analytic Techniques

Data sources include four of Nancy's lesson plans, four Zoom video recordings of the lessons, and four reflection reports. To analyze planned and enacted lessons, we used analytic techniques developed in our prior research (Buchbinder & McCrone, 2020), outlined here briefly.

We analyzed the *lesson plans* by first noting the percent of time planned for reasoning and proof integration. We also ranked each lesson plan on a three-point scale (high, medium, or low) on three dimensions: (1) the extent to which the plan focused on the intended proof theme, (2) the alignment between the objectives and the tasks, (3) how appropriate was the choice of technology for proof integration.

The videos of the *enacted lessons* were analyzed using the *Lesson Enactment Rubric* (Buchbinder & McCrone, 2020), aligned with the MKT-P framework. The rubric has three dimensions: quality of proof-specific language (KLAP), making the proof theme explicit to students (KCS-P), and actions for promoting student engagement with proof (KCT-P). Each lesson was ranked on a three-point scale (high, medium, low) on each of these three dimensions.

When completing the *reflection reports,* the PSTs watched the video on *Canvas* Learning Management System and used the commenting feature to write reflective comments for every 5-minutes, about 8-9 total comments per lesson. Using open coding (Strauss & Corbin, 1994) in conjunction with the noticing literature (e.g., Stockero, 2021; van Es, 2011), we identified four main categories of noticing in Nancy's comments. These are (1) *instructional decisions*, e.g., 'One teaching move that I liked during this lesson was creating a theme for the lesson."; (2) *student engagement*, e.g., "I was impressed that one of the students was able to see that"; (3) *technology*, e.g., "I think the transition from the *Prezi* presentation to *GeoGebr*a was pretty smooth," and (4) *time*, e.g., "I felt pretty crunched for time." We also coded whether Nancy reflected on-, for-, or back- on her teaching (see examples in the Results section).

Results

Abbreviated Summaries of Nancy's Proof-Oriented Lesson Plans

Lesson 1. *Direct proof and argument evaluation: Supplementary and Vertical angles.* Initial group discussion: "what makes a good two-column proof"? Expected responses: generality, mathematical correctness, but can follow different paths. Teacher-led exploration of Vertical angles theorem using *GeoGebra*; first with specific angle values, followed by a generalization. Students contribute ideas as the teacher writes the proof.

Lesson 2: *Conditional statements: Isosceles and Equilateral Triangles***.** Introduction of a conditional statement and its converse and using mathematical examples (*Prezi*). Student-led exploration in *GeoGebra* of three conditional statements about triangles. For each statement, students first determine if it is true or false (if false, construct a counterexample), then write a converse and determine whether the converse is true or false.

Lesson 3: *Quantification and the role of examples: Triangle Similarity Theorems.* Introduction of universal and existential statements; the role of examples in proving/disproving universal statements. A reminder of the three similarity theorems. Students work on one SSS similarity proof by typing their work on an individual slide in a shared *Google Slides* document. Next, students find counterexamples to three universal statements (e.g., All isosceles triangles are similar) and find confirming examples proving two existential statements (e.g., There exist two right triangles that are similar).

Lesson 4: *Indirect reasoning: Coordinate proofs.* The lesson is structured as a game, "The Quadrilateral Detective," where students use distance and slope formulas to determine what type of quadrilateral is given by a set of four coordinates. Next, students create two statements of the form "This quadrilateral cannot be ___, because otherwise ___," e.g., "The quadrilateral cannot be <u>a kite</u> because otherwise <u>it would have no pairs of parallel sides.</u>" After one teacher-led example, students work individually; write their proofs on paper, take a picture and paste it into a shared *Google Slides* document. Indirect reasoning is defined during the lesson summary as "a type of reasoning that shows that something is impossible since it leads to a contradiction."

As the description above shows, Nancy succeeded in integrating each of the four proof themes with the mathematical topics requested by Ms. Meyer in creative and engaging ways. Each lesson had 3-4 objectives, all focused on reasoning and proof, e.g., "Students will come up with counterexamples to disprove mathematical statements," and additional objectives related to student engagement in mathematical discussions. The tasks were closely aligned with the proof theme and with the mathematical content of the lesson. The lesson plans were written in a high level of detail. The percent of time planned for reasoning and proof in each lesson was above 67%, while the rest of the time was devoted to icebreakers in the beginning of the lesson and exit tickets at the end. Nancy's plans used a variety of technological tools: *GeoGebra, Prezi, Google Slides* to facilitate active student engagement.

Nancy's Enactment of the Four Proof-Oriented Lessons

Nancy's *enacted lessons* closely matched the planned ones in terms of content, but not the time. Nancy's lessons were planned for 30 minutes, but since the students had no other class afterward, they stayed between 34 to 56 minutes, allowing Nancy to finish all the planned activities. Table 1 summarizes the analysis of the enacted lessons. Throughout the lessons, Nancy used precise mathematical language and proof-specific vocabulary to discuss proof themes with the students. She used appropriate visual, symbolic, and verbal methods to make the main concepts and key ideas of the proof themes explicit to students. The percent of lesson time devoted to reasoning and proof was very high (above 80%), although the time devoted to the proof themes differed between the four lessons. The two proof themes: Role of Examples and Indirect Reasoning received much less class time and were less of the focal point of the lesson, which is consistent with the lesson plans. Nevertheless, Nancy still explicitly addressed these proof themes in her lessons. The first two enacted lessons - Direct Proof and Conditional Statements - were highly successful on almost all dimensions of the *Lesson Enactment Rubric*.

Table 1. Analysis of Nancy's Four Proof-oriented Enacted Lessons

Dimension on the *Lesson Enactment Rubric*	Lesson			
	1DP	2CS	3RE	4IR
Quality of proof-specific language (KLAP)	High	High	High	Medium
Explicating specific proof-theme (KCS-P)	High	High	Medium	Medium
Actions to promote student engagement (KCT-P)	Medium	High	High	High
Percent of time devoted to proof (KCT-P / KCS-P)	83%	84%	91%	82%
Percent of time devoted to the proof theme	83%	84%	30%	25%

Nancy's Reflective Noticing on the Enacted Lessons

Figure 1 shows the distribution of Nancy's categories on noticing in percent of the total number of comments, which varied by lesson. In the first lesson, Nancy mainly reflected on how her instructional decisions affected student engagement. In lessons two and three, the focus of reflection shifted away from instructional decisions towards technology and time management. This was when Nancy started having students themselves interact with technology (*GeoGebra*, *Google Slides*) providing space for productive struggle and free exploration. By doing so, Nancy had less control over the time spent on each planned activity and thus encountered challenges in coordinating technology, content, and time. The Indirect Reasoning lesson was most challenging for Nancy to enact, as evident in the percent of reflective comments focused on her instruction.

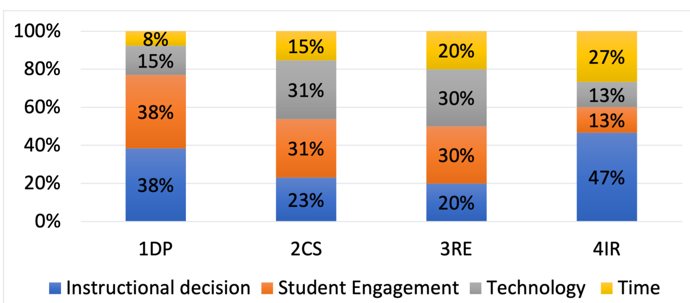

Figure 1. Distribution of Nancy's categories of noticing across four lessons

Table 2 shows the distribution of Nancy's reflective comments along with the four categories of noticing and the three types of reflection: *on, for* and *back*.

Table 2. Distribution of Nancy's categories of noticing and types of reflection

Categories of Noticing	Reflection on	Reflection for	Reflection back	Total
Instructional decisions	11	5	1	17
Student Engagement	6	6	2	14
Technology	6	4	1	11
Time	5	2	2	9
Total	28	17	6	51

Table 2 shows that the total number of Nancy's reflections-*on* (only) was much larger than the number of reflections-*for*, which, in turn, was larger than the total number of reflections-*back*. Most of Nancy's reflections-*on* focused on her instructional decisions. When reflecting-*for,* Nancy focused almost exclusively on instruction, students, and technology; when reflecting-*back* Nancy attended to all four categories.

The following excerpt illustrates the three types of reflection in Nancy's comments. In lesson #1, Nancy attempted to develop a set of criteria for a "good" proof with the students.

Nancy: What do you need [in order] to make our proof a good proof?

S1: Uhm, you need a proof.

Nancy: Yeah, so, we need a proof, right? So we need to be able to show that we can start with our given and then come up with our solution, right? So what are the parts are there in the proof? So we have two parts, right? So what are they to make up the two parts or the two columns?

When reflecting-*on* this exchange while watching the video, Nancy wrote: "one student said that you need a proof, when asked what makes up a "good" proof. This wasn't exactly what I was looking for. It was too general, but I tried to guide her response and make it more specific". Nancy followed by a comment reflecting-*for* the future: "What I should have done was ask the student directly and say something like "I like that idea, S1, what do you mean by that?" Then it would give her a chance to elaborate." In lesson #2, Nancy had a chance to act on her intentions and reflect-*back* on her improvement. She wrote: "I asked them what bisecting means. This is important because I wanted to make sure that they understood what the conditional statement was saying. […] I liked this part is because I asked this question as a follow up to a student's answer and that was something that I had mentioned wanting to work on after last time".

This progression shows how the three types of Nancy's reflections played out to support her gradual improvement of responding to student inputs and leading discussions about proof. Figure 2 shows the distribution of different types of Nancy's reflections in each lesson.

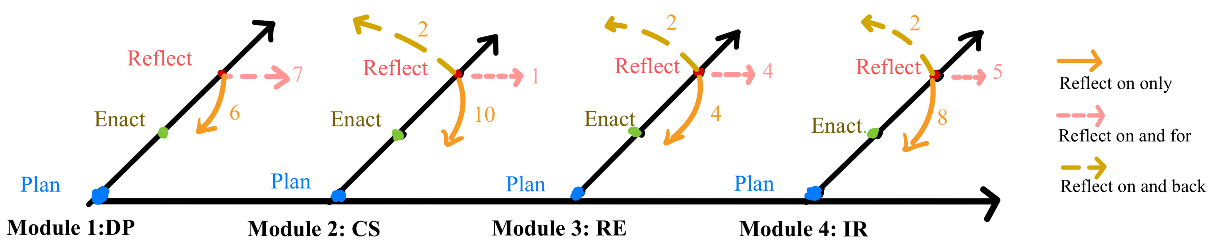

Figure 2. *Distribution of Nancy's types of reflection across four lessons*

Not surprisingly, the majority of Nancy's reflective comments were on the particular lesson she taught. Nancy learned from the teaching experience by making decisions on how to improve her teaching (reflecting-*for*) and kept herself accountable for these changes by reflecting-*back*. These processes supported Nancy's learning through reflecting on her own teaching practices.

Discussion

This paper examined the case of Nancy – a PSTs with a strong mathematical background and positive dispositions towards proof – as she progressed through a capstone course *Mathematical Reasoning and Proof for Secondary Teachers*. We attempted to trace how Nancy's MKT-P classroom practices evolved throughout the teaching experience component of the course. In particular, we focused on Nancy's ability to plan and enact proof-oriented lessons. Our analysis revealed that Nancy integrated the four proof themes with the topics from a regular geometry curriculum using creative and engaging activities, as evidenced in the description of her lesson plans. This is a non-trivial accomplishment, especially since the choices of the mathematical topics and the proof themes were outside of Nancy's control, and due to the shift online.

The analysis above shows that the Role of Examples and Indirect Reasoning proof-themes were most challenging for Nancy to integrate into the lesson plans and to enact (Table 1). This outcome concurs with the results of the previous study (Buchbinder & McCrone, 2020). The fact that the Indirect Reasoning lesson occurs towards the end of the course is probably a contributing factor, especially when the natural end-of-semester fatigue is exacerbated by the pandemic.

Not surprisingly, Nancy's enacted lessons differed from the planned ones (Stein, Remillard & Smith, 2007). When judging the enacted lessons, it is important to note that Nancy's educational coursework did not prepare her to teach online – she had to come up with teaching strategies and technological tools that were new to her and to the students. This willingness to take pedagogical risks makes Nancy's teaching performance even more impressive.

While reflecting on the video of her lessons, Nancy noticed four main aspects: instructional moves, student engagement, timing, and technology. The first three categories of noticing are consistent with those of Sherin and van Es (2005) and Stockero (2021). Nancy's focus on technology is understandable due to the unusual and unfamiliar circumstances of online teaching. As Nancy became more comfortable with the students and her own teaching, she tried new teaching approaches, e.g., having students explore conjectures in *GeoGebra* and naturally encountered new challenges, causing her noticing to shift towards technology and time.

Further examination revealed that Nancy used three types of reflections: *on* a particular lesson, *for* future practice, and *back* to past lessons. Collectively, these three types of reflection are characteristic of productive reflection (Jay & Johnson, 2002; Moore-Russo & Wilsey, 2014). Indeed, Nancy went through repeated cycles of identifying areas for improvement (reflect-*on*), devising a course of action (reflect-*for*), and checking her progress with respect to previous lessons (reflecting-*back*) (Figure 2). Developing purposeful and explicit reflective practices allowed Nancy to leverage her challenges into learning opportunities (Tekkumru-Kisa et al., 2020) and helped her to learn how to learn from teaching (Hiebert, Morris, & Glass, 2003).

Interpreting Nancy's teaching performance in terms of the MKT-P framework suggests that her areas of strength were the quality of proof-oriented language (KLAP) and actions for promoting student engagement (KCT) (Table 1). The areas of improvement, which Nancy reflected *–on*, *for,* and *back* were facilitating discussions (KCS) and using productive teaching moves (KCT), as evidenced in the data excerpt.

Nancy's case serves as a "proof of existence" that it is possible to support PSTs learning teach reasoning and proof through a structured educational experience of the kind provided by the capstone course described above. Nancy's challenges are not unique and sometimes are even more pronounced with less advanced PSTs (Buchbinder and McCrone, 2020). What we found unique and enlightening is how Nancy addressed those challenges and how her reflective practices seem to support her professional growth. Teacher educators can model and promote the use of such reflective practices to support PSTs' professional growth in other teacher preparation programs.

Acknowledgments

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Slope and the Differential Calculus of Two-Variable Functions

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We discuss results of a second research cycle on student understanding of the differential calculus of functions of two variables, focusing on the role that slope plays in a meaningful understanding of plane, tangent plane, total differential, and directional derivative. Results compare the performance of students in a section that used research-based activities and a corresponding pedagogical strategy, to that of students in a regular section. We show how students using the research-based activities were able to construct slope as an object they could build upon in order to understand other important notions of the differential calculus, while students in the regular section showed the same understandings of slope in the differential calculus as reported in previous studies.

Keywords: functions of two-variables, APOS, slope, differential calculus, multivariable calculus

Multivariable calculus is important to model different phenomena in science, mathematics, engineering, and other fields. However, its teaching and learning has not been as fully explored as that of one-variable functions (Martínez-Planell & Trigueros, 2021). In particular, there are relatively few publications dealing with the didactics of its differential calculus (e.g., Weber, 2015; Bajracharya et al., 2019; Moreno-Arotzena et al., 2020). Martínez-Planell, Trigueros, and McGee (2015, 2017) and Trigueros, Martínez-Planell, and McGee (2018) presented the results of a first cycle of research on student understanding of the differential calculus of two-variable functions. In these studies, they used an idea of Tall (1992) to emphasize the local linearity of differentiable two-variable functions. Martínez-Planell et al. (2015) started with a model of mental constructions (genetic decomposition) that builds upon the notion of slope in 3D, to describe how students may construct the idea of vertical change on a plane (see Figure 1). Then, this idea was used in the model to describe how students might construct the point-slopes equation of a plane, tangent plane, total differential, and directional derivatives. The model was tested with student interviews and results of the study led to refining the genetic decomposition and designing pedagogical activities to help student do the revised proposed constructions. A second research cycle was then undertaken to test the refined genetic decomposition. Here we report some of its results. Our research question is: what is the effect of a research-based activity set on student understanding of slope and the differential calculus of two-variable functions?

Theoretical Framework

We use Action-Process-Object-Schema (APOS) theory (Arnon et al., 2014). In APOS an Action is a transformation of a mathematical object that the individual perceives as external. This perception results from its relative isolation from other mathematical knowledge of the individual, who, as a consequence, will not be able to justify the Action. An Action could be the rigid application of a procedure, which may have been memorized. When an Action is repeated and the individual reflects on the Action, it might be interiorized into a Process. A Process is perceived as internal and this allows the individual to omit steps, anticipate results, and thus generate dynamical imagery of the Process, without having to explicitly perform it. The individual will also be able to justify the Process. When the individual is able to think of the

Process as a whole in itself, the Process is encapsulated into an Object and the individual is able to do Actions on it. A Schema is a coherent collection of Actions, Processes, Objects, and other previously constructed Schemas dealing with a specific mathematical notion.

The progression from Action, to Process, and then to Object conceptions, may appear as a dialectical progression where students may appear to go back and forth between stages, as they assimilate new problem situations to their existing Schemas, or accommodate the Schemas to deal with new problem situations. Hence, in order to classify an individual's conception of a mathematical notion, one needs to consider the individual's overall tendency on different problem situations involving the concept. A model proposed in terms of the structures and mechanisms of APOS of how a student may construct a particular mathematical notion is called a genetic decomposition (GD). A didactic strategy frequently used in APOS studies and the implementation of APOS-based activities is the ACE cycle (work in small groups of students, class discussion, and exercises for home). This strategy foments reflection, necessary for the construction of Processes and Objects.

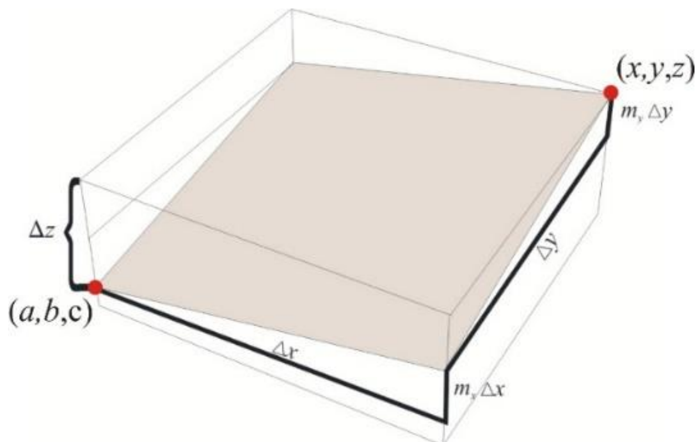

Figure 1. Vertical change on a plane $\Delta z = m_x \Delta x + m_y \Delta y$.

Methodology

In this study we used research-based activities resulting from a previous research cycle (Martínez-Planell et al., 2015) together with the ACE didactical strategy in a section (APOS section) taught by one of the researchers, while another section (regular section) taught by an experienced professor, not one of the researchers, used mainly lectures with the usual textbook exercises. Eleven students from each section volunteered to participate in post-semester semi-structured interviews, designed to test the GD. All students used the same textbook and had the same instructor (the professor of the regular section) in their previous calculus course. Participating students from both sections were chosen so that their grades in the previous calculus course were comparable. Each student interview had two parts, approximately of one hour each, conducted in different days. Interviews were transcribed, translated to English, individually analyzed by the researchers, discussed as a group, and differences were negotiated. The interviews were analyzed in terms of the constructions proposed in the GD. They were also graded as an aid in the search for patterns.

Results on Slopes in 3D

Students in the APOS section showed they had constructed a more robust understanding of slope in three-dimensional space than students in the regular section. Indeed, eight of the 11 students in the APOS section gave evidence of having constructed an Object conception of slope while only one of the 11 students in the regular section gave such evidence. Student A7 showed to have an Object conception of slope and exemplifies the typical response pattern of students in the APOS section as it regards slope. He was an average student in the sense that he had the 6th highest score in the interview (of 11 students) in the APOS section and the seventh highest grade in the previous calculus course among the 11 students in the same section. A7 could generalize a geometric conceptualization of slope (Moore-Russo, Conner, & Rugg, 2011) to 3D to find the slope of the line in bold in Figure 2:

> The slope of this line is m_x. On the line in bold the vertical change is 3, from 2 to 5, and the horizontal change is 1, from 1 to 2, so its slope is $m_x = \Delta z/\Delta x$ … which is 3.

Similarly, A7 was able to compute the slope in the *y* direction of the plane given in Figure2.

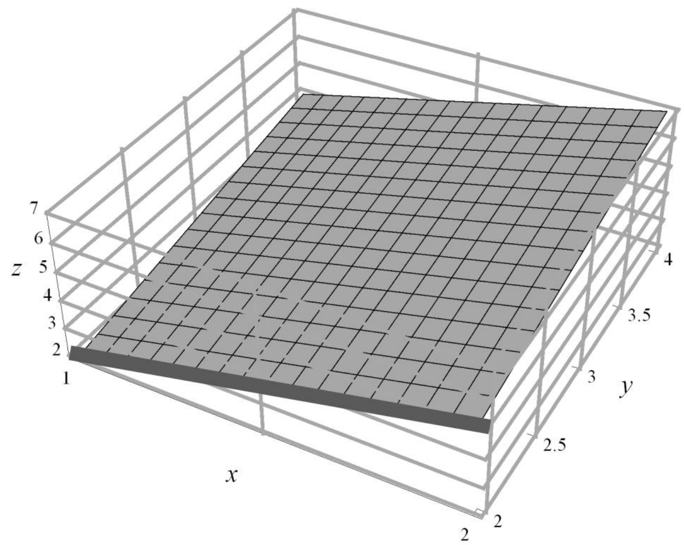

Figure 2. Plane for finding slopes in the x and y directions and vertical change for $\Delta x = 4$ and $\Delta y = 5$.

A7 also used the notion of slope to deal with problems related to vertical change on a plane, the equation of a plane, partial derivatives, directional derivatives, tangent plane, and total differential. This shows he had established significant connections between his notion of slope and other studied mathematical objects. When A7 was asked to find how much the *z* coordinate changes if, starting at any point on the plane, $\Delta x = 4$ and $\Delta y = 5$ [see Figure 2] he replied:

> We know m_x is 3, and $3 = \Delta z/\Delta x$, since Δ*x* is 4 so we have Δ*z* as 3×4 which is 12. We know also m_y is 1, and $1 = \Delta z/\Delta$y so this vertical change which is in the *y* direction is umm $\Delta z = 1 \times 5$ which is 5. So, the *z* coordinates increase as 12 plus 5 which is 17.

So, his construction of vertical change on a plane was based on his doing Actions on slope. In the case of partial derivative, A7 used slope to interpret geometrically partial derivatives and find their sign from the graph of the function. He also used it to approximate a partial derivative given a tabular representation or a contour diagram of the function. By being able to consider slope as independent of representation, this student gave evidence of his construction of a Process conception of slope. For example, when asked to find the sign of $\partial f/\partial y(3.5{,}0)$ given the graph of the function in Figure 3, he said:

In $\partial f/\partial y(3.5,0)$ we know that the *x* coordinate is fixed at 3.5, so I have the point here, now I draw the tangent line to the surface in the *y* direction, we can see this line is an increasing line so its slope is positive and it means the sign of $\partial f/\partial y(3.5,0)$ is positive.

When dealing with directional derivatives, A7 could decide the sign of $D_{\langle -2,1\rangle}f(4,0)$ with the function given by the graph in Figure 3, as he did for partial derivatives.

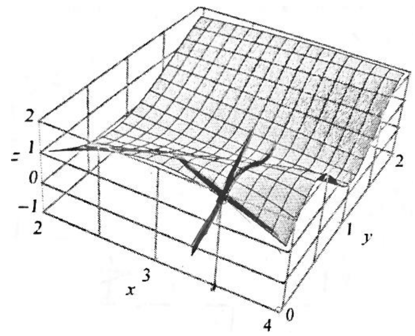

Figure 3. Surface for sign of partial derivative and A7's work

In another problem, he was given the graph in Figure 4, and told to suppose that point *P* moves towards point *Q* at a constant speed along the curve that joints them, then he was asked how may the graph of $D_{\langle 1,-1\rangle}f(P)$ look as a function of time. His response shows that he could generate the dynamical imagery necessary to answer the question. His responses demonstrated the construction of a Process conception of directional derivative and his possibility to consider the slope Process as a whole, which is consistent with an Object conception of slope.

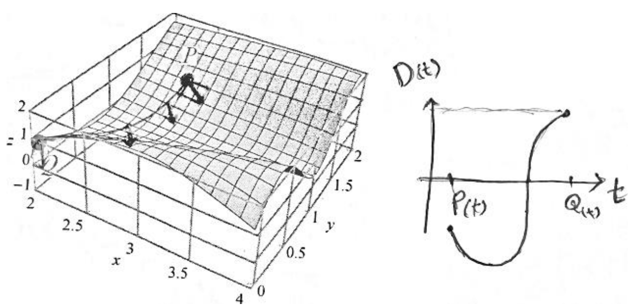

Figure 4. Directional derivative as P moves towards Q

Student A7: I have to draw the graph of the directional derivative respect to time. I first check the points umm they move from P to Q. So, I have my points umm now I check the direction vector for the directional derivative umm it's the vector $< 1, -1 >$ so I check the tangent slopes at different point in this fixed direction.

Interviewer: Okay, draw D.

Student A7: First the slopes are negative umm so the graph of D is negative at this interval then slopes are negative and more negative so the graph of D is decreasing until may be here. Then the slopes are going and going to 0 and somewhere between P and Q the tangent line in the given direction has slope 0 so the graph of D goes to be increasing and cuts the time axis.

Interviewer: Okay, continue to the point Q and draw your graph completely.

Student A7: After 0 umm the slopes are positive because the tangent lines in the given direction are increasing so the graph of D at this interval is increasing.

So, we may say that A7 demonstrated the encapsulation of slope when he acted on it in order to construct directional derivative. He also used slopes as a tool to relate vertical change on a plane, tangent plane, and total differential:

Student A7: [Given the graph of the tangent plane in Figure 5, he was asked about what he could say of the change in the value of the function, if *x* increases 0.02 units and *y* decreases 0.02 units] I first have to find m_x and m_y umm m_x is $\Delta z/\Delta x$ using this line in the x direction it's $\frac{1-0}{2-1}$ which is 1. To find m_y I use this line umm on this line *z* is from 0 to 3 and *y* is from 2 to 4 so m_y is $\frac{3-0}{3-2}$ which is 3

Interviewer: Ok, continue

Student A7: Here we have $d_x = 0.02$ and $d_y = -0.02$. We know also the formula $d_z = m_x d_x + m_y d_y$ which gives us the change in the z coordinates or the change in $f(x,y)$. [Note that he talks about "change in *z*", that is, vertical change on a plane, relating it to notation he next uses for the total differential] By plugging the changes in this formula, we have $d_z = 1(0.02) + 3(-0.02)$ so d_z is 0.02-0.06 which is -0.04

Student A7: [In the next problem, when asked for the total differential] I know the formula for the total differential is $d_f = m_x d_x + m_y d_y$ so it's $d_f = 1(d_x) + 3(d_y)$ and this is equal to $d_f = d_x + 3d_y$.

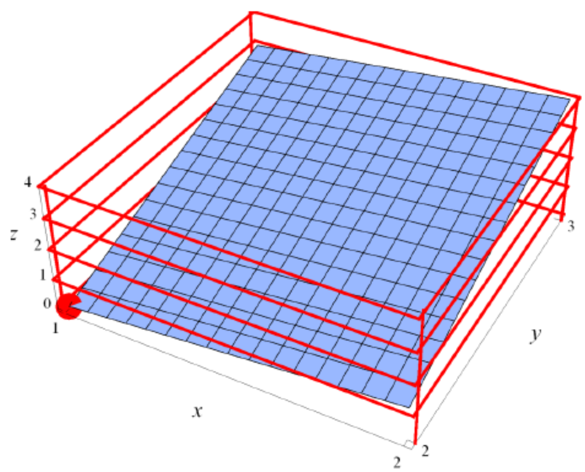

Figure 5. Tangent plane to the graph of a differentiable function $z = f(x,y)$ at a point $(1,2,0)$.

When examining A7's overall tendency when working in different problem situations involving the notion of slope, we noted that he referred to slope when using or discussing partial derivative, directional derivative, the equation of a plane, tangent plane, and total differential. He also seemed to think of slope in different representations. Further, he was able to use slope in his sense making and problem solving. As we found evidence showing that he was performing Actions on slope as an Object, we concluded that he had constructed an Object conception of slope which he used to construct his understanding of other concepts involved in the differential calculus of functions of two variables.

It is interesting to compare, in general, the behavior of students in the APOS and regular sections as it concerns slope. Five of the 11 students in the regular section showed not to have constructed the notion of slope in 3D, while all 11 students in the APOS section showed they had. The average score on four interview problems directly dealing with slope, was 95% for students in the APOS section, while for students in the regular section it was 53%. Given that the students in both sections had obtained comparable grades in their previous calculus course, where both groups had the same professor, one may consider that the emphasis on geometric interpretation given to the development of the two-variable function differential calculus in the APOS section helped those students to develop a more robust understanding of slope. In order to have a wider perspective, it was considered as worthwhile to compare the number of times that students acted on slope (i.e., used slope and explicitly referred to it either verbally or symbolically) when solving the interview problems. Results are shown in Table 1.

Table 1: Number of actions on slope per student and section

Section/student	1	2	3	4	5	6	7	8	9	10	11	Total
APOS	14	16	15	12	12	10	15	14	5	7	5	125
Regular	9	0	4	7	2	4	4	0	0	0	0	30

It is clear from the table's data that students in the APOS section acted in a fundamentally different way with regards to slope when constructing their understanding of the differential calculus of two-variable functions when compared to students in the regular section. The students in the regular section who did not construct slope in 3D responded similarly as students R5 and R9:

Student R5: [Figure 2] I don't know how to find the slope of a line in 3D. For a line in 3D umm its points are like (x, y, z) and have three components. I have never learned how to find the slope of a line in 3D.

Student R9: [Figure 2] The formula for the slope is $\frac{\Delta y}{\Delta x}$ and it's equal to $\frac{y_2-y_1}{x_2-x_1}$. I need two points to find the slope, a point here like $A(1,2)$ and umm the other one here $B(2,2)$. So, the slope will be $m = \frac{2-2}{2-1}$ which is 0.

Note that R5 had not reconstructed slope in the new three-dimensional context of two-variable functions. As shown by Moore-Russo et al. (2011) and McGee et al. (2015), this needs to be explicitly considered in class. Also, note that R9 unsuccessfully tried to directly generalize the formula for slope without doing a needed reconstruction. More interesting is to consider the case of R2, a regular section student who did construct slope in 3D, but seemed not to recur to it while solving problems. She was the second highest scoring student in the regular section in terms of her course score in the previous calculus course, and the third in terms of the interview instrument.

When asked to compute the slope of the line in bold in Figure 2, she initially showed some doubts but was eventually able to generalize slope to 3D:

It's a line in $3D$ umm I don't know how to compute the slope of a line in $3D$ because we have three variables x, y, and z in $3D$. Let me to find two points on the line in bold. We have the points $(1,2,2)$ and $(2,2,5)$ on the line, I have to check how their coordinates change. The y coordinate is fixed at 2 in both points, the x coordinate change from 1 to 2, and the z coordinate change from 0 to 3. So, I ignore y in my computation, and I use the formula $\frac{z_2-z_1}{x_2-x_1}$, by plugging in the formula I have $\frac{5-2}{2-1}$ which is 3 as the slope.

It has been observed that in a 3D context, students may be aware of rates of change in the x and y directions but not know how to combine them to obtain a rate of change in another direction (Yerushalmy, 1997; Weber, 2015). The following excerpts exemplify this observation and also show that R2 has yet to relate slope in 3D to directional derivative:

I am thinking how we can find the slope of a line in $3D$ if all the three coordinates x, y, and z change from first point to the second point. I have no idea for finding the slope of such points because in these cases the formulas $\frac{z_2-z_1}{x_2-x_1}$ and $\frac{z_2-z_1}{y_2-y_1}$ don't work…

[Later, when asked to determine the sign of a directional derivative given the graph of the function in Figure 3] The problem is we don't have the expression of $z = f(x, y)$ and I don't know what directional derivative means on the surface.

When asked for vertical change on a plane given the graph of the plane in Figure2 and $\Delta x = 4$ and $\Delta y = 5$, she did not use a simple argument based on slope like student A7, but rather used the more cumbersome procedure of computing the point-normal equation of the plane and then plugging in values. She did the same thing when given the graph of the tangent plane at a point (Figure 5), she had to approximate the change in the value of the function in a neighborhood of the point. When asked to approximate a partial derivative given a tabular representation of a function, she said "I need to find an algebraic expression of x and y such that these 9 points satisfy it". In problems dealing with the graphical interpretation of second order partial derivatives she said: "The problem here umm is that we are in 3D and actually we have three variables x, y, and z umm and we don't have any equation. The graph of f is very complex and it's impossible to guess an algebraic equation for it". These problems could be done by reflecting on the graphical meaning of slope at given point in a given direction in 3D, as the points move. That is, the problems required to consider the Process of slope as an Object to do Actions on it.

So, R2 showed she was able to solve some problems doing more complex computations than those involved in finding vertical change on a plane, while also showing her relying in algebraic rather than geometric reasoning for important ideas of the differential calculus. According to Duval (2006), it is important to distinguish a mathematical object from its specific representations. R2's dependence on algebraic representations all along the interview showed she still needed to construct geometrical representations to avoid such confusion. It seemed she needed more opportunities to think of slope and to apply it in a way that would enable her to fully understand ideas of the differential calculus of functions of two variables.

Conclusion

The results of this study indicate that students who used APOS-based activity sets with the ACE didactic strategy, developed a meaningful understanding of slope, that contributed to construct different differential calculus concepts for two-variable functions. For the most part, these students related slope to different important notions like, partial and directional derivatives, equation of a plane, tangent plane, and total differential. Students in the regular section showed the same kind of behavior that has been documented in the literature (McGee & Moore-Russo, 2015; Martínez-Planell et al., 2015, 2017; Trigueros et al., 2018; Weber, 2015). That is, for the most part, they showed dependence on algebraic representations in order to do computations, instead of comparing different representations. They also seemed to have constructed isolated notions of the differential calculus, resulting in the use of memorized algebraic calculations.

The study also shows the potential of APOS theory to foster students' deeper understanding of the differential calculus of two-variable functions and their possibility to relate the concepts involved in it. It also shows that through the use of research cycles as practiced in this theory it is possible to progressively improve the design of tasks to be used by students and, in this way, contribute to improving student understanding. Our future research activities include examining the data for students' construction of Processes of partial and directional derivatives, and the differential calculus Schema for functions of two variables.

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Unpacking Neutrality when Responding to Microaggressions in Mathematics Classrooms

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In this contributed report, we document results from the third and final round of interviews with 14 mathematics instructors. In the interviews, we presented two teaching scenarios about a racial microaggression and white-supremacy messaging, and asked what the instructors would do if they encountered such a scenario in their own classroom. Results from our study indicate that some of the participants would do nothing to address the scenario publicly in class. This neutrality resembled similar responses as those from our second round of interviews, but there was a nuance that differentiated the third interview results. That is, of the participants that opted not to address the scenario publicly, they also condemned the scenarios privately as something that was inappropriate, as opposed to avoiding and minimizing the situation. We find that doing nothing in response to a potentially problematic social justice scenario can be interpreted in two distinct ways.

Keywords: race, microaggression, teaching scenarios

With the push for increased diversity and inclusion efforts in higher education (U.S. Department of Education, 2016), university educators should be prepared to engage with and encourage the incorporation of diverse thoughts and experiences that our students bring to the classroom. Diversity not only leads to new ideas coming from different lived experiences, but as technology continues to improve in the hands of the newest generation, a diverse workforce in STEM may curtail the impact of biased systems such as facial recognition software (Garvie & Frankle, 2016; Nkonde, 2019), and machine learning (Garcia, 2016). Advocating for justice in the STEM classroom is the first step to allow students and instructors to recognize their own prejudices and biases. This is particularly important as women, people of color, and other marginalized groups of students are at a higher risk of failing introductory mathematics, being pushed out of STEM altogether (Koch & Drake, 2018; Weston et al., 2019). The idea of failure and marginalization is reinforced for these groups with every offensive comment made, which is why university instructors should be prepared to handle difficult conversations in the classroom and support the students that need it most.

As research has demonstrated, active learning classrooms have the potential to provide superior learning environments compared to non-active ones (Freeman et al., 2014; Theobald, 2020). However, with the use of active learning comes the potential for conflict in the classroom, as students are encouraged to engage with one another's ideas and interact much more compared to lecture classrooms. That is, within any setting of peer-to-peer interaction, problematic discussions around race, politics, gender, and other topics may arise. University instructors are not necessarily trained to handle these situations, which may lead to unproductive conversations and the marginalization of certain groups of students. Such problematic scenarios may present themselves as explicit acts of racism, sexism, homophobia, etc. Although, they may also occur as more subtle acts of hostility and enmity known as microaggressions.

As a research team composed of faculty, graduate and undergraduate students, our stance is that the university classroom should be a space where students learn how to engage with one another around controversial ideas. If university is not a place to challenge students' ways of reasoning and encourage civil discourse, then where is the right place? Research suggests that having these discussions in class can be difficult, but has the potential to foster the development of productive community engagement and tolerance (Camp, 2020; Hess & Gatti, 2010; Moran, 2009). With that in mind, we challenged a group of university instructors by presenting to them a set of potentially problematic teaching scenarios in which they were asked to respond as if they were the instructor. The goal of presenting these scenarios was to answer the following research question: *In what ways can a neutral stance by university mathematics instructors be interpreted when addressing racial microaggressions and supremacist attitudes in the classroom?*

Literature Review

Mathematics instructors at the college-level typically have undergraduate and graduate degrees in mathematics or closely related fields. This extensive background in mathematical content makes them experts in the discipline. However, there is often limited training on a major component of their roles at universities – teaching – where most doctoral students only take one course on undergraduate teaching (Baum & McPherson, 2019). Effective teaching requires knowledge of the content, strategies to teach the content, practices to assist with classroom management, and knowledge of learners and learning (Eggen & Kauchak, 2006). Student learning not only includes specific content but also the development of interpersonal and cross-cultural competencies, expected outcomes of a college degree.

Engagement strategies shown to promote student learning in mathematics classrooms, like the utilization of group work or active learning (Smith et al, 2021), increases the need for college faculty to have competent classroom management skills. Duek (2000) and D.W. Johnson and Johnson's (1989, 1991, 1992, 1994) research on productive cooperative learning environments supports the argument for additional training on classroom management practices. Two of the six elements of productive cooperative learning environments, value of heterogeneity and interpersonal communication, require skills to manage classroom discussion. As students engage in conversation with one another through group work, conversation may move to current events, like racial tensions or political conflict. When these emotionally charged conversations occur between students, faculty should be prepared to respond appropriately to ensure the classroom environment is operating as an inclusive space. Student belonging is an important element of inclusion, and ample research has linked belonging to persistence in higher education (Solorzano et al., 2000; Clark et al., 1999; Mercer et al., 2011; Torres, et al., 2010; Bair & Steele, 2010; Salvatore & Shelton, 2007).

In the context of this study, we are operating under the assumption that race and racism is inherent to society's systems, including the education system, and is difficult to address if not directly acknowledged (Delgado & Stefancic 2017; Bourdieu, 2018). Research suggests that there is a lack of noticing or acknowledgement of bias in the classroom for STEM educators (Boysen et al., 2012). For educators who do recognize the inequity and issues of racism in the classroom, they often struggle to devise a response in practice (Duncan-Andrade, 2009; Solórzano & Delgado-Bernal, 2001). They may not feel qualified to craft a response or confront larger issues in practice, even though they hold a critical stance on the issue at hand.

The prior research within the field of education suggest that faculty must respond to microaggression incidents in order to avoid “siding with the offender” and creating a hostile environment for the affected students, typically students of color, women, and people in the LGBT community (Boysen et al., 2012; Sue et al., 2007; Hernandez et al., 2010). Faculty can ameliorate the situation through a variety of actions, the first being a recognition of their own biases. This idea is often conveyed in critiques of modern liberalism in which neutrality and colorblindness are upheld as critical practices to eradicate issues of race and racism (Delgado & Stefancic, 2017). This neutral perspective generally does not address more covert acts of racism in society and can only address the most egregious and outward forms of racism. This also extends to arguments of the right to free speech, in which hate speech toward minoritized groups may be protected (Delgado & Stefancic, 2017), highlighting how societal structures allow racism and microaggressions to occur. The argument to ensure rights to free speech may even prevent the formation of close communities, which develop through mutual negotiation and dialogue (Wenger, 1999).

Researcher Positionality

As a data collection and analysis team, our personal identities represent diversity in gender, sexuality, race/ethnicity, and academic ranks. We recognize that the data collection team’s positionality likely impacted the rapport, comfort, and safety of participants to share their lived experiences and beliefs. Our personal experiences and positionality were also leveraged to interpret and understand the shared experiences of our participants. While inherently reductionist, we share some of the identities of the authorship team to help contextualize our positionality. Author 1 identifies as a cisgender straight Chicano, Author 2 as a cisgender queer white woman, Author 3 as a cisgender Asian and African American woman, Author 4 as a cisgender white queer man, Author 5 as non-binary queer Chicanx person, and Author 6 as a cisgender white queer man.

Methods

The data from this study draws on a set of interviews with 14 mathematics instructors (see table 1) from nine institutions who agreed to participate in a Professional Learning Community (PLC). The PLC activities started in September 2020, with virtual meetings occurring every two weeks, focusing on diversity and inclusion in undergraduate mathematics. The nine different institutions were all partners in the NSF-funded SEMINAL project which was examining departmental change efforts to infuse active learning in introductory mathematics courses. Participants were interviewed three times at the start, middle, and end of the year-long PLC. The interviews included questions about instructional practices, departmental support, and responses to a set of 4-5 teaching scenarios. The 1-hour long semi-structured interviews were conducted by members of the research team in pairs via Zoom and were audio and video recorded. The interview transcripts were automatically generated by Zoom, reviewed by the research team for clarity, and loaded into the qualitative coding software MAXQDA for analysis. The transcripts and audio recordings were linked within MAXQDA to ensure interpretability, tone, and pauses.

The analysis in this manuscript focuses on three different teaching scenarios. The first scenario, *Zoom Microaggression*, describes a situation in which a student posts a racial microaggression about another student in the chat during class and instructors were asked how they would respond. The Zoom microaggression scenario was presented in the first and third interview. The second scenario, *Classroom Microaggression (BLM)*, describes a situation where a student is wearing a Black Lives Matter hat, and another student reacts saying "All Lives Matter." The third scenario, *Classroom Microaggression (Alt-Right)* is a variation on the Classroom Microaggression (BLM), but in this scenario a student is wearing clothing in their mathematics classroom with the insignia of an Alt-right movement (Proud Boys, Neo-nazis).

Table 1. Participant Gender and Racial Identity as Perceived by the Researchers

Gender and Racial Identity	Pseudonym
white woman	Emma, Shea, Lacy, Kathleen, Crystal, Cassandra
white man	Thomas, Bill, Mark
Woman of Color	Shivya, Camila, Aadaya
Man of Color	Robert, Collin

As a research team, we conducted a thematic analysis of the interviews, which is "a method for identifying, analyzing, and reporting patterns (themes) within data" (Braun & Clarke, 2006, p. 79). We used a cyclical approach to our analysis, analyzing and reporting results from the first interviews (Machen et al., 2021), followed by the second interviews (Ralston et al., 2021), and we present new findings from the third interview in this manuscript. Prior analysis by Machen and colleagues (2021) identified five archetypes emerging from the first round of interviews and included the: Action Taker, Cautionary, Connector, Confidant, and Thinker. Some participants had overlapping archetypes based on their reactions to the microaggression scenario. During the second set of interviews a sixth archetype was identified (Ralston et al., 2021), the Neutral, as many of the faculty's responses indicated hesitation or an interest in keeping their actions neutral due to the concern of impinging on free speech. Analysis of the third set of interviews which included issues of free speech, politics, and covert racism, suggested that the actions (or lack thereof) by faculty were more nuanced than the Neutral category suggested from prior analysis.

Results

We found that the participants' responses were nuanced enough to merit the splitting of the Neutral archetype into two strands, *Apolitical* and *Conflicted*. As we were coding the responses to the third interview, we noticed that the existing Neutral archetype that was generated from the second round of interviews was not entirely capturing some of the participants' responses to the Far-Right scenario. For this reason, we went back to the Neutral codes from the BLM scenario and compared those with the responses that were being coded as Neutral for the third interview. Below we highlight some excerpts from the BLM scenario that are representative of the Apolitical archetype. We then provide excerpts of participants' responses that characterize the new framing of the Conflicted archetype.

The Apolitical archetype that emerged from the responses to the BLM scenario in the second interview generally reflected a sense of not wanting to take a side in the situation. A common motif was the idea that, as one participant put it, "everybody is entitled to their own view or their own opinion." Other participants would have liked more context and clarification on the scenario in order to determine how they would respond, as Collin suggested, "I've had the joke with my friends that we'll mockingly say 'all lives matter' to each other, just to mock the people who have that sort of opinion." Even without the additional context, Collin went on to say that he thinks intervening into the situation and saying something to the students is "more dangerous for me, as an instructor, to make a bigger deal of it than is being made by the students." This theme of causing more harm than good was echoed by other participants characterized with the Apolitical archetype as multiple instructors were apprehensive about taking attention away from the mathematics. Cassandra described the actions they would take to try and redirect the focus back to mathematics:

> *My initial reaction would be like 'hey that's not math content, take it outside' you know, or … 'we can work on math today, and you guys can talk about that later' or … 'this is not the time or the place for this conversation.' You know, and I would try to say something like, 'we all have different opinions about different things, but right now we're working on co-functions, so can we work on co-functions, please.' You know, just try to defuse the situation.*

Generally, the Apolitical archetype characterizes an approach that would prefer to move past the situation, either because the instructor does not see the offending students' actions arising to a level that warrants an intervention or having a discussion in class is not appropriate for a mathematics class. Four of the 14 participants responded to the BLM scenario with some action that was characterized as Apolitical; however, it is important to note that these are not fixed traits that we are assigning to the participants. In some cases, a participant might suggest one approach reflecting an Apolitical preference, but also later take an approach as a Confidant in that they would pull the offending student to the side to have a one-on-one conversation or with the aggressed student about what had occurred in class.

What we noticed with the Far-Right scenario represented a different stance altogether. That is, for the Apolitical archetype, the participants' actions could be categorized as not taking a side or not wanting to intervene because they did not see it as appropriate. In contrast, participant responses to the Far-Right scenario characterized a situation in which the participant would take a stance (in theory), but would not intervene, even though they knew that something should be done. The important distinction that we are making here is the idea that the responses reflecting the Conflicted archetype generally take an approach of *condemnation*, rather than one of *minimization*. Lacy's reaction to the scenario describes the Conflicted archetype well in that they know that this Far-Right scenario is problematic, but is unsure about how to respond given the context of the situation:

> *To do nothing is not acceptable. I think, in years past, to just ignore it and go on was acceptable. I don't think it's acceptable, [the instructor] needs to do something. What [they] need to do, I don't know because I don't know the makeup of the class … The only thing I can say is yes, it needs to be addressed, yes, something should be done. Even if it's after the fact, I think [addressing the scenario] should be done.*

Although Lacy takes a firm stance that something should be done and this scenario is unacceptable, she does not describe what she would do herself because she is unsure about the context. From our perspective, we interpret this as not taking action but still having a condemning stance on the issue, stating that this scenario is unacceptable. Another instructor had a different reason as to why they would not address the scenario in class,

> *The student who is wearing [the far-right clothing] is wearing it because they want to stir the pot, they want to offend people, they want attention right? So, if this was me, am I going to call this person out? No, because that's what the student wants, right? They want to make this space about them, and these ideas. So I don't think I would do something publicly … I don't think people that wear neo-Nazi t-shirts are unaware that that is offensive right? They do it because they want to offend people, they want attention.*

Bill's response was something that we initially interpreted as apolitical given that they did not explicitly condemn the scenario as being problematic. However, we interpret Bill's response as a condemnation in that Bill does not want to give the offending student a voice in the classroom to spread hate and neo-Nazi ideals. We also saw a Conflicted response from Kathleen, as she wanted to do something about this scenario, and offered one potential avenue to stop the student from wearing far-right clothing, but was grappling with her ideals of freedom of speech.

> *This is really tough because I believe in freedom of expression … I do think that these shirts are inappropriate. Because the shirt is specifically promoting hate … The university must have some kind of dress code and if the dress code mentioned something about hate speech or this kind of propaganda or promotion, then, absolutely … If it doesn't, I don't think that [the instructor] really has a right to tell the student what they can and cannot wear. However, if we're doing some kind of group work and somebody doesn't want to work with the student because they find what they're wearing is offensive, or it makes them feel uncomfortable, then I'm not going to force them to stay in a group.*

Three participants reflected a Conflicted response to the Far-Right scenario, with the Conflicted archetype also arising twice with the Zoom Microaggression scenario in the third interview. One participant, Bill, had a Conflicting response to both the Far-Right scenario and the Zoom microaggression scenario. Overall, four participants from the third interview provided responses that we coded as the Conflicted archetype. Interestingly, we did not characterize any of the responses from the third interview as Apolitical. In Table 2 we document the definitions of the two archetypes for posterity.

Table 2. Definitions of the Apolitical and Conflicted archetypes.

Apolitical	The "Apolitical" archetype reflects actions that avoid and dismiss the situation. Respondents that fall into this category often did not want to take a stance on the issue and leave it to the students to handle any potential conflict. Some characterized as Apolitical worried it might trigger an emotional reaction from students, and/or they did not feel capable of facilitating a meaningful discussion about social justice issues. As a result, actions represented by this archetype tend to minimize the situation to avoid bringing discussions about politics or racism into the classroom.
Conflicted	The "Conflicted" archetype reflects actions that avoid, but condemn the situation. Respondents that fall into this category often know that the situation is wrong and something should be done to resolve any potential conflict, but may not know the best approach to do so. We see the actions of those characterized by the Conflicted archetype as a minimum requirement of allyship with the marginalized and oppressed; knowing that an injustice has occurred is the first step to intervening and facilitating a productive conversation about social justice issues.

In the next section we provide a summary of the two new archetypes that emerged from the third interview and our thoughts as to why we saw the emergence of the Conflicted archetype, and absence of the Apolitical, in the third interview.

Discussion

Our research study involved presenting 14 university mathematics instructors with difficult teaching scenarios and asking how they would respond to those scenarios if they were teaching in the classroom. After the third and final interview with the research participants, we noticed a nuance in the instructors' responses to neutrality that we considered to be valuable to unpack. This nuance was particularly interesting in that the action that the instructors would take was, in fact, no action at all. That is, in the second interview, the participants that mentioned that they would not address the scenario in class often wanted to move past the scenario by downplaying the seriousness of the comments made, opting to rely on the idea that everyone is entitled to their own opinions or commenting that the mathematics classroom was not the space to have those types of conversations. This approach was referred to as the Apolitical archetype. In this manuscript, we highlight the emergence of a new archetype, the Conflicted, as a similar, but different approach. With those that fell into the Conflicted category, they also opted not to address the scenario in class, but wanted to make it clear that the scenario was unacceptable or something should be done as a way to condemn the action taken by the offending student. Our purpose with highlighting the difference between the Apolitical and Conflicted archetypes is to consider what it means not to address these difficult conversations in class from a critical perspective.

A particular point of interest based on our results is the absence of the Apolitical archetype in the third interview. We hypothesize that verbal microaggressions are easier to minimize and ignore compared to a physical manifestation or racial aggression such as the far-right messaging on the clothing. Furthermore, it may be easier for mathematics faculty, and particularly white faculty, to notice and condemn white supremacy compared to facilitating more nuanced discussion of race, racial microaggressions, and political movements. Moreover, what are the implications of approaching difficult scenarios as Conflicted compared to Apolitical? We consider this as an open question for future research, but for now we offer a few of our thoughts.

From our perspective, we see the BLM scenario as an easier scenario to downplay and ignore, due to the fact that it was a verbal microaggression and the potentially ambiguous nature of the context in which the microaggression occurred. However, we would like to address the microaggression for what it is, covert racism in the classroom. We see the responses that were categorized as Conflicted to be a productive step in that the instructors that opted for a "neutral" stance still condemned the overt act of white-supremacy, as teaching is inherently a political act (Kozleski & Waitoller, 2010). They knew that *something* should be done, even though they did not know exactly what that *something* was. We see the identification of how one might approach these difficult situations as an important professional development practice that all instructors should consider. Future work will address how all of the archetypes that emerged from the three rounds of interviews might impact our teaching practices, and we will explore what it means to bring conversations of racism, politics, and equity into the mathematics classroom.

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Responding to Students' Suggestions: Attribute Substitution and Heuristic Approaches

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Knowing how best to respond to students' mathematical inquiries is an important skill for all teachers to develop. A class of pre-service teachers (PSTs) was presented with a scripting task in which a student conjectured that 1/6.5 was "exactly in between" fractions 1/6 and 1/7. However, instead of addressing the student's inquiry directly, many of the PST's responses contained a variety of explanations for more general information about fractions and their various representations. We offer a classification of the responses using the ideas of attribute substitution along with the availability and representativeness heuristic.

Keywords: Fractions, Fraction Representation, Attribute Substitution, Representativeness Heuristic, Availability Heuristic

Introduction

An important aspect of classroom discourse is how teachers respond to students' mathematical inquiries. Alternative interpretations of content may naturally arise in the course of any lesson, and it is the responsibility of the teacher to respond to student ideas in a meaningful way. However, it is often the case that, rather than responding directly to a particular question, individuals respond to a different, but related question without realizing that they have done so.

Building on prior work by Marmur, Yan, and Zazkis (2019, 2020a), we presented pre-service teachers (PSTs) with a scripting task in which a student conjectured that 1/6.5 was "exactly in between" fractions 1/6 and 1/7. While many of the PSTs in our sample directly addressed this suggestion, several focused on mathematical information that was tangentially related to the task, such as fundamental ideas about fractions and fraction representations. In this report, we present a classification of PSTs' responses in terms of questions that were answered, rather than what was asked. We frame the responses using the notions of attribute substitution (Kahneman & Frederick, 2002) and the representativeness and availability heuristics described by Kahneman and Tversky (1972) and Tversky and Kahneman (1973).

On Fraction Representations

Students' struggles with fractions at the K-12 level are well-documented (e.g., Behr, Wachsmuth, Post, & Lesh, 1984; Clarke & Roche, 2009; Mack, 1990; Vamvakoussi, Christou, Mertens, & Van Dooren, 2011; Zhou, Peverly, & Xin, 2006). Both in-service and pre-service teachers have been shown to struggle with fractions in many of the same ways that students do, leading to an increase in research on prospective teachers' knowledge of fractions (Ball, 1988; Post, Cramer, Lesh, Harel, & Behr, 1993; Post, Harel, Behr, & Lesh, 1988; Sowder, Bedzuk, & Sowder, 1993; Zhou, et al., 2006). Despite the identification of at least five distinct conceptualizations of fractions, such as fraction as measure and fraction as ratio (Brousseau, Brousseau, & Warfield, 2004; Kieren, 1976, 1993; Lamon, 2007; Sowder, Philipp, Armstrong, & Schappelle, 1998), the part-whole conceptualization remains dominant in both students' and teachers' reasoning about fractions (Ball, 1988; Newton, 2008; Post, et al., 1993; Post, et al., 1988; Sowder, et al., 1993; Weller, Arnon, & Dubinsky, 2009; Zhou, et al., 2006).

Many of the difficulties learners experience with fractions stem from the variety of representations and ways of conceptualizing them (Čadež & Kolar, 2018; Charalambous & Pitta-

Pantazi, 2007; Pantziara & Philippou, 2012). Vamvakoussi and Vosniadou (2004) showed that middle school students may reason about equivalent fractions (such as 1/2 and 2/4) as if they were different numbers instead of different representations of the same number. Marmur, et al. (2019, 2020a) observed a similar result in pre-service teachers who rejected the value 1/6.5 as a candidate for a number between 1/6 and 1/7 but allowed the equivalent fraction 2/13, suggesting that it is not the number itself, but its non-standard representation, that presented a challenge for PSTs. Overall, it was demonstrated that the prospective elementary school teachers in their study experienced difficulty in assigning meaning to 6.5 when it appeared as the numerator or denominator of a fraction.

We extend this work on PSTs' perceptions of non-standard representations of fractions, and in particular, the ways they conceive of and interact with the representation 1/6.5. Our research aimed at addressing the following question:

How do prospective teachers respond to a student's idea that 1/6.5 is exactly in the middle of 1/6 and 1/7? In particular, how does the non-traditional representation shape their responses?

We presented a group of pre-service middle and high school teachers with a hypothetical classroom dialogue in which one student, Cory, asked what number will be "exactly in between" 1/6 and 1/7, and another student, Alex, conjectured that 1/6.5 was the desired value. As the PSTs extended the dialogue to address Alex's suggestion, many of them focused on other aspects of the scenario, responding by re-teaching foundational knowledge about fractions, trying to give meaning to the number 1/6.5, or addressing this representation's status as a fraction or as a number of any kind.

Theoretical Perspective

Attribute Substitution

We adopt the perspective of Kahneman and Frederick's (2002) *attribute substitution* model. Responding to related questions that were never asked is an example of attribute substitution: "We will say that judgment is mediated by a heuristic when an individual assesses a specified *target attribute* of a judgment object by substituting another property of that object -- the *heuristic attribute* -- which comes more readily to mind" (Kahneman & Frederick, 2002, p. 53).

In daily life, people often respond to questions with the answer to a different, but related, question. Consider, for example, the following exchange between a mother and her son:

Mother: Do you need help with your math homework tonight?
Son: It's not due until Friday.

The mother's question asked about the son's need for assistance on his assignment, but the son has instead responded to the related question "When is your homework due?" Despite the fact that this question was never asked, one can clearly infer the implied answer to the question that was asked. In the dialogue we presented above, the target attribute referred to by Kahneman and Frederick (2002) is that which the original question asked about: the son's need for assistance. The heuristic attribute is the property to which the response pertained: when the assignment is due. In this case, it is up to the mother to infer from her son's response that he does not need help, but this may not necessarily be the case. He may require help, but only on

Thursday night. The clarity of attribute substitutions in the response to the original question may depend on the particular situation and the particular relationship between the interlocutors.

This response pattern is consistent with Kahneman and Frederick's attribute substitution model. Attribute substitution refers to the cognitive process in which an individual unwittingly thinks about related attributes of an object or concept instead of the one which is under direct scrutiny.

Attribute Substitution and the Representativeness and Availability Heuristics

Research on attribute substitution has so far focused mainly on the representativeness heuristic (Grether, 1980; Gualtieri & Denison, 2018; Kahneman & Tversky, 1972; Tversky & Kahneman, 1974) and availability heuristic (Agans & Shaffer, 1994; MacLeod & Campbell, 1992; Pachur, Hertwig, & Steinmann, 2012; Tversky & Kahneman, 1973).

Chernoff (2012) described attribute substitution when examining prospective mathematics teachers' responses to tasks related to relative likelihoods (i.e., which outcomes are more likely than other outcomes in a given set). Although respondents were asked to identify which of two given answer keys to a multiple-choice test was probabilistically more likely to occur, several respondents were found to have responded to the related question, "Which of these two answer keys is more *representative* of the collection of possible answer keys?" Thus, while many respondents provided objectively incorrect responses to the question that was asked, it was determined that these individuals were still thinking mathematically – but were in fact responding to a question that was not asked using the representativeness heuristic.

Furthermore, when responding to questions about relative likelihoods, such as "Which string of outcomes of successive coin flips is more likely: HTTH or HHHH?" individuals may appeal to the representativeness heuristic rather than use their knowledge of probability (Chernoff, 2012). A generic outcome of the process of tossing a coin four times is one in which some number of the tosses result in H and the others result in T, thus a sequence of tosses that includes no T may be perceived as less representative of the event space than one resulting in some combination of H and T. Individuals reasoning according to the representativeness heuristic may respond that the sequence HTTH is more likely to occur than HHHH, as it looks more like a generic outcome. Rather than responding to the question about the relative likelihoods of these two events, these individuals provided a response to the related question, "Which of these two sequences is more representative of a typical element of the event space?"

The Study

Participants and Setting

Prospective secondary and middle school teachers (n=34) participated in this study. They were in the last term of their teaching certification program, enrolled in the methods course for teaching secondary and middle school mathematics. The participants held Bachelor's degrees in Mathematics or Science; the latter included a sufficient component of mathematics courses required for teaching certification.

The course provided participants with an opportunity to explore and extend their own mathematical thinking, while focusing on the learning and teaching of mathematics. During the course, as part of exploring instructional materials and didactical approaches, participants prepared for instruction by completing several scripting tasks (see section 4.2.1); their responses to one such task, Fraction in the Middle, are analysed in this report. Prior to completing the Task, the class discussion attended to various representations of fractions and operations with fractions

-- topics that initially appear in elementary school, but are revisited in middle school, as they are known to present difficulties for many learners. The Task was presented to participants as an individual assignment, which they were asked to complete within two weeks.

The Task

The Task presented below belongs to the genre of "Scripting tasks" developed in mathematics education to explore and strengthen teacher knowledge, while considering instructional situations. Scripting tasks used in prior research usually included a *prompt*, which is a beginning of a dialogue between a Teacher-character and Student-characters. In most cases the prompt introduced a student error (e.g., Zazkis, et al., 2013a), a student question (e.g., Marmur & Zazkis, 2018; Zazkis & Kontorovich, 2016), or a disagreement among students (e.g., Marmur, Moutinho, & Zazkis, 2020; Zazkis & Zazkis, 2014), to which the scriptwriters had to respond by continuing the dialog. The scripted dialogue demonstrated teachers' "awareness-in-action" (Mason, 1998, p. 255) by showing the envisioned response in practice, rather than in theory, to student errors, queries, or unusual ideas.

The "Fraction in the middle" Task

The "Fraction in the middle" Task included the prompt in Figure 1.

Teacher:	So we have discussed different ways to find a fraction between two fractions. Let us look at 1/6 and 1/7 . Can someone please summarise what ways were mentioned?
Alex:	[…]
Teacher:	Thanks. Let someone else continue.
Brook:	[…]
Cory:	But what number will be exactly in between?
Teacher:	What do you mean?
Cory:	Like, in the middle of these.
Alex:	It will be 1/6.5
Teacher:	This is a very interesting suggestion. Why do you think so?
Alex:	Clearly, 6.5 is exactly in the middle between 6 and 7, so 1/6.5 should be exactly in the middle between 1/6 and 1/7 .
Teacher:	[…]

Figure 1. The "Fraction in the Middle" Task Prompt

Continuing the dialogue – in particular, addressing a query of Cory (what number will be exactly in between) and an unexpected suggestion of Alex (it will be 1/6.5) as well as justification for it by drawing analogy to whole numbers – was the main part of the Task.

The prompt was developed building on the study by Marmur, et al. (2019), in which prospective elementary school teachers responded to a student conversation about the possibility of finding a number between any two numbers. While a student-character in the prompt developed by Marmur, et al. (2019) suggested that there were no numbers between 1/6 and 1/7, the reference to 1/6.5 appeared in several scripted conversations. The study examined several cases in which student characters grappled with the unconventional representation, either rejecting it as a viable number or seeking a way to make sense of it.

We were interested to learn how 1/6.5 would be interpreted and treated by prospective secondary teachers when it is referenced in the context of seeking the number in the middle of the two "pseudo-successive" fractions. We focused on a pair of pseudo-successive fractions, as such fractions are frequently perceived as consecutive numbers with no numbers in between (e.g., Vamvakoussi & Vosniadou, 2004). As in Marmur, et al (2019), we kept the numbers small enough to simplify the calculations, but chose numbers that would inhibit immediate conversion to decimals, as the use of decimal representation is known to simplify the identification of numbers in between (Van Hoof, Degrande, Ceulemans, Verschaffel, & Van Dooren, 2018).

Data Analysis

Participants' responses to the Task "Fraction in the Middle" comprise the data analyzed in this study. The analysis of the data began with a thorough reading of all the scripts to get a general sense of how the PSTs responded to the task. We then narrowed our focus to the PSTs' responses to Cory and Alex's discussion on what number would be "exactly in between" 1/6 and 1/7 and the ways PSTs addressed the suggestion of 1/6.5.

We initially noticed the wide variety of mathematical approaches and representations that were present in the PSTs' responses. The PSTs' mathematics included but were not limited to averaging algorithms, number lines, decimal representations, descriptions of graphs, and part-whole fraction models. On a second read through of the scripts, we focused on how the PSTs were responding to Cory's inquiry in particular. We noted that the responses provided by the PSTs were typically not mathematically incorrect, but they seemed either irrelevant to or only tangentially related to Cory's question and Alex's suggestion. Although some responses of the participants seemed to us inappropriate for the given task, they did seem appropriate for *a different* task related to fractions. Revisiting the data, we identified 10 questions which were implicitly addressed by the participants in their scripted instructional interactions and explanations. We then classified each of the explanations provided within the scripts as a response to one of these 10 questions. We further grouped these 10 questions under three themes, as some questions addressed related content.

Our unit of analysis was an explanation provided by a PST within a script. An explanation could have been provided by a teacher-character or by a student-character and accepted by a teacher. All of the PSTs provided multiple explanations within their scripts: for example, they might have begun with an explanation of the relative size difference between 1/6 and 1/7, thus addressing the pseudo-ordering of the fractions, then transitioned into a discussion of different algorithms for identifying fractions between a given pair of fractions. Instances like this were categorized as two distinct responses and were analysed separately. Thus, although our sample consists of 34 scripts, the total number of responses we considered in our analysis is 127.

Results

Overview

Again, each script contained multiple explanations, yielding a total of 127 explanations to 10 questions across the three themes as organized in Table 1. The three themes we identified were explanations related to prerequisite knowledge of fractions, explanations focused on ideas around the number 1/6.5, and explanations related to the middle of 1/6 and 1/7.

For this proposal, we briefly address the results of the first theme and then discuss these findings using through the lens of attribute substitution in the discussion.

Table 1. Themes and questions identified in responses.

Themes and related questions	Number of explanations
<u>1. Focus on pre-requisite knowledge related to fractions</u>	
How do you properly order fractions by relative size (i.e. 1/7<1/6)?	9
Why can't you compare fractions with different denominators?	9
How do you find a common denominator?	9
<u>2. Focus on number representations, specifically for 1/6.5</u>	
What is a fraction? Is it a single number of two numbers?	3
Why isn't 1/6.5 a rational number (issue with decimals in fraction)?	8
How do you manipulate a fraction containing a decimal?	8
<u>3. Focus on the middle</u>	
How do you find the middle of two fractions?	24
What is the middle of 1/6 and 1/7?	29
Is 1/6.5 the middle of 1/6 and 1/7?	20
Why isn't 1/6.5 the middle of 1/6 and 1/7?	6

Focus on pre-requisite knowledge related to fractions

Many of the explanations highlighted prerequisite knowledge related to fractions, where the PSTs used their response as an opportunity to re-teach elementary concepts. Focusing primarily on the fractions 1/6 and 1/7 given in the prompt, 27 of the 33 scripts contained explanations to three different questions related to fractions, specifically: "*How do you properly order 1/6 and 1/7?*", "*Why are fractions with different denominators not comparable?*", and "*How do you find the common denominator given two fractions?*"

Of the 27 explanations focused on pre-requisite knowledge, 9 attended to the pseudo-successive ordering of the fractions 1/6 and 1/7, highlighting the fact that 1/7 is smaller than 1/6. Another 9 explanations were provided for why fractions with different denominators cannot be compared. Most of these explanations contained visual representations including pizzas and number lines, often highlighting the inevitable size difference between the fractions and concluding in statements such as, *"1/6 and 1/7 aren't on the same number line, sort of like they are speaking different languages" (Student 19)*. Lastly, 9 explanations provided details for how to find the common denominator between two fractions, a skill we would consider pre-requisite for secondary mathematics.

Focus on Prerequisite Knowledge and the Availability Heuristic

We view the scripts in which the main attention was directed towards providing prerequisite fraction information as a consequence of the availability heuristic. The availability heuristic involves an attribute substitution by which an individual makes decisions about an event by recalling familiar instances of similar events (Tversky & Kahneman, 1973).

The number 1/6.5 presented our prospective teachers with an unfamiliar representation (fractions containing decimals) of familiar mathematical objects (fractions more generally). Rather than responding to Cory and Alex or addressing the value of the number or its unfamiliar representation, some PSTs chose to respond with information about fractions that was more easily retrievable, including familiar facts, representations, and commonly taught models. Several of these responses focused on the fact that 1/7 < 1/6 (despite the fact that 6 < 7) or on constructing a visual representation for fractions in standard form, like cutting a pizza into 6 pieces of equal area to represent 1/6 (part-whole model), taking one pencil away from three to represent 1/3 (set model), or placing fractions on a number line (measure model). These responses explained fundamental concepts and procedures, such as why the magnitudes of fractions with different denominators cannot be compared and the necessity for finding equivalent fractions with a common denominator as well as procedures for doing so.

Indeed, from a pedagogical perspective, these explanations resemble standard lessons on fractions and ways of thinking about them. The prospective teachers in our sample have likely experienced several lessons like these and discussed central ideas related to teaching and learning fractions in their coursework. As such, a familiar topic and strategy for a fraction-related lesson served for some scriptwriters as an available heuristic attribute.

Discussion and Conclusion

The results of our investigation highlight the difficulty of PSTs' experience when asked to respond to students' ideas that describe a particular strategy for finding a midpoint, and to unconventional representation of fractions.

With respect to the former, we argue that pre-service teachers should be given more practice responding to students' suggestions and focusing on their approaches and reasoning rather than on the correctness of the solution. Scriptwriting is a good opportunity to engage in such practice in a hypothetical situation, before teachers engage in "thinking on their feet" in actual instructional interactions.

With respect to the latter, we note the PSTs' uneasiness in dealing with 1/6.5 as a number. While treating this number as "illegal" or "not allowed" by prospective elementary school teachers was noted in prior research (Marmur, et al, 2020a), we were surprised to find a similar uneasiness and similar expressions used by prospective secondary school teachers, who in their mathematics courses had significant exposure to fractional forms involving all kinds of real numbers, such as $\frac{\pi}{2}$, or algebraic expressions in numerators or denominators.

In addition to extending research on unconventional representations, our contribution is in describing the pedagogical choices of PSTs in terms of attribute substitution. The concern that some teachers chose to devote major attention in their scripts to issues that are not central to the mathematics at hand was depicted in prior research as a "pedagogical shield". This included detailed discussions related to prerequisite concepts (Koichu & Zazkis, 2013b), considering examples that that are peripheral to the mathematical core of the task (Marmur & Zazkis, 2018) or limiting the scope of the issues under consideration (Zazkis & Kontorovich, 2016). That is, the "shield" metaphor was used as means designed by teachers to protect themselves from uncertainty in how to deal with the presented mathematical situation. We analysed such pedagogical choices in terms of availability and representativeness heuristics.

Prior research used attribute substitution in describing judgment in cases of uncertainty in probabilistic decisions. We extend the applicability of this construct to cases of uncertainty in making pedagogical decisions. Future research will examine additional heuristic attributes in instructional situations that explicitly or implicitly divert attention from responding to students' mathematical ideas.

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Students' Mathematical Reasoning With and About Representations

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In mathematics, problems are solved and communicated to others using various forms of representation. For this reason, many mathematics courses are designed to develop and foster student fluency with mathematical representations. However, it is equally important to understand how students naturally produce and use representations when solving problems. Using intentional questioning in task-based, semi-structured interviews, I sought to identify emergent themes relating to what representations students construct in solving mathematical tasks, how they use these representations, and the relationship between their conceptions of the tasks and their constructed representations. Two Calculus I students participated, individually, in an interview comprised of two novel tasks. Inductive coding of interview transcriptions and participant representations identified three purposes of representation construction: as direct interpretations, as auxiliary objects, and as constraints. Results suggest that tasks provide students with the opportunity to construct and reflect on their own representations and compare them to other available representations.

Keywords: Calculus, Mathematical Reasoning, Representations

The importance of students' ability to utilize and understand various representations when engaged in mathematical contexts has been heavily emphasized in policy and curriculum-standard documents (e.g., CCSSI, 2010; CUPM, 2015; NCTM, 2000). In all levels of mathematics preparation of students (e.g., PreK-12 to post-secondary), the guidance of these standard documents includes fostering students' ability to reason about, model, prove, and communicate mathematical ideas and concepts. Student construction of verbal or written utterances, models using manipulatives or figures, and formal or informal proof that communicate or convey mathematical ideas or concepts all exemplify some form of representation in mathematics (Goldin, 2014; Goldin & Kaput, 1996; Goldin & Shteingold, 2001).

In this study, I situate students' utilization and interpretation of representations within the context of mathematical reasoning; that is, "purposeful inference about mathematical entities or relationships" (Conner et al., 2014, p. 183; see also Moshman, 2004). Through a constructivist lens, I seek to describe two students', Seris and Samantha, reasoning with and about representations in a mathematical setting.

Background Literature

There exists a plethora of research investigating students' use of representations in mathematical settings. However, previous research has focused on students' use of representation in relation to specific mathematical content including algebra (Clement, 1982; Moon et al., 2013; Stylianou, 2011), arithmetic (Carraher, Carraher, & Schliemann, 1985), calculus (Mamolo & Zazkis, 2012; Stylianou, 2011), conic curves (Moon et al., 2013), and prime numbers (Zazkis & Liljedahl; 2004). Related literature also tends to investigate students' construction and held meanings of *specific* representations including algebraic functions or equations (Carraher, Carraher, & Schliemann, 1985; Clement, 1982; Mamolo & Zazkis, 2012; Stylianou, 2011), contextual situations and word problems (Carraher, Carraher, & Schliemann, 1985; Clement, 1982; Mamolo & Zazkis, 2012; Stylianou, 2011), graphs (Moon et al., 2013; Stylianou, 2011), tables (Stylianou, 2011), "transparent" representations (Zazkis & Liljedahl, 2004), and verbal definitions (Mamolo & Zazkis, 2012; Moon et al., 2013). Thus, modern research has yet to investigate the utilization and held meanings of students' naturally constructed representations in contexts not designed to directly elicit construction of traditional representations.

DiSessa et al. (1991) investigated students' reinvention of graphing that was centered around activity meant to promote students' invention (or reinvention) of the concept of cartesian graphing. In this activity, the goal for the students was to use technology to create a simulation of the motion of an object depicted in a given scenario, and then, using pencil and paper, construct a static motion picture that could be used to express, in as much detail as possible, their simulations of the original scenario. The use of a contextual problem, its origin void of any direct relation to those representations which our students have been conditioned culturally to use, may be the catalyst for opportunities to invent representations by the students in the study.

In relating expert mathematicians' and students' use of representations in problem solving situations, Stylianou (2011) identified emergent themes relating to use of representations in individual and sociocultural settings. Stylianou determined that, in individual problem-solving, participants used representations as means to understand information, as recording tools, as tools facilitating exploration, and as monitoring or evaluating devices. In social settings, participants further used representations as presentation tools, tools to negotiate, and as tools to co-construct meaning. Taken together, Stylianou's (2011) and DiSessa et al.'s (1991) studies provided motivation for the following research questions: (a) what representations do students construct when engaged in novel mathematical tasks, (b) how do students use their constructed representations when engaged in these tasks, and (c) how do the meanings students imbue in their constructed representations relate to their conceptions of the tasks at hand.

Methods

Participation was solicited from College Algebra and Calculus I courses at a mid-sized university in the Rocky Mountain region. These courses were chosen for solicitation based on the perceived diversity of student backgrounds and interests and because instruction in these courses often emphasizes the use of traditional representations in mathematical reasoning. Two students, Seris and Samantha (participant selected pseudonyms), participated in this study. Due to unanticipated restrictions on social interactions with the Covid-19 pandemic, no further participants were added to the study. Both Seris and Samantha were enrolled in a Calculus I course at the time of the interviews. Seris and Samantha each participated in a single one-on-one task-based clinical interview with the researcher. Each interview lasted approximately 75 minutes. Two tasks were utilized in these interviews.

Interview Task Design

Individual interviews were centered around two purposefully constructed tasks; each task was comprised of multiple parts and were unique in the prerequisite knowledge required and the types of representations they were predicted to elicit.

Task 1-Loveland Task. This task was developed from Saldanha and Thompson's (1998) research investigating students' covariational reasoning. The original task was created in GeoGebra and allowed a student to move the point associated with a car along a slider representing a road on which the car traveled. As the student varied the car's position on the slider, a secondary representation would display the relationship between the car's distance relative to reach of two cities positioned on either side of the road.

Modifications to the task by Stevens et al. (2017) added a third city and named each of the cities, situating the car's path of travel within a context relative to the region in which their study was conducted. Further, in their goal of constructing tasks specifically designed to elicit students' covariational reasoning, Stevens et al. removed the function allowing students to manipulate the cars position themselves and the secondary representation depicting the relationship between the car's distances between each of two cities. Students were then asked to construct the graph relating the cars distances relative to each of two cities themselves. The graphs that this non-trivial task yielded were utilized by Martsching (2019) to provide students with multiple representations to discuss relative to the same prompt. These on-screen representations were utilized to investigate students' eye-movements as they reasoned covariationally about each representation.

The Loveland Task was thus created void of any mention of traditional representations to encourage students to construct their own non-traditional representations to convey meaning, and further incorporated a speed element to make utilization of traditional representations more non-trivial. The context for the Loveland Task is provided below:

> Loveland Task: Sam is driving from Ft. Collins to Greeley. Before leaving, she notes that her map shows Loveland is about 13 miles due south of Ft. Collins, and Greeley is about 19.5 miles due east of Loveland. Sam's trip from Ft. Collins to Greeley can be described as follows: Sam begins driving due south from Ft. Collins at a constant speed of 40 miles per hour. After 12 minutes, Sam turns and begins driving due east at a constant speed of 45 miles per hour. After six minutes, she again turns south, driving for four minutes at 75 miles per hour. She then turns east and, after 20 minutes at 45 miles per hour, finally arrives at Greeley.

This task had multiple subtasks which were intended to build in complexity in a sequential manner. The themes of these subtasks were: a) construct a representation of the trip, b) construct a representation of the car's distance from a given city with respect to time, and c) construct a representation relating the car's distance between each of two cities.

Task 2-Compound Interest Task. The second task was inspired by my observations as a teacher and as a researcher of student difficulties in relating contextual, verbal, and figural representations involving percentages. This task was included in this study because of its prevalence in algebra courses at both the K-12 level and the undergraduate level and the perceived association between it and traditional symbolic and figural representations (e.g., $A = P\left(1 + \frac{r}{n}\right)^{nt}$, exponential models, etc.). This association with traditional representations in conjunction with explicit verbal reasoning was anticipated to cause perturbation with students' reasoning and reveal the prominence of their deeper contextual understanding.

> Compound Interest Task: Brian invests $1,000 into a savings account with an APR of 5%, compounded annually (i.e., at the end of each year, 5% of the account's current value is added to the account).

This task had two subtasks: a) discuss aloud how the value of the account changes over time and b) sketch a graph of the value of the account with respect to time. The explicit verbal representation was predicted to elicit focus on meaning of the contextual representation provided and prompt reasoning about discrete changes in the account's value, whereas figural representations where anticipated as allowing students to strip the task of its context and reason about the value of the account in traditional, pre-conditioned ways.

Data Collection

The video and audio data relating to participant verbal utterances and physical gestures were collected via a camera positioned over-the-shoulder and behind the participant. Primary video data relating to participant in-the-moment constructions of representations were collected via screen recording of a touchscreen tablet. Secondary audio data was collected via a portable audio recording device.

Both Seris and Samantha engaged in the Loveland Task first. However, the first subtask (construct a representation that could be used to convey the trip to someone else) yielded rich discussion and took longer than anticipated. Because of time restraints, some subtasks of the Loveland Task had to be omitted from the interviews. Further, based on Seris's reasoning during the Compound Interest Task, the order of the two subtasks was changed for Samantha so that she was asked to construct a graphical representation prior to constructing an explicit verbal representation of the context.

Analysis

Following the interviews with Seris and Samantha, the primary datasets for each participant (i.e., over-the-shoulder video and audio; screen-recording videos) were superimposed to display verbal, gestural, and written representations simultaneously. Audio data were transcribed in their entirety. The superimposed videos were reviewed by the researcher in conjunction with the transcriptions to familiarize the researcher with the data prior to coding.

While deductive coding utilizing the themes described by Stylianou (2010) may have been fruitful, in this study I was interested in students' construction of representations in solving mathematical problems. In particular, in this study a representation or a collection of representations were often the product or solution to a problem, rather than tools by which a problem was solved. Inductive coding was thus deemed a more appropriate tool for analyzing the data in this grounded theory study. The data sets were hence coded based on (a) instances in which each participant constructed or interacted with a representation, (b) the participant's held meaning of the representation of focus, and (c) the participant's use of the representation of focus.

Results

Through analysis of Seris' and Samantha's mathematical reasoning with and about representations in the context of the Loveland Task and the Compound interest Task, I determined that these students used representations in three distinct ways. Both Seris and Samantha constructed *Representations as Direct Interpretations* of the context provided. For the Loveland Task, Seris and Samantha constructed "maps" depicting the orientation of and distance between the three towns in addition to representations of the trips constructed with vector representation to depict directionality, speed, and time spent traveling. Seris and Samantha also demonstrated their construction and use of *Representations as Constraints*. This theme was recurring throughout the Loveland Task in Seris' and Samantha's "maps", Samantha's conclusion that subtasks c and d were impossible to complete, and in Samantha's modification of her interpretation of the context in the Compound Interest Task to match her constructed graph. Finally, both Seris and Samantha constructed *Representations as Auxiliary Objects*; that is, they constructed representations that were to be used in conjunction with one another, but that did not necessarily fulfill their intended purpose alone. This use of representation was evident in Seris's triangular map, his "time series", and in Samantha's use of equations and table in the Compound Interest Task.

Representations as Direct Interpretations

The purpose of each task was to construct a representation that could be used to communicate or depict the information of a given context, hence the representations constructed by participants were the end products of a solution rather than tools used to find a solution. To achieve the goal of communicating such a context, the most appropriate and readily available representation for participants was the context itself. Rather than quantifying the information given and reproducing it using traditional (or normative) mathematical representations (e.g., tables, graphs), participants constructed still images depicting the context itself as they perceived it. These representations served as direct interpretations of a context and sought to capture emergent components through non-traditional means.

Task 1, subtask a) asked students to create a representation that they could use to convey Sam's trip from Ft. Collins to Greeley to someone else. Seris began by constructing a representation depicting the relationship between towns themselves (see the top-left representation of Figure 1a), stating "so I guess with any problem like this, the first thing I would do is draw out kind of the basic elements that I know". Similarly, Samantha also depicted the orientation of the towns and the distances between each within a rectangular "map" (Figure 1c).

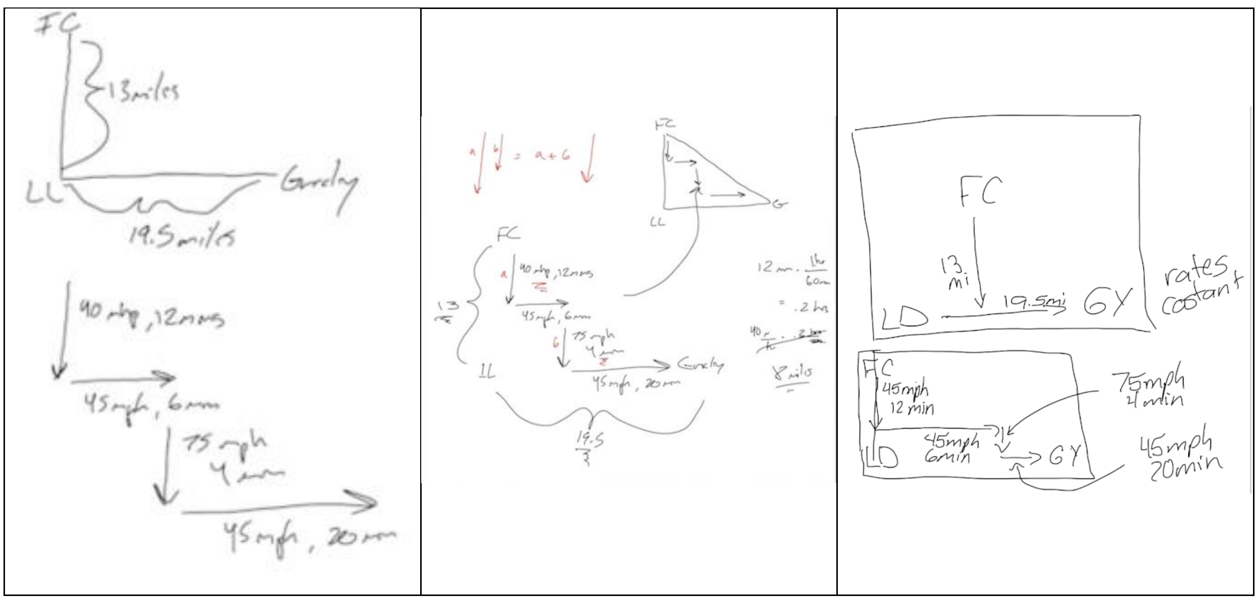

Figure 1. Seris's initially constructed representations (a; left), Seris's final constructed representations (b; center), and Samantha's final constructed representations (c; right) for the first subtask of the Loveland Task.

When depicting the trip itself, both Seris and Samantha utilized vectors to convey the speed and linear directionality of the car's travel. In addition to assigning directionality and units for speed, both Seris and Samantha included with each vector the time, in minutes, that the car spent traveling in a given direction at a given speed (see the bottom representations of Figure 1a and Figure 1c). Further, Seris stated that his original representation (see top left of Figure 1a) was "grounding to what these vectors actually mean".

When engaged in the Compound Interest Task, Seris was asked to first reason verbally about the prompt before constructing any written representations. He quickly determined that the money in the account only changed at the end of each compound period, at which time the account grew by 5% of its current value. Thus, when asked to construct a graphical representation, Seris paid little attention to symbolic manipulations, and immediately constructed a step function depicting the constant value of the account within each compound period. This instance was determined to be exemplary of Seris constructing his graph as a direct interpretation of the context of the task, as he constructed the trace of the function whilst simultaneously describing the emergent change in the accounts value over time.

Representations as Constraints

Whether implicit or explicit, contexts in which a problem is situated typically contain parameters that constrain or limit a solution strategy. Similarly, representations used to convey information can be validated or made more accurate by imposing characteristics that serve to represent these constraints.

Upon remembering that his goal for the Loveland Task was to convey the trip to someone else, Seris himself opted to construct the "map" seen at the top of Figure 1b. By situating the trip inside the depicted triangle that "constrain[s] the problem geometrically", Seris asserted that this depiction including both the vectors and the orientation of the towns depicted the trip in simplest terms to get from one town to another. When calculating how far the car had traveled during each leg of the journey to Greeley, Seris further stated that "you could use that as a way of understanding like, if you are leaving this triangle then I guess either it shows that her pathing is weird […] or it kind of constrains you and tells you like, okay, I did something wrong in this initial kind of calculation". To Seris, if any of his calculations meant that the car would travel outside of the triangle, then either the original context did not correctly describe the car's path, or his calculations were wrong.

In the third and fourth subtasks, asking participants to create a representation showing how the car's distance from Loveland changes over time and to create a representation showing how the car's distance from Ft. Collins and its distance from Loveland relate, respectively, Samantha claimed that the task could not be answered. To Samantha, the phrase *from Loveland* implied that the car had to have been *in* Loveland at some point, and that the distance being described was in fact the car's distance travelled as it travelled *from* Loveland. Like Seris's triangular map, Samantha's interpretation of the task at hand was a representation in and of itself that constrained her ability to solve the problem. To Samantha, the constraints that she'd created, although she constructed no tangible representation of them, implied that the task was an impossible one.

Representations as Auxiliary Objects

When constructing representations used to convey information from a given context, it may be challenging to create a single representation that adequately captures all of the information of interest, particular when the context is a dynamic one. In this case, secondary representations may serve as auxiliary objects to the primary representation of focus; that is, additional representations may be constructed to enhance or organize the information of another.

When Samantha related her two representations in Figure 1c, she stated that her second representation represented that Ft. Collins was directly opposed to being above some point in between Loveland and Greeley, and that her initial representation was "more like a compass". To Samantha, this compass oriented her construction and interpretation of her primary "map".

Seris, in stating that "maps are 2D so you can put them on a coordinate plane" situated the vectors of his trip on the coordinate plane shown in Figure 2a. In realizing that this representation alone did not convey how the car's distance from Ft. Collins changed with respect to time, Seris constructed what he referred to as a "time series" (top-right of Figure 2a). He justified this construction by recognizing that the velocities along each vector were held constant, enabling him to break the overall trip into "discrete coordinates" that, when related to the corresponding point in the time series, could be used to depict time spent travelling, horizontal distance travelled, and vertical distance travelled relative to Ft. Collins. In this way, Seris constructed the compass shown to illustrate that a $-y$ value corresponds to southward travel, whereas a $+x$ value corresponds to eastward travel. In representing how the car's distance from Ft. Collins and its distance from Loveland related during the trip, Seris drew the red and blue vectors shown in Figure 2b. Seris "zoomed in" on two corresponding vectors to illustrate how the distances from each of the two cities related at that point on the trip. Like the "time series", these vectors added a layer of information to Seris's original representation of the trip.

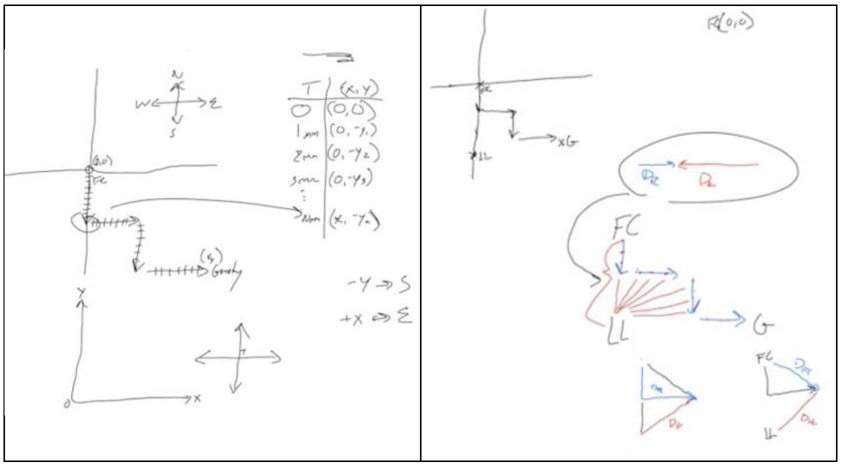

Figure 2. Seris's constructed representations (a; left) for subtask b of the Loveland Task, and Seris's constructed representations (b; right) for subtask d of the Loveland Task.

Discussion

Because each interview capped at 75 minutes, there was difficulty in fully completing both tasks with both participants. The restriction on time meant that only two tasks were able to be used in the study; due to COVID-19, restriction on contact also meant the study was limited to two participants. Future studies should investigate students' construction of non-traditional representations when solving contextual problems to determine if any additional purposes in representation construction arises.

The results of this study echo the discussion brought forth by Gravemeijer et al. (2017), who questioned what mathematical proficiency was needed by members of society. In their review of mathematical workplace tools, the authors described how workers utilized noncanonical graphs and problem-solving practices that differed significantly from what is taught and used in the classroom. Workers are motivated to use nontraditional representations as practical tools to arrive at efficient solutions and, like Seris and Samantha, tailored them to the affordances and constraints of the context from which they were derived. As was evident in the results of DiSessa et al.'s (1991) study, students provided with more time on the tasks in this study would likely have been able to move beyond the use of representations as auxiliary objects and may have the opportunity to reconstruct more meaningful representations. I believe that these results warrant investigation of the design of tasks and practices that both encourage students' construction of practical representations and promote reflection on the accuracy and efficiency with which their representations communicate mathematical ideas.

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Behavioral Nudges Improve Student Outcomes on Mastery-based Assessments in Precalculus

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STEM disciplines and especially mathematics are turning to alternative forms of instruction and assessment in order to improve learner outcomes. Here, we report on assessment outcomes in a mastery-based Precalculus course (N = 464) at a large, public research university. In particular, the impact of behavioral nudges on students' assessment scores is evaluated using regression with academic performance covariates. We find a positive effect of two types of nudges on student assessment scores that span the semester. These results provide evidence that behavioral nudges can help students make choices that ultimately lead to higher scores, and presumably improved learning, in Precalculus.

Keywords: Behavioral nudges, Mastery-based grading, Nudge theory, Precalculus, Regression

Unique among STEM disciplines, introductory math courses in the U.S. have come to hold a hegemonic place in the university curriculum (Reinholz et al., 2019). For most universities and STEM degree programs, students must minimally either pass one or two math courses or demonstrate "proficiency" through a mechanism like a placement exam or earned transfer credit. With math being a foundational discipline across degree programs and at the K-12 level, it is concerning that, historically, learners have had a fraught relationship with the subject (Solomon, 2008). As a discipline, math faces the serious challenges of inducing student anxiety and effecting negative student perceptions (Eden et al., 2013; Sonnert et al., 2020). Some student groups are affected by these challenges more than others; poor learning outcomes and negative changes in affect are felt disproportionately by women, low-income students, and students of color especially in courses prior to calculus (Larnell, 2016; Larnell et al., 2014; Martin, 2019; McGee & Bentley, 2017), and these groups of students are also more often pushed out of STEM degree programs (Seymour & Hunter, 2019). The Common Vision report succinctly stated that "the status quo is unacceptable" (Saxe & Braddy, 2015, p. 1) and called for removing barriers that keep students from progressing, especially for courses taken during the first two years.

In this milieu, mastery-based pedagogy has shown promise at addressing these challenges by centering student progress in relation to their demonstrated understanding of learning objectives (Marzano, 2011; Sadler, 2005; Zimmerman, 2017) and emphasizing the importance of learning through struggle and revision, collaboration over competition, and that all students can succeed when properly supported. Mastery-based pedagogy has been correlated with gains in student achievement in several disciplines (Beatty, 2013; Boesdorfer et al., 2018; Buckmiller et al., 2017; Diegelman-Parente, 2011), including math (Riordan & Noyce, 2001; Schoen et al., 2003; Tarr et al., 2008). Mastery-based pedagogy may provide a productive pathway for course transformations, but much remains to be understood about the process of course redesign for mastery and its impacts on students and instructors, especially in high-enrollment courses with large instructional teams.

Here, we report on the impact of an intervention designed to improve student behavior with respect to an important aspect of mastery-based pedagogy: the opportunity to make multiple attempts on course assessments and improve over time. The intervention is a short series of personalized reminder emails and was conceived of as a behavioral nudge, an economics concept describing subtle and low-cost changes in an environment that help to promote a desirable action. Our research question is: To what extent can behavioral nudges improve student outcomes on mastery-based assessments in Precalculus? We found that engagement with the reminder emails increased students' scores on 40% of assessment opportunities, even while controlling for relevant covariates.

Theoretical Framework

The definition of behavioral nudges is often drawn from the work of Richard Thaler and Cass Sunstein (2008) who summarized them as "choice architecture that alters people's behavior in a predictable way without forbidding any options or significantly changing their economic incentives" (p. 6). In the face of cognitive limitations, nudges can make some choices more salient; for example, stocking produce near the cashier at a grocery store improves the rate at which it is purchased (Kroese et al., 2016). Importantly, nudges are conceived of as low-cost, non-coercive, and easy to ignore, leaving autonomy with the person exposed to the nudge. Though ideas about nudges and nudge theory have been criticized for being vague and having wide applicability across too many domains (Kosters & Van der Heijden, 2015), small changes to learning environments have been shown to effect positive outcomes and student achievement in myriad educational settings with students, instructors, and even parents (Damgaard & Nielsen, 2018). In contrast with interventions that change default and passive behaviors (e.g., opting in versus opting out), the nudges employed in this study are informational, providing an addition to the decision environment that students can actively use (Damgaard & Nielsen, 2018).

Methods

Context

This research took place at the University of Michigan (U-M), a large, public university where the Precalculus course of interest (herein called Math 101) enrolls approximately 600 students annually across small sections of about 20 students each. Recently, the Math 101 teaching team began work to shift the course assessments from a traditional model of three high-stakes exams to a mastery-based model including approximately 10 repeatable assessments and standards for each letter grade (Bennett et al., 2022). The new assessment structure now has 55% of students' course grades determined by the repeatable mastery assessments and a further 15% from work done during class meetings and on assignments. These scores plus two short, low-stakes exams comprise students' course grades, and there is a guaranteed scale for the course: students know in advance what grade they will get for a given course average and work. This assessment structure replaces a model in which 95% of students' course grade was generated by three timed (and necessarily high stakes) exams and a scaled grade distribution.

In the study period, students could complete the mastery assessments for a grade up to twice per day. Most assessments included five problems and were scored out of 5 points; students earned partial or full credit for scores of 4 or 5, respectively, but zero credit for scores of 3 and lower. The final assessment had 10 possible points and a similar credit/no credit assignment. During the first term with mastery assessments, instructors observed that students tended to make their first attempt close to the due date and that they overall used few of the available attempts. Thus, an email-based nudge intervention was designed with the goals of 1) encouraging students to make their first attempts earlier (each assessment was open for 10 to 14 days), 2) giving students feedback about their current score and number of remaining attempts, and 3) adjusting norms so that, in contrast to typical perceptions of quizzes and exams, initially low scores were seen as okay. The intervention was implemented with ECoach, a software platform already in use in Math 101. ECoach is an award-winning web-based student "coach" that supports success in large courses through tailored feedback (IMS Global, 2021; Matz et al., 2021). ECoach uses student survey responses, prior performance metrics, and information about how students are doing (e.g., homework scores) to provide tailored messages and resources like advice about exam preparation.

The nudge intervention consisted of two short reminder emails tailored to each student's behavior and sent with ECoach for each of the 10 mastery-based assessments. The first reminder (Reminder 1) was generally sent five days before the due date to all students and included information about students' current score, their remaining number of attempts, and tailored feedback. For example, students who hadn't yet started were encouraged to make their first attempt at least three days ahead of the due date, and students who had already made several attempts but still had low scores were encouraged to consider getting help during office hours. The second reminder (Reminder 2) was generally sent two days before the due date but only to students who had not yet earned any credit for the assessment (i.e., a score of 3 or lower). Students again received information about their current score and number of remaining attempts, and the feedback was tailored based on whether the student was not passing because they hadn't started the assessment or because they hadn't yet earned a passing score. Together, these reminders reflect a combination of two techniques identified by Münscher and colleagues (2016) in their framework for choice architecture: the nudges assist with "decision information" by showing students their scores and how many attempts they have remaining, and the nudges serve as a reminder about due dates, which is a form of "decision assistance."

Data Collection and Analysis

In the semester of interest, 464 students enrolled in Math 101. A demographic profile is provided in Table 1 showing that, compared to the rest of U-M, Math 101 tends to have higher proportions of students who are female, first-generation, low income, and members of racial and ethnic groups historically excluded from STEM (Asai, 2020). We also collected several variables from the ECoach database including 1) students' maximum mastery assessment score for each assessment, 2) whether students read the reminder emails for each assessment, and 3) whether students read their other Math 101-related ECoach emails throughout the semester as a gauge of their general proclivity to read similar emails.

Further, we collected students' grade point average in other courses (GPAO), a relevant metric of college-level academic performance. GPAO is calculated as the cumulative college GPA through the given semester excluding the course of interest and thus acts as a measure of student performance in all their other courses (Huberth et al., 2015; Koester et al., 2016). Most Math 101 students (≥80% in this study period) are in their first year, so GPAO is essentially a measure of their first-year college GPA (excluding Math 101). We note that at U-M, GPAO has been found to outpace the predictive value of all other typical covariates (e.g., high school GPA and standardized exam scores) with respect to student academic performance, namely course grades. The data were imported into and analyzed using IBM SPSS Statistics (version 27). Following basic descriptive statistics, we used linear regression to evaluate the impact of reading the reminder emails on assessment scores using GPAO and the extent to which students read their other Math 101-related emails as covariates. This study was determined by U-M's Institutional Review Board to be exempt research.

Table 1. Summary of demographic information for students in Math 101 during the study period compared to all full-time U-M undergraduates in the same term. [a]Only binary sex information was available. [b]Students were considered continuing-generation if any parent had any college experience. [c]Students self-report estimated gross annual family income. We use a $50,000 cut off because it is approximately 200% of the federal poverty guidelines for families with four persons (Office of the Assistant Secretary for Planning and Evaluation, 2021).

Characteristic	Math 101		U-M
	n	%	%
Sex[a]			
Female	274	59	51
Male	190	41	49
Missing	0	0	1
Race			
Asian	28	6	23
Black	70	15	4
Hawaiian	1	<1	<1
Hispanic	52	11	7
Native American	1	<1	<1
Two or more	28	6	5
White	274	59	56
Missing	10	2	6
First or continuing generation status[b]			
Continuing generation	368	79	87
First generation	82	18	8
Missing	14	3	5
Estimated family annual income[c]			
≥$50K	257	55	62
<$50K	128	28	13
Missing	79	17	25

Limitations

One key limitation in using these data is that we know whether a student has *viewed* the reminder email but not actually whether they *read* the reminder email. That is, the ECoach database records the timestamp at which the reminder was viewed and there is no way based on the current data infrastructure to know the length of time that a student interacted with the reminder, though we were able to assess if students opened the same reminder multiple times. Throughout this study, then, we assume readership based on viewership and acknowledge that other methods of data collection would be needed to better understand the quality of student engagement with the reminders.

Results

Results from this pilot semester of email nudges are encouraging. Overall, the reminders were read by most eligible students (Figure 1). This pattern holds across both reminders which is somewhat surprising because it might be reasonable to suspect the group of students that is sent Reminder 2 (students who hadn't yet achieved a passing score two days before the due date) to be different in some academically relevant ways compared to the aggregate group that is sent Reminder 1. Readership drops over the course of the semester but levels out at approximately 60% by the fourth assessment, due about half-way through the term. Overall, students read these reminders at a slightly lower rate (M = 67%, SD = 27%) than that at which they read all their other Math 101 emails from the ECoach system (M = 72%, SD = 28%; $t(463) = 5.2$, $p < .001$).

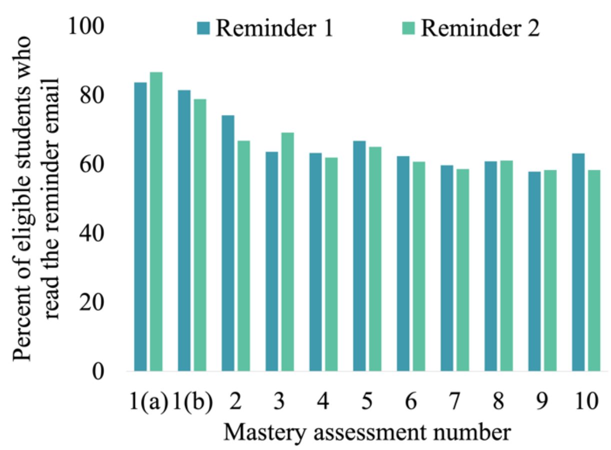

Figure 1. Percent of eligible students who read each reminder email. Because of a scheduling disruption at the beginning of the semester, there was a double set of reminders—1(a) and 1(b)—for the first assessment.

Students who read the reminder emails earned higher average scores on every assessment compared to students who did not for both Reminder 1 (Figure 2a) and Reminder 2 (Figure 2b). These score differences are statistically significant for 12 of the 20 total reminders (we used only the data from the first set (1a) of reminders for assessment #1; see Figure 1 for details).

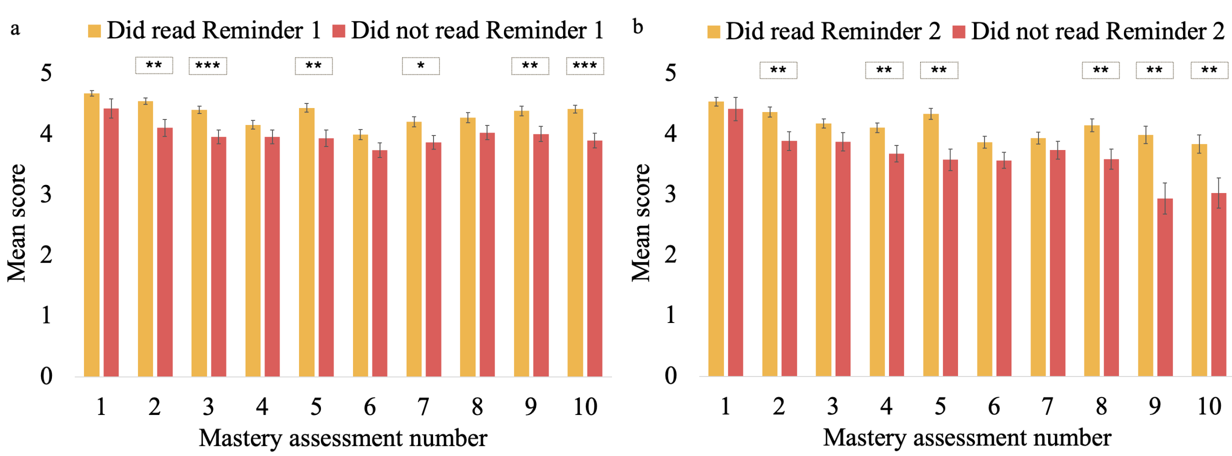

*Figure 2. Mean scores +/- standard error of the mean on each mastery assessment parsed by students who did and did not read Reminder 1 (a) and Reminder 2 (b). Comparisons for statistically significant differences were made by independent samples t-tests; *p < .05, **p < .01, ***p < .001.*

While it might be expected that students who read their email exhibit other behaviors that support earning higher scores, this result was stable for six of the reminder emails across four of the ten mastery assessments (#2, #5, #9, and #10) even after using linear regression to account for prior performance (with GPAO) and how much students read their Math 101 ECoach email in general (Table 2). In these cases, reading the reminder emails was associated with higher assessment scores by somewhere between 0.27 and 0.84 points on the 5-point assessment scale. Average variance inflation factors (VIF) close to 1 indicate that multicollinearity between covariates does not substantially bias the models, even considering the expected and observed relationship between reading the reminders and reading other course-related emails (correlations not shown). Inspection of the standardized residual plots for each model indicated that the assumptions of linear regression were met.

*Table 2. Linear regressions relating whether students read Reminder 1 or Reminder 2 emails and assessment scores. "Other email" refers to the percent of other eligible Math 101 emails sent through ECoach that students read. CI = confidence interval; LL = lower limit; UL = upper limit. Each model was significant with p < .001; *p < .05, **p < .01, ***p < .001.*

Covariate	*B*	95% CI for *B*		*SE B*	*β*
		LL	*UL*		
Assessment 2, Reminder 1 (n = 462, R^2_{adj} = 0.15, VIF_{avg} = 1.21)					
Constant	1.75	1.16	2.33	0.30	-
GPAO	0.65	0.48	0.82	0.09	0.34***
Other email	0.29	-0.12	0.70	0.21	0.07
Reminder 1	0.28	0.02	0.53	0.13	0.10*
Assessment 2, Reminder 2 (n = 311, R^2_{adj} = 0.13, VIF_{avg} = 1.18)					
Constant	1.72	0.99	2.45	0.37	-
GPAO	0.61	0.40	0.82	0.11	0.31***
Other email	0.25	-0.33	0.82	0.29	0.05
Reminder 2	0.38	0.05	0.71	0.17	0.13*
Assessment 5, Reminder 1 (n = 458, R^2_{adj} = 0.20, VIF_{avg} = 1.28)					
Constant	0.54	-0.14	1.23	0.35	-
GPAO	0.95	0.75	1.15	0.10	0.41***
Other email	0.22	-0.29	0.73	0.26	0.04
Reminder 1	0.38	0.09	0.66	0.15	0.13**
Assessment 5, Reminder 2 (n = 321, R^2_{adj} = 0.21, VIF_{avg} = 1.17)					
Constant	0.46	-0.34	1.26	0.41	-
GPAO	0.90	0.67	1.14	0.12	0.39***
Other email	0.28	-0.33	0.89	0.31	0.05
Reminder 2	0.52	0.17	0.87	0.18	0.16**

Assessment 9, Reminder 2 (n = 149, R^2_{adj} = 0.17, VIF_{avg} = 1.17)					
Constant	0.51	-0.80	1.83	0.67	-
GPAO	0.81	0.42	1.19	0.20	0.32***
Other email	-0.23	-1.25	0.78	0.51	-0.04
Reminder 2	0.84	0.26	1.42	0.30	0.24**
Assessment 10, Reminder 1 (n = 421, R^2_{adj} = 0.22, VIF_{avg} = 1.23)					
Constant	0.12	-0.65	0.89	0.39	-
GPAO	1.02	0.80	1.24	0.11	0.41***
Other email	0.42	-0.05	0.89	0.24	0.09
Reminder 1	0.27	0.01	0.52	0.13	0.10**

Discussion

Here, we find a positive effect of behavioral nudges on student performance on mastery-based assessments in Precalculus. These improvements are observed in assessments that span the semester, indicating some robustness to variation in student workload and the mathematical concepts and exercises at hand, and they are observed for both Reminder 1 and Reminder 2 though these nudges are targeted differently. Mastery-based pedagogy and similar assessment frameworks (e.g., standards-based grading and ungrading) are gaining attention in introductory STEM courses and specifically mathematics as a route for supporting positive student outcomes, both cognitive and affective. The results described here provide evidence that within a mastery-based framework, non-coercive behavioral nudges can be effective in helping students make choices that ultimately lead to higher scores and presumably improved learning.

Importantly, this study provides an example of successful personalized nudging. Improving the precision and effectiveness of behavioral nudges through more personalized default rules is an active area of research (Mills, 2020; Sunstein, 2013). With evidence that broadly applied homogeneous rules can have unintended effects and even harm specific subpopulations (Thunström et al., 2018), understanding the nature and extent of efficacious personalization in learning environments is important. There is also a relationship between this work and efforts to cultivate growth mindsets with students, especially for those in developmental courses (Mills & Mills, 2018). By design, mastery-based pedagogy asks learners to focus on evaluating their progress toward achieving learning goals, a key characteristic of students who exhibit intrinsic motivation to learn (Middleton & Spanias, 1999). Supporting students' growth orientation and intrinsic motivation to learn may be particularly crucial in math, and evidence that mastery-based pedagogy supports growth orientation is building (Collins et al., 2019; Prasad, 2020).

Future research will report on the quality of interactions that Precalculus students have with the reminders and insights into the mechanisms by which they affect outcomes. Certainly, student mindset about learning in math, and especially in mastery-based courses, is an important area for future work that can provide momentum toward the improvements in undergraduate education sought by the larger mathematical sciences community (Saxe & Braddy, 2015).

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Easy, Medium, and Hard: Structuring Space in 2D and 3D by Way of Linear Combinations

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Understanding linear combinations is at the core of linear algebra and impacts their understanding of basis and linear transformations. This research will focus on how students understand linear combinations after playing a video game created to help students link the algebraic and geometric representations of linear combinations. I found that having students reflect upon the game and create their own 3D version of the game illustrated which elements of 2D understanding could be translated into 3D. Also, students creation of easy, medium, and hard levels provided insight into how students progressively structure space.

Keywords: Linear Algebra, Linear Combinations, Game-based Learning

Understanding linear combinations in a variety of dimensions is core understanding linear algebra. In addition, it is important that students can reason through a variety of representations of multiple concepts including linear combinations (Hillel, 2000; Sierpinska, 2000; Larson & Zandieh, 2013). Well-designed games created with learning theories in mind can help students associate multiple representations and problem-solve (Ke & Clark, 2019). Vector Unknown (Zandieh et al., 2018; Mauntel et al, 2019; Mauntel et al., in press) is a game designed using the tenants of Realistic Mathematics Education (RME). The focus of the game is to help students connect geometric and algebraic representations of linear combinations with vector equations. This paper is part of a larger project where the goal is to analyze how students structure two-dimensional space space after playing the game *Vector Unknown,* and how this two-dimensional space structuring informs their structuring of three-dimensional space.

Literature Review

Vector Unknown is based on the Magic Carpet Ride task (Wawro et al., 2013) which is the first task in a sequence that cover linear combinations, span, linear independence. The task uses travel as a metaphor for taking linear combinations of vectors. Stewart and Thomas (2010) found that linear combinations were central to students' understanding of basis, span, and linear independence. Dogan-Dunlap (2010) utilized Sierpinska's modes of reasoning (2000) and found that students who were able utilize both geometric modes of thinking were also able to utilize analytic modes of reasoning. This research indicates that bridging the gap between geometric and algebraic representations of vectors and linear combinations can be beneficial for students learning linear algebra. Coordinating the ideas of linear independence and span is a key area of difficulty related to organizing linear combinations and transitioning from two-dimensional to three-dimensional space. These areas could be addressed using game-based learning as they can be represented in multiple ways including a visual mode that may difficult to represent on paper.

Well-designed games follow good learning principles (Gee, 2003). Gee (2003) postulates that one of the reasons video games are so popular is because they keep students pleasantly frustrated with well-ordered problems that engage them in complex reasoning. Furthermore, intermittent periods of reflection and gameplay have been shown to be successful in helping students learn from video game environments (Foster & Shah, 2015). For this reason, this research will involve having students play the game *Vector Unknown*, reflect upon the game, and then use this reflection to design their own three-dimensional space version of the game.

Study Context

The Game Vector Unknown

In this section I describe the game *Vector Unknown* (https://tinyurl.com/linearbunny) its creation, and describe some gameplay. The game *Vector Unknown* was developed as a coordination between math educators, experts in game-based learning, and programmed by a team of capstone students at Arizona State University to help students understand the connection between a vector equation and its geometric representation. Zandieh et al. (2018) designed the game *Vector Unknown* based upon a theoretical framework that intersects GBL with RME (Zandieh et al., 2018). The goal of the game is to have the player select two vectors, adjust the scalars, and press GO to guide the rabbit to the basket. In the first level (Figure 2), the game presents with a collection of four different vectors. Once players have chosen at least

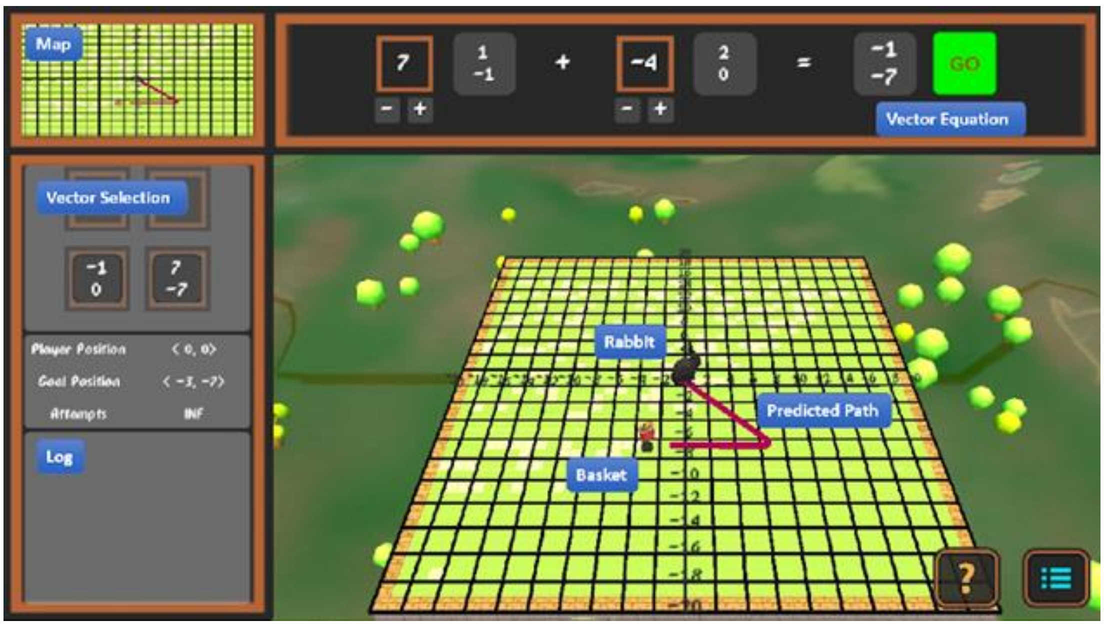

Figure 2. First level of the game *Vector Unknown* from Mauntel et al. (2019)

one vector, they can adjust the scalars in front of the vectors. Feedback received by the players includes a red predicted path generated from the rabbit illustrating each component of the vector equation. The result of the vector equation adjusts as the player adjusts the scalars in the vector equation. Above the log, players can view their position and the position of the basket location. The map in the upper left–hand coordinates provides an alternate view of the map. After the player presses GO, the rabbit moves along the path towards the basket and the log provides all vector equations used.

Prior Research on Vector Unknown

Mauntel et al. (in press) previously categorized game-play strategies from two main lenses: geometric or numeric. The geometric lens is when students focus on utilizing the graphical components of the game relating to the graph in particular the Predicted Path. Numeric thinking relates to when the player focuses more on the Vector Equation. One of the main goals of the game is to encourage the player to transition fluidly between the two different lenses. Mauntel et al. noticed several strategies that students employed while playing the game.

One of the strategies, quadrant-based reasoning, is when players utilize direction to anticipate to make their vector choices. The term quadrant is used because players start at the origin and often they would choose vectors that corresponded to the quadrant of the basket location. In the numeric lens this presented itself as a player choosing vectors that shared the same sign as the basket. In the geometric lens, students would often drag a vector and try it to see its direction to determine if it were in the same direction as the basket. Often players would transition between lenses for instance there are times when players would find the slope between two points by counting the change in *y* and change *x* on the Cartesian plane and then using this information to find the vector switching to a more numerical lens.

For the strategy focus on one vector, players would choose a single vector and then scale that vector as close to the basket as possible and then choose another vector to complete the trip to the basket. This theme presented itself in the numeric lens as scaling a vector until it was as close to the basket location in the vector equation as possible. In the geometric lens, players would scale the vector until the predicted path was close to the basket location, and then choose another vector to complete the trip. The fourth theme, focus on one coordinate, involved the players trying to use a vector or combination of vectors to reach a value or destination that matched the *x*- or *y*-coordinate of the basket location. This theme presented itself frequently when standard basis vectors were involved.

Theoretical Framing

My research will focus on analyzing student's mathematical activity in the sense of Freudenthal (1971, as cited by Gravemeijer & Terwell, 2000) as it relates to the taking linear combinations of vectors in two dimensions and the emerging activity that it generates. From this mathematical activity, I want to characterize a new type of emergent activity which I call structuring space use the term emerging activity as opposed to emerging models to indicate that tasks are not designed to guide the students to a fixed emergent model or formalize a particular concept, rather this research is focused on what models are possible to inform future development of an RME sequence. I call the students' emerging activity their structuring of space. After the students structure two-dimensional space, they will be asked to structure three-dimensional space. This transition is meant to induce a model of/model for transition as their structuring of two-dimensional space is utilized for structuring three-dimensional space. This research will utilize a realistic starting point of the video game Vector Unknown. Also, I will be looking at a how the student's structuring of two-dimensional space impacts their gameplay to gain more insight into how their ideas of structuring space relate to the game context. My research questions are:

1. How did students structure two-dimensional space with respect to linear combinations in relation to the game Vector Unknown?
2. How did students adapt their structure of two-dimensional space to a three-dimensional setting when designing a three-dimensional game based upon Vector Unknown?

This paper is part of a larger dissertation project and in this paper I will present a portion of my findings for research questions 1 and 2 for one group of students.

Interview Protocol

The interview took place over the course of four sessions. During the first session students played the game Vector Unknown individually. In the second section students were paired with another student based upon their gaming experience and were given a set of four vectors taken from the Vector Selection in the game and asked to list all possible locations to place the basket. Then, they were asked to create an easy, medium, and hard difficult set of vectors that could reach the basket location at (-3, 5). Finally, during the third session, participants were asked to design a three-dimensional space game based upon the two-dimensional space game by either creating an Easy, Medium, and Hard difficulty game with vectors and potential basket locations or design three sets of vectors (Easy, Medium, and Hard) that could be used to reach a basket placed at (3, -4, 5).

Participants, Data Sources, and Methods of Analysis

For this paper I will be looking at two students from a large Southern University. Neither student had taken linear algebra at the time of the interview. Students were asked to self-identify their race and gender. Both students identified as women, with one identifying her race as white (given the pseudonym Gabby) and the other identifying her race as other (given the pseudonym Delores). Also, both students played video games for less than 5 hours per week and thus were classified as non-gamers.

Interviews were conducted over Zoom, recorded to a secure server, and auto-transcribed using Zoom's automatic transcription service. The videos and associated transcripts for the second and third sessions were then reviewed for instances of structuring space in two and three dimensions respectively. These instances were then detailed using captured images and quotations from the transcripts edited for accuracy when necessary. The two-dimensional and three-dimensional structuring spaces were then reviewed for connections and cross-cutting themes consistent with grounded theory (Strauss & Corbin, 1990).

Findings

In this section I present several examples of structuring space induced by the game *Vector Unknown* and the post-play reflections. Delores and Gabby had several ways of thinking about linear combinations of vectors in two- and three-dimensional space. In two-dimensional space, Delores and Gabby focused on creating increasing more complex linear combinations of the vectors that would reach the basket of (-3,5). When designing the three-dimensional space game, they created easy, medium, and hard version of the game and chose to illustrate all the possible locations that could be reached by the vectors. With the basket location not fixed their method of creating different difficulties of the game was based upon creating basket locations that were more difficult to reach rather than creating different sets of vectors.

Easy, Medium, and Hard in two-dimensional space: Numerical Structuring leads to Vector Equations

Gabby and Delores's work in creating Easy, Medium, and Hard vectors led them think about vector equations and how they could be solved. Gabby and Delores designed their Easy vectors by thinking about the numerical relationships of the vectors in relation to the basket at <-3,5> (Figure 2). They wanted the vectors to sum to the basket location without having to scale. This activity resulted in breaking apart the vector <-3,5> in terms of x and y coordinates (for <0,5> and <-3,0>) or finding numbers that sum to -3 and 5, for example <-1,3> and <-2,2>. Gabby described their strategy for finding Medium vectors by comparing it to their strategy for finding easy vectors:

So that one would be more medium, I guess, because, instead of just having the numbers right there you actually have to multiply the numbers to get the ones you want. For medium, we can do like negative one zero and then zero negative one, so that way it's less addition and more multiplication.

Medium difficulty is the first instance that Gabby and Delores transition to using linear combinations of vectors that included addition and multiplication. Gabby's idea for the first two vectors led to Delores's idea for finding the remaining vectors by writing down a vector equation.

Easy	Medium	Hard
<-1,3>, <-2,2>, <0,5>, <-3,0>	<-1, 0>, <0,-1>, <-1,2>, <-1, 1>	<6,7>,<-9, -2> <4,7>, <9, 26>

Figure 2. Easy, Medium, and Hard Vectors Selections

Here Delores first started with trying to find two vectors that would add up to <-3,5> knowing that she wanted to use -2 as a weight (Figure 3). She eventually dropped the negative and through a process of guess and check was able to find vectors that worked. For the Hard set of vectors both Gabby and Delores agreed that larger numbers were appropriate. Delores created the pair of vectors <6,7> and <-9,-2> to sum to the basket location of <-3,5>. Gabby used a similar technique to Delores to design the Vectors <4,7> and <9,26> by choosing a scalar of -3, selecting the vector <4,7> and multiplying it by -3, and finally finding solving for each coordinate to find the vector <9,26>. Gabby's work did not indicate clearly if she had guessed and checked both vectors or if she chose -3 and <-4,7> and found then solved for <9,26>.

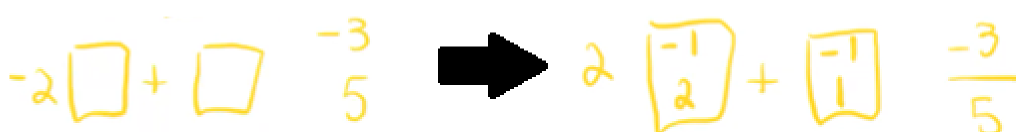

Figure 3. Vector Equations as drawn by Delores for Medium Difficulty

After designing Easy, Medium, and Hard vectors, the interviewer asked the participants how they would explain to a programmer a technique for creating vectors. They focused on describing their process for Medium and Hard difficulty vectors. Delores explained that you could create the vectors by "just take four random numbers and subtract them until I get my, until I get either the -3 or 5" (Figure 3). Here Delores is describing her process of trying random numbers in the vectors until you find a combination that works for the x and y coordinate. This indicates that Delores is thinking about using subtraction to identify individual coordinates instead of whole vectors. This corresponds strongly with the focus on one coordinate strategy in the Vector Unknown game.

Figure 3. Random Vector Equation

Gabby built on this idea and stated that "you would already have the point specified that you want like (-3,5) and then picking random numbers for Vector 1 … then the number you multiply it by and then solving for Vector 2." This suggests that Gabby has started thinking about Vector Equations as equations in themselves, subtracting whole vectors as opposed to just thinking in terms of individual coordinates.

Easy, Medium, and Hard in three-dimensional space: Working with Vector Equations

When asked to design a three-dimensional space game, Gabby and Delores chose to create Easy, Medium, and Hard games and illustrate all the possible locations where the basket could be located. This contrasts to the exercise in the previous session where they were designing easy, medium, and hard vectors to reach a specific basket location. The interviewer introduced them to GeoGebra three-dimensional space since neither student had worked in three-dimensional space previously. They began by designing by choosing random numbers for easy vectors and taking all combinations of the easy vectors for possible locations of the basket indicated by points A-F which were sums of the vectors <0,2,0>, <-1,2,3>, <2,0,1>, and <-4,3,-1> (Figure 4).

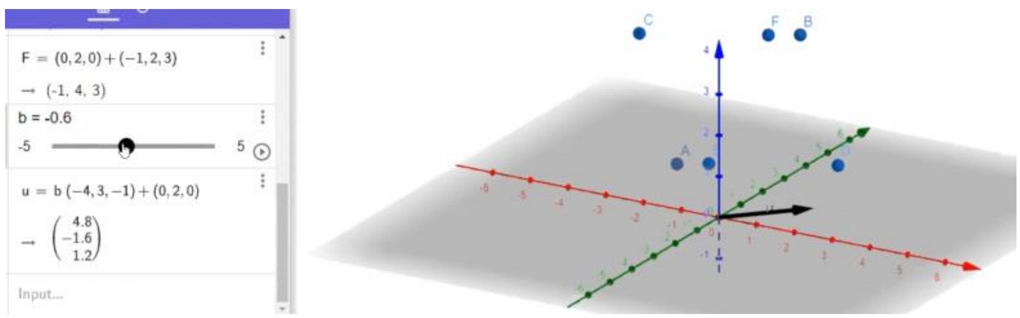

Figure 4. Creating the basket locations for the Easy and beginning goals for Medium

When they transitioned to Medium rather than change the vectors, they decided to change the possible basket locations by placing scalars in front of one vector in each sum that represented a basket in the Easy mode of the game. Figure 5 illustrates one basket location from easy and how it was adapted to medium and hard.

Basket Locations for Easy	Basket Locations for Medium	Basket Locations for Hard
(2,0,1) + (-1,2,3)	b (2,0,1) + (-1,2,3) (2,0,1) + b (-1,2,3)	b (2,0,1) + c (-1, 2, 3)

Figure 5. Easy, Medium, and Hard basket location comparison

Both students focused on how they could adapt expression that they had developed to find the basket location. Here Gabby and Delores used reasoning about the Easy, Medium, and Hard vectors where an increase in difficulty resulted in more scalar multiplication in the context of their linear combinations.

After creating Easy, Medium, and Hard difficulty, Gabby and Delores were asked to describe geometrically some of their basket locations:

Interviewer: What does G = (-4, 3, -1) + b (0,2,0) look like geometrically?
Delores: It would go all around the board because it has negatives and positives…[Delores adjusts her slider] It stays on one side of the board.

Gabby: I can't tell if the vector stays on one side because the graph is three-dimensional space. We cannot use the quadrants anymore.

Here Delores and Gabby are trying to build on different knowledge they used previously in the interview. Delores is trying to first use a mainly numerical argument using signs until she adjusts the sliders and discovers that the basket location seems to be on one side of the board. Gabby could not decide if she was convinced Delores's statement is correct. This indicates that Gabby is trying to adapt the notion of quadrants to three-dimensional space and what it means to be on "one side." This is important because one of the keyways that student's reason about the game in two-dimensional space was quadrant-based reasoning. If students were using quadrant-based reasoning to reason about three-dimensional space, they need a new way to reconceptualize quadrants. One possible way of dealing with this involves thinking about the controls of the video game and their relations to the scalars as suggested in the following conversation:

Interviewer: If you want to look at all the places you put a goal [basket] with the equation b <2, 0, 1> + c <-1, 2, 3>, can you describe it?
Gabby: b makes it move the vector move up and down and c makes it move side to side
Interviewer: So if I were to plot all the goals [baskets], what would it look like?
Delores: It would look like an L-shape and on the left side.

This indicates that thinking about the way the basket moves with the slides similar to the controls of the game might be a way to think about the location movement of the basket. Delores's comments highlight the point that while she has conceptualized some points in the linear combination b <2, 0, 1> + c <-1, 2, 3>, the basket locations that require moving both b and c eluded her description. This could be a result of the limited time of the session or because she thought there might be constraints on what scalars were used at what points. Either way it illustrates that points that involve changing the scalars on each vector simultaneously

Discussion

While students were reflecting upon the two-dimensional game and designing their own three-dimensional game, they provided valuable insight into how they organized linear combinations, solved vector equations, and conceptualized span (all possible basket locations). While this work only analyzed one group of students and thus lacks generalizability, it illustrates that certain aspects of two-dimensional space like quadrants, lines dividing space, and balancing multiple coordinates do necessarily translate easily for students to three-dimensions. In addition, the linear combination $av_1 + bv_2$ for vectors v_1 , v_2 and scalars a,b can prove difficult to understand and may need additional geometric representational support in order to be fully understood. Additionally, game-based scaffolds such as easy, medium, and hard type levels can provide insight into how students scaffold their conception of structuring space and taking linear combinations. This could be valuable as well-designed video games contain several scaffolds to create pleasantly frustrating experiences for the player (Gee, 2003). I can see having students analyze these scaffolds and design games with these scaffolds could be a rich area of research that allows students to structure their ideas and provide valuable insights into their thinking.

Future research includes interviewing more students and seeing how they structure space and transition from two-dimensional structuring to three-dimensional structuring. Also, additional analysis will compare the student's gameplay strategies from the their playthrough of the game with their structuring of space in two and three dimensions.

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Student Experiences with Stereotype Threat in Undergraduate Mathematics

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Mathematics education is a racialized and gendered space in the US, with white males historically holding dominion. One form of oppression manifests in the establishment and perpetuation of stereotypes involving innate mathematical ability. Students who identify as a stereotyped population often perform under extra pressure to not align their performance with stereotypes, and this extra pressure is known as stereotype threat. In this report, I amplify student voices from undergraduate introductory mathematics courses at multiple universities across the US and value their lived experience. We analyze a subset of free response survey data where three broad populations of experiencing stereotype threat emerged: Asian, Pacific Islander, or Desi American (or APIDA) students, Black or African American students, and women or female students. As students relate their identities to their own mathematical ability, our results show that their experiences with stereotype threat are distinct.

Keywords: Stereotype Threat, Student Experience, Identity, Precalculus, Calculus

Introduction and Background

While a multitude of students buckle under the stress of overlapping midterms, increasingly complex schedules, living away from home for the first time, and trying to make new friends, many are also laboring silently, negotiating their own identities as they learn and grow (Swann, 1987; Ting-Toomey, 2015). There are some identity groups that have stereotypes involving their ability to do mathematics imposed upon them, irrationally based on race or ethnicity, which effectively work to place some students of color in a hierarchical positionality (Shah, 2017). Members of stereotyped groups often feel extra pressure to not perpetuate these narratives by performing in a way that aligns with the stereotype (Spencer et al, 2016), and this pressure that hinders students' academic ability is referred to as stereotype threat (Steele & Aronson, 1995).

One way that mathematical stereotypes have shaped student experience is through this mechanism of stereotype threat, which is driven by implicit and explicit references to racial or gendered mathematical narratives. Stereotype threat has been shown to significantly affect test performance in women and minority groups (Nguen & Ryan, 2008; Spencer et al, 1999), create barriers for the success of Black students (Johnson-Ahorly, 2013), and negatively affect the ability of women and minority students to identify with school culture (Steele, 1997). This evidence leads to the striking realization that without effort to learn about and combat racial and gendered stereotypes, students of many identities will continue to experience inequities in opportunities to perform and succeed in mathematics classrooms.

Although mathematics is often referred to as a perfect, objective, race-free, universal language, the disparities found in women and historically oppressed students' achievement in mathematics compared to their white, male counterparts suggest otherwise (Voyer & Voyer, 2014, Jeynes, 2015). The perpetual enforcement of meritocratic ideals that exist within US society permeates the education system, enforcing the belief that anyone can do well at math if they just try hard enough. In a country founded on capitalism, meritocracy, and white supremacy, these ideologies have certainly bled into our educational institutions. They linger today in the assumptions we make about students, the narratives we have heard about their ability to do mathematics, the implicit biases we all have, and the inequities that emerge throughout undergraduate student experiences.

This exposition values undergraduate mathematics students' voices, as they report their experiences in introductory mathematics courses. Students tell stories of how they relate their identity with their ability to engage and succeed in mathematics and the implicit role that stereotype threat plays in their success in Precalculus through Calculus II courses (P2C2). Through the analysis in this study, I make a connection between undergraduate mathematics students' identity and their experience with stereotype threat by asking the research question: How are students experiencing stereotype threat in undergraduate P2C2 mathematics courses?

Methodology

In this study, I analyze data collected through a modified version of the Postsecondary Instructional Practice Survey for undergraduate students (S-PIPS), developed by Apkarian et al. (2019). The NSF-funded initiative called Student Engagement in Mathematics through an Institutional Network for Active Learning (SEMINAL) administered the S-PIPS survey to nine different institutions over the course of three semesters: Spring 2018, Fall 2018, and Spring 2019. The dataset for this study is comprised of 12,188 responses from students in Precalculus, Calculus I and Calculus II courses across the US.

One free-response item on the S-PIPS survey asks, "is there anything about who you are (or your identity) that affects your ability to do or learn mathematics?" Out of the 16,523 total responses to the survey, there were 5,787 responses to this item that specifically targets student's relation of their identity to ability in mathematics. These responses were analyzed using MAXQDA, a qualitative data analysis program that affords the ability to code responses using an open, axial coding method described by Miles, Huberman, and Saldana (2014). Through this thematic organization of emergent categories and super categories, a system of codes was developed to illuminate the patterns hidden within the thousands of responses. A subset of these categorized themes are presented in **Figure 1**, to illustrate the extent of my coding process for the focus of this study.

Parent Code	Code	Sub-Code
Race (40)	Asian (22)	Good at Math / Threat
		Seek Help
		Family/Society Pressure
	Black / African American (18)	Prejudice
		Pressure / Prove a Point
		Representation
Woman/Female (25)	Inferiority (9)	
	Male Dominated Space (9)	
	Extra Work (6)	
	Representation/Inclusion (5)	

Figure 1. Parent code, code, and sub-codes for Race and Woman/Female themes. A number after a code indicates the number of unique responses with that code.

Since the dataset is combined responses of undergraduate students from several universities all with very different individual stories, it is important to note the demographic analysis of our dataset as a whole. Respondents are college students from 9 universities identifying their own gender as: 51% men, 41% women, <1% gender fluid, <1% transgender, and about 7% other. Racial demographics break down to 48% white, 13% Black or African American, 18% APIDA, 18% Hispanic or Latino, <4% other, with many students choosing two or more ethnicities. Attaining enough data to analyze responses qualitatively is inherently difficult in a space where white males hold dominion, but the diversity and cardinality of this dataset helps populate the next section with quotes from many unique students from all over the US.

Results

Gender identity and race play a significant role in the results of the qualitative analysis of the free response question, "Is there something about who you are that has impacted the way you experience mathematics here at [institution]?" Within those broad categories, there are emergent themes that reference common stereotypes and harmful narratives regarding racial and gender identities.

Being "Good at Math" is a Burden

One of these harmful narratives is regarding "Asians" (Central/East Asian, Pacific Islander, Desi American (APIDA) students that are presumed to be "Asian" in the US) and their supposed innate proficiency at mathematics. While "being good at math" can be viewed objectively as a positive trait, this analysis shows that there are multiple ways expressed by APIDA students of how this stereotype is actually harmful. While APIDA is a pan-ethnic term that collects several large umbrella racial descriptors, it is important to note that there exist hundreds of identities, cultures, and languages that fall into this complex group of students.

This analysis shows that not all APIDA students are experiencing stereotype threat in the same way, especially with subtle differences in the language that they use to describe their experience. For example, some students appear to lend some accuracy to the narrative that Asians are good at math by saying, "I am an Asian so I should be good at math. But I'm not." Another student reflected on how they were, "wishing the Asian stereotype was true." Another student recalled a previous semester of their math course when they, "heard a student say that all Asians are good at math." This last quote exposes the relevance to stereotype threat in current undergraduate mathematics culture when students experience racism in their classroom, rather than solely being aware of a popular narrative. While these minor differences in perception of how the narrative is accepted, rejected, or experienced for these individuals, there are multiple themes that emerged that show how qualitatively different APIDA student experience can be when it comes to stereotype threat.

Awareness of the stereotype and bringing it up in response to a question that asks to relate their identity to their ability in mathematics suggests that the expectation and pressure to perform a certain way are on these students' minds. Cognitively, these students could be at a disadvantage from paying some amount of attention to the potential judgement of their performance. On top of this constant awareness, several students' responses paralleled the following sentiment: "as an Asian, I feel as if I am expected to be good at math, but honestly I suck." Another student mentions how they have, "beaten the stereotype by being an Asian who is terrible at math." The prevalence of low self-efficacy and self-doubt caused by individuals comparing themselves to the stereotype of being good at math appears in more students' stories, as they describe how they are an, "Asian who is not good at math so I feel perpetually judged."

This theme of prejudice, fear of judgement, and pressure to succeed also goes beyond the individual, the classroom, and the math department. Intersectionality of identities and their relationship to mathematical ability also begins to take shape in the analysis, as this student explains,

> There is always this stress that East Asians have to [perform in math] well. It is an unsaid law from your parents, family, and society though they say they do not. Furthermore, coming from a junior college with academic excellence in mathematics, I have more stress to do it well.

APIDA student experiences with family, culture, and intersecting identities are shaped in and out of the classroom, and even appears to influence interpersonal relationships when a student claims,

> Yes, so I am a Asian, and I feel like all my other Asian friends are good at math that they do not study much but they understand everything. Thus, it gives me somewhat pressure in [passing] mathematics at [institution].

The final example for APIDA students is a result that was unique to this group, due to the nature of prejudice they faced from students and faculty alike. Through multiple students' stories, the analysis revealed that when students are assumed to be adept at math, or have greater math skills in comparison to their peers, they feel unable to ask for help in mathematics. One student spoke to their experience regarding this phenomenon: "I guess the stereotype of Asians being smart at math is something I think a lot about. With this, I feel less encouraged to actually seek for help if I don't understand something." While they explained how stereotype threat hindered their ability to seek help, another student's experience aligns with this sentiment and additionally speaks out against the well-known narrative:

> As an Asian student, people will infer that I am good at Math. So a lot of them are shocked when they [find out] I need help. Actually, I do need help and not every Asian is good at Math.

In a society that has accepted broad, over-generalized narratives about how groups of people learn and experience mathematics by the way they look, act, talk, or behave, these stories demonstrate the personal, affective impact on so many P2C2 students across the US.

Underestimating Black and African American Students

Black and African American students share the burden of prejudice and implicit bias, but in a qualitatively different way than the APIDA students that responded to this survey. This analysis uncovers multiple themes across the students who identified as Black and African American, who were treated as a single group in this analysis in order to amplify their unique voices and develop their story as respectfully and thoroughly as possible.

The first theme that was a glaring difference for Black and African American students was a severe lack of representation in higher education, specifically in their calculus classes. NameB1 says, "not having enough African Americans students in my upper division classes with me is very saddening and discouraging!" While other students shared similar frustration, NameB2 added to the detail of how this isolation can affect them in a social dimension and, in turn, affect their ability to succeed in mathematics:

> As a black person at [mostly white institution], there is often a feeling of isolation. This leads to less opportunities to work in a group setting on math problems outside of the classrooms.

Along with under-representation, students also expressed that they felt significant prejudice in mathematical contexts. One student explains that, “being black, people automatically underestimate me”, and another student follows with, “sometimes I feel as if being black and go to use the resources such as the [Math and Sciences Resource Center] have limitation with how people look at me when I ask and need extra help.” So in these cases, the actions and reactions of how others treat these students make them feel as though they are believed to have some lack of ability to do or understand mathematics. These feelings can build over time and translate into pressure to succeed, such as when this student explains, “being African-American I can feel pressure to do good to prove a point.” Another student explains that, “being an African-American student in the classroom often times I feel pressure being one of the only student or few students in the classroom, leading me to take a more regressive role in the course dialog and overall participation.” Feeling pressure to prove a point could add a cognitive load onto these students that is not present for some of their peers, and is compounded by their intersecting identities such as this young African American female:

> As an African American female, it can be uncomfortable entering spaces as the double minority: being one of few African American students in the [calculus] class as well as being an African American woman. Attempting to receive help often ends with a continued confusion of the material and an added question of if the wasted time was due to my lack of ability to grasp the material, the helper’s (whether it be the instructor or recitation instructor) lack of teaching skills to properly help, or if their inability to help my understanding is linked with a sexist or racist issue.

This response indicates an effect that is unique to her experience in this dataset: any interaction with a teacher or recitation instructor that transpires in an unsatisfactory way always has the possibility of racism or sexism as some part of the cause, even if that was not the case. After so many inappropriate or negative interactions over a lifetime of schooling, black students may be faced with the need to question the intent and the reality of their experiences with people in positions of power and positions to help gain access and opportunity for succeeding in mathematics. This student’s story also exposes the dynamic landscape students must traverse while negotiating multiple intersecting marginalized identities, such as women of color.

Mathematical Inferiority of the “Lesser Sex”

Women who responded to this survey also shared deep and sensitive experiences as undergraduate mathematics students in a male-dominated space. Some of them feel intimidated from the lack of representation and mentioned in several ways that the dominance of male presence in their class affected the way they learn or do mathematics. This analysis revealed that not only was the presence of significantly more men in their math class problematic in terms of lack of representation, but also how some men act towards women in math classes. One student explains that, “being a woman in mathematics courses people often underestimate your abilities and belittle your accomplishments”. This quote opens the door to an unfortunate truth of the disparity between how women and men experience mathematics courses so differently: women are assumed to be inferior to men in mathematics. Another quote adds to the frustration these women feel from their experience in math courses: “I am a woman in STEM, so often my fellow male students will treat me like I’m stupid (even when I help them time and time again with homework).” So even when these women demonstrate their capability their peers, they still must fight against prejudice of their mathematical ability. Women respondents also shared experiences of rude behavior that was unique to their plight in this analysis, such as several women corroborating the sentiments that “sometimes guys just talk over me when I’m trying to explain a problem because I’m a woman”, and that, “… the men in my recitation class tend to overlook me because of my gender.”

Women respondents also find themselves experiencing the compounding effects of identities that intersect multiple marginalized groups, such as a Black woman who felt that the double minority status of their identity has them notice that, “people tend to be shocked when I frequently answer questions during lecture.” The underestimation and prejudice that Black students experienced in this analysis stacked on top of the historic marginalization of women in the sciences creates a critically problematic experience in their mathematics courses. Unfortunately, these issues do not begin or end in the mathematics classroom. Rather, students come from backgrounds and educational experiences that shape the way they perceive the world and their success relative to others: “Being a girl and a minority who attended public school has been a detriment and given me a disadvantage compared to a lot of other people.” While this could be perceived as a dig at public education in general, it could also reflect injustices and inequities that plague the US through the defunding of public education, gerrymandered school districts, and a general disregard for schools that need the most support.

The extra pressure stereotyped groups feel to perform in a certain way does not escape women in mathematics. In fact, the results of this analysis show that women were affected by stereotype threat in a different way than other groups. While other students mentioned proving themselves, women were the only group that stated in multiple occasions the idea that the general belief of the inferiority of women in STEM motivated them in some way to become advocates for other women. This sentiment is echoed in this student’s experience, where she not only wants to meet expectations, but break them and show the next generation that they should not fit the mold:

> I believe that being female in a male dominant field has made me want to succeed more than anything in mathematics. Not only that, though, I want to be at the top of my class just to prove that women are capable. I want to inspire other girls to realize that math is fun and easy. Math should not be something to be intimidated by.

Advocacy for equitable experiences in mathematics and motivation to succeed is important, but the stereotype threat that women experience still inhibits them from performing the way they should, having their contributions taken seriously, and being perceived as equals in mathematical spaces. So, while a positive motivation to break the cycle of problematic narratives can be beneficial, women are still greatly affected by stereotype threat in the mathematics classroom.

Discussion

The results indicate that dozens of minoritized, stereotyped students experienced struggles related to stereotype threat, but that different groups experienced stereotype threat in qualitatively different ways. All three large demographic groups that emerged through analysis (APIDA students, Black students, and Women) expressed their frustration with extra pressure to perform a certain way, encountered prejudice of their mathematical abilities based on race or sex, and experienced negative interactions with others in the mathematical community. However, the way that identities were tied to specific ways of experiencing these hardships suggests that stereotype threat is not the same for every stereotyped group. The thread of continuity through all these intersecting, dynamic, and unique experiences is the unseen psychological, physiological, and cognitive battles some students are constantly fighting behind the typical outward-facing roles and identities they assume when in school. While there is no realistic way of separating identity from the dynamically intersectional social sphere that juggles power, rightful presence, and proving self-worth, beginning to form an understanding of the nuances of each groups' unique experiences empowers and emboldens the identities of students who need the most support.

Unpack Student Identity-Related Experiences

It is important to note the fluidity and dynamic nature of identity, especially when studying experiences in mathematics education directly to identity. For example, students have the agency to reject false narratives about their own abilities in mathematics. However, their identity is also shaped by the way people think and talk about them, informed in some degree by their implicit biases or perhaps belief in a narrative about innate mathematical ability. As the results of this study show, these reflexive influences appear to have a particular significance in the mathematics classroom due to the relevance of mathematics to the "Asians are good at math" narrative.

Black and African American students related their identities directly to how their abilities in mathematics are perceived by others. They also affiliated who they are by birth in relation to mathematics, which is certainly related to how narratives around mathematical ability are constructed and perpetuated. Through visible physical features such as dark skin or high cheek bones, people will utilize these features to inform themselves about the identity of a student. The identity of the students in this study appears to be what is directly tied to the shaping of narratives and beliefs that have such an incredible impact on experience in mathematics courses. Black students are recognized as Black, and their abilities in mathematics are scrutinized, connecting their identity to their ability in mathematics. How other people talk about their students and how other students act and talk about each other feed into an existing narrative that Black students are aware of and actively fighting against throughout their mathematical career.

Throughout this analysis, women respondents often referred to the position of power that their male counterparts unrightfully place themselves in. This identity is shaped through an othering of women in science and mathematics, historically bestowing the majority perspective and representation to the men in the class. The responses from women in this study indicate that men are known to disregard contributions from women, believe their mathematical abilities to be inferior to their women counterparts, and even talk over them in an acute demonstration of superiority. Women have long been marginalized in this hierarchical manner regarding ability, and this extends into the mathematics classroom as well.

Conclusion

As long as the imperfect, racist, classist, sexist, ableist society that shapes the people and policies that run US institutions remain, they will continue to be a detriment to the historically oppressed identities within US higher education. Furthermore, as long as there are narratives that tie broad-sweeping identities to specific abilities in science and mathematics, some students will be fighting an internal battle of not living up to certain ways of being or performing while in a mathematics classroom. This constant, yet often silent, battle is part of what shapes the experiences of so many students in US undergraduate mathematics courses today. As I have shown in this paper, the subtle ways in which different groups experience stereotype threat differently can inform instructors and fellow students how to address these stereotyped students' specific needs.

Honorable Mention

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On the Composition of Even and Odd functions

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We attend to the composition of even and odd functions, as featured in imagined dialogues between a teacher and students, composed by sixteen teachers in a professional development program. Data were analyzed as aimed at addressing students' intellectual needs, with particular attention to the need for causality and the need for certainty. The results indicate that participants bring into account a myriad of explanations that either complement or replace algebraic definitions of even and odd functions. We provide possible explanations for such a phenomenon in the discussion of the findings.

Keywords: even and odd functions, intellectual needs, visualization, teacher education

The concept of a function is central in mathematics and is central in mathematics education research. Given the breadth of the function concept, researchers focused on particular families of functions, such as linear (e.g., Glen & Zazkis, 2020; Knuth, 2000; Moschkovich, Schoenfeld, & Arcavi, 1993), quadratic (e.g., Zaslavsky, 1997), trigonometric (e.g., Weber, 2005) or exponential (e.g., Confrey, & Smith, 1995). Our research also attends to a particular "cross-cutting family", that of even and odd functions, which are found within polynomial, rational or trigonometric functions, among others.

Brief Background

Our study focuses on the composition of even and odd functions. We are not aware of any research that attends to this explicit focus. As such, in this section we turn to prior research on the related topics: (a) function composition and (b) even and odd functions.

On Composition of Functions

Function composition is one of the standard topics in the undergraduate mathematics curriculum that causes great difficulty for students (Ayers, Davis, Dubinsky, & Lewin, 1988). A possible source of this difficulty is that composing functions necessitates the switch between the process and object views of a function. In order to form a composition $f \circ g$, one needs to start with two mental objects for f and g and to call one process at a time. Then "to form a new mental process that consists of first performing the process of g, generating a result, and then performing the process of f on that result. The new process is interiorized, and the resulting mental representation is encapsulated as the new function $f \circ g$" (p. 247). Ayers et al. (1988) suggested that programming experiences help students construct the concept of function composition as they necessitated operating on functions as cognitive objects. This suggestion is echoed by Vidakovic (1996), who argued that a computer environment might enhance students' ability to understand the concepts of function composition and inverse function.

Meel's (2003) study found that prospective teachers had difficulty establishing the composition of two functions – more than half of the participants treated the composition as a product. In the study of Lucus (2006), teachers were asked to describe the main ideas involved when they teach composition of functions. The participants focused on "how to do this, how to substitute" (p. 101), omitted the definition of function composition, and did not attend to the domains of functions to be composed. Consequently, the participants did not complete the task of composing functions correctly when the range of one function did not correspond to the domain of another.

Steketee and Scher (2012) advocated for the use of multiple representations in the teaching of composition of functions. They suggested that such an approach “can help students generalize the concept of composition, overcome the tendency to tie function concepts to a single representation, and develop a more robust and mathematically rigorous understanding of the topic” (p. 267).

On Even and Odd Functions

A function $f(x)$ is defined to be even when $f(-x) = f(x)$ for all x in the domain; and $f(x)$ is odd when $f(-x) = -f(x)$ for all x in the domain. These definitions imply that the domain of even and odd functions is symmetrical around the origin. Graphically, even functions have a reflectional symmetry with respect to the y-axis and odd functions have a 180-degree rotational symmetry with respect to the origin. Given the analytic and graphical formulations, the topic of even and odd functions may serve as a “conceptual intersection between symmetry, which is encountered most often in visual contexts such as geometry, and functions, which are often explored through analytic means” (Zazkis, 2014, p. 32).

Even and odd functions are important in many areas of mathematical analysis, particularly in studies of power series and Fourier series. The literature on the teaching and learning of even and odd functions, however, is limited. Rasslan and Vinner (1997) found that when the concept of even and odd functions was introduced by the power function $f(x) = x^n$ where n is a natural number, students related the concept of even and odd function with the even and odd exponent of a polynomial function. In fact, if a polynomial or rational function has only odd exponents, then it is odd (for example, $f(x) = x^5 + x^3$); and if it has only even exponents, then it is even (for example, $f(x) = x^6 + x^2 - x^{-2}$). However, an exclusive reliance on exponents is unhelpful when a combination of even and odd exponents is present.

Sinitsky, Leikin, and Zazkis (2011) noted that the terminology of even and odd for functions could be related to the Maclaurin series expansion, where for even functions it consists of only even powers of the variable, and similarly for odd functions is consists only of odd powers of the variable. The authors also suggested that students’ difficulties with even and odd functions may come from their former conceptualization of even and odd numbers, as students tended to borrow and transfer the properties of even or odd numbers to even or odd function.

In addressing arithmetic operations with even and odd functions, Sinitsky et al. (2011) suggested that “Exploring the composition of even and odd functions is an appropriate task for students” (p. 34). The Task developed in our study explicitly focuses on the composition of even and odd functions, starting with particular examples of student interpretations. In particular, we are interested in how teachers determine the evenness and oddness of a function which is a composition of even and odd functions, and how they guide students in this endeavor.

Theoretical Underpinnings: Intellectual Needs

Every instance of mathematical knowledge, explicitly or implicitly, serves a purpose. However, as Harel (2010, 2008) argued, the purpose of introducing a new fragment of mathematical knowledge is gravely overlooked in mathematics curricula at all grade levels. Important questions, such as how to intellectually necessitate the transition from trial and error to analytic and abstract thinking, and how to intellectually necessitate the transition from empirical proof scheme to deductive proof, are rarely addressed in the classroom.

Intellectual need refers to the perturbational stage in the process of justifying how and why a particular piece of knowledge came into being (Harel, 2013). The notion of *intellectual need* consists of the following five categories – certainty, causality, computation, communication and structure – of which we focus here on the first two.

- *The need for certainty* refers to the need to prove, to remove doubts. It is the need to determine whether an assertion is a fact or a conjecture.
- *The need for causality* refers to one's desire to explain the causes of a phenomenon, to understand what makes a phenomenon the way it is.

"Humans' instinctual desire to explain phenomena in their environments serves as a cognitive primitive to mathematical justification" (Harel, 2013, p. 126). In the history of mathematics, however, a long debate during the sixteenth and seventeenth centuries focused on the distinction between achieving *certainty* and finding *causality* in mathematics. Some mathematical solutions and proofs are non-causal – they provide certainty but do not identify a cause for the observed phenomenon. Therefore, they offer little insight into why a result is true. Proofs by contradiction, for example, show that a given assumption leads to an absurdity, yet do not explain the causal relationship between the premise and conclusion (Harel, 2013).

Moreover, the empirical approach to proof, that is, the use of multiple examples to achieve certainty, often leaves the causes unexplained. In fact, Harel (2013) argues that the transition from undesirable proof schemes, especially the empirical proof schemes, to deductive proof schemes is dependent upon students' attention shift from certainty to cause.

In our study, we address the following research questions:

1) In what ways do teachers explain the behavior of composition of even and odd functions?

2) How can teachers' explanations related to composite functions be interpreted through the lens of *intellectual needs*?

The Study

Participants and Setting

Sixteen practicing secondary mathematics teachers participated in this study. At the time of data collection, they were enrolled in a professional development program in mathematics education. Their mathematics background and experience varied significantly, but all held degrees in Mathematics or Science and had at least three years of teaching experience. The particular course aimed at strengthening the participants' mathematical knowledge and investigation skills while focusing on topics that are not usually attended to in school, but which do not require mathematical knowledge beyond school curriculum.

The topic of even and odd functions was one such topic chosen for exploration. At a class meeting the participants were either reminded of or recalled the definitions of even and odd functions from their undergraduate studies and connected these to the graphical representation of functions. The participants revisited various examples, and discussed properties in discord with expectations carried forward from experience with even and odd numbers. For example, the sum of two odd functions is an odd function, and the sum of an odd function and an even function is neither odd nor even. Following in-class discussion, the participants were presented with the task related to function composition, described in the next section.

The Task, the Data and Data Analysis

The Task presented below belongs to the genre of "scripting tasks" developed in mathematics education to explore and strengthen teacher knowledge while considering instructional situations. Initially, scriptwriting was introduced in mathematics teacher education as a lesson play, a task in which participants script interaction between an imaginary teacher-character and student-character(s) (Zazkis, Liljedahl, & Sinclair, 2009; Zazkis, Sinclair, Liljedahl, 2013). Juxtaposed to a classical lesson plan describing merely content and activities, the lesson play reveals how a teaching-learning interaction unfolds. In later research, the idea of a lesson play was extended to an activity of writing an imaginary dialog that is not necessarily restricted to a lesson, referred to as scriptwriting. When used in teacher education, scriptwriting is a tool related to "approximations of practice" (Grossman, Hammerness, & McDonald, 2009), which "include opportunities to rehearse and enact discrete components of complex practice in settings of reduced complexity" (p. 283).

Prior research that analyzed scripting tasks described multiple affordances of scripting tasks for teachers, teacher educators, and researchers. In particular, as teachers script-writers imagine themselves in a role of a teacher, it highlights how they imagine the unfolding of a lesson or of any interaction with students (e.g., Marmur & Zazkis, 2018).

The following prompt was developed as the beginning of an instructional dialogue. The participants were asked to continue the dialogue addressing the student-characters' suggestions.

Teacher:	Is $f(x) = \sin(\frac{1}{x^2})$ even or odd?
Alex:	I know $f(x) = \sin(x)$ is an odd function. Therefore, for any function $g(x)$, $\sin(g(x))$ is an odd function.
Brian:	I totally agree with Alex. With the same logic, $\lvert sin(x) \rvert$ is an even function because $\lvert x \rvert$ is an even function.
Teacher:	Let's talk about these ideas....

Figure 1. Prompt for the Task

In addition to continuing the dialogue, the participants were asked to include commentary on their chosen approaches, addressing mathematics and pedagogy involved in their composed scripts, and reflect on their experience of completing the task.

The responses of the teacher-character in a scripted dialogue shed light on a teacher-scriptwriter's mathematical and pedagogical knowledge. In particular, the prompt includes one correct claim ($\lvert\sin(x)\rvert$ is an even function) and one incorrect claim (for any function $g(x)$, $\sin\big(g(x)\big)$ is an odd function). Both claims are based on the same reasoning, extrapolating from the oddness of $f(x) = \sin(x)$ and evenness of $f(x) = \lvert x \rvert$. However, this faulty reasoning results in one correct and one incorrect conclusion. We were interested in how the scriptwriters would explain this unexpected result in the voices of their characters.

The scripts for the dialogue between the teacher- and student-characters, along with accompanying commentary comprise the data for our study. The claims provided by a teacher-character or by one of the student-characters for evenness or oddness of a composition of even and odd functions, served as a unit of analysis. As the writers attributed the claims to either student-characters or the teacher-character, we considered the teacher's claims as well as claims provided by students accepted by a teacher in the script as reflecting the scriptwriter's choices for what is an acceptable explanation.

Having repeatedly read the data, we compiled all the data excerpts in which claims of interlocutors recognized functions as being either even or odd and attempted to justify the observation. We then considered what students' intellectual needs were aimed to be addressed by the provided examples and explanations.

Results and Analysis

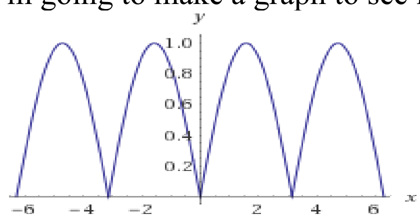
As mentioned, we focused on claims that attend to evenness or oddness of a function composition. We identified several clusters of explanations by the main focus: (1) algebraic manipulation vs. graphical representation, (2) consideration of dominance, and (3) consideration of examples. While causality was our main focus initially, we also considered whether the provided approach attended to other intellectual needs, in particular the need for certainty. We briefly exemplify below clusters (1) and (2).

Algebraic Manipulation vs. Graphical Representation

Evenness or oddness of a function can be determined by considering whether it satisfies the algebraic definition. However, the teachers in scripts did not guide student-characters towards the algebraic determination. Even when the algebraic justification was provided, alternatives were sought. Excerpt A exemplifies this occurrence.

A.1 *Jess:* The inside function is $\sin(x)$ and the outside function is $|x|$. If we test the input $-x$ into the inside function, we get $-\sin(x)$. But taking the absolute value of that gives us $\sin(x)$. Therefore $|\sin(x)|$ is even, just like you said, Brian.

A.2 *Sam*: That makes sense. I'm going to make a graph to see if you're right.



A.3 *Alex*: That is definitely even.

A.4 *Teacher*: Excellent. What do we conclude?

In Excerpt A, Jess proved that $|\sin(x)|$ is an even function algebraically [A.1]. In fact, she not only connected the definition of an even function to the given function, but she also explained how the input $(-x)$ affects the consideration of evenness. We note that Jess' claim was an attempt to address the need for causality. Nevertheless, Sam suggested to make a graph [A.2] and check if the graph is symmetrical about the y-axis. That is, the algebraic proof has not addressed the need for certainty about what has been proven, but the graph, which is "definitely even" [A.3] by observation, satisfied the certainty. Alex's conclusion was implicitly reinforced by the teacher [A.4]. While checking answers was consistently encouraged by teachers, our interpretation of the provided conclusion is that algebraic justification was insufficient in drawing the desired conclusion.

Considerations of "Dominance"

Determining the evenness or oddness of $f(x) = \sin\left(\frac{1}{x^2}\right)$ – a function that is a composition of even and odd functions – pointed to an interesting general question, which function is "dominant" in determining the evenness or oddness of the composite function

The metaphorical reference to "dominance" appeared in discussions in several scripts. However, mathematically, using the language of dominance, evenness is the dominant factor in specifying the evenness or oddness of function composed of even and odd functions. Furthermore, the outcome does not depend on the order in which the functions are composed. This explains why Brian's claim was correct – the function that has been specified as the dominant factor is even, regardless of its order in the composition. However, the similarity of these two claims (focusing on the outsider function) could be very confusing. The situation described in the task invited the scriptwriters to address students' reasoning, regardless of the correctness of their conclusion.

The exchange in Excerpt B follows the teacher's invitation to consider composite functions of even and odd functions.

B.1 *Student 1*: How about $g(x) = (\cos(x))^3 - \cos(x)$?
B.2 *Teacher:* Okay. If you test $g(-x)$ what do you get?
B.3 *Student 1*: $g(-x) = (\cos(-x))^3 - \cos(-x) = (\cos(x))^3 - \cos(x) = g(x)$
B.4 *Student 2*: So, $g(x)$ is even!
B.5 *Alex*: Yes, because an even function inside any other function still is even.
B.6 *Teacher*: Very good. Now we still must discuss Brian's claim that $|\sin(x)|$ is even.
B.7 *Brian*: No, I don't think that anymore! It's odd because $\sin(x)$ is odd. It's the inside function that matters.

Following the teachers' invitation to consider $g(-x)$ [B.2] led to the conclusion that the suggested composite function was even [B.4]. However, Brian was "stuck" in his thinking that the *order* of composition is the factor that determines parity, so he replaced his initial idea, at least temporarily, with "it's the inside function that matters" [B.7]. As such, concluding from the example, the dominance in distinguishing parity of the composite functions was assigned to the "inside function". The implied dominance assured causality of the conclusion. Eventually, Brian's conjecture was refuted by considering the graph, which assured certainty and further causality was not sought.

Some script-writers used the language of dominance with examples from genetics to focus on even functions as dominant factor. This is exemplified in Excerpt C.

C.1 *Alex:* Look at Teacher's example, $f(x) = \sin(\frac{1}{x^2})$. It's a composite function made of an even function x^2, inside an odd function $\sin(x)$. Yet, when I graph the entire composite function it is even.
C.2 *Brian:* Well maybe, even functions are dominant functions within composite functions?
C.3 *Teacher:* What do you mean?
C.4 *Brian:* Well in biology we learned about dominant and recessive genes, where the dominant gene always overrides the recessive gene. Maybe even functions within a composite function are more powerful over odd functions. So as long as there is an even function involved, the resulting composite function would be even. My example from earlier, $|\sin(x)|$, would follow this.

Brian drew the analogy between the role of even functions in composite functions and the role of dominant genes [C.4]. In a way, it is the dominance of even function that "overrides" an odd function in the composite function. The analogy appeared to address the need for causality. As the script continued, a variety of examples was considered and the dominant role of even functions in a composition with odd functions was confirmed.

Discussion

In this study we explored how secondary mathematics teachers explain the behavior of a composition of even and odd functions. The teachers responded to a task in which they were invited to craft a script for an imaginary dialogue between a teacher and students, discussing student ideas about the parity of a composite function. We focused on how the scriptwriters' explanations can be interpreted through the lens of *intellectual need* (Harel, 2013).

The results indicate the prevailing attention of the scriptwriters to the graphs of composite functions, which appeared to address *the need for certainty*. *The need for causality* was addressed by demonstrating how switching signs of the input affects the result of the composition in considering the algebraic definitions of even and odd functions. However, algebraic considerations were consistently supplemented by attention to graphs. In addition, causality was sought by considering what function is "dominant" in the composition.

Several decades ago, Artigue's (1992) and Hitt's studies (1998) showed that both mathematics students and future mathematics teachers tended to avoid the graphical representation of the concept of function while preferring the algebraic expression. In particular, researchers noted not only the preference towards algebraic methods, but also students' "reluctance to visualize" (e.g., Eisenberg & Dreyfus, 1991, Lowrie, 2000). However, in a more recent study, Zazkis (2014) noted that "the script-writers showed evidence of being comfortable with both analytic and graphical modes of representing odd and even functions, drawing appropriate connections between the two" (p. 42).

The results of our analysis appear to be in discord with prior research. The participants, in the voices of their characters, who used the algebraic approach initially, attended to the definitions of even and odd functions but avoided claiming definite conclusions. They sought confirmation by considering graphs. It appeared that algebraic manipulation was perceived as unreliable; a confident conclusion, appealing to the *need for certainty,* was derived only by attending to graphical representation. The increasing availability of digital graphing provides easily accessible demonstrations that substitute the reliance on formalism. The ease of access and immediate availability of a graph are the reason for the participants' preference and dependence on the visual.

Conclusion

Our study contributes to research on learning and teaching functions, focusing on the unexplored niche of even and odd functions and their composition. The behaviour of a composite function appeared unexpected to many of the participants, yet they sought and employed various approaches to verify and explain the surprising result.

When considering a real-valued function, we attend to the correspondence that connects x and y as well as the domain for which the function is defined, though the name of a function is often used as a shortcut. For example, students often claimed that "the sine function is odd" or "sin (anything) is odd". In such claims, which we consider incomplete rather than incorrect, the implied interpretation attends to the function $f(x) = \sin(x)$ where the domain is the real numbers, $\mathbb{R}$. However, the name "sine" does not apply to one particular function, but to a family of functions, given that the domain can be any subset of $\mathbb{R}$. While some features of $f(x) = \sin(x)$, such as periodicity or range, are applicable to a wide range of functions in the sine family, other features are not preserved. For example, while $f(x) = \sin(x)$ is an odd function and $f(x) = \sin(x^2)$ is an even function, $f(x) = \sin(x)$ where $0 < x < \pi/3$ is a function that is neither even nor odd. Considering the function name without attention to the argument or the domain points to a reduced abstraction level which could be a source of errors.

Our results point to the need to re-examine the role of visualization, in particular graphs of functions, in considering function properties. Additionally, we call instructors' attention to the potential dangers in referring to functions exclusively by their name. The sine function, for example, is odd or even? Most learners will likely answer that the function is odd, while such a question cannot be answered without specifying the argument and the domain.

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Moving Beyond Solving Equations: Characterizing Elementary Pre-service Teachers' Development of Algebraic Reasoning

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Although elementary pre-service teachers (EPSTs) often have experience with high school algebra topics, early algebra (or algebra designed for elementary learners, commonly involving story problems and diagrams) may be novel to them. In this study, we seek to examine EPSTs' initial algebraic reasoning and shifts in that reasoning after completing a college course in early algebra. Specifically, we examine their algebraic reasoning related to linear equations, story problems, and diagrams by analyzing three aligned problems from a pre- and post-test. We found that, although EPSTs' initial strategies with linear equations were deductive, their work with story problems and diagrams was not. After the course, however, deductive strategies were more commonly identified across all representations. We discuss the implications for the design of mathematical experiences to support EPSTs' algebraic reasoning across representations.

Keywords: Algebraic reasoning, Elementary pre-service teachers, Early algebra, Diagrams, Deductive reasoning

Algebra knowledge and algebraic thinking are critical for success in calculus and beyond. However, for many students, high school algebra is where their mathematics development falters or even ends (Kaput, 1998; Loveless, 2013). Moreover, difficulties with algebra have historically been more pronounced among students of color and other marginalized populations (Gamoran & Hannigan, 2000; Matthews & Fuchs, 2020; Moses & Cobb, 2001). Introducing algebra in elementary grades has been theorized as an approach that will better prepare students for high school algebra (e.g., Carpenter & Levi, 2000). However, those advocating for this approach emphasize that algebraic reasoning in the elementary grades should be fundamentally different from algebra in high school (Carraher et al., 2008). To distinguish algebraic reasoning in elementary grades from algebra in high school, we use the label *early algebra* (e.g., Blanton et al., 2018).

Despite early algebra's potential to improve student outcomes, there currently is a lack of systematic development of algebra ideas in the elementary grades (Blanton et al., 2019). Preparing teachers to teach early algebra could help address this issue. Given the recency of the early algebra movement, elementary teachers and elementary pre-service teachers (EPSTs) often have not had experiences with early algebra in their own schooling (Hohensee, 2017), which may complicate their teaching of these topics to elementary students. We conjecture that EPSTs' required undergraduate teacher education mathematics content courses can provide opportunities to support them in developing their algebraic reasoning and early algebra meanings, thereby potentially increasing the traction of early algebra in schools more widely. In this report, we advance toward this goal by characterizing EPSTs' algebraic reasoning with linear equations, story problems, and diagrams prior to and after experiencing such an undergraduate mathematics content course focused on topics of early algebra.

Theoretical Framework: Algebraic Reasoning and Early Algebra

Researchers (e.g., Blanton et al., 2018; Kaput, 2008; Stephens et al., 2017) have identified the following three broad concepts as being fundamental parts of algebraic reasoning: (1) generalized arithmetic; (2) equivalence, expressions, equations, and inequalities; and (3) functional thinking, which involves representing relationships between two quantities. These are concepts that are seen as cutting across high school and early algebra.

Importantly, early algebra researchers view algebraic reasoning as not dependent upon a representational modality; rather, generalized arithmetic, equivalence, and functional relationships can occur in a variety of representational forms. For this reason, while high school algebra often emphasizes symbolic representations (like *equations*), early algebra can focus on representing the same ideas through *story problems* (Empson et al., 2011) and *diagrams* (Carraher et al., 2006). Although early algebra includes many of the same concepts covered in high school algebra, such as solving for unknowns (e.g., Smith & Thompson, 2008), and exploring variables and functions (Carraher et al., 2008), the algebraic representations used are one of the key distinctions between high school and early algebra.

An additional aspect of mathematical reasoning that played an important role in our study was the extent to which students employed deductive reasoning, part of what Jeannotte and Kieran (2017) conceptualized as the structural aspect of mathematical reasoning. We characterize deductive reasoning as systematic and logical, featuring activity that maintains algebraic equivalence as students solve a problem. For example, and as exemplified in our results, we consider a student removing equivalent quantities in a story problem to determine an unknown amount as a deductive strategy. We characterize non-deductive strategies as those that do not apply a systematic and logical approach to algebraic equivalence. For example, we consider guessing-and-checking strategies to both story problems and algebraic equations as a non-deductive strategy. We differentiated between students' deductive and non-deductive strategies as we addressed our research questions: *What are ways EPSTs reason about linear equations, story problems, and diagrams prior to taking an undergraduate content course on early algebra? What shifts in EPSTs' strategies occur after taking an undergraduate content course focused on their developing algebraic thinking and early algebra meanings?*

Methods

In this section we first describe our participants and the setting of the course. We then describe the three problems we analyze and our data analysis efforts.

Participants and Setting

Participants were EPSTs enrolled in a required undergraduate mathematics content course in a four-year teacher education program at a university in the Mid-Atlantic region of the United States. The participants were sophomores and juniors. We invited all 51 students to participate in a pre- and post-assessment. 45 EPSTs engaged in the pre-test and 28 engaged in the post-test. In the pre-test, we asked participants to provide information about the previous math courses they had taken, with 44 responding to this prompt. All 44 EPSTs indicated they had taken Geometry or Algebra 2. Further, 39 (89%) indicated they had taken Precalculus, 15 (34%) indicated they had taken Calculus, and 21 (48%) indicated they had taken a course in statistics. Hence, the students had varied but numerous experiences with high school algebra content.

The early algebra content course associated with this study was the third of three required semester-long mathematics content courses in the elementary teacher preparation program. The first two courses covered whole numbers, decimals, number operations, rational numbers, ratios, and proportions. The pre-test was conducted at the beginning of the early algebra course (i.e., after participants had passed the first two courses and before instruction began in the third course). The course covered topics including representing unknowns, variables, equations, and functional relationships using discrete, linear, and strip diagrams to represent quantities and relationships between quantities. Students were asked to consider novel ways to represent and reason about situations they had previously represented and reasoned about with algebra symbols. For example, in the early algebra course, we encouraged EPSTs to determine the solutions to story problems by constructing and operating on strip diagrams, rather than writing equations and solving them algebraically. We conducted the post-test at the end of the course.

Data Source

We used the same assessment for the pre- and post-test. The four-problem assessment was administered online through a Desmos activity. For this report, we analyzed three of the problems in alignment with our research questions. To assess students' reasoning about *linear equations*, Problem 1 (P1) asked participants to solve $(3x + 2) + (x + 4) = 5x + 3$. Problem 2 (P2) asked participants to consider the following *story problem*: "Charles buys 6 bags of apples and \$10 worth of oranges. Julien buys 3 boxes of strawberries and \$10 worth of oranges. Charles and Julien spend the same amount of money. What can you say about the cost of a bag of apples and the cost of a box of strawberries?" For this problem, we provided space for students to type a response and/or draw using Desmos's tools, stating, "Feel free to draw a picture and/or type your thinking." Problem 3 (P3) entailed a strip *diagram* representation of a story about two friends who together spent as much money for fruit as a third friend (Figure 1). Although the equation from P1 could apply to this story, we asked participants to write on the provided diagram to solve this problem.

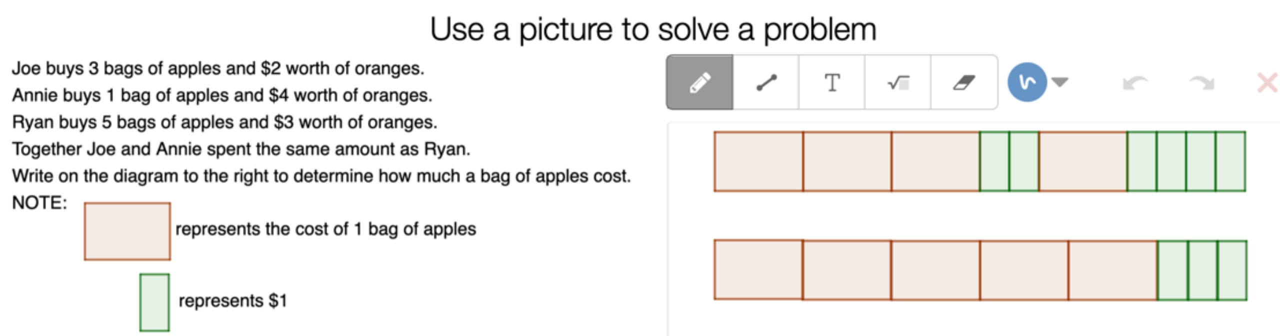

Figure 1. Problem statement and diagram in P3.

Data Analysis

To answer our research questions, we engaged in conventional and directed content analysis (Mayring, 2015). We began by individually reading through EPSTs' responses and making notes about ways they reasoned about equations, story problems, and diagrams within each problem. We then met to compare observations and develop codes that could apply across problems. We used both a priori codes (e.g., deductive reasoning; Jeannotte & Kieran, 2017) and emergent codes (e.g., binomial multiplication), as per Miles et al. (2014). Once we agreed on features for coding, each author independently coded nine pre-test and six post-test responses for each problem. Interrater reliability for codes ranged from 80-100%. After reliability was established and consensus reached for all responses in the samples, one author coded the remaining responses for each problem. Difficult decisions were brought to the team for discussion. For brevity's sake, we do not provide complete descriptions of the codes, instead presenting them as needed to help the reader understand the results.

Results

In the sections that follow, we present results that respond to our research questions, 1) characterizing the ways EPSTs reason about linear equations, story problems, and diagrams prior to taking an undergraduate content course on early algebra and 2) characterizing the shifts in EPST reasoning with these representations after taking this course. To do this, we present our analysis of the features of algebraic reasoning in EPST responses to the three questions (P1, P2, and P3) on the pre-test and post-test.

Reasoning with Linear Equations (P1)

Our first finding is that EPSTs largely engaged in deductive strategies with linear equations in their work on P1 both before and after the early algebra course. Specifically, 35 of 45 pre-test responses (78%) and 23 of 28 post-test responses (82%) involved a strategy that maintained algebraic equivalence in the solution process, suggesting that most EPSTs could recognize and symbolically operate on linear equations both prior to and after instruction (see Figure 2, left, for one example). Among the responses that did not apply a deductive strategy with the equation, 7 responses on the pre-test (16%) and 3 responses on the post-test (11%) approached the problem via binomial multiplication instead of a binomial addition (see Figure 2, right, for an example). We interpret the popularity of these two strategies as indicative of the salience of EPSTs' high school algebra experiences coming into the study (e.g., experiences FOIL-ing in high school). It also suggests the stability of these experiences during interventions that targeted early algebra.

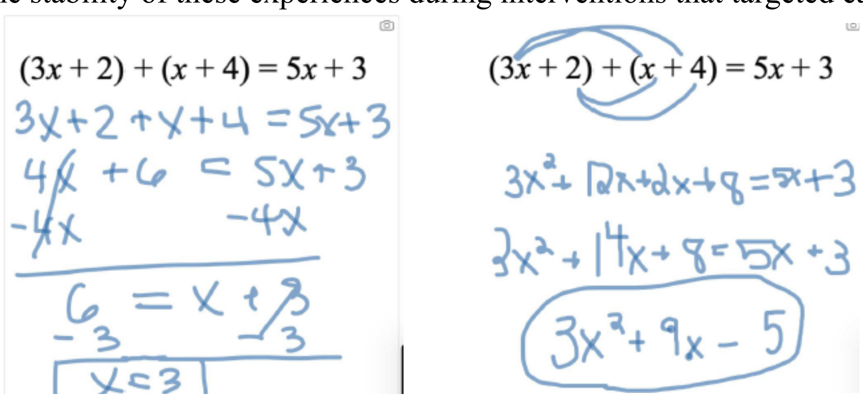

Figure 2. (left) An example of deductive reasoning with a linear equation and (right) an example of interpreting binomial addition as multiplication in a linear equation.

Reasoning with Story Problems (P2)

Whereas EPSTs' strategies with linear equations were consistent and deductive in the pre- and post-tests, EPSTs demonstrated a notable shift in their algebraic reasoning with story problems. Prior to completion of the early algebra course, EPSTs' responses to P2 included 1) infrequent justification of conclusions and 2) frequent exclusively typed or symbolically represented responses. With respect to the first feature, only 8 of the 42 total EPST pre-test responses (19%) included what we coded as justification (a clear, generalized connection between the given information and their conclusion) to support an accurate comparison between the quantities. We interpret justification as central to a logical solution, so this feature also indicates a lack of evidence for EPSTs' deductive strategies with the story problem. With respect to the second feature, Table 1 shows an overview of coded representations. We note 21 of the 42 total pre-test responses (50%) provided no representational support beyond words. Furthermore, consistent with EPSTs' high school algebra experiences, responses on the pre-test that did include a representation most frequently included algebra symbols (13 of 42 responses, 31%).

Table 1. Number of responses coded for each representation to support their solutions to the story problem in P2. (Note: if a student included more than one representation, each was counted independently.)

Representation	Pre-test (n = 42)	Post-test (n = 26)
Equal sized components	2 (5%)	10 (39%)
Non-equal sized components	8 (19%)	8 (31%)
Algebra symbols	13 (31%)	3 (12%)
No representation	21 (50%)	7 (27%)

After EPSTs' completion of the course, responses to the story problem in P2 included 1) frequent justification and 2) frequent diagrams, particularly those with equal sized components. Whereas only 19% of EPSTs responses were coded as providing a justification in the pre-test as described above, we coded 15 of 26 responses (58%) on the post-test as providing a justification. Additionally, 8 of the 26 responses presented diagrams with non-equal sized components (31%, Figure 3, right) reflecting the problem's quantities and an additional 10 featured diagrams with equal sized components (39%, Figure 3, left) reflecting not only the problem's quantities but also their magnitudes and relationships. Taken together, the combined increases in both justification and diagram use after completing the course suggests the impact of deliberate instruction of early algebra topics on algebraic reasoning with deductive features for EPSTs.

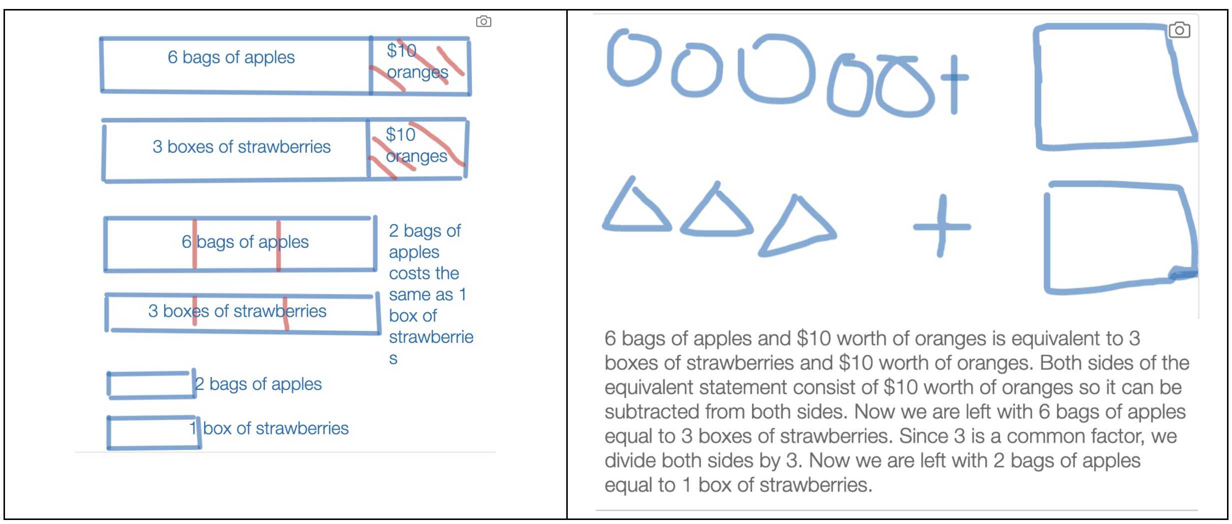

Figure 3. Correct and justified responses involving (left) equal sized components and (right) non-equal sized components.

Reasoning with Strip Diagrams (P3)

We also observed shifts in EPSTs' strategies toward deductive reasoning when working with strip diagrams in responses to P3. Table 2 presents a full summary of these findings. Prior to instruction, we note that EPSTs tended to implement verification strategies (15 out of 40 total responses, 38%) and partitioning strategies (13 out of 40 responses, 33%). Verification strategies involved checking a solution, which could stand alone or be combined with other strategies (see Figure 4, left); partitioning strategies involved breaking apart at least one of the red components without any deductive strategy elements, suggesting measurement or estimation (see Figure 4, center). Furthermore, while five EPSTs (13%) used an equation representing the situation to deductively determine a solution, only one EPST (3%) provided a strategy indicative of deductive reasoning with the diagram itself; such findings are consistent with Hohensee's (2017) finding that EPSTs lack experiences with early algebra in their own school experiences.

Table 2. Students' strategies for P3 when using a diagram to solve a problem. (Note: if a student used more than one strategy, each was counted independently.)

Strategy	Pre-test (n = 40)	Post-test (n = 25)
Verification	15 (38%)	6 (24%)
Partitioning	13 (33%)	3 (4%)
Inaccurate equation	2 (5%)	0 (0%)
Deduce with equation	5 (13%)	4 (16%)
Deduce with diagram	1 (3%)	14 (52%)
No or unclear strategy shown	9 (23%)	3 (12%)

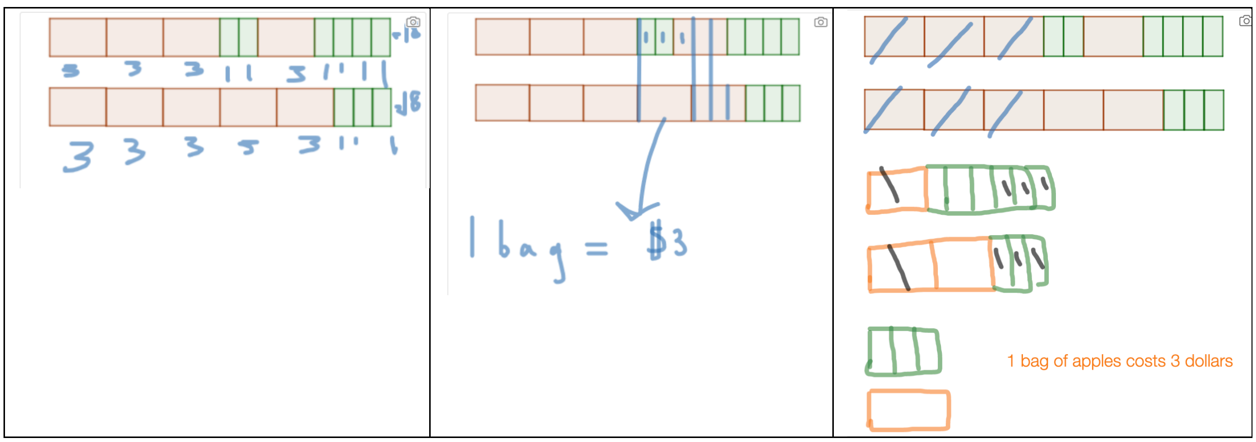

Figure 4. (left) An example of a verification strategy, (center) an example of a partitioning strategy, and (right) an example of a deductive strategy with the diagram.

After instruction, EPSTs' responses indicate a strong shift toward strategies that involved deductive reasoning with the diagram. Deductive reasoning with the diagram showed a logical process to group or eliminate quantities to determine a solution. For example, the student's work in Figure 4 (right) shows first removing 3 bags of apples from each diagram by crossing out three orange rectangles, before re-drawing two new strip diagrams. The student then removed 1 bag of apples by crossing out one orange rectangle and $3 by crossing out three green rectangles. They then re-draw two new strip diagrams representing the cost of 1 bag of apples. Whereas only one student leveraged such a strategy in the pre-test, 14 out of 25 post-test responses (52%) featured this strategy in the post-test.

A Note on Solution Accuracy in P1, P2, P3

While our analysis has demonstrated features of EPSTs' initial algebraic reasoning and shifts in that reasoning after an undergraduate course on early algebra, we also wish to provide additional context about the accuracy of students' solutions across the pre- and post-test. These findings are detailed in Table 3. We highlight two key observations from this table. First, prior to early algebra instruction, the accuracy of EPSTs' conclusions was relatively high (e.g., 80% of pre-test responses in P3 reached the solution of $3). Second, the proportion of EPSTs providing accurate responses on the post-test was slightly higher than on the pre-test (e.g., 88% of post-test responses in P3 reached the solution of $3). This finding suggests that the accuracy of responses remained high and moderately improved alongside the noticeable changes in features of EPSTs' algebraic reasoning with respect to diagrams and story problems early algebra instruction.

Table 3. Number of accurate responses / number of EPST responses (as a percent) to P1, P2, and P3 on pre- and post-test.

Subheading	P1	P2	P3
Pre-Test, Accurate	35 / 45 (78%)	27 / 42 (64%)	32 / 40 (80%)
Post-Test, Accurate	25 / 28 (89%)	21 / 26 (81%)	22 / 25 (88%)

Discussion

Addressing our first research question, we note that prior to taking an undergraduate content course focused on early algebra, EPSTs' reasoning was largely focused on algebraic symbols and manipulations with limited evidence of deductive strategies in story problem contexts and with diagrams. Addressing our second research question, we noticed that justification in responding to story problem contexts (P2) and deductive strategies with the diagram (P3) increased substantially after completing the course. Further, as observed in P2, more students used non-symbolic representations to support their thinking. We hypothesize diagrams may support EPSTs' justifications (and thus, deductive reasoning) in story problems. This is supported by the prevalence of diagrammatic deductive strategies in P3 in the post-test as well, suggesting that the course supported ESPTs' ability to translate their algebraic reasoning to diagram representations. We interpret these findings to suggest positive impacts of the early algebra course on the students' algebraic reasoning across a wider variety of representational modalities.

We note that the students' prior school experiences likely created a ceiling effect in terms of the number of correct responses; as all students had multiple experiences solving equations and story problems, they were able to determine correct solutions prior to instruction. However, we contend the course fostered the EPSTs' early algebra meanings as they developed ways of representing and reasoning that will support their future teaching of elementary students.

Our findings support the claim that, prior to a course on early algebra, many EPSTs may possess understandings of algebra focused on symbols and manipulations that may be challenging to transfer to their future teaching of story problems and diagrams. We also provide evidence to support the claim that undergraduate courses focused on EPSTs developing early algebra meanings can support their content knowledge needed to teach such a course. The findings from the pre- and post-test establish the need for an undergraduate mathematics content course for EPSTs focusing on early algebra. Such interventions are critical if we intend for EPSTs to become teachers who support their future students' success in algebra as part of their pathway to advanced STEM courses and future STEM careers.

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Student Engagement with Interactive Instructional Media in Calculus

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Many topics in calculus require students to reason about situations quantitatively and covariationally. Furthermore, such quantitative constructions are often the foundation of mathematical abstractions that enable students to construct robust and productive understandings of concepts throughout the course. With targeted design, dynamic media and virtual manipulatives can support students exploring the interrelationships of varying quantities and to leverage this reasoning to solve problems that support concept formation. In this paper, we report on student interactions with a variety of curricular resources involving such dynamic and interactive design to promote productive understandings of the central concepts in calculus. Our findings suggest that although strategic use of virtual manipulatives can support students' quantitative reasoning and conceptual development, even highly constrained activities allowed students to focus on irrelevant features and seek memorized procedures. Interviewer guidance was generally critical to the productivity of the students' problem-solving activity and concept development.

Keywords: calculus, virtual manipulatives, formative assessment, quantitative reasoning

Studies have shown that covariational and quantitative reasoning supports students in developing productive understandings of calculus concepts (Thompson & Carlson, 2017; Carlson et al., 2002). However, when confronted with calculus tasks, students may revert to executing routine procedures rather than attending to the quantitative and covariational aspects of the task. Moreover, it can be difficult, if not impossible, for students and instructors to explore the effects of varying quantities in a calculus context using pencil-and-paper constructions or static imagery. We developed a variety of instructional materials—including iClicker activities and accompanying online quizzes—which rely on dynamic imagery, animations, and interactive virtual manipulatives to demonstrate calculus concepts. We describe students' interactions with these resources and the extent to which these resources supported students in developing productive ways of reasoning about the central concepts in calculus.

Background/Literature Review

Quantitative and Covariational Reasoning

Several scholars have demonstrated the affordances of *quantitative reasoning* (Smith & Thompson, 2007; Thompson, 1990, 2011) and *covariational reasoning* (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson, 1994b) for constructing a robust understanding of a variety of ideas in algebra, precalculus, and calculus. Quantitative reasoning is comprised of the mental actions involved in conceptualizing situations in terms of *quantities* and *quantitative relationships*. A *quantity* is an attribute, or quality, of an object that admits a measurement process (Thompson, 1990). Conceiving a process by which one might measure a quantity often involves an operation on two or more previously defined quantities. Such a *quantitative* *operatios* may result in a new quantity, which itself represents the measure of an attribute, may be unitized and partitioned, and related to other quantities. (Thompson, 1990, p. 12).

Covariational reasoning refers to the mental actions involved in coordinating the values of two or more varying quantities while attending to how these values change in relation to each other (Carlson et al., 2002). Productive covariational reasoning entails coupling quantities to form a *multiplicative object* of their measures in variation (Thompson & Carlson, 2017) in which one recognizes that for every possible value that a given quantity can assume, the other quantity also has a value (Saldanha & Thompson, 1998).

Physical and Virtual Manipulatives

Physical and virtual manipulatives have been used in mathematics education to support students' construction of abstract mathematical concepts (Moyer et al., 2002) and to advance their ability to reason quantitatively and covariationally (Thompson, 2002). Moyer et al. (2002) describe a virtual manipulative as a "web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (p. 373). Much of the research literature on the use of virtual manipulatives in particular, and multimedia learning resources generally, is informed by information processing models of cognition and seeks to identify universal principles of their design that contribute to students' encoding and retention of information (Pampel, 2017). We additionally focus on features of the students' engagement with the manipulatives that contributed to their mathematical reasoning (e.g. quantitative and covariational reasoning) or their construction of targeted mathematical meanings (e.g. approximation metaphors for limits, rate as a proportional correspondence between changes in two quantities, etc.).

Methods

The purpose of this study was to examine calculus students' interactions with instructional media including a variety of illustrations, animations, and virtual manipulatives. We conducted semi-structured task-based clinical interviews (Hunting, 1997) with each participant individually via video conferencing software. All students enrolled in Calculus I classes at a large public university in Spring 2021 were recruited for this study, and five students agreed to participate. Of these, two students participated in one interview, one student participated in two interviews, and two students participated in six interviews. Each interview focused on one of five different calculus concepts and was conducted shortly after that concept had been covered in their class.

The first portion of each interview consisted of the student answering multiple choice and numerical response questions designed for use during instruction (i.e., iClicker activities). Some questions were part of a sequence intended to help develop students' understandings of a particular concept, while others were summative in nature. If a student had already seen the activity in their calculus class, they were encouraged to recreate their problem-solving rather than remember which answers were correct from class. Students were asked to explain their answers and show their work.

During the second portion of each interview, the students completed online quizzes based on interactions with a virtual manipulative designed to focus attention on the relevant quantities and quantitative relationships. Students were instructed to complete each question of the quiz as if they would without the interviewer present, pausing after each question for the interviewer to ask them to explain their thinking or decisions. Interviewers focused specifically on students' interactions with the virtual manipulatives, particularly on how students decided to move the manipulatives, how they interpreted the results of their actions, and how they used this to answer the quiz questions. At the end of each activity in the interview, students were asked what they thought the main ideas of the activity were, what they thought they were supposed to learn from the activity, and if they found any aspects of the activity particularly helpful or unhelpful in learning about the topic.

Each author reviewed the video of all interviews and noted instances or absences of the following for each student: quantitative interpretations, strategic decision making, abstraction, and the ability to parse questions. Moreover, each author focused on perceived primary sources from the interview materials which yielded these instances or absences, namely: virtual manipulatives, animations, or representations; previous questions or scaffolding; context and applications; prior knowledge; and textual descriptions and aids for students' parsing questions. In particular, we documented which of these instances yielded a notable change in students' patterns of actions and reasoning throughout the interviews.

Results

Routine engagement

Many of the students initially engaged in most of the iClicker and quiz sequences with the expectation that they should know and implement specific procedures to successfully answer the questions. If students did not immediately know how to answer a question, they often applied whatever strategy or procedure they thought most appropriate so that they might receive partial credit. When the interviewer presented and discussed the solution to an interview problem, students often interpreted the feedback as a minor correction to their initial attempt, which they should remember for future use on similar problems. In these cases, students attributed incorrect responses either to insignificant mistakes or to a momentary inability to recall a correct procedure. In most of these cases, however, the research team's assessment was that the student's difficulty with the problem was due to more significant limitations in their current understanding of the targeted concept. We interpret students' tendencies to associate rehearsed procedures to particular types of tasks and attribution of incorrect responses to lapses in memory or minor procedural errors as evidence that the students were largely not approaching the tasks from a problem-solving perspective necessary to reason quantitatively and thus to develop productive meanings from the activity.

Throughout most of her interviews, Kellen claimed that she was generally able to complete individual steps of calculus problems but "Sometimes in calculus, I have trouble piecing all of the information together." She did not, however, refer to specific concepts or methods she might connect or how that association might inform her interpretation of the problem. In her interview about related rates, Kellen started, "I think I'm supposed to take some type of derivative, and I'm confused as to what part.… I know what it's asking for, but I get a little confused as to how to get there." She often attended to surface-level features of the problem statement to determine the appropriate procedures to execute. For example, when presented with a related rates problem involving a triangle, Kellen said, "I'm assuming I can just do trig, using like the Pythagorean Theorem and trig," without attending to the quantities involved and the nature of variation in this particular situation. For Kellen, the triangle itself served as a cue to apply either trigonometric ratios or the Pythagorean Theorem. Perceiving the context in terms of quantities and quantitative relationships might have enabled Kellen to purposefully construct an equation to express the relationship between the relevant quantities and to recognize the need to implicitly differentiate this equation with respect to elapsed time and manipulate the resulting equation to solve for and evaluate the requested rate of change.

Kellen tended to quickly answer the multiple-choice questions presented during the interviews, doing little or no calculations. When asked to explain her answers, Kellen would often recount her process of eliminating choices she thought were incorrect and selecting what she perceived as the most plausible answer from those that remained. She was also quick to accept explanations for why her answers were incorrect and why the correct answers were indeed correct. For example, in response to an explanation about an iClicker question that she had answered incorrectly, "Estimate $f(2.1)$, assuming that $f(2) = 1$ and $f'(2) = 3$, using the Linear Approximation," she said "I know we have to have a derivative times a difference because we're looking for the area between 2.1 and 2, which is really accurate…. We're not looking for a larger number. We're looking for a smaller approximation." When the interviewer revealed the correct answer, Kellen quickly stated that she accepted the interviewer's claim this answer was correct without being provided an explanation by the interviewer. When asked to then explain the meaning of the correct answer in her own words, she responded by describing the appearance of addition to increase an approximation value, "You would add $f(2)$ to the $f'(2)(2.1 - 2)$ because... so you're looking for, if $f(2)$ is one, you're probably going to need a little bit more than one, $f(2.1)$. So that's why I can see you would add."

Quantitative reasoning

The iClicker and Quiz questions were designed to require students to interact with the associated illustrations, animations, and interactions in ways that focus attention on the relevant quantities and quantitative relationships. In some cases, this design was sufficiently robust that students interacted in the intended ways even when only routinely engaged. In other cases, the interviewer was able to ask guiding questions that prompted such interaction.

The virtual manipulatives did seem to support Kellen in enacting more quantitative reasoning as opposed to a focus on procedures. The quiz over related rates involved two virtual manipulatives. The first context involved a common problem involving a person walking away from a lamp post and the related rates of change in the their distance from the post and length of their shadow. Students could select "Walk" and play an animation of the person waling at a fixed speed or click and drag the figure to manually move it to different locations. In either case, the manipulative illustrated the two relevant similar triangles and a single distance scale from the lamp post, but it did not display a numerical measurement for the individual's position, x, or length of their shadow, s. Before reading the first question, Kellen played the animation, then paused it and manually moved the individual. Kellen then read the first question requesting the relationship between Δx and Δs. Kellen manually moved the figure to 0 meters, and successively moved the figure to the right 2 meters at a time, concluding, "Every time there is a change in x, the change in s is increasing, but Δx and Δs are not equal." She then selected the correct answer choice. Kellen explained that she observed how the shadow changed as she moved the figure, "as he kept going [i.e., as she successively moved the figure to the right by increments of 2], the shadow isn't necessarily the same change every time. There's always an increase in the change every single time." Although her description was not entirely correct, Kellen attended to all of the quantities, changes in quantities, and covariational relationships needed to successfully answer the question and interpret the task situation as intended quantitatively.

Pseudo-empirical abstraction

The iClicker and Quiz questions were designed to require students to reason in ways that promote the desired mathematical abstractions for the lesson. While routine engagement was never sufficient to support such abstraction, when students did engage in problem-solving activity, they did demonstrate aspects of the desired generalized mathematical reasoning.

In a quiz based on a virtual manipulative illustrating quantities involved in a Riemann sum approximation of the energy required to lift a 1 kg mass to geosynchronous orbit, Jarrod was supported in making critical initial pseudo-empirical abstractions of the results of his interactions with the manipulative. This manipulative displayed the quantities in a contextual diagram alongside a graph of force of Earth's gravity on the mass as a function of the distance r from the center of the earth. The manipulative included three sliders in which students could choose the following: the locations of evaluation points on a continuum from a left sum to a right sum, the “number of segments” (subintervals), and the “subinterval to evaluate.” The diagram and graph then illustrated energy and area associated with the corresponding term of the appropriate Riemann sum. Although Jarrod’s class had recently covered Riemann sums, he did not initially recognize the notation used in the quiz and was confused about what the various slider variables represented in the context and the graph. He answered the first two questions based on unit analysis, computing values that would merely have the same units as the quantities requested. When asked to find the force used in the second term of L_{10}, Jarrod did turn to a more meaningful interpretation of the relevant quantities, but pointed to the first two intervals in both the picture and graph and multiplied his change Δr by 2 indicating the combined length of the first two intervals. He divided the value of the energy for only the second interval by this distance to obtain the incorrect force, despite the illustration highlighting the correct distance in both the picture and graph.

When asked to compute M_7, however, Jarrod scrolled the sliders for “number of segments” and “segment to evaluate,” observing their effects in the diagrams. He asked if the question was asking for “all of the [energy components] added up.” Upon confirmation, he scrolled to the first segment and said “I could write this one down,” scrolled to the second, “write this one down and like add it,” then scrolled through the remaining segments writing down the displayed energy values and added them together. He explained this represented “the whole area of all of like the rectangles… It’s the total energy needed to, uh, the total energy needed to get that object into space in that many intervals.” When asked what the individual terms represented, he explained it was just the energy needed for that interval, not cumulative from one to the next, a departure from his initial incorrect interpretation. When asked what proportion of the trip used half of the energy required, he noted that the first term in M_7 was nearly half of the total, and correctly estimated $1/7^{\text{th}}$ of the trip, justifying “as you get further the gravity of earth is gonna affect you less and if you’re close it’s gonna affect you more so it’s going to take more energy to push past it.” Jarrod summarized the purpose of using an integral as,

> I think specifically like with energy and force it's because… if you want to calculate energy using force, the force has to be constant. And so, it makes it where you can make the force constant by dividing it into different intervals. And so you can kind of work like that for like anything if you need to.

When asked how he could make his estimate as accurate as possible, Jarrod indicated a left sum would generate an overestimate and a right sum would generate an underestimate.

> I think midpoint maybe the best just because like there are parts that go over but there's also parts that go under because it's in the middle… the ones that are over and under kind of makeup for themselves. I would bring it to the largest number of segments and I'd set it to a midpoint. For like the like the total energy, I would like add up each segment energy.

Role of the instructor

As noted in the previous sections, the primary challenge to the instructional effectiveness of the iClicker and Quiz sequences was students' disinclination to problem-solving engagement. In many cases the interviewer was able to ask questions that disrupted this routine engagement, resulting in the desired quantitative reasoning and pseudo-empirical abstraction.

When interacting with the quiz and manipulative covering Riemann sums described above, Kellen correctly calculated the distance from the surface of the Earth to the satellite. Proceeding to the next question, "How far is each distance Δr (in m) used in the computation for L_{10}?" Kellen first observed that each segment would have the same length, then attempted to visually estimate the width-measures of the rectangles corresponding to L_{10}. "I'm looking at how the graph is broken up… I just wanted to see if the radius [the variable *r* for the distance from the center of the Earth] was an even number [i.e. an easily determined whole-number] but it appears not to be." The interviewer prompted her to reference her previous answers and attempt another solution method. Kellen sought a formula to answer the question, "So, we know that our energy is equal to the force times the change in distance, which is going to be that Δr in this case." Kellen then focused on the current selected segment and attempted to read the height of the rectangle from the graph to represent the approximate force with which she could divide the energy approximation for that segment to solve for the distance Δr. Again, unable to read the numerical value accurately from the graph, Kellen conceded to being stuck.

After the interviewer reminded her of her previous observation that L_{10} separates the total distance interval into 10 equal parts, she responded "Oh so I could easily use a different part if I wanted to." Kellen then moved the slider to select the second segment, which highlighted a rectangle with a height of 4. "So here, you have an even 4 … so 4 - with an energy of 14.38 mega Joules over 4 is going to give a Δr." Kellen then answered the question after performing a unit conversion.

We note that, in this interaction, the interviewer's two responses merely prompted Kellen to further reflect on what she knew, and to persist in her problem solving. Kellen's responses to the prompts included incorporating a quantitative operation into her activity (deriving the radius change from the equation $E = F \cdot d$ as well as the height-width product constructing each rectangle), and then manipulating the slider to find a rectangle that provided her with a suitable height for her computation. This slider manipulation followed from her observation that each interval had the same size, and so she only needed to find one width (segment length) in order to calculate all of them. We consider this interaction to be akin to a teaching interaction possible in a classroom implementation of this activity, and note that the actions of the interviewer supported Kellen's completion of a solution by building from the reasoning that she had already brought to the task. As such, the actions of the interviewer supported Kellen's engagement in further quantitative reasoning, whereas the task merely provided an opportunity to engage in quantitative reasoning, possibly constraining her activity at best.

Discussion

At the heart of our study is an investigation of the ways that students might bring forth and possibly alter their mental organizations of mathematical activity during their engagement with our tasks, hopefully working to support students' reasoning quantitatively and covariationally. By exploring Kellen's task interpretation, her goal setting process, and the reasoning that she brought forth when solving these tasks, we gain initial insights into the different ways that students might interact with our designed materials. We also use her interactions with the interviewer as an opportunity to discuss the importance of accounting for both student and teacher interactions when designing interventions.

As seen through Kellen's routine engagement, her primary problem-solving approach aligned with test taking strategies (e.g. eliminating wrong answer choices) and attempts to recall particular steps taken in accordance with cues from the problem statement (e.g. presence of a triangle in a problem suggests use of the Pythagorean theorem, related rates problems involve taking derivatives). Accordingly, her engagement with quantities was subservient (at times nonexistent) to her primary routine. This constrained her activity when questions required nontrivial considerations of quantities at play, which Kellen described as an inability to "[piece] all of the information together".

One fundamental goal in the development of these tasks was supporting students' engagement in quantitative reasoning. We note that some tasks, particularly for the virtual manipulatives, corresponded to her engagement with quantities. Despite this, an important result from analyzing Kellen's work is that her activity was indeed constrained in the absence of engagement with quantities, but this constraining itself was insufficient to support her learning of our desired mathematical concepts, or her adoption of particular ways of reasoning. We highlight that the didactic expectations and ways of interacting in class that students bring to bear when solving tasks can fundamentally influence their goal-oriented activity. One manifestation of this for Kellen was her quick verbal acceptance of a stated right answer and its justification from the interviewer with minimal or no evidence of reflection on what contributed to her selecting an incorrect response, including a lack of reflection on ways that the correct answer reflected the coordination of the quantities at play in the task. We point out that this kind of interaction might occur in classroom assessment contexts, where students might reflect little on *why* they got incorrect answers instead focusing on *what* the correct answers were. This is in line with Kellen's routine engagement, as if she thought the activities themselves to operate much like an in-class quiz.

As such, we consider that while tasks can serve to constrain and orient students' activity, teaching interactions can support students' mental activity within the confines of attempting the tasks. Though not an aspect of her routine engagement, Kellen did demonstrate the potential to productively consider quantities, particularly in moments of interaction where the interviewer supported such attention. Though the original goal of the tasks (individually and wholistically) was to support students' engagement with quantities and their learning through abstraction, we found that Kellen's engagements in these mental processes were supported by the interviewer in moments where her activity was constrained by the task. This is perhaps unsurprising, as instructors' teaching actions provide an integral aspect of the interactions within a classroom environment. We find that, for our purposes, identifying this distinction between interactions that constrained Kellen's activity versus interactions that supported her adoption of particular cognitive processes to be important.

With our eyes particularly towards relevant features of design work, we consider it important that future iterations of these materials attend not just student interactions with the task, but also the teacher's role in implementation. More generally, we consider accounting for instructors' views of the tasks, goals for instruction, and means of implementation to be vital components of meaningful design research made evident by our renewed awareness of the idiosyncratic nature of student interactions with tasks through analysis of Kellen's work. Our future iterations will specifically incorporate a greater awareness of the instructors' roles in implementation of these materials.

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Active Learning in the Language Diverse Undergraduate Math Classroom

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Active learning is generally considered to be an equitable teaching approach in math education. More post-secondary math classrooms have adopted active learning pedagogies to improve student learning and promote equity. However, given active learning's emphasis on verbal communication and peer interaction, it is important to consider how bilingual students experience active learning. This study espouses a sociopolitical perspective to examine the experiences of bilingual students in active learning undergraduate math classrooms. Data was collected from interviews with twenty-eight bilingual students for whom English is an additional language. Findings suggest that most students participated less than their peers during group discussions, were often positioned as peripheral members of their groups, and were not able to use their home language as a resource in their learning. Findings suggest the need for more research on implementing active learning so that it is equitable in language diverse classrooms.

Keywords: active learning, group work, language diverse classrooms, equity

Introduction

At the post-secondary level, math classrooms are becoming more reflective of the multicultural landscape that we live in (Durand-Guerrier et al., 2016). Undergraduate math classrooms in the United States are often no longer monolingual spaces that serve predominately native English speakers. It is becoming more common that these classrooms are rich, multilingual spaces where students use different cultural, linguistic, and experiential knowledge to make sense of mathematics. This calls into question whether teaching approaches developed within the context of monolingual classrooms are still meeting the needs of bi/multilingual college math students today.

Active learning is a teaching approach that is also becoming more common in post-secondary math classrooms. Ample amounts of research suggests that active approaches to teaching improve student learning (Freeman et al., 2014). Furthermore, Ballen (2020) discusses the general consensus in the literature that active learning promotes equity in undergraduate STEM classrooms. To illustrate this, Theobald et al. (2020) performed a meta-analysis of the post-secondary literature on active learning. Findings suggest that on average active learning reduces the achievement gap for underrepresented students in STEM courses by 33%. Laursen et al. (2014) demonstrated using data from over 100 undergraduate math courses that active learning in the form of IBL helped "level the playing field" for women. (p. 412). Generalizing from this work, researchers and practitioners often treat active learning as a silver bullet for equity in post-secondary classrooms.

However, some scholars have questioned active learning's impact on the experiences of underrepresented students (Henning et al. 2019; Esmonde, 2009; Takeuchi et al., 2019). At the post-secondary level, Johnson et al. (2020) found that male students outperformed females on a content assessment test when inquiry-oriented instruction was used in abstract algebra. Henning et al.'s (2019) work suggests that during peer engagement there may be a "heightened awareness of social identities" which can impact underrepresented students' participation, STEM self-efficacy, and sense of belonging (p. 1).

Active learning emphasizes communicating mathematically and participating in the classroom (Moschkovich, 2002). Students in active learning classrooms are also expected to regularly interact with their peers and make sense of mathematics collectively. Given this emphasis on communication and peer interaction, there is a need to better understand active learning's impact on classroom equity in language diverse spaces.

At the undergraduate level, a lack of research exists exploring this need. However, at the K-12 level, math education scholars have advocated for active learning – specifically group learning – as a tool for supporting students with diverse language needs (Boaler & Stapes, 2008; Gutiérrez, 2002; Cohen & Lotan, 2014). For example, Complex Instruction suggests that placing students in mixed language-ability groups supports the language development of emerging bilingual students through interactions with native speakers (Cohen & Lotan, 2014). Gutiérrez (2002) also demonstrates how group work enriched learning for high school English language learners by utilizing their home language.

At the same time, some K-12 researchers have documented inequities that bilingual students experienced under more active approaches to teaching (Civil 2014; Takeuchi et al., 2019; Takeuchi, 2016). For example, Planas and Setati (2009) describe how immigrant students in Barcelona took on identities as active doers of mathematics during small group discussions, however they generally adopted more "passive listener" identities in whole class discussions (p. 41).

Theoretical Framework

This paper adopts a sociopolitical perspective (Gutiérrez, 2013), by centralizing issues of power and identity. This perspective highlights the ways that power and other sociopolitical structures privilege certain identities in the math classroom, while marginalizing others (Langer-Osuna and Esmonde, 2007). In active learning settings, classroom interactions have the potential to facilitate learning, but must also be understood as existing in and being shaped by classroom power structures. This has implications on how students are positioned and their opportunities for learning (Adiredja & Andrews-Larsen, 2018). In this sense, power can be made visible by examining the ways that students position themselves or are positioned by others in the classroom (Martin-Beltran, 2013).

A sociopolitical perspective also views language as political, as some languages – like English – are assigned "higher status" over others (Civil, 2008). For instance, English is often seen as the language of "access and power" (Setati & Adler, 2000, p. 247), whereas Spanish is often used as a marker of poverty (Moschkovich, 2007). Because of this politicization of language, it is also inextricably connected to who is afforded power and access to participation in the classroom. This can be seen in Takeuchi (2016). This study documents different power imbalances that arise when bilingual students are assigned to groups with monolingual students. For instance, bilingual students were less likely to participate and had less access to being positioned as an expert or leader when they were placed in teacher selected groups.

To think about the role of language and power in the classroom, Planas and Civil (2013) presented the language-as-resource, language-as-political framework. This framework recognizes that on one hand, language is a resource for students. That is, the metaphor language-as-resource represents the potential that language has to facilitate mathematical learning. On the other hand, language-as-political recognizes that not all languages carry the same status in the classroom and can operate to privilege some students while marginalizing others. Planas and

Civil (2013) demonstrated how students were impacted by dominant language norms in the classroom which impacted their participation.

By leveraging the work that has been done at the K-12 level and by viewing language through a sociopolitical lens, this paper focuses on exploring the experiences of bilingual students in active learning post-secondary math classrooms. In better understanding students' experiences, this papers also explores to what extent active learning served as an equitable teaching approach in language diverse classrooms.

Research Question

This paper addresses the following research question: *How did bilingual students experience active learning in undergraduate math classrooms?*

Methods

Data Source

The data presented here are part of a larger, mixed-methods dissertation study which explores the experiences of bilingual students in undergraduate pre-calculus and calculus courses. This study took place at a large, public, research university that has been designated as a Hispanic-Serving Institution. For the qualitative part of the study, semi-structured interviews were conducted with twenty-eight bilingual students for whom English is an additional language. Nineteen of these students were international students. The languages that were represented among the student participants included: Arabic, Bangla, Chinese, Farsi, Hindi, Korean, Spanish, Uzbek, and Vietnamese.

Each interview lasted between 60-90 minutes. The interviews focused on understanding students' general experiences in college math courses, and their experiences with active learning and group work. This data was collected during the Covid-19 pandemic. As such, students attended different modes of instruction, i.e., students were either enrolled in in-person courses, online courses, or hybrid courses. In online course settings, group work was facilitated through breakout rooms. During the interviews, students in online courses often reflected about their experiences with group work over Zoom and their experiences in in-person courses the previous year.

Analysis

All interviews were transcribed in full and pseudonyms were given to each participant. Although interviews were transcribed verbatim, the quotes presented in this paper have been modified slightly for clarity. No modification impacted the meaning or feel of any of the sentences in the quote. The transcripts were then carefully studied and coded using open and axial coding as part of thematic analysis (Braun & Clark, 2006). This analysis first focused on identifying themes pertaining to students' experiences in active learning (e.g., experiencing microaggressions, not having your ideas understood). These themes were then further analyzed and compared to gain greater, more nuanced understanding of student's experiences.

Results

Three students in this study reported having a positive experience with active learning and group work. One student attributed this to her ability to advocate for herself while working with peers. The other two students found group work to be a less intimidating space for seeking help rather than going to the instructor. However, most participants in this study described the challenges and inequities they experienced in active learning environments as a student with a diverse language background. This often resulted in students participating less in the classroom even though they were actively engaged in the mathematics. For example, Juzhen, an international student from China, shared his experience trying to participate in group discussions in his calculus II course:

> I can understand what they're saying. I just can't join in the speaking. When they talk, if I want to make a response, I have to come up with a sentence in my brain and then speak it, but the native speakers, they will just talk without hesitation /…/ I can understand the conversation but sometimes, maybe it's kind of hard to join the conversation.

From my analysis of student interviews, four themes emerged regarding the inequities that students experienced in active learning classrooms: (1) feeling that they were not understood by classmates because of language, (2) experiencing microaggressions while working with others, (3) being positioned as a peripheral or marginalized member of the group, and (4) not being able to use their home language as a resource in their learning. First, many students reported feeling like their peers were not as good at understanding "what they were trying to convey" during mathematical discussions compared to their instructors. They felt that students often lacked "experience communicating with international students" and "people whose first language was not English". Second, because active learning opened the classroom space up for students to interact, several students reported experiencing microaggressions during these interactions. These microaggressions were based on their country of origin or their accent in English. In this paper, I will focus my discussion on (3) and (4).

Student Positioning During Group Work

Analysis of the data suggests that students in this study were often positioned as peripheral or marginalized members of their groups. First, several students shared stories of being positioned as passive listeners in their groups, rather than as an intellectual authority or leader. For instance, Anayeli, whose first language is Spanish, described how her groupmates were "not really willing to listen to [her] input". She recalled one instance during an in-class activity where she knew how to solve a problem because she had encountered something similar before. However, the student who was positioned as the group's leader was not receptive to Anayeli's ideas. Instead, he decided that the group should use his method of solving the problem, which did not end up leading to a correct solution. This left Anayeli feeling "frustrated that people don't listen to [her]". When asked what role Anayeli typically takes up during group work, she explained:

> They tell me, oh, we should probably solve it like this and then I'll try it that way. And then if it doesn't work, I'll suggest a different way. I think that sometimes I would also like to kind of lead. But it's just not always possible, you know.

This quote highlights a mismatch in the way Anayeli would like to be positioned, as an intellectual authority in the group, and the positioning that she had access to, as someone who takes directives from others.

In the interview, Anayeli described her classroom as being predominately composed of male students. Later in the interview, I asked her if she thought that being a woman impacted the experiences in group work. Anayeli explained “Well like there's the other girl in my group. And she also kind of just disregarded my ideas. So, really, it could probably be like the fact that I have an accent, maybe, because she doesn’t have an accent.” Here, Anayeli identifies having an accent as a reason she felt that her ideas were devalued by her group members. Therefore, the language background that Anayeli brought into the classroom impacted her experiences during group work and caused her to be marginalized by her peers.

Several students were also positioned as inhibiting the group’s productivity. This often occurred when instructors created incentives for students to engage in group work. For example, this can be seen in the case of Jason, who is a native Chinese speaker from Trinidad and Tobago. In his math class, students were allowed to leave class early when the group finished their work. Jason describes the pressure he felt to make sure he could explain his group’s solution because he did not want to inhibit them from leaving early.

> I just feel kind of ashamed when we doing a question and I'm like one of the person in that group that doesn't really engage much /…/ And I’m just their getting the answers, and just listening, and when it comes to the explanation, they're like, oh, make sure you understand this. That’s what’s going through my head like, make sure I understand this and I can explain it so we can leave \...\ I felt like I was kind of a burden.

Similarly, Carina, a student from Guatemala, was continually positioned as “slowing down” her group during in-class activities. She attributed this to the fact that the activities were graded and worth a significant amount of points. Carina describes numerous experiences where her classmates were rude and belittling to her when she asked questions. Below are two examples of this:

> I would be in a group, sometimes I wouldn't know how to say a word. And so sometimes my English would go like really bad, like where they wouldn't know what I was talking about. And they're like, I'm sorry I don't understand what you're trying to say. And they're like okay, well just maybe forget your question and let's just actually get to the problem. And I'm like well I am trying to go to the problem, but I don't know how to phrase my words to explain the problem.

> Sometimes one student was like I'm sorry but we really need to get going but some others were like, dude, stop, we can't do this. And then that time I would feel like I'm so worthless.

These quotes depict the ways that Carina was marginalized from her group for asking questions. Rather than engaging with her questions, her peers positioned the questions as distractions or not relevant to the task. Carina described her groupmates as “selfish” and “uninterested” in understanding her or helping her learning. In reflecting on her experiences as a bilingual student, Carina shared: “I am glad I’m a Spanish speaker but sometimes I'm not when students or people feel that they have the upper hand.” This quote highlights how language is political in the classroom, positioning some students as having the “upper hand” because they speak English.

Using Students’ Home Language as a Resource

Another theme that emerged from the data was that students were not able to use their home language as a resource in the active learning classroom. The majority of students in this study reported leveraging their home language to make sense of mathematics outside of the classroom, i.e., forming study groups with peers that share a common home language, finding resources in their home language, etc. However, inside the classroom, students expressed the need to only use English to "go with the status quo".

For example, Gabriela, an international student from Peru, connected with another Spanish speaker in her class. They often worked on math together outside of class using both English and Spanish. The use of both languages helped facilitate their learning. However, when Gabriela was placed in a group with her friend during class, they only spoke English: "When I'm in a group of three, me, my friend and someone else, we have to speak in English to make them feel included, so it's not like an option for us." Making sure group members are included is a valid concern. However, for Gabriela, there is little room for her to leverage her linguistic resources, as an aspect of both her past experiences and current classroom suggest to her that only English should be used.

To further illustrate, Anthony is a domestic student whose first language and the language he prefers to do math in is Spanish. When asked to describe his experience in the classroom as a bilingual student, he first responded "Well, first I have to pretend I'm monolingual". This quote suggests that a part of Anthony's identity as a bilingual student is not being affirmed in the classroom.

Furthermore, when asked if he was ever paired with group mates that spoke Spanish in his math course, he responded:

> *Interviewer:* So, do you ever get grouped with Spanish speakers when you work in groups in math?
> *Anthony:* Yeah. I've had that happen, but they don't want to speak Spanish.
> *Interviewer:* Why is that?
> Anthony: I don't know, I feel like they have that thing about like that you should be only speaking English. If not, they're going to see you as a weird person or something. So, at the beginning of my university life, I tried to speak Spanish with other people, but I noticed that nobody wants to speak Spanish. So, then I just started doing the same thing. I was like okay I'm not speaking Spanish. So, even though we knew each other speaks Spanish we just pretend to be fully English speakers.

This exchange also captures the tension between language-as-resource and language-as-political in the classroom. For Anthony, Spanish is an important tool for checking his mathematical understanding: "When I'm really thinking about something [in math], I tend to go back to Spanish as a way to verify what I'm thinking in English is correct". However, because students felt stigmatized when speaking Spanish, Anthony was unable to activate this resources during group work.

Discussion and Implications

Most bilingual students in this study did not have equitable experiences in the active learning undergraduate math classroom. In particular, the data suggests that group work generally did not facilitate equitable opportunities for participation and learning. Results indicated that bilingual students reported participating less in group discussions, not being positioned as legitimate members of the group, and not having their ideas taken up by peers. Furthermore, active learning spaces often did not affirm students' bilingual identities and allow them to use their native language as a tool in mathematical sensemaking. During interviews, students identified their language background or their accent in English as reasons they felt marginalized. This exemplifies the political nature of language in the classroom, which can serve to privilege some students by marginalizing others. In this study, viewing language as political allowed for a deeper understanding of classroom equity in active learning spaces.

Although students generally did not have positive experiences, the argument of this paper is not that active learning should be abandoned in multilingual classrooms. Nor is the argument that active learning is not an equitable teaching strategy. Findings do suggest, however, the need for more research on effectively implementing active learning in these spaces. This research should examine how active learning can be implemented so that it is more humanizing, affirming, and allows bilingual students to leverage the resources that they bring.

Students in this study provided suggestions for improving the implementation of active learning. For example, as discussed in the previous section, some instructor's used incentives to encourage student engagement and active participation in group work (i.e., allowing students to leave early, assigning points, etc.). However, this had the unintentional outcome of negatively impacting the group dynamics, as the group's goals shifted from group learning to group productivity (Webb, 1995). Students described having better experiences with group work when the tasks were low stakes. Students suggested making group activities for participation points only. Other suggestions from my dissertation data include having students stay in the same group for longer periods of time. Students expressed that this helped them feel more comfortable communicating. Students also preferred to be paired with other international or language diverse students. The shared experience of having a language diverse background helped make group work a more affirming environment.

Finally, I argue that this paper makes a contributions to the field in the following ways: (1) It addresses the lack of research on bilingual students' experiences in undergraduate math classrooms. (2) It examines the impact of active learning on historically marginalized students. (3) It illustrates the ways that language is political in the undergraduate classroom. (4) Finally, it provides a few recommendations for more effectually implementing active learning to affirm and support bilingual students in undergraduate mathematics.

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Instructor Beliefs and Practices at the Periphery of STEM

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As a practice, modeling is beneficial for students. For students to have the opportunity to do modeling, instructors must choose to incorporate it into their courses, a decision based on the instructors' beliefs about modeling in and out of the classroom. To expand applicability and generalizability of results and theories, to expand the focus of mathematics education research to domains trending mathematically, and to work towards incorporating modeling into other classrooms, we interviewed 10 STEM instructors in domains atypical to the current literature base. Analysis indicated this demographic of STEM instructors held beliefs about modeling in and out of the classroom similar and different to those documented about typical STEM instructors. However, similar beliefs are more nuanced than previously reported.

Keywords: Teacher Beliefs, Modeling, Integrated STEM.

Mathematical modeling (hereafter: modeling) is beneficial for students for multiple reasons: from developing general competence towards creative problem solving to helping acquire, learn, and keep mathematical concepts by providing motivation for and relevance of mathematical studies (Blum & Niss, 1991). However, facilitating modeling tasks is challenging, and many STEM instructors cannot easily find time in their courses to dedicate to modeling. Because of this, one obstacle in incorporating modeling tasks into STEM students' coursework is persuading STEM instructors that doing so is achievable and worthwhile.

In general, instructors' judgements about their pedagogical practices arise from their beliefs (Pajares, 1993), a link that has been documented by many researchers in many contexts.: k-12 science (Haney, Lumpe, Czerniak, & Egan, 2002), K-12 mathematics (Bray, 2011; Clark et al., 2014; Jacobson, 2017; Yurekli, Stein, Correnti, & Kisa, 2020), college sciences (Gibbons et al., 2018; Pelch & McConnell, 2016), and engineering (Borrego et al., 2013). The first step in addressing STEM instructor's beliefs about a topic, such as modeling, is to first document their differing beliefs about the construct (Nathan et al, 2010). Thus, a better understanding of modeling's place might be in the curriculum, can inform efforts to persuade STEM instructors that teaching with modeling is an achievable objective.

Across STEM fields, research has documented instructors' beliefs about modeling (as a professional and educational activity), the integration of mathematics into their courses, and the characteristics of successful STEM students where STEM was typically taken to mean physical sciences, engineering, and computer science. However, other fields, such as psychology, biology, and economics, are becoming more mathematical. For fields whose roots are not mathematical, or even statistical, there is still much to be learned about the how these professionals conceptualize modeling, and how they view the role of modeling in their course work. Articulating an inclusive, and empirically informed, account of what constitutes modeling can provide novel perspectives about modeling instruction absent from the literature. Such perspectives can better inform the teaching and learning of modeling by expanding the contexts in which modeling is studied and potentially include a demographics of students not currently accounted for in literature. That is to say, their perspectives are important to include because they are STEM professionals who teach STEM students. Indeed, even the NSF classifies anthropology, psychology, and economics as STEM fields (NCSES, 2014).

This study lays the groundwork for describing an inclusive view of STEM instructors' beliefs about modeling. The goal of this paper is to extend what is known about STEM instructor's beliefs about modeling in STEM majors course work by providing the perspective of STEM instructors not currently accounted for in the current literature base.

Literature Review

Some studies exposed the contrasting views held by STEM instructors with regards to their field's relationship to mathematics which insinuate a instructor's conceptions about modeling's place in their courses. Holmberg and Bernhard (2017) interviewed 22 university instructors who taught content related to Laplace transforms. Some instructors believed that mathematics, physics, and technology are inseparable; others verbalized the opposite view, that these fields are not related at all. Nathan et al. (2010) developed a measure of STEM instructor's beliefs about engineering students' success. They studied differences in beliefs and practices between STEM high school instructors with masters' degrees and instructors using an integrated curriculum. Instructors with master's degrees were least likely to identify sources for engineering support, least likely to claim their class was integrated with STEM, and more likely to agree that students needed to be high achieving to be successful in a STEM career. Bergsten, Engelbrecht, and Kågesten (2015) interviewed two professional engineers about their views of conceptual and procedural mathematics skills in engineering education and practice. One engineer, Robert, from Sweden worked in technical physics and electrical engineering. The other engineer, Ben, was a civil engineer from South Africa. Both engineers held that conceptual mathematics skills are the most important for engineering education. However, Robert emphasized the connection between conceptual and procedural actions, while Ben stated that procedural mathematics skills are not necessary. Bergsten et al. (2015) conjectured that this difference was due to the engineers' differing fields and backgrounds. These studies empirically showcase two ideas about modeling and curriculum present in Kaiser (2017), the idea that different fields view applied mathematics and pure mathematics as either separate and should be taught separately or inseparable from the subject and was an inherent part of other sciences and should not be taught separately.

Other literature focuses on STEM instructors' beliefs about characteristics of successful STEM students, particularly their beliefs about their students as learners of science, mathematics, and, the intersection, modeling. Faulkner and Herman (2016) interviewed engineering and computer science instructors about the skills students needed to be considered mathematically mature. Results indicated that the instructors valued algebraic fluency, quick computations, symbol sense, ability to use online tools to solve mathematics problems, confidence, and other modeling skills are necessary for a student to be called mathematically mature. Similarly, Gandhi-Lee et al., (2015) interviewed biology, chemistry, computer science, engineering, geoscience, health science, mathematics, and physics instructors. Their participants held that to be successful, students must be curious, independent problem solvers, with positive attitudes. Additionally, these instructors identified mathematics overall as a roadblock to success, and specifically identified algebra as the minimum requirement for success.

The field has also documented how professional engineers, and instructors of engineering and mathematics view modeling as a construct (see Drakes, 2012; Frejd & Bergsten, 2018; Gainsburg, 2013). Instructors in these fields, as well as secondary and post-secondary science teachers, have not reached consensus when describing the role of mathematics in their classes. STEM instructors more broadly, including computer science, health and geo sciences, biology, and chemistry have well-considered characterizations for student qualities they believed contributed to success. The field has yet to learn how disciplines at the periphery of the definition of STEM, such as geography, psychology, anthropology, and economics, conceptualize modeling and how those STEM instructors view the role modeling plays in the education of their STEM students. It is thus unknown whether these results generalize to the broader population. Attending to the beliefs and perceptions of the peripheral STEM disciplines will strengthen applicability and generalizability of results and theories expressed in the current literature, will expand the focus of mathematics education research to incorporate domains that are trending mathematical, and will work toward meeting a societal need by getting more modeling into these other classrooms. With these goals in mind, the purpose of this study is to answer the question: what is the role of modeling in the education of undergraduate STEM majors, according to non-traditional STEM instructors and how do their accounts comport with existing research on traditional STEM instructors?

Instructors' beliefs have commonly been studied using a combination of qualitative and quantitative methods (see Bray, 2011; Gibbons et al., 2018; Haney et al., 2002; Nathan et al., 2010; Pelch & McConnell, 2016). Typically, observations are analyzed qualitatively to study instructor practices while instructors' beliefs are measured using surveys and statistical models are used to test associations between beliefs and practices (Philipp, 2007). This approach has been critiqued in the broader higher-education literature for the underlying assumption that there is a clear causal relationship among instructors' conceptions, practices, and student learning (Devlin, 2006). Thus, qualitative methods, such as thematic analysis, are preferred when studying individuals' beliefs (e.g., Bergsten et al., 2015; Drakes, 2012; Faulkner & Herman, 2016; Frejd & Bergsten, 2018; Holmberg & Bernhard, 2017). However, a balance must be struck; overly broad characterizations lose descriptive power necessary for explaining individuals' instructional choices. Consequently, studying the relationship between teacher beliefs and instructor practices necessitates a fine grain size to allow for local causal models that are consistent within participants and their circumstances (Speer, 2008).

Theoretical-Methodological Lens

We adopt the stance that a person's identity, personality, desires, and importantly, their beliefs are embedded within the stories they tell, an assumption of narrative inquiry (Loong, 2019). Following Pajares (1993), we take beliefs to be knowledge a person holds that is either descriptive, evaluative, prescriptive, or any combination of the three (Pajares, 1993). Following Polkinghorn (1995), we constitute a story as a narrative preserving "the complexity of human action with its interrelationship of temporal sequence, human motivation, chance happenings, and changing interpersonal and environmental contexts." (p.4). Thus, a story is more than a description of what happened at a point in time, it has also an underlying structure connecting the events through choices made by the storyteller. The structure, or plot, aids in identifying how individuals connect the events in their lives as precursors for and consequences of the choices they make. This orientation affords a view of STEM instructors' stories as embedding their beliefs about the role of modeling in the education of STEM majors, as follows:

A STEM instructor has beliefs about modeling (even if they do not use the label "modeling"), which include beliefs about modeling in all contexts including their research, their industry jobs (if applicable), and their teaching. An instructor can have an experience, a notable instance salient to them, that may affect their beliefs about modeling. Consequently, analyzing the stories STEM instructors tell about their experiences with modeling in their personal-professional lives and their teaching will afford inferences as to the nature of those beliefs. Descriptive-analytic accounts of the instructors' stories also articulate explanatory mechanisms for how individuals came to hold their beliefs, data useful waypoint for future research.

Methods

We conducted this study at a large southwestern university in the USA. We selected instructors from fields that the NSF (NCSES, 2014) has identified as STEM fields yet are not typically represented in modeling or mathematics education literature: economics, anthropology, geography, and psychology. After identifying majors associated with these fields, we recruited instructors who had recently taught courses for those majors. The population has two advantages for addressing the research questions. First, STEM instructors who are professionals are more likely to have experience with modeling in their undergraduate studies, graduate studies, research work, or industry job. Second, insisting the STEM instructor primarily teaches STEM majors increases the likelihood that the instructor has considered the role of modeling in the education of majors in their classes. In total, the 10 participants of this study were: two economists, two anthropologists, three geographers, and three psychologists.

Data were collected through episodic narrative interviews (Mueller, 2019) conducted over zoom. The episodic narrative interview is a fusion of three other qualitative methods: semi-structured interviews, narrative interviews, and episodic interviews. This approach enabled cross-participant comparisons, provided a strategy for looking at experience-focused narratives which allowed for the participants' views of salience to be prioritized, and allowed for exploration of the target phenomenon (Mueller, 2019). In this way, experience-centered narratives of research and teaching were prioritized while also targeting the scope of each interview toward instructors' beliefs about modeling, generally, and in the classroom through their own salient experiences. Then the salient-to-participants aspects could be inferred and compared across cases and to the extant literature.

In episodic interviews, the interviewer typically starts the interview by asking a question that defines the phenomenon of interest, and then follows with a question to elicit an episode from the interviewee's everyday life in where the phenomenon of interest would take place. The interviewer then asks questions about the phenomenon of interest within that evoked situation (e.g., Romaioli & Contarello, 2019). Similarly, episodic narrative interviews are constructed to funnel the interviewee's story towards the phenomenon of interest (Mueller, 2019). We organized the interview around two sub-stories, building one cohesive story to state and explains the instructors' beliefs about modeling in their classrooms. The first sub-story focused on the professor experiences with modeling outside of teaching. The second sub-story focused on the instructors' experiences with modeling while teaching.

We used analytic techniques informed by narrative inquiry (rather than the more common coding techniques that are appropriate when a pertinent, codified framework exists, which our review of the literature did not reveal). We began with thematic analysis to identify major themes, understood to be patterns within the data (Braun & Clarke, 2006) salient to the participants. We then used emplotment analysis (Polkinghorn, 1995) to probe and then reconstruct the data. Questions such as how does modeling fit into your class as a whole? and How does modeling fit into your students' major (course)work? elicited responses that intimated the instructors' beliefs about the role modeling in the education of STEM majors.

The data were analyzed at the latent level (Braun & Clarke, 2006) with the grain size of analysis being finished thoughts. A finished thought was one or multiple statements about the same topic. A new thought was started when there is a turn in topic of the interview. As informed by Braun and Clarke (2006), the analysis was conducted in five phases: becoming familiar with the data, generating initial codes, looking for themes within the initial codes, reviewing those themes for refinement, and defining the themes. This analysis produced a list of major themes that emerged from the data when focusing on the participants conceptions of modeling in and out of the classroom and curricula. The results of this part of analysis gave a group of codes providing overarching ideas about modeling in and out of the classroom. For example, one idea about modeling was modeling is statistical, and an idea about modeling in the classroom was students are not mathematically prepared for it. To personalize a participant's particular set of beliefs, we identified each participants' core beliefs, where core beliefs are the set of beliefs held by a person that was showcased through multiple instances. This was done to get a nuanced view of the participants beliefs, as a participant could have answered an interview question with a polite response and not necessarily their truly held beliefs. This was done by using the Max Maps feature in the qualitative data software MAXQA. The feature visualized the highest-frequency codes present in a single interview. Codes were transferred to the participants' maps only when they were at least partially constitutive of a core belief.

To account for important background information, significant experiences with modeling in daily life, and significant experiences with modeling while teaching, we used ideas from emplotment from narrative analysis to construct individual narratives about each professor. The first step in emplotment is to identify the end goal. In this analysis, the end goal was each participants' set of core beliefs. The next step is to hypothesize a plot which is then tested against the data. This is done by asking questions like: do any of the major events conflict with this current plot structure? If a major event from the data does conflict, then changes are made to the plot to accommodate and the new revised plot is tested against the data again. While testing the plot against the data, one must ask if each event is pertinent. If an event is not pertinent to the plot, that data is culled from the story in a process called narrative smoothing (Kaasila, 2007; Polkinghorn, 1995). This process is undertaken until a cogent plot emerges that considers all of the pertinent events, this forms the plot outline. This plot outline is then filled in with data elements to form the final coherent story (Kaasila, 2007; Polkinghorn, 1995).

This analysis produced individualized narratives for each participant that tie their core beliefs with important background, and significant experiences with modeling both in daily life and while teaching. It is important to note that the beliefs about modeling in and out of the classroom highlighted by this analysis are not the only ideas about modeling held by the participants in this study. For the purpose of this paper, we present the most salient beliefs about modeling in and out of the classroom at the time of the interview. This helps with identifying beliefs that are most important to the participants but does not identify all the beliefs the instructor might hold. After the construction of these individual narratives, we reanalyzed the narratives, again using techniques from thematic analysis (Braun & Clarke, 2006), to look for overlap between the themes we identified and those present in the literature. This analysis allowed us to make broad-stroke comparisons of our participants' beliefs and the existing literature.

Results

The literature suggests there are differing ideas about the role of mathematics, and thus of modeling, in science, technology, and engineering (STE) courses. The two big ideas about the role of mathematics in STE is that of inseparability (STE cannot be taught without also teaching mathematics) and isolation (STE and mathematics are taught in their own courses). Both sentiments were also found among the core beliefs of our participants. Karter, an economist, and River, a geologist, were insistent that mathematics was inseparable from their courses' content. For example, Karter said

Karter: I tell them [his students] economics and mathematics are inseparable. So, there is no way. If you come to this class and you think that I'm just going to chit chat and not write an equation or any numbers on the board, and then you better drop out of this classroom. I let them know, they know what is coming ahead of them. But, then I motivate them.

In contrast, Phoenix, an anthropologist, talked about mathematics as a tool to be taught in another class.

Phoenix: Well, I'm very glad to have them [economics majors] in my class, but it's really for the economics program to teach them that part of it [the mathematics behind some theories]. Like I said, anthropology is a high-level discipline meaning it's at a high level of abstraction. If they want to do economics, they really do need to learn, they need economic analysis classes. But that's not my job. That's not my job and truthfully, I'm not really qualified to do it.

We do not claim that differing conceptions of STEM integration originate in the participants' fields. The contrasts seem to be rooted in the instructors' salient experiences with mathematics and modeling in their professional lives and teaching. For example, Karter told a compelling story of studying mathematics in his youth and explicated many examples of using modeling in his career. In contrast, Phoenix did not share any salient experiences with modeling in his research or while teaching.

Gandhi-Lee et al. (2015) and Faulkner and Herman (2016) showcased how STEM instructors believed that mathematics proficiency was important for students in STEM fields. More specifically, Gandhi-Lee et al. (2015)'s participants talked about mathematics as a roadblock, and how algebra was the minimum for mathematical preparedness. Almost all participants (9 of 10) in our study explicitly voiced a similar sentiment, that mathematical preparedness of students was a roadblock to implementing modeling tasks in the classroom. One anthropologist, called Phoenix, did not hold this core belief, did not communicate salient examples of using modeling or any mathematics in the classroom. While almost all voiced level of mathematical preparedness as a roadblock to including modeling in their courses, we observed a level of idiosyncrasy in how each participant operationalized preparedness. Predictably, some instructors operationalized mathematical preparedness to mean algebraic fluency or a certain level of proficiency in calculus. For example, Quinn, a psychologist, explained that he had to scale back the difficulty of the mathematical analysis in an in-class experiment.

Quinn: I, over the years, I still do that in the class but I've kind of scaled back the complexity and difficulty of the exercises. I've found that I just needed to and the main reason… Where a math problem that I thought should be pretty simple if you've just taken college level, I don't know, algebra for example. It wasn't anything too crazy that I gave the students. Even then some students had difficulty with it. Not all, some students did just great but I felt that I needed to kind of scale back the complexity of those problems over time but still using them.

Haven, a psychologist, suggested that mathematical preparedness meant fluency with graphical representations and their meanings. She recounted a conversation with one of her graduate students working on a research problem. The student was having difficulty labeling the scatter plot that would illustrate their hypothesis for statistical testing. While recalling this conversation, Haven lamented that students, more generally, were not skilled with graphical representations. She explained that graphical expressions were most important for her students because specifics about what statistical models to run could be looked up later.

Haven: If you can't figure out what's the label on our scatter plot, if you can't figure out what the Y and X axis should be labeled, take a step back and think through what you're doing. I guess that's not an issue of what buttons to click or what the test is called. I was like, "I can tell you what the test is called." Once you get that to me, that's the work I want to see you doing is thinking through graphically how to depict the data. Once you do that work, then you also know what to Google.

Both psychologists teach STEM majors in a psychology department. However, their operationalization of mathematical preparedness was quite different. This is partially due to the courses they teach. Quinn's classes tend to lend themselves more to mathematical exploration than Haven's. This might also be partially due to their differing backgrounds. Both have experience studying mathematics as students themselves, but the types of mathematics are vastly different. Haven spoke mostly of studying mathematics in her statistics and methods courses, while Quinn studied mathematics and physics at both the undergraduate and graduate level. Overall, there was broad consensus that mathematics was a roadblock for students, and that this roadblock was common in their fields. However, when delving deeper, there was no clear pattern of what was meant by mathematical preparedness based on discipline.

Discussion

Our study builds on and extends a synthesis of literature describing beliefs held by STEM instructors about modeling in the classroom through documenting perspectives of instructors from STEM fields not typically included in the literature. This work was necessary as more fields on the periphery of STEM come to rely more heavily on mathematics and statistics.

Overall, there were sentiments held in common about modeling in and out of the classroom shared among these non-traditional STEM instructors and traditional STEM instructors, such as mathematics knowledge being a barrier to implementing modeling in the classroom. However, what is meant as mathematical preparedness seems to be idiosyncratic to the individual professor and partially dependent upon the specific course content. This implies that discipline-level analysis may not be an appropriate grain size for investigating the mathematical barriers students face. We do not claim that there are no differences according to discipline, but we observed differences within disciplines indicating that using discipline to differentiate participants may be too broad. Future research endeavors to uncover instructors' beliefs and practices about modeling in STEM may wish to be cautious when constituting discipline as an independent variable because idiosyncrasies of the instructor's beliefs imply idiosyncrasy of instructional decisions. Additionally, beliefs about mathematics' role in instruction was also as mixed as it was in Traditional-Stem focused literature (Holmberg & Bernhard, 2017; Kaiser, 2017; Nathan et al., 2010). Because instructor's beliefs influence their pedagogical decisions, this differing view of mathematics' role in instruction must be accounted for in future work on persuading STEM instructors modeling is a doable and worthwhile endeavor, but we do not recommend accounting for it at a discipline-based level.

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The Influence of Graduate Student Instructors' Mathematical Meanings for Teaching Sine Function on their Enacted Teaching Practices

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Prior research suggests that a teacher's mathematical meanings for an idea constitute their image of the mathematics they teach (Thompson, 2013), their pedagogical decisions, and the language they use to cultivate similar images in students' thinking (Thompson & Thompson, 1996). As such, it seems reasonable that there is a link between teachers' meanings for the ideas they teach and their instruction, including teachers' instructional goals, choice of tasks, and the questions they pose to students. This report presents results from clinical interviews that illustrate the relationship between graduate student instructors' (GSIs) mathematical meanings for teaching (MMT) and their enacted teaching practices. More specifically, these results reveal and contrast two GSIs': i) meanings for sine function; ii) goals for students' learning of sine function; and iii) tasks for supporting students' learning of sine function.

Keywords: Mathematical Meanings for Teaching, Teaching Practices, Quantitative Reasoning

Introduction and Review of Literature

In 2016, Thompson proposed using the construct *Mathematical Meanings for Teaching (MMT)* rather than *Mathematical Knowledge for Teaching (MKT)* to make explicit that he was using the word *knowledge* in the sense of Piaget. To Piaget, knowledge and meaning were largely synonymous and grounded in the knower's schemes (Montangero & Maurice-Naville, 1997). Thompson also proposed using the word meaning since a meaning is attributed to a person and knowledge is less personal and disjoint from the knower (Thompson, 2016). Investigations into teachers' meanings (Carlson & Bas Ader, 2019; Thompson, 2013) rather than teachers' declarative knowledge (Schilling, Blunk, & Hill, 2007) provide math educators with insight into teachers' image of the mathematics they teach and intend students to learn and ultimately what they might say or do while teaching (Thompson, 2013; 2016). The affordances of attending to teachers' meanings vs. teachers' knowledge may be best described through an example. If two teachers, Teacher A and Teacher B, were asked to determine the value of sin θ in *Figure 1* below, both Teacher A and Teacher B may correctly answer $sin\ (\theta) = \frac{1.78}{3.96} = 0.45$, indicating that they know how to answer this question. However, this correct answer provides math educators with limited insight into the thinking the teachers used to determine their answers and how the teachers might convey this idea to students.

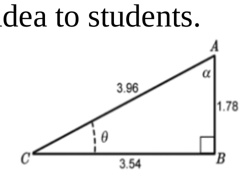


Figure 1. The value of Sine

Instead, a shift in focus to what teachers mean by $sin\ (\theta)\ =\ 0.45$ can uncover what a teacher might say or do in a classroom, and thus, the meaning that a student might construct. For instance, imagine that Teacher A uses the commonly applied trig ratios (SOHCAHTOA) to answer this question. With this way of thinking, Teacher A might convey a meaning to students that $sin\ (\theta)$ is not a function of θ, rather a ratio of sides of a triangle. In contrast, imagine that

Teacher B's meaning for $sin\ (\theta)$ is a function grounded in quantitative reasoning (see next section for example). Regardless of the nature of a teacher's meaning for sine function, they will likely have the goal to convey their meanings to students (Silverman & Thompson, 2008). As such, uncovering teachers' meanings for an idea is essential for understanding a teacher's instructional goals. Although attention to teachers' mathematical meanings for teaching has been repeatedly called for (Byerley & Thompson, 2017; Musgrave & Carlson, 2016, Thompson, 2013, 2016, Tallman & Frank, 2018), few researchers have investigated teachers' MMT. This report addresses the following question: In what ways does a teacher's MMT sine function influence their choice of tasks to be used in class and their goals for students' learning?

Theoretical Perspective

Quantitative reasoning- the analysis of a situation into a quantitative structure, a network of quantities and quantitative relationships (Thompson, 1990, 1993, 2011), has been identified as a critical way of thinking that supports students' development of coherent meanings for angle measure and trigonometric functions (Hertel & Cullen, 2011; Moore, 2012, 2014; Tallman, 2015; Thompson, 2008). A quantity is a quality of something that one has conceived as admitting some measurement process (Thompson, 1990). Quantities exist in the mind of the individual conceiving them. To comprehend a quantity, an individual's conception of "something" must be elaborated to the point that they "see" characteristics of the object that are admissible to the process of quantification (ibid). Quantification is a process of direct or indirect measurement which results in a value. A quantity's value is the numerical result of a quantification process. Numerical operations are used to calculate a quantity's value; however, numerical operations differ from quantitative operations. A quantitative operation is the conception of two quantities taken to produce a new quantity. Put another way, a quantitative operation is a description of how quantities come to exist (ibid).

A Quantitative Meaning for Sine Function

Moore (2014) described a productive meaning for sine grounded in quantitative reasoning. Namely, sine is a function that relates the measure of an angle (measured from the 3 o'clock position) and the vertical distance (of the terminal point of the angle) above the horizontal diameter measured in radii. As such, the value of $\sin(\theta)$ represents how many times as large the vertical distance of a terminal point above the horizontal diameter is compared to the circle's radius when the terminal ray of an angle is rotated to the point that is θ radians counterclockwise from the 3 o'clock position. Thus, for example (see *Figure 2)*, the point (0.8, $sin\ (0.8) \approx 0.717$) conveys that for an arc length of 0.8 radii counterclockwise from the 3 o'clock position on any circle, the terminal point's corresponding vertical distance above the circle's horizontal diameter is $sin\ (0.8)$ or 0.717 times as large as that circle's radius.

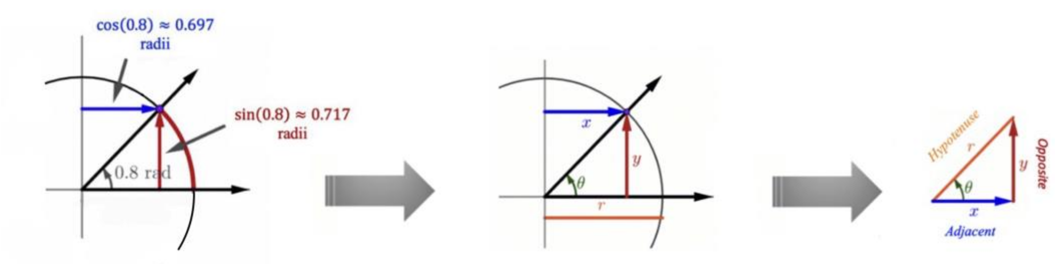


Figure 2. Relating unit circle trig to right triangle trig- Adapted from the Pathways Precalculus Curriculum

A coherent meaning for the sine function involves coordinating two varying quantities' values while attending to how the quantities' values vary in tandem. Moreover, when a student or teacher attends to how the value of $\sin(\theta)$ changes as the angle measure, θ varies, they are engaging in covariational reasoning (Carlson et al., 2002). A coherent meaning for trigonometric functions also includes robust connections between right triangle trigonometry and unit circle trigonometry. The commonly used trigonometric ratios (SOHCAHTOA) can be viewed as using the hypotenuse (radius) as a unit of measure for the legs of the right triangle. As one example, the output of the sine function can be viewed as the length of the triangle's leg opposite the angle whose measure is input to the sine function, measured in units of the hypotenuse (see *Figure 2*).

Methods: Subjects, Data Collection, and Analysis

This study's purpose is twofold (1) to explore GSIs' mathematical meanings for teaching sine and cosine function and (2) to investigate the impact of GSIs' MMT on their enacted teaching practices. To accomplish this goal, the first author conducted clinical interviews (Clement, 2000) with two GSIs who were in their first year of teaching using research-based Pathways Precalculus curriculum materials at a large, public, PhD-granting university in the United States. The purpose of the clinical interview was to investigate the instructors' MMT sine function, including each instructor's goals for students' learning, images of key ways of thinking they wanted students to engage in, and choice of tasks. Before the first clinical interview, each GSI was asked to select a task they considered vital to discuss with students during their sine and cosine function lesson. During the interview, each participant was asked to describe the task they selected, their goals for student learning (relative to a lesson they would be teaching on sine and cosine function), and their image of a coherent understanding for sine and cosine function. Each GSI was also asked to respond to two mathematical tasks designed to elicit their meanings for sine and cosine function.

The analysis of the clinical interviews occurred in three phases, as described in Simon (2019). The first phase involved listening to the interviews to generate hypotheses about the teachers' ways of thinking. The second phase involved a more in-depth, line-by-line conceptual analysis (Thompson, 2008) to describe aspects of the teachers' MMT for sine and cosine. The third and final phase of this analysis involved identifying themes in the conceptual analysis to characterize the teachers' MMT sine and cosine.

Results

During the first clinical interview, the participants, Razi and Wilma, were asked to complete two tasks designed to elicit their meanings for sine and cosine. The estimation task shown in *Figure 3* was posed to the GSIs to reveal the nature of their meanings for sine and cosine function and the degree to which they engage in and value quantitative reasoning. To answer the estimation task, the instructors needed to reason about the meaning of the measure of $\sin(\theta)$ and $\cos(\theta)$ rather than solely performing a computation. As described earlier, one conceptualizes the sine function quantitatively if she conceives of sine as a function that relates the measure of an angle (measured from the 3 o'clock position) and the relative size of the vertical distance of the terminal point to the radius of the circle.

Wilma's response to the estimation task revealed her meaning for the sine function as the "terminal point's vertical distance above the horizontal diameter, measured in radius lengths." She estimated the value of sine function for the angle measure θ to be 0.8 (see Figure 3). Wilma also expressed that 0.8 means that the “orange line is 0.8 times as big as a radius” or “the orange line is 80% of a radius”.

During class you ask students to estimate the value of sin(θ) and cos(θ) using the diagram below. Gabriel, a student in your class, tells you that it isn’t possible to answer the question since there are no numbers. Is this student correct? How will you respond?

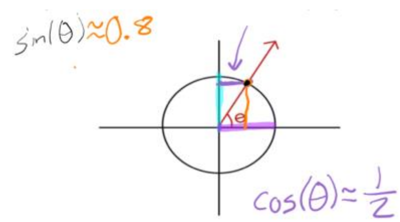


Figure 3. Wilma’s solution to the estimation task

I interpret Wilma’s meaning for sine to be grounded in quantitative reasoning. When describing the sine function, she initially identified the attribute of a circle to measure (the vertical distance of the terminal point above the horizontal diameter), a unit of measure (radius length), and a measurement process (multiplicatively comparing the length of the orange line to the length of the radius). Wilma’s description of 0.8 as a measure of how many times as large the orange line is compared to the radius indicates that her meaning for sine entails a quantitative operation. Wilma’s approach of multiplicatively comparing the vertical distance of the terminal point above the horizontal diameter to the radius resulted in a measure of a quantity in an identified unit (the terminal point’s vertical distance measured in radius lengths). Moreover, Wilma’s response to the *Estimation Task* suggests that she conceptualized the value of the sine function for a particular angle measure as the relative size of the terminal point’s vertical distance above the horizontal diameter to the length of the circle’s radius.

In contrast to Wilma’s stable meaning, Razi expressed multiple meanings for the sine function throughout the clinical interview. As one example, Razi described the value of the sine function as a *y*-coordinate on the unit circle, the proportion of the y-coordinate to the radius, opposite over hypotenuse, and a height. In his response to the estimation task, Razi expressed the former two meanings and described the value of sine as "the length of the y-coordinate in terms of the radius length".

At first glance, one might interpret Razi’s description of sine as “the length of the y-coordinate in terms of the radius length” as conveying a quantitative meaning for the sine function. However, a closer analysis of Razi’s response to the estimation task suggests that Razi was not thinking about measuring quantities. In particular, Razi described his estimate for the value of sine as “a proportion” and “the ratio between the y-coordinate on the circle and the radius”. In this instance, I interpreted Razi’s meaning for the value of the sine function to be that of a numerical operation since Razi appeared to be comparing numerical values rather than the relative size of two quantities. The interview continued by the researcher, prompting him to explain how the idea of proportion was related to his meaning for sine function.

Razi: Oh, I get tongue tied in this all the time, I mean the amount that I have to multiply the denominator by in order to get the numerator.

Interviewer: Okay, so is that what you want students to think about when you say proportion?

Razi: Another thing I like to think about is how many times does the denominator go into the numerator. That’s the classic. Umm… yeah, those two ones. Those are the ones I rely on.

Interviewer: So, do you want students to think of sine as the number you have to multiply the denominator by to get the numerator?

Razi: No, no, I said this earlier I want students to think that sine of an angle corresponds to a place on the unit circle. Which at the same time could be considered a proportion, but I think the unit circle with the x-y coordinate is more important.

Razi's description in the above excerpt provides further evidence that he conceptualized the value of sine as the result of a numerical operation. Razi's description for proportion focused on multiplicatively comparing the numerator and denominator of a ratio. However, it does not appear as though Razi conceptualized the values in the numerator and denominator as measures of quantities. Moreover, Razi's expression "that sine of an angle corresponds to a place on the unit circle" indicates that Razi was thinking about the value of sine as a location rather than a measure of vertical distance.

Razi and Wilma's Goals for Students' Learning of Sine and Cosine

During the interview, Wilma repeatedly stated that she wanted her students to think about quantities and how the quantities values changed together when reasoning about the sine and cosine functions. Wilma's goals for students' learning, as expressed during the first interview (see excerpt below) represent the ways of thinking she wanted her students to use. Wilma expressed that she wanted her students to think about how the vertical distance (blue vertical line in *Figure 4*) and horizontal distance (red horizontal line in *Figure 4*) of the terminal point varies as the measure of the angle "swept out" varies.

Wilma: So, I think…goals for students learning umm I guess like I talked about before, imagining the covariation of the angle measure and either the horizontal distance or vertical distance. Like, as the angle measure changes, or as that terminal ray sweeps out more of the circle, how do the blue line and the red line change? Umm… because I feel like that is a really helpful way to think about sine and cosine.

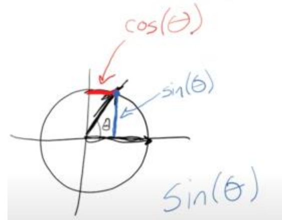


Figure 4. Wilma's Description of Sine and Cosine

When prompted to explain her goals for her students' learning, Wilma conveyed that she wanted her students to conceptualize the values of the sine and cosine functions as measures of distance. It is noteworthy that her goal for student thinking aligned with the thinking she used to respond to the estimation task. Similarly, Razi's description of his goals for students' learning (see excerpt below) was consistent with the meanings he expressed for the sine function. Razi explained that he wanted students to think of the value of the sine function as a coordinate on the unit circle.

Razi: I want them to think that like if I input an angle that it corresponds to or if I have a angle on the unit circle or something that it should have an x,y coordinate in the plane that it corresponds to. And if I want to find out information about any of those points the cosine and sine function are pretty good tools.

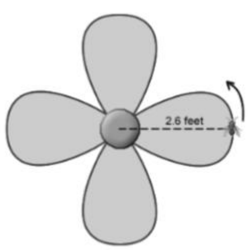
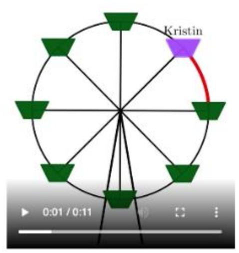
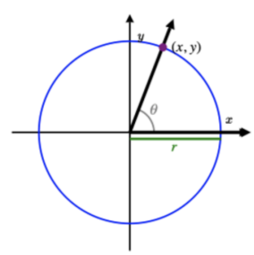
Interviewer: Okay

Razi: I honestly think that if they understand that, that the angle corresponds to a point on the circle then that's the pretty big one.

Razi and Wilma's Choice of Tasks to be Used During a Lesson on Sine and Cosine

During the first clinical interview, each instructor was asked to present a task they felt was most important to discuss with students during an upcoming lesson on sine and cosine function. This section displays the tasks and discusses the meaning the instructors intended students to construct while interacting with the task(s). During the interview, Wilma presented the *Bug on a Fan* task (see Table 1) appearing in the Pathways Precalculus curriculum materials (Carlson & Oehrtman, 2010). The first part of the task prompted students to identify quantities that vary as the fan blade rotates. The prompt was followed by a request to explain how the bug's vertical distance above the horizontal diameter changes as the bug travels through various positions around the circle (i.e., from the 3 o'clock position to the 12 o'clock position). This task concludes with the students creating a graph to illustrate how the bug's vertical distance above the horizontal diameter covaries with the measure of the angle, θ, swept out by the bug's fan blade.

Table 1. A table showing Wilma and Razi's Chosen Task(s)

A bug sits on the end of a fan blade as the blade rotates in the counterclockwise direction. The bug is 2.6 feet from the center of the fan and is located at the 3 o'clock position as the blade begins to turn.	Kristin boards a Ferris wheel at the 3 o'clock position and rides the Ferris wheel for one full rotation (as shown below). The radius of the Ferris wheel is 14 meters. Let *s* represent the varying number of meters Kristin has traveled along the circular path sine the ride started.	The vertex of the angle below is at (0,0) and a circle with a radius of *r* units is centered at the vertex of the angle. The angle has a measure of θ radians. What are the coordinates of the terminal point, (x, y)? (Your answers should be expressions in terms of r and θ.
		
Wilma's Chosen Task	Razi's Chosen Task 1	Razi's Chosen Task 2

Wilma's discussion of the Bug on a Fan task primarily focused on her plan to ask students to attend to the quantities involved in the task and how the quantities vary together. Wilma expressed that she wanted to provide students with ample time to think about what quantities they could measure to track the bug's position on the fan. During the interview, Wilma also expressed that she intended to use an animation during her discussion of this task to provide students with an image of how the bug's vertical distance above the horizontal diameter changes as the bug travels around the fan.

The tasks Razi chose are also shown in Table 1. The Ferris Wheel task also included three parts. In the first part, students were asked to write an expression in terms of s to represent the number of radians Kristin swept out since the ride started. The remaining two parts of the Ferris Wheel task asked students to write an expression in terms of s to represent Kristin's height above the center of the Ferris wheel measured in radius lengths or meters, respectively. It is noteworthy that the Ferris Wheel Task included an animation of Kristin traveling along the Ferris Wheel; however, Razi did not express a plan to use the animation in class.

Razi's descriptions of the two tasks he intended to use in class (shown in Table 1) were primarily focused on describing how to determine the coordinates of a point on a circle and converting the values of those points when the radius of the circle was not one. It is noteworthy that Razi initially expressed that the value of the sine function represents a height in radians. However, when Razi was asked about the critical ways of thinking he wanted his students to engage in, he expressed a goal for students to "use the sine function to get information about wherever I am on the circle".

Discussion and Conclusions

The results of this study illustrate the influence of an instructor's MMT sine function on their goals for student learning and choice of tasks. During the clinical interview, Razi and Wilma expressed different meanings for the sine function. Wilma described the sine function as a function that relates angle measures to the vertical distance of a ray's terminal point above the horizontal diameter measured in radii. Wilma also expressed that the value of sine represents how many times as large the vertical distance of the terminal point is to the radius. When reasoning about the sine function, Wilma also consistently described how the vertical distance of the terminal point varied as the measure of the angle of interest varied. As such, it appears as though Wilma's meaning for sine function is grounded in quantitative and covariational reasoning.

In contrast, Razi consistently expressed that sine was a function that relates an angle measure from the unit circle to the y-coordinate on the unit circle. Although Razi consistently expressed that sin(θ) gives a y-coordinate on the unit circle, he was inconsistent in describing what the y-coordinate represents. For instance, when the radius of a circle is one, Razi expressed that the value of the sine function is the y-coordinate on the unit circle. However, when discussing tasks in which the radius of a circle was not one, Razi described the value of sine as representing (1) a ratio between the y-coordinate of the terminal point and the radius of the circle, or (2) "opposite over hypotenuse" (in reference to the commonly applied right triangle trig ratios SOHCAHTOA). It is noteworthy that Razi's meaning for sine included trigonometric ratios; however, he consistently referred to the sine function as relating an angle measure on the unit circle and a location in the cartesian plane (y-coordinate on the unit circle).

Both instructors' meanings for sine function were germane to the tasks they chose to discuss in class, their image of the meanings they wanted to elicit from students when discussing the tasks, and their goals for student learning. For example, when Wilma was prompted to describe her goals for students' learning, she expressed that she wanted students to think about quantities and how the vertical distance of the terminal point varied as the measure of an angle varied. In alignment with her goals and meanings for the sine function, the tasks Wilma used when teaching prompted students to engage in these ways of thinking. Similarly, Razi expressed that he wanted students to think about the sine function as a function that inputs an angle measure on the unit circle and outputs the y-coordinate on a circle. In alignment with his goals and meanings for the sine function, Razi selected tasks that prompted students to determine the location of a point on a circle. The influence of both teachers' meanings on their goals for student learning and choice of tasks is not surprising as it supports other's findings that teachers' MMT includes their image of how an idea is learned and tasks they envision for achieving this learning (Silverman & Thompson, 2008; Tallman 2015; Thompson 2013; 2016).

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Active-Learning Strategies That Suggest Ingresses for Math Graduate Student Instructors' Use of Student-Centered Teaching

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In the context of a peer-mentoring program for graduate student instructors (GSIs) who were teaching during their first year as instructor of record, we examined what active-learning (AL) strategies could provide ingresses for mathematics GSIs' use of student-centered teaching. Mentors observed Novices approximately six times during their first-year teaching, collected observation data including a standardized list of AL strategies, and conducted critical feedback conversations with each Novice after every observation. Data includes the number of AL strategies observed and discussed as well as pre- and post-survey data from Novices and Mentors on their perceived use of AL strategies. We found that Novice GSIs were primarily observed implementing Quick Polls, Think-Pair-Share, and Conceptually-Based Teacher Questioning. Each of these strategies were utilized by Novices differently throughout their first-year teaching. Our results provide insights for PD facilitators who work with GSIs, and for novice collegiate mathematics instructors.

Keywords: Active-Learning Strategies, Mathematics Graduate Student Instructors, Student-Centered Teaching

In collegiate mathematics classrooms, the instructor and their students interact around mathematics content in a fast-paced, (not entirely) predictable manner. The instructor is expected to make quick decisions about facilitating mathematics discussions, when/how to ask and answer questions, and how to utilize the class time to support students' progress. In this setting, our intention is to explore ways to provide professional development (PD) and support for novice college mathematics instructors encouraging their effective use of active-learning (AL) strategies. However, implementing AL strategies can seem overwhelming, especially for novice instructors during teaching. Thus, we endeavor to help a specific subset of novice college mathematics instructors, namely graduate student instructors (GSIs)[1], to efficiently navigate their use of instructional practices. This often-overlooked subset of instructors is a vital workforce, tasked with teaching hundreds of thousands of undergraduate students in foundational mathematics courses each year (Belnap & Allred, 2009). College mathematics instructors often have their first experience teaching as instructors of record as GSIs, with little or no pedagogical education, while being provided minimal support structures (Speer et al., 2005). As such, in our project, PD facilitators[2] work with peer mentors to help GSIs learn *when and how* to use AL strategies among other best teaching practices for collegiate mathematics teaching. It can be challenging as AL strategies often include teaching approaches that GSIs may have little-to-no experience utilizing or observing in their own learning experiences. Focusing on this subset of best teaching practices, we consider, what specific AL strategies provide natural paths for GSIs to improve their undergraduate mathematics teaching? Moreover, *how* can we cultivate AL strategies with GSIs for their future teaching?

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[1] GSI is used instead of TA (Teaching Assistant) because GSI specifically means graduate students who are instructors of record: responsible for every day classroom decisions and assessments in math courses.
[2] Mathematics faculty members who support GSIs in their learning to teach.

To summarize, we provide the following analogy. Since GSIs may need to learn about AL strategies in tandem with their first teaching assignment, we envision the vast collection of possible AL strategies that one could choose from as contained in an *AL Sea*. A role of a coach (e.g., a PD facilitator or mentor), in this analogy, is to help GSIs get comfortable kayaking tributaries so they can access the AL Sea, navigating it effectively. To access the AL Sea, GSIs could begin by kayaking along tributaries aligned with a specific AL strategy. Coaches could guide GSIs in navigating these tributaries, if they knew which of the tributaries may be most accessible to a majority of GSIs (e.g., easy to implement, not a steep learning curve, potentially applicable to a range of undergraduate mathematics learning environments or goals). GSIs kayaking along a tributary can begin getting comfortable in the water of AL strategies while getting closer to accessing a wealth of strategies in the AL Sea. We propose that these tributaries (specific AL strategies) could serve as natural *ingresses* for GSIs to begin navigating their AL strategy journey, leading to the potential use of additional strategies and more effective use of current strategies that foster improved student learning outcomes. Throughout this paper we consider, "what specific AL strategies provide natural ingresses for GSIs into the use of student-centered teaching?"

To identify the natural AL ingresses, we implemented a peer-mentoring program for novice GSIs[3], hereafter referred to as *Novices*, at three universities over five-years (Cite Grants Here). Through this peer-mentoring program, Mentor GSIs[4], hereafter referred to as *Mentors,* guided Novices during their first year teaching undergraduate mathematics and statistics courses. From observation data, follow up conversations between the Novice and Mentor after each observation, and survey data, we address our overarching research question, *what specific AL strategies provided natural ingresses into student-centered teaching for Novices?* To unpack this overarching question, we considered the following subquestions:

RQ1: What AL strategies did Mentors frequently observe and discuss with Novices?

RQ2: What type and frequency of AL strategies did Novices and Mentors report using?

To answer RQ1, we identified which AL strategies Mentors documented when observing Novices' classes, and which AL strategies Mentors discussed with Novices during post-observation conversations. We also analyzed the times during the academic year when AL strategies were observed and discussed. To answer RQ2, we analyzed responses to a self-reported survey from Novices and Mentors about their use of AL strategies.

Related Literature

Active Learning in Research

One foundational axiom across most AL definitions is that an AL strategy actively engages students in the content (Freeman, 2014; Anthony, 1996, Laursen & Rasmussen, 2019). Contextualizing the verb "engage" is where definitions diverge. Fundamentally, how a teacher engages students is content-specific, but most AL definitions lean heavily on contrasting AL with traditional lecture (Laursen et al., 2014; Freeman, 2014). In other words, the antithesis of engagement has been articulated as traditional lecture. Although it can be helpful to unpack non-examples when defining something, as Dewey (1938) argued one cannot define an educational approach simply by saying it is not another. Kyriacou and Marshall (1989) argued that there are two independent dimensions of AL (1) learning activities promoting collaboration and autonomy

[3] Defined as mathematics and statistics GSIs in their first or second year of teaching.

[4] Defined as GSIs who taught for at least 2yrs, and who were trained to mentor Novices.

and (2) meaningful learning experience for each student. Further, they said that the structure of an activity does not guarantee mental engagement: An AL "activity can foster either an active mental experience or a passive mental experience, just as a passive learning activity can foster either an active mental experience by passive or active learning activities, must be necessarily constructive (p. 350)." Thus, generating activities alone is not sufficient to avoid the passive mental experience that is often assumed to be the major shortcoming of the traditional lecture.

Context of Study

We consider the PD facilitator to be a seasoned kayaker who teaches newer but not completely novice kayakers (Mentors), who then teach the Novices, providing them with the information and support needed for beginners to learn the necessary basic knowledge and skills to enable them to navigate their craft safely and effectively. We implemented and facilitated the peer-mentoring program at three universities to support Novices and generate *communities of practice* (Wenger et al., 2002) among the GSIs. The peer-mentoring program prepared experienced GSIs to become Mentors through a semester-long professional development with the PD Facilitator that met for 15hrs over the course of a semester (Yee et al., 2021).

Mentors used a GSI observation protocol (GSIOP: Rogers et al., 2019) to observe each Novice teach (3x per semester) in foundational mathematics course (e.g., PreCalculus, Quantitative Reasoning, Introductory Statistics, and Calculus); Novices were mentored for two semesters. The GSIOP informed Mentors' feedback for the Novices (Yee et al., 2021) and focused Mentors' attention on AL strategies for the Novices to consider integrating into their teaching practices. Mentors submitted the number and types of AL strategies they observed, and those that they discussed at follow-up meetings with their Novices. In addition, Mentors and Novices responded to beginning- and end-of-academic-year surveys that included items regarding their perceptions of their use of AL strategies (14 strategies compiled by the Eberly Center from Carnegie Mellon University & higher education researchers; Faust & Paulson, 1998; Meyers & Jones, 1993). We provided both Mentors and Novices with guidance and examples of each of the 14 AL strategies (named in Table 1). Some terms were adjusted to make them more understandable to mathematics GSIs who may not be familiar with the educational vernacular. For example, instead of listing a broad category like "inquiry-based learning" we focused specifically on "conceptually-based teacher questioning" (C-BTQ) because it operationalized a classroom practice that Mentors could observe. It is important to note here that teacher questioning alone is not an AL strategy (Faust & Paulson, 1998). We included C-BTQ so that Mentors could concretely see examples of inquiry-based learning used with C-BTQ, where Novices elicited responses from all students.

Data Collection & Analysis

Data from two years at three universities included 293 total observations. We coded the observations by the semester in which they took place (first or second), and the timing in which they took place (beginning, middle, or end of semester). An AL strategy was *observed* if Mentors noticed it being used during the class period. Frequency of a single AL strategy used in a single class was not recorded (frequency of use did not imply improved quality). An AL strategy was *discussed* if the Mentor and Novice talked about the method during 1-on-1 follow-up conversations (e.g., ways to improve the use of an observed AL strategy or suggestions for future AL strategy use). To answer RQ2, we considered Novices' and Mentors' self-perceived use of the AL strategies and the observation data. We analyzed pre-post survey data where we asked three questions about the AL strategies: (1) their familiarity with the strategies, (2) how useful they perceived them to be, and (3) how often they used them in their own teaching. We report on our analysis of responses to the third survey question, triangulating our RQ1 findings.

Findings

Across the 293 observations, Mentors noted 278 instances of AL strategies in use. In 127 cases (43.3%) no AL strategies were observed. In the remaining 166 cases, Mentors observed an average of 1.63 AL strategies. Mentors and Novices discussed potential use of AL strategies 403 times, an average of about 1.4 strategies per follow-up meeting. Our findings elevate three of the fourteen AL Strategies studied, described as follows: *Think-Pair-Share*—students answer a question individually, synthesize a joint solution with a partner, and then share/discuss it with the class. *Quick Poll*—all students vote in response to a question. Votes can be tallied using technology or quickly by counting. *C-BTQ*—GSIs' questions are used in tandem with tasks for students to investigate, discover and/or apply concepts for themselves.

RQ1: Observed and Discussed AL Strategies

Table 1. Observed and Discussed AL strategies Descriptive Statistics

AL Strategy	Number Observed	Percent Observed	Number Discussed	Percent Discussed	Observed to Discussed Ratio
C-BTQ	68	24.46%	51	12.66%	1.3333
Quick Poll	64	23.02%	116	28.78%	0.5517
Think-Pair-Share	45	16.19%	108	26.80%	0.4167
Peer Review	20	7.19%	16	3.97%	1.2500
Self-Assessment Quiz	16	5.76%	18	4.47%	0.8889
Set It Up	16	5.76%	24	5.96%	0.6667
Brainstorming	14	5.04%	29	7.20%	0.4828
Minute Paper	9	3.24%	8	1.99%	1.1250
Jigsaw	2	0.72%	5	1.24%	0.4000
Muddiest Point	1	0.36%	17	4.22%	0.0588
Concept Maps	1	0.36%	7	1.74%	0.1429
Case Studies	1	0.36%	1	0.25%	1.0000
Other Strategies	21	7.55%	3	0.74%	7.0000

Note: *Other Strategies* means places where Mentors wrote in a Strategy that was not on our original list. Also, two AL Strategies (*Application Card* & *Role Playing*) from our original list were not observed or discussed.

Table 1 presents the number of times each AL strategy was observed and discussed, arranged in descending order of observed. We note that such limited use of 80% of the 14 strategies may indicate further training is needed for such techniques or that they are not natural ingresses for novices implementing AL strategies in collegiate math courses. To compare the data, we created a percentage from each strategy observed (or discussed) out of the total number of strategies observed (or discussed) for that period of mentoring. The right-most column (Table 1) shows some AL strategies were observed more often than they were discussed (ratio > 1), while others were discussed more often than observed (ratio < 1). The *Muddiest Point* strategy was observed only once but discussed 17 times (ratio of 0.0588), suggesting it may be a more difficult strategy for Novices to implement without further guidance. In a more even ratio, we see *Minute Papers* were observed being used 9 times and discussed 8 times (ratio of 1.125) and *C-BTQ* was observed four times to every three times it was discussed (ratio of 1.3333). Recall, *C-BTQ* required more than *teacher questioning*, but conceptual questions engaging students in a task.

The three most frequently observed AL strategies (*C-BTQ* 24.46%, *Quick Poll* 23.02%, *Think-Pair-Share* 16.19%) were the same three most frequently discussed (*Quick Poll* 28.78%, *Think-Pair-Share* 26.80%, *C-BTQ* 12.66%). They were not in the same order, but there is a 55% drop in use for the fourth most-observed strategy, *Peer Review*, (7.19%) and a 43% drop in use for fourth highest discussed strategy, *Brainstorming* (7.20%). Hence, the three observed and discussed AL strategies form a cluster and are separated noticeably from the remaining AL strategies. These results suggest the top three observed and discussed AL strategies require further analysis as they are likely to be viable ingresses for Novices. Based on these findings, we wondered if there was a connection between those methods discussed by Mentors after an initial observation and those observed subsequently. To investigate this latter possibility, we analyzed relationships between the discussion of these three strategies and their use in observations.

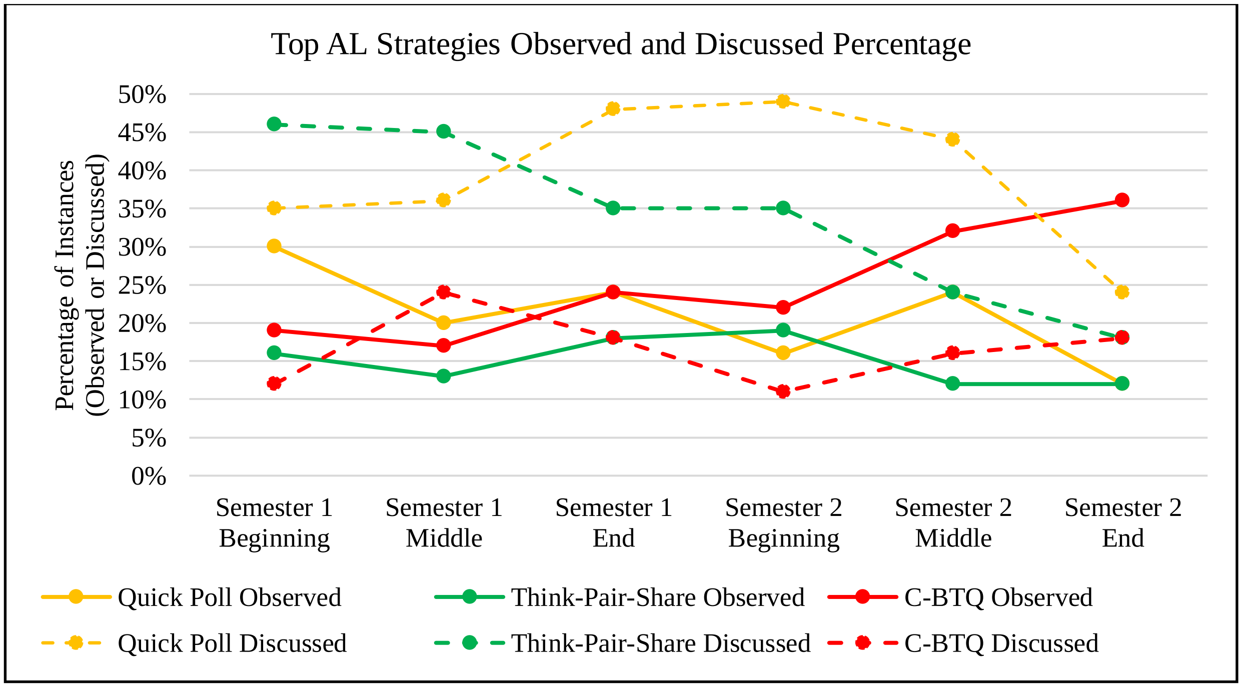

Figure 1: Top three Active Learning strategies by percent of time frame's method

For each of the strategies, we divided the data by when in the semesters the observations were conducted. We then computed the percentage of instances where the strategies were observed or discussed (Figure 1). We did not expect that discussed strategies would be implemented in a subsequent observation because there were weeks between subsequent observations during which time different learning objectives and instructional goals would also affect the Novices' choices of strategies. We were curious if there would be an increased likelihood that Novices would use strategies that had been discussed by their Mentors.

Looking across time (Figure 1), *Quick Poll* and *Think-Pair-Share* had discussed percentages consistently higher than observed. *Think-Pair-Share* had the largest differential between discussed and observed percentages at the beginning of first semester, while *Quick Poll* had the largest difference between observed and discussed percentages at the start of second semester. However, *C-BTQ* had observed percentages greater than discussed most semesters with the largest difference at the end of second semester, where *C-BTQ* observed percentage continued to increase. Thus, *Think-Pair-Share* had the greatest percentile difference with discussed greater than observed at the beginning of first semester, *Quick Poll* had the greatest percentile difference with discussed greater than observed at the beginning of second semester, and *C-BTQ* had the greatest percentile difference with observed greater than discussed at the end of second semester.

RQ2: AL Strategies and Survey Data

Of the 66 Novices and 16 Mentors who participated over 2 years, survey response rates were 56% and 100% from Novices and Mentors, respectively. These GSIs indicated their use of AL strategies by selecting Never, 1-2 times per semester, 3-6 times per semester, 7-12 times per semester, and Almost Every Class Session. We classified *regular use* if a GSI selected 7-12 times per semester or Almost Every Class Session (use of a strategy 7 or more times per semester translated to an average of at least once per two-week period). We defined the *most regularly used* AL strategies as those used at least once every two weeks by at least 50% of the respondents. Because we asked GSIs to indicate how frequently they used each of AL strategy, we compiled their responses to consider which strategies they reported using most often. This data supplements the observation data by capturing semester-long data. Table 2 shows which AL strategies at least 50% of the respondents said they used at least once every two weeks (*regular use*) and Mentors' Pre-Survey responses were comparable to Novices' Post-Survey responses.

Table 2. AL Strategies Novices and Mentors Cited Using Regularly

Novices' Pre-Survey	Novices' Post- & Mentors' Pre-Survey	Mentors Post-Survey		
1. Think-Pair-Share 2. Quick Poll	1. C-BTQ 2. Quick Poll 3. Think-Pair-Share	1. Quick Poll 2. C-BTQ 3. Set It Up	4. Think-Pair-Share 5. Brainstorming 6. Muddiest Point	7. Self-Assessment Quiz 8. Peer Review

We note that 50% or more of Mentors said they regularly used eight AL strategies, a four-fold increase from Novices' pre-survey responses. Table 2 also shows 50% or more of beginning Novices said they regularly used only two AL strategies (*Think-Pair-Share*, *Quick Poll*). By the end of their first-year teaching, Novices cited three strategies (adding *C-BTQ*). These three AL strategies were also the most frequently observed and discussed (*Quick Poll*, *Think-Pair-Share*, *C-BTQ*, Table 3). Thus, the pre-post survey findings (Table 2) corroborate the most frequented AL strategies observed and discussed by the Mentors (Table 1).

Discussion

Table 3. Summary of Findings for AL Ingresses for Novices GSIs

AL Ingress	**Observed: Discussed (Table 1)**	**AL Self-Report (Table 2)**	**Observed AL Trends (Figure 1)**	**Discussed AL Trends (Figure 1)**
Quick Poll	0.5517 (More discussed)	Novices reported using it right away	Initially observed often, and dipped down until middle of 2nd semester	Initially, consistently ~30%; highest in middle of 2nd semester
Think-Pair-Share	0.4167 (More discussed)	Novices reported using it right away	Consistently observed through both semesters	Discussed at beginning of each semester; decreased from there
C-BTQ	1.3333 (More observed)	Reported used more often 2nd semester	Nearly doubled observed use by mid 2nd semester	Not often discussed until middle of 2nd semester

We identified and triangulated three AL strategies (*Quick Poll*, *Think-Pair-Share*, and *C-BTQ*) to answer our over-arching question, *what specific AL strategies provided natural ingresses for*

GSIs into student-centered teaching? Our results provide a path for PD facilitators who work with GSIs, and for novice collegiate mathematics instructors. In the context of a peer-mentoring program utilizing regular observations and post-observation feedback (Rogers et al., 2019; Yee et al., 2021), we found that Novices were primarily observed implementing *Quick Polls, Think-Pair-Share*, and *C-BTQ*. Each of these strategies were utilized by Novices differently throughout their first-year teaching (as summarized in Table 3).

Implications

Our results build upon prior work when examining instructors' use of Muddiest Point Simpson-Beck (2011) argued that studying only one AL strategy may yield misleading results. In our study the *Muddiest Point* was observed only once but recommended 17 times. We interpret this result to mean that the *Muddiest Point* may not be a viable ingress for these Novices. Moreover, Angelo, and Cross (1993) argued AL strategies need to appeal to your intuition, should not be a burden, and provide benefits to you and your students, aligning with the role of Novices' natural ingresses. Among the 14 AL strategies we trained Mentors to observe and discuss, three rose to the top, and each AL strategy had unique characteristics in its observed and discussed use by Mentors (Table 3), suggesting that future studies could be designed examine longitudinal use of these three strategies and potential connections to student-learning outcomes.

One strategy, *C-BTQ* provided insights into a specific timeframe we should expect for its use to burgeon. Unlike *Quick Poll* and *Think-Pair-Share*, the ability to implement *C-BTQ* is likely a reflection of teaching skill and experience, rather than a teaching technique. *C-BTQ* requires a deeper understanding of the curriculum and pedagogical content knowledge (Shulman, 1986). To know what significant questions to pose (and how) for discovery and task-oriented learning, the instructor must be anticipating what significant topics are coming up. Thus, it was not surprising that the use of *C-BTQ* was often observed more in Novices' second semester teaching. This result could imply that PD programs interested in increasing collegiate mathematics instructors' use of *C-BTQ* may need to plan for more than one semester to support Novices' PD.

Conclusion

Initially, we asked how can PD providers cultivate AL strategies with GSIs for future teaching? We found one method that demonstrated growth in frequently observed and discussed AL strategies was providing time and ongoing PD via peer mentoring during that first critical year of teaching. Due to the variety of intentions and resources for mathematics departmental programs to educate current and future GSIs, there is inequity in PD opportunities because novice GSIs do not have equal access to the pedagogical support for using "best practices" as listed within the Instructional Practices Guide (MAA, 2018). Combating this inequity, we examined what natural ingresses Novices can first navigate with the support of a peer mentor to gain access to the larger set of AL strategies (the AL Sea). Of the three ingresses we identified, how and when they are used by Novices varies and so our findings provide guidance within each ingress differently. Ideally, these ingresses will provide a means for GSIs to access and have positive experiences using AL strategies, resulting in student-centered instruction to foster GSIs' long-term PD.

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Using Concept Images as a Framework for the Concept of Confidence Intervals

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Both the American Psychological Association (APA) and American Statistical Association (ASA) recommend that researchers move toward reporting interval estimates, such as confidence intervals. While previous literature in psychology and statistics education has documented many non-normative conceptions about confidence intervals, little is known about how individuals construct knowledge of confidence intervals nor what constitutes robust understanding of the concept of confidence intervals. This paper defines a formal concept image for the concept of a confidence interval. The formal concept image is based on the underlying statistical theory for deriving a confidence interval. The statistical theory is followed by a presentation of a formal concept image for confidence intervals and the interpretation of the interval and its level. The paper concludes with a brief statement of the affordances and limitations of the formal concept image.

Keywords: Statistics, Confidence Intervals, Concept Image

While a number of misconception studies[1] inform statistics education research about *what* individuals can communicate about interpretations and meanings of confidence intervals, little is known about *how* individuals construct their knowledge of confidence intervals. For example, previous studies have explored the knowledge retained about interpretations of confidence intervals (e.g., Belia et al., 2005; Crooks, 2014; Crooks et al., 2019; Fidler, 2005), interpretations of confidence intervals by learners (e.g., Andrade et al., 2014; Andrade & Fernández, 2016; Fidler, 2005; Grant & Nathan, 2008; Henriques, 2016; Kalinowski et al., 2018), and aids for instruction of confidence intervals (e.g., Bertie & Farrington, 2003; Gordon & Gordon, 2020; Hagtvedt et al., 2008). These studies included a diverse group of students (pre-service teachers, psychology undergraduate and graduate students, and general population undergraduate students) at different levels of statistical experience, with the aim of identifying common misconceptions subjects had about confidence intervals, but little work had been done to understand how these misconceptions form. As a result, there is no existing theoretical framework describing the required knowledge and understanding for a well-developed and connected understanding of confidence intervals – including how to interpret the necessary components. This paper describes a framework, in the form of a formal concept image, that can be used to elicit and study the conceptual knowledge of confidence intervals and uncover, from a cognitive perspective, the concepts and connections individuals have developed around the concept of confidence intervals.

Theoretical Perspective

This paper defines a formal concept image for the concept of a confidence interval based on the underlying statistical theory for deriving a confidence interval, and the interpretation of the interval and its level. A *concept image* is defined as the complete cognitive structure associated with a concept, including all mental pictures and associated properties and processes (Tall & Vinner, 1981, p. 152). This definition resulted from the difficulty in "[distinguishing] between

[1] The authors prefer the words *conception* or *resource* to *misconception*. The authors use the word misconception to reflect the intentions and meanings of the original studies cited.

the mathematical concepts as formally defined and the cognitive processes by which they are conceived" (Tall & Vinner, 1981, p. 151). In particular, Tall and Vinner discussed the difference between formally defined mathematical concepts (*formal concept definition*) and an individual's definition of mathematical concepts (*personal concept definition*). An individual's personal concept definition may deviate from a formal concept definition, and may vary because of a particular *evoked concept image*: "a portion of the concept image which is activated at a particular time" (Tall & Vinner, 1981, p. 152). This paper presents a formal concept image for the concept of CIs to be used as a foundation to analyze individuals' personal concept images. It is the authors' hope that the formal concept image defined here will be a foundational framework for future research.

Statistical Theory

The formal concept image describing the cognitive structures underlying understanding of confidence intervals was developed from the statistical theory for deriving confidence intervals and the related curricular concepts. A condensed summary of the statistical theory used as the basis for the formal concept image is provided in this section. Formally, *confidence intervals*[2] are defined as "interval estimators, together with a measure of confidence (usually a confidence coefficient)" (Casella & Berger, 2002, p. 419). In statistics, an *estimator* is a function, $T(x_1, x_2 \ldots x_n)$, of a random variable used to estimate an unknown value of a parameter, θ. In the case of a point *estimator*, this is represented by a single function of a random variable from a yet-to-be-collected sample. In the case of an interval *estimator*, this is constructed through two functions of a sample, $L(x_1, \ldots, x_n)$ and $U(x_1, \ldots, x_n)$, that satisfy $L(\boldsymbol{x}) \leq U(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathcal{X}$, the random interval $[L(\boldsymbol{x}), U(\boldsymbol{x})]$. A confidence interval, however, takes the idea of an interval estimator one step further by defining a *coverage probability*, $(1-\alpha)$, which is a term used for the probability that the random will contain the unknown value of the parameter, $P_\theta(\theta \in [L(\boldsymbol{x}), U(\boldsymbol{x})]) = (1-\alpha)$ or $P([L(\boldsymbol{x}), U(\boldsymbol{x})]|\theta) = (1-\alpha)$. The formula for the confidence interval estimator for a confidence interval for the population mean, μ, with an unknown population variance σ^2

$$\bar{X} \pm t_{n-1,\alpha/2} * \frac{s}{\sqrt{n}} \tag{1}$$

is derived an example of this statistical theory. Given that certain conditions are met, Equation 1 is derived starting with the *pivotal quantity*, T, which allows the researcher to model the probabilities associated with the estimators ($\bar{X}$, the sample mean, and *s*, the sample standard deviation) with known distributions not determined by either of the unknown parameters (μ and σ^2):

$$T = \frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_{n-1} \tag{2}$$

Using the *t*-distribution, find the endpoints of the region of the distribution that would satisfy the specified coverage probability (1 - α):

$$P\left(-t_{\alpha/2,n-1} < T < t_{\alpha/2,n-1}\right) = 1-\alpha \tag{3}$$

Substituting Equation 2 into Equation 3 and solving for μ, the confidence interval estimator is derived in a way that ensures, probabilistically, the desired coverage probability is achieved:

$$P\left(\bar{X} - t_{\alpha/2,n-1} * {S}/{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2,n-1} * {S}/{\sqrt{n}}\right) = 1-\alpha \tag{4}$$

[2] The theory of confidence intervals discussed from the frequentist perspective, referring to the belief that probabilities represent long-run frequencies of repeated random experiments.

Thus, Equation 1 is derived from functions of random variables: meaning Equation 1 represents a random interval.

When using Equation 1 in practice, however, the researcher has already collected (or is planning to collect) a sample. Once a sample has been collected, Equation 1 no longer represents a random interval because $\bar{X}$ and s are no longer estimators. Instead, $\bar{X}$ and s are point *estimates* as they are now a realized value of a function of a random variable used to estimate the unknown value of the parameter, μ. Since $\bar{X}$ and s are now fixed values from an actualized sample from a random variable, the probability statement represented by Equation 4 is no longer true because there are no longer any random values to which probability can be assigned. The difference in an *estimate* and *estimator* determines how and why a confidence interval is interpreted.

As an estimate, Equation 4 is either true or false: the value of the parameter, μ, is either in the interval (probability of occurrence is 1) or is not in the interval (probability of occurrence is 0). Since the value of the parameter is unknown, the individual calculating the interval cannot know whether the interval is one of the $100(1-\alpha)\%$ that captures the value of the parameter or one of the $100(\alpha)\%$ that does not capture the value of the parameter. Therefore, there was confidence level *probability* that the value of the parameter would be within the random interval, but there is confidence level *confidence* that the value of the parameter is within the actualized interval. *Confidence* refers to the probability associated with the random process – not the probability that the value of unknown parameter is within the actualized interval. Thus, in communicating the meanings of a constructed confidence interval, interpretations need to be clear about the difference between the process that was used to derive the *estimator* for the confidence interval and the *estimate* that has been calculated.

Formal Concept Image for Confidence Intervals

The theoretical underpinnings of a confidence interval presented above require learners to simultaneously coordinate meanings and ways of thinking across multiple curricular concepts in statistics. This section contains a description of a formal concept image for confidence intervals along with two of the "sub"-formal concept images that make up the larger formal concept image for confidence intervals. The goal is to provide a formal framework based on necessary cognitive structures from which individuals' personal concept definitions of the concept of confidence interval, of the interpretation of a confidence interval, and of the interpretation of a confidence level can be understood. This formal concept image is not intended to serve as a learning progression for concept of confidence intervals, rather as a guide to understand current knowledge based on prior learning, rather than a framework to analyze current learning.

Hypothesized concept clusters

It is hypothesized that individuals need to coordinate many statistics curricular concepts[3] when they construct their personal concept image for the concept of confidence intervals. Specifically, the model contains five concept clusters representing the pre-requisite and co-requisite knowledge needed for a robust understanding of the concept of confidence intervals: 1) parameter/statistic, 2) random process, 3) estimator/estimate, 4) sampling distribution, and 5) coverage probability. Requisite curricular concepts for statistical inference start with understanding the underlying purpose of inference: to use information provided in a sample to learn more about a characteristic or attribute of a population. The information typically derived

[3] The following descriptions are a brief summary of the curricular concepts within each concept cluster, for those seeking formal definitions of each of the statistics terms see AUTHOR (2020).

from a sample is summarized into a statistic, which should be an unbiased estimator of the unknown value of a parameter (a summarization of information gathered from a population).

Parameter/Statistic. Within this concept cluster, individuals need to understand the difference between the unknown value of the parameter, which is a summarization of a characteristic of the population, and the *actualized value of the statistic*, which is the point estimate calculated from a collected sample. Before an individual can begin to formalize conceptions of parameter and statistic, the individual must understand the difference between a population and a sample. Furthermore, the individual needs to coordinate information about 1) the population to which results can be generalized, 2) a sample, 3) a statistic, and 4) a parameter. Thus, the statistics curricular concepts within this cluster are: (a) observational unit, (b) datum, (c) variable, (d) types of variables, (e) population, (f) parameter, (g) sample, and (h) statistic.

Random Process. The concept cluster, *Random Process*, encompasses understanding the random process. A random process is a process that can be repeated many times under identical conditions for which any one instance is unknown, but for which the proportion of times a particular outcome occurs in the long run can be predicted (Tintle et al., 2021). This process helps explain the variability that exists in the population and sample data and can aid in developing a model for the long-run probabilities of particular outcomes. Understanding the difference between the assignment of probability to the model of the outcomes of a random variable generated from a random process and the lack of probability associated with collected data (a particular instance of the random process) is pivotal in understanding the idea of random process. The purpose of this concept cluster is to describe the ideas of probability and random processes (models designed to explain a real-world situation) that underlie the field of statistics. The following curricular concepts may require simultaneous or prior development of the Parameter/Statistic concept cluster: (a) random variable, (b) random process, (c) outcome, (d) relative frequency, (e) independence, (f) probability, and (g) sampling methods.

Estimator/Estimate. Understanding the interpretation of an actualized confidence interval is reliant on distinguishing between random variables and fixed (but unknown) values, which forms the basis of the concept cluster, *Estimator/Estimate*. A common conception of a confidence interval is the belief that the confidence level is the probability that the fixed, but unknown, actual value of the parameter is between the endpoints of the interval. This interpretation implies the random interval is a fixed interval and the fixed actual value of the parameter a random value of the parameter. Both are incorrect. The endpoints of the interval are random values based on the random statistic, which is based on the model of a random process from real-world situations. The actual value of the parameter is not known to the researcher, which explains the reason for inference, but the parameter of interest is defined. Understanding the distinction between the fixed unknown parameter and the random endpoints of the interval is not trivial and difficulties with this distinction may exist due to incorrect or incomplete knowledge of probability and random processes. Thus, the concepts being coordinated within this concept cluster are: (a) random variable, (b) actualized variable, (c) estimator, and (d) estimate. In addition to these formal concept image-specific concepts, an individual needs to coordinate the Parameter/Statistic and Random Process concept clusters.

Sampling Distribution. The *Sampling Distribution* concept cluster encompasses many concepts including the sampling distribution. The connection between confidence intervals and sampling distributions is very complex. The sampling distribution models the natural variability in statistics from samples generated by repeated sampling from a population (or repeated instances of a random process). The natural variability of the random statistics can be modeled by either theoretically-defined or simulated models, both of which provide a structure to understand how statistics generated by repeated sampling vary with respect to the associated parameter. This model forms the basis of the theoretical derivation of the formulas for confidence intervals. The Sampling Distribution concept cluster may incorporate: 1) properties of distributions (measures of center, measures of variability, shape, probability); 2) differences between distributions of populations, distributions of sample, and distributions of statistics (sampling distributions); and 3) characteristics of sampling variability (i.e. between-sample variability). Thus, the concepts that are being coordinated within this concept cluster are: (a) mean, (b) variability, (c) standard deviation, (d) distribution of a population and a sample, (e) theoretical and empirical probability distributions, (f) shape of a distribution, (g) sampling distribution of a statistic (including the mean, standard deviation, and standard error of the sampling distribution for a statistic), and (h) sample size.

Coverage probability. The *Coverage Probability* concept cluster is the foundation of the theory of a confidence interval. Within this concept cluster, the individual is hypothesized to be coordinating the ideas of Parameter/Statistic, Sampling Distribution, Random Process, and Estimator/Estimate but specifically within the context of a confidence interval. The Coverage Probability concept cluster is hypothesized to contain the ideas of confidence level, the confidence interval estimator, and the confidence interval estimate. Therefore, the specific concepts that an individual may be coordinating are: (a) confidence interval estimator, (b) coverage probability, (c) confidence level, and (d) confidence interval estimate.

Formal Concept Image for Confidence Intervals

The formal concept image for confidence intervals appears to have sub-formal concept images (gray diamonds, Figure 1), only two of which are described in this paper: interpretation of a confidence interval and interpretation of a confidence level. The curricular concept clusters discussed previously combine in different ways to create the sub-formal concept images (tan hexagons, Figure 1). Note that some of the curricular clusters are hypothesized to be dependent on each other, but those connections are not shown in the figure.

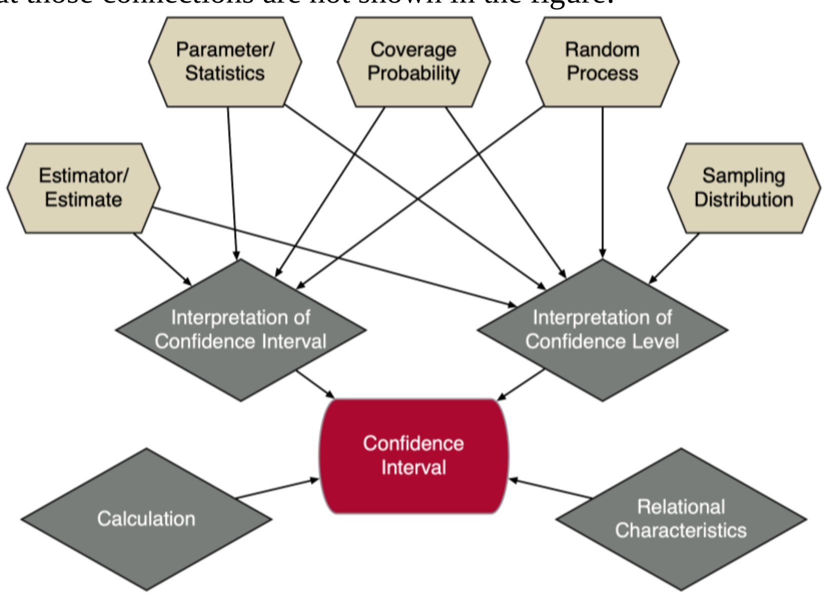

Figure 1: Formal Concept Image for the concept of Confidence Intervals

Formal Concept Image for Interpreting a Confidence Interval. The commonly accepted *interpretation of the confidence interval*[4] is: "We are 100(1-α)% confident that the calculated interval contains the true value of the parameter." *Confident* in this sentence implies that the user is confident in the random process of sampling and calculating confidence intervals rather than the probability that any one actualized confidence interval contained the value of the unknown parameter. An individual needs to coordinate several concept clusters in order to unpack the meaning of this sentence:

- $\mathbf{100(1-\alpha)\%}$ refers to the coverage probability that the interval estimator will capture the unknown value of the parameter of interest and requires the Coverage Probability concept cluster.
- **Confident** refers to the random process and coverage probability used to derive the confidence interval and requires the Random Process and Coverage Probability concept clusters.
- **Parameter** refers to a fixed, but unknown, value that summarizes a variable of interest (or attribute) within a population and is contained within the Parameter/Statistic concept clusters.
- **Between** is used to denote the actual value of the parameter is fixed and the interval "captured" the unknown value of the parameter not that the unknown value of the parameter was moving (see for more information: Callaert, 2007; Foster, 2014). This requires understanding the Estimator/Estimate and Random Process concept clusters.
- **(Lower limit) and (Upper limit)** are possible values for the unknown value of the parameter which requires the Parameter/Statistic and Estimator/Estimate concept clusters.

Formal Concept Image for Interpreting a Confidence Level. Traditionally, the interpretation of the confidence level is "Approximately $100(1-\alpha)\%$ of all samples of size n from a given population are expected to produce $100(1-\alpha)\%$ confidence intervals that will contain the actual value of the parameter of interest." Like the interpretation of a confidence interval, this interpretation is hiding several concepts that need to be unpacked:

- **Approximately** refers to the long-run probability derived from an estimated sampling distribution and the difference between theoretical and empirical distributions. This requires the concept clusters Sampling Distribution and Random Process.
- $\mathbf{100(1-\alpha)\%}$ refers to the Coverage Probability concept cluster of the given confidence interval.
- **All samples of size n from a given population** requires the concept clusters Parameter/Statistic, Sampling Distribution, and Random Process to understand the ideas of gathering all of the samples possible from a population.
- **Will produce** refers to the Random Process and Estimator/Estimate concept clusters, which allow an individual to understand the difference between an estimator and an estimate.

[4] The interpretation of a confidence interval is not without its own set of controversies. Morey et al. (2016) argued that the frequentist assumptions of traditional (Neyman) confidence intervals do not lend themselves to interpretation once an actualized interval has been created. While Morey et al. identified a difficult aspect of the interpretation of a confidence interval, the general consensus of the textbooks sampled appeared to side with some form of the interpretation: "We are 100(1-α)% confident that the calculated interval contains the true value of the parameter" (e.g., Agresti et al., 2017; De Veaux et al., 2018; Larson & Farber, 2019; McClave & Sincich, 2017; Triola & Iossi, 2018).

- $\mathbf{100(1-\alpha)\%}$ **confidence intervals** refers to the specific confidence interval of a given confidence level that had been constructed from each sample. This requires the Random Process and Estimator/Estimate concept clusters.
- **Will contain** refers to the ideas that the actual value of the parameter is fixed, and the interval is random. This requires the Estimator/Estimate concept cluster.
- **Parameter of interest** requires the Parameter/Statistic concept cluster.

The final part of the sentence that requires unpacking is the "$100(1-\alpha)\%$ of all samples … $100(1-\alpha)\%$ confidence intervals." These two percentages should be the same for the sentence to hold true. It is still a true sentence if the first "$100(1-\alpha)\%$" was larger than the second "$100(1-\alpha)\%$": "approximately 98% of all samples of size n from a given population will construct 95% confidence intervals that will contain the actual value of the parameter of interest." In this case, the researcher has overestimated the proportion of samples (98% instead of 95%) that should produce 95% confidence intervals that contain the actual value of the parameter of interest.

Affordances and Limitations

The formal concept image presented in this paper provided a theoretical framework for a foundational study into the cognitive structures required for a robust understanding of confidence intervals, allowing the analysis of the study data to be grounded in statistical content. A limitation of this formal concept image as an analytic framework was its specificity and focus on curricular concepts. The results from the study demonstrated that student personal concept images about interpretations of confidence interval and confidence level are more connected than defined through the formal concept image. Future iterations of a formal concept image for the concept of confidence intervals may need to demonstrate connections among the concept clusters prior to the development of "sub"-formal concept images for the interpretation of the confidence interval and the interpretation of the confidence level. Further research into applying the formal concept image to individuals' personal concept image for the concept of confidence intervals and into how individuals' personal concept images affect their knowledge of a confidence interval, interpretation of a confidence interval, and interpretation of a confidence level.

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The Implications of the Difference Between Estimators and Estimates in the Meaning of Confidence Intervals: Brody and the Jamie's Colleague Task

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Recently professional associations have called for increased use of confidence intervals in reporting inferential results. Little is known, however, about the knowledge structures necessary for a robust understanding of confidence intervals (CIs). This paper presents a case that demonstrates an important connection within a formal concept image for the concept of CIs. Understanding the interpretations of CIs and confidence levels (C-Lvls) requires understanding the probabilistic difference in a CI estimator and a CI estimate. Several tasks were developed to explore participants' demonstrated knowledge of this difference and its connection to the interpretations of CIs and C-Lvls. The case provides evidence of 1) the necessity of a hypothesized curricular concept cluster including the concepts of CI estimator/ estimate and 2) the importance of its connection to both the interpretation of CIs and C-Lvls.

Keywords: Statistics, Confidence Intervals, Confidence Levels, Estimator/Estimate, Case Study

News reporting of data and models reached unprecedented levels in 2020 as the world battled the coronavirus pandemic. Making sense of this information requires global citizens to be knowledgeable consumers of data and results of statistical inference. The two main frequentist[1] statistical inference techniques are hypothesis testing and confidence intervals (CIs). Of these, hypothesis testing and in particular, the use of the *p*-value cutoff, $p < 0.05$ for publication of scientific results, is ubiquitous (Hubbard, 2016, 2019). Both the American Statistical Association and the American Psychological Association, however, have issued statements cautioning against the over-reliance of *p*-values in scientific research (see: Wasserstein & Lazar, 2016; Wilkinson, 1999) and strongly suggest reporting point and interval estimates and/or effect sizes accompanied by a measure of uncertainty, such as a standard error or interval (Wasserstein et al., 2019). In fact, scientists already use interval estimates in reporting of modeling: for example, in predictions about the path and intensity of a storm or the spread of a disease or a contaminant spilled into a water source.

Communicating and understanding the uncertainty associated with CIs requires a robust understanding of the interpretation of 1. the CI and 2. the confidence level (C-Lvl). Complicating matters, the idea of CIs is a highly complex set of concepts within statistical inference requiring an individual to understand fine details in the derivation of the confidence interval procedure. In essence, there is a fundamental difference between an *estimator*, a function of a random variable used to estimate an unknown value of a parameter, and an *estimate*, a realized value of a function of a random variable (i.e. a value calculated from a collected sample) used to estimate an unknown value of a parameter. Probabilities lie with the estimator and become fixed when an estimate has been calculated. This is the theoretical underpinning of the interpretation of CIs. The authors hypothesize that developing a deep understanding of the difference in the estimator and estimate, particularly with respect to CIs, will help individuals be better consumers and producers of statistics. This paper presents an interesting case, selected from a larger study, that

[1] This paper discusses only frequentist CIs. Future work should study Bayesian credible intervals and bootstrap percentile intervals.

provides evidence of the high-level of connections among concepts necessary for a robust understanding of CIs presented in a hypothesized concept image for the concept of CIs. In particular, the participant discussed in this paper was the only participant in the larger study (n=11) to show development of knowledge associated with differentiating between an *estimator* and an *estimate*.

At present, the authors are unaware of research about students' knowledge of the difference between an *estimator* and an *estimate*. Current research has explored the knowledge retained about interpretations of CIs (e.g., Belia et al., 2005; Crooks et al., 2019) and interpretations of the conclusion and p-values from hypothesis testing (see: Castro Sotos et al., 2007). These studies illuminated what they call misconceptions[2], but little work had been done to understand how these conceptions form[3].

Theoretical Perspective

The theoretical perspective and underlying framework for this case study is based in the development of a concept image for the concept of confidence intervals. A *concept image* is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152) The construct of a concept image helps researchers distinguish between formally defined mathematical concepts, *formal concept images*, and a *personal concept image*, an individual's interpretation and coordination of concepts and images, which may or may not align with the formal concept definition (Tall & Vinner, 1981, p. 151). Roland and Kaplan (2022) present a formal concept image for the concept of confidence intervals. They hypothesize that the concept of a CI requires a complex coordination of several "sub" concept images and statistical concept clusters. This formal concept image was derived from the formal statistical definitions and curricular concepts required to understand the formal definitions. Roland and Kaplan created *statistical concept clusters* to describe the potential clusters of statistics curricular concepts. Since this theoretical framework is grounded in the statistical theory, a brief review of the necessary statistical theory will be discussed prior to describing the concept image of the concept of confidence interval.

Statistical Theoretical Background

There are two typical expressions of the frequentist *interpretation of the confidence level*: 1) approximately 100(1-α)% of all possible 100(1-α)% CIs should capture the unknown value of the parameter and/or 2) prior to collecting data, the probability of a 100(1-α)% CI capturing the value of the unknown parameter was 100(1-α)%. These interpretations are fairly straightforward because both describe the coverage probability either in terms of long-run probability or the probability associated with a random sample from a random variable. The commonly accepted frequentist *interpretation of the confidence interval*[4] is "We are 100(1 -α)% confident that the calculated interval contains the true value of the parameter." *Confident* in this sentence implies

[2] The authors prefer the words *conception* or *resource* to *misconception*. The authors use the word misconception to reflect the intentions and meanings of the original studies cited.

[3] For current literature on how learners conceptualize CIs see Andrade et al. (2014), Andrade and Fernández (2016), Fidler (2005), Grant and Nathan (2008), and Henriques (2016).

[4] While Morey et al. (2016) make a reasonable argument that the frequentist assumptions of traditional (Neyman) CIs do not lend themselves to interpretation once an actualized interval has been created, the general consensus of textbooks appears to include interpretations similar to that presented here (e.g., Agresti et al., 2017; De Veaux et al., 2018; Larson & Farber, 2019; McClave & Sincich, 2017; Triola & Iossi, 2018).

that we are confident in the random process of sampling and calculating CIs rather than the probability that any one actualized CI contains the value of the unknown parameter.

The derivation of a CI is based on the assumption of the existence of random components (point *estimators*, such as $\bar{X}$ and *s*) and fixed, but unknown components (parameters, such as μ and σ). The use of point estimators allows the researcher to leverage theoretical probability models to create an interval *estimator* (specifically, a confidence interval *estimator*) with random endpoints. The width of the interval estimator corresponds to the probability the CI estimator will capture the true parameter, which is called the *coverage probability* $(1-\alpha)$. Roland and Kaplan (2022) provide a detailed example, a CI for the population mean (μ) with an unknown population variance (σ^2), to demonstrate this statistical theory. Summarizing the example, the endpoints of the region of the distribution that would satisfy the desired coverage probability (1 - α) and solving for μ, the confidence interval estimator is derived ensuring, probabilistically, the coverage probability:

$$P\left(\bar{X} - t_{\alpha/_{2},n-1} * {}^{S}/_{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/_{2},n-1} * {}^{S}/_{\sqrt{n}}\right) = 1-\alpha \tag{1}$$

Recall that $\bar{X}$ and *s* are functions of random variables, meaning the equation represents a random interval with an associated coverage probability, called the confidence level (C-Lvl). Since $\bar{X}$ and *s* are now considered *estimates*, however, the equation no longer has meaning because there are no random components in the equation. As an interval estimate, the statement is either true or false: the value of the parameter, μ, is either in the interval (P(parameter) = 1) or is not in the interval (P(parameter) = 0). Since the value of the parameter is unknown, the individual calculating the interval cannot know whether the interval is one of the $100(1-\alpha)\%$ that captures the value of the parameter or one of the $100(\alpha)\%$ that does not capture the value of the parameter. Therefore, there was C-Lvl *probability* that the value of the parameter would be within the random interval, but there is C-Lvl *confidence* that the value of the parameter is within the actualized interval (the interval estimate based on the collected sample data).

The Formal Concept Image of the Concept of Confidence Interval

There are five statistical concept clusters (tan hexagons in Figure 1) defined in Roland and Kaplan (2022): 1) Parameter/Statistic, 2) Random Process, 3) Sampling Distribution, 4) Coverage Probability, and 5) Estimator/Estimate. Figure 1 summarizes part of the overall formal concept image of a confidence interval (red box), including two "sub"-concept images (gray diamonds): 1) interpretation of a CI and 2) interpretation of a C-Lvl. This case study explores the evidence for the existence of the Estimator/Estimate concept cluster, and its connection to the interpretations of a CI and C-Lvl.

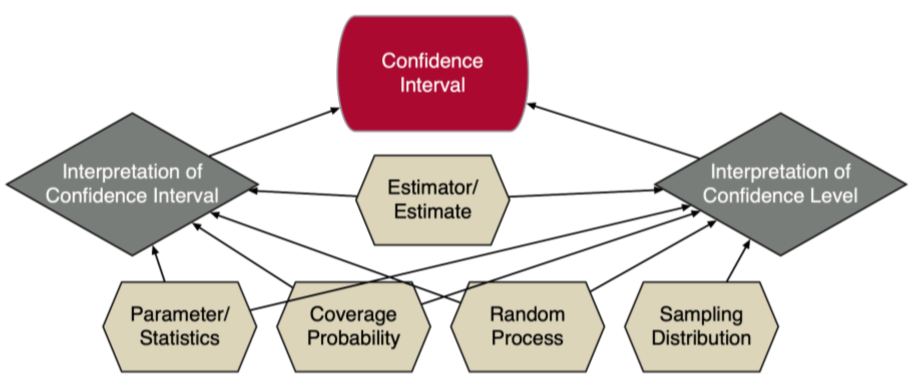

Figure 1: Modified Concept Image for the Concept of Confidence Intervals

The statistical concept cluster, Estimator/Estimate describes the statistical curricular concepts needed to understand the difference between the estimator and the estimate and the implications of the differences in the interpretation of an actualized CI and the confidence level. Thus, the concepts being coordinated within this concept cluster are: (a) random variable, (b) actualized variable, (c) estimator, and (d) estimate. To demonstrate knowledge within the Estimator/ Estimate concept cluster, an individual should be able to explain how the difference between the estimator and the estimate affects the interpretation of the CI and the interpretation of the C-Lvl. Using the case of Brody as an example, this paper demonstrates the connections between the interpretation of the CI and CL reliant on the Estimator/Estimate cluster within the formal concept image of the concept of confidence intervals.

Methods

Brody[5] is an interesting case from a larger study. He, along with the other participants, were students at a large research-focused institution in the Southeastern part of the United States. Brody was a senior undergraduate dual major in statistics and theology enrolled in the statistics capstone course in the Fall 2019 semester, when data were collected. The senior capstone course for statistics majors is 3-credit year-long seminar course centered around a data-based project. Brody also reported he had completed most of the requirements for his degree in statistics, including a 2-semester mathematical statistics sequence.

Brody engaged in three clinical interviews each of approximately one hour in length over the course of the semester. The first two interviews consisted of open-middle tasks (Bell & Burkhardt, 2002; Yeo, 2017), all with correct answers. The interviews were mostly conversational in nature and were designed to elicit, among other things, the conceptualizations participants had about the interpretation of CI and interpretation of the C-Lvl. The third interview piloted an activity designed to introduce students to the concept of CIs by developing deep connections between the C-Lvl and the derivation of CIs. The data presented in this paper center around the curricular concepts within the Estimator/Estimate concept cluster. When asked to define the word *confident* in the interpretation of a CI in the first interview, several participants explained that confident did not mean probability with a statement exhibiting the fundamental difference between the estimator and the estimate: *there is no longer 95% probability, the interval either contains the parameter or it does not*. Based on these statements, the authors created three tasks to explore the concept cluster:

1. *Task Statement 1*: There is a 93% probability that the actual mean monthly rent for students at HTSU is within the interval $\bar{x} \pm \left(t^*_{n-1}\frac{s}{\sqrt{n}}\right)$.
2. *Task Statement 2*: The process used to generate confidence intervals will capture the actual mean monthly rent for students at HTSU approximately 93% of the time.
3. *Jamie's Colleague*: Jamie[6] talked to a fellow reporter about constructing 93% confidence intervals. Jamie's colleague said that prior to collecting his sample, there is a 93% probability that the confidence interval will capture the actual mean monthly rent of all HTSU students. The colleague continued the explanation by saying that once Jamie collected a sample, the probability of the interval ($705, $793) actually containing the mean monthly rent of all HTSU students is now either 0 or 1.

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[5] a pseudonym, demographic information and preferred pronouns were not gathered at the time of the study, the use of the gendered pronoun is based on the researcher's assumption of outward projection of gender

[6] In the previous task, Jamie, a recurring character in the interview, had constructed a 93% CI for the mean monthly rent for students at a fictional university, Hill Top State University and was found to be ($705, $793).

Task Statement 1 is a true statement about the CI estimator presented in a non-traditional way. The sentence is Equation 1 in narrative form. Task Statement 2 is a true statement about the interpretation of the C-Lvl, also in a non-traditional way. As it is not the typical statement used to interpret the C-Lvl, participants may not have seen it before. These statements set up the ideas summarized in the *Jamie's Colleague* task. The colleague is correct in identifying that probability does exists prior to collecting data (discussing the CI estimator), but does not once a sample has been collected (discussing the CI estimate). This task describes the reason why the statement provided by participants in the first interview is correct. Since little was known about the connections students make when conceptualizing and reasoning about CIs, a *generative* research approach, defined by Clement as "generat[ing] new observation categories and new elements of a theoretical model in the form of descriptions of mental structures or processes that can explain the data" (2000, pp. 332–333) was used. Methods proposed by Powell et al. (2003), which are similar to grounded theory-inspired methods, were used as a guide to analyze the task-based clinical interviews.

Results

In the first interview, Brody said the statement of the interpretation of a CI typically taught is "formulaic." He continued his critique of this interpretation explaining the incorrect conceptualization people often make: understanding this sentence as reference to a probability that the parameter is within the CI. Brody used the explanation that the probability of the actualized interval either capturing (1) or not capturing (0) the actual value of the parameter to explain how the interpretation of the [actualized] interval does not a include probability. He saw the word confident as indicating a CI was used and was quite clear that confident did not mean probability. While this is correct, Brody was very uncomfortable with any use of the word "probability" with respect to interpreting the CI. Brody initially rejected Task Statement 1 as incorrect. When questioned about difference between whether the sample has or has not already been collected, Brody was able to make the distinction:

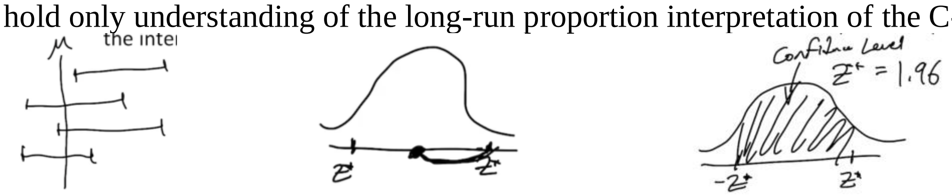
> Yes, in that if you've already collected it, then … the $\bar{x}$ is no longer a random or it would still be a random variable, but you know the value of it. Whereas with if you haven't done yet, obviously, you don't know the value.

He was, however, still uncomfortable with the use of the word probability. He continued by saying "You could look at probability differently on what is the probability of obtaining a sample monthly rent. …But I don't think it changes the probability for the actual mean monthly rent."

Brody was also able to articulate the interpretation of the C-Lvl clearly. He seemed to prefer the use of visuals with his explanations and drew many different pictures to demonstrate his depth of knowledge of the concept of CIs. Over the course of the first and second interviews, Brody drew an image that is a common display used to demonstrate the long-run proportion of C-Lvl% CIs that capture the parameter, similar to Figure 2a. This particular drawing was used to illustrate Task Statement 2. He described that 93% of the confidence intervals (the horizontal lines) would capture (indicated by the horizontal lines that cross the vertical line) the parameter (the vertical line labeled μ). He continued this explanation by drawing Figure 2b. Previously in the interview, Brody drew Figure 2c, which visualizes the region in the theoretical sampling distribution where 95% of the statistics calculated from repeated sampling from the same population are within a given critical value ($\pm z^*$in Figure 2c) of the unknown value of the parameter. Having already discussed Figure 2c, Brody explained:

I think of how we look at the distribution, we'd expect …93% [of the statistics to fall] within these like these like z^* or t^*[7]. And we'll have a margin of error that's this long [drawing the line connecting the dot with the z^*]. And so 93% of the sample values will include whatever is at the center [pointing to the dot], right? And, obviously, approximately, if we do anything less than infinite sampling, and we're not going to get an exact representation.

These visualizations and explanations demonstrate the interpretation of the C-Lvl, but only describe the long-run behavior and the lack of probability of capturing the parameter for an actualized CI. This imagery does not make a connection between the long-run proportion and the theoretical reason for its truth: the difference in the estimator (and associated coverage probability) and the estimate. Brody had not discussed any connection between the estimates and the probability associated with the estimator. Brody's response to the inclusion of the word probability, as discussed above, indicates that his explanation of the interpretation of C-Lvl did not include knowledge of the difference between an estimator and an estimate. Rather, Brody seemed to hold only understanding of the long-run proportion interpretation of the C-Lvl.



a b c

Figure 2: Brody's drawings used to explain the interpretation of a confidence level

Brody had demonstrated his isolated knowledge about the interpretation of the CI and interpretation of the C-Lvl, separate from any probability implications. At the end of his discussion about Task Statement 1, Brody began to make connections between his interpretations of a CI and a C-Lvl and the idea that probability could be discussed prior to collecting a sample.

In approaching the *Jamie's Colleague* task, Brody stated, "It's actually kind of makes sense. And I don't know if it's just because the way I was taught, I think, is that like, instinctively you go, it's like, oh, like it has to be 0 or 1." To punctuate this point, Brody said,

So, I guess it would be true that it's not until you set those parameters [Brody's gesture indicates he means the bounds of the interval, not statistical parameters] that it either is or it isn't. … But if we haven't taken our sample yet, there is still a chance that it could be or that could not be and so I think your colleagues right. I don't know. This is the most in depth I've actually explored this, I haven't really thought about it. … Because I've always been taught that like, it's not a probability because it's set already and that like that's true. … But when it's not set, I think this [referring to *Jamie's Colleague* task] is correct.

Brody acknowledged this as a new thought process and different from his statements after Task Statement 2. By directing him back to Figure 2c and reminding him of his previous statements about Figure 2c, Brody continued:

If 95% of these are in this range, that means 95% would give me intervals that include the true parameter. And since this is the sampling distribution for a sample mean of size, whatever we're using, there's a 95% chance that a random value chosen from this distribution would be on the interval [referring to the area in the sampling distribution that contains central 95% of the statistics]. And so yeah, there is a 95% chance that my

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[7] Brody had already demonstrated he knew the context required the *t*-distribution. He tended to use the standard normal since he already knew (without calculation) the critical values.

> sample yields a confidence interval that includes the true proportion, or true mean… But we can only say that before we do the sample.

At the conclusion of this task, Brody was able to connect the long-run proportion of CIs that capture the parameter (Figure 2a) with the percentage of statistics within a given region of the sampling distribution (Figures 2b and c) with the Estimator/Estimate concept cluster. Thus, the *Jamie's Colleague* task ultimately helped Brody develop and demonstrate the missing connection between the concept cluster Estimator/Estimate and the sub-concept images for the interpretation of the confidence interval and the interpretation of the confidence level.

Conclusion

Brody was the only participant in the larger study who demonstrated robust conceptualizations of the concept of a CI, the concept of the interpretation of a CI, and the concept of the interpretation of a C-Lvl. From his first interview, it was clear Brody had a deep conceptual understanding of the concept of CIs, the concept of the interpretation of a CI, and the concept of the interpretation of the C-Lvl. These conceptualizations, however, seemed to be isolated: there did not appear to be connections between the concept of a CI and how to interpret the C-Lvl and the CI. Brody used visualizations to demonstrate his knowledge of these concepts and often used imagery that are standard textbook practice for explaining interpretations of CIs and C-Lvls. As Brody demonstrated, these images and statement like "confidence does not refer to probability because the parameter is either in the interval or it is not in the interval" when discussing or explaining the interpretation of a CI do not necessarily aid in helping students develop the necessary connections for robust understanding. The is/is not in the interval statement requires just as much unpacking as do the correct interpretation of the CI and the C-Lvl. In fact, the meaning of the is/is not statement is perhaps more difficult to unpack than either of the interpretations studied. The face-value interpretation of this statement is fairly easy to explain:

> If I were to toss a coin, there is a 50% chance that it lands on heads. Once I toss the coin, even without revealing the outcome, there is no longer a 50% chance it landed on heads. It either landed on heads (P(head) = 1) or it landed on tails (P(head) = 0).

Unfortunately, this explanation does not clarify the underlying theoretical connections between the is/is not statement and the difference between the estimator and the estimate. Many participants in the larger study made statements similar to the is/is not statement, but it was unclear from their demonstrated knowledge whether they fully understood this statement. *Jamie's Colleague* was invented to determine the strength of the connections. Brody was the only participant who was able to make the necessary connections.

The use of the *Jamie's Colleague* task helped Brody make important connections among his well-developed conceptualizations of the interpretations of CIs and C-Lvls. This final connection, described by the curricular concept cluster Estimator/Estimate was needed to solidify the distinction between the use of *confident* rather than *probability* in the interpretation of CIs. Further, this provides conceptual understanding of the theoretical derivation of the confidence interval estimator, particularly with respect to the coverage probability. The long-run proportion of CIs that capture a parameter Generally, Brody's demonstrated conceptualizations were unusual in the larger study, even among peers who had taken the same courses. Finding more students with his level of conceptualization and identifying how they generated their robust understanding of CIs will aid future instruction of CIs.

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Values and Norms of Mathematical Definitions

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Definitions play an important role in mathematics by stipulating objects of interest to mathematicians in order to facilitate theory building. Nevertheless, limited research has examined how mathematicians approach writing definitions or the values they seek to uphold through definitions. Based on interviews with nine mathematicians, we highlight key mathematical values upheld through definitions as well as values related to defining that mathematicians do and do not attempt to emphasize through their instruction.

Keywords: Definitions, Norms, Values, Mathematical Culture

Definitions are central to mathematical activity (Edwards & Ward, 2004) as they specify objects used for proof activity. Nevertheless, limited research has directly examined how mathematicians approach writing definitions or how mathematicians view the role of definitions in perpetuating mathematical norms and upholding mathematical values. We examine these aspects as well as consider how these norms and values relate to instruction of definitions.

Literature Review

Extensive research has examined various facets of proof understanding and instruction. Examples include examining the purpose of proofs (e.g., Weber, 2002), the logic of proofs (e.g., Selden & Selden, 1995), formats of instruction (e.g., Weber, 2004), differences in proof-production in different disciplines (Dawkins & Karunakaran, 2016), how students understand proofs (e.g., Stylianides & Stylianides, 2009) and differences in students' and mathematicians' approaches to proof (e.g., Weber & Alcock, 2004). However, limited research has focused specifically on the role of definitions in proof or mathematics more generally.

Most literature on definitions has focused on students' difficulties. Edwards and Ward (2004; 2008) highlighted students' reliance on extracted definitions, where students tried to work from examples to create a definition for themselves rather than work with the stipulated definitions (written to define a concept) that they were given, as well as students' tendency to act as though their concept image was more important than the concept definition. Similarly, Alcock and Simpson (2011) highlighted students' inconsistency in applying given stipulated definitions for increasing and decreasing to particular sequences. Implicit in these works is the mathematical community's belief in the importance of stipulating definitions appropriately. Nevertheless, work directly asking mathematicians about their process for making or teaching definitions is limited.

Theoretical Perspective

In order to examine how mathematicians' views of definitions align with their broader views of mathematics, we used the theoretical lens of mathematical values created by Dawkins and Weber (2017) in the context of proof. In their paper, they highlighted four mathematical values upheld by norms of proof as addressed in the literature. These values were:

1. Mathematical truth is a priori
2. Mathematical knowledge and justification should be independent of (non-mathematical) contexts, including time and author
3. Proofs should increase mathematicians' understanding

4. Mathematicians desire a consistent set of norms and practices

While Values 1 and 2 are related, Value 1 emphasizes the logical and objective nature of mathematics, while Value 2 especially focuses on the role of the author being irrelevant to final presentations of mathematical knowledge. For our purposes, we are modifying Value 3 to say *definitions* should increase mathematicians' understanding, with the idea that definitions also contribute to how mathematicians understand theory. Value 4 highlights the notion of a shared mathematical culture that views mathematical knowledge as possessing a standard form. We used this framework to organize norms of definitions as described by participants to see whether these four values (or others) were supported by mathematicians' views of the role of definitions.

Methods

Data was collected from interviews conducted remotely with nine mathematicians. These participants were a subset of those recruited for a previous study targeting mathematicians who had taught at least one abstract algebra or category theory course in the last five years. (All names are gender-neutral pseudonyms.) For this paper, the first two interview questions about how they approach writing a new definition and the purpose of definitions in math are the most relevant, although some participants made relevant comments about definitions later in the interview when discussing isomorphism and homomorphism, which were also coded.

Two independent researchers used thematic analysis (Braun & Clarke, 2006) which included multiple iterations of coding (Anfara, Brown, & Mangione, 2002). First, two transcripts were open-coded using descriptive coding (Saldaña, 2016), noting themes about the construction or use of mathematical definitions highlighted by participants. Then, the two researchers discussed the open codes and produced a first round of focused codes, which were created by grouping the original open codes into categories. Next, all nine transcripts were coded with these focused codes and additional codes were developed for any relevant themes appearing more than once in later transcripts. Any discrepancies in coding were discussed by the researchers until agreement was reached. The focused codes were then examined to come up with axial codes, which led to the adoption of the theoretical framework from Dawkins and Weber (2017) for categorizing codes. Finally, focused and axial codes were revised and transcripts were recoded as necessary.

Results

Following the norms and values of proof laid out in Dawkins and Weber (2017), our themes (axial codes) are grouped into four similar values with focused codes (italicized) as evidence.

Value 1: Mathematical truth is a priori

In order for mathematics to be rigorous, proof must be based on stipulated definitions that are unambiguous (Dawkins & Weber, 2017; Edwards & Ward, 2008). These codes are related to this norm of proof in that they highlight that there is a norm of precision in writing mathematical definitions and that this precision is necessary for using definitions to support arguments.

Mathematical communication norm of precision. The code *axiomatic system/work with definitions* was given to five participants who referred to definitions as part of the axiomatic method or as the building blocks of a particular theory in mathematics. Avery said:

> And definitions [are] a critical part of the axiomatic method. This last spring, I taught my students an axiomatic geometry course, and so the first couple of weeks is pretty heavy on just the structure of logic. Day two, we defined the axiomatic method. We have our

> undefined terms. We have our axioms. We have our definitions…the definitions are those things we define for this theory because we're going to talk about them all the time.

Like others given this code, Avery believes mathematical definitions are important to the axiomatic method and students need to understand how to work with them for this reason.

Eight participants highlighted the norm of mathematical definitions being unambiguous (*clarity/no ambiguity in communication*). Several of these participants emphasized this lack of ambiguity is important for communicating with other mathematicians. For instance, Greer said:

> I think fundamentally the purpose [of definitions] is to make clear what you're talking about. I do have trouble conversing with mathematicians who don't make precise definitions because I can't understand their arguments because I don't have this precise set of rules I'm allowed to work by.

Other participants tended to use similar language when talking about this topic, describing definitions as an agreement of rules between the author and the reader. Five participants noted the norm of rigor in definitions, often without clarifying exactly what that means in a math setting, and were given the code *rigor/precision*. This included phrases like "it would be defined properly, rigorously" (Avery) or "it's fundamental for rigorous mathematics to use clearly stated definitions" (Dallas). These participants may have chosen not to elaborate because they assumed the interviewer, as a member of the mathematical community, would know what was meant.

Norms perpetuated through teaching. The code *norm of precision perpetuated through teaching* arose in all nine interviews. Here, Emerson described their approach to teaching definitions in proof courses such as abstract algebra:

> I think the … right place to start proof in formal methods is to talk about definitions. And so I spend the first couple of days just practicing definitions. So why is this definition good or bad? You don't want to define something in terms of what it's not. You should define it in terms of what it is. It needs to be unambiguous. And it should be something that you can use. If it's not clear enough from what you call a definition, how you're going to be able to use this quote "definition" in a proof then it's not a good definition.

Notice Emerson believes that a proper understanding of what a good definition is aids discussing the role of definitions, which then supports teaching proof.

Value 2: Mathematical knowledge and justification should be independent of (non-mathematical) contexts, including time and author

The second value of proof in Dawkins and Weber (2017) is that mathematical knowledge and justification should be independent of non-mathematical contexts. Related to Value 1, many participants spoke about being "on the same page" (Cameron) with regards to definitions. However, only Avery was explicit that definitions should be written in a way that avoids multiple interpretations based on the readers' experience:

> As different people have sort of different personal, cultural connotations that they have, we do have to make sure that we are talking about the same things. So we need a rigorous definition because, I mean, if you take a word like group…that has a very specific meaning in mathematics, but just in standard English, there are lots of ways you could use that word, sometimes good, sometimes bad…And that's the thing is we have to be prepared that someone's life experience might have prepared them to hate a certain word, and therefore, we need to make sure that we see exactly what it is.

Notice Avery focused on personal, context-dependent ways of interpreting mathematical definitions as being excluded by writing a proper definition.

Value 2.5: Mathematical knowledge and justification are generated by people with agency

We propose an addition to Dawkins and Weber's framework that highlights that definitions are specifically chosen by mathematicians both to communicate results effectively and for purposes of exploration in research.

Definitions written purposefully by people. All nine participants stressed that definitions are written by people who have agency and actively choose how to frame the definition. This idea was given the code *people make definitions*. For example, Hayden said:

> I mean, often, I guess I'm working in something for a while before some of the definitions are crystallized. The ones that become the sort of bones on which I want to sort of build the story.… I think of them as part of building a story rather than something that exists a priori.

This is in contrast to Dawkins and Weber's Value 1 that mathematical truth is a priori, although still compatible since the emphasis is on a tool for discovering truth rather than the conclusion. Many quotes coded in this way emphasized that mathematical definitions are actually quite flexible and are crafted specifically for a purpose. The latter idea was coded as *write definitions purposefully for utility,* which also occurred in all nine interviews. For example, Greer said, "I was trained to think of the definition as the set of properties needed to prove a theorem" and then later when talking about students, "It's hard to get into this conversation with students about [alternate] definitions because I think of them as so flexible because I think of them as just designed to make the theorems true." Again, we see that mathematical definitions are often written in such a way that makes them easy to use in particular results, though Greer acknowledges that this idea is not necessarily easy to communicate to students.

Definitions created to explore in research. Many participants pointed out the importance of writing definitions to aid the research process, whether through generalizing examples or homing in on a particular topic. Blair was given the broad code of *definitions as a way to explore* for saying things like "When you're originally mapping out areas that you're not even necessarily sure what consequences are going to look like, definitions are ways of focusing on particular areas of that map and then being able to explore the territory around you." Blair clearly invoked an exploration metaphor, though the manner of exploration was not explicit.

Others were clearer about mechanisms for exploration. Seven participants talked about the fact that *definitions arise from/relate to examples* in their research. For instance, Finley said:

> I guess it comes about in the way that it would for most mathematicians—that you have some examples in mind and you recognize that they have common features. And at some point, you try to write down those common features, and those become the definition.

Seven participants similarly mentioned that *definitions permit inspection of relevant features*, often noting the separable axioms embedded in definitions. For instance, Blair noted:

> So when you're researching different ideas and different structures within existing frameworks, certain adjectives and hypotheses and settings begin to present themselves as being useful of specifically naming and studying. And so as those come up, you go back and create definitions that pull out the relevant features from whatever you're thinking about so that you can explore just those structures with those features.

These codes all highlight the idea that while mathematical definitions may arise from existing mathematical objects or examples, they are specifically constructed by mathematicians.

Value 3: Definitions should increase mathematicians' understanding

This value is a slight reframing of that in Dawkins and Weber (2017) in order to center on definitions. While this value emphasizes mathematicians' understanding, we found that the norms that served to uphold this value focused specifically on characteristics of good communication.

Good mathematical communication helps explain concepts. This norm centers on ways that definitions are written to aid understanding and positioned to enhance clarity. Five participants highlighted students' need for more than memorized definitions to understand a concept—rather they expected students to have intuition or use multiple representations (*need more than definitions to understand concepts*). Dallas gave a number of ways students should be able to engage with a concept:

> A lot of the goal with teaching, …you want to help them to kind of build an intuition for these different concepts. Certainly, …they need to understand the formal definition, be able to work with it. But…[I] also want to kind of convey how do we actually think about what does it mean when two groups are isomorphic or two rings are isomorphic. And here, kind of the sameness concept is… part of the—sort of, the intuition.

Notice Dallas expects students to have some sense of what isomorphism is besides reciting a definition, though being able to recall some version of the definition is implicitly important.

Similarly, six participants viewed undergraduates as requiring scaffolding and more intuitive explanations of definitions than graduate students or mathematicians (*definitions scaffolded for undergrads*). Reasons for this included students' unfamiliarity with notation or being conceptually unready for rigorous definitions to be used without explanation. Indy said:

> For graduate students, … [y]ou don't have to necessarily motivate the definitions as much so you can just say, "We're going to consider this kind of definition or this definition." Or you could just say, "Because we're going to get some really nice results, and then, we'll see cases where that applies." … Whereas, with the undergraduates, I think you really have to sort of motivate things a little bit more.

Indy suggests that more explanations are necessary for undergraduates because they are less familiar with why mathematicians work the way that they do.

Formatting and notation choices aid communication. This norm focuses on ways that mathematicians utilize standard formatting and notation for definitions to make communication easier. Five participants addressed standards of written mathematical communication, such as the layout of research papers, and their norms for formatting and ordering (*norms of mathematical communication through formatting choices*). Avery emphasized the use of different typefaces for showcasing important definitions:

> For the important definitions…I take…a textbook approach, in which case you will have "Definition 1.3", we'll write it out, the word or words that are sort of the focus here, they'll be bolded…Other terms that aren't as important, I'll…take a less formal approach that perhaps in the prose there'll be a sentence that describes it, and then I'll maybe italicize the word…to kind of emphasize that we're trying to define this word right now.

Thus, changes in text style facilitate the reader's understanding by highlighting what's important.

*Conventions between mathematicians***,** noted by six participants, addressed mathematicians' adoption of conventions in their papers and when teaching; specifically, choices need to be clearly communicated, especially when multiple choices could be made. For instance, Blair noted their choice of a different convention on whether rings had to be commutative:

> In a class, I'll try to pick a convention and have it stick consistently through that class. And ideally, it will be the same conventions that the textbook shows…if I really disagree with the textbook, I will either not choose that textbook, …or I'll go out of my way to constantly saying like, "Author of textbook, definition" every time I talk about that and

> then, versus, "Our definition" of something…I did do that in my last abstract algebra class. I wanted to talk about rings not necessarily being commutative. And I believe, in the textbook we were using, very quickly, the rings all became commutative with unity.

Notice the choice of definitional convention influences the types of assumptions that can be made, which can influence students' understanding of the concepts.

Eight participants emphasized efficient communication through the norm *definitions as shorthand*. For instance, Emerson contrasted naming a property "*" with having a word name:

> A name just gives you—just makes the paper shorter because then you can refer back to this thing every time, rather than having to say you have some condition you call star [(*)]. …It's a lot nicer to read and easier on the brain if you have a word for it.

Similarly, Finley focused on how encapsulating a concept with a standard definition makes it easier to process new information: "I mean, it's just kind of a way of preserving space in your working memory, I guess." This seems to suggest that definitions make easier building blocks for arguments than unencapsulated, multifaceted concepts.

Value 4: Mathematicians desire a consistent set of norms and practices

The norms discussed here related to norms of communication within mathematics and how communication differs with people outside that community. Three participants highlighted the norm that *mathematical communication has consistency*, which focused on mathematicians having a standard way of communicating that should not be changed lightly. Avery gave an example of someone trying to defy this norm:

> I regrettably…got involved in some stupid social media debate on ResearchGate where people were talking about, are the rational numbers bigger than the integers…one author sort of had the argument that…rational numbers are like infinity times infinity. And we're trying to convince them that's still infinity…They wouldn't accept that the question was closed because they were, in their essence, trying to change what cardinality meant, without actually saying that.

This participant was focused on how definitions, once defined and accepted, should not be redefined at will; new concepts should be described with new terms. Implicitly, defying this norm would cause communication problems because people would not know which meaning was intended. This also demonstrates that violations of norms can clarify the form of norms. This idea arose indirectly through instances of violations like the failure of consistency noted above. However, it also arose more explicitly with the four participants given the code *norms irrelevant to laypeople*, which occurred in discussions about how to explain isomorphism and/or homomorphism to a layperson. For instance, Dallas said "So if I were trying to describe isomorphism to say someone in a different department… I might just be really vague and say they're roughly the same," suggesting that ambiguity was acceptable for laypeople, which was quite different from the typical mathematical norm of precision in communication.

Discussion and Conclusions

This study aimed to determine mathematicians' norms of mathematical definitions, both in the writing of definitions and their use, and how these norms relate to their values. Analysis of interviews showed these norms largely aligned with the four values upheld by norms of proof conjectured by Dawkins and Weber (2017), with the exception that mathematical definitions showcase mathematicians' agency more than a-contextual knowledge. Specifically, the mathematical norm of precision (Value 1), various aspects of clear communication (Value 3), and a desire for consistent norms (Value 4) all parallel ways that proof upholds these same values. However, participants focused less on definitions as a-contextual objects (Value 2) and more on the (changeable) context that leads to their generation (Value 2.5). Indeed, participants noted definitions are written purposefully for known or exploratory purposes. Thus, while mathematical truth is a priori, the definitions one chooses to get at this truth are not. This is not a contradiction to Value 2 of Dawkins and Weber–the purpose of definitions merely relates better to a different mathematical value, that of mathematical agency.

Of special note, while we might naturally think about precision as a norm in mathematics and expect it to be cited by mathematicians, the range of ways in which communication was centered was illuminating. In particular, definitions encapsulate a large amount of information and thus can be used as a shorthand in both writing and in memory. However, while experts seem to view these definitions as enhancing their ability to work with new ideas, it is less clear whether they expect students to able to use definitions in the same way. Research suggests students may not automatically have those skills (Weber & Alcock, 2004). Even while highlighting this important role of definitions, many participants noted that students and researchers need more than just definitions to understand a concept, such as specific examples or multiple intuitive representations.

Many norms found in these interviews are critical to doing good mathematical research and to communicating mathematical results. Thus, it is important that these norms are communicated to students, and many of those interviewed mentioned that they try to do so. For instance, all nine participants indicated that they communicate the norm of precision to students. This included encouraging clear and explanatory writing, talking about how definitions are situated in an axiomatic system, and discussing what makes a good mathematical definition. They also seemed to attend to aspects of communication in teaching, such as noting the need for more than definitions to understand concepts and for additional scaffolding to understand mathematical definitions. Some attempted to provide this scaffolding through examples and intuitive explanations, though future research should examine more details of how this scaffolding is provided. In contrast, there was limited emphasis in the interviews on communicating to students the norm of agency in creating definitions. Some participants mentioned adopting conventions in class that are different from those in the textbook, and that this difference must be clearly communicated to students. However, others indicated that students are likely unaware of the fact that mathematicians choose definitions and the flexibility that results from this. Furthermore, they seemed less likely to try to communicate this norm to their students, which may skew students' view of what mathematicians do. Edwards and Ward (2004) noted that some upper-level undergraduate students do not always categorize mathematical definitions as stipulated but rather see them as extracted from what we know about a concept, suggesting a perceived lack of agency for mathematicians. Future research should examine how, if at all, students' views of math and mathematicians might be impacted by explicit conversations around mathematical agency or when such interventions would be most impactful for students considering advanced mathematical study.

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College Algebra Students' Definitions of 'Simple Mistakes' Through A Causal Attribution Lens

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Mathematics students sometimes refer to making "simple mistakes" in analyzing or explaining their past performance. We interviewed eight college algebra students to learn whether they classified their mistakes on a first test as simple or not-simple and how they would direct their studying efforts in the future based on these classifications. Using Attribution Theory, we created categories of students based on how their causal attributions impacted their definition of a simple mistake.

Keywords: Error Analysis, Simple Mistakes, College Algebra, Attribution Theory

We know too little of how mathematical learning for college students happens outside the classroom and how it might be influenced. We do know that study habits, study attitudes, and study motivation, which are underdeveloped when students arrive to their first college math course, are strongly correlated with collegiate academic performance and can predict collegiate performance at a rate comparable to test scores and prior grades (Crede & Kuncel, 2008). Some authors have called for instructors to explicitly help students learn how to study math while teaching math itself, but this is not common practice (Lewis, 2014; Mireles et al., 2011), and attempts to do this have not always led to significant results (Bogardus, 2007; Hight, 1993).

Changing student behavior requires students to see a need for change and understand the change will be worth the effort. This is of primary importance in first-year college math courses, since these students are more likely to be struggling with confidence in mathematics and questioning whether to persist (Brainard & Carlin, 1998; Ellis et al., 2016; Seymour & Hunter, 2019). This study was motivated by our prior research in using self-reflective activities in the College Algebra classroom to help students process how their efforts leading up to an exam may have impacted their performance. On one self-reflection, multiple students indicated they were not satisfied with a recent exam score (50-70%) because they made "simple mistakes" (Ryals, Hill-Lindsay, Burks, & Pilgrim, 2000). We define a "simple mistake" as one that could be made accidentally, would likely not be repeated, or violates a mathematical convention rather than a rule while a "not-simple mistake" arises from a lack of conceptual understanding (Ryals et al., 2000). We hypothesize that 1) students' definitions may not agree with ours and 2) their definitions and their classifications of their mistakes may provide evidence for why and how they do or do not modify their future study behavior. Therefore, we posed the following research questions:

1. What are the different student distinctions between simple and not-simple mistakes?
2. How do those distinctions compare from student to student and to the researchers' distinction?

3. How do students say they will change their behavior after identifying their mistakes as simple or not-simple?

Literature and Analytic Framework

Math educators have long studied the impact of students' beliefs about the nature of mathematics (Carlson et al., 1999; Carter & Norwood, 1997; Kloosterman et al., 1996; Schoenfeld, 1989; Suthar et al., 2010), but the impact of students' beliefs about their own characteristics and behavior on performance and persistence in mathematics has been given less attention. Instructors and researchers may ask "why" do students perform well or poorly, but their interpretation of causes do not necessarily agree with students' perceived causes, or causal attributions. Causal attributions are explanations or reasons for behaviors or outcomes (Heider, 1958; Weiner, 1972). Motivation is closely tied to these attributions (Borkowski et al., 1990), and the potential exists for instructors to impact attributions, thereby changing students' effort allocations and persistence in mathematics (Perry & Hamm, 2017). Causal attributions are directly linked to affect and future expectations, significantly impacting future behavior (Kelley & Michela, 1980). For example, a student who attributes failure on a test to a lack of ability may experience shame or hopelessness, may expect similar results on future exams, and be unlikely to change their approach to the course; whereas a student attributing failure to a lack of effort may study more or seek help in the future with the expectation they can change future outcomes. Ability and effort have been commonly viewed as dominant attributions (Graham, 1991), but Weiner and others have expanded the list of common attributions to include variables such as task difficulty, luck, and strategy (Watkins & Astilla, 1984; Weiner, 2000).

Weiner theorizes that each attribution has three dimensions, locus (internal/external), stability (stable/unstable), and control (controllable/uncontrollable), which are underlying properties of the attributions that can help explain how that attribution will impact affect and behavior. The attributions effort and ability, for example, are both internal, while task difficulty, something determined by the instructor, is external. Individuals are more likely to attribute success to internal factors and failure to external factors (Wolleat et al., 1980). While both effort and ability are internal attributions, ability is typically perceived as stable, or at least more stable than effort. Stable attributions for failure can lead to hopelessness because they indicate past failures are likely to be repeated. While external factors are inherently uncontrollable by the student, internal factors, particularly those that are unstable, can be perceived to be either controllable or uncontrollable. Effort, for example, is perceived to be controllable, while ability often is not.

We have several decades of research linking attributions and causal dimensions to performance, behavior, or affect. High performance is consistently associated with not attributing failure to low ability (Bempechat et al., 1996; Shores & Smith, 2010) which is supported by a well-established link between self-efficacy and mathematics performance (Hackett & Betz, 1989). Confidence in one's potential to succeed in the long-run can increase effort and persistence in the face of failure (Pajares & Kranzler, 1995; Stevens et al., 2004). Shores and Smith (2010) note that attributions to unstable or controllable factors can lead students to believe that past failure does not dictate future performance. In contrast, attributing failure to stable, uncontrollable factors, such as low ability, often results in reduced effort in the short term and a lack of persistence in mathematics (DeBoer, 1985; Shores & Smith, 2010; Weiner, 2010). There is a difference between confidence in overall mathematical ability, which can improve with effort and/or assistance, and confidence in the ability to complete a particular mathematical task in the moment (Hackett & Betz, 1989; Bandura, 1986). Over-confidence in the ability to complete a particular task can lead to insufficient effort, decreasing performance.

Methods

Data Collection

Study participants were enrolled in College Algebra for STEM-intending students at a large public institution and were also enrolled in a co-course designed for students whom the university identified through multiple measures as having weaker prerequisite backgrounds. We invited all students present in class to participate in a 30-minute interview after the first test was graded and returned. Eight students signed up and were interviewed. In the interviews, students were asked about each error they made (not just each problem they missed). We asked them to identify the type of error (such as a copy error or arithmetic mistake) and then classify it as simple or not-simple and explain. After all errors were addressed, students were asked for their overall definitions of simple and not-simple mistakes and were asked how they would modify their study behavior if the majority of mistakes on a failed test were simple versus not-simple.

Analysis

In order to address Research Questions 1 and 2, we first inductively coded for students' definitions of simple mistakes. We went from error to error on each student's transcript and used the constant comparative method (Glaser, 1965) to develop new codes and refine existing ones. When coding a particular error, we considered student's comments about their other errors to help interpret their meaning. After coding all eight transcripts, we wrote a description of the defining characteristics between simple and not-simple for each student. We did not rely solely on the student's overall definition of a simple mistake because they often omitted some of the attributions they had made for individual errors in these overall statements, likely due to availability bias and relying more heavily on recently discussed errors (Tversky and Kahneman, 1973).

We were surprised by how often students referred to some cause of the error rather than the type of error and also by their emotional response to the error; we had not explicitly asked about causes or emotions. This led to the choice of attribution theory as an analytical framework. We began coding for student's *perceived* causal attributions, starting with Weiner's four main attributions (ability, effort, task difficulty, and luck), and added other distinct attributions (anxiety, running out of time) or created subcategories (study effort, metacognitive effort during test) as they emerged from the data. We noticed specific attributions were not always consistently described on the control and stability dimensions, a phenomena Weiner uses to justify shifting focus to the three dimensions rather than the causes themselves (2012). Therefore, we began coding for the three causal dimensions: controllability, stability, and locus, rather than focusing on the specific attribution.

For each student, we compared our codes for the simple versus not-simple distinction and the attributions. We looked for patterns across the errors and summarized each student's pattern of attributions for simple and not-simple mistakes. We then did a cross-case analysis. We used a matrix to compare and contrast these summaries for each pair of students. From this arose the major themes we will discuss in the results below.

For Research Question 3, we coded the behaviors mentioned in responses to the final interview questions and compared these behaviors across students to determine if there was any pattern between categorizing past errors as simple and modifications to future study behavior.

Results

We will first address Research Questions 1 & 2 together and end by answering Question 3 separately, as there were separate prompts in our interview protocol addressing Question 3.

Relationship between Stability and Control

For a majority of participants, there was a strong link between unstable and controllable attributions. These students did not attribute their errors to unstable, external causes like bad luck or the teacher choosing a particularly difficult test problem. For these students, some aspect of control was a distinguishing factor of their definition of a simple mistake.

Control during the test. Of the students that addressed controllability, TY, R, and MM defined simple mistakes as those that were controllable on the test (i.e. an in-the-moment error due to lack of metacognitive effort during the test) and not necessarily controllable prior to the test (i.e. an error in understanding due to lack of study effort). TY explained, “simple is more like when like you know what you’re doing and how to do it but like you kind of just like mess up in that moment.” This conception of a simple mistake was most consistent with ours.

This group of students associated not-simple mistakes very closely with a fundamental lack of understanding (as opposed to not knowing how to perform a procedure or not having something memorized). One problem asked students to rationalize a denominator. Multiple students, including MM, remembered they were supposed to multiply by the conjugate, but used the same, rather than the opposite sign. MM called this a not-simple error, explaining, “I didn’t really understand how to do that. Like, it wasn’t such a small error in my mind, even though it is on paper. But to me it was not a small error.” On the only problem with an error that R classified as not-simple, he had still received 7 out of 8 points. While missing a single point would be considered a simple error for other students, R was very critical of the fact he had had to attempt this problem multiple times before finding a correct approach. He believed that not knowing immediately which strategy to use implied he did not understand the material, and therefore it was not a simple mistake.

Control at any point in time. Where TY, R, and MM saw simple mistakes as attributable to factors that were controllable during the test, two other students, AY and RG, classified mistakes as simple if they were attributed to controllable factors at any point in time, including before and after the test. RG classified all nine of his mistakes as simple. He suggested some simple mistakes could have been prevented *during the test*, such as when he lost points for not showing his work. He also often suggested a mistake was simple if he could now understand his mistake and solve the problem *during the interview*: “I feel like if I was to do [the] problem again, now I learned. We went over it. You told me. Not really the kind of mistake that I would make twice.” (This is despite the fact that the instructor had gone through this type of problem immediately prior to the test, and he did make this error twice on the test.) We contrast this with TY, who called a particular mistake not-simple, saying, “I remember doing like this problem right before like doing the test but I remember like going over it but like when I did it on the test I just forgot." RG also attributed other simple mistakes to a lack of effort *prior to the test*: "If I would have looked at these questions again other than the one time I did it...I probably would have got that one right." We noticed a frequency of what we termed "if/coulda" language that was common for RG and AY, such as "If I had" or "I could have." This language alludes ultimately to the perceived unstable nature of their errors and hope for the future, and specifically for RG and AY, indicates a belief they ultimately have control over the outcome.

Both RG and AY downplayed the amount of effort that would have been required to be successful and associated simple mistakes with minimal additional effort. RG stated explicitly, "I think that a simple mistake is something that can be fixed without a, without a, uh, large amount of time devoted to fixing the problem." AY suggested that simple mistakes are due to a lack of memorization. She never mentioned studying explicitly, and we suspect she equates studying mathematics with memorization. When discussing a problem where she incorrectly added exponents rather than multiplying them, she explained, "I know there's some instances where you're supposed to add them but I don't remember which instances...I feel like [my mistake is] simple because it's memorizing rules of exponents."

Did not address control. We now discuss those students for whom control was not a distinguishing factor between simple and not-simple mistakes, J and C. Similar to RG and AY, C and J frequently used "if/coulda" language, but in contrast, when using this language, C and J did not necessarily indicate they believed these possibilities were within their control, unlike RG and AY. C frequently said "If I had studied..." or "If I had practiced…," attributing multiple errors to a lack of practice or lack of memorizing, but some of these errors she labeled simple and others not-simple. On the surface, these appear to be effort attributions (controllable), but she began and ended the interview explicitly stating she doesn't "know how to study for math, so [she] didn't study" indicating (uncontrollable) lack of ability. C missed a problem asking solely about vocabulary. She explained that she did study the definitions, but didn't know how to study them correctly. "I think I did go over them but there's different like, um, definitions so I was like, 'What? These are not the definitions I studied.' And I Googled the definitions. I didn't get it from this course, so it's probably different."

While the large majority of attributions made by all participants were internal, C and J used "if/coulda" language frequently with external attributions. C classified one error as simple, saying "for these like I guess I just feel like if I like had more time…" It was common for the external attributions to be associated with the frequency of exposure to certain types of problems (as part of course activities). C justified classifying one error as not-simple saying, "Yeah, like I feel like - like I don't really remember seeing that, like, in the homework, or it probably was but it wasn't as common [as another problem where she classified a similar error as simple]." Similarly, she had classified two seemingly similar mistakes differently, and when prompted to explain the difference, she said "When I was studying, I did problems from, like, number 3, but...I didn't do problems that would be square root times square root." We contrast this with TY, who did not consider how frequently or recently she had practiced a problem essential to her definition of a simple mistake. TY classified a mistake as not-simple and explained: "I remember doing this problem right before like doing the test...but when I did it on the test I just forgot."

For both C and J, the distinction between simple and not-simple mistakes is not primarily about control but about the complexity of the problem itself and the student's perceived probability of being able to solve it correctly or at least more completely in the future. C explained her reason for classifying a particular error as simple: "It was a simple, like question. It wasn't, it didn't ask me to do much." C made the same sign error MM made when attempting to rationalize a denominator, but C classified her error as simple, explaining "...if I would have remembered like that it's the opposite like that, I would have gotten it right." To C, this is something she could have corrected fairly easily, and could, with little effort, remember in the future, and therefore it is a simple mistake. J had multiple similar attributions for errors she classified as simple, such as "cause if I just read it correctly I would have gotten it right." On a different problem, J said her error was simple "because if I would have remembered the equation and then remembered the steps, it would have been simple." We note that she did not remember during the test, but she still classified this error as simple, reiterating the importance of this "if/coulda" influence on her simple/not-simple distinction. J and C labeled any error as simple that may not have occurred if any reasonable changes to circumstances, whether within or beyond their control, could have prevented the error.

Control changes depending on time frame. MF was the only participant whose simple mistake definition did not reflect patterns in causal attributions. Regardless of whether the mistake was simple or not, she framed her mistakes as uncontrollable in the past, yet controllable in the future. For example, she claims some errors were made because she did not know what to study, but speaks hopefully about the future when stating how she will change her future behavior to prevent similar mistakes: "I mean, it's just memorizing just definitions of terms, and then on the test, it wasn't the correct form that I was looking for, but, um, hopefully for the next test, I'll definitely look at the vocabulary more, and then that way I won't have this issue again".

Like C and J, MF's simple mistake distinction was based on the likelihood of making that mistake again in the future. For example, if the problem required only one or two steps or if the mistake could have been caught with minimal monitoring, then she believed it was likely she would not make that mistake again, and therefore it was a simple mistake.

Modifications to Study Behavior

At the end of the interview, students were asked how they might modify their studying in response to two hypothetical situations. In the first, they missed 50% of a test's problems due primarily to simple mistakes, and the second, they missed 50% due primarily to not-simple mistakes. Five of the students explained that making a lot of not-simple mistakes would mean they needed to seek outside help (such as from a tutor or their instructor during office hours), but that they could learn to prevent simple mistakes on their own. There was an important difference, however, in the goal of seeking outside help. Some described a need for procedural assistance, such as J. She explained she would need to ask for help at the learning center because "it's like I don't know how to do it at all." In contrast, some students would seek outside assistance to deepen their conceptual understanding. TY said that if she was making a lot of not-simple mistakes, it's because "I'm not understanding something in it." Though R did not indicate he would seek outside help for not-simple mistakes, he said "I would work on harder problems. Or I would like work on like fewer problems but like understanding like in depth every step and like what you're actually doing in the step."

A common approach to address simple mistakes was repetition or a revisiting of problems they had already worked and believed they already knew how to solve. For some, this repetition was closely associated with memory. This was especially true for the students who did not limit simple mistakes to those that could be controlled during the test. RG, for example, said that to avoid simple mistakes, he needed “familiarization I guess. Lookin’ over the stuff.” For J, direct memorization of formulas was a strategy to combat simple mistakes. C and MF suggested they needed to actually work and “write down” more problems (as opposed to avoiding practicing particularly challenging problems and/or only relying on studying provided solutions).

Discussion and Implications

Our analysis uncovered clear distinctions between students’ definitions of simple mistakes when compared to our definition or other students’ definitions. While we see a simple mistake as something made once, in-the-moment, some of our participants have a much wider umbrella for simple mistakes. Some include any mistake on problems they believe they previously knew or currently know how to solve, and others also include problems they believe they could or would have a decent probability of solving, if just one of many possible variables was or had been different. To some, the distinction between simple and not-simple mistakes has more to do with the amount of effort required to learn to solve the problem or the perceived difficulty of the problem itself. In practice, this means that instructors and tutors cannot assume that students attributing their underperformance to “simple mistakes” means the student made a bunch of copy errors. It is quite possible that those “simple mistakes” were made on problems that the student cannot solve and/or never knew how to solve.

What is more consistent across these different groups is an approach for how to prevent certain types of mistakes in the future. Most students argued they can address simple mistakes on their own, with more repetition, practice, or memorization, versus needing assistance with problems on which they made not-simple mistakes. This is crucial for instructors and tutors to consider when assisting a student in making changes to their study approach. Some of our participants had a wide net for simple mistakes, but did not indicate a need to seek help or deepen their conceptual understanding in the future when trying to avoid these mistakes.

We posit that students whose definition of a simple mistake was limited to those controllable during the test (R, TY, MM) are the most adaptive, as they recognize mistakes due to flawed studying or misunderstandings are not-simple. Students such as RG and AY, who consider a mistake simple if they knew how to solve the problem prior to or after the test, are not motivated to understand why they made the mistake during the test and do not think critically about how to prevent these types of errors in the future. Finally, students like C and J, who do not use control to distinguish simple mistakes, are at a distinct disadvantage. Their focus is on the perceived instability of their simple mistakes and their hope is that circumstances, possibly beyond their control, will change in the future and lead to their improved performance even without making changes to their future behavior. We recommend instructors challenge the last two groups of students to compare results across tests, highlighting inconsistencies between expectations and outcomes from one test to the next. Interventions aimed at retraining students’ causal styles towards controllable, unstable causes have been shown to increase performance and persistence (Perry & Hamm, 2017), and we suspect classroom discussions surrounding simple mistakes could be an effective intervention to help move students toward controllable, unstable causes.

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Department's Openness to Change. A Study from Calculus Instructors' Perceptions

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I studied factors that predict the department's openness to change (DOC). I used survey data collected from STEM instructors using the Survey of Climate for Instructional Improvement (SCII) instrument. Using Mathematics Instructors' responses, I first conducted an exploratory factor analysis. I extracted six factors, including DOC. Then, considering the components of the SCII, the extracted factors, and the elements that influence change according to the literature, I conducted a multiple regression analysis assuming collegiality, leadership, and participation in professional development could be predictors of DOC. Based on the literature and the extracted factors, I also added self-efficacy as a possible predictor. In this analysis, leadership is the strongest predictor, followed by collegiality and self-efficacy. Professional development is not a predictor of DOC.

Keywords: change, collegiality, leadership, self-efficacy, and professional development.

Change in STEM higher education has been a focus of study over the last decades (Henderson et al., 2011; Reinholz et al., 2020, 2021). There are mainly significant efforts for supporting instructors (Kezar et al., 2015; Reinholz et al., 2021; Tirosh & Graeber, 2003). Furthermore, different perspectives of change (Corbo et al., 2016; Kezar, 2013) have called for also considering contextual factors that may influence instructional practices.

Academic departments are critical sites for change, and more research on changes in the mathematics department is needed (Reinholz et al., 2020). Research in mathematics departments would contribute to understanding unique aspects of change in these settings and contexts (Reinholz & Andrews, 2020). Also, studying departmental culture for change is essential since faculty are highly impacted by their department (Corbo et al., 2016; Henderson et al., 2011). As part of the efforts for understanding how change occurs in departments, Walter et al. (2016, 2021) designed and applied the Survey of Climate for Instructional Improvement (SCII) to elicit "organizational climate for instructional practices" (2016, p. 411).

The SCII's design assumes that culture and climate are different, although commonly, they have been referenced indistinctively (Walter et al., 2016, 2021; Wang & Degol, 2016). Whereas culture refers to values, beliefs, and rituals; climate refers to "shared, subjective experiences of organizational members that have consequences for organizational functioning and performance" (Walter et al., 2021, p. 167). Furthermore, climate includes the atmosphere of the organization, it is multidimensional (Wang & Degol, 2016), and it is "more malleable to change than culture" (Walter et al., 2021, p. 167). In this way, as an instrument that gathers information about organizational climate, the SCII is applied to collect instructors' perceptions of the organization they belong to according to seven components: rewards, resources, professional development (PD), collegiality, academic freedom, and autonomy, leadership, and shared perceptions about students and teaching.

Viewing departments as crucial units of change (Quardokus & Henderson, 2015; Reinholz & Apkarian, 2018), I concentrated my study on the mathematics department's openness to change (DOC). In any change process, openness to change and resistance to change are considered essential factors (Hussain et al., 2018). Commonly openness to change has been considered the individual's willingness to accept and implement a specific change (Sinval et al., 2021; Wanberg

& Banas, 2000). In the organizational context, this idea can be transferred to consider, for instance, the department's acceptance to change or the department instructors' willingness to implement changes. The SCII includes questions regarding the department chair's, the department instructors', and the department's willingness or flexibility to implement different strategies for improving teaching. Thus, this study started with the assumption that the SCII could also be used for studying DOC. Using the instructors' perceptions, I looked to understand the extent to which some factors predict the openness or willingness of the mathematics department to improve teaching by implementing different strategies or providing various resources to instructors.

Here I focus on three components of the SCII, collegiality, leadership, and professional development, that are also commonly cited factors for instructional change (Berebitsky & Andrews-Larson, 2017; Kilinc et al., 2020; Melo Moreno et al., 2020; Sleegers et al., 2014). Thus, the research question was: to what extent do collegiality, leadership, and PD predict DOC? While not part of the SCII, the survey did include some items on the instructor's self-efficacy. As some literature does point to this being an important factor that may influence change (Tschannen-Moran & Woolfolk Hoy, 2001; Wheatley, 2002), secondary analysis added instructor's self-efficacy into the primary model.

Framework

In the SCII, collegiality is defined as the instructors' feelings of belongingness to a community of colleagues (Walter et al., 2016, 2021). As part of the community, an instructor can have different types of interactions with their colleagues that will help to improve their practice (Males et al., 2010). Those interactions and the sense of community between colleagues may positively influence DOC. Solid interactions between colleagues are essential for different approaches to changing mathematics teaching practices in K-12 education, such as lesson study or collaborative action research (Sánchez Robayo, 2020). In high education, Reinholz et al. (2021) found that Communities of Practice was the most common theory used in research about change in STEM. In particular, communities of practice have been used to design theories of change for specific attempts of change (Reinholz & Andrews, 2020). Communities of practice refer to groups of people (communities) with a common goal, a mutual compromise, share a concern, and belong to a specific enterprise (Wenger, 1998). In this way, participating in the community gains a particular connotation and represents individual learning and community strengthening.

Collegiality can take place as part of a formal learning opportunity or as an on-the-job learning opportunity. As an on-the-job learning opportunity, interactions between colleagues can vary from low to a high depth depending on the focus of the conversation (Cobb & Jackson, 2011; Coburn et al., 2012). For instance, a low-depth interaction is a quick exchange about a deadline, whereas a high-depth interaction could be a conversation about the nature of student learning (Coburn et al., 2012). Walter et al. (2021) found that experienced faculty build stronger and more diverse networks of colleagues, facilitating instructional improvement in higher education. Similarly, instructors who belong to a social network with strong ties, deep interactions, and more experienced teachers develop a flexible instructional approach to adjust it to changing conditions (Berebitsky & Andrews-Larson, 2017; Coburn et al., 2012).

The idea of leadership has been extended, and currently, there are types of leadership such as distributed leadership (Sleegers et al., 2014) or shared or collective leadership (Kezar, 2013) in high education. In general, these variations assume leadership beyond the scope of an individual. For instance, leadership is a property of a group, network, or community rather than of an individual in the case of the distributed version (Bianchini et al., 2014). However, generally, leadership has been considered an individual enterprise, in which the leader influences a group of people to achieve a common goal (Hussain et al., 2018). In the SCII, leadership is defined as "policies, actions, or expectations established by the formal leader of the department that communicate the value of teaching and instructional improvement" (Walter et al., 2021, p. 171). In particular, Walter et al. (2021) consider leadership a catalyst of change in higher education, and they found that department chairs are essential for instructional improvement. In this way, the leader's actions and perceptions may determine the department's willingness to implement different approaches or resources.

PD refers to a set of activities designed to train staff members to improve their performance in their role through learning about their profession (Ball & Bass, 2002; Ball & Cohen, 1999; Ball & Forzani, 2010; Desimone, 2009; Korthagen, 2017; Opfer & Pedder, 2011; Osman & Warner, 2020; Shabani, 2016; Silver, 2009; Tirosh & Graeber, 2003; Walter et al., 2021). At different educational levels, PD is considered a driver for change in instructional practices (Ainley & Carstens, 2018; Ball & Cohen, 1999; Ball & Even, 2009; Durksen et al., 2017; Frank et al., 2011; Franklin, 2015; Guskey, 2002; Osman & Warner, 2020). In addition, through PD, instructors can extend their knowledge and skills to address challenges (Walter et al., 2021). Thus, department instructors' participation in PD may influence the department's flexibility at the moment to implement some change.

Self-efficacy is defined as one's perceptions of their capabilities to perform a task (Ainley & Carstens, 2018; Skaalvik & Bong, 2003; Skaalvik & Skaalvik, 2007). It is a belief about what the person can do rather than their attributes; thus, self-efficacy is more about what individuals believe they can do independently of their skills and abilities. Teacher self-efficacy refers to teachers' beliefs of their ability to influence student learning, to provide effective learning environments, or to enact a teaching behavior (Agudelo-Valderrama et al., 2007; Ainley & Carstens, 2018; Fackler & Malmberg, 2016; Rodríguez et al., 2009; Skaalvik & Skaalvik, 2007; Tschannen-Moran & Woolfolk Hoy, 2001; Wheatley, 2002). Instructors with high self-efficacy are more receptive and more likely to implement changes (Bellibaş et al., 2020; Guskey, 1988; Piwowarski, 2010); thus, instructors with high self-efficacy in the same department may shape a receptive department for change.

Methods

The data came from the SCII applied to different STEM instructors from different colleges in the US. Along the design and validation process, the designers confirmed five factors to study organizational climate: leadership, resources, collegiality, respect for teaching, and organizational support. Although the survey also included questions about change; change and precisely the department's openness to change was not considered a factor. Thus, I started conducting an exploratory factor analysis.

The exploratory factor analysis

First, I restricted the dataset selecting only mathematics instructors and keeping survey items related to department climate and context, instructors' self-efficacy questions, frequency of different actions related to teaching, and participation in PD.

I transferred the data from excel to SPSS, and once I manage the missing values, I did all the analysis in SPSS. First, I confirmed the sample had an acceptable Kaiser-Meyen-Olkin measure of sample adequacy (KMO= 0.915) and a significant Barlett's Test of Sphericity ($\chi^2(528)$=14403.938; *p=.000*). Since data were not normally distributed, I followed Samuels'

(2016) advice of using Principal Axis Factoring as the extraction method since I intended to deal with the data for further analysis rather than developing an instrument. I also used Promax rotation as the recommended rotation method for correlated factors (Osborne, 2015). According to Samuels (2016), the threshold for the commonalities is 0.2; thus, I removed items with lower commonalities. Over the process, I removed 14 items because they had low commonalities. Six factors were extracted and retained once I compared them using parallel analysis (https://analytics.gonzaga.edu/parallelengine/). **Error! Reference source not found.** shows the final factors with their reliability (Cronbach's alpha measure).

Table 1. Factors from EFA

Factor	Eigenvalue (% variance)	Reliability	Number of items	Representative item
Leadership	8.890 (26.939%)	0.921	7	The department chair is receptive to ideas about how to improve teaching in the department
Instructors' self-efficacy	4.194 (12.709%)	0.858	9	How confident are you in ability to ask open, stimulating questions?
Departmental openness to change	1.969 (5.966%)	0.807	5	Instructors in my department value teaching development services available on campus as a way to improve their teaching.
Organizational support	1.905 (5.772%)	0.797	5	Instructors in my department have adequate departmental funding to support teaching improvement.
Collegiality	1.683 (5.101%)	0.884	3	Instructors in my department frequently talk with one another about their teaching
Participation in PD	1.321 (4.003%)	0.692	3	Have you ever participated in half-day workshop(s)?

Although participation in PD had low reliability, I decided to keep that factor in the model because the literature highlights it as one of the strongest mechanisms for change.

The multiple regression analysis

Once I got those factors, I conducted a multiple linear regression to predict departmental openness to change based on leadership, collegiality, and participation in PD. Following is the model:

$$DOC = \beta_0 + \beta_1 C + \beta_2 L + \beta_3 PD + \varepsilon$$

Where

DOC is the department's openness to change

C is collegiality

L is leadership

PD is participation in PD

For running the model, I found the mean of the questions that belong to the same factor. In the case of PD, I recoded the answers as follows:

Table 2. New codes for questions about PD

Question	Original value	New code

Have you ever participated in any of the following types of teaching-related professional development?	2- yes	
Half-day workshop(s)	2	1
Full-day or longer workshop(s)	2	2
Attending a teaching-focused conference	2	3

In all the cases, the answer "no" had the original value '1' and was recoded as '0'. Using these new codes, I summed up the values of the questions for the factor PD.

Once I run the model, I included the factor self-efficacy in the model. Then, as I did with the other factors (*DOC*, *C*, *L*), I calculated the mean of the questions that composed that factor.

Results

Since the research question is focused on a predictive relationship, I will focus the results in the multiple regression analysis calculated to predict the department's openness to change. Table 3 shows the means of instructors' responses to the items of each factor (*DOC*, *C*, and *L*). The items were Likert scale questions ranged from one to six, with one being the lowest positive score, and six being the highest from strongly disagree to strongly agree. The table also shows the mean of the sum of PD values after recoding.

Table 3. Descriptive Statistics

	Mean	Std. Deviation	N
DOC	3.8857	1.07173	1084
sumPD	4.2583	2.17381	1084
Mean_leadership	4.4040	1.09169	1084
Mean_Collegiality	4.8495	1.01377	1084

A backward multiple regression analysis was conducted to identify predictive variables based on their contribution to the model. Pearson correlation revealed no correlation between *DOC* and *PD* ($r(1082)=0.063$, $p=0.020$). *PD* was removed during the multiple regression analysis obtaining a significant regression equation for two predictors: *L* and *C*. Table 4 shows beta coefficients for these two variables that were statistically significant related to *DOC*, *L* ($r(1082)=0.712$, $p=0.000$), and *C* ($r(1082)=0.569$, $p=0.000$).

Table 4. Multiple regression coefficients

Variables	β coefficients	Std. error	t	Sig
(Constant)	-.056	.112	-.501	.616
Leadership	.559	.023	24.865	<.001
Collegiality	.305	.024	12.579	<.001

A significant regression equation was found ($F(2,1081)=715.210$, $p<0.001$) with $R=.755$, $R^2=.570$, and 57% of variance explained (adjusted $R^2=.569$). A subsequent stepwise multiple regression analysis starting with the predictor leadership yielded the same regression equation.

Overall, leadership is the best predictor of *DOC*. It is an expected result since leadership has been considered as one of the strongest facilitators for change in higher education, and leaders are seen as change agents (Kezar, 2013) in particular, in top-down changes. This means leaders are considered the ones who promote and lead changes initiated by the top of the organization's hierarchy and attempt to affect faculty. In particular, this result confirms the essential role of department chairs in the department climate for change (Henderson et al., 2011; Walter et al., 2016, 2021). However, as Knaub et al. (2018) highlight, not everyone can be a leader, particularly in the context of change. Organizations need individuals who can adapt easily to change (Wanberg & Banas, 2000). In particular, leaders as change agents should know how to create conditions for a change initiative and work with others (Quardokus & Henderson, 2015). Based on the questions that compose the leadership factor, department chairs who: are flexible, view changes in teaching positively, accept the possible struggles that come with change, recognize possibilities for improvement, and acknowledge the potentialities of the department in that process will have a significant role to set a department that is prepared for implementing change.

Collegiality is also a predictor of *DOC,* although it is not the strongest. As a collective stance, the community and its relationships are a strong influential element in determining how the collectivity will scope challenges. In this case, the opportunity to discuss with colleagues about teaching, the use of different resources, or challenges that arise in classrooms seem to influence the department's willingness to accept different strategies in teaching. In particular, collegiality as an on-the-job learning opportunity creates the scenario for having a mathematics department willing to introduce different teaching strategies or resources.

Surprisingly, *PD* is not a predictor of *DOC*. Since *PD* is an approach that seeks instructional improvement viewing the instructor as a learner, it is expected this factor influences the instructor, their practice, and the department on a broader level. However, some authors have reported that PD has a very limited impact on teaching practices (Cobb & Jackson, 2011; Guskey, 2002) and unsuccessful results on teacher change (Parise & Spillane, 2010; Simon, 2013). Thus, although the impact of PD in teacher learning is undeniable, there is a doubt about the effect of PD on change at the instructional and departmental levels.

Additionally, a backward multiple regression was conducted to identify if instructors' self-efficacy was also a predictor of DOC. In this case, also a significant regression equation was found (F(3, 1086)=498.123, p<0.001) with R=.761, R^2=.579, and 58% of variance explained (adjusted R^2=.578). The following table shows the coefficients:

Table 5, Multiple regression coefficients including SE variable

Variables	β coefficients	Std. error	t	Sig
(Constant)	-.660	.165	-3.998	<.001
Leadership	.548	.022	24.601	<.001
Collegiality	.304	.024	12.724	<.001
Self-efficacy	.201	.041	4.949	<.001

Self-efficacy is a significant predictor of *DOC* (p<.001). Although not in high proportion, the model increases the percentage of variance explained, which means self-efficacy as a predictor improves the regression equation.

Discussion

The SCII proved to be a useful tool for investigating the extent to which faculty view their departments as open to change. In addition to the factors already identified within the SCII (Walter et al., 2016, 2021), the EFA conducted here demonstrated that the SCII has a considerable potential to study even more specific factors than those considered in the instrument's design. For instance, the factor DOC arose with high reliability (α=0.807).

Commonly, the study of change in educational settings has concentrated on the process of implementation or afterward; the SCII and, in particular, the idea of DOC would enable to identify the grounds for studying change not just during the implementation but also before. Investigating DOC allowed to identify specific elements as part of the department climate to analyze the state of a department to implement new strategies or resources to improve teaching. In this way, I found three significant predictors: leadership, collegiality, and self-efficacy. As the type of individual leadership, it lies in the department chair. The study reveals that an open-minded department chair willing to manage the difficulties that could come with changes in teaching strategies or with the introduction of different sources could increment DOC. Meanwhile, collegiality is an expected predictor since it shapes the interactions between the community members. Finally, self-efficacy has a subjective nature and predicts DOC, maybe by increasing the instructors' trustiness in their ability to face challenges.

Although this study confirmed the influence of individual leadership in a change in the department, particularly in DOC, for future work, it could be worthy to identify since a descriptive view, how the individual leadership may influence DOC. Similarly, other types of leadership, such as shared leadership (Kezar, 2013), seem to have great potential to influence DOC. Inquiring how these collective views of leadership influence change would also bring essential steps to understand change phenomena.

There is an open question regarding the influence of PD in DOC. On one way, PD is one of the components of the SCII. Also, many studies in K-12 education focus on the relationship between PD and instructional improvement; some of them included characteristics of effectiveness to produce deep and significant changes in mathematics teaching practices (e.g., Boström & Palm, 2020). On the other way, the relationship between PD and change in teaching practices is not clearly defined. Although that result could sound striking, it is important to recall the finding from Opfer and Pedder (2011). They establish that the literature about PD's effects has the epistemological fallacy of assuming some measures of teacher change as teacher learning.

There is also another open question about how self-efficacy, which is a variable with an individual nature, influences DOC. One possibility could rely on what has been called teachers' collective efficacy, which refers to teachers' beliefs about their collective capacity to do a task (Klassen et al., 2011; Skaalvik & Skaalvik, 2007). Instructors' self-efficacy could influence their confidence as a collective producing a better disposition to implement a change in the department.

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Teacher Actions to Foster Creativity in Calculus

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Fostering mathematical creativity in the classroom requires intentional actions on the part of the instructor. We examine the teaching actions that students in a creativity-based Calculus I course report as contributing to their sense of creativity. Based on interview data, we found the four overall types of teaching actions: Task-Related, Teaching-Centered, Inquiry Teaching, and Holistic Teaching. We discuss subtypes as well as concrete actions, to provide actionable steps practitioners can take to foster students' creativity.

Keywords: creativity, Calculus I, teacher actions

Educational efforts to prepare students for Science, Technology, Engineering, and Mathematics (STEM) related jobs need to be furthered to prepare the STEM workforce to tackle ill-defined problems with no clear solution paths (Wilson et al., 2017). Fostering creativity in mathematics classrooms help develops this (Leikin, 2014), but at the tertiary level such classroom experiences are often restricted to pre-service teachers (Shriki, 2010) or mathematics majors in upper-level courses beyond Calculus I (Zazkis & Holton, 2009). By the time mathematics is exposed as a creative subject, we have lost many potential STEM majors, due to the ways in which calculus serves as a gateway course (Moreno & Muller, 1999). We argue that creativity must be fostered as early as Calculus I.

The role of teachers in fostering students' mathematical creativity is crucial. Moore-Russo and Demler (2019), citing Aiken (1973) stated that "teachers [are] the keys to unlocking creativity in the classroom" (p. 1). This is furthered by Hershkovitz, Peled, and Littler's (2009) statement: "creativity in mathematics classroom[s] can be improved through appropriate teaching methods" (p. 255). However, most discussion of how to foster creativity has been theoretical and/or at the K-12 level (e.g., Levenson, 2011, 2013). There is a need for actions tertiary instructors can take that are grounded in empirical data. In this paper, we explore teacher actions, as reported by undergraduate students, that fostered their sense of creativity in a creativity-based Calculus I course. Using qualitative coding, we present types and subtypes of actions that students discussed in interviews to provide practical actions that instructors can take.

Conceptual Framework & Background Literature

Creativity

We take a stance that mathematical creativity is relative to the student (Liljedahl & Sriraman, 2006; Zazkis & Holton, 2009); if any item (process or product) is new to the student then that is an act of creativity. This perspective differs from absolute creativity, which requires an item to be new to the field of mathematics (Kaufman & Beghetto, 2013; Leikin, 2009) to be considered creative. This discussion of relative versus absolute creativity is also reflected in the psychological literature, discussed as Big-C creativity (absolute) versus little-c creativity (relative) (Beghetto & Kaufman, 2007; Levenson, 2011). In educational settings, where the goal is to support students, relative creativity may be most relevant. We therefore take a developmental perspective on creativity: that a person's creativity can and does develop over time (not fixed). The main focus of investigation then is a person's process and actions, as opposed to the product created (Kozbelt, Beghetto, & Runco, 2010). Given that a person's creativity can change, this implies creativity can be honed by an instructor's processes and actions.

Teacher Actions

We consider a teaching action to be any act (physical, written, or verbal) in or out of the classroom that can be attributed directly to the teacher. Teaching actions share similarities to teaching practices (e.g. Ponte & Chapman, 2006). According to Ponte and Quaresma (2016), these actions can be "framed by two basic elements: the tasks proposed to the students, and the communication processes that take place in the classroom" (p. 52). Teacher actions have been one of the important constructs of research studies within inquiry-based and inquiry-oriented teaching. For example, Kuster et al. (2018) identified "four primary components of inquiry-oriented instruction: 1) generating student ways of reasoning, 2) building on student contributions, 3) developing a shared understanding, and 4) connecting to standard mathematical language and notation" (p. 2). For each component, Kuster et al. provided examples of instructional actions. For instance, when generating student ways of reasoning, Kuster et al. stated a description in action, including: "The teacher explicitly asks students to share their approaches to the tasks and the reasoning the students used to complete those tasks" (p. 6). In this paper, we follow a similar trajectory of categorizing creativity-oriented instruction through providing specific and explicit actions as reported by students that fostered their mathematical creativity.

Actions to Foster Creativity: K-12 & Tertiary

Actions to foster mathematical creativity proposed in the literature at the tertiary level are largely theoretical or conjectural (Sriraman, 2005). At the lower grades, Levenson (2011, 2013) explored fostering mathematical creativity at the 5th and 6th grades. Levenson (2013) found concrete actions that fostered creativity: "choosing appropriate tasks, fostering a safe environment where students can challenge norms without fear of repercussion; playing the role of expert participant by providing a breakdown of the mathematics behind a process; and setting the pace, allowing for incubation" (p. 273). Both Sriraman's theoretical principles and Levenson's empirical findings agree with some of the educational psychology literature, including Cropley's (1997, 2018) nine categories of fostering creativity. Levenson (2011, 2013) is one of the few that identified concrete actions grounded in data of fostering creativity.

In the context of tertiary education, Zazkis and Holton (2009) spoke about one crucial aspect of fostering creativity: "for a student to be creative, the instructor has to provide a problem where creativity can be shown" (p. 346). They included tasks in specific topics (e.g., graph and number theory). The authors suggested other tasks, including Watson and Mason's (2005) learner-generated examples, Leikin's (2013) multiple-solution tasks, and Shriki's (2008) explicitly valuing creating new mathematical concepts. They concluded with some teaching actions that "attend to the 'flow' of students' thinking, rather than setting the boundaries of formal mathematical curriculum" (p. 361). Shriki (2010), focusing on pre-service teacher education, furthered the research by allowing students to create new geometrical concepts and questions over the course of six weeks. They starkly stated, "Refraining from development of creativity in the classroom conveys the impression that mathematics is merely a set of skills and rules to memorize, and in doing so, many students' natural curiosity and enthusiasm for mathematics might vanish" (2010, p. 161-162).

We aim to complement these existing, mostly theoretical or conjectural work, by supplying ways of fostering creativity through practical actions that instructors could apply in their tertiary Calculus I courses. Given the need for actions grounded in data, coupled with the need for more creativity in calculus (Ryals & Keene, 2017), our research question is: *What are the teaching actions that calculus students have reported as fostering their sense of creativity?*

Methods

Participants were 34 undergraduates enrolled in Calculus I courses across various institutions in the United States. This study is part of a larger NSF-sponsored research project investigating fostering mathematical creativity in calculus. Participants were spread across two cohorts: Spring 2019 (C1) and Spring 2020 (C2). Twenty-four students identified as female (four bi-racial, five Latina, four Black, two AAPI, one Persian, eight White), nine as male (one bi-racial, one AAPI, one Latino/Hispanic, six White), and one as non-binary (White).

Instructors who taught these courses took part in a weekly online professional development about fostering creativity in Calculus I as part of a larger NSF-funded grant, where they designed and implemented at least six creativity-based tasks (El Turkey et al., 2020) and used a reflection rubric for creativity (Karakok et al., 2020) in their classroom.

Data Collection, Sources, & Analysis

We conducted semi-structured interviews with all participants. Interviews lasted 45-90 minutes long and took place over video conferencing software. The researcher asked questions such as "Did you feel creative in this course?", "Why and when do think you were creative?", "What have you learned about your mathematical creativity from this course?", and "What aspects of this course contributed to your or your classmates' creativity in the course?" Interviews were transcribed using a third-party transcription service and uploaded to the software nVivo™ for further analysis.

Because our theoretical stance was students' relative mathematical creativity, we examined each student's response to "Did you feel creative in your Calculus 1 course?" If they answered "yes," we looked at the reasons they gave for why they felt creative. If they answered "no," we looked at actions stated in the follow-up questions about their peers' creativity in the classroom. This was to account for students who did not see themselves or may not have felt comfortable stating they were creative during the interview but perceived a fellow classmate as creative. Two authors used holistic coding (Saldaña, 2016, p. 142) to identify (1) teacher actions: any student utterance about what the teacher did in reference to themselves or the class as a whole and (2) instances of creativity: any specific mention of creativity or answers to questions pertaining to mathematical creativity. In doing so, we identified teacher actions that fostered creativity.

For the first round of coding, we looked at the intersection of creativity and teacher actions, specifically if a student referenced a teaching action when talking about their own creativity. Two authors created teaching action nodes using a combination of descriptive and in vivo coding (Saldaña, 2016) and created 40 nodes. From those nodes, we used the process of theming (Saldaña, 2016) and organized them into five initial types based on emerging themes: *Task-Related, Assessment-Related, Teacher-Centered, Inquiry Teaching,* and *Holistic Teaching*. Within each of these types, we then read over all 211 instances of teaching actions mentioned and identified further sub-types. Due to the low number of instances and similarities in theme, we subsumed *Assessment-Related* under *Task-Related* as a subtype, resulting in four types.

Results

We created four overall types of teaching actions from analyzed student utterances related to fostering their own creativity. These types are separated into two overall categories, as inspired by Ponte and Quaresma (2016): actions that are part of the design of tasks and assessments, and actions that are part of the implementation of those tasks in class (See Table 1). These are teaching actions that the students reported as fostering their mathematical creativity in calculus; this does not consider teachers' perceptions of fostering creativity.

Table 1. Teacher actions reported to foster creativity, by type and frequency.

Type		Subtypes (with excerpts of teaching actions)
Design	Task-Related (76)	*Design High-Cognitive Tasks:* assign create new definitions, theorems, functions, and problems; assign solve created problems; assign problems that require making connections; give higher-order-thinking tasks
		Include Meta-Task Properties: provide group-worthy tasks, assign writing assignments such as journaling/reflections, revising homework
		Assessment-Related: assess open-ended questions, assess journaling assignments, assess creating new theorems, assess purposefully, de-emphasize correctness in assessment, does not assess drafts
Implementation	Teacher-Centered (18)	*Teacher Answers:* give different ways to solve problems, use online lecture videos, teach how topics are connected, review material
		Teacher Guides: foster understanding, persist to foster understanding, guide to correct answer, allow students to quietly work
	Inquiry Teaching (50)	*Allow for Discussion:* allow for discussion in class, allow for students to build (on other's thinking), divide class into groups
		Allow to Present: allow to present in class
		Teacher Active Listening: be aware of students' actions, encourage participation, inquire into students' thinking

Holistic Teaching (67)	*Encourage Mathematical Behavior:* prompt and encourage different approaches or divergent thinking; de-emphasize correctness in class; allow students freedom of time; use of the Creativity-in-Progress Rubric on Problem Solving tool (Karakok et al., 2020)
	Attend to Emotional Space: explicitly encourage students in their creativity, show excitement after student contributions, respect differences in the classroom

Within our dataset, the most frequently reported types were *Task-Related*, *Inquiry Teaching*, and *Holistic Teaching*. We focus our results on these three, as each had over 50 reported instances (out of 211). *Teacher-Centered* had the fewest: 18 instances. We defined *Teacher-Centered* as any action that was mostly focused on the instructor, whether it be verifying correctness or connecting topics. Given the low number and that they primarily consisted of typical teaching actions that many instructors already undertake, we do not expand on it here.

Task-Related

We defined *Task-Related* as any action that mentions properties of a mathematical content task that were (re-)designed, evaluated, or assessed by the instructor. Actions of this type were split into three subtypes. The first sub-type, *Design High-Cognitive Demand Tasks*, were made of teacher actions that were essentially about designing tasks with high cognitive demand (Stein & Smith, 1998). For example, Abbie (White woman, C1) discussed problem posing as fostering her creativity: "We had to create and solve most of our own problems based on problems in the textbook." Bryan (White woman, C1) also talked about the choice of assigning word problems that provide a "need to understand the concept in order to solve the problem." She went on to talk about making a connection with "information we've already learned in previous things and applying it here or just being very um inventive and creative about how to solve the problem."

The next sub-type, *Include Meta-Task Properties*, were about overall properties of the task itself that were not necessarily part of high-cognitive demand tasks. These tasks were not content-focused, but rather prompted the students to think about their problem-solving processes or reflect on how others thought about content. For example, Sal (Biracial Filipina American woman, C2) stated that "I know we had one class about thinking creatively and how to approach a math problem in a creative situation or asking, are there? Is it possible to approach a problem with a different mindset or different mannerisms?" This also involved re-evaluating homework or getting together in groups to discuss homework that had been done, in order to "understand how we went through the process to get our answer" (Breezy, Mexican man, C2).

The last sub-type, *Assessment-Related*, consisted of any mentions of how an assignment was graded or of tasks on quizzes or exams. For example, Bryan shared how homework was graded on a complete/incomplete scale. Murphy (White woman, C1) also said that the instructor would "give them [problems] back to us with like a little bit more notes on it just saying like oh that's interesting. Like the way that you found that and if you like you could have done this or this and that was good." We note that there were very few reported instances of Assessment-Related actions (11). Moreover, most of the instances (7) were in reference to one instructor. We explore what this lack of students reporting Assessment-related actions could mean in the Discussion.

Inquiry Teaching

Inquiry Teaching was defined as any action that can be linked to inquiry-oriented (or -based) instruction. We center the definition of active learning on two previous studies: Shultz and Herbst (2020) and Kuster et al. (2018). Shultz and Herbst (2020) created the INQUIRE (inquiry-oriented instruction review) instrument with four constructs about in-class teacher instruction: "interactive lecture, hinting without telling, group work, and student presentations" (p. 532). With these understandings of inquiry-oriented instruction, we saw the following three sub-types in our coding: *Allow for Discussion, Allow to Present* and *Teacher Active Listening*.

Allow for Discussion actions were about student discussion and characterized by group work, including actions as basic as *divide the class into groups* for deeper attention to student thinking such as *allow for students to build knowledge.* JCRU (Black woman, C2), in response to what contributed to her creativity, stated: "we all are able to learn from each other and build off of other people's ideas. Or take it and add a little something or maybe change it a little bit to make it work and make sense for us."

Allow to Present actions were also about student discourse but in front of the whole class. For example, interactive lecture was one way that Amelia (White woman, C2) reported her instructor contributed to her sense of creativity. The instructor "would always ask, like, how someone solved something. And then she would ask if someone solved it in a different way. And so she kind of wanted us to think differently and have different solutions."

Teacher Active Listening actions were where the teacher attended seriously to student voices. Aon (Black woman, C2) said her professor was aware of students' actions in the classroom:

> We did the problem, but we did it in a different way. And she compared the two ways and told me, "what's good about this way? What's good about this way?" So I feel that right there, her showing us, the students, different ways to do different problems showed her creativity and showed how, like she's able to adapt to everyone's learning style, learning that everyone won't be able to learn something as quickly in one way.

Holistic Teaching

Holistic Teaching consists of any teaching actions that do not require a response from students yet psychologically build an environment for fostering creativity. Within this type, there were two subtypes: *Encourage Mathematical Behavior*, which encourages mathematical behaviors that lead to creativity, as well as *Attend to Emotional Space* which attends to the student's affect. For example, Jmenard (White man, C2) shared the impact of his instructor promoting students' different responses on the growth of his sense of creativity:

> We spent like a whole 10 minutes, just everybody coming up with their own, you know, thoughts of what are different examples and then we went through them all together and she didn't tell anyone that they were wrong. She was just like, 'Well, I don't think that is one. Can you try to prove me wrong' or, 'Yeah, that is one. Can you prove [to] me why you're right.' Or, you know, so and so forth. So you're not in a situation where there's only one and one acceptable answer…So then that's where the creativity side gets in, because you're not worried about just doing a bunch of homework and applying the formula and just essentially robotic type learning. To where it's so I can actually learn the material, get comfortable with it. And then when it comes time to applying it, I can apply it any which way I want. And as long as I can convince the teacher that I'm right, then I'm right, you know?

Attend to Emotional Space actions attended to students' emotional states, by including supportive actions. As an example, Jennifer (White woman, C2) discussed how her instructor encouraged creativity: "He always tried to show the different ways that we would solve stuff, because he knew that some of us did do it that way and some of us did this way. So he always encouraged that even if it wasn't the most conventional method to use he would always encourage everyone's methods." She also shared that "he would always get excited whenever we would answer the questions and everything. And that whenever we would be understanding [the content]. So, it was just nice to see that he was rooting for us, (laughing) I guess all the time."

Holistic Teaching was made up of actions where instructors relinquished some control of the classroom. Some of the creativity-fostering actions reported involved uncertainty in solutions or understanding and acknowledging that teachers can learn from students. For example, Frost (Native American/Latinx/white man, C1) stated:

> [T]here's never…any negative outcome whenever you try and fail with him. It's like 'Oh you tried this. That's a good idea. But maybe next time like or what else could you do in this situation?' and he'll make you think through it and then eventually you'll solve it yourself.

This shows the impact of how students feeling there are no negative impacts to their attempts can lead to positive attitudes and mathematical practices that feed into feeling creative.

Discussion

In summary, we themed teaching actions that Calculus I students reported fostered their sense of creativity: *Task-Related, Teacher-Centered, Inquiry Teaching,* and *Holistic Teaching*. We further split these into subtypes, to show the breadth of actions teachers can take to foster students' creativity. We provided practical actions, as examples within these types and sub-types, to provide actionable recommendations for instructors looking to further creativity in their Calculus I classrooms. We note that these actions are from the students' (not instructor) perspective. We did not corroborate whether instructors in fact did these actions. Given our lens of creativity and that the individual themselves is the ultimate judge of whether they felt creative (independent of their reporting of it to us), students' judgments of what their instructor did that actively contributed to their sense of creativity are most valuable.

A number of the *Task-Related* teaching actions corroborate findings within the mathematical creativity literature (e.g., Levenson, 2013) and our own previous findings regarding creativity task features (El Turkey et al., 2020), e.g., different approaches leading to one answer, posing problems and questions, allows for originality/novelty, and uncertainty. That these are reflected in what students report back (versus the instructor/task designer perspective) and that moreover stand out in memory for them confirms that students are noticing their importance.

Our work confirms Hassi and Laursen's (2015) findings that "…inquiry, collaborative problem solving, and class discussions seemed to foster students' creativity and flexibility, growth that also improved their learning in other classes, and in everyday life" (p. 17). Next steps include exploring the further impact of these teaching actions on students, namely on their affect. We will link specific teaching actions to specific affective or other desired outcomes, so that instructors can choose which teaching actions to focus on, given their classroom goals. Another step is to use the themes from this paper to analyze classroom video data to identify what the teachers in fact did. This work may serve as a basis for developing a Creativity-Fostering Teaching Guide for practitioners. Our results suggest that teaching to foster creativity does not require a complete redesign of a class but can be done in little changes by incorporating some of the actions listed here.

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Students' Conceptual Understanding of Normalization of Vectors from $\mathbb{R}^2$ to $\mathbb{C}^2$

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Interdisciplinary studies illuminate ways mathematics is incorporated into core STEM courses. Vector normalization is a crosscutting idea that appears in several mathematics and physics courses. The research question pursued in this study is: how do quantum physics students reason about normalization of vectors from $\mathbb{R}^2$ and $\mathbb{C}^2$, before and after quantum mechanics instruction? The data are analyzed using the theory of coordination classes (diSessa & Sherin, 1998). Results focus on students' thinking as they normalize different types of vectors: (A) a real vector and (B) a complex vector before instruction; and (C) a complex vector after instruction. Analysis identifies the ideas students coordinate when problem solving, which problem aspects students attend to, and how students take up or disregard ideas while they problem solve.

Keywords: linear algebra, quantum mechanics, normalization, coordination class theory.

Interdisciplinary studies have a significant role within mathematics education research because they illuminate the ways mathematics is incorporated into core science, technology, engineering, and mathematics (STEM) courses. For example, many physics programs require students to take mathematics courses in integral and vector calculus, differential equations, and linear algebra. In these courses, problem solving and reasoning involve a coordination of discipline-specific and mathematical knowledge (e.g., Christensen & Thompson, 2012; Hu & Rebello, 2013; Uhden et al., 2012; Wagner et al., 2011; Wittman & Black, 2015).

Physics and mathematics education researchers have started investigating student thinking around how linear algebra concepts are used in quantum mechanics. These areas of research include eignetheory (Dreyfus et al. 2017; Her & Loverude, 2020; Wawro et al., 2019), notation (Gire & Price, 2015; Wawro et al., 2020), expectation values (Schermerhorn et al., 2019), basis and change of basis (Serbin, et al., 2021; Close et al., 2013), and boundary conditions (Ryan & Schermerhorn, 2020). One mathematical concept that is essential to quantum mechanics is vector normalization, a crosscutting idea that also appears in several other mathematics and physics courses. Despite students encountering normalization several times in their undergraduate studies, students' understanding of normalization has been relatively uninvestigated. In this study, we pursue the research question: *how do quantum physics students reason about normalization of vectors from $\mathbb{R}^2$ and $\mathbb{C}^2$, before and after quantum mechanics instruction?*

Literature Review

Research on students' understanding of norms and normalization at the undergraduate level is sparse. There is some education literature where these ideas are relevant, such as student understanding of unit vectors. Barniol and Zavala (2011) asked physics students at a Mexican university to draw a unit vector in the direction of a vector drawn from the origin to the point (2,2) on the Cartesian plane. 22% of students gave a correct answer, 25% drew a vector from the origin to the point (1,1), and 14% drew the component vectors $2\hat{\imath}$ and $2\hat{\jmath}$; these findings were confirmed in a subsequent study (Barniol & Zavala, 2014). Vega et al. (2016) investigated students' abilities to draw unit vectors representing the motion of a particle moving in a two-dimensional plane, where the unit vectors were in terms of polar unit vectors $\hat{r}$ and $\hat{\theta}$. Many students drew vectors that did not satisfy the definition of a unit vector. The authors found four fundamental ideas to correctly solve this problem, namely that unit vectors: are vectors, have a length or magnitude of one, point in the increasing direction of the corresponding coordinate, and are dimensionless. This shows that finding a unit vector (i.e., normalizing) is not trivial.

Theoretical perspective

We analyze the data using the theory of coordination classes (diSessa & Sherin, 1998). Derived from a knowledge-in-pieces perspective (diSessa, 1993), a coordination class (CC) is a model for student understanding of a concept based on a networked system (or coordination) of context-specific knowledge elements. The theory of CC was developed as a means to define conceptual understanding and conceptual change (diSessa & Sherin, 1998) for the analysis of thinking and learning. To account for how information is determined from observation and from inference, a CC is divided into two parts: readout strategies and a causal net.

The first component of a CC, Readout strategies, relates to how information is observed from the surrounding world, while the second component, a causal net, deals with inferences. Readout strategies are the processes by which the information is drawn out from the external source. diSessa and Sherin note that for many quantities in physics, the readout strategies mostly involve determining the value of the quantity in a given situation (1998). A readout strategy for a CC around the concept of “fastness” could mean attending to what is more (Parnafes, 2007).

The second structural component of the CC, the causal net, represents the network of knowledge elements that connect observation to targeted information. This network can include small elements such as phenomenological primitives (diSessa, 1993) or more complex resources (Buteler & Coleoni, 2016; Wittmann, 2002). The causal net is often intertwined with the readout strategies. How students read information out of a given task can determine which elements are invoked and/or the causal net might influence strategies for answering a question.

In addition to usefulness in describing conceptual understanding, CC theory was designed to map conceptual change as new information is integrated into understanding or previously held ideas are deemed inapplicable. Incorporation is the process of recruiting knowledge elements in a CC. New CCs can be constructed by an individual through reorganizing and extending existing readout strategies and causal nets from known CCs and other prior knowledge (diSessa & Sherin, 1998). Complimentary to incorporation, displacement is the acknowledgement that certain elements or ideas are not applicable or helpful in a particular context.

Methods

To analyze the impact of a quantum mechanics course on the ways students conceptualize normalization, interviews[1] were conducted at two universities identified as University A and University C. Both universities offered a spins-first approach following the textbook *Quantum Mechanics: A Paradigms Approach* (McIntyre et al., 2012) and incorporated student-centered activities which encouraged discussion in the classroom. University A is a large, public, research-intensive university in the northwestern United States. The course enrolled 35 students and met seven hours a week for three weeks after a week-long preface on matrix methods. Linear algebra was a prerequisite course. University C is a medium-sized, public, research university in the northeastern US. The course met three hours a week for 15 weeks and enrolled 17 students. Following the textbook, eigentheory was first introduced in the context of spins before learning about wave functions in a continuous positions space. The second round of interviews was

[1] All interviews were conducted in-person prior to the COVID-19 global pandemic.

conducted within a few weeks of the conclusion of the spins content. As the prerequisite, students could take differential equations and linear algebra as one combined or two separate courses, and students saw some linear algebra in a mathematical methods for physics course.

Both the pre-interview and post-interview were semi-structured (Bernard, 1988), containing targeted questions related to thinking about normalization. Interviews were conducted using a think-aloud protocol and students were encouraged to discuss their choices when problem solving. The main stem for the interview questions are given in Figure 1. In the first interview (pre-interview), students were given two different vectors. Each was written on a piece of paper with the statement "Normalize the following vector." The first vector did not include any complex terms. After students completed the normalization, they were asked what it means to normalize and why they chose that procedure to normalize v. Students were then presented with a second vector included complex terms in both components. This vector was changed at University C to align with a physics convention of having real-valued first components and complex-valued second components. Additionally, the vector components were chosen so that students who did not use the complex conjugate[2] would arrive at $\|w\| = 0$ and need to reconcile the connection of length with normalization. The second interview (post-interview) focused on a vector in $\mathbb{C}^2$. Students were asked out loud to "Normalize a vector whose components are 3 and 2i" so they could choose the representation of the vector. Students were again asked about their meaning for normalization and about their procedure.

At University A, interviews were conducted with nine and eight students during the preface and at the course's end, respectively; six students participated in both interviews. At University C, eight and nine students were interviewed in the first and eighth weeks, respectively; seven students participated in both interviews. All interviews were videotaped and transcribed. Written work was collected and scanned. Participants were assigned pseudonyms "A#" or "C#" to identify them from a roster of all students in the courses. Students were not asked for their pronouns, so we use the gender-neutral singular pronoun "they" throughout the paper.

Data analysis identified the knowledge elements (units of ideas) as students reasoned about normalization. CC theory is used to analyze (a) information students invoke when normalizing vectors and describing normalization, and (b) compare snapshots of students' CCs to determine how the CCs are impacted by learning quantum mechanics. The second focus places interest on how knowledge of complex terms is incorporated into a student's conceptual framework.

Interview transcriptions were analyzed synchronously with the video data to account for students' written work. Analysis specifically attended to ways students conceptualized or

	University A	University C
Pre-interview	Normalize the following vector: $v = \begin{bmatrix}5\\2\end{bmatrix}$	
	Normalize the following vector: $w = \begin{bmatrix}3+2i\\4-i\end{bmatrix}$	Normalize the following vector: $w = \begin{bmatrix}3\\3i\end{bmatrix}$
Post-interview	Please normalize a vector whose components are 3 and 2i.	

Figure 1: Overview of interview questions presented at University A and University C. The questions given during the first week of the course were written out on paper. The question given post-spins content was asked verbally.

[2]For $v = [v_1\ v_2]\epsilon\mathbb{R}^2$, $\|v\| = \sqrt{\langle v|v\rangle} = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2}$, but for vectors $w = [w_1\ w_2]\epsilon\mathbb{C}^2$, $\|w\| = \sqrt{\langle w|w\rangle} = \sqrt{w_1\overline{w}_1 + w_2\overline{w}_2}$. The conjugation in the inner product for $\mathbb{C}^2$ produces nonnegative magnitudes in $\mathbb{R}$ for $w \neq 0$.

characterized vectors, vector representation, complex quantities, mathematical norms, and normalization. We performed inductive open coding (Miles et al., 2013) to identify knowledge elements, which were student ideas (complete thoughts or utterances expressing an idea), representational choices, or procedural choices.

The initial coding of the knowledge elements incorporated the language used by students. For example, the two student utterances *"*that will give us the length, and we can divide that length to normalize the vector*"* and *"*So, I'm just going to divide *v* by the magnitude of *v"* resulted in two similar codes: "Dividing a vector by its length normalizes the vector" and "Normalization is vector divided by its magnitude," varying only by the concepts of magnitude and length. Codes based on representational choices or calculation were also framed around student statements. A student who said "I'm going to do this in Dirac Notation" and a student who rewrote the vector in Dirac notation[3] would both be coded as "A vector can be expressed using Dirac notation." To interpret students' readout strategies, the initial review of the transcripts involved identifying what elements or aspects students attended to when problem solving.

Results

We present students' coordination classes as they normalize different types of vectors: (A) a real vector and (B) a complex vector during pre-interviews; and (C) a complex vector during post-interviews. Using a CC perspective, we identify the ideas students coordinate (causal net), which aspects of the problem statement students attend to (readouts), and the ways in which students take up or disregard ideas while they problem solve (incorporation and displacement).

Pre-interviews for normalizing a real vector

Overall, students were successful in normalizing a real vector. All but one student read out the coefficients of the vector and coordinated that the coefficients were squared and added. Four of seventeen students explicitly identified the dot product. Others bypassed the dot product by directly correlating magnitude or length with the square root of the sum of squared components.

We first present the CC established by A11 to highlight the method of data analysis, then expand discussion to other students. A11 correctly approached the procedure for normalization:

> ***A11:*** Normalize the following vector. Uh, well, we just, we solve for the dot product, which is v-dot-v, which would be five-squared plus two-squared. And we want to square root that, and divide by it. So, you'd have v divided by the square root of v-dot-v.

A11 immediately connected normalization and the dot product. We identify a knowledge element "Normalization involves a dot product." They then coordinated this element with their reading out of the components, an element we label "Dot product adds the components squared." Lastly, they take the result of the calculation and divide it into the original vector, which is identified by the element "Normalization involves dividing by the square root of the dot product." The progression of ideas in the causal net is shown by (1)-(3) in Figure 2.

When asked to "explain why you chose that process," A11 introduced the idea of magnitude and unit vectors, and reiterates several of the earlier connections.

> ***A11:*** Oh, so, to find the magnitude of v, square root of that [v-dot-v] … To normalize it, I took v and I divided by the magnitude of v. It's taking the vector and dividing by its length. ... What's the symbol for a unit vector? … [the] hat kinda tells you it's normalized.

Figure 2 shows the causal net with the elements from the student's additional explanation. Since the student references the dot product they wrote in their initial calculation, we can draw a

[3] Dirac notation uses bras, $\langle v|$, and kets, $|v\rangle$, to represent row and column vectors, respectively.

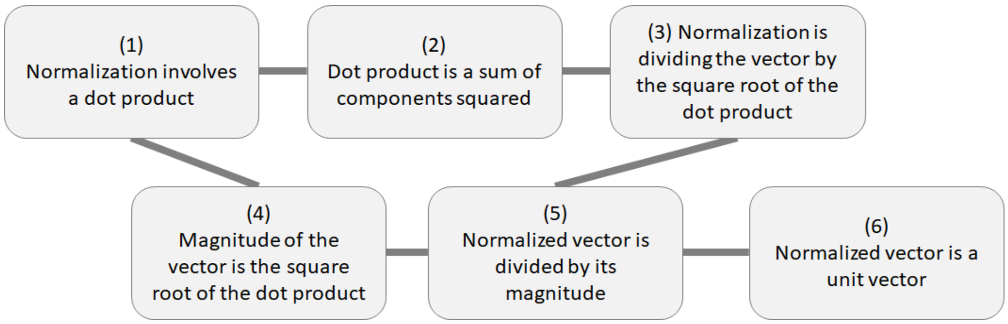

Figure 2. A depiction of A11 CC after they are asked to explain their process. Elements 1-3 were part of their original solution., while the remaining elements appear during further explanation.

connection between their first element and "Magnitude is the square root of the dot product." They coordinate this information with the element "Normalized vector is divided by its [magnitude/length]." The subsequent discussion of division, as the student finalizes their response, is why we connect this element with their initial element (3) in Figure 2.

Our analysis of A11's initial solution provides a look at the sequence of knowledge elements that led from the observed prompt to the determined output. As the student explained their process in response to the interviewer, other elements were identified ((4)-(6) in Figure 2).

Three students approached normalization geometrically, either through graphing on a set of Cartesian axes or by representing a triangle, as exemplified by C7 below.

> ***C7:*** 5 units along one basis vector and two units along the other basis vector which is orthogonal. [draws triangle] … the magnitude of this entire vector would be 5^2 plus 2^2.

C7 coordinated an earlier element "Normalization makes magnitude 1" with elements: "Vector forms a triangle," "Components are amounts along basis directions" and "Magnitude is sum of components squared." Reading out the components as a number of units, C7 identified the magnitude in a physicalized space, albeit incorrectly since they did not include a square root.

C7 was the only student in the pre-interviews to invoke the use of an arbitrary constant.

> ***C7 (continued)***: But we need to normalize that so if I say constant C times 5^2 plus 2^2 equals... C times 29 is one. … So, C is equal to 1 [over] 29, that way it is equal to one. [Adds 1/29 to expanded vector] But then how do I get// Oh, I could say that's 5 over the sqrt of 29 all of that squared plus 2 over the sqrt of 29. All that squared equals one.

A traditional physics approach is to include a *normalization constant* within a given vector and then solve $v \cdot v = 1$ to determine the constant. Inconsistent with the physics approach, C7 applied a constant after the dot product was calculated (Figure 3). We identify a coordination between "Normalize involves multiplying by a constant" and "Constant times the magnitude is one." These connections result in the determination of the constant and a normalized vector.

Three other students read out "normalize" from the question statement, and activate a CC for finding the normal vector. Two of these students end up displacing the incorrect elements by separating the ideas of finding a normal vector and normalization as involving magnitude.

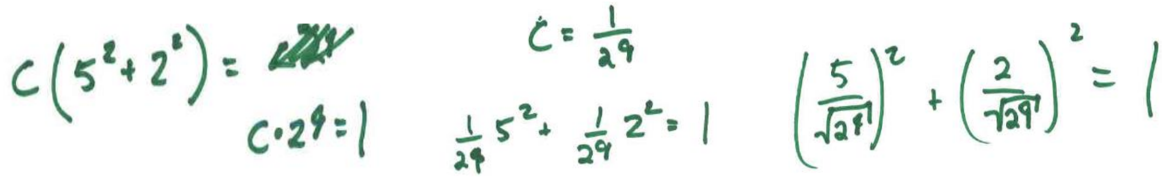

Figure 3. C7's written work where they multiply a constant by the result of their dot product. They then solve for the constant as part of finding the normalized vector.

Pre-interviews for normalizing a complex vector

The additional readout of an imaginary number requires a shift in thinking to include complex conjugates. Most commonly, 10 of 16 students[4] did not incorporate elements for complex vectors. The causal nets were similar to those for the first task. Students commonly invoked the summation of squared components and the division by the square root. As an example, A11 applies the same strategy as with the real vector.

> ***A11:*** So, you do 3 + 2i squared, uh, plus 4 - i squared, square root of that. That's going to equal the w magnitude. [Calculates to $\sqrt{20+20i}$] … divide by that.

A11 begins the same process of summing the components squared as they did with the dot product for the initial vector. They take the square root and identify the result as the magnitude, consistent with the knowledge element "Magnitude of the vector is the square root of the dot product." Lastly, they divide the vector, w, by the magnitude. Figure 4 shows the relevant knowledge elements that were identified overlapping with the student's initial causal net.

Six students identify the imaginary component and change their strategies to incorporate the idea of a dot product of the vector with its complex conjugate, as shown by C7 below.

> ***C7:*** I think I am going to be using a complex conjugate but I am not sure. So first of all, to get the magnitude of w, I am going to do w vector times w vector star. … that is 3+3i times 3-3i right. That is equal to ... 18 and then I have got to square root the whole thing.

After being presented with a complex vector, C7 identified the need for a complex conjugate, w^*. They changed the sign of the i term and carried out a dot product between w and w^*. They determined the magnitude as the result of the calculation, the square root of 18. The elements used by C7 up to this point are consistent with other students' causal nets for incorporating complex terms. However, C7 went on to use a constant and again invoked the elements "Normalized vector has a magnitude of one" and "Constant times multiplied vectors is one."

Post-Interviews with a complex vector

Following instruction in quantum mechanics, all students invoked the complex conjugate and only one of seventeen students incorrectly normalized the vector. The majority carry out normalization by independently finding the magnitude of the vector and then dividing by it.

The most common method was calculating the inner product of a vector with its complex conjugate, then dividing by the square root of the result. A11 now exemplifies this process.

> ***A11:*** Ok, so normalize it. ... v dot v ... would be, uh, 3^2 plus negative $(2i)^2$ [laughs] because now, we know how to do that, and that's 9 plus 4 and that's going to be 13. So, you're going to have v normalized, is going to be 1 over root 13, … you take v, and you multiply by its complex conjugate, ... so this would be 3 times 3 plus 2i times negative 2i.

In addition to elements related to the dot product, A11 referenced the complex conjugate in their

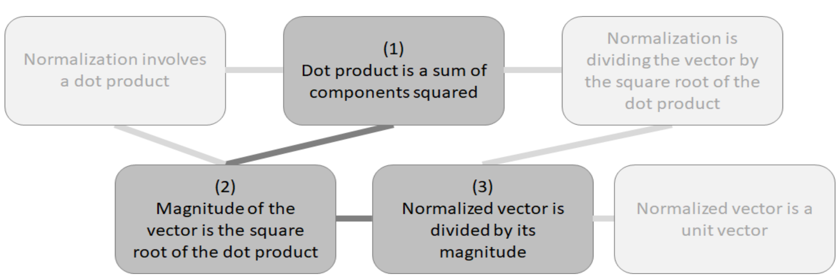

FIGURE 4. A11's causal net for normalizing a complex vector, which incorporates the same knowledge elements.

[4] One student did not get the question with the complex vector, since they did not complete the first task.

explanation. Here, we identify “Dot product is between vector and complex conjugate” and “Complex conjugate adds a negative sign to the imaginary term.”

Three of the four students using Dirac notation were also among the four to apply a normalization constant. As part of the post-interviews, C7 accurately applied the constant to the vector (not the dot product), consistent with other students who applied this method.

> ***C7:*** Now, to normalize … there needs to be some scalar times these two [kets] such that when I square the components of them and add them together, I'll get a total of one. ... v bra times v ket is equal to one, alright? So complex conjugate times [writing]. All that times the normal thing [initial vector]. So, C 3 up ket plus C 2i minus ket. (Figure 5)

In C7’s work, we identify the elements “Normalization involves a scalar” and “Multiplying by scalar means sum of the components squared is 1.” C7 then sets the Dirac inner product equal to one and writes out both the bra and ket in terms of the basis vectors as shown in Figure 5. They then invoke elements “Bra is the complex conjugate,” consistent with the application of Dirac notation in physics, and “Complex conjugate changes the sign of the imaginary terms.”

Discussion

Vector normalization is a cross-cutting concept that spans both mathematics and physics instruction. This work informs on physics students' conceptualizations of normalization of both real and complex vectors. The use of interviews before and after relevant instruction explore the way the concept of normalization changes following instruction in quantum mechanics.

Pre-interviews establish a baseline for thinking about normalization, in which all but one student normalized the vector correctly. Students commonly coordinated elements related to dot products, the sum of squared components, dividing by a square root, and magnitude or length of a vector. The transition to the complex vector resulted in about two-thirds of students invoking the same knowledge elements, and thereby utilizing the same CC. Students that correctly normalized, incorporated additional elements related to complex vectors into their causal net, such as needing a dot product of the vector with its complex conjugate.

In the post-interviews, all but one student correctly normalized the complex vector. By the end of quantum mechanics, students successfully incorporated elements related to complex vectors into their causal nets. The identified CC became more consistent with the standard approaches for normalization within both physics and mathematics.

The results provide information to mathematics instructors about how mathematics is applied in a physics course. For example, the use of a constant to normalize a vector is a common practice in quantum mechanics that is taken up by students following quantum instruction. This research further reports common knowledge elements used by students to construct a process for normalization by way of coordination class theory from physics education research.

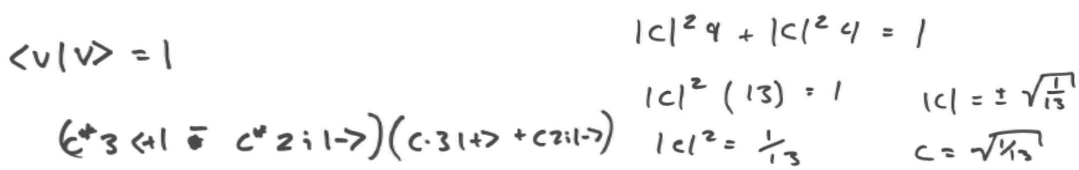

Figure 5. C7’s work using a normalization constant. C7 applies C with the complex conjugate vector but later choose C to be real and positive, consistent with physics conventions.*

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Ways that Student Reasoning about Linear Algebra Concepts Can Support Flexibility in Solving Quantum Mechanics Problems

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Reasoning with mathematics plays an important role in solving quantum mechanics problems. In addition to understanding mathematical concepts and procedures, physics students often connect the content areas by mathematizing physical constructs in terms of their associated mathematical structures and by interpreting mathematical entities in terms of the physical context. In this study, we investigate undergraduate physics students' reasoning about linear algebra in two quantum mechanics problems. Through analysis of interview data from twelve students, results show that student reasoning about orthonormal bases, change of basis, and inner products informed their flexibility in choosing problem-solving approaches. We illustrate the results with student reasoning examples and provide directions for future research.

Keywords: Linear algebra, Quantum mechanics, Student reasoning, Flexibility, Problem-solving

Mathematics and physics have an interconnected, reflexive relationship. Physical problems motivated the origins of several mathematics concepts, and the mathematization of physical phenomena often enables the development of physical theory. According to Uhden et al. (2012), "the role of mathematics in physics has multiple aspects: it serves as a tool (pragmatic perspective), it acts as a language (communicative function) and it provides a way of logical deductive reasoning (structural function)" (p. 486). Due to the entangled nature of mathematics and physics, it is essential for undergraduate physics students to reason with mathematics as they solve physical problems. They often make connections between concepts and procedures learned in mathematics and physics courses, and these sometimes vary between the two disciplines.

In this paper, we examine undergraduate physics students' mathematical reasoning used as they solve two probability problems in the context of quantum mechanics (see Figure 1). We address the following research question: *How do undergraduate physics students reason with mathematical concepts and procedures as they solve quantum mechanics problems?* In particular, our research goal was to investigate what linear algebra reasoning students leveraged in their solutions and explanations regarding the quantum mechanics problems shown in Figure 1. Our results demonstrate how students draw on their understanding of linear algebra to inform their flexibility in choosing an appropriate problem-solving approach.

Consider the quantum state vector $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$.

a) Calculate the probabilities that the spin component is up or down along the z-axis.
b) Calculate the probabilities that the spin component is up or down along the y-axis.

Figure 1. The quantum mechanics problems addressed in this study.

Brief Physics Background

To assist the reader in following student work, we summarize relevant content. Spin is a measure of a particle's intrinsic angular momentum. This observable is mathematically represented by an operator such as S_z (where the z indicates the particle's axis of rotation). Each state of the physical system is associated with a vector, denoted as a ket $|\psi\rangle$. The eigenstates corresponding to possible measurements of an observable create an orthonormal basis for the associated Hilbert space. The eigenstates for the spin-½ operator S_z are $|+\rangle$ and $|-\rangle$, which correspond to the measurements $\hbar/2$ and $-\hbar/2$, respectively. Any quantum state $|\psi\rangle$ in this system is a linear combination of the eigenstates $|\psi\rangle = a|+\rangle + b|-\rangle$ for $a, b \in \mathbb{C}$. The complex conjugate transpose of a ket $|\psi\rangle$ is a bra symbolized as $\langle\psi| = a^*\langle+| + b^*\langle-|$. The probabilistic interpretation of superposition in quantum mechanics implies $|\psi\rangle$ will sometimes have attributes that resemble those of either $|+\rangle$ and sometimes those of $|-\rangle$. If the particle is in a state $|\psi\rangle$, the measurement of its spin along the *z*-axis will yield one of the eigenvalues $\hbar/2$ and $-\hbar/2$ with probability proportional to the modulus square of the projection of $|\psi\rangle$ along the eigenvector $|+\rangle$ or $|-\rangle$, respectively. The state changes from $|\psi\rangle$ to $|+\rangle$ or $|-\rangle$ as a result of the measurement.

Problem A (Figure 1) asks for the probability of obtaining $\hbar/2$ or $-\hbar/2$ in a measurement of observable S_z on a system in state $|\psi\rangle$. The solution is calculated by $P_{\pm} = |\langle\pm|\psi\rangle|^2$, where $\langle\pm|\psi\rangle$ is an inner product of one of the *z*-basis vectors and psi. The solution for problem B (see Figure 1) is calculated by $P_{\pm,y} = |{}_y\langle\pm|\psi\rangle|^2$, where ${}_y\langle\pm|\psi\rangle$ is an inner product between one of the *y*-basis vectors and psi. To complete problem B, a change of basis is involved because $|\psi\rangle$ is written in terms of the *z*-basis, but the prompt asks for the probability that the spin component is up or down along the *y*-axis. The two main approaches are to either change $|\psi\rangle$ to be written in terms of the *y*-basis (denoted $|\pm\rangle_y$) and calculate $P_{\pm,y} = |{}_y\langle\pm|\psi\rangle|^2$, or change the *y*-basis vectors to be written in terms of the *z*-basis and calculate $P_{\pm,y} = \left|\left((1/\sqrt{2}\langle+| \mp (i/\sqrt{2}\langle-|\right)|\psi\rangle\right|^2$. In either approach, one would need to utilize the equations $|\pm\rangle_y = (1/\sqrt{2}\,|+\rangle \pm (i/\sqrt{2}|-\rangle$.

Literature Review and Theoretical Framework

Although some research does exist on student understanding of relevant linear algebra concepts such as basis and change of basis (e.g., Adiredja & Zandieh, 2017; Bagley & Rabin, 2016; Hillel, 2000; Stewart & Thomas, 2010), the more pertinent literature for this study is the relationship between mathematics and physics and how students reason about that interplay (e.g., Christensen & Thompson, 2012; Karakok, 2019; Redish, 2006). For example, Schermerhorn et al. (2019) investigated physics students' reasoning about basis and change of basis in the context of calculating expectation value problems. They found that a challenge for most students was choosing an appropriate basis in which to express the matrices and vectors involved in the calculation. Wan et al. (2019) discussed how structural features of quantum notations can foster or hinder students' reasoning about inner products and quantum probabilities. They found that Dirac notation brackets helped students make sense of inner products of energy eigenstates and state vectors. In examining metarepresentational competence, Wawro et al. (2020) found that "students' rich understanding of linear algebra and quantum mechanics includes and is aided by their understanding and flexible use of different notational systems" (p. 020112-2).

As Caballero et al. (2015) noted, researchers have documented that reasoning about mathematics in physics contexts can be a complex endeavor for students. This complexity is partly due to the need to connect mathematics and physics via mathematizing and interpreting. Karam (2014) defined *mathematizing* as the "process of constructing a mathematical representation for a physical situation (in the broad sense). This process can be seen as a translation from the physical world (e.g., observations and experiments) into mathematical structures (e.g., numbers, functions, and vectors)" (p. 5-6, parentheses in original). The notion of *interpreting* involves making sense of mathematical symbols and structures in terms of the physical phenomena they represent or correspond to (Uhden et al., 2012). Mathematization and interpretation are central aspects of Uhden et al.'s (2012) model for the use of mathematics in physics, in which one structures physical phenomena, performs varying degrees of mathematization, uses technical skills to reason with mathematics, and interprets mathematical structures in terms of the corresponding physical phenomena. We draw on Uhden et al.'s theory as it highlights the entanglement of mathematics and physics that students navigate.

We use Uhden et al.'s and Karam's (2014) technical and structural skills to analyze students' mathematical reasoning on quantum mechanics problems and illustrate how their structural and technical skills support their flexibility in choosing problem-solving approaches. *Technical skills* involve using knowledge of mathematical concepts and procedures as a tool to solve physics problems; this use of mathematical skills is independent of connections to physics. Karam (2014) elucidated two types of technical skills: procedural and conceptual. *Technical-procedural skills* involve using mathematics to perform manipulations or procedures. This is akin to procedural knowledge (Hiebert & Lefevre, 1986; Star, 2005) encompassing "knowledge of procedures that is associated with comprehension, flexibility, and critical judgment" (Star, 2005, p. 408). We focus on the aspect of *flexibility,* a central facet of students' decision-making when choosing a particular problem-solving approach. It "incorporates knowledge of multiple ways to solve problems and when to use them" (Rittle-Johnson & Star, 2007, p. 562). *Technical-conceptual skills* involve giving conceptual explanations of mathematical rules and procedures, akin to Hiebert and Lefevre's (1986) conceptual knowledge. *Structural skills* incorporate reasoning about the interconnectedness of mathematics and physics and are "based on the capacity of employing the mathematical knowledge for structuring physical situations" (Pietrocola, 2008, p. 7). *Structural-mathematizing skills* involve translating from the physical world to mathematical structures and formulas, and this "involves not only a significant understanding of mathematical concepts and theories, but also the ability of abstracting, idealizing and modelling physical phenomena" (Karam et al., 2010, p. 2). *Structural-interpreting skills* involve making sense of mathematical structures in terms of the physical phenomena with which they correspond.

Methods

The participants were 12 undergraduate physics students, of which eight (pseudonyms A#) were enrolled in a junior-level Quantum Mechanics course at University A, a large research institution in the Northwest US. The other four participants (pseudonyms C#) were enrolled in a senior-level Quantum Mechanics course at University C, a medium-sized research institution in the Northeast US. Semi-structured interviews (Bernard, 1988) with each participant were conducted with the broad goal of gaining insight into how students reason with linear algebra concepts in quantum mechanics contexts. The interviews were recorded, transcribed, and written work was retained. We analyzed the participants' responses to the problems shown in Figure 1.

We performed inductive open coding (Miles et al., 2013) on each transcript, capturing what knowledge or skill the student implicitly used or explicitly described as they engaged with the problems. The authors independently coded four students' transcripts, compared, and created a primary code list, which the first author used to code the remaining transcripts. When new codes emerged from interpreting the remaining transcripts, they were added to the original list. The transcripts were coded again to ensure that no segments were missed or miscoded. We then performed deductive coding by assigning the codes one of four a priori parent codes: structural-mathematizing, structural-interpreting, technical-conceptual, or technical-procedural, which derive from Uhden et al.'s (2012) and Karam's (2014) structural and technical skills, as well as Hiebert and Lefevre's (1986) and Star's (2005) conceptual and procedural knowledge.

We explored the students' reasoning behind their decisions to use certain problem-solving approaches. To perform this analysis, we identified segments of the transcripts where the student justified their choice in using a particular approach and assigned those segments the code, "Flexibility in Choosing Approach." We aggregated all the transcript segments labeled with this code and identified which part of the problems the students were working on as they decided which approach to use. We identified three[1] such places in the students' work: calculating $|\langle\pm|\psi\rangle|^2$, $|_y\langle+|\psi\rangle|^2$, and $|_y\langle-|\psi\rangle|^2$. To see which technical and structural skills the students used as they chose their problem-solving approach or reflected on their choice, we identified all the other codes that were assigned to those transcript segments. We then decided which coded technical and structural skills were relevant to the students' flexibility in choosing an approach.

Results

Our primary finding was that students' technical and structural skills related to reasoning about linear algebra concepts supported their flexibility in choosing an appropriate approach to the problems. Particularly, (1) students' technical and structural skills related to reasoning with inner products and orthonormal bases supported their flexibility in choosing an approach for calculating $|\langle\pm|\psi\rangle|^2$, and (2) their technical and structural skills related to reasoning with basis and change of basis supported their flexibility in choosing an approach for calculating $|_y\langle+|\psi\rangle|^2$.

Students' Technical and Structural Skills Related to Reasoning with Inner Products and Orthonormal Bases Supported their Flexibility in their Problem-solving Approaches

For Problem A, there are two main approaches that could be used to calculate the probability that the spin component of angular momentum was up or down along the *z*-axis: that is, to compute $P_\pm = |\langle\pm|\psi\rangle|^2$. The first approach involved calculating $P_+ = |\langle+|\psi\rangle|^2$ by substituting $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ into the inner product to get $P_+ = \left|\langle+|(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle)\right|^2$, using the distributive and commutative properties to find $P_+ = \left|\frac{3}{\sqrt{13}}\langle+|+\rangle + \frac{2i}{\sqrt{13}}\langle+|-\rangle\right|^2$, using known properties $\langle+|+\rangle = 1$ and $\langle+|-\rangle = 0$ to reduce the equation to $P_+ = \left|3/\sqrt{13}\right|^2$, and simplifying to $P_+ = 9/13$. The procedure for determining the complementary probability $P_- = |\langle-|\psi\rangle|^2$ can be performed similarly to conclude that $P_- = \left|2i/\sqrt{13}\right|^2 = 4/13$. Alternatively, the second approach allows students to skip most of the procedures in the former approach: students could square the norm of the coefficient of $|+\rangle$ or $|-\rangle$, respectively, in $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ to find P_+ and P_-. We include this description of the approaches that the students considered using because their reasoning here includes the acts of considering various options and using their understandings of the mathematical or physical concepts to decide on an approach.

First, students' technical and structural skills related to *inner products* supported their flexibility in choosing this problem-solving approach. For instance, A8 explained:

> Because this is written along the *z*-axis, I'm assuming that we're working in the *z*-basis here, standard representation. Then you do… norm squared of plus with psi [$|\langle+|\psi\rangle|^2$], and by the same rule I talked about earlier, about how we have this just plus plus minus minus equals 1 [$\langle\pm|\pm\rangle$ =1], then all you get here is plus and minus. Literally just pull out these same coefficients, so you get 3 over root 13 squared and 2 over root 13 squared.

A8's technical-conceptual skill of knowing that the vectors in the inner product were expressed in terms of the same basis allowed them to take advantage of properties of the *z*-basis. To be able to skip the steps in the first approach of evaluating the inner product, $\langle+|(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle)$, students first used the structural skill of interpreting the mathematical symbols to recognize that

[1] For brevity, we only discuss the first two instances in this paper. They correspond to the two Results subsections.

the vectors in the inner product were linear combinations of vectors from the same basis. Their technical-conceptual skill of recognizing that vectors in an inner product must be in terms of the same basis then allowed them to use the inner products of *z*-basis vectors.

Second, the students' technical and structural skills related to *properties of orthonormal bases* supported their flexibility in choosing to use the second efficient approach to solving problem A. The students used structural-mathematizing skills to recognize that $|+\rangle$ and $|-\rangle$ comprise an orthonormal basis which allowed them to use technical skills involved in using the inner products of orthogonal basis vectors: $\langle\pm|\pm\rangle = 1, \langle\pm|\mp\rangle = 0$. For example, when asked about how they found their answer, A11 explained their reasoning about this idea:

> Since this is a basis, uh, plus with a plus is equal to 1 [$\langle+|+\rangle = 1$], whereas plus with a minus is equal to 0 [$\langle+|-\rangle = 0$]. So, if I was to distribute a plus [$\langle+|$] out to all of these, this would give us zero automatically because they're orthogonal. This would go to 1, so I square that. Same thing with the other way, because minus plus [$\langle-|+\rangle$] is equal to 0.

Reasoning about orthonormal bases, namely that $\langle\pm|\pm\rangle = 1$ and $\langle\pm|\mp\rangle = 0$, allowed students to anticipate that evaluating inner products by distributing $\langle+|$ to $(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle)$ and $\langle-|$ to $(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle)$ would leave only the coefficient of $|+\rangle$ and $|-\rangle$, respectively. This allowed them to skip these steps and instead calculate the probabilities by squaring the norm of the coefficients of $|+\rangle$ and $|-\rangle$. Overall, students' technical and structural skills related to reasoning with inner products and orthonormality supported their flexibility for calculating $P_\pm = |\langle\pm|\psi\rangle|^2$.

Students' Technical and Structural Skills Related to Reasoning with Basis and Change of Basis Supported their Flexibility in their Problem-Solving Approaches

To calculate $|_y\langle+|\psi\rangle|^2$, the students recognized a need to perform a change of basis, and they had the choice to either change $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ to be written in terms of the *y*-basis or change $|+\rangle_y$ to be written in terms of the *z*-basis. The students' technical and structural skills supported their decision in choosing their problem-solving approach. In particular, the students' technical and structural skills related to reasoning about the *basis that the vectors in the inner product were expressed in terms of* and about the *properties of orthonormal bases* supported their choice to perform a change of basis. Their technical and structural skills also supported their flexibility in deciding which of the two aforementioned change of basis approaches to use.

The students' technical and structural skills related to reasoning with basis and inner products supported their choice to perform a change of basis. For example, A13 explained:

> You'd either have to change this [$|\psi\rangle$] to *y*-basis to fit this, which would not be fun probably, or change your *y*-basis to *z*-basis… psi is in a completely different basis, so you can't just multiply out in- when they're in different bases, so you have to switch bases.

As exemplified in A13's reasoning, the students used their structural skill of interpreting that "psi is in a completely different basis." They recognized that $|\psi\rangle$ is a linear combination of *z*-basis vectors, which does not match the basis expression of $_y\langle+|$, the other vector in the inner product. The students then used their technical-conceptual skills to recognize that the vectors in the inner product needed to be expressed in terms of the same basis for them to be able to perform the inner product. For instance, C6 claimed, "you can't do anything until you're in the same basis." The students then used the structural-mathematizing skill of recognizing that a change of basis is necessary to be able to perform that inner product. Thus, the students' technical and structural skills related to reasoning about the basis of the vectors in the inner product supported their choice to perform a change of basis.

Reasoning about the orthonormality of the bases and the associated inner product values also supported their choice to perform a change of basis. Some students discussed how changing basis made the calculations simpler due to orthonormality of the bases. For instance, A21 said:

> I wanna be able to read off those coefficients really easily and do this in bra-ket notation if these are in the same uh basis. If I'm expressing plus y [${}_y\langle +|$] in the *z*-basis then I can make all those assumptions about one, you know, the pluses and the minuses, the cross terms are gonna be zero. But if I were to do this in the *y*-basis… like write this out as like let's say y plus, but against all of this [$\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$], then I can't make any assumptions about that, so I don't really know how to calculate that in bra-ket language.

A21 claimed it was necessary to change basis to make assumptions about the inner products of the various (orthonormal) basis elements, such as $\langle \pm|\mp\rangle = 0$. A21 suggested that leaving the inner product with the vectors expressed in terms of different bases would not allow them to use $\langle \pm|\mp\rangle = 0$. Thus, the need to use the orthonormality property informed their choice to perform a change of basis. C5 also claimed a change of basis was necessary for "inner products to be nice:"

> Because my state vector was given in the *z*-basis, if I'm doing the inner product of the positive *y* with that, I need that to be written in the *z*-basis, or to do those inner products to be nice. So I guess the plus and plus gives you one. The plus and minus gives you zero.

Taking advantage of the orthonormal basis properties motivated the students' selection of the change of basis approach. The students used their structural-mathematizing and technical-conceptual skills to leverage that the *y*-basis and the *z*-basis are both orthonormal. These structural and technical skills informed their choice of approach and therefore their flexibility.

In addition to informing their decision to perform a change of basis, the students' structural and technical skills also supported their flexibility in choosing a change of basis approach: either changing $|\psi\rangle$ to be a linear combination of *y*-basis vectors or changing ${}_y\langle +|$ to be a linear combination of *z*-basis vectors. A8 performed the former after acknowledging both approaches:

> There are two ways to go about it, um, one of them is to put this vector in some phi prime that's in the *y*-basis, and then just do y plus phi prime y [${}_y\langle +|\psi'\rangle_y$]…it follows the same rules as this. Um, the other possibility is to do, is to take the spin up *y* and go to whatever it is in the *z*-basis, cause we have this in the *z*-basis. Um, they're both equivalent.

Expressing $|\psi\rangle$ as a linear combination of *y*-basis kets allowed A8 to square the norms of coefficients of *y*-basis kets. A8 recognized that both methods were "equivalent" and yielded the same probability result. A8 reflected on their choice of approach and compared the efficiency of the two methods: "The other method is probably faster if you think of it. Actually, I don't know if it's really faster. You just save so much time on this side, if you do it this way." In summary, A8's structural-mathematizing and interpreting skills supported their flexibility in recognizing the two approaches, comparing their efficiency, and choosing one for solving the problem.

Most students chose to use the approach of changing ${}_y\langle +|$ to be a linear combination of *z*-basis vectors, and their structural and technical skills supported their flexibility in doing so. Some students chose this one due to computational ease. For instance, A6 explained:

> You can't do anything until you're in the same basis…I needed plus *y* in the *z* basis, because this was in the *z* basis. If I really wanted to, I could have changed [$|\psi\rangle$] to the *y* basis. Um, this [${}_y\langle +|$] is a lot easier because we had the spins sheet, so I changed this from the *y* basis to the *z* basis here, so then both of them were in the *z* basis.

Students could use the equation $|+\rangle_y = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}i|-\rangle$, which made the change of basis procedure "a lot easier," only involving substitution and not the solution of a system of equations. Students' technical-procedural skills of using substitution and given equations supported their flexibility by allowing them to compare the efficiency of possible approaches. A13 also acknowledged that changing $|\psi\rangle$ to be in terms of the *y*-basis vectors "would not be fun probably," so they chose to change the basis that ${}_y\langle+|$ was expressed in terms of, instead. A11 similarly recognized that changing $|\psi\rangle$ to be in terms of the *y*-basis vectors would involve more work, explaining, "I didn't really want to have to deal with that math." These students' structural-mathematizing skill of recognizing that changing ${}_y\langle+|$ to be in terms of the *z*-basis vectors was easier than changing $|\psi\rangle$ to be in terms of the *y*-basis vectors supported their flexibility in choosing a way to change basis. Overall, the students' structural and technical skills related to reasoning about change of basis via substituting and using equations supported their flexibility.

Discussion

Given the entanglement of mathematics and physics, it is essential for undergraduate physics students to learn to reason with mathematics as they address physical problems. This is complex for students as it involves connecting their mathematics and physics reasoning via interpreting and mathematizing. We leveraged Uhden et al.'s (2012) and Karam's (2014) framework of students' technical (conceptual and procedural) and structural (mathematizing and interpreting) skills to investigate the flexibility of physics students' reasoning about mathematics in relation to physics content addressed in two quantum mechanics problems. Through our analysis of interview data from twelve physics students, we found that the students' technical and structural skills related to reasoning about linear algebra concepts supported their flexibility in choosing a problem-solving approach. Flexibility is an essential aspect of problem-solving, as it involves being aware of multiple approaches to solve a problem and choosing an appropriate one. Other researchers have demonstrated how conceptual knowledge can support procedural flexibility (e.g., Rittle-Johnson et al., 2015). This finding is furthered in our study. When students draw on their mathematical knowledge to inform their approach for solving these quantum mechanical problems, it relies on their understanding of how the mathematics and physics concepts are intertwined. Students' use of conceptual understanding in their work on these problems is more complex than just reasoning about mathematical concepts and procedures because students must reason about them in relation to their corresponding physical concepts. Thus, not only are physics and mathematics are entangled; students' reasoning about them is also entangled.

With respect to teaching implications, our research helps raise the mathematics community's awareness of what concepts from mathematics courses are used and in what way by students in physics courses. Our study revealed the centrality of: basis, orthogonality, normality, change of basis, algebraic substitution, simplification of vector equations or system of equations, and inner product in the solution process for a quantum mechanical problem. While those are core to a linear algebra course, additional mathematical concepts were central, namely the probabilistic relationship of mutually exclusive events, distribution, and operations with complex numbers. These concepts are relevant for larger systems, as well as other quantum mechanical observables. Future research can address how students use their understanding of linear algebra in quantum mechanical contexts in which the mathematical concepts correspond to physical phenomena.

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Prospective Teachers Connecting Group Axioms with the Structure of the Identity Function

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I examined the development of three Prospective Secondary Mathematics Teachers' (PSMTs) understandings of connections between concepts in abstract algebra and secondary algebra. I investigated how PSMTs deepened their understanding of the identity function and unified various identities under the overarching identity structure as they engaged in a Mathematics for Secondary Teachers course. Analysis revealed that one PSMT conceptualized the identity function as both a function that leaves its input alone and the result of composing inverse functions. The other two conceptualized the identity function as x, the output of $f(f^{-1}(x))$, instead of a function. The PSMTs unified the additive identity, multiplicative identity, and identity function as instantiations of the same overarching concept by reflecting on the group axioms. I discuss the opportunities given to the PSMTs during instruction that contributed to their development of these understandings. I conclude with implications for PSMT preparation.

Keywords: Abstract algebra, Prospective teachers, Mathematical knowledge for teaching

Abstract Algebra is an almost universally required course for prospective secondary mathematics teachers (PSMTs, Blair et al., 2013), but teachers might not perceive connections between properties of algebraic structures in secondary algebra and abstract algebra contexts (e.g., Cofer, 2015; Wasserman, 2017). Thus, there is a need to make more explicit the ways in which teachers use their knowledge of abstract algebra to better understand the algebra content they teach. Wasserman (2018) theorized that for knowledge of abstract algebra to be useful in teachers' instruction, it must first help reshape the teacher's understanding of the content they teach. Thus, this study examines the reshaping of PSMTs' mathematical understandings that occurs as they reason about content connections between secondary algebra and abstract algebra.

Teachers' knowledge of abstract algebra could potentially influence their instruction on inverses (Wasserman, 2016). The structure of inverses is derived from that of a group. An *identity element* e in a group $(G,*)$ satisfies the property that $a * e = e * a = a$ for any element a in G (referred to as the identity group axiom). In a group, there exists an *inverse element* a^{-1} for every element $a \in G$, such that $a * a^{-1} = a^{-1} * a = e$ (referred to as the inverse group axiom). Some groups commonly found in secondary contexts include $(\mathbb{Z},+)$, $(\mathbb{Q}^*,\cdot)$, and the set of invertible functions (with domain restrictions) under function composition. Understanding the structure of a group might help one make sense of the structure of additive, multiplicative, and function inverses, as well as the structure of various identities. Each type of inverse is an inverse element from a group and is associated with an identity and binary operation. Thus, every element f of the group of invertible functions under composition has an inverse f^{-1} such that $f \circ f^{-1}(x) = f^{-1} \circ f(x) = i(x)$, where i is the *identity function* defined as $i: X \to X$ such that $i(x) = x$. The application of group axioms is involved in the algebraic cancellation used in equation solving, which illustrates content connections in secondary and abstract algebra. These content connections were covered in a Mathematics for Secondary Teachers course. In this study, I explore how PSMTs in that course reasoned about the group axioms to make connections among the additive, multiplicative, and compositional identities. I address the research questions: (1) *How do PSMTs reason about the structure of the additive identity, multiplicative identity, and identity function? (2) What aspects of the course helped them develop those understandings?*

Literature Review and Theoretical Background

Several studies have addressed teachers' or undergraduate students' understandings of inverses, focusing on their attention to the corresponding binary operation and identity. Zazkis and Kontorovich (2016) explored prospective teachers' explanations of the superscript -1, which is the same notation used for inverse functions and multiplicative inverses (reciprocals). Most PSMTs in their study perceived the superscript -1 as a symbol that refers to different unrelated meanings that are context-dependent. They did not use their knowledge of group structure to reason about the similar inverse meanings implied by the superscript -1. Furthermore, Bagley et al. (2015) found all ten of their participating undergraduate students "said the result of composition of a function and its inverse should be 1" (p. 36), implying they associated inverse functions with the multiplicative identity. They found that when students resolved that inaccurate idea, they used "do-nothing function reasoning" or "net-do nothing function reasoning," which respectively involve understanding the composition of inverse functions as a function that does nothing to its input argument (i.e., the identity function) or as a process of an inverse function undoing what a function does to an input. They suggested these were productive ways of reasoning about the composition of inverse functions. Wasserman's (2017) study also focused on teachers' attention to the binary operation and identity that was associated with inverse functions. He investigated how secondary teachers, who had recently taken Abstract Algebra, situated inverse functions within their group structure by attending to the identity function and composition; only two of seven teachers could do so. These studies illustrate the complexities teachers may experience in distinguishing inverses by reasoning about their associated identity.

This study is framed by Wasserman's (2018) theory of Knowledge of Nonlocal Mathematics for Teaching. Wasserman (2018) distinguished types of mathematics as local or nonlocal, relative to the mathematics one teaches and used this distinction to consider how teachers' understanding of nonlocal (advanced) mathematics influences their understanding and teaching of local mathematics. Wasserman argued for teachers' understanding of nonlocal mathematics (e.g, abstract algebra) to influence their instruction, the teachers need to first make connections between the nonlocal and local content that *reshape* their understanding of the mathematics they teach. I conceptualize this reshaping of knowledge according to Lee's (2018) construct of a transformative transition which involves *extending* the contexts in which one's understanding is situated, *deepening* one's understanding of a concept, *unifying* disparate concepts under an overarching concept, and *strengthening* the connections between their existing understandings of multiple mathematical concepts. I use the deepening and unifying categories to analyze how PSMTs reshape their understandings of identities used in secondary algebra.

For documenting this increase in depth of students' understandings, Lee (2018) used Action-Process-Object-Schema (APOS) theory (Arnon et al., 2014), which provides a lens for how students can deepen their understanding of a concept through transitioning from having an action to a process to an object level conception. As students begin to understand a mathematical concept, they manipulate either physical objects or previously constructed mental objects to form actions. A student first performs an action explicitly and then repeats and reflects on this action. Once a student can mentally imagine the steps of this action without having to explicitly carry out all of the steps of the action, the student has interiorized the action into a process. A student then encapsulates a process into an object by performing actions on the process as if the process was a static object. A schema is a collection of actions, processes, and objects and the connections between those structures. I focus on the PSMTs' development of Identity Schemas, which comprise conceptions of additive, multiplicative, and compositional identities and the relations among them. I use Lee's (2018) construct of unifying to analyze the development of this schema in which the connections between identities become more coherent. A learner can unify their existing understandings under an overarching mathematical object by increasing the extent to which seemingly unrelated mathematical concepts become more coherent. This reshaping of a mental schema can be analyzed using Piaget and Garcia's (1983/1989) triad. At the intra-object level of unifying, the learner focuses on constructs in their mind individually in isolation from each other. The inter-object level of unifying involves finding similarities between some of the concepts but not articulating the overarching idea that the concepts embody. The trans-object level of unifying is characterized by the learner recognizing the concepts all as instantiations of the same overarching idea and thus connecting the concepts into a coherent schema. In this paper, I use these deepening and unifying categories in Lee's (2018) Extend-Deepen-Unify-Strengthen (EDUS) framework as mechanisms to document how PSMTs' understandings of identities are reshaped in a Mathematics for Secondary Teachers course.

Methods

Data were collected in a senior-level Mathematics for Secondary Teachers course at a large research university in the eastern US. The class consisted of four PSMTs, of which three (Amelia, Christina, and Derek) participated in this study. The participants had previously taken an introductory undergraduate course in Abstract Algebra. Clinical interviews (Clement, 2000) were conducted with each participant before and after instructional units covering ring and group axioms in relation to inverses and equation solving. These interviews elicited evidence of the PSMTs' understandings of inverses, identities, binary operations, and the relations between them. These interviews had a pre-post format, so I could observe changes in their understanding.

To address research question (1), I used Lee's (2018) EDUS framework to analyze changes in each participant's understanding of inverse, identity, and binary operation. I characterized the depth of the students' understanding of a single concept (e.g., identity function) using the levels of Action, Process, and Object. A student with an action conception can compose two inverse functions and get x as the resulting output but does not perceive the identity function's identity structure. A student with a process conception can compose two inverse functions and anticipate the resulting output to be the identity function internally in their mind. A student with an object conception can perceive the identity function as a function in its own right and can act on it via function composition. I also classified the extent to which their understandings of different types of the same concept (e.g., additive, multiplicative, and function identities) were unified into a coherent schema using Piaget and Garcia's (1983/1989) triad of schema development. At the intra-object level, a student focuses on additive, multiplicative, and function identities in isolation from each other and does not recognize similarities among them. At the inter-object level, a student can articulate some similarities between the identities. At the trans-object level, a student can articulate the identity structure shared by these different identities, i.e., that two inverse elements operated together equals the identity or that an element operated with an identity equals that element. I documented changes in these classified levels of the PSMTs' understanding between the pre- and post-interviews to find how they were deepened or unified.

To address research question (2), I collected video and audio data of each class from the two instructional units. I analyzed class episodes in which the PSMTs had opportunities to deepen or unify their understandings of identities as they received instruction on abstract algebra concepts and their connections to secondary mathematics. Through this analysis, I identified aspects of instruction (e.g., the instructor's discussion prompts or tasks) in the Mathematics for Secondary Teachers course that supported the PSMTs' in deepening and unifying their understandings.

Results

The PSMTs Deepened Their Understanding of the Identity Function

The PSMTs deepened their understanding of the identity function during the Mathematics for Secondary Teachers course, although to different depths. During the pre-interviews, Amelia and Derek described the composition of inverse functions as one or zero and thus had not yet constructed a conception of the identity function. For example, Amelia anticipated the composition of inverse functions to be 0 and was surprised when the composition yielded x. She explained because "they negate each other, it would make sense for it to be zero." She thus related the composition of inverse functions with the additive identity.

Amelia, Christina, and Derek then developed Action conceptions (Arnon et al., 2014) of the identity function as they verified that two given functions were inverses of each other. Once they reflected on x as the result of the composition of inverse functions, they could anticipate that result in subsequent tasks and use that to determine whether a pair of functions were inverses of each other. They thus developed a Process (Arnon et al., 2014) understanding of the identity function as the result of composing inverse functions. For example, Derek constructed a Process conception of the identity function during a pre-interview. After Derek found the inverse of a given function, the interviewer asked follow-up questions:

Interviewer: How does this inverse relate to the inverse definition you gave?

Derek: … so in functions, your like, I guess the thing you're doing is composition of functions … and $f(x) = x$ is kind of your identity in functions. So when they're put together, you want it to be equal to the identity, which would be x. So you want to get the equation $f(x) = x$ when you do those two things together…

Interviewer: Why, again, do we want it to be equal to x there?

Derek: That's the identity function, that we've referred to as the identity of functions is just y equals x, so we want to have that kind of form when we do the composition of functions.

Derek's responses demonstrated his Process conception of identity function as the result of composing inverse functions, as he could mentally anticipate the result of the composition. Derek also developed an Object conception (Arnon et al., 2014) of the identity function, as he could perform actions, such as composition, on the identity function as if it were a static entity. Derek demonstrated an understanding of the structure of the identity function as the function that maps an input to the same output, as well as the function that is the result of a composition of inverse functions. For example, Derek said, "The identity function is just a function that equals x, so it's something that if we plug in, it just doesn't change the function, I guess, so if we like plug in an x, we just pop out y, which it happens to be x, so it just like leaves it alone." He described that if one plugs an input into the identity function, the identity function does not change it and "leaves it alone," meaning the input of the function is the same as the output.

Amelia and Christina developed a process conception of the identity function as the result of composing inverse functions. They conceptualized the identity function as a variable x rather than a function, which seemed to inhibit them from constructing an Object conception of the identity function because they were unsure of how to compose it with other functions. Whenever they reasoned about why x was the identity function, they related it to the property that it was the resulting output of the composition of inverse functions ($f(f^{-1}(x)) = x$). For example, Amelia described the identity of the set of invertible functions, saying "For invertible functions with composition, plugging them into each other would give you back x as the identity… when you compose them with one another you get out x, so if f inverse is actually the right inverse for f, then you'd get out an x." She thus demonstrated a Process conception, but she did not show evidence of having constructed an Object conception of the identity function by the post-interviews. She expressed uncertainty about whether the identity x was a function or a variable. She tried to show that the identity function satisfied the definition of identity, explaining:

Amelia: The inverse would just be f inverse and the identity would just be x, the identity function, since it is under composition, and so we know, f, f inverse, x. But I'm not really sure. Like going back to the definition of an identity, because earlier, I would do it like a plus 0 equals 0 plus a, which equals a. I'm not sure how that would work like with the definition similar to that for functions and composition of functions.

Interviewer: Okay, so the identity x, is that just like the variable, or is that a function?

Amelia: It's the variable, or it is a function. I'm not sure.

Amelia tried to relate the additive identity structure ($a + 0 = 0 + a = a$) to the structure of the identity function, but she was unsure of how to do that. Thus, she did not know how to show that the identity function satisfied the definition of identity that for any function f in the given group, $f \circ i = i \circ f = f$, where i is the identity function defined as $i(x) = x$. Given that she consistently referred to the identity function as just x, she conceptualized the identity function as the variable x, not a function. Christina similarly expressed uncertainty regarding the identity function, explaining, "I think identity might not be the function. I think it's the variable." Amelia and Christina did not know how to compose the identity function with another function. Thus, they had not yet constructed an Object conception of the identity function.

In summary, Amelia and Christina developed process conceptions of the identity function as the result of composing inverse functions, but they did not recognize how the identity function satisfied the definition of identity that for any function f in the given group, $f \circ i = i \circ f = f$, where i is the identity function defined as $i(x) = x$. They thus conceptualized the identity as x, the resulting output of $f(f^{-1}(x))$, instead of the function $i(x) = x$, while Derek conceptualized the identity as a function itself. He could articulate how it satisfied the identity group axiom and the inverse group axiom that the identity function equals the composition of inverse functions.

The PSMTs Unified Additive, Multiplicative, and Function Identities

Amelia, Christina, and Derek all unified additive, multiplicative, and function identities under the same overarching identity concept throughout the semester. They originally exhibited an inter-object level understanding of the connections among the additive and multiplicative identities, but they conflated the multiplicative identity and the identity function, particularly when composing inverse functions. For example, Derek explained his reasoning about $e^{\ln(2)}$ as "e to the natural log cancels out, and that just essentially becomes one, and same with natural log of e is just one…I guess multiplicatively they're inverses because if you multiply… they come back to an identity of one." Derek reasoned about the structure of operating inverse elements together to get the identity, so he connected the idea of "e to the natural log canceling out" to the idea of multiplicative inverses "coming back to like an identity of one" instead of the identity function. He thus confused the multiplicative identity with the identity function. Derek had not yet mentally constructed the concept of the identity function, so he could not yet form connections between it and the multiplicative identity to develop a coherent schema of identities. Overall, the students' Identity Schemas were not yet coherent during the pre-interviews because they had not yet unified the identities as instantiations of the same concept.

Once the students each developed a conception of the identity function, they recognized the structure of the identity function as the result of composing inverse functions and perceived it as similar to the structure of the additive and multiplicative identities. For instance, when Amelia defined an inverse during the post-interviews, she said, "An inverse is an operation or a function that when computed will bring that portion of like the equation *to the identity… for a function it would be the identity function, or like the additive identity, or the multiplicative identity.*" Christina similarly could identify the identities of various groups. For the groups $(Q^*,\cdot)$, $(\mathbb{Z},+)$, and the set of invertible functions with the same domain and range under the operation of function composition, she identified their respective identities as 1, 0, and x. Furthermore, after Derek mentally constructed the identity function during Interview 2, he could identify the similar structure of different types of identities. For example, when Derek compared multiplicative and function inverses, he identified similarities in the structure of their corresponding identities, explaining, "they're similar in the sense that they take it back to the identity… if we had a and it was operated with a inverse, it'd give you back to identity in that set, and f inverse does the same for f." Derek recognized that the multiplicative identity and the identity function are both the result of an element of a set being operated with its inverse. He made connections between the different identities associated with different inverses by reasoning about their similar structure of being the result of operating inverse elements together. Once Derek recognized that the identity associated with inverse functions was $i(x) = x$, he could unify $0, 1,$ and $i(x) = x$ as instantiations of the same overarching concept. Overall, these students recognized $0, 1,$ and the identity function as instantiations of the same overarching identity concept. They could articulate the shared identity structure of the additive, multiplicative, and compositional identities as the result of operating two inverse elements in a group together, which is indicative of a trans-object level (Piaget & Garcia, 1983/1989) understanding of these identities. Thus, the PSMTs all unified their understandings of these different types of identities under the overarching identity concept and thereby developed a more connected and coherent Identity Schema.

How Instruction in the Course Supported the PSMTs in Developing these Understandings

The PSMTs had opportunities to develop these understandings of the identity function during class. The instructor led class discussions about the inverse group axiom and the identity group axiom to lead the PSMTs to reason about the structure of the identity function. Reasoning about the structure of operating inverses together to get the identity (i.e., $f^{-1}\big(f(x)\big) = f^{-1}\big(f(x)\big) = x$) first led the PSMTs to conceptualize x as the identity that resulted from the composition of inverse functions. The class later proved that the identity function defined by $i(x) = x$ satisfied the inverse group axiom, (i.e., $f \circ f^{-1} = f^{-1} \circ f = i$), by showing these three functions had the same output for each input. Thus, the class discussed the identity function as an actual function. The instructor often referred to "getting back to the identity function" after composing a function with its inverse. This likely contributed to the PSMTs' development of a Process conception of the identity function as the result of composing inverse functions. The class also discussed how the identity function satisfied the identity group axiom, "i is the identity if for all f in some set S, $i \circ f = f \circ i = f$," by showing that $(i \circ f)(x) = i\big(f(x)\big) = f(x)$. This likely supported Derek in developing an Object conception of the identity function, and this might have supported Amelia and Christina in transitioning toward their development of an Object conception, as well.

The PSMTs had opportunities to unify the additive identity, multiplicative identity, and identity function as they reflected on their shared identity structure during class. They reasoned about their identity structure as the result of operating two inverses together, particularly in the context of equation solving. The instructor led the PSMTs to consider the motion of crossing out while solving equations and had them identify what structures were used that allowed them to cross terms out. Discussing how "getting rid of" terms while solving equations involves the use of the additive identity, multiplicative identity, or identity function gave the PSMTs an opportunity to identify similarities in the structure of the identities corresponding to different binary operations. The instructor often juxtaposed the identity function with the additive and multiplicative identities to lead the PSMTs to think about the identity was associated with inverse functions. This gave the PSMTs an opportunity to recognize the similar structure of the various identities as the result of operating two inverse elements together. The PSMTs' development of the connections among these concepts during class involved reasoning about the inverse group axiom and reflecting on the similar structure of each of the identities, $0, 1$, and $i(x) = x$. This likely helped the PSMTs recognize $0, 1$, and $i(x) = x$ as instantiations of the same identity concept and unify them into a coherent Identity Schema.

Discussion

Wasserman (2018) suggested that teachers' abstract algebra knowledge can potentially become useful for teaching if it helps reshape their understandings of the algebra content they teach. Drawing on Lee's (2018) extend-deepen-unify-strengthen framework, I explored this reshaping of PSMTs' understandings as they reflected on connections between group axioms and the structure of the identity function. I examined how PSMTs deepened their understanding of the identity function and unified various identities under the overarching identity structure as they engaged in a Mathematics for Secondary Teachers course. The first finding revealed that the PSMTs deepened their understandings of the identity function from not having a conception of it to developing process or object conceptions. One PSMT conceptualized the identity function as a function that leaves its input alone and as the result of composing inverse functions. The other two conceptualized the identity function as x, the output of $f(f^{-1}(x))$, instead of a function. The participants' conceptions of the identity function are similar to the do-nothing function (DNF) reasoning or net do-nothing function (net-DNF) reasoning documented by Bagley et al. (2015). These authors claimed, "the process of pushing an arbitrary element through the function and its inverse and observing that the net result is no change… can allow students to come to see the result of composition of a function… and its inverse as a function…in its own right" (p. 45). Derek was successful in reflecting on the result of composing inverse functions to develop an Object conception of the identity function (i.e., do-nothing function). However, Amelia and Christina reflected on the result of composing inverse functions and demonstrated net-DNF reasoning, but they conceptualized the output x as the identity instead. Thus, the findings of this study provide additional insight into students' conceptions of the identity resulting from a composition of functions as the function $f \circ f^{-1} = i$ or the output $f\left(f^{-1}(x)\right) = x$. Future research can address how students transition from Process to Object conceptions of the identity function.

Another finding was that the PSMTs unified the additive identity, multiplicative identity, and identity function as instantiations of the same overarching concept by reasoning with the group axioms. They all could identify that the different identities were the result of operating two inverse elements together with the group's corresponding binary operation. Reasoning about how the inverse group axiom applied to various inverses and identities in secondary algebra contexts allowed them to make these connections among the different identities and thereby develop a more coherent Identity Schema. The PSMTs had opportunities in their course to develop this understanding. Implementing class discussions of how the cancellation procedure in equation solving involves the use of the various identities helped the PSMTs identify similarities in their structure. I, therefore, suggest that teacher educators should ask PSMTs to explicitly identify which properties from abstract algebraic structures are used in each step solving an equation.

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Epistemological Tensions in Applying Culturally Relevant Pedagogy to Undergraduate Mathematics

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Communities of practice are rich places to learn new practices and refine existing practices. Mathematics schooling has been racialized and inequitable to minoritized groups and learning how to teach mathematics in culturally relevant ways is one potential way to address the needs of students who have been systemically ignored. We found from analysis of ten interviews with mathematics instructors at Hispanic-Serving institutions that two central barriers to enacting culturally relevant pedagogy in undergraduate mathematics instruction are a lack of communicating about students' identities and holding epistemologies that mathematics is a culture-free discipline. For the community of mathematics instructors to incorporate more culturally relevant pedagogy, first, it needs to become safer to talk about students' racial and gendered identities and, second, culturally relevant tasks that leverage mathematical tools in nontrivial ways need to be developed.

Keywords: Hispanic-Serving Institutions, Professional obligations

Mathematics has been given a high-status position as an academic subject in the United States school curriculum (Berry et al., 2014; Schoenfeld, 2004). Students' mathematical performances have influence over their future academic and economic opportunities (U.S. Department of Education, 2008) and their social access and mobility (Schoenfeld, 2004), and can be interpreted as an indication of one's intelligence (Shah, 2013). Scholarship in mathematics education has revealed how K-16 mathematics education is inherently racialized and inequitable and has a long history of ignoring the needs of minoritized groups (Berry et al. 2014; Gutiérrez, 2017; Martin, 2019). Mathematics functioning as an important subject for access and social status, yet systematically discriminating based on race, is a toxic combination. Given this perpetuated inequity, *Culturally Relevant Pedagogy* (CRP; Ladson-Billings, 1995; 2014) has been regarded by education researchers as a way to empower minoritized students and update the mathematics curriculum (Brown-Jeffy & Cooper, 2011; Morrison et al., 2008).

Ladson-Billings (2014) described CRP as teaching that helps students achieve three outcomes: (1) *academic success*, "the intellectual growth that students experience as a result of classroom instruction and learning experiences"; (2) *cultural competence*, "the ability to help students appreciate and celebrate their cultures of origin while gaining knowledge of and fluency in at least one other culture"; and (3) *sociopolitical consciousness,* "the ability to take learning beyond the confines of the classroom using school knowledge and skills to identify, analyze, and solve real-world problems" (p. 75). These are also known as the three *tenets* of CRP. She explained that undertaking this way of teaching is a long-term commitment to give students the education they deserve and to create the future society we want to live in (Ladson-Billings, 2006).

Enacting all three tenets of CRP requires attention to students' cultures and identities. For example, helping students understand how mathematics relates to their own culture and real-world situations relevant to them requires that the instructor knows what cultures students identify with and what sociopolitical issues are relevant to them. Identity research in mathematics education has drastically increased in recent years (Darraugh, 2016). However, it has largely been split into research done on the identities of learners and on the identities of instructors (Darraugh, 2016; Graven & Heyd-Metzuyanim, 2019), without much attention to how instructors are thinking about or incorporating student identities in their practice. We contend that understanding of how instructors consider student identities in their instruction is key to understanding if or why they enact CRP. We do not define identity ourselves, but explore how instructors at Hispanic-serving institutions (HSIs) perceive identity as playing a role in their instruction.

Instructors at HSIs are well-positioned to serve a critical mass of historically underserved students but can also replicate the existing inequities if they rely on their disciplinary enculturation to shape their pedagogical choices. Institutions with at least 25% Hispanic full time student populations can apply for HSI status (U.S. Department of Education, 2021). Hubbard and Stage (2009) found few differences between HSI faculty and their faculty counterparts at predominantly White institutions in terms of perceptions and attitudes towards teaching and students. It is unclear whether or how instructors are attempting to serve their increasingly diverse student populations. Shifts in demographics (Laden, 2004) have left many institutions without clear direction about the implications of being an HSI. This study investigates CRP enactment at HSIs and what barriers might currently exist to its implementation in mathematics classrooms.

Theoretical Framing

An integral part of changing and developing one's teaching practices is involvement in a *community of practice* (Wenger, 1998). We employ this framework to give a lens as to how mathematics instructors might change their practice to incorporate more CRP, or what might be preventing such change. The community of practice framework begins with the premise that learning is fundamentally a social activity that involves participation in the context of a given community. Wenger (1998) continues that communities of practice are not only places for individuals to learn but are especially rich places for new insights to be transformed into knowledge. Participants can bring their own unique experiences to generate the creation of knowledge within a community: "A history of mutual engagement around a joint enterprise is an ideal context for this kind of leading-edge learning" (Wenger, 1998, p. 214). As instructors become increasingly aware of their shifting student populations and the inequities that students experience, there is an opportunity to learn (as communities of mathematics instructors) new ways to serve those populations. However, little is known about if or how mathematics instructors at HSIs think about students' cultures and identities.

With this community of practice framing of learning, we ask the following research questions: (1) *To what extent are the three tenets of CRP being used by mathematics instructors at HSIs?* and (2) *What barriers might communities of mathematics instructors at HSIs face to enacting CRP?*

In a community of practice, meaning is negotiated in two primary ways. First, community members make meaning through *participation* by taking part in the actions and interactions associated with the social community and, second, through *reifications* of those participatory actions. Participation happens by doing, acting, teaching, and interacting, while reification refers to the product of giving concrete form to the shared experiences. For example, a lesson plan, a definition of CRP, or an abstract formula could all be reifications. Reifications are like the tip of the iceberg, where the iceberg represents what the community counts as meaningful. For change to happen in a community of practice, the way meaning is negotiated must change. We suspect that two substantial barriers in the community of undergraduate mathematics instructors correspond to the duality of participation and reification.

Ignoring Culture in Identities and Mathematics Reifications

Discussions around some aspects of identity can be inadequate to nonexistent. This is not unique to mathematics or even education, but deeply ingrained in history. In the United States, legal documents used the language of "colorblindness", intending to indicate that they treated humans as equal regardless of their color (Annamma et al., 2015). While initially intended to portray that the policies should be applied equally *regardless* of race, the language began to be used to indicate that the policies should instead *ignore* race (Annamma et al., 2015). In education, this color-evasiveness[1] in school policies has led to policies at odds with factors that center on race such as where students go to school, access to resources, and categorization into certain academic programs (Wells, 2014). The extent to which instructors currently recognize or ignore racial identities seems like valuable information for HSIs to better serve the large percentage of Hispanic and other minoritized student populations.

The second potential difficulty is how the mathematics community's reifications reflect an abstractness that some mathematicians see as culture-free. Non-applied mathematics, or 'pure' mathematics, is often about removing context and finding the most generalizable universal rules. In doing higher level pure mathematics, not only do mathematicians remove the real-world application of the problem, they also generalize to other non-Euclidean spaces (e.g., hyperbolic geometry). Framed by topology, a coffee cup and a donut have identical properties. Rabin et al. (2021) facilitated a collaboration between chemistry, biology, physics, and mathematics instructors to try to streamline how mathematics is presented and applied in sciences. Doing the work, the science instructors noted that "mathematicians' thinking is too abstract, context-free, or ungrounded in reality" (Rabin et al., 2021, p. 8). Thus, there could be a tension between the context-free reifications of mathematics problems and focusing on students' contexts to teach the mathematics that would be applicable to issues relevant to their lives.

Methods

We used a stratified random sampling technique to identify the institutions we invited to participate in this study. This paper is based on a larger study of 40 interviews with STEM instructors, ten of them with mathematics instructors from eight different HSIs. Participants represent a range of institution categories (associate's=4, master's=1, doctoral=5) and locations (East=1, Midwest=1, Southeast=1, Southwest=5, and West=2). The mathematics instructors

[1] Preferable term to color-blindness, which equates a form of racism to an unavoidable disability (Annamma et al., 2015).

self-identified their gender (women=6, men=4) and race (Caucasian=2, Indian=1, Latin White=1, Latina=1, South Korean=1, White[2]=4).

We coded the transcripts for enactments of the three tenets of CRP (academic success, cultural competence, and sociopolitical consciousness), evidence that identity did or did not play a role in their practice, and evidence of the epistemology that their field was culture-free, as defined in Table 1.

Table 1. Code names, descriptions, % agreement, and kappa scores (Cohen, 1960).

Code	Description	% Agreement	κ
CRP: Academic Success	Expresses the intent or willingness to take into account students' backgrounds, cultures or identity to set them up for academic success.	82.2%	0.63
CRP: Cultural Competence	Expresses the intent or willingness to set students up to understand things relevant to their own or other students' cultures.	93.3%	0.37
CRP: Sociopolitical Consciousness	Expresses the intent or willingness to help students use the discipline to develop the skills to critically "identify, analyze, and solve real-world problems" that tie to sociopolitical issues that directly impact the students being taught.	100%	undefined
Culture of no culture	Evidence that they believe that culture is somehow absent or neutral in their discipline.	93.5%	0.66
Identity: YES	There exists evidence that they intend or would like to take into account student identity in their teaching practice. Identity was operationalized as how instructors' interpreted it.	50.2%	0.42
Identity: NO	Considers student identity as not playing a role in their practice.	84.4%	0.61

The first author coded the full set of interviews. The codes largely applied to responses to the questions, "Do the identities of students who enroll in that course influence your approach or the way you teach it? If so, how?" and, "How would you describe the culture or climate for students in your department in terms of supporting their identities?" To check reliability, the two remaining authors coded responses to those two questions for 20 of the 40 interviews. Our agreement was at least moderate for all codes (Landis & Koch, 1977) except CRP: Sociopolitical consciousness and CRP: Cultural competence because there were so few instances.

Results

[2] In this publication, we capitalize all races, including White, emphasizing that there is no default race and that they are all social constructs with associated sets of cultural practices.

Instructing with Culturally Relevant Pedagogy

We coded these examples of academic success as CRP only when instructors specified that they did them on account of students' backgrounds, cultures, or identities. For example, altering assessment for all students would not be considered enactment of CRP. The instructor said he changed how he viewed and conducted assessment while explaining that his students are 50% Hispanic and often parents. As an example of how he changed assessment practices, instead of giving a poor participation grade to a student who kept missing class, he instead tried to proactively reach out to the student and communicate about what was going on. He made this change after switching from a non-HSI to an HSI.

We found that, of the ten mathematics instructors we interviewed, eight enacted CRP in ways consistent with the academic success tenet, two with the cultural competence tenet, and one with the sociopolitical consciousness tenet (see Table 2). The tenet of academic success was enacted through speaking to students in Spanish, making sure they had resources available (e.g., textbooks, calculators, online materials), maintaining high academic standards, taking personal responsibility for students' success, directing students to extra tutoring, reviewing more background content, recording lectures and making class notes available, modifying examples to be more inclusive, changing assessment, and making sure that minority students' voices were heard.

Table 2. Percentage (and number) of participants that gave evidence of enacting tenets of CRP.

Evidence of	Mathematics (*n*=10)	Total (*n*=40)
Culturally relevant pedagogy: Academic Success	80% (8)	75% (30)
Culturally relevant pedagogy: Cultural Competence	20% (2)	23% (9)
Culturally relevant pedagogy: Sociopolitical Consciousness	10% (1)	3% (1)

The two examples of the cultural competence tenet were showing the film Hidden Figures to familiarize students with more representation and talking about how politics spilled over into class. The example of the sociopolitical consciousness tenet came from a mathematics for future teachers course in which students read a special education article with the goal of helping them think critically about how what they were learning would apply differently to a marginalized group of people.

Barriers to Cultural Competency and Sociopolitical Consciousness

We found two main barriers to undergraduate mathematics instructors learning to teach with CRP, especially pertaining to the cultural competency and sociopolitical consciousness tenets: (1) lack of communication around student identity and (2) limited perspectives on what the mathematical community counts as disciplinary knowledge.

Barrier 1: Communication around student identity. For communities of practice to learn, they need to negotiate meanings with each other. However, we found a cautiousness around the language needed to communicate about race. When asked, six out of ten instructors stated explicitly that student identity does not play a role in their teaching. Yet we found evidence that seven of the ten instructors did incorporate identity in their teaching. Three of the instructors did so while explicitly denying it.

The only three student identities instructors explicitly said *do not* impact their teaching were gender, race, and nationality. Some instructors did say or show that gender and race influenced their teaching, but for many instructors, it seemed like they had a concern with making assumptions based on race and gender and preferred labels that had more direct implications. Other labels instructors used when they explained ways they changed their teaching on account of students' identities included first generation, English-language learner, international student, socioeconomic status, mathematics experience, caregiver, and age. For example, an instructor said, "I think of them as individuals, but not necessarily as identities, if that makes sense. So given the fact as far as you know, their color, their gender or anything else, you know, it's all the same to me in a sense when I teach." Moments later, however, she explained how if students were not native English language speakers it would impact how much work students would have to do to decipher the context of statistics problems.

Even when instructors said race played a role, they also emphasized the importance of not making assumptions based on these labels. Samuel, a mathematics educator, said, "Things like race, gender identity, first generation, they all sort of get boiled into building this relationship with the individual students. I don't want to [...] assume that because they're a first-generation student, they're gonna struggle with X, Y, Z." He did not want to make assumptions about the student based on the community that student might belong to, because he did not know how that individual had experienced the things associated with that community.

Barrier 2: Mathematics as culture-free. Two of the ten mathematics instructors viewed mathematics as a culture-free discipline. Instructors were never explicitly asked about their opinions on whether their discipline was culture-free or not, so it is possible that more instructors felt similarly. One other instructor expressed that, while she believed mathematics was culture-dependent, other instructors in her department did not. To instructors with this culture-free epistemology, mathematical knowledge could be viewed independent of race and gender. For example, Francesca said, "We don't see a color or sex or anything. We see a raw being that you can teach mathematics to. Mathematics has no color." She continued, "You look at something like the ocean and the color currents in it. And you know that those are all governed by mathematics. And, you know, many, many forces. And that's just nothing about ethnicity in it." Kyung Joon similarly expressed that the sciences are more neutral than other subjects.

Holding this culture-free epistemology does not appear to be wholly incompatible with enacting CRP or incorporating student identity into their teaching. Both instructors that expressed culture-free epistemological beliefs concerning mathematics also enacted the academic success tenet of CRP. Francesca held high academic standards for students especially when she taught racially diverse students and took personal responsibility for each students' success. Kyung Joon did extra review of background content for students who were struggling financially or had poor high school preparation. We found evidence that Francesca incorporated student identity into her teaching while we did not for Kyung Joon. Francesca said that students of different ethnicities had minds that understood things differently and all were capable of understanding - it was a matter of finding the right way to explain it so a student would be able to understand.

However, instructors holding culture-free epistemologies might be more hesitant to enact the cultural competence and sociopolitical consciousness tenets of CRP. Neither Francesca or Kyung

Joon found it necessary to enact any aspects of those two tenets. Kyung Joon explained, “If the math professor used K-pop as an example, or some Korean barbecue, they wouldn't make the math problem easier or more understandable. To me it's not really that relevant in math courses.” He said that not only is it not relevant to use such cultural examples, but also it can come off as insulting. Thus, we found that if instructors believed mathematics is culture-free, they did not see a need to teach how the content is relevant to students’ individual cultures nor how mathematics might be used as a tool to critique or solve sociopolitical issues relevant to students.

Implications and Conclusion

We found evidence that many practitioners in the community of mathematics instructors at HSIs already enact the academic success tenet of CRP, and there are two primary barriers to enacting the other two tenets. First, to communicate about students’ cultural contexts in a community of practice, conversations about aspects of identity must become more normalized. That instructors have difficulty talking about students’ racial identities is not evidence that instructors do not care about the cultural context of their students. Rather, it is a misguided but well-intentioned color-evasiveness or a desire to avoid essentialization of students (Gutiérrez, 2013). Race is politically charged, and we live in an era where the wrong publicly stated comment can ruin careers (Ronson, 2016). The implication for the mathematics community is that it must become a lower-risk place to talk about race and the role it has in doing mathematics. Dr. Loretta Ross advocates for calling people in rather than calling them out (Bennett, 2020). She proposes the idea that publicly shaming is a toxic, non-productive response. Instead, she advocates leaning into the discomfort of having a conversation and talking to the person directly with compassion and assuming they bring good intentions.

The second barrier to enacting CRP was perceiving mathematics as culture-free, corresponding to the issue that reifications of mathematics often reinforce that mathematics is free of culture. While holding these culture-free epistemologies did not prevent enacting the CRP tenet of supporting students’ academic success, the two instructors did not see reason to teach how mathematics could be relevant or applicable to students’ cultures or to sociopolitical issues relevant to them. For instructors like Kjung Joon who only could think of cultural applications that were surface-level, content-specific resources might make a difference in their enactment of CRP.

The findings suggest three implications for communities to start creating and sharing culturally relevant content that goes beyond helping students achieve in traditional ways. First, mathematics instructors can improve communication with each other about students’ racialized and gendered identities. Doing so is messy and political, so when instructors, departments, and universities inevitably say problematic things, they need to learn how to teach each other rather than shame each other. Second, if mathematics instructors are more comfortable talking about more concrete student characteristics (e.g., first generation, low socio-economic status), those identifiers could still be useful for understanding what issues might be culturally relevant. For example, calculus can be used to understand issues of minimum wage, student loans, or COVID growth/decline rates; these are nontrivial applications that can be taught before instructors work to communicate about gender and race. Moreover, if mathematics instructors would like to avoid making assumptions about a given individual, instructors can design teaching that is culturally relevant for a population of students in their course without assuming anything about the identity or culture of any specific student. For example, something related to Hispanic culture broadly would contribute to the cultural relevance of the course for everyone, whether they identify with the aspect of Hispanic culture involved. Third, a community effort must go into developing content that is relevant to both the students' lives and the mathematical tools learned in standard courses like the calculus series. With the practical time constraints on full- and part-time faculty, it would be unrealistic to expect everyone to create materials to enact CRP on their own.

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Key Aspects in the Development of a Quantitative Understanding of Definite Integrals

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This work draws on the framework of Emergent Quantitative Models to identify how calculus students might develop a quantitative understanding of definite integrals that supports them in modeling activity when the differential from is not a multiplicative product between a rate of change and a differential quantity (e.g. gravitational force). To characterize the mental activity that supports the productive development of a quantitative understanding of definite integrals I engaged students in an eight-week teaching experiment.

Keywords: emergent models, quantitative reasoning, calculus, definite integrals

Introduction

Research has shown the majority of students primarily reason about the definite integral in terms of either prototypical imagery (e.g. area beneath a function, above an x-axis, and between two boundary lines at $x = a$ and $x = b$), or purely algorithmic and non-quantitative ways (e.g. antiderivative). Over the past two decades, a growing body of research has identified that connecting definite integrals to Riemann sums and quantitative reasoning provides students with robust ways to reason about contextual tasks (e. g. Merrideth and Marongelle, 2008; Jones, 2015; Sealey, 2014). These studies have primarily focused on constructs in which the differential form is a local Riemann product—a multiplicative product between a rate of change and a differential quantity. However, limiting students' definite integral reasoning to local Riemann product structures has been identified as potentially inadequate for a successful transition to other STEM coursework in which the integrand does not naturally decompose into a rate of change or density, such as gravitational force (Oehrtman, 2015). This study aims to contribute to the mathematics education field by offering insight into the question:

> *How can calculus students develop an understanding of definite integration that supports the quantitative reasoning necessary to productively engage in definite integral tasks in which the differential form does not naturally decompose into a local Riemann Product?*

Literature Review & Frameworks

Quantitative reasoning encompasses the mental actions in the conceptualization of quantities and relationships between quantities. By quantities, I mean the measurable qualities of objects which are formed by individuals engaging in a dialectic between an object, an attribute of that object which is of interest, and a way in which to measure that attribute to solve a problem (Thompson, 1990, 2012). As part of the experimental design, I drew on Realistic Mathematics Education which views modeling as an integral aspect of what it means to do mathematics and sees "mathematics as a human activity" where experientially real problems play a crucial role. Realistic Mathematics Education is a domain-specific instructional theory originating out of the Netherlands with three central design heuristics: guided reinvention (Freudenthal, 1973), didactical phenomenology (Freudenthal, 1986), and emergent models (Gravemeijer, 1999).

In line with Gravemeijer's approach to modeling as an emergent process, Simmons and Oehrtman (2017, 2019) described an Emergent Quantitative Models framework for students' reasoning about definite integrals. This framework extends previous characterizations of students' reasoning about definite integrals when the differential form does not naturally decompose into local Riemann Product and can be utilized to understand, for example, how students reason about physics-based tasks. The framework relies on three conceptual models, basic, local, and global, which students draw on when reasoning about definite integrals.

> The basic model represents the quantitative relationship which would apply to the situation if the quantities involved were constant values, the local model is a localized version of the basic model applied to a sub region of the original situation (typically within a partition), and the global model is derived from an accumulation process applied to the local model, whose underlying quantitative reasoning is encoded in the differential form (Simmons & Oehrtman, 2019).

Through an analysis of students' emergent models while productively engaging with physics-based integration tasks, Simmons & Oehrtman characterized a *Quantitatively Based Summation* (QBS) symbolic form (Sherin, 2001; Jones, 2013) for the definite integral. This symbolic form associates the symbolic template $\int_{[A]}^{[\boldsymbol{B}]}[C]$ with an underlying conceptual schema in which A and B are the values representing the beginning and end of the measure for the quantity defined to be the differential respectively and C represents an algebraic representation of the local model which shares a quantitative structure with the basic model of the desired quantity. The construction of the symbolic template for the differential form $[C]$ involves a recognition that variation of a quantity makes the basic model inappropriate, a partitioning and accumulation process based on a parts of a whole symbolic form, and coordinates of the accuracy of global and local models as dependent upon the magnitude of a differential quantity.

Methodology

The epistemological underpinning of this study is that of Piaget's genetic epistemology (1972; Piaget & Duckworth, 1970). From this perspective, knowledge is an adaptive construct of human minds which is actively created consistent within an individual's conceptual structure through interactions with the outside world. Because the QBS symbolic form was identified as productive for students' reasoning for contextual integration tasks, this study aimed to engender the development of this conception through an eight-week teaching experiment (Steffe & Thompson, 2000). Unlike a clinical interview, where the goal is to capture students' understanding at a particular instant in time, the goal of a teaching experiment is to characterize the development of students' schemes as those understandings evolve to test hypothesized learning trajectories.

For the teaching experiment, I recruited six freshman students during their first-semester calculus course at a large southwestern university. The interviews began two weeks before the introduction of Riemann sums. Due to the longitudinal nature of planned follow-up studies, I requested all Fall 2020 introductory calculus instructors submit recommendations for students appropriate for a year-long study based on the engagement in the course. Due to the large time commitment, students were compensated $20 per hour for participation. Students were matched into groups of two based on availability.

Due to social distancing restrictions in Fall 2020, I interviewed and recorded study participants through the Zoom platform. There were two types of interviews throughout the eight-week teaching experiment: paired (approximately one hour) and individual (approximately a half-hour). Both types of interviews were planned to take place twice a week for a total of 3 hours per week in interviews per participant resulting in over 70 hours of video recordings. Most weeks all interviews took place as planned. At the end of the teaching experiment, I engaged participants in a series of task-based clinical interviews in which they were asked to model physics-based tasks in which the differential form was not a Riemann product. During paired interviews participants worked on tasks while talking with one another through zoom and writing on a collaborative online whiteboard, AWWApp.com. I asked clarifying and directive questions while transitioning between responsive/intuitive and analytical interactions as the students progressed through the tasks.

The data was analyzed in two phases: ongoing and retrospective. After each interview, I refined my models of the participants' understandings. This involved taking notes during and after each group interview, noting significant interactions, and reviewing clips before conducting the follow-up individual interviews. Follow-up interviews provided an opportunity to test my hypothesized models of participants' reasoning, build a more coherent image of their evolving schemes, and let students elaborate on constructs they might not have provided enough detail on in group interviews. When I developed specific hypotheses regarding students' reasoning that would not be investigated through the normal course of the hypothetical learning trajectory, I designed and introduced supplementary prompts and tasks which I provided to groups and/or individuals on an as-needed basis. After the teaching experiment, I reanalyzed all data using constant comparative analysis (Clement, 2000) with the MaxQDA analytic software. This analysis included refining the hypothetical learning trajectory and conceptual analysis based on additional passes through the teaching experiment data which characterized participants' emergent quantitative models. Follow-up reviews of the dataset were performed to identify episodes that supported, or refuted, my evolving image of the participants' schemes until I felt that the data was no longer able to provide additional nuance. I concluded with a cross-comparison between the emergent quantitative models of the different groups to identify commonalities and distinctions between participants.

As part of the data collection, there were limitations to conducting the teaching experiment online. First, the need for participants to have access to a computer, high-speed internet, and a web camera placed a potential handicap to the generalizability of the results. Participants needed to have access to a private computer with internet access for at least three hours a week which means it is likely participants in this study were of above-average socioeconomic status. Online interviews also limited the ways I could capture students' reasoning that would normally be evident in gestures, demeanor, and written work. By only viewing students' upper bodies I often missed out on slight hand movements, fidgeting, and quick scribbles made as they were problem-solving. Additionally, slow internet connections sometimes resulted in choppiness in videos and an inability to clearly identify what the participant was relaying. Finally, due to recruitment, it is possible the participants in might not be an accurate representation of the general calculus population. These students were handpicked by their professors as being highly engaged and likely to succeed through to the next calculus course. There also may have been implicit biases in the instructors' recommendations which affected the outcome of the final participant population.

Results

For brevity, this paper will focus on the key aspects of the conceptual development of quantitative understanding of definite integrals for a single group, Ashley and Adam, over the first two major tasks of the six-task teaching experiment sequence which adds nuance to the Emergent Models framework as participants' emergent models were in development.

Ashley was a Caucasian female statistics major pursuing a minor in music. Adam was a Caucasian male architectural engineering major. Both Ashley and Adam took a calculus course in high school and agreed that it felt like they had "missed the last half of calculus" due to

COVID which included integration topics. They both demonstrated an inclination to rely on procedural knowledge without giving much thought to the underlying mechanics in their initial clinical interviews. Ashley and Adam worked well together and were open about their thinking throughout the teaching experiment. As a result, I developed a detailed image of the precise development of their basic, local, and global models (and relationships between models) as they constructed a quantitative understanding of definite integrals.

The Development of a Gross Basic Model and Global Model

In the Curiosity Rover task, students were presented with the goal-orientated activity, in four parts, of identifying whether a rover would complete its mission on Mars. The task provided specific readings for rates of dust accumulation at different geographic locations and had a limitation that the rover could not continue operating when it had over 400 mg of dust on its solar cells. Students were provided a GeoGebra applet with a slider that presented data for specific sites which included location, total distance along the path from the landing site, composition of the Martian surface, and the corresponding rate of dust accumulation. The Curiosity Rover task was chosen and adapted due to its ability to provide a meaningful context which motivated the need to identify an overestimate using an accumulation of local estimates. The conception the Curiosity Rover task aimed to engender most closely represents a finite Riemann Sum, however, no formal summation notation or language was introduced.

I presented Ashley and Adam with prompts requesting rates of dust accumulation at different sites, the measured distance between sites, and how one could identifying the approximate amount of dust on the rover's solar cells as it traveled between sites. Early in the task, I prompted Ashley and Adam to identify an overestimate for the amount of dust that would accumulate on the rover as it traveled between two neighboring locations which did not share identical rates of dust accumulation. As they were discussing the task Ashley observed,

> *Ashley:* So, if we're traveling. The rover is traveling. And, at the beginning, it's kind of getting dust at six milligrams per kilometer, but slowly, by the end of it, it's only getting dust at 3.5 milligrams per kilometer. So, this rate is going to be decreasing. Like, if we were to draw a graph. You know what I'm saying? So, I think it's wanting us to overestimate. Like, for example, a gigantic overestimate for this problem would be 6 times 30. If we just say, well, it just keeps the 6 milligrams rate the entire time it travels, times 30, which is the kilometers, you know, we would get like, what 180? That would be like a massive overestimate because we know it changes because the soil changes. And then like, obviously, an underestimate would be the opposite to do it by 30 times 3.5.

That is, Ashley extended her basic model of [rate of dust accumulation]·[distance]=[total amount of dust] to what I describe as a *gross basic model*. A gross basic model represents directly applying a quantitative relationship that holds for constant quantities (a basic model) to a quantitative relationship in which one, or more, of the quantities is varying. An important aspect of a gross basic model is the recognition that the quantity obtained is only an approximation, and that the varying quantity within the gross basic model must be bounded (either above or below depending on the desired approximation).

From here, Ashley was able to construct a global model for the total dust on the rover along longer paths through the additive accumulation of gross basic models; "So basically, we can take from Darwin to Cooperstown [the next segment along the rover's path], find our overestimate of that, and then add it," and Adam agreed "yeah, it's just a bunch of intervals."

Increasing the Accuracy of a Global Model Through Refinement

A key aspect of the hypothetical learning trajectory was to have students construct their conception of a definite integral as a tool that would aid in precise approximation when one, or more, quantities of a basic model is varying. To that end, the Curiosity Rover task was designed so that additional data points provided more accurate underestimates and overestimates of the total dust obtained along the rover's journey. The initial applet displayed data for 7 major sites, a subsequent applet adjustment allowed students to see data for the midpoints between sites, and a final applet, with an accompanying spreadsheet, provided additional readings every 2.5 kilometers for a total of 65 data points. Until the third applet, it would remain unclear if the rover would be able to complete its journey on Mars.

After Ashley and Adam computed overestimates and underestimates for the initial 7 data points, resulting in a range of 295.75mg to 471.25mg of dust on the solar panels, they reported that they would not feel safe sending the rover to Mars. In response, I provided the second applet with a comment that an intern managed to obtained more data. When asked if this would help, Adam suggested that it would because they could calculate "another set of intervals… how we did intervals last time, right. Yeah, it's the exact same thing, just with more of them, because we can use the midpoints." Ashley added, "yeah, we could make an even more precise over and underestimate, and see how far away from 400 that is… to see how much our recommendations need to change." This anticipation that the difference between the overestimate and underestimate could be reduced by additional data would serve as a crucial component of Ashley and Adam's development of a local model in the next major task.

The anticipation of an increasingly more accurate global model through additional datapoints was put to the test when a computational error resulted in a larger overestimate value than 471.25mg. Adam was dissatisfied because the increase in the number of partitions "didn't really narrow the interval [difference between the underestimate and overestimate]" and that "the smaller value [the underestimate] should have gotten a bit bigger and the bigger value [the overestimate] should have gotten smaller." That is, Adam was already operating with the expectation that more intervals equate to a more accurate result and that this phenomenon should act "kind of like limits that should approach the actual [value]." After writing out the expression on the shared whiteboard, Ashley identified Adam's error and they moved on to the final phase of the Curiosity Rover task. To make quick computations Ashley and Adam multiplied each rate of dust accumulation by 2.5km (the distance between each reading) and added those values together to obtain a single answer of 381.7mg. Adam conjectured, "All right, so that is, umm. Is that just an estimate of the actual?...Yeah, cause that's not an over-under." However, because the question prompts continued to situate students' goal-oriented activity toward identifying both an under and overestimate, Ashley remained perturbed. Drawing on her earlier correction of Adam's overestimate, she recognized that the last term in their computations would represent an underestimate for the total amount of dust on the rover if it only traveled the last 2.5 km of the journey. She explained,

> *Ashley:* Because if we add up, okay, let's see, two minus, okay 65. Because 65 times 2.5 is 162.5, and see, we have 65 cells here and our total distance is only 160. So, whenever we just look at this, this is too much distance. That's why you have to subtract one or the other to get your over and underestimate….So the underestimate is gonna be 364.7, and the over will be 381.25. Since we've narrowed it to that interval, we know that the rover

should be able to handle the load of the milligrams of dust.… That's exciting! This is exciting, our rover works! At least according to our spreadsheet math.

The Development of a Local Model from a Gross Basic Model

In the second major task of the teaching experiment I tasked Ashley and Adam with identifying over and underestimates for the total fluid force exerted on both rectangular and trapezoidal dams. The Fluid Force on a Dam task was situated so that identifying over and underestimates was in service of providing parameters that allowed a superior to minimize the total cost of the dam. I included this task early in the teaching experiment sequence because (1) it provided students an opportunity to reason about non-Riemann product quantitative structures as their basic-local-global models were in development, and (2) provide a challenge to those students who may have already constructed a scheme for integration that was based in antidifferentiation or consistent with schemes for the differential as a Riemann product.

After engaging Ashley and Adam with a supplementary activity aimed at familiarizing them with the basic models involved in fluid force, I prompted them to identify estimates for the total fluid force that would be exerted on a rectangular dam. Unsurprisingly, Ashley and Adam directly applied a gross basic model to the entire dam to obtain over and underestimate values. Following this, I requested Ashley and Adam improve their estimations based on a supervisor's feedback to partition the dam into two pieces. This prompt was aimed at extending Ashley and Adam' gross basic models from discrete to continuous data in which they perform the partitioning and measurement process. That is, this prompt encouraged Ashley and Adam to explicitly construct a local model to measure the force on each half of the dam which required them to reason about the quantitative relationships involved in their basic and global models. After dividing the dam into two, Adam attempted to add the overestimate of the fluid force on the top half of the dam, to the overestimate for the fluid force on the entire dam. However, with an expectation of the global model being more accurate with more data and upper bound provided by the gross basic model estimate, Ashley observed, "If we add the same thing [the value of the whole dam] it [the total force] gets bigger instead of smaller." This expectation positioned Ashley to determine that the overestimate for local models should only reflect the largest pressure multiplied by the local elements area.

The Development of a Generalized Local Model

After correctly identifying overestimates and underestimates Adam actively anticipated the next steps in the sequence, commenting, "So, I guess just to kind of look ahead. Are we just going to keep breaking this up into smaller parts and adding these together to get narrower?... We did that before." In response, I had Ashley and Adam identify over and underestimates for a partition with 5 elements, before asking them to find the number of pieces they would need to break the dam into to identify estimates within 50,000 Newtons of the actual value. This request would require a significant shift in reasoning, as the correct answer was well over 8000 peices. Intuitively, Adam recognized that it would "be an insane amount of intervals to add" and went on to suggest, "I was thinking, I guess if there's some way to relate depth and force kind of like we already have but an equation, then we could use that equation to kind of get a more accurate feel for the estimations," and Ashley asked, "Do you want to write a formula… using a tool to compute it?" As Ashley and Adam began trying to construct a formula Ashley observed, "we know we're just multiplying area times depth times 9800, right? And then we're adding that again. We're just going to keep going for each interval that we decide to do." That is, Ashley recognized that each element within a global model shared the same quantitative structure. I provided Ashley and Adam a GeoGebra applet to compute their desired sum using their local model expression. During their initial 'guesses' Adam and Ashley explicitly worked out the value for area before wondering aloud if there was a way to automatically compute height, I informed them that the calculator was designed to accept Δx as the height. The timing of the introduction of this notation allowed Ashley and Adam to coordinate that the Δx element within a local model was a quantity that remained constant across elements of a global model, but whose magnitude was dependent upon the number of elements within the global model (i.e. the value varies across different partitionings). I characterized this mental activity of generalizing quantitative structure across elements of a global model as a *generalized local model*.

The most obvious outside behavior Ashley and Adam engage in as part of their development of a generalized local model was the algebraic representation of this activity. However, as a generalized local model become a more engrained aspect of their emergent model system, they drew on this same mental activity to anticipate the applicability of a partition for a global model. That is, in later tasks within the teaching experiment, if they could not notice meaningful variation across a fixed partition quantity, then they looked for alternative partitionings.

Discussion

As a result of this teaching experiment, I contribute two constructs to the Emergent Quantitative Models framework, which offer explanatory power and instructional implications for the development of a QBS conception of integration. I created the *gross basic model* as a part of the experimental task design to engender students' goal-oriented activity towards identifying estimations for a whole through the direct application of a basic model. In support of generating a global model as an accumulation of elements of the same quantitative type as the basic model, I engaged students in the act of progressive addition of gross basic models to create a global whole. This positioned students to create a local model, as a new construct, through an accommodation of applying a gross basic estimate to a partitioned element of a global model.

A *generalized local model* is a result of the mental activity students engage in as they generalize the structure of a local model across elements of their global model and was identified as a critical element in the evolution of students' emergent models. This construct is a result of the mental activity students engage in as they make comparisons across elements of their global model to coordinate which quantitative components vary and which remain fixed. In this study, the genesis of a generalized local model emerged from students' need to create an explicit formula for their local (or global) model which would allow them to identify the value of any element of a partition, for any size partition, in service of identifying the number of partitions necessary to be within a given tolerance. The mental activity required for such an activity is cognitively distinct from that of computing explicit values for partitions through a measurement process and is at the crux of the difficulty in students' ability to productively model complex quantitative situations using definite integrals.

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A Reemergence of a Non-Quantitative Interpretation of the Differential for Definite Integrals

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This work is drawn from a larger study on student conceptions in a Calculus course designed to promote quantitative understandings of definite integrals. For one student, we investigate the evolution of conceptions of definite integral applications. We pay particular attention to quantitative and non-quantitative interpretations of the differential as they emerge or disappear in unexpected ways. Implications for instruction and future research are considered.

Keywords: calculus, definite integral, differential, quantitative reasoning

Introduction & Literature Review

Students' reasoning about differentials has been of continued interest to mathematics and physics education research due to its importance in the fundamental understanding of derivative and integral concepts in STEM. In the area of integration particular interest has been paid to whether students interpret differentials as simply a cue for an antiderivative process (e.g. Merrideth and Marrongelle, 2008; Jones, 2013), as an infinitesimal quantity (Ely, 2017), as a width of a representative rectangle used in identifying graphical area (e.g. Jones, 2013), or as a physical quantity (e.g. Thompson & Silverman, 2008; Merrideth and Marrongelle, 2008; Sealey, 2014; Author, 2017, 2019). Physics education researchers Von Korff and Rebello distinguished different categories for differentials based on size, macro (Δx) and micro (infinitesimals dx) (2012), as well as different types of differentials, change and amount (2014). Amount differentials are distinct from a typical $x_2 - x_1$ computational change approach due to some quantities not being productively conceived of in terms of changes, such as mass.

In most learning trajectories and textbook curricula, definite integral applications are initially limited to position-velocity-acceleration contexts, and other meaningful quantitative interpretations are placed well after learning the fundamental theorem of calculus and computational methods. This promotes the dominance of antiderivative and area under a curve conceptions amongst calculus students. Based on evidence promoting the productivity of quantitative interpretations of the differential form (e.g. Jones, 2015; Wagner, 2018; Author, under review), in recent years, emphasis has been placed on the study and development of calculus curricula which promotes a quantitative understanding of differentials, leading us to an obvious research question;

RQ: How do students reason about the differential after engaging in curriculum designed to produce a quantitative understanding of the definite integral?

Theoretical Perspective and Methods

Six students were recruited from an accelerated eight-week asynchronous online introductory calculus course at a large southwestern university. The learning trajectory for the calculus course was designed to engage students in the mental activity of developing a quantitative understanding of definite integrals distinct from area under a curve and antiderivative conceptions (Author, under review). To that end, modeling using accumulation with summations and definite integrals was covered early in the coursework, while topics related to antiderivatives and computational practices were not covered until the final two weeks (Table 1).

Drawing upon the heuristics of Realistic Mathematics Education (Gravemeijer, 1994, 1999), the course materials were designed to engage students in experientially real tasks which extended beyond simple reproductions of instructor examples, called "Your Turn" activities. Each week students were provided with approximately 3 hours of lecture videos. Within these lecture videos there were embedded "Your Turn" activities that accounted for approximately 60% of the course material. These activities were required to be turned in weekly as a part of their course grade. "Your Turn" activities were routinely referenced when new topics were introduced to allow students to reflect on their previous problem-solving activity as they learned new skills (e.g. Recall Your Turn X.X). For example, the lecture videos introduced the concept of using a definite integral to model work against gravity for building a cement column, and students were tasked with identifying the work against gravity to build a pyramid or to lift a chain to the top of the building. Once students learned the fundamental theorem of calculus they revisited all tasks from the accumulation section to rework those solutions by hand.

Table 1. Course learning trajectory topics broken down by week covered.

Week 1	Some Basics: Introduction, Quantities, and The Spread of Disease Accumulation: Introduction to Accumulation, Approximations using Sums.
Week 2	Accumulation: Continuously Varying Rates, Limits, Modeling with Definite Integrals – Geometry
Week 3	Accumulation: Modeling with Definite Integrals – Density, Work/Energy, Force, Reinterpreting Accumulation as Area Under a Curve Rates of Change: Approximating Instantaneous Rates of Change; Limits
Week 4	Rates of Change: The Derivative Rules, Techniques
Week 5	Rates of Change: Modeling - Basic Applications, Related Rates, Limiting Values, Differential Equations
Week 6	Rates of Change: Modeling: Extreme Values, IVT and Monotonicity, Concavity, Drawing Graphs, Applied Optimization.
Week 7	Bringing It All Together: Fundamental Theorem of Calculus Part 1, Antiderivatives, Fundamental Theorem of Calculus Part 2
Week 8	Bringing It All Together: Accumulation Functions

Data collected for this study included an introductory questionnaire which identified a baseline for students' understandings for calculus concepts, all written notes for the course (including "Your Turns"), quizzes, exams, written homework, hour-long task-based clinical interviews mid-semester (immediately following the accumulation section in Week 3), and a series of task-based clinical interviews post-final exam. The interviews were conducted through Zoom using an online collaborative whiteboard. Additionally, there were 25 required short surveys throughout the semester, one at the end of each subsection module, in which students provided immediate feedback regarding their evolving understanding of calculus constructs. Students who participated in the study were offered extra credit in the course, worth 1% of the course grade, as well as a chance to win a $10 gift card. Data was analyzed using the constant comparative method to construct a model of the participants' image of definite integrals at various stages of the course (Clement, 2000). Particular attention was paid to the mid-semester, and post-final exam task-based clinical interviews as these interactions provided the most nuanced data to inform our models of students evolving schemes for the definite integral and differential quantity.

Results

In this paper, we focus on a single student, John, from the larger dataset. John was a white, male, sophomore Chemical Engineering major who reported having taken at least half a semester of university calculus before enrolling in the Summer 2020 course. John reported that low quiz scores prevented him from achieving a satisfactory grade. In John's preliminary survey he did not provide quantitative interpretations of the definite integral notation, including the differential (Figure 1).

What does the expression $\int_3^7 f(x)dx$ mean?

the interval of the derivative of f(x) where x =3, 4, 5, 6, 7.

In the expression $\int_3^7 f(x)dx$, what does the dx stand for?

the derivative of

In the expression $\int_3^7 f(x)dx$, what does the $f(x)dx$ stand for?

the derivative of f(x) in relation to x

Figure 1: John's preliminary calculus survey responses.

During his mid-semester interview, John was presented with four tasks. The first task was posed as a general inquiry to explore John's image of definite integrals in the abstract, while questions 2-4 situated definite integral tasks within specific contexts. Throughout John's mid-semester interview he repeatedly demonstrated that his primary conception of a definite integral is an area under a curve. John viewed a definite integral as an infinite summation of rectangles with height the function value at a particular point, and width an infinitesimally small width of the horizontal axis. The aim of this summation was to identify a 'really good' approximation for the total area under the rate function. John's view of infinitesimally thin rectangles coordinated values for differentials that could be considered smaller than the size of a single atom with a non-finite summation process. For John the differential form $f(x) \cdot dx$ represented the height times the width of a single rectangle and when asked if you could leave off the differential off the differential form, John objected, "Oh no. Because that would just tell you, let's see $f(x)$ on the interval a to b… that would just be talking about the height of the rectangles, which you can see by the graph. You don't calculate area without having these widths here."

Watson is filling a huge beaker with water from a faucet. He is playfully turning the faucet up and down, so that the water's flow rate is continually changing. There is a flow meter on the faucet that tells him this flow rate $r(t)$, in ounces/sec, where t is measured in seconds after Watson first turns on the faucet.

What does $\int_3^8 r(t)dt$ mean?

Figure 2: Mid-semester task 2 – flow rate.

When presented with the Flow Rate task (Figure 2), John quickly identified that the definite integral measured the total ounces of water in the beaker which accumulated between 3 and 8 seconds. John justified his conclusion through a graphical area interpretation, stating,

> *John:* The area under the curve, which is the rate that the beaker is filling up with water and ounces per second, if we take the area underneath that, that tells us the total amount of ounces put into the beaker over the interval of three seconds to eight seconds after turning the faucet on.

John went on to describe that he was able to identify what the definite integral was computing through unit cancelation that he had learned in Organic Chemistry. In particular, $r(t)$ was measured in units of oz/sec and dt was "just seconds," so their product, $r(t) \cdot dt$, must be a measurement of oz. That is, John was intentional with regards to assigning the differential (dt) a quantitative interpretation, the width of a rectangle, as well as its associated unit which corresponded to the independent variable of time.

During the school year, the population density of a certain college town can be measured as a radial density,

$$\rho(r) = \frac{25}{\sqrt{1+r^2}} \ (in\ thousands\ of\ people\ per\ mile^2)$$

where r represents the distance (in miles) from the University. Write a definite integral that will identify how many people live within 4 miles of the University.

Figure 3: Mid-semester task 4 – population density.

In the following task, which required the construction of a linear rate of change function in service of writing a definite integral, John continued to associate an appropriate unit to the differential quantity while reasoning through a graphical area interpretation of the context. Task 4 of the mid-semester interview (Figure 3) was chosen because in prior studies, students with a primarily area under a curve or antiderivative conception of the definite integral were often unsuccessful in productively producing a correct definite integral expression due to the radial density function being provided in terms of a radial length, while the natural differential quantity for a Riemann product structure would be in terms of area. In this task, John was asked to identify the total number of people within 4 miles of a university center when provided a two-dimensional, one-variable population density function. John's initial strategy for solving Task 4 was to write an integral of the form

$$\int_0^4 \rho(r) \cdot \pi r^2 \cdot dr = \int_{[no\ radius]}^{[4\ miles]} [radial\ density] \cdot [area\ at\ radius\ r] \cdot dr$$

Notably, when not reasoning in terms of area under a curve, John's associated schemes for the summative properties of a definite integral were transferred to the integrand rather than the entire differential form. When writing out the units associated with each term John ascribed $\rho(r)$ the units $people/mi^2$, πr^2 the units mi^2, and the differential term dr no units. When the interviewer brought John's attention to the fact that in his solutions to the previous three prompts he had ascribed a unit to the differential quantity, John immediately correlated the differential quantity with a width of a rectangle;

> *John:* That's 100% correct… so if I'm going to be consistent. That should just be the width of like, er yeah, the width of how much it should be. So why did I put πr^2 there? This is a thinker. I believe what I wrote is correct but.… Okay, I'm sticking by what I said before, dr is definitely the width of the rectangle for the area underneath a curve.

However, John had not initially reasoned about this task in terms of area under a curve which caused him to reevaluate his differential form. When attempting to reinterpret the context with an image of the differential quantity as a width of a rectangle, John scratched out the πr^2 leaving just the radial density to act as the height of the rectangles. Although bothered by the fact that the units in his new expression, $\int_0^4 \rho(r)dr$, still did not produce 'people' as expected, John was unable to make further headway on the task.

Approximately two months later (two weeks into his next calculus course) John participated in a set of 3 post-semester interviews in which he was presented with additional contextual definite integral tasks. The first task asked participants to identify the total fluid force acting on a dam, while the second centered on finding the total kinetic energy of a rotating rod. Despite John's inclusion of a differential during his mid-semester interview, his post-semester interview solutions did not include such a quantity. It was hypothesized that the physical contexts of the tasks were in opposition to John's tendency to ascribe to the differential the quantitative interpretation of 'rectangle's width' impeding the need for its inclusion within the differential form. At the end of the second interview, the interviewer directly asked about the omission of the differential. John observed that "there should definitely be a dr," and attached the notation to the end of his definite integral expression. When asked if there should be units attached to the dr John replied "I don't think so. I think it's just telling the person looking at it that it's with respect to r," and then hedged "or it adopts the unit of whatever variable it is." When asked to explain John continued,

> *John:* I've never thought of it as having a unit. That would mess with our units at the end. I've never seen it do that. But the other train of thought I was thinking was because it's with respect to r, and r has a unit of meters… but I've never seen it mess with units before.

John's reversal of units being associated with a differential was unexpected. To investigate further, the interviewer provided John a much simpler context of distance as an integral of velocity over time, which could be more readily adapted to an area under a curve interpretation (Figure 4).

When fully wound, a toy car will travel in a straight line for just over 17 seconds. Its speed t seconds after it is released is $v(t) = 3\sin\left(\frac{t^2}{100}\right)$ m/s. What distance will the car travel during the first 10 seconds?

Figure 4: Post-semester task 3 – toy car.

Once again, John left off the differential quantity when writing a definite integral expression to the toy car task, initially providing the solution $\int_0^{10} 3\sin\left(\frac{t^2}{100}\right) t$. The inclusion of the extra t in the differential form was "because otherwise we'd just get velocity if we didn't multiply by another *t*." When asked to describe how the definite integral was able to use the expression in the differential form to find the total distance traveled, John noted "there definitely should be a *dt* there," appending the notation to the end of his expression. Like in the previous post-semester interviews John did not ascribe a quantitative meaning to the differential, even when the interviewer presented him with a graphical representation of the velocity function.

Looking for an indication as to why John no longer considered *dt* to be a quantity, even if just the width of a rectangle, the interviewer related the notation of the differential with the derivative language of 'change in *t*', and associated notation df/dt, which prompted John to re-associate the notation with the "size of the partition" that was the Δt in the summation representation of accumulation. Attempting to reconcile both meanings John reflected,

> *John:* That's just a wicked small number…. I've always just negated it. Because, I remember thinking about it this summer, about how it does have some effect but not really much. On like computing it at least.

Discussion

The case of John provides a glimpse into an unexpected outcome of engaging in curriculum designed to engender a quantitative understanding of the definite integral. Although during the mid-semester interviews John did not demonstrate a fully formed quantitative conception of definite integrals, he provided clear quantitative interpretations for the differential form as an area of a rectangle and ascribed units of measurement associated with each quantity of that product. When these infinitesimally thin rectangles were accumulated they would provide the area beneath a rate of change curve. That is, during the mid-semester interview John demonstrated a consistent area under a curve conception along with an image of computing the area using a Riemann sum approach which was reinforced by his organic chemistry unital analysis. For tasks that did not naturally lend themselves to a graphical interpretation (e.g. radial density) John was unsuccessful in constructing a definite integral which would model the situation. Because the course curriculum de-emphasized the correlation between area under a curve and definite integrals, only briefly covering the topic at the end of the accumulation section as motivation to transition to investigating rates of change, it is safe to assume that John entered the course with some pre-existing schemes for the definite integral as area under a curve.

The most interesting aspect of John's development was the lack of inclusion, or meaning, given to the differential at the conclusion of the semester even when directly presented with a graphical curve. John's recorded coursework notes did not provide enough detail into how his image of definite integrals and differentials adapted over time or when he began to negate any quantitative meaning for the differential quantity. Additionally, due to the asynchronous nature of the course and John's free admittance to relying heavily on an outside tutor throughout the semester, it was not possible to pinpoint the precise source of his conceptions. Even with these limitations, John's case provides potential instructional and research implications. First, similar to other studies, John's reliance on graphical imagery to reason about definite integrals limited his ability to reason about quantitative situations in which the integrand is not inherently a rate of change. Second, John's decision to 'negate' the differential because it is such a small number leads to implications for students reasoning about differentials as infinitesimal quantities. If students interpret phrases such as "negligible" as "unimportant" or "infinitesimal" as "insignificant" then they might be prone to disregarding the differential as having a quantitative interpretation which could prevent them from being successful in constructing definite integral expressions.

In John's case, even with a heavy emphasis on modeling using definite integrals in his course curriculum, an area under a curve conception remained persistent through his mid-semester interview, potentially hindering his ability to adapt to a more general quantitative interpretation of the definite integral. By John's post-semester interviews he only demonstrated reasoning about the differential as a meaningful quantity when interpreting the differential in terms of a macro-sized change differential through an interviewer's direct intervention. With the continued rise of students who take calculus courses in high school, before retaking the course in university, such a finding suggests that more research be conducted to identify how coursework designed to engage students in quantitative understandings of calculus concepts interplays with the primarily procedural and computational skills learned in their secondary education.

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Constructing Linear Systems with Particular Kinds of Solution Sets

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Systems of equations are a core topic in linear algebra courses. Solving systems with no or infinitely many solutions tends to be less intuitive for students. In this study, we examined two students' reasoning about the relationship between the structure of a system of linear equations and its solution set, particularly when creating systems with a certain number of equations and unknowns. Using data from a paired teaching experiment, we found that both students favored the notion of parallel planes, geometrically and numerically, in the case of a system having no solution or infinitely many solutions. We also found that algebraic or numerical approaches were used as the main way of developing systems with a unique solution, especially in systems with more than two equations and two unknowns. Throughout the tasks, one student generally used geometric approaches and the other toward algebraic and numerical approaches.

Keywords: solution sets, linear systems of equations, Realistic Mathematics Education, paired teaching experiment, linear algebra

Systems of linear (SLE)equations are a fundamental topic of introductory linear algebra courses. Systems are one way to model relationships among multiple quantities (Smith & Thompson, 2007). Students from applied science, technology, engineering, and mathematics make up a dominant portion of those enrolled in linear algebra in the U.S. and Canada, and applications related to linear systems are an important component of their learning in this course (Andrews-Larson et al., in press). In applied contexts, SLE often do not have a unique solution, so in this work we focus on students' reasoning about SLE with non-unique solutions.

Students at a variety of levels have exhibited greater levels of success solving systems of linear equations with unique solutions as compared to solving inconsistent systems or systems with an infinite number of solutions (Harel, 2017; Huntley et al., 2007; Oktaç, 2018), and this has been linked to procedural and rule-based approaches to students' solving processes. Oktaç (2018), for example, found that interpreting the result "x=x" was not intuitive for students. Such approaches tend to be unhelpful to students when linking algebraic and geometric representations of systems and their solution sets. It has more recently been documented that, in the context of introductory linear algebra, though students are relatively successful at rewriting systems as augmented matrices and row reducing with technological assistance, many students experience a disconnect in reinterpreting solutions in relation to the original system (Zandieh & Andrews-Larson, 2019). In this paper, we identify resources in students' reasoning that may help address this disconnect, drawing on data from a teaching experiment. Our research question is: How did students reason about the relationship between the structure of a system of linear equations and its solution set, including their reasoning about the graphical representation of the solution set?

Theoretical Framing

In this paper we focus on how students symbolize systems of linear equations that have one, infinitely many or no solutions, and how the students relate their symbolizations to various graphical representations. In doing this we follow Larson and Zandieh (2013) who describe a system of equations as one way to symbolize a relationship between variables and scalars. They compare and contrast this symbolization to others such as a vector equation (a linear combination of vectors equaling another vector) or a linear transformation stated in function notation. Collectively they refer to these as interpretations of the matrix equation $A\vec{x}=\vec{b}$. Paired with these symbolic expressions are different graphical representations. For example, a system of two equations with two unknowns is depicted graphically as two lines with the intersection of the lines denoting the solution. In contrast a linear combination graphically is indicated by scalars stretching each of two vectors with the vectors being added in a tip to tail manner.

In a later paper, Zandieh and Andrews-Larson (2019) extend their work to symbolizations of augmented matrices and discuss how the solution(s) to a system of equations may be expressed using both implicit and explicit notation. For example, the statement of a system of equations to be solved is already an implicit description of the solution space. In this paper we focus on two students' growing recognition of how the components of the symbolic expression that implicitly describes the solution set (i.e., the system of linear equations) relates to the graphical inscriptions that illustrate the solution space.

Methods

This proposal reports findings from a paired teaching experiment (PTE) conducted as part of a larger NSF-funded project aimed at extending inquiry-oriented curricula in linear algebra. The task sequence was designed to support students' reinvention of ways to reason about solution sets to systems of linear equations (SLE). We conducted a task-based PTE with two students for four consecutive days on Zoom to see how students reason about solutions to SLE related to the task and how their reasoning evolved throughout the sequence of tasks (Steffe & Thompson, 2000). The participants we call Student R and Student L were undergraduate math majors also studying to be secondary teachers; one was a white woman and one a white man, respectively. They had no experience with linear algebra but had taken Calculus I at the time of the PTE. Two authors conducted the PTE with one leading the interview and the other taking notes and asking clarifying questions. We present findings from the last day of the PTE, in which students started working individually on the given tasks and shared their initial thoughts with each other. The leading interviewer prompted with questions about their thinking in the moment and leveraged their ideas to help them advance their mathematical thinking. While this proposal emphasizes the last day of the PTE, relevant results from the first three days are included in the task description to give readers the basic context of the instructional sequence.

The Day 1 and 2 tasks were designed to support students in reinventing the notion of large or infinite solution sets corresponding to a SLE. On Day 1, the students worked with a constraint of meal plans that included three variables, B, L, D (# of breakfasts, lunches, and dinners), where B+L+D=210. They used ordered *n-tuples* to organize a large set of solutions satisfying the constraint. On Day 2, the students made predictions about how the set of solutions to B+L+D=210 and 5B+7L+10D=1500 (an added meal plan constraint) would look geometrically. The students agreed the graphs of the two equations would intersect more than once because they found several solutions to both equations (for more details, see Smith et al., 2021b). Day 3 focused on graphing the two meal plan constraints in GeoGebra. This was the first time the students used GeoGebra in the PTE. Based on this work they concluded that each constraint constitutes a plane and the solutions found on Day 2 are on the same line where the two planes intersect. They were guided to the next task called "Intersections of Three Planes" (Wawro et al., 2013). Given the new SLE where the third equation is the sum of the first two equations, the students were asked to find the closest graphic from options (a) through (f), focusing on the intersection of the three planes. At first, they were not allowed to do any calculations nor use GeoGebra. The options consisted of graphics such as three planes having no intersection and intersecting at a point, line, or plane, as shown in Figure 1. Results from these tasks will be discussed in the next section.

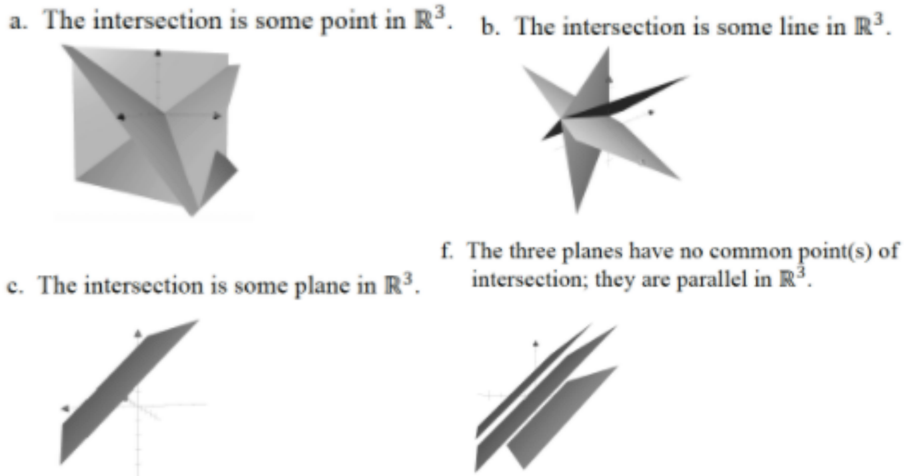

Figure 1. Examples of options in the Intersections of Three Planes task (Wawro et al., 2013).

On Day 4, the students continued to work on the Intersection of Three Planes task with a different approach. Using GeoGebra as an aid, they engaged in the activity to construct a SLE that looks like one of the options' graphics. The students started manipulating the given SLE (where the third equation is the sum of the first two) to satisfy the graphics. The students then worked on a task involving creating SLE that met a specified number of equations, unknowns, and solutions described in a table, called the Example Generation task. In this paper, we use abbreviations to report these more succinctly. For example, 2E2U refers to 2 equations and 2 unknowns and 3E2U refers to 3 equations and 2 unknowns. This task also included a prompt to make generalizations about SLE regarding the number of solutions, equations, and unknowns.

Our data sources include Zoom video recordings, students' written work, and field notes. The Day 3 and 4 video recordings were transcribed in spreadsheets. Two of the authors watched and reviewed the videos and transcripts and created reflective notes focusing on the students' reasoning about creating SLE with specific solution sets using an emergent coding method (Glaser & Strauss, 2017). We paid special attention to what students were referring to when reasoning about solutions, including students' attention to specific traits of the SLE as written and traits of its graph. We discussed the students' quotes and gestures that offered insight into their mathematical reasoning about SLE. Based on the findings, we were able to describe the differences and similarities between students' reasoning about SLE in relation to specified solution sets.

Findings

We found that Student R and Student L developed geometric and numeric strategies, respectively, for reasoning about parallel planes, and leveraged these strategies heavily in constructing systems of linear equations with specific types of solution sets. In this section, we illustrate how this strategy emerged from students' efforts to construct a system with particular visual characteristics in GeoGebra and provide evidence regarding how students leveraged these strategies to construct systems with no solution, unique solution, and infinite solutions -- even as the numbers of equations and unknowns in these systems varied.

Intersections of Three Planes Task

When working on the modified version of the Intersections of Three Planes task, students looked at scenario (a) in which three planes intersect at a point and initially noted that they did not think it was possible. Student R drew on her knowledge of systems of line equations by drawing three lines as a way to represent three planes to see if they could intersect in a point.

R: I didn't even think (a) was possible at first, but I drew out a picture of what I thought it could be. And I was like, maybe, I guess it is possible for planes to intersect in a point. But I thought only lines could intersect in a point.

Int (interviewer): Can you show what you drew? [Student R shows Figure 2] ... Tell me how you were thinking about what you drew.

R: I thought, these are lines technically, but I was like, even if they were extending, if they were all going different directions, then I just drew three random lines that all intersected only in the middle.

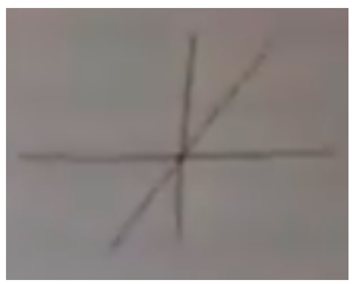

Figure 2. Student R's two-dimensional reasoning about three planes intersecting at a point.

Student (R) drew on her prior knowledge to make sense of a new type of intersection. She conjectured, through the lens of intersecting lines as shown in Figure 2, that it could be possible for three planes to intersect at a point. Both students revisited this idea in the Example Generation task.

The rest of this section will highlight students' initial reasoning with parallel planes that they used later in the Example Generation task. When starting the modified Intersections of Three Planes task, Student R stated that she initially thought the parallel option *"would be easier because… if you just double the values, then it'll be the same plane. But maybe if you manipulate the number that it's set equal to, to not be double, if you doubled the coefficients of the variables… [then] I think we can make the planes parallel."* She pointed out that every value cannot be multiplied by the same number, but parallel planes might be achievable by changing the constant by a different factor from the coefficients. While working on making three parallel planes, Student R referred to the "slopes" of the equations in terms of the coefficients in each. She reasoned that to make parallel lines they needed to create equivalent slopes in the equations, then expanded that reasoning to planes.

Int: So why do you think you have to make them the same for all of them?

R: If we want the slope to be the same, they all have to be multiplied by the same. I can't think of the words… We want the slopes to be there's a word that I can't think of, related…

L: Parallel. Yeah, related.

R: So like, if you change the coefficient of one number, it's no longer gonna be parallel because then the slopes of the equations aren't the same anymore.

Int: So like, what if your original equation was x plus three y plus z and you're trying to make one parallel by changing the coefficients? [Referring to L's eq7: x+3y+z=6]

L: ... like 2x+6y+2z is equal to like, whatever. I believe that would make it as well… Then, so then 2x+6y+2z=6.

Int: So you doubled one side but not the other. So it kept the slope the same.

L: Right. Exactly.

Both students reasoned that all the coefficients in an equation must be multiplied by the same number to have the same "slope" as the original equation. They said that if one coefficient is not multiplied, then the plane will no longer be parallel to the original. Once again, it was mentioned that the 'right side' (or constant) cannot be changed by the same amount as the coefficients.

The students used the parallel planes they had just developed to recreate a system consisting of two parallel planes with a third plane intersecting both. When asked about the number of solutions to the SLE, Student R relied on the fact that two of the planes are parallel and said "*Zero. Because two of the planes are parallel. And those are never going to have a solution that makes them both true.*" Student L came to the same conclusion but by reasoning numerically, stating, "*And also because you have x+y+z is two, and x+y+z is eight. So that doesn't really make sense either, because you can't have three numbers that equal two and also equal eight, when you add them together.*"

Overall, we found that students' previous knowledge regarding systems was essential to students' development of SLE in relation to graphical representations. Students wondered about the possibility of a unique solution for a system with three planes. Because they had never seen it previously, they explored this possibility by drawing on their knowledge of a system of equations in a 2-dimensional setting. The students developed initial reasoning for parallel planes, seemingly relying on what they knew about parallel lines. This reasoning about parallel lines and planes became a way for students to reason about systems with infinitely many and no solutions.

Example Generation Task

Constructing systems with no solutions. In this section, we will describe our findings regarding students' reasoning while creating SLE with no solutions. The students' work on the example generation task began in the context of 2E2U (two equations and two unknowns). In this case, the students started with one equation, x+y=3, and created another equation by multiplying each coefficient by two to get, 2x+2y=3. As they explained with planes in the previous task, the constant cannot be multiplied by the same number as the coefficients, or else the second equation is actually the same line as the first. When students subsequently worked to construct an example of a system with 3E2U with no solution, they added another parallel line to their previous example. The interviewer probed on whether this was the only way a SLE with no solution could be constructed.

Int: So would it have to be parallel?

R: I guess not. We could do like we just did with the last one we did with three variables. Maybe. That also has no solutions.

L: If it wasn't parallel, wouldn't they eventually come to a point or something?

R: Well, equation one and equation two are parallel. But if I change the third equation, instead of it being parallel as well, if I make it not parallel, then it's kind of like what we just did. How there's no solution because these two lines are parallel.

Student R mentioned their previous example in which there were two parallel planes and one plane intersecting the two, except with lines this time. The students used similar thinking to create a system of 3E3U with no solution with parallel planes.

Constructing systems with infinitely many solutions. When constructing systems of equations with infinitely many solutions, the pair began by working on the case with 2E2U. Student R created two copies of the same line by multiplying x+y=3 (equation one) by 2 and explaining, "*If we wanted infinitely many solutions, we can just do like 2x+2y=6 with equation one. And then they're the exact same line.*" The students used a similar process to construct a system of 3E2U and voiced that multiplying the second and third equations by some value is the only way they understand creating a system of lines with infinitely many solutions. Student L wondered, "*Is there? I'm not sure if there's another way to have infinitely many solutions*." When shifting to the cases with 2E3U and 3E3U, the pair leveraged their previous work, and created equations representing the same planes.

Constructing systems with unique solutions. In working to construct a system of 2E2U with a unique solution, Student L used a heavily numeric approach and started with one equation, found a viable solution to that equation, then created a second equation in which the previously viable solution satisfies. In other words, Student L started with the equation x+y=3 and a solution of x=2, y=1 and developed 2x+y=5 as a second equation. He explained, *"So I just used x+y is equal to three. And then I said, okay, x is equal to two y is equal to one. So, then I was just like, well, two times two is four plus that one is five. So that should give us the one solution. It'll be two, one."* The students only referenced the geometry of this SLE by describing (2, 1) as the point of intersection. In moving to 3E2U, Student L did something similar by building another equation based on a selected solution that satisfies the two originally developed equations: *"Just like how we did it with the, the two equations and two unknowns. I was just thinking, well, we have x+y is equal to three. And then if x is two, y is one. So then, in 2x+y, it would make it five. And then if you were to add like another one to just like another coefficient to that x, it would make it 3x+y. And now you just go to seven."* Once again, neither student explicitly referenced geometry as they built this SLE.

In shifting to the case with 2E3U, student L stuck with his numerical approach, presumably because it had been successful previously. Student R thought more geometrically, arguing that it was impossible for the system to have a single solution because two planes cannot intersect at a single point. She explained, "*When we were doing the algebra part of it, I was more like maybe there is only one solution. But because when you look at it, it's hard to just tell from the numbers. There's no way just to look at it and be like there's a million solutions. I know what they all are. But when you look at the graph, it becomes clear that there's no way that these two planes are going to ever intersect in just a point because they're planes. And there's only two of them."* Student L tried a guess-and-check method, but he later accepted Student R's justification because he could not find a system that had only one solution. In the case of 3E3U, the students questioned whether it was possible to have a single solution, thus revisiting their previous query. Their intuition told them that it was possible. They constructed the SLE in GeoGebra by creating two planes that intersect in a line and then added a plane that intersected that line at one point.

Making Generalizations

At the end of the interview, the interviewer asked students for generalizations about systems and their solutions. We found that both students attended to ideas they developed when reasoning about parallel planes, such as looking at coefficients and constants.

R: I would say pay attention to the **coefficients** and maybe look for like **how the equations could be maybe parallel… or how they're intersecting**… Parallel will tell you that there would be no solutions for the system.

L: I would also say, make sure to pay attention to the **constant**. So like whatever you're setting it equal to, x+y is equal to three. So for instance, it had no solutions, we know that the constant is just going to stay the same, because the lines are going to be parallel… The constant is going to change based on how the other equations change.

Student R eventually came to reason about the number of equations and unknowns, something that had not been deeply discussed previously. She stated, *"In the beginning, we were trying to find a solution to that system of equations that had only two equations and three unknowns. Now I know in the future, don't waste your time looking, doing substitution. It's not gonna work… Maybe because there's* ***more unknowns than equations****, you don't have enough information to use substitution or anything because there's not going to be a single solution. I don't know if that's true."* It could be that the design of the Example Generation task led Student R to connect that the reason two planes could not have one solution was because there were fewer equations than unknowns (along with her geometric reasoning), leading her to conjecture that there can never be one solution to a system with that trait.

Discussion

On Day 4 of the PTE, Student R generally oriented her reasoning around a geometric approach to make sense of no, unique, and infinitely many solutions. She predicted how the graphs of solutions would look as she modified the coefficients in the SLE and used the same strategy when constructing SLE in the Example Generation Task. On the other hand, Student L leveraged a numerical approach to reason about solutions to various SLE. He began by choosing specific tuples satisfying a first equation or concluded no solution by examining the coefficients and constants in equations. Then, the parallel graphics became sensible to him. The two students' different ways of reasoning seemed to reflect their interpretation of solution to SLE: It is the intersection of graphs [R] and the point(s) in the intersections of graphs that satisfies all equations in the SLE [L]. In the case of 2E3U, Student R's geometric approach allowed her to conclude that this type of system can never have a unique solution. She subsequently conjectured that this might be true in all cases where the number of equations is less than the number of variables. Throughout the task sequence, Student R's geometric reasoning was more useful in some cases, and Student L's numerical reasoning in other cases, but both students relied on their understanding of parallel lines and planes in many of the infinitely many and no solutions cases. This study demonstrates ways students can reason about solutions to SLE without having learned row reduction, pointing out potential areas for connections between students' prior knowledge and what is to be learned.

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Function Metaphors Seen in Undergraduate-Mathematics Sign Language

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In this study, two Deaf instructors and two American Sign Language (ASL) interpreters sign undergraduate mathematics terms and definitions. This study characterizes the types of metaphors present when the participants signed about the notion of function. We used a theoretical framework for characterizing the metaphor clusters students used in the unified notion for function to categorize the signed metaphors. We found the Mapping cluster to be the most common metaphor cluster used, and compared the different uses of an indicative sign of this cluster, MATCH, to show the differently conveyed meanings.

Keywords: deaf, sign language, metaphor, function

Deaf students are overlooked in many equity-based research projects along with other students with different abilities, specifically in STEM Education and Mathematics Education research (Schneiderwind & Johnson, 2020). American Sign Language (ASL) is the native language of the Deaf Community in the United States and most of Canada. It is not a one-to-one translation of the English language and maintains its own grammar structures and cultural significance (Lepic, 2015). Because of this, ASL is an important area to investigate in order to further the goals of increasing equitable research with regards to d/Deaf and Hard of Hearing (DHH) students. ASL also has regional differences throughout the states.

The structure of ASL lends itself to different affordances than spoken language because of its three-dimensional modality (Lepic, 2015). Thus, the study of ASL communication in mathematics not only has potential benefits for DHH students, but it also has potential benefits for non-DHH education in expanding our insight into language use and communication issues about mathematical concepts. In this paper we draw on research into metaphor in spoken English language and consider how these constructs might apply to ASL. This has overlap with other research on iconicity of signs in mathematics contexts.

Background Literature and Theoretical Perspective

Iconicity and Metaphor in Sign Language

Mathematics Education research on gestures has influenced research about sign language in mathematics education (e.g., Krause, 2019). McNeill (2005) created a classification of deictic, iconic, and metaphorical dimensions of gestures. Metaphorical gestures are used in synthesis with spoken word and are different from iconic gestures in that they convey abstract ideas rather than imagery. Some researchers have studied the use of iconicity and/or metaphor in ASL specifically (e.g., Krause 2019; Taub, 2001).

In a previous study, Smith identified categories of iconicity present in sign language of key undergraduate mathematics terms like function, limit, derivative, rate of change, slope, span, linear independence, concavity, and continuous function (Smith, 2021). Meir and Tkachman (2014) define iconicity as “a relationship of resemblance or similarity between the two aspects of a sign: its form and its meaning”. An iconic sign in some way represents the meaning of the concept being signed. Krause (2019) found that iconicity of signs can influence how students using ASL will conceptualize mathematical concepts. Smith extended Krause’s constructs of iconicity present in elementary students' mathematical signs by creating subcategories of innerlanguage iconicity and iconic-symbolic reference for the purpose of looking at undergraduate mathematical signs. The subcategories are conceptually-linked and English-linked (innerlanguage iconicity), and initialized, notational, and graphical (iconic-symbolic reference). In this study, the iconicity of signs is considered while examining the metaphor surrounding a specific sign: FUNCTION.

Metaphor in Mathematics Education

A number of researchers have studied conceptual metaphors as a way of analyzing student thinking about mathematical concepts (Adiredja & Zandieh 2020; Lakoff & Núñez 2000; Oehrtman 2009; Zandieh & Knapp 2006; Zandieh et al. 2017).

Zandieh et al. (2017) studied how ten linear algebra students viewed the concept of linear transformation and how it is related to their concept of function from high school. They found that the conceptualization could be described by three mathematical structures: properties, computations, and metaphorical clusters. Zandieh et al. found that students used metaphorical language to express a unified notion of function from their high school definition of function to the notion of linear transformation. For the purpose of this study, the primary use of the framework looks at the five metaphorical clusters that could be identified: Input/Output, Travelling, Morphing, Mapping, and Machine.

The Input/Output metaphor cluster involves language related to "putting in" an input and "taking out" an output. The travelling metaphor cluster involves language related to "moving" an entity or "sending" an entity to a location. The Morphing metaphor cluster involves language related to morphing an entity from a beginning state to an ending state. The Mapping metaphor cluster involves language related to a mapping, correspondence, or relation between entities. The Machine metaphor cluster involves language related to a machine, device, or process that changes one entity into another.

Research Question

What types of metaphors can be found in how undergraduate mathematics Deaf instructors and ASL interpreters sign about the concept of function? Are there any metaphors for function that seem to be more common?

Methods

The participants of this study are a part of the previous study analyzing iconicity of key undergraduate mathematics terms. The participants include two ASL interpreters of collegiate mathematics and two Deaf instructors (one previous, one current). They come from three different regions of the country for the purpose of collecting diversified sign language data.

The four participants of this study (all residing in different states and given pseudonyms) consist of two ASL interpreters and two Deaf mathematics students/instructors. Martha and James are the two ASL-interpreter participants. Martha has been interpreting for 35 years and works as a staff member at a community college known for having a large population of deaf students. James has been interpreting ASL for 27 years and is hired through an agency to interpret at a large Southwestern university.

Andrew and Thomas are the two Deaf participants who have been undergraduate mathematics students and worked as TAs in a STEM program. Andrew and Thomas have both taken and taught undergraduate mathematics courses in ASL at large universities. While Andrew has taught in ASL to deaf students, Thomas has taught ASL that was interpreted for hearing students.

The participants were interviewed with a series of general experience questions in the first half of the interview and shown term/definition cards to sign in the second half of the interview. Each participant was asked to show their primary sign and other signs they have seen for the main term on the cards. Then, they were asked to sign the definition that was written on the card. The ASL interpreters more closely signed the words written on the card, while the Deaf instructors took more liberties in elaborating on the definitions, providing more context for how they conceptualize the mathematical idea (and how they might explain the concept to students).

For the purpose of analyzing function metaphor, the pieces of data that were coded included the excerpts of signed definition for the *function* term card and excerpts surrounding any time the participant signed FUNCTION. Thus, we can establish what metaphors might be evoked when the concept of function is specifically being prompted or discussed. First, the excerpts were glossed (an informal way of writing ASL). Glossing is typically used as a mechanism for teaching English speakers ASL, and a sign is given an associated English word and written in all capital letters. Words are not always consistently used in the glossing of ASL, but are helpful for transcribing and coding ASL data. The excerpts were then translated to English. The metaphor clusters from the codes in Zandieh et al. (2017) were used to code the English translation and indicator signs were coded for their associated clusters. Then, a discussion between the researchers solidified the metaphor codes.

A follow-up email exchange with Andrew was used to clarify the intended use of a specific sign, MATCH, that was used by all four participants in different contexts (and by three participants in the context of describing *function.* His explanation was used to take another look at the use of the sign, MATCH, and note the differences in use and the implications of those differences.

It is important to note that the participants were asked to sign an English definition which can influence the type of metaphor being signed. When the study was originally designed, the term cards were designed for the purpose of seeing how specific words or concepts were signed. However, the participants took it upon themselves to either deviate from the written definition or provide further explanation for the terms in question, thus providing more information about how the participants conceive of the mathematics concepts.

Function Term Card:
Function: A function consists of three parts: Domain, Range, and Rule. The domain is the values the independent quantity may assume. The Range is the values the dependent quantity may assume. The Rule assigns to each value of the independent quantity exactly one value of the dependent quantity.

Results

All four participants used the same sign for *function* in the interviews. Its iconicity is classified as an initialized sign due to the use of the handshape F and can be seen in Figure 1.

Figure 1. James signing FUNCTION.

Metaphor Clusters

The most common metaphor cluster used was Mapping used by all four of the participants in their signing of the function term card definition. The least common metaphor cluster used by the participants was Machine (with no participant showing signs related to function as a process). The term card uses the language "The rule *assigns* to each value of the independent quantity exactly one value of the dependent quantity". This correspondence language is indicative of the Mapping cluster, and could have influenced the Mapping cluster to be more prominent in the participants' signing of the function card. However, only one participant could be identified as specifically using the sign, ASSIGN, in their function definition. Examples of signs that were indicative of metaphor clusters are given in Table 1. The Travelling metaphor signs, CLOSE/NEAR and MOVE were used to represent a function "moving" graphically towards a specific point (e.g., when signing about a limit). The Morphing metaphor signs BECOME and CHANGE were used in the context of an x value or input value "becoming" or "changing into" a y value or output value.

Table 1. This table shows example signs presented for each of the five metaphor clusters. When words are capitalized, they represent a sign.

Signs indicating metaphor cluster			
Metaphor	Example	Example 2	Example 3
Input/Output	IN/INPUT	OUT/OUTPUT	PUT-IN/ TAKE-OUT
Travelling	CLOSE/ NEAR (repeated)	MOVE	GOES-TO
Mapping	MATCH	LINK/RELATE	ASSIGN (give-to)
Morphing	BECOME	CHANGE	

Examples of Metaphor Signs

In his extended description of *function*, Thomas uses the Input/Output metaphor cluster to explain how one "puts in" an input into the function and "takes/gets out" an output (Figure 2).

Figure 2. Thomas signs PUT-IN (top 2 pictures) and TAKE-OUT (bottom 2 pictures).

Use of Match in Mapping Metaphor

All four of the participants used the sign, MATCH, at some point during the term and definition card signing portion of the interview. However, it was not always used to illustrate the same concept and was not prompted by a specific English word. Three out of four of the participants used the sign as a metaphor describing *function*. We can see an example of how MATCH is used in an English translation of Andrew's signing about *function*:

> I pick one element from the left set and calculate to get a number in the other set. I **match** every element from the left set to an element in the right set. I can only **match** it to 1 and only 1 number from the right set. I call the left set the domain and the right set the range.

Although MATCH would be characterized as part of a Mapping metaphor cluster in this English translation, the sign MATCH can mean many things. In a post-interview email discussion with Andrew regarding the use of the sign, MATCH, and its possible uses when describing *function*, he mentions:

> The usual words that go with that sign are "match", "fit", "-mate", "make a good pair", et cetera. If you inflect it, it can mean "accommodate", "adapt", "make changes for something or someone who cannot or will not change", "not a good a fit", "mismatch", and so on… it is a "pure" ASL sign in the sense that fluent signers rarely mouth anything for that sign… This makes it difficult to translate to English clearly.

He goes on to describe the importance in how the sign, MATCH, is used to convey different meanings, "It is important to clarify the subtlety of matching elements in two lists (a simple relation) versus a function (no multiple outputs, every element of the domain must have a match)." This subtlety can be illustrated by the different ways Martha and Andrew use the sign, MATCH.

In Figure 3, Martha (ASL interpreter) is shown using the sign, MATCH, to describe a simple relation of two lists, the domain and range. She had previously designated a space for Domain on her right side (left side from audience's perspective) and the Range on her left side. She then uses the handshape for 1's on both hands to illustrate 1 element from each set and uses the sign MATCH to relate the two elements.

Figure 3. Martha uses the sign MATCH to show a simple relation of matching two elements together.

In contrast, in Figure 4 and 5, Andrew sets up Domain (his left side) and Range (his right side) and uses the sign, MATCH, modified with directionality to show that elements of the range are *dependently* related to the independent elements of the Domain.

Figure 4. Andrew establishes a set space for domain (left picture) and a space for the range (right picture).

Figure 5. Andrew uses the sign MATCH to show directionality of element relation.

During the email discussion, Andrew sent a video where two more signs were presented for the sign for *function*. In Figure 6, Andrew shows the alternative signs.

Figure 6. Andrew shows two alternative signs for function.

Andrew described the sign on the left was described as an "English sign", however, it is a modification of the initialized sign for *function*, where the top hand forms the handshape of the letter F. The sign on the right was shown as a more recently developed sign used at NTID (National Technical Institute for the Deaf) at RIT that modifies the sign for *change.* Andrew described this sign as meaning "changer", and it is displayed almost exactly as Andrew's primary sign for derivative. This sign can be seen to contain the metaphor cluster Morphing, as well as displays innerlanguage iconicity to CHANGE, and for Andrew, DERIVATIVE.

Discussion

The most common metaphor cluster present in the function sign card is Mapping, however, the word that is associated with that metaphor, *assign*, is not the most common sign associated with the mapping metaphor clusters present in the participants' signing of that term card. This implies that the participants are familiar with signing Mapping metaphor clusters outside of specific prompting of the English language. While all four participants used the sign, MATCH, while signing terms and definitions, only three of the participants used it in the signing of the function definition. The other participant used it only in the context of linear combination as his sign for *combination*. Even within the mapping metaphor, the inflection of MATCH changed the meaning behind the signing, thus altering the conveyed definition of function.

During the post-interview discussion with Andrew regarding the sign, MATCH, he mentions it as a "pure" ASL sign. This means that exactly equating it to the English word "match" is not a completely accurate way to describe the meaning behind the sign. As he mentions, there are many different English words or phrases that can be associated with that sign, including the word "combine/combination" that shows itself in other mathematical contexts: computation, algebra, linear algebra etc. The importance in how MATCH is used in a metaphorical cluster when signing about function rests on the modification or inflection of the sign. Signs can be modified or inflected to show directionality of action, plurality, and many other features of a sign. For example, the sign, GIVE-TO, can be inflected to show "give to me" where the sign goes towards you or "give to her" where the sign goes towards the intended party. The difference in inflection in the sign MATCH between Martha and Andrew illustrates that slight differences in this sign can convey different meanings: a simple relation between lists of elements or a function mapping from an independent to a dependent set.

While Mapping metaphor cluster is the most common metaphor cluster used in the four participants' signing about function, this study shows the variety of metaphor clusters also present in their signing. The participants illustrate the Travelling metaphor cluster when describing shapes and behaviors of the graphs of function, typically when describing how a function looks or behaves as it approaches a specific point or limit. The Morphing metaphor is used as a supporting metaphor cluster for how one entity is changed into another entity by a function, typically in extended explanation after using a Mapping metaphor.

The sign choice and metaphor use in signing definitions of mathematical concepts is not the only important feature of mathematical signing to focus on. The inflection of the signs can alter the meaning, and future studies can dive deeper into these differences. Iconicity and metaphor of sign language give a window into how Deaf students, instructors, or ASL interpreters are conceptualizing mathematical concepts. The differences in metaphor surrounding the sign for *function* show us how sign language is being used by Deaf instructors and ASL interpreters in the classroom and how that signing can influence the types of conceptual metaphors students might encounter.

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“$A=2\pi rh$ is the Surface Area for a Cylinder”: Figurative and Operative Thought with Formulas

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Researchers studying students’ quantitative reasoning argue that students’ reasoning covariationally is essential for their construction of productive meanings for various ideas related to relationships (e.g., linear, quadratic, trigonometric) and representations (e.g., graphs, tables). In this report, I build on that literature by considering a unique feature of geometric formulas, the idea that one formula can be used to re-present multiple (geometric) relationships. To do so, I use The Formula Task *in exploratory teaching sessions and a teaching experiment; seven undergraduate students were given the formula “$A=2\pi rh$” and consider if/how the formula can be used to re-present relationships in various dynamic geometric contexts. In analyzing students’ responses, I use the ideas of figurative and operative thought to make sense of the ways that students reason about and construct formulas. This report provides implications for how students understanding formulas as ways to re-present situations vs. quantitative relationships impact their interpretation of formulas.*

Keywords: Cognitive Research, Teaching Experiment, Precalculus, Formulas

In the quantitative and covariational reasoning literature, there is evidence that supports that students’ reasoning covariationally is essential for their construction of productive meanings for rate of change (Carlson et al., 2002), trigonometric relationships (Moore, 2014), and numerous other topics (e.g., Ellis et al., 2013; Johnson, 2015, 2013; Trigueros & Jacobs, 2008). Many of these studies include students reasoning with dynamic contexts to construct various representations. In this report, I build on that literature, specifically the ideas of figurative and operative thought, by considering the idea that one formula can be used to re-present multiple (geometric) relationships (I use *re-present* to emphasize an image being presented again a new context (see von Glasersfeld, 1982, who describes re-present similarly as images that result absent of perceptual material). To do so, I use *The Formula Task*, in which students are given the formula “$A=2\pi rh$” and asked (i) to describe a situation that the formula can re-present, and (ii) when given various dynamic geometric contexts, if/how the formula can be used to re-present that relationship. This study is part of a larger study consisting of exploratory teaching sessions and a semester-long teaching experiment (Steffe & Thompson, 2000) conducted with seven total pre-service secondary mathematics teachers (heretofore students) aimed at understanding and developing students’ meanings for formulas. To analyze the data from *The Formula Task*, I consider the extent to which students’ activities relied on figurative or operative thought (Piaget, 1974). I report on the results of four students’ activities on the task. I then provide insights into how these students’ reasonings show that students constructing or interpreting a formula as a re-presentation of a relationship between quantities, rather than as an individual feature associated with a particular shape, is a powerful way for students to reason.

Background Literature

Researchers have identified students’ difficulties with reasoning about formulas. For example, students reverse symbols re-presenting variables’ measures (e.g., Clement et al., 1981, 1981) and treat symbols as static, or fixed given referents (e.g., Dubinsky, 1991; Gravemeijer et al., 2000; Musgrave & Thompson, 2014). These difficulties are problematic for undergraduate students, particularly those in Calculus and Differential Equations sequences, in which students need to reason symbolically and re-present and reason with changing quantities via formulas and equations. One way that researchers have recently considered supporting students' productive meanings for formulas is to incorporate opportunities for covariational reasoning—reasoning about two quantities change together (Carlson et al., 2002)—and dynamic situations. Along with the examples of covariational reasoning research mentioned in the introduction, some researchers have specifically considered dynamic geometric environments to support symbolization. For example, Fonger et al. (2016) used the context of an area of a rectangle growing in proportion with middle school students to explore and symbolize quadratic growth, and Panorkou (2020, 2021) has studied students' meanings for volume and area using dynamic geometric environments. In designing the tasks used in the exploratory teaching sessions and teaching experiment of this study (including *The Formula Task* described in this report), the literature on these dynamic geometric situations and covariational reasoning informed how we might support students' meaningful construction of formulas as re-presenting quantitative relationships.

One primary finding in the literature relevant to this report is the distinction between figurative and operative thought (Piaget, 1974, 2001; Steffe, 1991). In essence, figurative thought is associated with thought that foregrounds sensorimotor actions that are subordinate to perceptual (figurative) properties, and operative thought is not constrained by sensorimotor experience (see Moore (2016)). Building on the work of static and expert shaping thinking (Moore & Thompson, 2015), Moore et al. (2019) took up this distinction to make sense of students' graphing actions. For instance, they described a student struggling to re-present a dynamic relationship between two quantities that resulted in a trace that traveled from right to left, because "it's backwards" to the usual left to right graphing activities typically done in the classroom. In this way, the students' actions were constrained by the sensorimotor experience of a graph's trace rather than foregrounding the quantitative relationship re-presented by that trace.

In this report, I use the findings from *The Formula Task* to describe how the constructs of figurative and operative thought can be used to construct viable models of students' actions when constructing formulas. Specifically, I considered how Thompson's (1985) distinction between figurative and operative thought could relate to geometric formulas. Thompson (1985, p. 195) said, "Any set of schemata can be characterized as figurative or operative, depending upon whether one is portraying it as background for its controlling schemata or as foreground for the schemata that it controls." In this study on formulas, the set of schemata does not rely on perceptual actions taken on perceptual material that can be operated on, such as tracing right to left as Moore et al. (2019) described, or repeating partitioning activities across contexts as Liang & Moore (2020) described. Rather, the schemata relies on associations of a symbol (or collection of symbols) in a formula. Thus, I sought to answer what meanings for formulas do students have when they see "$A=2\pi rh$"? Those meanings could potentially be serving as controlling schemata in the background if they are tied to attributes/shapes (e.g., (surface) area of cylinder, circle, spherical cap), in which case the student would be engaging in figurative thought. Alternatively, if those meanings include the formula as re-presenting quantitative relationships that could be re-presented in a variety of contexts (e.g., a linear relationship between height and area that could be re-presented in a rectangle, cylinder, spherical cap, etc. depending on what quantities the symbols re-present in the situation), then the student would be engaging in operative thought. The results in this paper detail particular illustrations of how students engage in figurative and operative thought when reasoning with the formula $A=2\pi rh$.

Methods

The larger study was split into two parts: a round of 3-5 individual exploratory teaching sessions with four pre-service mathematics teachers (heretofore, students) and semester-long individual teaching experiments with three students (Steffe & Thompson, 2000). All the students were either in their first or second semester of secondary mathematics teacher program at a large public university in the southeastern U.S. Each student had completed a Calculus sequence and at least two other upper-level mathematics courses (e.g., linear algebra, differential equations) with at least a C in the course. Moreover, each student was either enrolled in or had completed a spring semester content course exploring secondary mathematics topics through a quantitative and covariational reasoning lens inspired by the *Pathways Curriculum* (Carlson et al., 2015).

For the exploratory teaching sessions, the teacher-researcher (TR) individually interviewed all four students who expressed interest in the study after contacting the entire class about the study. During the interviews, the TR presented the students with tasks via a semi-structured clinical interview style pre-interview and a series of 3-5 interviews (Clement, 2000). These interviews lasted about two hours each, and this report focuses on Charlotte, Kimberley, and Alexandria's final task of the sessions, *The Formula Task* (details in the next section).

For the subsequent teaching experiment, three students were selected from a different semester of the aforementioned secondary content course by identifying students that exhibited different ways of reasoning on a modified version of the *MMTSM* assessment (Thompson, 2012) and a clinical interview. The students in the teaching experiment individually met with the TR and a witness-researcher (WR) approximately weekly for a semester. In this article, I report on Lily's activities on *The Formula Task*, the last task of the teaching experiment.

The analysis done for this report involved ongoing and retrospective analysis through which the TR built these models of *students' mathematics* (Steffe & Thompson, 2000). It was during the retrospective analysis of the exploratory teaching sessions that the ideas of figurative and operative thought became a viable way to distinguish between students' reasoning about formulas in *The Formula Task*. The TR then went through the data again to identify different instances of figurative and operative thought, definitions that were carried through in both ongoing and retrospective analysis for the teaching experiment. The resulting definitions are presented in the results along with sample illustrations.

The Formula Task

In *The Formula Task* (Figure 1), students receive the following prompt: "Describe a situation in which the formula $A=2\pi rh$ describes a relationship between quantities. How does your situation describe that relationship?". After the students gave their initial responses, they were presented with the dynamic situations presented in Figure 1 one by one: a cylinder (height varying), a rectangle (width varying), a parallelogram (one of the side lengths varying), a cone (height varying), and a spherical cap (with radius of sphere constant, spherical cap varying up until a hemisphere). All these situations except for the cone situation entail relationships that can be re-presented by the formula $A=2\pi rh$, with A re-presenting (surface) area, and h re-presenting a certain length quantity (height, width, length, etc.) and r re-presenting a radius.

The purpose of this task is to analyze the images (quantitative, covariational, or otherwise) students evoke from $A=2\pi rh$, and to determine the importance of the particular structure, ordering, and use of the letters/numerals to their construction of situations. In this way, I consider the role of students figurative and operative thought. For instance, a student with a figurative meaning may conceive the formula as the normative formula for the surface area of an open cylinder, and only an open cylinder; the student would not describe any other situation with that formula. Alternatively, a student with an operative meaning may have a meaning for a formula as re-presenting a linear relationship that can be identified in various contexts. Specific results are detailed in the next section.

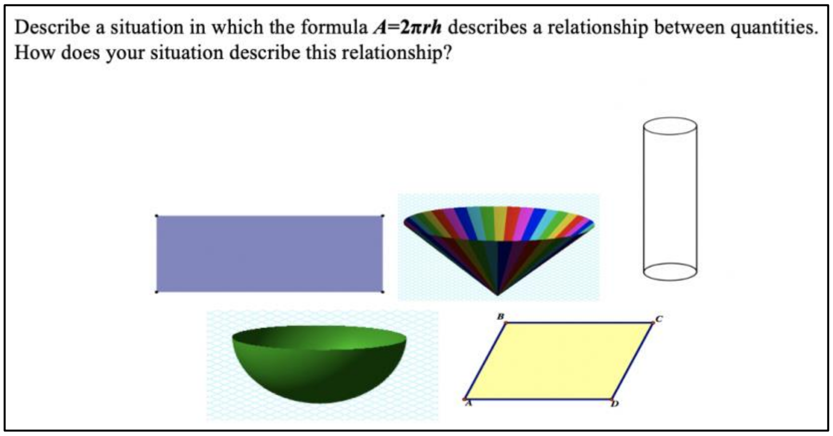

Figure 1. The initial prompt in The Formula Task *and images of the follow-up shapes that the students were presented with in a dynamic geometric environment.*

Results

The results section is organized into four sections. Recall that *The Formula Task* asked students to answer both *if* $A=2\pi rh$ was an appropriate relationship for a given and *how* (or *why not*). In the results below, students' figurative and operative thought is distinguished based on whether the students were reasoning about the formula as a whole (first two sections) or a more nuanced referencing of particular (groups of) symbols in the formula (last two sections).

Figurative Thought with the Formula as a Whole

In the first example of figurative thought, a student considers a formula as an attribute of one and only one shape. For $A=2\pi rh$, a student may conceive the formula as the normative formula for the surface area of an open cylinder, and only an open cylinder; any other context could not be described using the given formula. For example, after concluding that the formula $A=2\pi rh$ was appropriate for the cylinder context (using the normative definitions for each of the symbols), Charlotte said there was not a way for her to make sense of writing $A=2\pi rh$ for the (surface) area of a parallelogram or cone; her argument for the cone is in the following excerpt.

Charlotte: No, because if it were the area, then, it would-it would be, I would just be saying like it was this shape [*pointing to the cylinder shape*], but that shape is this shape with those cut out [*forming a cone shape within the cylinder shape*], so I can't say that."

As seen above, Charlotte's justifications for the appropriateness of writing $A=2\pi rh$ was rooted in comparing the shapes to one another rather than focusing on a potential quantitative meaning the formula could re-present across different contexts.

Alexandria had similar reasoning with the spherical cap. When the TR confirmed that, in fact, $A=2\pi rh$ was an appropriate formula given that *A=surface area of the spherical cap, h=height of the spherical cap*, and *r=the radius of the sphere from which the spherical cap is formed*, she was perturbed. She stated, "That really bothers me because that doesn't make sense to me... That's the surface area for a cylinder, but that's [*the spherical cap*] not a cylinder").

Like Charlotte, Alexandria's justification for a formula being appropriate for a context relied more on the shape being considered than the relationship between quantities re-presented in the formula. For both students, the formula was uniquely associated with the cylindrical shape.

Operative Thought with the Formula as a Whole

To engage in operative thought with a formula as a whole, a student considers the formula as re-presenting a relationship between quantities identified in a context. As an example, we return to Alexandria and her work with the spherical cap. Although her initial reaction was that the formula could not re-present the area of the spherical cap, when she engaged in covariational reasoning, she made sense of the implications that formula meant in terms of the relationship between the height and surface area of the spherical cap. Considering successive equal changes in height for the spherical cap (see Figure 2), she stated "that each change in area is the same as before." Although she was uncomfortable with this conclusion because of she had difficulty comparing changes in area in the spherical cap, she returned to the formula and said, "If *r* is staying the same, then you have a linear relationship, which is constant, by definition, so it would have to be." Thus, rather than relying on associating a formula with a particular shape like she did at first, Alexandria considered what quantities the formulas re-presented in each context and considered the (linear) relationship that the formula was re-presenting in each case. Her going through the activity of comparing the changes in area within and across the two contexts indicated that her focus shifted from associating a formula with a shape (i.e., a cylinder), a figurative association, to reasoning with a formula as re-presenting relationships between quantities in the given context (i.e., a spherical cap), which is operative thought.

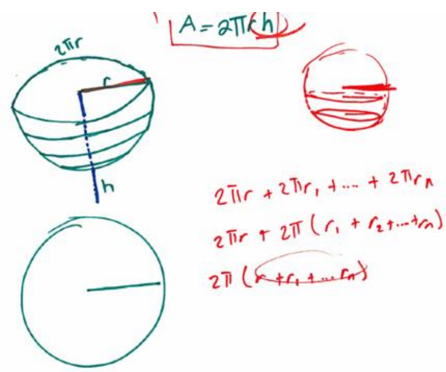

Figure 2. Alexandria's equal changes in height marked on the sphere for The Formula Task.

Figurative Thought with Quantities Re-presented within the Formula

In this second example of figurative thought, a student considers a formula for a shape as comprised of quantities/formulas that are identified in that shape. In this case of $A=2\pi rh$, students may associate $2\pi r$ with circles, and thus any context that contains a circle (e.g., cylinder, cone, sphere) may be associated with the formula in one way or another. For instance, Kimberley considered the 2π in the formula $A=2\pi rh$ and thought both the cylinder and cone were potential candidates, but not the parallelogram or the rectangle. Her thoughts on the parallelogram context, particularly her interpretation of the 2π in the formula, are in the following excerpt.

Kimberley: I don't know why-you wouldn't have pi for the area of a parallelogram. I guess the *r* is whatever you want it to be, but you wouldn't have a two pi.

TR: So why wouldn't you have a two pi?

Kimberley: Because I don't know what it would represent in a parallelogram. Like, in a circle, it's because you can like divide a circle into two pi radii, but you don't have anything even here that you could do that with.

In the excerpt above, Kimberley argues that 2π is associated with measurements of radii in a circle, but because there is no circle present, there is nothing with which to associate the 2π

(Note: It would be possible to re-present the height of parallelogram with 2π (i.e., $2\pi r$)). On the other hand, the cylinder and cone contexts do contain circles, and thus Kimberley used the circles to justify her conclusions about why A=2πrh could re-present these contexts. The following excerpt shows her reasoning. (It is important to note that Kimberley remained unsure whether $2\pi r$, πr^2, $2\pi r^2$ was the formula for calculating the area of a circle, so she used them interchangeably throughout the interview. At this point, she had settled on $2\pi r^2$.)

Kimberley: I mean, they'd obviously, either one of these [*cylinder of cone*] makes more sense than like a parallelogram or a rectangle.

TR: Okay, and why does it make more sense for these?

Kimberley: Because you got a circle and you know that, like, to find the area of a circle, you're gonna have two pi included.

TR: Okay. Right, and because these [*cylinder and cone drawings*] both have circles, then this formula [$A=2\pi rh$] kinda makes more sense for those?

Kimberley: Mhm.

Although figurative associations such as associating πr^2 with the area of a circle can be useful in constructing a quantitative structure in a context, it is important to note the effects of maintaining figurative associations in constructing formulas. Namely, the student may attempt to identify perceptual features in the context and then attempt to incorporate formulas associated with those figurative elements in their construction of a formula, resulting in a non-quantitative formula. For instance, after the previous dialogue, Kimberley identified both the cone and cylinder as including circles and a varying height, thought that only one should be able to be associated with the formula A=2πrh, and struggled to decide which one was appropriate. She continued reasoning figuratively with formulas by attending to the triangle shape that she noticed in the cone (but not the cylinder). At one point, Kimberley tried to write a formula for the cone using the formula for the area of the circle combined with the area of a triangle: $A=2\pi r^2 \cdot \frac{1}{2}bh$, crossing out the b for the base of the triangle, because the base is "just the circle part of it" which the $2\pi r^2$ already accounted for, leaving her with $A=\pi r^2 h$ as the formula for finding the surface area of a cone (Figure 3). Note that all her reasoning relied on associating formulas with shapes.

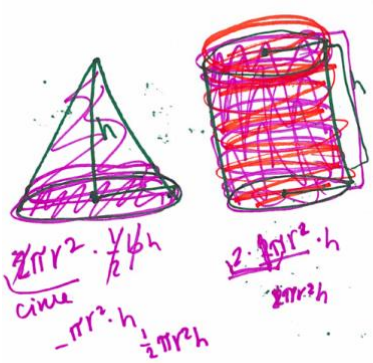

Figure 3. Kimberley's constructed formulas for a cone and cylinder.

Operative Thought with Quantities Re-presented within the Formula

To engage in operative thought with quantities re-presented within the formula, a student considers the formula as consisting of quantitative operations between quantities identified in a context. For the formula $A=2\pi rh$, these quantitative operations could include a quantitative operation (i.e., multiplication) between a quantity re-presented by $2\pi r$ and another quantity re-presented by h that would result in the measure of a quantity re-presented by A. This way of thinking is how Lily thought about the formula $A=2\pi rh$. More specifically, the formula pointed to covariational relationships between quantities; she could anticipate the relationship between the values of the amounts of change between quantities using her formula. For instance, when explaining why she anticipated that the amounts of change in area would be equal for equal changes in height, she stated, "[I]f you add like this much more to the height, you would multiply the height, that equal change in height, times two pi r, and that'd be the equal change in area." She could assimilate this covariational relationship with several different geometric shapes. For example, Figure 4 shows her comparing the cylinder and cone contexts by identifying and comparing changes in surface area (A) for equal changes in height (h), concluding that between the cylinder and cone, the cylinder was the "winner" (i.e., the context in which $A=2\pi rh$ was appropriate) because she identified equal changes in surface area for equal changes in height (unlike with the cone). In this way, Lily engaged in operative thought across situations fluidly.

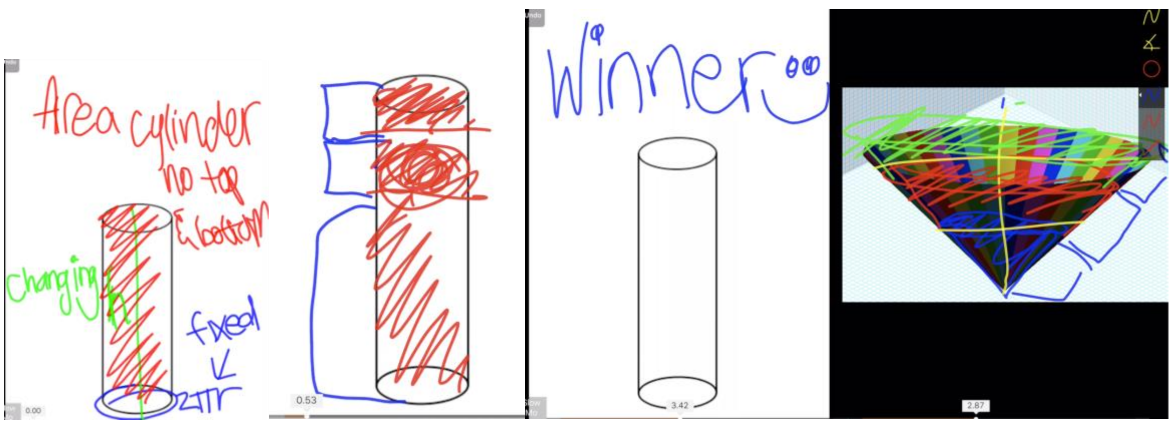

Figure 4. Lily's work on the Cylinder and Cone in The Formula Task.

Discussion

This report includes different ways in which figurative and operative thought might occur, depending on what symbols, groups of symbols, or formulas the students are considering. It also showed that, similar to Moore & Thompson's (2015) expert shape thinking, figurative thought may not always result in problematic conclusions (in the case of Charlotte and the cone), but that there is potential that figurative thought could result in ways of thinking about formulas that do not rely on re-presenting quantitative relationships (in the case of Kimberley's construction of a formula for the surface area of a cone). Lastly, although both Lily and Alexandria's examples of operative thought included covariational reasoning, the definition of operative thought deals with students' capacity to reason quantitatively (not necessarily covariationally) about the relationships rather than relying of associations of formulas with figurative materials (shapes). Alexandria, nevertheless, showed how opportunities for covariational reasoning with perceptual materials might support operative thought. These results are important for researchers to consider when thinking about how undergraduate students' meanings for formulas impact ideas such as what it means to take the derivative or antiderivative of formulas.

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Characterizing Quantitative Structures Students Establish for Real-World Scenarios

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In this report, we characterize a spectrum of mathematical structures of real-world situations. Using data from teaching experiments with undergraduate STEM majors and theories from quantitative reasoning, covariational reasoning, and multi variational reasoning, we build second order accounts of modelers' reasoning with and about conceived quantities. Through these accounts we illustrate four different kinds of structures as means for describing key aspects of how modelers develop their models.

Keywords: Mathematical Modeling, Quantitative Reasoning, Covariational Reasoning, Multivariational Reasoning

Typically, mathematical modeling involves translating real world scenarios into mathematical representations. The mathematical representations can take the form of mathematical expressions, tables and graphs depicting how variables vary with one another (or not), and even diagrams depicting the dynamics of the scenario (e.g. stock-flow diagrams). While tables, graphs, and figures are useful for representing the real-world scenario, the ultimate goal is producing a mathematical expression that is consistent with their previous representations and reasonings about the scenario *and also* representative of the real-world scenario. Quantitative relations govern the mathematical model of a real-world scenario. That is, a mathematical representation of a real-world scenario can be understood as an expression of the relationships among conceived quantities. Therefore, it would make sense to view mathematical modeling through the lens of quantities and relations among quantities (Thompson, 2011; Larsen, 2013; Czocher & Hardison, 2019) in order to find ways to help guide students towards a mathematical expression. However, reasoning with and about quantities doesn't necessarily yield a mathematical expression consistent with the modeler's reasoning *and* representative of the scenario as an end result (Czocher & Hardison, 2019). In milieu of this, we ask: what is the nature of the quantitative relations students establish of real-world scenarios.

Theoretical Orientation

Our research lies within the cognitive perspective of mathematical modeling (Kaiser, 2017). In this perspective, mathematical modeling is considered to be the cognitive processes involved in constructing a mathematical model of real-world scenarios. We define a mathematical model to be the external representation of the relations among the quantities a modeler conceived as relevant to model a real-world scenario. We define mathematical modeling activity as the mental activities involved in creating a mathematical representation of a real-world situation.

Thompson (2011) defines quantification as the "process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship its unit" (p. 37). In that sense, a quantity is a mental construct of a measurable attribute of an object. Quantitative reasoning involves conceiving and reasoning about conceived quantities. Reasoning about conceived quantity can entail operating on conceived quantities and reasoning about how the quantities can vary. Thompson (1994) defines quantitative operation as the "mental operation by which one conceives a new quantity in relation to one or more already-conceived quantities" (p.10). As a result of a quantitative operation a relationship is created: the quantities operated upon along with the quantitative operation in relation to the result of operating (Thompson, 1994). Examples of quantitative operations include combining two quantities additively or multiplicatively and comparing two quantities additively or multiplicatively. Scholars address the following ways individuals can reason about how quantities vary: variational reasoning (Thompson & Carlson, 2017). Co-variational reasoning (Carlson et al, 2002; Thompson & Carlson,2017), and multivariational reasoning (Jones, 2018; Jones & Jeppson, 2020, Panorkou & Germia, 2020).

While variational reasoning involves reasoning about varying quantities independently (Thompson & Carlson, 2017), co-variational reasoning involves "coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al, 2002, p.354). Carlson et al (2002) contributed a framework for the mental actions involved in covariational reasoning. Later, Thompson and Carlson (2017) proposed six major mental operations involved in covariational reasoning. These mental operations are: *no coordination of values*, *pre coordination of values* (envisioning asynchronous changes in variables), *gross coordination of values* (envisioning the general increase/decrease in variables' values), *coordination of values* (coordinating the amounts of change of each quantity), *chunky continuous variation* (envisioning change in variables happening simultaneously but in discrete chunks), *smooth continuous variation* (envisioning change happening simultaneously but smoothly).

Scholars have extended the work of covariational reasoning to multivariational reasoning, which is reasoning about more than two quantities changing in conjunction with each other (Jones 2018; Jones & Jeppson,2020). Jones and Jeppson (2020) identified the following mental actions attendant to multivariational reasoning: recognizing dependence/independence, reduce into isolated covariations, covariational reasoning, switch variables/constants, imagining simultaneous changes in inputs, coordinating multiple simultaneous changes, coordinating qualitative amounts of change, coordinating numeric amounts of change, articulate the type of relationship, identifying the order of effect between variables, and recognize a chain of influence.

Borrowing ideas from the aforementioned theoretical underpinnings, we define establishing structure for a real-world scenario to be creating a network of quantitative relations among the quantities the modeler conceives and recognizes as relevant to modeling the scenario. By network of quantitative relations, we mean the system of quantitative relations that was created as a result of reasoning about and operating on conceived quantities. In this paper we address the question: What is the nature of the quantitative structures students establish for real world scenarios?

Methods

We present data from a set of three teaching experiments (Steffe & Thompson, 2000) conducted with undergraduate STEM majors at a large university. The overall goal of the teaching experiment was to investigate the role of quantitative and co/multi-variational reasoning in students' conception of real-world situations as they attempted to model those scenarios. Baxil, Pai, and Szeth, each participated 10 interview sessions; each session was approximately 1 hour long. Baxil and Pai were enrolled in differential equations and Szeth was repeating the course. During the interviews, in addition to asking students the meanings they attributed to each symbol they introduced, participants were also probed to unpack the reasonings behind certain decisions they made during their mathematical modeling activity. In this report we present data from the following sessions: Baxil and the Fruit Ripening Task, Szeth and the Disease transmission Task, and Pai and the CI8 Account task. We focus on these sessions because they illustrate the finding that quantitative structures established by modelers may have differing natures.

- *Fruit Ripening:* There is a surprising effect in nature where a tree or bush will suddenly ripen all of its fruit or vegetables, without any visible signal. This is our first example of a positive biological feedback loop. If we look at an apple tree, with many apples, seemingly overnight they all go from unripe to ripe to overripe. This will begin with the first apple to ripen. Once ripe, it gives off a gas known as ethylene (C_2H_4) through its skin. When exposed to this gas, the apples near to it also ripen. Once ripe, they too produce ethylene, which continues to ripen the rest of the tree in an effect much like a wave. This feedback loop is often used in fruit production, with apples being exposed to manufactured ethylene gas to make them ripen faster. Develop a mathematical model that captures the dynamics of the ripeness of the fruit.
- *Disease Transmission*: Suppose a disease is spread by contact between sick and well members of the community. If members of the community move about freely among each other, develop a mathematical model that informs us about the dynamics of how the disease would spread through the population.
- *The CI8 Account*: The competing Amtrak Trust has introduced a modification to City bank's SI8, which they call the CI8 account. Like the SI8 account, the CI8 earns 8% of the "initial investment". However, at the end of each year Amtrak Trust recalculates the “initial investment” of the CI8 account to include all the interest that the customer has earned up to that point. Create an expression that gives the value of the CI8 account at any time t (Castillo-Garsow, 2010).

We retrospectively analyzed the data via building second-order models (Steffe & Thompson, 2000) of students’ reasoning. Since we did not have direct access to participants’ mental activities, the second-order models we constructed are inferences made from the students’ observable activities such as language, verbal descriptions and discourse, written work, and their mathematically salient gestures. Our retrospective analysis consisted of five rounds of data analysis to arrive at examples that illustrate the different natures of the structures present in students’ conception of real-world situations. First, we watched the subset of videos without interruption to observe students’ ways of reasoning about conceived quantities. Second, we paid close attention how they transformed these reasonings into a mathematical expression, or not. Third, we distinguished sessions where a normatively acceptable mathematical expressions were created from those where it wasn’t the case. Fourth, we sought to distinguish the sessions where acceptable mathematical expressions weren’t created according to the level of sophistication in mental actions attendant to co/multivariational reasoning. Fourth, we created annotated transcripts of such scenarios that provided rich description of the modelers mathematical modeling activity. Finally, we built and refined the explanatory models of participants structural conception of real-world scenarios.

Findings

Qualitative Structure – Baxil and Fruit Ripening

Baxil operationalized ripeness of the fruit as “readiness to eat” the fruit. He indicated that the ripeness of the apple is dependent on “rate and time”. By rate, Baxil meant the rate at which the apple will become “ready to eat,” as illustrated below:

Baxil: What I’m assuming is the rate of the apple are ready to eat because not all the apple will be always ready to eat, so I assume that, and the time will be keep increasing because if it keep too short, it's not ready. If it's right about time, it's ready, but if it's too long, it's not ready, as well, so it will be the time between ready and too ready. I would say it's same thing because it's any... It's changing maybe every minute or hours, but I think it's continuously again.

He conceived of the situation to be that not all apples will be ready to eat at the same time and the apples' readiness to eat depends on time. From above, it is evident that Baxil was able to coordinate the directions of changes of time and "readiness to eat", thus engaging in gross coordination of values. He reasoned that as time goes on the "readiness to eat" of the apple will increase and then after a certain time the "readiness to eat" will start decreasing. For Baxil, the "readiness to eat" will start decreasing because an overripe fruit cannot be consumed. This is evident in the excerpt below. He also envisioned this change happening smoothly and continuously (see Figure 1).

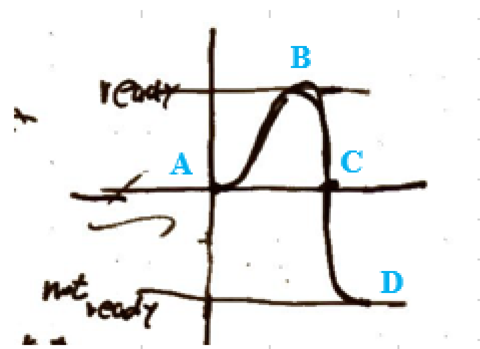


Figure 1. Baxil's graph for "readiness to eat" vs. time

Baxil: It would be negative because it's not ready to eat, so I am assuming it start at 0, and to here is ready to eat [B]. And if I'm here [C], it's not ready to, and that the graph will be from here [C] to here [D], then decreasing, I think, and it will be... because this the time those two... from here to here is ready to eat because one is increasing because giving time to ready to eat, and from here[B] to here is the time you can eat [D], I think. I'm going to assume, and then after you can eat, then you cannot eat anything because it's... I don't know what that word, but it was expired.

Here Baxil explains how the "readiness to eat" of the fruit changes with time. He was mostly involved with the gross coordination of the quantities "readiness to eat" and time and did not produce a mathematical expression that captured the dynamics of the ripening fruit. We call structures of this nature – where no more than the gross coordination of quantities is involved – as *qualitative structures*.

Emerging Quantitative Structure – Pai and Disease Transmission

After reading the Disease Transmission task, Pai immediately reasoned that the account would not change at a constant rate (linear growth) because "the value of what's going to be multiplied by 0.08 changes". This indicates that Pai envisions the amount by which the account grows each period changes. He then wrote down what the values of the account at the end of the first, second, and third year as follows: $S_1 = S_0 + 0.08(S_0)(t)$, $S_2 = S_1 + 0.08(S_1)(t)$, $S_3 = S_2 + 0.08(S_2)(t)$, where S_n is the value of the account at the end of the n^{th} compounding period and $t = 1$ year. Pai continued to reason as:

Pai: Because each year grows 8% times the initial level of the account balance, which is the prior year ending balance. Since the prior year, S_1, is greater than, essentially, S_0. S_2 is going to be taking the S_1 value and adding 8% of that value to it. It will just keep increasing…But if you take the entire account balance over time, it's going to grow at a faster and faster rate."

As he was reasoning he drew the graph of Account balance vs time (Figure 2). Upon probing to discuss the numerical amounts of change, Pai started with \$100.00 and used his expressions from above to find the accounts' value at the end of the first and second year as \$108, and \$116.64, respectively. Thus, Pai was coordinating the numeric amounts of change to the account balance with time. He further articulated that the change of the account size during each compounding period would continue to increase. Although Pai was able to coordinate the amounts of change of the value of the account and time, he was uncertain how to write a mathematical expression that would give the value of the account at any time t. We call structures of this nature – where a modeler has coordinated the amounts of change of the conceived quantities but hasn't translated that into a mathematical expression yet – as *emerging quantitative structures*.

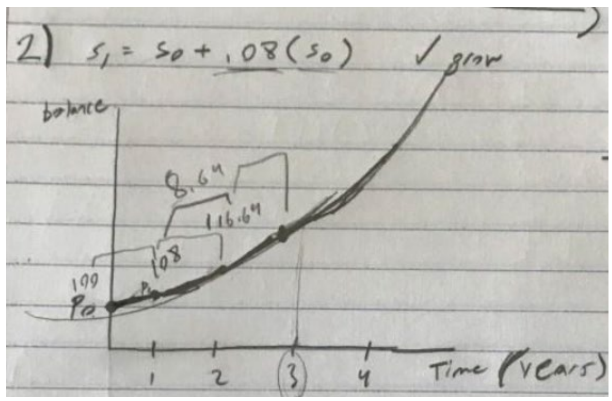


Figure 2. Pai's graph of account balance vs time in years

Quantitative Structure – Szeth and Disease Transmission

Szeth initially wrote down $P(t) = S(t)H(t)$, where $S(t)$ is the sick people, $H(t)$, is the healthy people and $P(t)$ represents the population that becomes sick. After realizing that *P(t)* and $S(t)$ measure the same thing, he changed his expression to be $S'(t) = S(t)H(t)$. When asked why he did so, he reasoned as:

Szeth: Yeah. The big, I guess, driving force was, like I said, these two variables felt like the same thing to me which then the equation doesn't make sense that way. So I was thinking of, well, should I try and change this one [pointing at $P(t)$] or should I try and change this one [pointing at $S(t)$]? So I quickly just look through the wording of the problem, and in the last sentence it says how the disease would spread through the population. So the spreading, that sounds like to me like a rate, how quickly it would spread out, slowly it spreads. So then that led me to change what the equation is equal to. It's equal to the spreading or how quickly people get sick, and then that's based on the interactions between healthy and sick.

As in the except above, Szeth was trying to determine if he should be changing $P(t)$ or $S(t)$ since having them as is doesn't make any sense". When re-reading the task, he realized that since he wants to know how the disease spreads, the left-hand side of the equation should be a rate rather than an amount. He then changed $P(t)$ to be $S'(t)$ because the spread of the disease is dependent upon the interactions between healthy and sick people. So, Szeth's final answer was $S'(t) = S(t)H(t)$. When probed, he indicated that his model assumes that all healthy people who come in contact with the sick people, get sick. When the interviewer asked him how he would modify it to account for only a portion of healthy people who come in contact with the sick people will get sick, Szeth wrote $S'(t) = \alpha S(t)H(t)$ where α is the percentage of interactions that lead to people getting sick. In this scenario, Szeth was able to construct the quantity *the spread of disease* through operating on the quantities $S(t)$, $H(t)$, and α, under two different assumptions. Not only was he able to recognize the dependence among the quantities $S'(t)$, $H(t)$, $S(t)$, *P(t)*, and *α,* but he was also able to translate this dependence into a mathematical expression. We call structures of this nature – where the network of quantitative relations is translated into a mathematical expression – as *quantitative structures*.

Pseudo Quantitative Structure: Baxil and Fruit Ripening

Baxil was asked to draw a graph of the gas produced vs time. Baxil, while drawing his graph (Figure 3), reasoned as "I would say increasing slowly at the beginning, then increasing faster as they are ready to eat because after you're ready to eat, it will produce more instead it didn't ripe yet." The interviewer probed his rationale for why the ethelyne gas production would be faster as the fruit ripens. Baxil explained "When you're not ready to eat, it's just like a little bit amount of the gas, I would think, but after it's ready, it goes faster because everywhere have the gas". Baxil engaged in coordination of three interdependent quantities (amount of ethelyne gas produced, gas production, and time), while maintaining pairwise coordination between amount of gas vs. time and gas production vs. time, and production of gas vs. amount of gas. We can say that Baxil has established a qualitative structure of the situation.

Figure 3. Baxil's graph for ethelyn production vs time

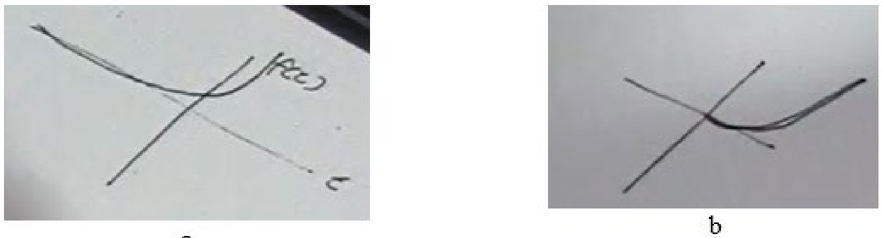
The interviewer then asked Baxil to write an expression for the amount of ethelyne gas produced. He wrote down two expressions and was trying to figure which suited the situation most.

i. Amount of gas produced by the apple which is ready to eat = $e^{rate\ of\ gas*\ time}$
ii. Amount of gas produced by the apple which is ready to eat = $rate^{\ time}$

In expression (i), Baxil conceived of rate of gas to be the "percentage of gas inside the apple". By that, Baxil meant the ripeness to ethylene conversion rate. Whereas in the second expression, he indicated that rate would be "the rate of gas that affect the (ripeness of the apple)". Baxil further indicated that the amount of gas, as represented in the first expression, would be increasing slowly. Whereas in the second expression, the amount of gas would increase quickly. This interpretation was evident in his following explanation:

Baxil: May I make an example like the raw apple there is a little bit of gas like I say 10% of them I guess, so it might be a 20% of them and the next there is something like that and there is a 40% then a 60% it doesn't add to 100% that's the second equation thinking and for the first equation I was thinking if it is 10% the rate won't be changing... I mean not the rate the like the amount then I say like its 10% it might be and depend on the tense it will be increasing by one-tenth, two-tenth, third-tenth, four-tenth... something like that.

The interviewer asked him to draw graphs of the two scenarios, he drew figure a to represent expression 1 and b to represent expression 2. While he attributed the amount of gas in the first one to be increasing slowly, and in the second to be increasing faster, he drew a steeper graph for the first one (Figure 4a).



a b

Figure 4. Baxil's graphs for ethylene production vs time

Here Baxil conceived two distinct measurable attributes of the same object, apple. One was by how much gas is produced by the apple and the other being by how much the gas affects the ripeness of the apple. As a result, he createded two expressions that, despite being mathematically equivalent, behaved different to him in terms of quantities and quantitative operations. He settled on expression (ii) as his final model because, according to him, in the second expression the gas is produced quicker which is most suited for the given situation. Baxil's expression modeling the amount of ethylene gas produced was normatively correct. However, his reasoning evidenced a few kinds of inconsistences. First, Baxil did not justify why he thought the second expression produced ethelyn gas faster than the first. Second, he produced graphs that are inconsistent with the aforementioned reasoning. We call structures of this nature – where the qualitative structure is mapped into an acceptable expression but for incorrect reasons – as a *pseudo quantitative structures*.

Discussion

In this report, we have illustrated four different kinds of structures students may establish for real-world scenarios. For completion, we suggest that it is possible that the modeler does not establish (a quantitative) structure for the real-world scenario. That is, the modeler may have conceived the quantities but has not recognized interdependencies among those quantities, thus explaining their absence from the structural network. We acknowledge that the types of structures reported in this paper are not exhaustive. In addition, the presence of these structures may be limited to students' modeling activities for dynamic systems. These distinct kinds of quantitative structures can be used as a researcher tool to describe the degree of the formality of the network of quantitative relations students established on real-world situations. These descriptions may be used to analyze where the student is in her model developing activity, with quantitative and co/multivariational reasoning as the backbone, complementing existing research on mathematical modeling processes. This in result may provide insights into how educators can guide students into creating a mathematical expression – the favorable outcome of mathematical modeling - as the mathematical representation of a real-world situation.

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“The reason why I didn't like [math] before is because I never felt creative”: Affective Outcomes from Teaching Actions to Foster Mathematical Creativity in Calculus 1

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In this paper, we describe the student-reported affective outcomes from teaching actions of professors involved in a professional development experience to explicitly value creativity in their Calculus 1 courses. Using the four main teaching themes that emerged (Task-Related, Inquiry Teaching, Teacher-Centered, and Holistic Teaching), we further explored the data for affective outcomes resulting from teaching actions that foster student creativity. We observed five distinct affective outcomes: Enjoyment, Confidence, Comfort, Negative then Positive Feelings, and Negative Feelings. Enjoyment and Confidence were the most reported affective outcomes from the creativity-fostering teaching actions. Particularly, Enjoyment was reported the most from Holistic Teaching and Task-Related teaching actions; Confidence was reported the most from Holistic Teaching actions among all the types. Finally, we offer concrete creativity-based teaching actions that have the capacity to build students' mathematical enjoyment and confidence.

Keywords: mathematical creativity, affect, confidence, enjoyment, teaching actions

Learners bring their prior experiences into the classroom that influence how they learn mathematics, who should learn it, and how they feel about learning it; this impacts students' persistence in mathematics and other STEM fields (Ellis et al., 2016). For example, Laursen et al. (2014) found that teaching pedagogies impacted students' affective gains in confidence, intent to pursue more mathematics classes and attitude about mathematics. Furthermore, despite having similar grades, students in Laursen et al.'s study reported different learning gains by gender and pedagogy. That is, these students had a weaker sense of mastery that is not reflective of actual content knowledge. Indeed, affective outcomes such as confidence impact persistence through Calculus II and STEM in general (Ellis & Cooper, 2016). There are numerous aspects of a course that could impact students' affect. In this paper, we concentrate on the teaching actions aspect of a creativity-based undergraduate Calculus I course, because creativity is often overlooked in this course (Ryals & Keene, 2017). We conclude with teaching actions that have the greatest potential to positively impact students' enjoyment of and confidence in doing mathematics.

Background Literature & Theoretical Perspective

We utilize a relativistic perspective of mathematical creativity (Liljedahl & Sriraman, 2006). Through the four “C” model of creativity, we situate our definition of mathematical creativity as “mini-c,” defined as “subjective self-discoveries—the novel and personally meaningful insights and interpretations inherent in the learning process” (Kaufman & Beghetto, 2013, p. 230). This notion of self-discovery necessitates a phenomenological perspective (Abakpa et al., 2017; van Manen, 1990) – we asked participants to define creativity in their own words, report if they felt creative according to their definitions, and list aspects of the course that impacted their levels of creativity. We focus on students' perspectives because actions to foster mathematical creativity have mainly been posited as theory or conjecture (e.g., Sriraman, 2005); actions based on empirical work have rarely been analyzed at the tertiary level (e.g., Levenson, 2011 for 5th and 6th grades).

Affect encompasses a wide range of constructs that involve feeling (McLeod, 1988), including attitudes, emotions, engagement, and so forth (Middleton et al., 2017). Researchers have long tried to develop constructs that distinguish various forms of affect. McLeod (1988) discussed the importance of beliefs, attitudes, and emotions as a trio; he differentiated among them using the dimensions of magnitude, direction, duration, level of awareness, and level. More recently, Middleton et al., (2017) distinguished between trait-like versus state-like affect: the former refers to affect that is longer in duration, relatively stable, and thus not amenable to change easily (e.g., beliefs), while the latter is shorter in duration, oftentimes in reaction to an event, and can be volatile in intensity. This "in-the-moment" nature suggests state-like affect may be more open to influence by the environment and teacher.

We approach affect broadly as many questions remain over the robustness of definitions and various forms of affect (Grootenboer & Marshman, 2016). Following the recommendations in the literature (Hannula, 2002; Schindler & Bakker, 2020), we define an *affective outcome* as any emotions, beliefs, attitudes—whether they are state-like or trait-like— that the students referenced when speaking about the teaching actions they felt fostered their mathematical creativity. In this paper, we address the research question: what affective outcomes do students report from teaching actions of instructors involved in professional development to explicitly value creativity in Calculus I?

Methods

Participants and Setting

Within a larger NSF-funded project investigating fostering mathematical creativity in Calculus I, this study focuses on semi-structured interviews conducted with 34 undergraduate Calculus I students. The larger research project consists of 3 total cohorts of instructor participants from various universities in the U.S. We report only on the two completed cohorts. The research team interviewed 12 students from Cohort 1 (Spring 2019) instructors and 22 from Cohort 2 (Spring 2020). Because different students have different educational experiences or opportunities, we provide students' self-reported gender and racial categories (Adiredja et al., 2015) along with their instructors' self-reported gender and racial categories. Twenty-four students self-identified as female (four bi-racial, five Latina, four Black, two AAPI[1], one Persian, eight White), nine as male (one bi-racial, one AAPI, one Latino/Hispanic, six White), and one as non-binary (White). These students' instructors participated in an online professional learning community in which fostering creativity in Calculus was the emphasis. Nine total instructors have completed participation in the project: three from Cohort 1 and six from Cohort 2. Six self-identified as female (two Latina or Hispanic, three White, one Black) and three as male (one AAPI, two White).

Data Collection, Coding, and Analysis

Participating students were interviewed once by one of the authors for 45-90 minutes towards the end of their Calculus course, prior to taking their final exam. We asked students questions such as "Did you feel creative in this course?", "Why and when do think you were creative?",

[1] Asian Americans or Pacific Islanders

"What have you learned about your mathematical creativity from this course?", and "What aspects of this course contributed to your or your classmates' creativity in the course?" First, we used responses to this last question to code the transcribed interview data for students' references to teaching actions when talking about their own or others' creativity. We created nodes using a combination of descriptive and in vivo coding (Saldaña, 2016). From those nodes, we used the process of theming (Saldaña, 2016) to categorize the teaching actions. The themes that encompass the teaching actions are described in Satyam et al. (submitted) as follows:

- *Task-Related*: any action that mentions properties of a mathematical content task (re-) designed, evaluated, or assessed by the instructor.
- *Teacher-Centered*: any action that was mostly focused on the instructor, whether it be verifying correctness or connecting topics.
- *Inquiry Teaching*: any action that can be linked to inquiry-oriented (or -based) instruction.
- *Holistic Teaching*: any teaching actions that do not require a response from students yet psychologically builds an environment for fostering creativity.

Each teaching theme has sub-types and associated concrete teaching actions. For more details on each theme, sub-types and teaching actions, see Satyam et al. (submitted).

We used nVivo™ (a qualitative analysis computer software) to isolate all student references coded with any of the teaching themes and performed a secondary coding for students' self-reported affective outcomes. Therefore, the affective themes that we have categorized came directly from student-reported teaching actions that contributed to their or their peers' creativity. To code the affective outcomes, we took the same coding approach as for the teaching actions: creating nodes using descriptive and in vivo coding followed by theming. To identify themes, we organized the nodes into groups using constant comparison (Glaser & Strauss, 1967).

Results

Five affective themes surfaced and were titled: *Enjoyment, Confidence, Comfort, Negative then Positive Feelings, and Negative Feelings.* At times, students reported several affective outcomes in one utterance. In those cases, all affective themes appropriate were used to code the student's words. Below, we expand on how we coded for each affective theme and provide interview excerpts. The corresponding teaching action is given in quotation marks and teaching theme in parentheses. The underlined portions indicate the context that situate the quote into the affective theme; the underlined and italicized words show the phrases associated with the theme. Conversational fillers such as "um", "like", "so", "I guess", or "you know" were removed.

Enjoyment

The *Enjoyment* theme includes utterances that reflected students' enjoyment, excitement, interest, appreciation, entertainment, or satisfaction due to the professor's creativity-based teaching action. Moments where students were stimulated or inspired by their professor were also coded into this theme. For example, Optimus's (White male student with Hispanic female instructor) response which stemmed from the teaching action "assign open-ended questions" (*Task-Related*) is shared below.

> I think the most creative I felt was when I did that C++ program to do my homework. <u>*It felt nice* to just do a different way and approach from a completely different angle.</u> I think it gives you a <u>different level of *satisfaction*</u> because it's not like the same mundane objective. And getting those results, you just don't get that satisfaction. <u>*It was cool* for me</u>

to figure something out finally on my own. It's one thing that really turns me off about math; it's like, "damn, I'm learning about like just something some smart dude said" and you know, I don't understand how we got to this point…I'm just spitting out whatever he said. I don't know why. I don't know what it really means…Nobody told me to make the C++ program, nobody told me how to put it together, *it was very satisfying* when I did get the result I wanted.

Confidence

Students' references that mentioned confidence, success in the class, feeling good about themselves, self-efficacy (making use of ideas for next time or feeling they can figure out a problem in the future), or persistence were coded under this theme. Aon (African American/ Black/Nigerian female with Hispanic female instructor) expressed the sentiment below from the teaching actions "allow for freedom in time" (*Holistic Teaching)* and "teach how topics are connected" (*Teacher-Centered).*

I remember the first the very first day of class, she already gave us the problem and I was like "oh, my gosh like, I don't know this." But then as the class went on, it was like, "wow, I see why she taught us this, because it connects to this…[S]he thought that teaching us something else before something else would really help us connect when we learn that next topic. And it really did. Just being very abstract with it really helped me be creative when it came to math. Because *I feel like today, if you give me a problem, I'll be able to think of different ways [it can be, went about]*.

Comfort

Students reported they felt no pressure in being right or wrong, comfortable, encouraged, and their mistakes were valued. They also reported that they were not made to feel dumb and that they did not feel rushed. They felt the classes they were in grew closer as a group and felt like different backgrounds, including nationalities, state residencies, and educational systems were appreciated. All these types of references were coded into this theme. Amelia's (White female with White female instructor) quote fits into this theme because she discusses the comfort in not having to perform quickly with respect to the teaching action of "prompt and encourage different approaches or divergent thinking" (*Holistic Teaching*).

I think that's kind of the reason why I didn't like it before is because I never felt creative. I just felt like I had to do these steps and give these answers. And now understanding that it's all right to take different steps. Before it was always you have to take the quickest steps to get to the answer the quickest, and you have to do everything quickly. And now *I like how it's not rushed.*

Negative then Positive Feelings

There were instances of students reporting initial negative feelings and then a shift to a positive feeling. Experiences below like Sal's (biracial Filipina American female with Latinx female instructor) were coded into this theme. Her quote was coded with the teaching actions "assign writing" (*Task-Related)*; "allow to present in class" and "allow for discussion in class" (*Inquiry Teaching*); "divide class into groups for collaboration" (*Inquiry Teaching*); and "respect differences in the classroom" (*Holistic Teaching*).

[The instructor assigned] very reflective, open-ended questions that necessarily aren't calculus related. I think she just wants this to show…there is a possibility to approach [calculus] differently than what she's teaching or than what may be one of your peers is

> doing… I think she also just wanted to address the fact that everyone’s minds work differently…whether it be more creative or more like critical thinking or more analytical…So people might be moving at different speeds or might be just thinking and approaching of, approaching certain calculations differently…*In the beginning, I’ll be honest, I didn’t really exactly see a point to it.* Just because, it was like very early on in the semester, and I was just like, like, "I wonder *why we aren’t doing math*." But I’m like, "OK, that’s fine. I understand these reflective questions much more than I do calculus. So that’s all right." *[I]n the beginning, I definitely was a little lost in the intent* that she had. But then *looking back* on it, *I definitely see it has helped.* And honestly, *it’s helped us grow closer as a class, I feel, because it was a very good… bonding moment for everyone* because it kind of *forced us to talk,* in a way, and *get to know each other and kind of share our ideas and perspectives*. So that *definitely helped.*

Negative Feelings

Feelings of annoyance, struggle, frustration, or being overwhelmed were coded into the *Negative Feelings* theme. Additionally, comments regarding a negative change in belief in their mathematical skill level were captured in this theme. Bryan (White female with White male instructor) mentioned the negative feelings towards the end of the semester when the instructor made “use of Karakok et al.’s (2020) Creativity-in-Progress Reflection (CPR) on Problem Solving tool” (*Holistic Teaching*).

> At this point, I feel like [using the CPR was] …*one more thing I have to do* and *it doesn’t mean as much to me* because I have seen a little bit improvement on what I rate myself, but sometimes I feel like either *I don’t understand how to use it* or I just feel like *it doesn’t necessarily apply*. Um, and so then *I find it a little bit annoying* to be doing it and also sometimes I just forget *because I forget to do it.* In the beginning it was very helpful and I did think it was good to do that.

Teaching Actions and Affective Outcomes Overlaps

We used nVivo™ to run overlaps of students’ data between the creativity-fostering *Teaching Actions* and *Affect* because we were interested in uncovering the teaching actions that had the greatest number of reported affective outcomes. Table 1 below shows the counts of the quotes that were coded with both the *Affect* listed in column 1 and the *Teaching Action* in row 1. Note that these are not the counts for number of students. That is, one student could have several quotes referring to *Enjoyment* & *Inquiry Teaching.* Table 1 is organized by frequency of the codes for both the *Teaching Actions* (most to least from left to right in row 1) and *Affect* (most to least from top to bottom in column 1). For example, *Enjoyment* is the most reported *Affect*, and *Holistic Teaching* is the most reported *Teaching Action.*

For the purposes of this paper, we will look at the three largest counts in Table 1: *Enjoyment* & *Holistic Teaching* (18), *Enjoyment* & *Task-Related Teaching Actions* (18), and *Confidence* & *Holistic Teaching* (17). Within these overlaps, we look at the most reported creativity-based teaching action to uncover which could be most encouraged to foster these affective outcomes.

In examining the *Holistic Teaching* actions that made students feel creative while also feeling *Enjoyment* or excitement, the action that had the greatest number of references was “prompt and encourage different approaches or divergent thinking.” In the *Enjoyment* and *Task-Related* intersection, students reported enjoyment came mostly from tasks that were “open-ended (i.e., that can be solved in multiple ways).” The top two *Holistic Teaching* actions that students reported affected their *Confidence* positively were “de-emphasize correctness in class” and “use of CPR”.

Table 1. Number of Affect Utterances by Teaching Action Theme

		Teaching Action				
		Holistic Teaching	Task-Related	Inquiry Teaching	Teacher-Centered	TOTAL
Affect	Enjoyment	18	18	14	2	52
	Confidence	17	14	8	8	47
	Comfort	14	7	6	4	31
	Neg then Pos	5	8	3	1	17
	Negative	3	3	1	0	7
	TOTAL	57	50	32	15	

Conclusion

These student experiences point to the promise of incorporating creativity tasks in Calculus I to increase students’ enjoyment, confidence, comfort, and transitional feelings of this gatekeeper class. There were also negative affective outcomes reported, but as Table 1 shows, they were comparatively fewer than the other affect themes. Table 1 also shows that *Enjoyment* and *Confidence* were the most reported affective outcomes from the instructors’ teaching actions. Considering students’ references to *Enjoyment* and *Confidence* together, the creativity-based teacher actions that most promote both affective outcomes are:

- prompt and encourage different approaches or divergent thinking
- de-emphasize correctness in class
- show excitement after student contributions, and
- explicitly encourage students in their creativity.

It appears these four teaching actions have the most potential for practitioners not only for fostering student’s creativity, but also encouraging students’ enjoyment of and confidence in the course. Ellis et al. (2016) found that all mathematically-capable students in their sample of 1,524 lost confidence over the course of their Calculus I course. Thus, encouraging students’ confidence is particularly important for STEM students since those with less confidence are less likely to continue on in STEM (Nugent et al., 2015). As teaching is shown to be a major influence on students’ persistence in school subjects (Rasmussen & Ellis, 2013; Regan et al., 2015), these four creativity-based teaching actions have major implications on persistence in Calculus I. We offer some examples of how to incorporate these teaching actions in the course.

The instructors in our study “prompt and encourage different approaches or divergent thinking,” by soliciting different ways from individual or groups of students. Eb (Asian-American female with Black female instructor) reported on the comfort of choice in different methods or approaches to open-ended questions:

> …definitely the questions that were a little bit more open-ended and not just solve it and find a particular answer. Especially the ones where depending on how you solve it or which identities you’re using, you might come up with something that looks different at the end but it means the same thing, or as long as you solved it correctly using correct rules, you’ll come up with an answer that should be correct and there might be multiple

answers to that. In that case, I really like that because I could pick and choose which one I'd want to use or which ones which ones I'm most comfortable with.

For specific open-ended tasks that can be designed to "prompt and encourage different approaches or divergent thinking" see El Turkey et al. (submitted).

With regard to the "de-emphasize correctness in class" teaching action, students reported the emphasis on the freedom to mess up during the learning process due to a lack of grading for correctness. For example, Ensigo (Mexican female with White male instructor) shared:

> My creative abilities in this class have been a lot better than they had been in calculus one in high school just because…here…it doesn't matter if we have the right or wrong answer. [W]hat I learned about my personal creative abilities is that I have more of a freedom to you know mess up. And it being OK because they're not looking for the right answer.

One way instructors can promote the freedom to mess up is to grade certain assignments for completion and save grading for correctness for more summative assessments.

Students from this study reported that instructors' excitement moved them, as it showed the instructors' investment in their learning. Jennifer (White Female with White Male instructor) said, "[H]e would always get excited whenever we would answer the questions and…whenever we would be understanding. It was just nice to see that he was like rooting for us, (laughing) all the time." We can see that these seemingly small actions can have big impacts on students.

Lastly, instructors can encourage creativity by explicitly acknowledging that creativity is a mathematical skill. Clare (White female with White male instructor) reported:

> [C]reativity...I learned it's a thing. I learned that in math there are ways to be creative. And I think I've started understanding that and using it. But I also understand now why creativity and math is so important because the creative solutions or the creative people are the ones that are the most helpful and are making those innovative discoveries…[T]he exam questions are so open ended and…when we would go through [the answers], no one would necessarily have the same way of going about it. I think that's what helped me understand that there is a creativity level.

This also helps to address the myth that creativity and mathematics have an empty intersection.

Though this study shows existence of positive affective outcomes from explicitly teaching for creativity, we need to collect more data to generalize to the greater undergraduate Calculus I student population in the U.S. Furthermore, in the future, we want to take a more in-depth analysis of affective outcomes by social identity categories such as gender, race/ethnicity, or their intersections. It is important to tease out the experiences of students and women of color because they often report negative affective outcomes that impact their ability to succeed or persist in STEM (Leyva et al., 2021; McGee & Martin, 2011, Trytten et al., 2012). We also want to highlight teaching actions that may promote positive affective outcomes by social identities. Preliminary analysis shows that learning in a course that explicitly fosters mathematical creativity is not a zero-sum game that benefits one group; as seen above, students from many different social identities have reported positive affective outcomes.

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Integral Feedback: An Analysis of Students' Experiences of Calculus Enrichment Lab Sessions

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The calculus sequence has outsized importance with regards to the experiences of students in STEM degrees. As such, non-traditional formats have been introduced to calculus classrooms in attempts to enhance student experience. This work conducts a critical case which analyzes the value of a newly implemented Calculus II course paired with an applied lab component by explicitly seeking to understand what components of the newly designed course were salient to students' experiences and how those align with the motivations behind the course creation. Informed by interpretive grounded theory, we identified three higher-order concepts related to students' experiences in the course: instructor impact, student feelings of confidence in the course and content, and accessible exposure to various applications. We conclude with suggestions for how these themes can provide foundations for further work in understanding student experiences of structural calculus innovations.

Keywords: Calculus II, Student Experience, Lab Calculus, STEM

Despite an overwhelming national need for individuals with STEM degrees (Olson & Riordan, 2012), STEM graduates remain a relative rarity; the National Center for Education Statistics indicates that in the academic year 2015-16, only 18% of bachelor's degrees awarded were in STEM fields (National Center for Education Statistics, 2019). Attempting to explain this gap in the supply of STEM graduates and demand for a STEM-focused workforce has led researchers to identify mathematics- and specifically the calculus sequence - as an important influence on whether STEM-intending students will continue in their chosen STEM field, or whether they will switch out (or be pushed out) of STEM (Steen, 1988; Ellis et al., 2016). The need to ensure that the calculus sequence is oriented toward positive student experiences is key in ensuring both that universities are awarding the quantity of STEM degrees required by advances in STEM industries, and that students of diverse identities are structurally enabled and encouraged to pursue fields that they are interested in. This study examines a course that was a part of a broader campus-wide initiative to improve student experiences in introductory STEM courses at Dunshire University (pseudonym), especially the experiences of students who were not already successful at Dunshire. This course, called Enrichment Lab Sessions, sought to connect calculus students with upper level mathematics and a research mathematician.

The idea of reforming the way that mathematics is taught is not a new one; in 1986, the National Science Foundation put out a report calling for academic institutions to increase the involvement and investment of university faculty in the mathematics education of their undergraduate students (National Science Foundation, 1986). Several such investments targeting reform in the course structure and instructional strategies used in calculus-level courses have shown promise for increasing student learning and engagement (Moore et al., 1987; Young et al., 2011; Kogan & Laursen, 2013), and among them is the idea of having both a lecture and recitation component to a standard calculus classroom (Anderson & Loftsgaarden, 1987; et al., 2015). As educators and administrators contemplate and implement structural reform within the calculus sequence, it is important that they think critically about the impacts of such reform. While studies often look at the "success" of such reforms as being quantifiable boosts in earned grade or pass rate (i.e. Anderson & Loftsgaarden, 1987; Young et al., 2011; Vestal et al., 2015), fostering critical analysis of these programs requires the centering of the narratives of individual student experience when assessing program effectiveness and providing direction for future adaptations. Teaching and learning are simultaneous functions, and thus listening to and understanding student feedback is key in successful program adaptation. This qualitative analysis of how students experienced an Enrichment Lab Sessions (henceforth referred to as ELS) provides such a centering, and enables student-focused and student-informed adaptation of similar corequisite structural reforms. We address the following research question: *How do enrolled students experience and describe the Enrichment Lab Sessions?* In our discussion of these results, we attend to *how the motivations for creating ELS relate to student experiences of it.*

Theoretical Framing

This study is rooted in constructivism (Narayan et al., 2013; Bazeley, 2021) in which each participant in the focus group has a uniquely constructed student experience of ELS. These students are positioned as key informants of the impactful aspects of the instructional innovation of ELS, which aligns with previous work regarding student voice as a necessary component of program evaluation and development (Robinson, 2007). Attending to student voice allows us to contrast how students talked about their ELS experience with the targeted goals of the department and university, and thus takes a critical stance on those goals and the programs themselves through their relevance (or irrelevance) to the students' own experiences. The goal of this analysis is to learn from these students' experiences, and not to generalize on how all Calculus II lab courses are experienced. This case study considers what students found impactful about this course, and how these impactful attributes align or misalign with the motivations for its creation.

Research Design and Methodology

Data from this analysis comes from the broader NSF-funded study *Progress through Calculus*. The overall *Progress through Calculus* project included data collected from twelve institutions, and included classroom observations and analysis of course materials, individual interviews, surveys, and focus groups. As part of this data collection, a focus group with students enrolled in the ELS at Dunshire University occurred during the Spring 2019 semester and was conducted by a research team specifically examining issues of diversity, equity, and inclusion within the calculus sequence. Dunshire is a highly selective private university located in the Southern United States with approximately 6,500 undergraduate students, of whom 41% identify as white, 21% identify as Asian, 9% identify as Black, 7% identify as Hispanic/Latinx, and 10% identify as international students.

Dunshire's implementation of the ELS program was a direct result of an administrative call for departments across the university to enhance the experience of first-year students. Dunshire's mathematics department chose to invest into creating ELS. ELS added a lab component to a traditional lecture section of Calculus II, creating a course structure in which students were in a lecture with their primary instructor, Tara, 3 hours per week, and participated in a lab section 2 hours per week. While most lecture-lab or lecture-recitation structures rely on Graduate Teaching Assistants (TAs) to facilitate the lab or recitation component of such a structure, the lab component of ELS was led by an established research faculty member in the mathematics department, Hank.

Interviews with administrators and faculty confirmed that a prime motivator for the development of ELS was to encourage personal and professional connection between undergraduate students and faculty members- connections which can lead to increased student persistence through heightened social and academic integration within their department or broader institution (Seymour & Hewitt, 1997; Tinto, 1997; Wolniak et al., 2012). These connections can be especially salient for individuals identifying with populations that have been traditionally underrepresented in STEM fields, such as first-generation college students and students from low socioeconomic backgrounds (DiGregorio, 2018).

An additional motivator for the development of ELS was to engage students with mathematical content that existed outside of a traditional Calculus II course structure. In the context of the labs, students were encouraged to learn about STEM field-relevant examples that leveraged the math that they were learning in the lecture portion of the class. In addition to enhancing students' mathematical understanding and providing reason for furthering that understanding through connection with outside STEM concepts (Furner & Kumar, 2007; Young et al., 2011), connecting mathematics to students' lives outside of an educational setting is also important as a measure in working towards equity and engaging students critically with mathematics (Martin, 2003; Díez-Palomar et al., 2006).

Recruitment and Participants

In order to recruit participants for this study, Tara sent an email to students who were enrolled in ELS at the time of the study requesting participation. Each student was assigned a pseudonym (Ana, Audrey, Bella, Cassidy, Madiha, Paulo, Rebecca, and Sarah). While demographic information was not collected on these participants, each participant did self-disclose various elements of their identities and backgrounds with the researchers in the context of the focus group. The focus group contained: five economics majors, one math major, one economics and math dual major, and one environmental science major. In addition, six identified as women, two as international students from Brazil, and one as a student with a dis/ability.

Eight students in total were a part of the roughly one-hour semi-structured focus group (National Defense Research Institute, 2009). The focus group centered on students' experiences in ELS with particular attention paid to how their identities may have related to these experiences, and allowed room for additional interpersonal exchange and attendance to student opinion on ELS. Within the context of the focus group, the interviewers attended to *who* was speaking, and explicitly asked for notions of agreement and disagreement with stated perspectives, and was intentional about making space for less vocal individuals to share. The focus group was audio-recorded and transcribed for analysis. We recognize that the identities of the research team inherently impact the lens through which data and analyses are filtered. The authors of this study represent variation in gender, sexuality, first-generation status, and academic rank, and all identify as white, neurotypical, able-bodied individuals, holding or pursuing advanced degrees in mathematics. Data analysis for this project was informed by the tenets of interpretive grounded theory (Corbin & Strauss, 1990; Sebastian, 2019). This involved the development and refinement of themes from the data through line-by-line open coding, memo-ing, axial coding, and selective coding of the themes.

Results

Three main themes related to how students experience the ELS were identified: *Instructor Impact on Student Experience, Student Feelings of Confidence in the Course and Content,* and *Accessible Exposure to a Variety of Applications*. A total of ten sub-themes, which exist under three higher-order concepts, arose from the data as being impactful to student experience. In this section, we describe each of the refined themes and provide evidence for their development.

Instructor Impact on Student Experience

The higher-order concept "Instructor Impact on Student Experience" stemmed from ways in which students attended to how the ELS instructors impacted students' experiences of the course. This came through strongly; many participants began the focus group by comparing Tara and Hank's instructional styles to those of TAs they have had in past math classes. Five subthemes emerged and are described below.

Instructor responsibility for learning. When describing past mathematics experiences at Dunshire, several of the students emphasized that they felt that they were responsible for their own learning. Comparatively, students described ways in which Tara specifically took responsibility for student learning, both through her words and actions. Cassidy described an experience in which her ELS section did poorly on an exam, and Tara responded by taking responsibility and telling the class that perhaps the exam was difficult due to how she wrote it, and not necessarily due to the students' lack of knowledge. Tara also took responsibility for students' learning by prioritizing students' learning above simplistic assurance of content coverage. Madiha mentioned how Tara "[is] definitely much better in making sure that you actually learned rather than just checking the topic off."

Welcoming response to questions. Audrey, Paulo, and Madiha all attended in several ways to how Tara's positive responses to student questions impacted their experience. Paulo specifically attended to Tara's smile in response to questions as being impactful to his time in ELS. Audrey highlighted Tara's positive verbal response to questions as being important as well, saying that "with being able to ask questions and everything, even if it might be a simpler question, she's like 'oh, that's such a good question.'" Madiha expands on the importance of verbal affirmation by saying that if ELS has been instructed by someone who wasn't Tara but "who answered the questions also" in a similar way, then "it would definitely have some sort of a similar experience," implying that welcoming responses to questions were so impactful that they would shape how Madiha experienced ELS whether the instructor was Tara or someone else.

Community of care leading to inclusion. "Community of care" is a term borrowed from DiGregorio (2018), in which faculty create an environment in which "everyone feels safe, supported, and encouraged to express [their] views and concerns" (Kardia & Saunders, 1997). For the students in ELS, ways in which Tara and/or Hank created a community of care and thus an inclusive environment spanned a range of actions and traits. For Ana and Paulo, who are both international students originally from Brazil, a community of care and consequent feelings of inclusion stemmed from Tara's meeting them where they were at in their coursework. Several other students articulated "openness" as being an important trait that Hank and Tara brought to their ELS experience, for "making people feel included" (Bella), "making [students] want to work harder" (Rebecca), and for reducing intimidation in instructor interactions (Sarah). In all of these ways, Tara and Hank worked consciously or unconsciously toward creating a community of care within ELS, and thus enabled a more inclusive environment.

Instructor enthusiasm. With Tara specifically, Madiha, Cassidy, and Ana all mentioned how Tara's enthusiasm for both the content and for student learning was positively impactful to their experience. Cassidy explained from the first day of class Tara had been "so positive" and wanted to "have fun." Ana built directly off of Cassidy's statement by saying that "because we see her actual excitement with math, we feel more excited." In this way, Tara's enthusiasm shaped in some ways Ana's own affect toward the class and the content. Madiha also highlighted that Tara "was super eager about it, like she was actually enjoying math," and that this translated to greater enjoyment for the students. Tara's enthusiasm impacted student experience by influencing the enthusiasm that students themselves felt for the course and for the material.

Instructor as a role model. This subtheme pertained particularly to Tara, and how she functioned within participants' minds as a female role model in a STEM field. Madiha encompassed both her own experience and that of her female friends in ELS by stating that she and "a lot of [her female friends] were inspired by the fact that she is so good and she's in the STEM field and she's female." Other participants discussed how seeing Tara as a role model impacted their experience individually by way of mathematical affect or desire to pursue STEM. Rebecca mentioned that she "[doesn't] see a lot of people who look like me sitting in my big lecture halls. So I think seeing a female [teach ELS], it's really inspiring," a sentiment also shared by Ana. Audrey recognized that "having a female professor… has helped me consider a math major whereas I previously wasn't." In these comments, Tara's status as a role model extended beyond just impacting experience within the ELS classroom, but served to shape how these student participants see themselves as women in their respective fields.

Student Feelings of Confidence in the Course and Content

In this higher-order concept were ways in which students expressed feeling secure about the structure of the class and about their own competency in the material. In contrast to previous courses, many expressed how having a clear outline for the structure and grading system of ELS, the way in which assessments and assignments were lower-stakes, and the mathematical knowledge gained from Tara's lectures were impactful to their experience. The subthemes observed within this higher order concept are described below.

Agency attributed to instructor. The subtheme of *Agency Attributed to Instructor* encompassed ways in which the perceived agency of Tara and/or Hank in determining grades, creating assignments, and disseminating content impacted their experiences of the course. This was often presented in comparison to previous classes that the participants had experienced which had been taught by TAs. Bella noted that the content coverage expectations that TAs are subject to "forces some [TAs] to not teach all of the material," while Tara was able to "take the time to get through all of the material" in a way that still allowed for a positive environment. For Cassidy, the agency that Tara had as a lecturer in ELS was impactful because it allowed for more directed homework and a more subsequently positive testing experience. She appreciated that homework "is [Tara's] problems, because they're a lot closer to what the test is than a textbook." Because of ELS's novelty, Tara and Hank had the agency to determine grades irrelevant of other Calculus II sections, and it is this agency that impacted student experience in the above ways.

Clarity in grading. Students felt further confidence in the course that stemmed from an enhanced knowledge and clarity regarding how their overall course grades and exam grades were determined. Madiha noted that the grading is not "complicated" nor "hard to explain," as it was for previous mathematics courses. Ana also highlighted how this clarity in grading policy allowed her to understand better where she stood quantitatively in the class. This clarity, for Audrey, allowed her to better prioritize the time she spent on coursework, making this clarity in policy impactful to both her experience within ELS and to her overall time-management in the broader context of her education.

Lower-stakes format. A majority of the focus group participants made comments relevant to how the various ways in which ELS presented them with a lower-stakes grade environment made for a more positive experience by way of reduced stress and anxiety related to the course and content. Paulo called the replacement of quizzes with weekly problem sets a "better format"; Audrey agreed, and notes that having problem sets is "so much less stress," and highlighted that this is in direct contrast with Calculus II's "difficult and scary" reputation at Dunshire. Ana and Madiha also noted that removing high-pressure quizzes from the Calculus lab format reduced anxiety around the labs themselves.

Accessible Exposure to a Variety of Applications

Showcasing a variety of applications for Calculus II was a primary motivator for the development of ELS, as mentioned earlier in this paper. Of particular interest within this higher-order concept is the fact that both positive and negative responses emerged regarding both *Content Applications* and *Accessibility of Material*, as described below.

Content applications. One target of the ELS lab component was to expose students to a wide variety of applications for the material that they were learning in their lectures, and students' comments reflected the impact of this attribute of ELS. A majority of students appreciated and enjoyed the variety presented by ELS; Cassidy and Paulo saw the variety of applications as having educational value through giving her "a little bit of exposure" (Cassidy) to subjects she might otherwise not see. In contrast, Bella noted that she felt that the content applications presented were irrelevant and had a negative impact on her experience, indicating she was "not interested" in them and that they didn't apply to her field. The contrast of Bella's perspective with that of the other focus group participants exemplifies how incorporating a variety of content applications impacted students' experiences in different ways.

Accessibility of ELS material. ELS's design was intentional about connecting Calculus II material to those applications in ways that were accessible to students. Rebecca noted that the videos, posted by Hank to be watched prior to the ELS lab, were particularly helpful in allowing students time to "get into those topics and...understand them with the videos." She and Sarah also noted that Hank made the content applications more accessible by breaking that material down during class time "so it's very simple" (Sarah). Bella, however, did not experience this accessibility in the context of the ELS labs; instead, she saw the ELS content applications material as going beyond her "math background" in a way disconnected with Calculus II content. In this way, the (in)accessibility of the ELS material was seen as a negative impact to Bella, while other students noted that the accessibility of the ELS material had a positive impact.

Discussion

This analysis has resulted in an in-depth case study of how eight students experienced the addition of a lab component to a Calculus II course, and highlights ten components of ELS that were positively or negatively impactful to their experiences. From these components, each student constructed their own perceptions of their experience, and built off of one another's perspectives in the focus group setting. These robust themes provide indication of what may be important to student experiences in other non-traditional mathematics course formats. As the need to innovate to ensure mathematics courses are relevant and reflective of best practices is tried and tested, it becomes increasingly important that student's perceptions of their own experiences are elevated and used as valuable tools of assessment.

The emergent themes illustrate many unintentional consequences of ELS's development. The two original goals of ELS were to enable students to connect with a tenured professor in mathematics and to expose students to a variety of applications for the Calculus II content they were learning. While student attention to *Content Applications* and *Accessibility of ELS Material* align with the latter of the goals, the former did not prove to be consciously impactful to students' constructions of their experiences. While students did discuss Hank's "openness" and ability to teach complicated material effectively, there was no mention that they felt that their experience was at all impacted by his specific position within the department, but by his personality and teaching style. Hank was not mentioned exceptionally often in the focus group.

Instead, it was Tara who emerged as being exceptionally impactful to students' constructed experiences of ELS. Tara was mentioned specifically as contributing to all components created under the higher-order concept of Instructor Impact on Student Experience. Her prominence in *Instructor as a Role Model* is particularly salient because of *who* was contributing to this discussion; every focus group participant who identified as a woman noted that Tara functioned in some way as a role model for them within their STEM fields, with several citing ways in which Tara's presented femininity had helped them see themselves in fields that they previously hadn't. The experience of the women in this focus group was universally and positively impacted by Tara's presented gender identity. This speaks to previous work on the importance of representation and the role of shared-identity role models in faculty members within higher education, and particularly for women in STEM fields (Herrmann et al., 2016; González-Pérez, et al., 2020). The findings presented in this paper certainly appear to support the idea that female role models within STEM have a positive impact on the experiences of female students, and set the stage, in the particular context of experimentally structured mathematics courses, for additionally inquiry into how having an instructor with a shared gender identity affects women students' experience in the course.

The higher-order concept *Student Feelings of Confidence in the Course and Content* and its associated themes arose as another seemingly unintentional component of student experience. While ELS was not necessarily designed to reduce student feelings of uncertainty surrounding their grades, the course structure, or their own content knowledge, this does appear to have been a significant factor in how they experienced ELS. Because ELS was a new, largely experimental course, it was very intentionally crafted in terms of structure. Consequently, grading and coursework policies were perhaps more clearly defined than they might be in other course contexts. Clear definitions, in place from the beginning of the course, may have allowed for ease in communicating these definitions around grading and coursework to students, resulting in positive responses from students in regard to *Clarity of Grading* and *Lower-Stakes Format.* While this is purely speculation on the reasoning behind why the themes within *Student Feelings of Confidence in the Course and Content* were so prominent, it does undoubtedly speak to the ways in which the intentionality with which a course is developed can have unintentional positive impacts.

This paper has sought to elevate student perspectives, while simultaneously acknowledging that these themes provide merely an introduction into the kinds of student-centered topics that ought to be considered when designing, implementing, or assessing an alternative-format course. More work is necessary- particularly work aggregated to emphasize the experiences of individuals from traditionally marginalized populations- to affirm these themes as relevant to student experience of mathematics courses with lab components outside of ELS, and to build upon them and enhance knowledge about what students attend to as important to their experience of such courses.

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Developing Geometric Reasoning of the Relationship of the Cauchy Riemann Equations and Differentiation

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This study details the embodied, symbolic, and formal reasoning of two undergraduate students as they attempted to develop geometric reasoning about the Cauchy-Riemann equations with the aid of Geometer's Sketchpad. Participants took part in a task-based interview designed to encourage them to shift between embodied, symbolic, and formal reasoning, and to make connections between the Cauchy-Riemann equations and the amplitwist concept. Results suggest that the participants gradually refined the idea that complex-differentiable functions map small circles to small circles although they did not recognize a distinction between real-linear and complex-linear functions. The shifting of student reasoning was prompted in various ways. In particular, cognitive dissonance seemed to both progress and prevent the participants from refining their conjecture about complex-differentiable functions mapping small circles to small circles; eventually, it led them to doubt their conjecture entirely. Pedagogical suggestions are provided as a result of this data analysis.

Keywords: Cauchy-Riemann equations, Embodied, Symbolic, Formal, Geometer's Sketchpad

Introduction

The National Council of Teachers of Mathematics, the Mathematical Association of America (MAA), and other mathematical organizations have a long history of stressing the importance of connecting algebraic and geometric reasoning in K-16 mathematics. The MAA *Committee on the Undergraduate Program in Mathematics Curriculum Guide to Majors in the Mathematical Sciences* states that "geometry and visualization are different ways of thinking and provide an equally important perspective … [which] complement[s] algebraic thinking … [and] remain[s] important in more advanced courses" (Zorn, 2015, p. 12). Complex analysis is a course that is inherently geometric, though many textbook authors focus on presenting symbolic and formal representations of the concepts (Oehrtman et al., 2019) and omit geometric interpretations. Thus, students enrolled in complex analysis may not have sufficient opportunities to explore geometric interpretations of complex analysis concepts. We summarize how researchers have successfully aided in the development of students' geometric reasoning of both arithmetic and analytic concepts of complex analysis, and in this similar research, we explored the question: In what ways does a team of two undergraduate students reason about and explore the Cauchy-Riemann (C-R) equations and differentiability of a complex-valued function with the aid of a dynamic geometric environment (DGE)? Specifically, we explored the process by which a team of two undergraduate students attempted to develop a geometric interpretation of the relationship of the C-R equations and differentiability of a function with the aid of *Geometers' Sketchpad* (*GSP*). The participants were familiar with the theorem: Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the first-order partial derivative of the function u and v exist and satisfy the Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (Zill & Shanahan, 2015, p. 131). Overall, we found the team connected complex-differentiability, local linearity, the Jacobian matrix, and non-conformality by shifting between embodied, symbolic, and formal reasoning. Additionally, they used *GSP* to make and test conjectures as they deepened their reasoning and navigated between symbolic, embodied, and formal worlds.

Literature Review

Education research literature on the teaching and learning of complex analysis has flourished for the past two decades. Research participants for such work include high school students (Panaoura et al., 2012; Soto-Johnson, 2014), in-service mathematics teachers (Karakok et al., 2015), collegiate students (Danenhower, 2006; Dittman et al., 2016; Hancock, 2019; Nemirovsky et al., 2012; Soto-Johnson & Hancock, 2019; Soto-Johnson & Troup, 2014; Troup, 2019; Troup et al., 2017), and mathematicians (Hanke, 2020; Oehrtman et al., 2019; Soto-Johnson et al., 2015). An overwhelming theme of this research is how research participants conceive of complex analysis concepts geometrically. Some of these studies explore how gesture or body movement can assist students in developing a geometric understanding of the arithmetic of complex numbers or complex-valued functions. Other studies explore how DGEs such as *GSP* can facilitate students' geometric reasoning of the arithmetic of complex numbers or of analytic concepts such as function behavior or the derivative of complex-valued functions. Researchers of these studies also consider gesture in their studies, as it serves as a source of evidence indicating that participants may be engaged in geometric reasoning.

Many of these studies leverage Needham's (1998) geometric interpretation of the derivative of a complex-valued function as an *amplitwist* which has three characteristics: (1) small circles are dilated by $|f'(z)|$, (2) small circles are rotated by $Arg(f'(z))$, and (3) small circles map to small circles, i.e., differentiation is a local property. In their seminal work, Troup et al. (2017) describe how *GSP* helped their undergraduate research participants to move away from perceiving of the derivative of a complex-valued function as the slope of the tangent line. As the students reasoned about the derivative of a linear complex-valued function, they discovered that the function $f(z) = az + b$, where *a* and *b* are complex numbers, rotates and dilates a pre-image by the argument and magnitude of the derivative, which is the complex number *a*. Unfortunately, the undergraduate students did not initially show evidence of generalizing their geometric interpretation to non-linear complex-valued functions because the derivative of a linear complex-valued function is a constant. This was overcome with the aid of *GSP,* where they discovered that the derivative describes the rotation and dilation of an image with respect to its preimage under the function. However, there was no evidence that they fully recognized that the derivative is a local property, just as with differentiation of real-valued functions. In a follow-up study, Troup (2019) addressed this problem by having students who had already discovered the rotation and dilation characteristics of the derivative of a complex-valued function work in reverse. Instead of providing students an algebraic function, he provided the students with the *GSP* mapping of the rational function $f(z) = \frac{2z+1}{(z+i)(1-z)}$ and the students were asked to determine the function. By attending to small discs, the students determined where the function was non-differentiable and constructed the algebraic formula. Soto-Johnson and Hancock (2019) obtained similar results when they integrated the *GSP* labs into the classroom. Thus, both research and practice suggest that DGEs can support students' geometric reasoning about complex analysis concepts.

Theoretical Perspective

We adopted Tall's three worlds of mathematical thinking (Tall, 2013) to analyze and interpret our data to continue this prior work. These three worlds refer to three stages of development in mathematical reasoning: conceptual-embodied (embodied world), proceptual-symbolic (symbolic world), and axiomatic-formal (formal world). Tall (2007) specifies that the embodied stage of development describes how we allow ourselves to see a single representation of an entire concept and the "idea of embodiment conceptualized through thought experiments based on perception and reflection on the properties of objects" (2007, p. 2). The symbolic stage involves performing actions with little conscious effort after practicing them repeatedly and the ability to use mathematical symbols to express ideas which can represent "both a process to be carried out or the thinkable concept produced by that process" (2007, p. 2). The formal stage is largely based on formal proof and definitions. Tall states, "the fundamental shift to the axiomatic-formal world occurs through a shift in attention from the focus on properties that belong to known objects to properties formulated as concept definitions to define mathematical objects" (2007, p. 3). In this study, through guided exploration, participants utilized *GSP* as an interactive technology with the aim that they would develop a geometric interpretation of the C-R equations. By design, participants moved between symbolic, embodied, and formal reasoning to deepen their reasoning. We acknowledge and use physical experiences of the learner as evidence for knowledge rather than unobservable evidence. This includes *GSP* interactions where learners can visualize, create, and manipulate a physical representation for points, lines, mappings, etc. as they attempt to abstract their understanding of concepts in complex analysis.

Methods

Setting and Participants

Participants were recruited as volunteers from the 11 undergraduate students who had completed an introduction to complex analysis class at a western university. Two students, David and Will (both pseudonyms), participated in a two-hour task-based interview involving *GSP*. Will was an applied mathematics major, David was a pure mathematics major, and both students were seniors at the time of the interview. Neither participant had previous experience with *GSP*.

Task-Based Interview

The tasks for the interview were designed to motivate shifting in different directions between Tall's worlds, given that research suggests that moving from symbolic to embodied reasoning offers different affordances than moving from embodied to symbolic reasoning. The first two tasks were intended to motivate a shift from symbolic to embodied reasoning, the third was intended to encourage a shift from embodied to symbolic reasoning, and the fourth was intended to offer an opportunity to use the previous experiences with symbolic and/or embodied reasoning to shift to formal reasoning. Additionally, the lead researcher conducted the interview and probed as needed based on the students' comments and interactions with *GSP*.

The first task was partially intended to help teach the participants how to use the program. Thus, they were provided with detailed instructions on how to construct the transformation $f(z) = ax + by + i(cx + dy)$, where $z = x + iy$. We chose this form of the function because it renders the partial derivatives for the real and imaginary parts apparent, facilitating potential connections with the C-R equations. Participants might observe that a real-linear transformation can be characterized as "a stretch in [a particular direction], another stretch perpendicular to it, and finally a twist" (Needham, 2009, p. 208), which results in an infinitesimal circle mapped to an infinitesimal ellipse under these transformations. This family of real-linear functions also facilitates connections to the Jacobian matrix and how it relates to the geometric behavior revealed via *GSP*. In particular, the family of functions above can also be expressed as a matrix transformation $f(z) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$. Geometrically, these real-linear functions map circles to ellipses (if the matrix is invertible; otherwise, circles will map to lines or points). If the C-R equations are satisfied, the Jacobian matrix $\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{dx} & \frac{\partial v}{\partial y} \end{bmatrix}$ becomes $\begin{bmatrix} \mathrm{A} & -\mathrm{B} \\ \mathrm{B} & \mathrm{A} \end{bmatrix}$ for some values of A and B. If this transformation is complex-differentiable, that is, locally complex-linear, the matrix transformation must be a composition of a rotation and a dilation, and nothing else. Thus, if the C-R equations are satisfied, the two stretch factors are ensured to be equal, so that such functions map infinitesimal circles to infinitesimal circles rather than ellipses. The total effect is that the C-R equations ensure that the transformation is a rotation and a dilation. Therefore, connections to the Jacobian offer another possible path to the amplitwist concept. Given the values of $a, b, c,$ and d can be varied with *GSP*, this family of functions includes all possible complex-linear functions $f(z) = az + b$, which are complex-differentiable and satisfy the C-R equations. This was intended to mirror previous research.

In the second task, the participants constructed the function $f(z) = z^2 - 4\bar{z}$, which has real part $f_a(x, y) = x^2 - y^2 - 4x$ and imaginary part $f_b(x, y) = 2xy + 4y$. After the participants constructed this function, they observed its geometric behavior, attempted to determine points where the function is complex-differentiable to calculate the partial derivatives, explored how these values are related to the geometry, and determined at what points, if any, the C-R equations are satisfied. The function for this task was purposefully non-analytic, so participants experimented with non-linear examples where the C-R equations are *not* satisfied. For the third task, the interviewer constructed the function $f(z) = z^2$ in *GSP* ahead of time and the equation was hidden from participants. The interviewer directed them to explore and make observations about the geometry of this function in *GSP* without knowing the function. Like previous tasks, the interviewer asked the students to determine at what points, if any, the function is complex-differentiable, to calculate the partial derivatives at a particular point, and to identify points where they believed the C-R equations are satisfied. Unlike previous tasks where the students could use both the algebraic formula and the geometric representation to answer these questions, here, they only had use of the geometric data. Finally, the participants were asked to try to determine an algebraic formula for this function based on the geometric data. The third task was intended to encourage participants to move from the embodied world to the symbolic world. In the fourth and final task, the students were asked to connect what they learned from the previous tasks to the theorem stating that if the real and imaginary parts of a complex-valued function are real-differentiable at a point, then the complex-valued function itself is complex-differentiable if and only if the C-R equations hold at that point. As such, participants were asked to use their collective embodied and symbolic experiences to make sense of the theorem relating the C-R equations to complex-differentiability. Thus, this task was intended to encourage a shift from either embodied or symbolic (or a combination thereof) into the formal world.

Data Collection and Analysis

Data were collected via a video recorder and screen-capture software, two mediums which both recorded video and audio data. The participants were seated side-by-side at a desk in front of a laptop computer. The video recorder was placed on the opposite side of the desk facing toward them, intended to capture the physical gestures of the participants. The screen-capture software simultaneously recorded all the actions the participants took on screen with *GSP*. As such, our transcribed data includes utterances by each participant, sensorimotor actions such as gesture, and the technological actions they performed within *GSP,* including mouse and touchpad actions. The research team then analyzed this transcription for occurrences of where the participants engaged in symbolic, embodied, or formal reasoning and shifted their reasoning. We also documented episodes related to the participants' characterization of the C-R equations themselves, of complex-differentiability, or connections drawn between these two concepts.

Results

We found that the nature in which David and Will shifted between worlds in their attempts to characterize the geometric interpretation of the C-R equations and differentiability of a complex-valued function was influenced or prompted by the following: their *GSP* explorations, board work, their intuition, the interviewer's probing, David's and Will's interactions, and moments of cognitive dissonance.

David's and Will's development of the idea of local linearity appeared to be driven by a combination of symbolic and embodied reasoning. During Task 1, David initially conjectured that a function is complex-differentiable if "the output moves smoothly with the input" and Will conjectured that a function is complex-differentiable if "it's not jumping around." After trying to no avail to write this function symbolically as a complex-linear function $f(z) = az + b$, the participants realized that the function they were investigating was not complex-differentiable. In *GSP*, they changed the parameter values so that the function satisfied the C-R equations and observed that the output was a circle rather than an oval. Thus, with a blend of embodied and formal reasoning, they formed a conjecture that complex-differentiable functions map circles to circles. They returned to this idea when they investigated $f(z) = z^2$ via embodied reasoning in *GSP* in Task 2. The participants appeared to realize in Task 2 that their previous characterization of differentiability as a smoothly moving output better depicts continuity than differentiability. They noticed that $f(z) = z^2$ did not map a circle around the origin to a circle and still shortly afterward articulated that complex-differentiability means that circles map to circles. Noticing, via embodied reasoning, that one of their circles mapped to a triangular shape, David claimed that the function is nowhere differentiable. Unsure about this conjecture, Will continued interacting with *GSP*, made embodied observations, and redirected the team to determine at which points the function is complex-differentiable. This motivated the participants to consider mapping smaller and smaller circles, where they noticed, via embodied reasoning, that the output becomes more circular as the input becomes smaller as shown in Figure 1 and 2. After this observation, the participants operated under the conjecture that complex differentiability means that small circles map to small circles, rather than just circles mapping to circles.

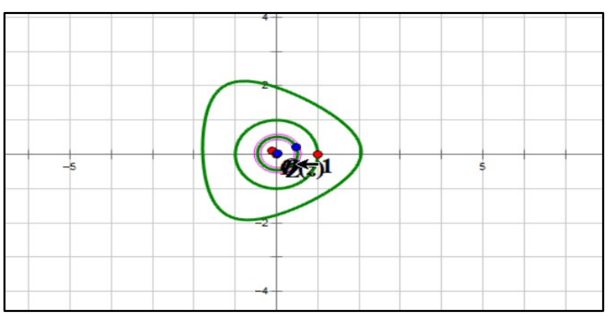

Figure 1: Exploring the mapping of smaller and smaller circles.

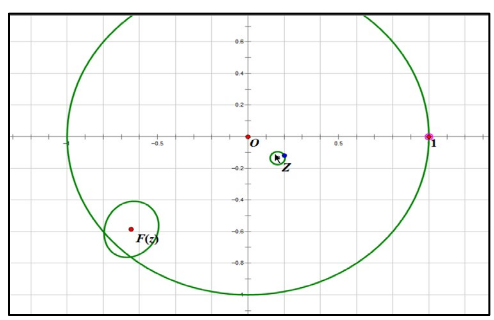

Figure 2: "More circular" output $f(z)$.

Our second finding emerged while the participants explored Task 1 in the embodied *GSP* world. The participants claimed their current function was linear because the function could be represented as a matrix, and believed that, because the function was linear, they could write it in the symbolic form $f(z) = Az + B$. The interviewer then asked them to write the function in this form. Shifting to the symbolic world, David wrote the parameter values in matrix form and converted to the equivalent linear expressions $u = ax + by$ and $v = cx + dy$. David struggled to find workable values for A and B, as the function was not complex-linear. David's confidence in his symbolic manipulations created cognitive dissonance between their previous embodied investigation and the current symbolic board work. This cognitive dissonance further caused David to express a disconnect between his symbolic and his formal reasoning, when he claimed that all matrices are associated with linear functions. He did not recall that this theorem applied specifically for transformations from $\mathbb{R}^n \to \mathbb{R}^m$. As such, this could be an instantiation of *Thinking Real, Doing Complex* (Danenhower, 2000), where David applied the theorem to transformations from $\mathbb{C} \to \mathbb{C}$ without acknowledging the distinction between real-linear and complex-linear functions. His symbolic and formal conclusion appeared to be that their function could not be written as $f(z) = az + b$, and therefore not every linear function is differentiable. However, he remained certain that there was a correspondence between 2 by 2 matrices and linear transformations and could not resolve this seeming contradiction.

In Task 3, only given the embodied representation of $f(z) = z^2$ in *GSP*, the participants seemed to shift between embodied and symbolic reasoning, specifically utilizing embodied reasoning in service of a symbolic goal. David first observed the non-conformality of the function and how it "wants to wrap around zero," and then stated a symbolic goal of determining the function formula. To determine the function, the team transitioned back to embodied reasoning, as they conjectured that they might still be able to make the output a circle. Much of the participants' embodied reasoning here seemed driven by an embodied-formal version of the symbolic-formal epsilon concept: small circles map to small circles. Similarly, Will also made the embodied observation that the function was non-conformal at zero. In Task 4, David noted a conflict between the idea developed in previous tasks that "functions are complex-differentiable if and only if they map small circles to small circles," and their observation from Task 3 that "$f(z) = z^2$ is nonconformal at zero and thus does not send a small circle around zero to another small circle." Although he originally thought that functions that are complex-differentiable have something to do with the preservation of angles or circles under the transformation, here he concluded that he had no idea how to characterize the relationship between the C-R equations and the geometry of complex-differentiable functions. As the interview was concluding shortly after, they abandoned some of their previous incomplete but salient conjectures.

Discussion and Teaching Implications

Although the participants did not formalize a geometric interpretation of the C-R equations and complex differentiability by the end of the interview, they made progress which can be leveraged in the classroom. The shifts between worlds were related to the participants' mathematical knowledge and conjectures in three different ways, as summarized in Table 1. First, shifting from symbolic and embodied reasoning to an embodied-formal blend, the participants recognized the importance of small circles mapping to small circles to determine local linearity. This occurred when they realized that their symbolic reasoning did not align with their *GSP* output. In classrooms, labs based on these tasks might lead students to draw similar conclusions through guided moments of cognitive dissonance. The participants' interactions in *GSP* with $f(z) = z^2$ may have motivated David and Will to refine their conjecture to focus specifically on small circles. David noticed that a circle mapped to a triangular shape, which did not fit their conjecture of circles mapping to circles. However, Will doubted this conclusion and noticed that smaller circles did map to circles, at which point they modified their conjecture.

Table 1. Summary of findings.

Participant Source(s)	Participant Destination(s)	Prompt/Trigger	Participant Result
Symbolic/ Embodied	Embodied-Formal	Dissonance between board work and *GSP* output	Differentiability = circles map to circles
Embodied/ Symbolic	Symbolic/ Formal	Dissonance between inability to write $f(z) = az + b$ and (assumed) theorem that states a correspondence between linear transformations and 2 by 2 matrices	Wrote function on whiteboard as a matrix and attempted to determine values $f(z) = az + b$. This failed.
Symbolic/ Embodied-Formal	Embodied-Formal	Dissonance between the fact $f(z) = z^2$ is differentiable at zero but $f(z) = z^2$ doesn't map circles to circles at zero	Maybe differentiability ≠ circles map to circles

Second, David and Will transitioned from embodied and symbolic to symbolic and formal reasoning, in their attempt to show that they could write their *GSP* function as a 2 by 2 matrix, and therefore also as $f(z) = az + b$. This shift occurred after they recalled an isomorphism between 2 by 2 matrices and linear functions, though they did not realize that this theorem only holds for linear transformations from $\mathbb{R}^n \rightarrow \mathbb{R}^m$ and not for complex-linear functions from $\mathbb{C} \rightarrow \mathbb{C}$. We should additionally point out that many of the theorems in linear algebra apply specifically to real-linear functions, but not necessarily to complex-linear functions, as in the case of the theorem our participants attempted to recall. While it is encouraging that David recalled a theorem from linear algebra establishing the correspondence between matrices and real-linear functions, neither participant distinguished real-linear functions from complex-linear functions during the interview. Such a student conception is an opportunity for instructors to help students recognize this difference. In our study, we gave examples of functions that are real-differentiable but not complex-differentiable. Perhaps providing examples of functions that are real-linear but not complex-linear could aid to further distinguish between these two concepts.

Third, in Task 3, the participants utilized symbolic reasoning in conjunction with an embodied-formal blend to draw the embodied-formal conclusion about the non-conformality of $f(z) = z^2$ at zero. They considered this observation as a potential contradiction to their idea that small circles should map to small circles at complex-differentiable points. Instead, they started investigating the preservation of angles under the transformation, but they could not explain a perceived contradiction. It seems this task motivated the participants to connect to ideas embodying conformality, but they did not provide evidence of noticing a connection to the derivative value of the nonconformal point of $f(z) = z^2$.

Our results reinforce the importance of classroom discussions about the geometric implications of a derivative value of zero, the particular case where infinitesimal circles do not necessarily map to other infinitesimal circles. Future research could investigate how such classroom discussions can facilitate a complete geometric interpretation of the C-R equations. Another line of inquiry is to investigate mapping squares instead of circles as seen in Needham (1998) to leverage conformal mapping ideas.

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Analyzing the Structure of the Non-examples Contained in the Instructional Example Space for Function in Abstract Algebra

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The concept of function is critical in mathematics in general and abstract algebra in particular. We observe, however, that much of the research on functions in abstract algebra has focused on specific kinds of functions, including binary operation, homomorphism, and isomorphism. Research focusing on the function concept itself – and such fundamental properties as well-definedness – is exceptionally scarce, particular in abstract algebra settings. To this end, in this short report we describe our investigation of the instructional example space for function in abstract algebra, with particular attention to the non-examples related to the fundamental, definitive function characteristic of well-definedness. By conducting a textbook analysis and semi-structured interviews with mathematicians, we impose a productive refinement of the non-examples of function with well-definedness issues by defining and illustrating the categories of 'equivalent representations' and 'multiple rules.' We conclude with a discussion of the theoretical and practical applications of these categories.

Keywords: function, abstract algebra, example space, non-examples

The function concept is critically important in mathematics and, accordingly, is a core topic in the secondary and undergraduate mathematics curriculum (e.g., Bagley, Rasmussen, & Zandieh, 2015; Oehrtman, Carlson, & Thompson, 2008). In abstract algebra, functions play a fundamental role in the development of such topics as homomorphism, isomorphism, and binary operations. While there has been a fair amount of research examining students' reasoning about functions in abstract algebra, nearly all of it has examined particular types of functions (e.g., Brown et al., 1997; Larsen, 2009; Leron, Hazzan, & Zazkis, 1995; Melhuish et al. 2020; Rupnow, 2019). Aspects of the function concept itself, such as what students might need to attend to when determining what is and is not a function, have been relatively unexplored.

The definitive function property of well-definedness[1] is of particular importance. Melhuish and colleagues (2020), for example, reported that only 2 of their 18 participants' concept definitions of function addressed well-definedness. They offered an illustrative example of the potential implications of such a definition in which a student classified a non-example of function as a homomorphism. This calls attention to the need to better understand well-definedness and how students reason about it. We note, however, that well-definedness is currently not well-documented or coherently characterized in the literature, particularly in the context of advanced courses like abstract algebra. Additionally, research on the constituent parts of function concept itself (as opposed to specific types of functions, like isomorphisms) is scarce. We propose that a productive way to begin addressing this issue is to develop a clearer image of the various examples of functions and non-functions that students encounter and are expected to reason about in an introductory abstract algebra course. To this end, here we investigate what the key function property of well-definedness entails by examining the contents and structure of the instructional example space (Watson & Mason, 2005; Zazkis & Leikin, 2008) for function in

[1] The notion of *everywhere-definedness* is also of fundamental importance to the function concept. Due to space constraints, in this short report we focus only on well-definedness.

abstract algebra. Our research question is: what examples of function are students expected to reason about in introductory abstract algebra, and how might we productively classify them?

Literature Review

The overwhelming majority of function research focuses on a covariational approach to functions (e.g., Carlson, 1998; Carlson et al., 2002; Oehrtman, Carlson, & Thompson, 2008). However, covariation is often not relevant in an abstract algebra setting because it "superimposes an ordinal system on function, which does not underlie many of the discrete structures in abstract algebra" (Melhuish & Fagan, 2018, p. 22). Thus, the majority of research on functions in the math education literature does not account for the ways in which students must reason about functions in abstract algebra, highlighting the need for research on the function concept in abstract algebra settings.

We focus here on the fundamental, definitive notion of well-definedness (also called 'univalence'), which states that each element of a function's domain can map to no more than one element of the codomain. The majority of studies of function-related ideas in abstract algebra acknowledge well-definedness but do not directly examine these properties in detail. For example, Melhuish and colleagues (2020) prompted abstract algebra students to state their personal concept definition for function and list several examples. While the researchers noted whether or not well-definedness was included in some form in the students' definitions, they did not investigate students' reasoning with or conceptions of it. Many other studies have also generally called attention to the importance of the underlying function concept for understanding such topics as binary operation (e.g., Brown et al., 1997; Melhuish, Ellis, & Hicks, 2020; Melhuish & Fagan, 2018), homomorphism (e.g., Hausberger, 2017; Rupnow, 2021), and isomorphism (e.g., Leron, Hazzan, & Zazkis, 1995; Nardi, 2000) but have similarly stopped short of explicitly addressing well-definedness.

There are, however, a few studies that do address well-definedness more directly. We note two themes from these studies. First, the concept's nuance creates some difficulties for students. As Melhuish and Fagan (2018) explained, "students may not identify all required properties in their concept images and, for example, miss the requirement of well-definedness" (p. 23). Additionally, students struggle to articulate what it means and why it is important (e.g., Even, 1993; Even & Tirosh, 1995) and typically associate it with procedural conceptions of the vertical line test (e.g., Clement, 2001; Kabael, 2011; Thomas, 2003). Second, students have difficulties adapting well-definedness (and the vertical line test) to functions whose domains are not the real numbers (e.g., Dorko, 2017; Even & Tirosh, 1995). We note that the vertical line test is of limited use in abstract algebra as many functions do not usually lend themselves to a useful graphical illustration (which is required for the vertical line test). Thus, much of the literature on well-definedness focuses on students' use and understanding of a procedure that is of very limited use in abstract algebra. We also found no studies directly examining the notion of well-definedness and its use in abstract algebra settings. Thus, in this paper we elaborate the notion of well-definedness to provide additional insight into the nature of the function concept in abstract algebra.

Theoretical Perspective

We use examples of the function concept to gain insight into the nature of the concept of well-definedness. Following Watson and Mason (2005), we interpret 'example' inclusively to mean any specific illustration of an abstract mathematical principle, concept, or idea. This might include exercises, diagrams, and, importantly for this study, non-examples. Non-examples are particularly insightful because they "demonstrate the boundaries or necessary conditions of a concept" (Watson & Mason, 2005, p. 65). They showcase the essential aspects and features of definitions (such as the feature of well-definedness in the definition of function) by illustrating what happens when these features are not satisfied (therefore highlighting their importance). In this short report, we use non-examples as a means to investigate the fundamental notion of well-definedness.

We operationalize examples in this study by employing Watson and Mason's (2005) notion of *example space*— that is, the *content* and *structure* of the examples that are associated with a particular concept. Researchers have found that constructing models of both students' and experts' example spaces affords valuable insights into their thinking because one's example spaces "mirror their understanding of particular mathematical concepts" (Zazkis & Leikin, 2008, p. 131). Watson and Mason (2005), distinguishing between between different kinds of example spaces, defined the *conventional example space* as the collection of examples "as generally understood by mathematicians and as displayed in textbooks, into which the teacher hopes to induct his or her students" (Watson & Mason, 2005, p. 76). Zazkis and Leikin (2008) proposed a useful refinement of the conventional example space, distinguishing between *expert example spaces* and *instructional example spaces*. Expert example spaces display the "rich variety of expert knowledge" whereas instructional example spaces involve what is "displayed in textbooks" and used in instruction (Zazkis & Leikin, 2008, p. 132). Watson and Mason (2005) noted that one of the ways in which students can extend their personal example space—and thus also extend their understanding of the associated concept—is by reasoning about the contents of the conventional example space. This calls attention to the potential for examining the instructional example space for function in abstract algebra to identify its key aspects.

Example spaces, including the instructional example space, are not just lists of examples (the *contents*), but also include the categories by which the (non-)examples in these lists might be productively organized (the *structure*). Indeed, we note that many papers that operationalize example spaces are fine-grained analyses of collections of examples; the analyses are then based upon researchers' perceptions and inferences of how these collections are organized. A key point here is that the *structure* of the instructional example space is consists of researchers' inferences – based upon the input of experts – about how the *contents* might be productively and coherently organized. Researchers can base these inferences upon the explanations and rationale that an experts use to describe particular (non-)examples. Inferences might include, for instance, researchers' views of (1) the purpose served by an example (such as the attributes that make it exemplary), or (2) important distinctions between (non-)examples in a given collection (and what aspects of the associated topic these distinctions might correspond to). In particular, in this short report we infer a *structure* for the instructional example space for function by considering the explanations and rationale offered by abstract algebra instructors (textbook authors and other algebraists) to describe the non-examples that comprise its *contents*.

Methods

We employed two methodologies to examine the instructional example space for function in abstract algebra. First, we conducted a *textbook analysis* because (1) the instructional example space, by definition, contains the examples in textbooks, and (2) textbook analyses can provide insight into "how experts in a field … define and frame foundational concepts" (Lockwood, Reed, & Caughman, 2017, p. 389). Accordingly, while the primary purpose was to identify the non-examples in the instructional example space (the *contents*), we were also attentive to insights in the textbooks regarding how experts might organize these non-examples (the *structure*). In total, we collected data from 14 abstract algebra textbooks. Four of the textbooks came from Melhuish (2019), who identified the four most popular abstract algebra textbooks used in the United States, while others were compiled from the textbooks used in introductory abstract algebra courses at the top 25 ranked universities in the United States (National University Rankings, n.d.). Lastly, we included five textbooks from our personal textbook libraries. (A listing of the textbooks we analyzed is included in a section called 'Bibliography of Textbooks' that appears after the references.) We then created a list of terms (informed by the literature and our knowledge of abstract algebra) related to function (e.g., map, correspondence, well-definedness, etc) and examples of function that arise in abstract algebra (e.g., binary operation, homomorphism, isomorphism). We then collected the sections in each textbook related to these terms via a digital PDF file of the textbook (when available) or by scanning the desired sections from a hard copy of the textbook. To analyze the data, we followed Creswell's (2012) method for identifying and interpreting themes in qualitative data, beginning by making notes on first impressions of the data. There were two kinds of excerpts we sought to identify: (1) those containing non-examples of function, and (2) the authors' associated descriptions and explanations related to a given non-example. We then administered codes that described particular characteristics of these non-examples and looked for new codes that arose as well as elaborations to existing codes; these additional codes were continually refined and revised as coding progressed.

Second, we conducted a series of *semi-structured interviews* (Fylan, 2005) with mathematicians as a way to follow up on conjectures we developed—as well as points that needed clarification—from the textbook analysis. Semi-structured interviews were important for our objectives because they allow the interviewer to "address aspects that are important to individual participants" (Fylan, 2005, p. 66) and thus allowed us to flexibly pursue emerging themes we inferred related to the structure of the instructional example space. Indeed, the primary purpose of these interviews was to gain insight into the *structure* of the instructional example space (though we were also open to identifying additional *contents* as well). The five mathematicians who participated (whom we refer to as Professors A, B, C, D, and E) were all tenured or tenure-track faculty members at a midwestern Research 1 university who had taught an abstract algebra course in the last five years. The prompts used in this interview were informed by the textbook analysis. The interviews were about an hour to an hour-and-a-half in length and allowed the first author (the interviewer) to ask clarifying questions about themes and comments that emerged in the textbook analysis and group interview. All mathematicians participated in at least one individual interview; Professor B participated in two and Professors A and E participated in three. To analyze the data from these interviews, we again used Creswell's (2012) method. One distinction, though, was that this analysis was more targeted and made use of the codes from the textbook analysis. Iterating Creswell's (2012) procedures enabled us to clarify, refine, and elaborate these existing codes. These emerging hypotheses about the key structural elements of the non-examples in the instructional example space were then iteratively elaborated and refined.

Results

We classified a non-example as having a well-definedness issue if there was an element of the (proposed) domain for which there were at least two corresponding images contained in the (proposed) codomain. For example, consider the rule $f: \mathbb{Z}_6 \to \mathbb{Z}_7$ given by $f([a]_6) = [a+1]_7$ (this is a special case of an exercise in Hodge, Schliker, & Sundstrom, 2014, p. 134). Then $f([0]_6) = [0+1]_7 = [1]_7$ and $f([6]_6) = [6+1]_7 = [7]_7 = [0]_7$. Since $[1]_7 \neq [0]_7$ in $\mathbb{Z}_7$, we have that $f([0]_6) \neq f([6]_6)$ but $[0]_6 = [6]_6$ in $\mathbb{Z}_6$. Thus, f is not well-defined. Other non-examples that we classified in this category are displayed in Table 1.

Table 1. Non-examples of function with well-definedness issues.[2]

Non-example	Source
1. $\phi: \mathbb{Q} \to \mathbb{Z}$ given by $\phi\left(\frac{a}{b}\right) = a + b$	Gallian (2017, p. 21)
2. A is the union of two subsets A_1 and A_2 f from A to the set $\{0, 1\}$ where f maps elements of A_1 to 0 and elements of A_2 to 1	Dummit & Foote (2004, p. 1-2)
3. $f: \mathbb{Z}_4 \to \mathbb{Z}_6$ given by $f([x]_4) = [x]_6$	Beachy & Blair (2019, p. 57)
4. □ defined on the set $\mathbb{R}$ by: $a□b$ is the number whose square is ab	Pinter (2010, p. 20)

We further refine these non-examples based upon a distinction in the way the experts in our study discussed them. For example, when commenting on non-example 1 (Table 1), Professor A noted that "there is a clean procedure … the formula as written, it looks like it's cut and dried and well-defined. Um, it's not, but it looks like you're getting, an unambiguous output." Consider also non-example 4, which, we note, is equivalent to the proposed correspondence $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $f(a, b) = \pm\sqrt{ab}$. Making this identification, Professor A further noted that, unlike non-example 1, "there's no way you could write this formula [for f] down and as you were writing it, think your output is unambiguous." He concluded that, compared to non-example 1, non-example 4, "really does demand a different treatment." Observations of this kind formed the foundation for our refinement of the intstructional example space. We represent this distinction by introducing the categories of *equivalent representations* and *multiple rules*, which we elaborate in the subsections that follow.

Equivalent Representations

We characterize the *equivalent representations* non-examples as non-examples of function in which (1) elements in the domain can be represented in multiple ways, and (2) these equivalent representations are mapped to different outputs. Textbook authors attended to this distinction as well. Beachy and Blair (2019), for example, noted that "problems arise when the element x can be described in more than one way, and the rule or formula for $f(x)$ depends on how x is written" (p. 56). Indeed, if "there are multiple ways to represent elements in the domain (like in $\mathbb{Z}_n$ or $\mathbb{Q}$), then we need to know whether our mapping is well-defined before we worry about any other properties the mapping might possess" (Hodge, Schlicker, & Sundstrom, 2014, p. 129).

Non-examples 1 and 3 (Table 1) are both examples of proposed correspondences in the *equivalent representations* category. On non-example 3 (on Table 1), for instance, Gallian (2017) explained that ϕ "does not define a function since $1/2 = 2/4$ but $\phi(1/2) \neq \phi(2/4)$" (p. 21). Similarly, Professor D noted that "one half and two fourths, you get different

[2] Some of these non-examples were slightly modified for clarity and simplicity.

answers. So if you get different answers for the same input, it's not a function. […] Any rational number that you pick, has non-unique representations as a fraction." Notice that this comment underscores how different (yet equivalent) representations in the domain (e.g. "one half and two fourths," "for the same input," "non-unique representation") are mapped to different elements in the codomain ("you get different answers"). Other mathematicians made this point as well. For example:

- Professor E: "the function is deliberately taking, a particular presentation. [It] takes a, a particular presentation of the rationals … That's the issue … that's a problem. Like if you're going to, if you're gonna use a representative … then you have to be extra careful."
- Professor C "students tend to look at fractions as a fixed thing and not […] a representative of an equivalence class."

We also note that attending to this kind of well-definedness issues—related to *equivalent representations* of elements in the domain—is of particular importance in abstract algebra because, as Professor B explained, "a large, uh, an important, uh, aspect of abstract algebra is to construct things, by means of, equivalence relations. And, uh, so, the validity of your constructions, depends on checking, that equivalent things are used in the same way." This is a key issue in proofs, especially those involving functions on quotient structures (e.g., $\mathbb{Q}$, $\mathbb{Z}/n\mathbb{Z}$).

Multiple Rules

The *multiple rules* category includes non-examples in which the definition of the rule involves two or more choices of images in the specified codomain for a single element of the domain. For example, we classify non-examples 2 and 4 (Table 1) in the *multiple rules* category. Regarding non-example 4, we note that this rule could also be stated as $f(a, b) = \pm\sqrt{ab}$, and thus the input (2,8), for example, has two outputs (4 and -4) caused by the '$\pm$' part of the rule (and not, for example, by the representation of the input (2,8) in the domain $\mathbb{R} \times \mathbb{R}$). Regarding non-example 2, Dummit & Foote (2004) explained that "this unambiguously defines f unless A_1 and A_2 have elements in common (in which case it is not clear whether these elements should map to 0 or to 1)" (pp. 1-2). Professor B – in a comment that inspired the name of this category – pointed out that "the definition has, two possible values on the intersection of $[A_1]$ and $[A_2]$ … you have to clarify which value you're gonna choose … that's a problem with the *multiple values of the rule*. The primary distinction we inferred between *multiple rules* and *equivalent representations* involves the nature of a 'choice.' For example:

- Professor A: "Where is the *choice* taking place? Is it in your input? Or is it, uh, in the execution of the rule?"
- Professor B: "They are two different types of problems … your [proposed] function could be, um, not well-defined because, the value in the domain is not well-defined, or that you have to make a *choice* in the value of the domain. Or they could be, not well-defined because the value of the output is not well-defined and you have to make a *choice* of that value of the output."

We therefore infer that *equivalent representations* is characterized by a choice in the *domain* (e.g. "it is in your input?," "choice in the value of the domain"), whereas *equivalent* *representations* is characterized by a choice in the *codomain* (e.g. "value of the output") caused by the rule (e.g. "execution of the rule").

Discussion

Using a textbook analysis and semi-structured interviews with mathematicians, we focused on answering the following research question: what examples of function are students expected to reason about in introductory abstract algebra, and how might we productively classify them? Our answer to this question centers on our elaboration of the fundamental notion of well-definedness – these elaborations are summarized in Table 3.

Table 3. Summary of the structure of the instructional example space for function in abstract algebra.

Well-definedness	Description	Non-example	Familiarity
Equivalent Representations	There is at least one element in the domain that has different, equivalent representations; the rule assigns these to different images in the codomain.	$\phi: \mathbb{Q} \to \mathbb{Q}$ given by $\phi\left(\frac{a}{b}\right) = a + b$	Relatively unfamiliar to students
Multiple Rules	There exists at least one element in the domain that gets assigned different images in the codomain due to the rule being ambiguous.	$f: \mathbb{R}^+ \to \mathbb{R}$ given by $f(x) = \pm\sqrt{x}$	Somewhat familiar from previous courses

These elaborations are important for several reasons. First, non-examples in the *multiple rules* category are more likely to resonate with students' experiences with functions in previous courses, while those in the *equivalent representations* category are relatively unfamiliar to introductory abstract algebra students. Thus, this elaboration is important because (1) it points out aspects of the function concept that students must attend to in order to successfully reason about functions in abstract algebra, while also (2) providing a viable explanation for why students struggle with functions (they have limited, if any, experience with *equivalent representations*, the category that is most prevalent in abstract algebra). Second, a practical suggestion that emerges from this analysis is that productive learning experiences for students should involve non-examples of each of these four kinds. This elaboration can therefore inform instructional design and selection of non-examples for abstract algebra lessons. However, though our analysis establishes that students should gain experience with non-examples in each category, it does not address or provide insight into the nature or instructional context that might make these experiences impactful. That is, it remains unanswered how these categories might be used to support students' development of a productive conception of function in abstract algebra. It is also still unclear in what contexts and for what purposes these non-examples should appear in an instructional sequence. We propose that these nuanced questions represent productive lines for future research via task-based clinical interviews, conceptual analyses, and hypothetical learning trajectories.

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A Quantitative Critical Analysis of Instructional Practices and Math Identity

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Abstract: The growing interest in the implementation of active learning practices necessitates a critical inquiry into how students with identities that are traditionally marginalized in mathematical spaces differentially experience these practices. In this work, we draw on critical quantitative theories to analyze how shifts in math identities in precalculus and calculus courses are mediated by intersectional gender identities in regard to the active learning instructional practices of math engagement, peer collaboration, instructor inquiry, and participation. Using a generalized linear model, we identified that (1) math engagement was the strongest instructional practice linked with positive shift in math identity overall, (2) intersectional gender identities linked with sexuality and First-Generation college status were significant contributors to explain the model variation, and (3) high levels of math engagement mediated against a loss in math identity for Women, and high levels of peer collaboration mediated against a loss in math identity for Indigenous students.

Keywords: active learning, equity, quantitative methods, critical theory, precalculus, calculus

There is a growing body of literature investigating the impacts of active learning classroom practices on students' experience and success, in general, and specifically within college math classrooms (Freeman et al., 2014; Hyland, van Kampen, & Nolan, 2021; Rämö, Lahdenperä, & Häsä, 2021). There is also a growing body of literature documenting the marginalizing experiences and exclusionary systems within mathematics (Battey et al., in press; Ellis, Fosdick, & Rasmussen, 2016; Leyva et al., 2021). For example, Leyva and colleagues (2021) document discouraging and marginalizing instruction in college precalculus and calculus classes, emphasizing the ways that instruction perpetuates whiteness and patriarchy. Battey et al. (in press) connect these discouraging and marginalizing instructional events to additional emotional and cognitive labor undertaken by Students of Color and white Women, which negatively impacts the labor they have available to learn the content of precalculus and calculus. At the intersection of these bodies of literature are a small number of studies examining how active learning practices differentially impact students with marginalized identities. While a recent meta-analysis by Theobald and colleagues (2020) provides evidence that active learning narrows the racial opportunity gap in STEM, Johnson et al. (2020) found that active learning instruction in Abstract Algebra courses introduced a gender performance difference in content assessment, with men in active learning courses outperforming women, but men and women performing similarly within lecture-based courses.

How students with marginalized identities differentially experience active learning in college math classes remains an open question. There are numerous reasons why answering such an important question is difficult. One challenge lies within defining what we mean by active learning, what we measure related to student experience or success, and what student identities are attended to. A second challenge appears when using those definitions to answer the question in a robust way, given that for many identity groups, marginalization within math is related to a lack of representation in math classrooms. In this study, we draw on a large-scale dataset of student responses related to their experiences in college precalculus and calculus to investigate: *what aspects of active learning are beneficial for students' math identity development, for whom, and are they differentially beneficial for certain students?* Specifically, our analysis centers on how gender mediates active learning classroom practices related to mathematical identity development with a particular focus on the intersections of identities. Our decision to center the experiences of women and gender non-conforming students with an attention to intersectional identities within womanhood is driven by literature (e.g., Ellis et al., 2016; Johnson et al., 2020; Laursen et al., 2014; Leyva, 2017), by preliminary analysis of our data, and by the lived experiences of the author team.

Literature Review

To answer our research question, we must operationalize how we conceptualize math identity, intersectionality, and classroom practices linked to active learning.

Definition of student success: Math identity

To measure student success, we focus on math identity, which Martin (2006) defines as "the dispositions and deeply held beliefs that individuals develop, within their overall self-concept, about their ability to participate and perform effectively in mathematical contexts and to use math to change the conditions of their lives" (p. 206). Other research has further detailed math identity as including (1) self-perceptions of mathematical ability and capacity to participate in the math environment; (2) sense of belonging in the math environment including feelings of wanting to be in that space, being an important part of the space and one's perception of others recognizing them as belonging in the space; and (3) the role of social discourses within institutions, individuals, society, etc. that shape (either support or constrain) students' mathematical identities (Adiredja & Andrews-Larson, 2017; Good et al., 2012; Mendick, 2006; Leyva, 2016, 2017, 2021; Rainey et al., 2018, Voigt, 2020).

Defining student identities: Intersectional lens

Intersectionality draws from the work of Critical Race Theory and Black Feminist Thought to highlight how aspects of a person's social and political identities combine to create overlapping forms of privilege and marginalization (Crenshaw, 1990). Attending to intersectionality in STEM education research can be challenging as participant recruitment and sample size may limit possible analysis. Laursen and Rasmussen (2019) call on researchers to "[design] studies that have the statistical power needed to unpack average gains or outcomes in more intersectional ways" (p. 140). Our study design allows us to respond to this call by investigating the perceptions of active learning classroom practices among women students while also attending to other identities within womanhood, including race, sexual identity, and first-generation status. It is necessary to, when methodologically possible, attend to differences within groups (such as within women students) to avoid having the experience of the majority group (in our sample: white, straight women whose parents have completed college) speak for all women students. Further, intersectional theory centers how holding multiple marginalized identities, such as Queer and woman, compound one another and so the marginalization of being a Queer woman is different from being a Queer man and different from being a straight woman (Crenshaw, 1990).

Definition of active learning: Four pillars of inquiry instruction

We draw on the four pillars of Inquiry-Based Mathematics Education (IBME) (Laursen & Rasmussen, 2019) to define active learning and identify which active learning classroom practices relate to students' math identity development. Inquiry, as a branch of active learning, centers both the reinvention and creation of math not previously internalized by the student, and scaffolds prior mathematical work and knowledge to "build to big ideas." IBME is composed of four pillars: (1) students engage deeply with coherent and meaningful mathematical tasks, (2) students collaboratively process mathematical ideas, (3) instructors inquire into student thinking, and (4) instructors foster equity in their design and facilitation choices. While the participants in this study were not all enrolled in IBME classrooms, these pillars describe the instructional practices being reported by students to varying degrees in the courses in our study.

Methodological Approach: Critical theories related to quantitative analysis

For many identity groups, marginalization within math is linked to a lack of representation in math classrooms, thus the number of students from such groups (e.g., Students of Color, Queer students, First-Generation Students) is relatively small within introductory math classrooms. This has resulted in few quantitative studies documenting the experiences of students from marginalized populations in college math, meaning these experiences are often ignored or, potentially more harmful, inferred based on the experiences of other marginalized populations, specifically (predominantly white) women. This "exclusion and misrepresentation of [Students of Color] in education research" is often attributed to methodological limitations (Teranishi, 2007, p. 38). In this paper, we bring the potential power of quantitative methods to examine the experiences of students from marginalized identities in precalculus and calculus classes. We do this informed by two perspectives for conducting quantitative analyses from a critical lens: Data Feminism and QuantCrit.

Data Feminism

Building from various works in Feminist thought and Intersectionality theory, D'Ignazio and Klein (2020) characterize Data Feminism as "a way of thinking about data, both their uses and their limits, that is informed by direct experience, by a commitment to action, and by intersectional feminist thought" (What Is Data Feminism?, para. 9). Thus, data maintain power hierarchies as well as present an opportunity to challenge sociohistorical power differentials. In particular, Data Feminism recognizes the unequal distribution of power within and because of higher education institutions. Controlling educational spaces allows dominant groups to systematically exclude certain groups and explicitly or implicitly benefit their own social, political, and economic interests. Pervasive mathematical discourses continue to uphold white, patriarchal values which marginalize women and students of color within these spaces (Leyva, 2017; Martin, 2009; Mendick, 2006). Data Feminism emphasizes utilizing data to acknowledge important and often unrecognized counternarratives to dominant discourse. This extends across all arenas of quantitative research including the content, the form of communication, and data processes such as collection and analysis. In terms of content, Data Feminism suggests challenging dominant discourses by dismantling gender and racial binaries beyond man/woman and Black/white and illuminating often subjugated knowledge from various voices. Communicating data within a Data Feminist lens includes leveraging emotion and other aspects of the human experience rather than "valorizing the neutrality ideal and trying to expunge all human traces from a data product." In terms of processes and analysis, Data Feminism urges researchers to carefully consider and convey the social, cultural, historical, institutional, and material contexts underpinning and encompassing the data. Overall, a Data Feminist lens values various forms of knowledge, particularly highlighting the need for including those most frequently marginalized within the data context, and analyzing and presenting data in a way that reflects the humanity ingrained within each data point.

QuantCrit

There has been a recent movement to introduce more quantitative methods into Critical Race Theory research, creating the field known as QuantCrit. Ladson-Billings and Tate (1995) characterize Critical Race Theory as a means to unveil the institutional and structural racism embedded in US schooling systems. With this perspective, scholars are encouraged to center the lived experiences of the oppressed and document the human biases, programmatic structures and institutional policies that may lead to negative experiences, negative academic identities, and hence, negative student outcomes for Students of Color. Garcia et al. (2018) note that simply using quantitative methods for critical research is not enough; one must constantly reflect-on and engage with the “historical, social, political, and economic structures and power relations at any given point in time” (2018, pg. 150). Gillborn et al. (2010) offer a way to strive for this engagement by outlining five central tenets of QuantCrit: (1) racism is a complex, deeply rooted aspect of society that is challenging to quantify; (2) numbers are not neutral and can promote deficit perspectives; (3) categories are human creations and so units of measures and forms of analysis must be critically evaluated; (4) data cannot speak for itself and critical analyses should be informed by experiential knowledge of marginalized groups; and (5) statistical analyses have power only because of their potential to support social justice.

Data Analysis

Data from this analysis comes from the NSF-funded study SEMINAL, which administered the Student Postsecondary Instructional Practices surveys (Apkarian et al., 2019) at 12 universities across four academic terms. A general linear regression model was fit to the data (n=19,192) using R software. Our outcome variable was a *change in mathematical identity* (confidence, interest, enjoyment, and ability) that was computed as a difference between student reports of their perceptions of math identity at the beginning of the course and near the end of the course. Our predictor variables of interest included: university (12 sites), course level (precalc, calc 1, calc 2), four measures of active learning instructional practices (math engagement, peer collaboration, instructor inquiry, and participation/community) that were determined based on prior factor analysis (Creager et al, submitted) and aligned with the four pillars of inquiry instruction, and social markers for race (white, Asian, Black or African-American, Hispanic or Latinx, Indigenous, Multiracial, Middle Eastern or North African), gender (cisman, cisWoman, Gender non-conforming), sexuality (straight, Asexual, Bisexual, Gay, Lesbian, Queer, Straight+), First-generation college status, and pairwise interaction effects for each of these variables. We used AIC model selection to distinguish among models describing the relationship between shifts in mathematical identity and our predictor variables. The best-fit model, carrying 94% of the cumulative model weight is presented in equation 1.

$$\Delta\text{math identity} = \begin{array}{l}\text{Intercept + university + course level +} \\ \text{math eng. + peer collab. + inst. inquiry + participation +} \\ \text{gender + race + sexuality + first-gen. + gender:sexuality + gender:first-gen. +} \\ \text{gender:math eng. + race:peer collab.}\end{array} \quad (1)$$

In building this model, we had to identify a control group for the regression analysis. We chose the control group to be those with the most privileged identities in math, because we want to highlight the educational debt owed to and the differential experiences of students with marginalized identities. Though we understand why we have to use a comparison group for a regression model, we did not want this to be a comparison that played into deficit narratives. To avoid this, we focused our analysis on changes in mathematical identity to situate comparison within a student experience as opposed to examining differences in academic outcome variables. Furthermore, we move beyond differences between groups to understand the differential impact in relation to classroom practices. We believe doing so avoids deficit narratives and gap-gazing by contextualizing how instructional practices interact with social markers which draws attention to how mathematics courses are serving or failing to serve particular students.

Results

In this section we present the results of the regression analysis. Informed by QuantCrit and Data Feminism, these data should not speak for themselves, the analyses must be informed by experiential knowledge, and the presentation of data can and should leverage emotion. We dedicate the written space here to present the results and for brief discussion. We look forward to more space to contextualize and interpret this data in presentation format and to bring in qualitative data from students' survey responses.

Change in math identity ranged from [-5, 5] with a mean of -0.084 and standard deviation of 0.21. A summary of the fitted model is presented in Table 1. The instructional practices of *math engagement* (beta = 0.51, p<.001) and *instructor inquiry* (beta=0.03, p=.002) were positive predictors for changes in math identity across all students. The instructional practice of *peer collaboration* (beta= -0.10, p<.001) was a negative predictor and *participation* (beta= 0.01, p=0.34) was not statistically significant. We also identified that course level was a statically significant predictor of changes in math identity compared to calculus 1 for both precalculus (beta=0.18, p<.001) and calculus 2 (beta= -0.12, p<.001).

Table 1. Regression Model for Change in Math Identity

	Estimate	CI- 2.5%	CI- 97.5 %	Std. Error	t-value	p	sig
(Intercept)	-1.78	-1.89	-1.66	0.06	-30.28	<.001	***
Instructional Practices							
Math Engagement	0.52	0.49	0.55	0.02	30.55	<.001	***
Peer Collaboration	-0.10	-0.14	-0.07	0.02	-6.67	<.001	***
Instructor Inquiry	0.04	0.01	0.06	0.01	3.06	0.002	**
Participation	0.01	-0.01	0.04	0.01	0.95	0.342	
Course level							
Calc 2	-0.12	-0.16	-0.08	0.02	-5.45	<.001	***
Precalculus	0.19	0.15	0.23	0.02	8.73	<.001	***

Social Markers							
cisWoman	-0.59	-0.73	-0.45	0.07	-8.38	<.001	***
Gender non-conforming	-0.47	-1.48	0.55	0.52	-0.91	0.365	
Asian	0.17	-0.02	0.35	0.10	1.72	0.086	.
Black or AA	0.05	-0.13	0.24	0.10	0.57	0.570	
Hispanic or Latinx	-0.08	-0.24	0.07	0.08	-1.04	0.300	
Middle Eastern or N. African	-0.17	-0.47	0.12	0.15	-1.17	0.244	
Multi-racial	-0.10	-0.26	0.05	0.08	-1.29	0.197	
Indigenous	-0.82	-1.45	-0.20	0.32	-2.59	0.010	**
Asexual	-0.06	-0.19	0.06	0.06	-1.00	0.317	
Bisexual	-0.02	-0.19	0.14	0.08	-0.25	0.803	
Gay	-0.07	-0.22	0.07	0.08	-0.99	0.325	
Lesbian	0.06	-2.03	2.16	1.07	0.06	0.954	
Straight+	-0.07	-0.35	0.21	0.14	-0.47	0.637	
Queer	0.36	0.04	0.68	0.17	2.17	0.030	*
First-Generation	-0.01	-0.06	0.04	0.03	-0.32	0.748	
Intersectional Social Markers *(Sig only)*							
cisWoman x Bisexual	0.20	0.01	0.39	0.10	2.02	0.043	*
cisWoman x First-Generation	-0.09	-0.16	-0.02	0.04	-2.36	0.018	*
GNC x Straight+	-3.40	-4.99	-1.81	0.81	-4.19	<.001	***
Instructional Practices x Social Markers *(Sig only)*							
Math Engagement x cisWoman	0.16	0.12	0.20	0.02	8.28	<.001	***
Peer Collaboration x Indigenous	0.24	0.03	0.45	0.11	2.22	0.026	*
Site *(Omitted)*							

The estimate for the intercept (beta=-1.77, p<.001) indicates that on average white straight non-first-generation men report a decrease in math identity while holding all other predictors constant. Examining the predictive social marker, being a woman was associated with a further decrease in math identity (beta=-0.59, p<.001) as was identifying as Indigenous (beta=-0.82, p<.009). These results highlight how systems of oppression based on race and gender are acting on students in math classrooms to communicate concepts of who is a mathematician. Being a Queer student was a mitigating factor and associated with less of a decrease in math identity (beta=0.35, p=.03). This means that a white, Queer, non-first-generation man still has an overall decrease in math identity, but would have a -1.77+0.35 shift in math identity rather than a -1.77 overall decrease. Furthermore, there were intersectional identities that had statically significant impacts on the outcome variable which included Bisexual women (beta=0.19, p=.04), Gender non-conforming straight+ person (beta=-3.39, p<.001), and First-generation women (beta=-.09, p=-.02). These results point to the nuanced nature of intersectional identities especially at the intersection of gender and sexuality and gender and first-generation college status. Interestingly, in our data the interactions between gender and race were not significant in the model; however, this does not negate the impact of the lived experience of students but may highlight the limited nature of our instrument and outcome variables.

There were two interaction effects of identity social markers with the instructional practices that contributed to the model. These indicate that the instructional practices had a differential impact on shifts in math identity based on the social marker. The interaction between math engagement and gender was significant, whereby higher levels of math engagement had a positive association with increases in math identity for women (beta=0.15, p<.001). This relationship is presented in Figure 1a. The interaction between race and peer collaboration was significant, whereby higher levels of peer collaboration had a positive association with increase in math identity for Indigenous students (beta=0.23, p<0.3). This relationship is presented in Figure 1b.

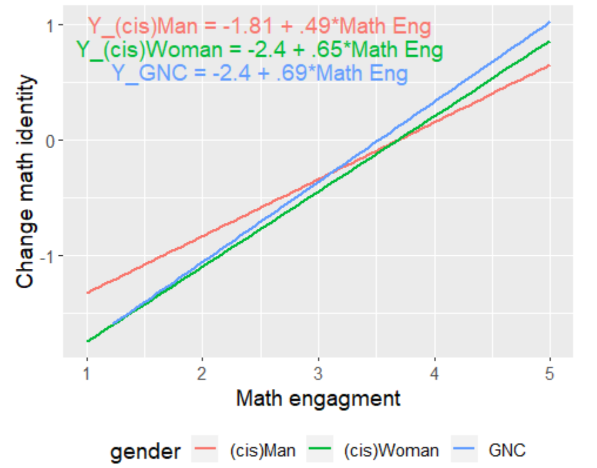

Figure 1a: Regression line for math engagement on change in math identity for women, men, and gender non-conforming students.

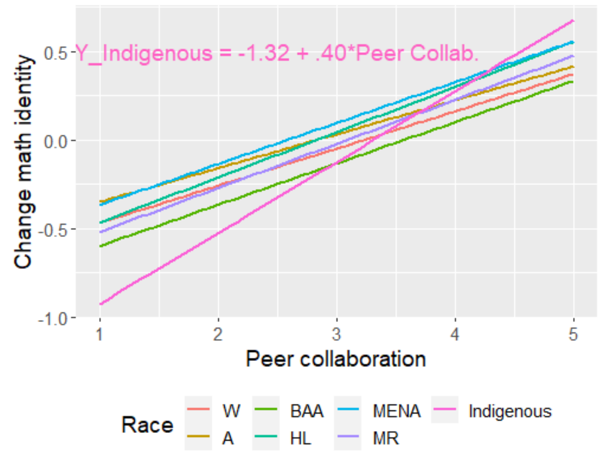

Figure 1b: Regression line for peer collaboration on change in math identity for white, Black or African American, Middle Eastern or North African, Indigenous, Asian, Hispanic or Latinx, and multi-racial.

Discussion

In this study we have performed a critical quantitative analysis to identify how classroom practices related to active learning can support the math identity development of women students in precalculus and calculus courses, while attending to intersectional identities related to race, sexual orientation, and first-generation status. Our findings indicate that not all of the pillars of active learning positively contribute to math identity development, and there are differences across course levels of precalculus and calculus. Math identity in general decreases in these introductory math courses, highlighting a need for critical self-reflection as a field. Our model development further highlights the need to attend to intersectional identities, as gender alone was not a significant predictor to account for the variation in the model, but including the interaction effects of gender with sexuality and gender with first-generation college status aided to the model fit. Furthermore, in an effort to avoid the pitfalls of gap-gazing (Gutiérrez, 2008) and align with the tenets of QuantCrit and Data Feminism, we examined how the instructional practices differentially impacted change in math identity. This analysis suggests that higher levels of math engagement can positively support women, and higher levels of peer collaboration positively contributed to the identity development of Indigenous students. Further analysis should examine the causal mechanism underlying these effects to work against a homogeneous view that active learning is beneficial for all students.

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Students' Validations of Constructive Existence Proofs: There's More than Meets the Eye

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Undergraduate students are expected to construct and comprehend constructive existence proofs; yet, these proofs are notoriously difficult for students. This study investigates students' thinking about these proofs by asking students to validate arguments for the existence of a mathematical object. We share salient ways that the students described the mathematical arguments, and how, if at all, this impacted their view of the arguments as proofs.

Keywords: Existence proofs, proof framework, proof validation, explanatory proof

Existence proofs are a prominent part of undergraduate students' advanced mathematics courses. These proofs argue the existence of a mathematical object by explicitly producing the desired result or providing an algorithm for its production (Brown, 2017). Yet, these proofs are notoriously difficult for students (Brown, 2017; De Guzmán et al., 1998; Samper et al., 2016; Schaub, 2021).

Our classroom experiences have highlighted one difficulty that students often have with constructive existence proofs. When attempting to prove the existence of a mathematical object, many students produce an argument with a flaw in the overall structure of the argument, or what's sometimes referred to as the proof framework (Selden & Selden, 2003). The students' argument assumes the existence of the object by starting with a desired property and follows by solving for the object (two examples of such an argument are given in the Methods Section, see Figure 1 and 2). We believe that students can and will engage in meaning-making, and so, the prevalence of this error made us wonder what mathematical reasoning students engage in as they produce or endorse an argument of this nature.

Scholars have identified various reasons for why students have difficulties with proofs that we see as especially relevant for constructive existence proofs. To start, students find the mathematical language used in proof challenging (Moore, 1994), including unpacking the logical statement that they wish to prove (Selden & Selden, 1995) and understanding the nuanced ways that mathematical objects are introduced and used (Lew & Mejía-Ramos, 2019). Additionally, there are several studies that show students might view an invalid proof with a flaw in the framework as a proof because students sometimes ignore the argument's structure or doubt that it matters (Selden & Selden, 2003; Weber, 2009, 2010). Instead, students may focus on surface features such as the computations offered in the argument (Inglis & Alcock, 2012; Selden & Selden, 2003). Our study fills a gap in the research literature on how students think about constructive existence proofs, and in doing so, we gain some insight into why these proofs can be difficult for students.

Theoretical Grounding

For this report, we use Stylianides' (2007) characterization of proof in school mathematics: "Proof is a mathematical argument, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:

1. It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;

2. It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and
3. It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community" (p. 291).

We note that what constitutes a proof is dependent on the audience's accepted statements, modes of argumentation, and representation. In this report, we specify when we consider the students' evaluation of an argument versus the mathematics community's. We reserve the term valid (or invalid) proof for what we view as the larger mathematics community's acceptance (or rejection) of the mathematical argument as a proof of a given statement. Additionally, we expand on Stylianides' (2007) definition by making explicit that the associated mathematical claim plays a role in the second characteristic: when evaluating a mathematical argument, one should consider whether the argument uses acceptable modes of argumentation that are for (or against) a known mathematical claim. When we say that an argument was (or was not) viewed as a proof, we mean that the argument was (or was not) viewed as a proof of the given claim.

Scholars have distinguished between proofs that only convince and proofs that also explain (i.e., explanatory proofs) (e.g., Bartlo, 2013; Lockwood et al., 2020; Weber, 2010). In this study, we view explanatory proofs as audience-dependent since they may not always give the same insight to people with different experience levels. Additionally, we take the explanatory nature of a proof as closely tied to the activity of constructing the proof. A reader may find a proof to be explanatory if they gain insight into the informal reasoning that was used to create it. A prover might then include part of their problem-solving process in their proof in attempts to produce an explanatory proof for a particular reader.

In this study, we explore students' thinking about, and validation of, arguments for an existence claim. One argument is an invalid proof because it features an error in the framework while the other argument is a valid constructive existence proof. In particular, we investigate: How do students make sense of arguments for the existence of a mathematical object?

Methods

The data for this study comes from 16 semi-structured interviews from students at two universities enrolled in three different Introduction to Proofs classes with different instructors as part of the NSF-funded project *Advancing Students' Proof Practices in Mathematics through Inquiry, Reinvention, and Engagement* project (NSF DUE #1916490). One of the research goals of the interviews was to understand how the students reasoned about arguments for the existence of a mathematical object. This part of the interview typically lasted between 30 and 40 minutes. The interview tasks were either in the context of groups (N=11) or functions (N=5) depending on the interviewee's course context.

Both authors were present for each of the interviews. The interviews were facilitated remotely via Zoom and a shared Google Doc and were video-recorded capturing the students' gestures and typed work. We refer to individual participants with a code that indicates their proof validation for each argument and the context of the interview tasks. For instance, for the label "NPG-7", the "N" represents the participant did *not* view the first argument as a proof, the "P" represents they viewed the second argument as a proof, the "G" represents the arguments were in the group context, and the "7" represents they were the seventh interviewee. We use "F" to represent the tasks in the function context and "U" to represent the one instance in which a student was ultimately undecided about the validity.

Interview Tasks

There were three parts to the interview task. First, we asked students to (re)interpret a given existence statement. Then we asked them to describe their sense making and validation of the Invalid Proof. Last, we asked them to describe their sense making and validation of the Valid Proof. See Figure 1 and 2 for the statements and the Invalid and Valid Proofs, which we constructed based on our experience with students[1] and our mathematical knowledge, respectively.

Group Context

Statment: Consider the group G with operation $*$. Let b be in G. For every a in G, there exists an x in G such that $a * x = b$.

Invalid Proof:	*Valid Proof:*
Let G be a group with operation $*$. Let b be in G. Let a be in G. Then, $a * x = b$ Implies that $a^{-1} * (a * x) = a^{-1} * b$ Which implies that $(a^{-1} * a) * x = a^{-1} * b$ Which implies that $e * x = a^{-1} * b$ Which implies that $x = a^{-1} * b$. By closure x is in G since a^{-1} is in G and b is in G.	Let G be a group with operation $*$. Let b be in G. Let a be in G. Choose $x = a^{-1} * b$. By closure, x is in G since a^{-1} is in G and b is in G. Then, $a * x = a * (a^{-1} * b) = (a * a^{-1}) * b = e * b = b$.

Figure 1. Group version of the statements and two arguments.

Function Context

Statement: Let f be a real-valued function defined by $f(x) = mx + b$. For all real numbers m and b with $m \neq 0$, there exists a real number t such that $f(t) = 0$.

Invalid Proof:	*Valid Proof:*
Let f be a real-valued function defined by $f(x) = mx + b$. Let m, b be in $\mathbb{R}$ such that $m \neq 0$. Then, $f(t) = 0$ Implies that $mt + b = 0$ Which implies that $mt = -b$ Which implies that $t = -b/m$. Since $m \neq 0$, t is a real number.	Let f be a real-valued function defined by $f(x) = mx + b$. Let m, b be in R such that $m \neq 0$. Choose $t = -b/m$. Since $m \neq 0$, t is a real number. Then, $f(t) = f(-b/m) = m(-b/m) + b = -b + b = 0$.

Figure 2. Function version of the statement and two arguments.

Data Analysis

Our data analysis process was consistent with a thematic analysis (Braun & Clarke, 2006). Together, we engaged in a cyclic process examining each interview transcript and corresponding

[1] It was relatively common during the interviews for students to comment that the argument of the Invalid Proof was how they would construct a proof of the given statement.

video for the students' sense-making of the existence statement and two corresponding arguments. To do so, we focused on the guiding questions: (a) How did the student describe their interpretation of the statement?, (b) How did the student initially reason about the Valid/Invalid Proofs? (How, if at all, did this change over time?), and (c) How did the student discuss the two arguments in comparison to each other? For each student, we discussed our answers to these guiding questions until we came to an agreement on how to interpret the students' sense-making. We documented our answers to the questions with relevant quotes and our shared-interpretation of the quotes in an analytic memo. After creating each analytic memo, we then compared the reasoning to the previous participants' reasonings. During this comparison, we generated and refined a list of the ways in which the participants described the arguments and how, if at all, this impacted their view of the arguments as proofs. We offer these results next.

Results

The Invalid Proof Shows *How* to Find the Mathematical Object

Most of the participants (N=12) indicated that the Invalid Proof showed how to find the mathematical object. For instance, PNG-3 explained that it described "the process of finding that element [x] for which that [a*x=b] is true". These students reasoned about the argument in at least one of the following ways (we will expand on the first two):

1. The argument was a proof since it showed the existence of the desired object by explaining why such an object fit the desired property,
2. The argument was a proof since it fit the logical structure of the statement,
3. The argument was not a proof since it assumed the existence of the desired object, or
4. The argument was not a proof since it had only solved for the desired object but also needed to show the desired property was true.

Among the students who viewed the first argument as a proof, many thought that the algebraic work it provided elaborated on the reasoning behind why the claimed object fit the desired property. One student, PPF-13, noted that while both arguments were logical, the Invalid Proof provided a "more logical step by step process" for how to find the object. Additionally, this elaborated reasoning led many students to observe that the Invalid Proof was more appropriate for a novice reader as one student, PPF-16, said, "I could see where somebody at a lower math level would appreciate having that". This suggests that students viewed the Invalid Proof as being explanatory for less experienced students since it illustrated the problem-solving process for constructing the claimed object. Some of these students also addressed the logical implications for constructing the argument in this way. For instance,

> "No, they don't start with the thing that they're trying to prove. They're trying to prove that x needs to be an element in G. Right. That's what we're trying to prove, that there exists an x in G such that the statement is true" (PPG-11).

For this student, the primary goal of the proof was showing that the desired object was a group element and the property $a * x = b$ was simply the characteristic of that object. Another student, PPF-16, explained that "assuming that there exists a real number t that makes the function equal to zero" was appropriate because "we know that whatever number we're going to solve t to be is always going to make that [$f(t)$] zero". To this student, the prover could show that there existed a real number t such that $f(t) = 0$ by solving the desired equation for t because the value that they solved for must be the value that makes $f(t) = 0$ true.

To some other students, the Invalid Proof was a proof since it used the appropriate logical structure for showing how to find the mathematical object. PUG-4 explained that she preferred the first argument because it not only made sense to her, but she could also see the statement connecting to the argument. She said:

> "I think it makes more sense mathematically. But also in context of the problem that we're supposed to show 'for every a there exists an x…', you know? So, I like that it gets, you know, it starts here, this is the equation that we're given. This is what we have and then here's the x that exists and that's like proof. That's why I like it, because it's kind of in the order of the proof..."

We interpret her comment to mean that she saw the algebraic steps as fitting the logical structure of the given statement. Another student, PNG-6, went further in articulating why she viewed the Invalid Proof as a proof. When asked how she saw the argument connected to the statement, she explained that the prover showed that "no matter what the a is, we're able to figure out an x, that will equal b when you do that [operation]." To these students, it was logically correct show how to find a desired mathematical object to prove its existence.

The Valid Proof Shows an Instance of the Mathematical Object

All the students in our study (N=16) identified that the Valid Proof introduced an instance of the mathematical object. For instance, when explaining the prover's logic, NPG-8 said "they're just choosing an x, which is essentially what the problem is asking for. It's just saying, just find, just find one of them" (NPG-8). We found the students in our study reasoned about the argument in at least one of the following ways (we will elaborate on the first two):

1. The argument was not a proof because it assumed the existence of the object instead of deducing its existence,
2. The argument was a proof with a jump in explanation when the mathematical object is introduced,
3. The argument was not a proof because it only showed that the object met the desired property but not how they found the object,
4. The argument was a proof since it only needed to show the existence of one element that met the desired property, or
5. The argument was a proof even without the line that argued the mathematical object is in the desired set.

While all students saw the Valid Proof as identifying a particular instance of the mathematical object, some students did not view it as a proof. Four students explained the error of the Valid Proof was that it assumed the conclusion by starting with the desired mathematical object. For instance, PUG-4 debated whether or not the Valid Proof is a proof, explaining: "because it starts with the x - that's a little confusing to be like 'oh, for every a, an x exists' but that's what they're saying right here, this is the x that exists". For this student, like the other three, the prover should not start with the desired object (e.g., $x = a^{-1} * b$) because this is what needed to be deduced.

While the rest of the students (N = 11) did view the Valid proof as a proof, a subset of these students saw introducing the mathematical object at the start of proof as a jump in explanation but one that was logical. For example, "The second [argument], just like tosses that up there. So, they maybe did some side work or maybe were just able to see it. But, uh, but they kind of leave a lot of their reasoning off the page" (PPG-11).

We see students' acknowledgement that the prover omitted relevant reasoning from the reader by introducing an object up front as thinking that the proof only convinced rather than also explained to the reader how to find the object.

The Arguments are Structured in Different Ways

Many of the students in our study identified the different logical structures of the arguments (N=15). Most of these students described the two arguments as "opposite" from one another, explaining how the first deduced the desired mathematical object while the second began with it. For instance, when comparing the two arguments PNG-3 stated that "this one [the Invalid Proof] shows how they got to that x and this one [the Valid Proof] just like here's the x and you get b when you multiply by a". We found these students as reasoning about the argument in at least one of the following ways (we will elaborate on the first two):

1. Only the Invalid Proof had an effective framework since it was the only one that concluded the mathematical object instead of assuming it at the start,
2. Both arguments functioned as proofs since the Invalid Proof was more explanatory to certain readers while Valid Proof only convinced,
3. Neither of the arguments functioned as proofs since neither had acceptable or complete frameworks, or
4. Only the Valid Proof had an effective framework since it was the only one that concluded the desired property held for an instance of the mathematical object.

Only one of the two frameworks worked to prove the existence of the desired object for some students. Students who viewed the Invalid Proof as a proof suggested it had an effective framework since it was the only one that concluded the mathematical object instead of assuming it at the start. These students tended to be the ones who thought a) the Invalid Proof was a proof because it explained how to find the mathematical object and b) the Valid Proof was not a proof because it assumed the existence of the object instead of deducing its existence. PNG-3 explained:

"Well, we're trying to conclude that there exists an x. Okay yeah, so I don't think proof two [the Valid proof] is [a proof], because that is the conclusion, that there exists an x such that $a * x = b$. Whereas in this one [the Invalid Proof] we're definitely looking for that x, in the first one. Because I think that the conclusion should be that there exists an x in G such that $a * x = b$, and our givens are that a is in G and that b is in G, essentially and that's it."

She later confirmed that to her the Valid Proof assumed the conclusion by introducing the element $x = a^{-1} * b$ at the beginning of the argument.

For some students, both frameworks worked to prove the existence of the desired object, though in different ways. In particular, they saw a) the Invalid Proof as a proof since it showed the existence of the desired object by explaining why such an object fit the desired property and b) the Valid Proof as a proof with a jump in explanation when the mathematical object is introduced. We interpret this as them seeing one proof as being more explanatory to certain readers, while another may only convince since for these students, the first proof provided additional information that the second proof left out. In other words, while they saw both arguments as proofs, the students interpreted the arguments as being written for different audiences. For instance, PPF-13 explained:

> "So, this one [the Invalid Proof] feels like it's more like explain it like your five and the other one [the Valid Proof] is more like assume they already know what it means to go from there and that they're both valid, but they're both different ways of doing it."

To PPF-13, and the other two students whose thinking fit in this category, both arguments accomplished the task of showing the object's existence, they just did so in different ways. As PPG-11 stated: "It's just the construction of it [the two arguments] that's different [...] They [the provers] both understand that x needs to be equal to $a^{-1} * b$ whatever a is."

Conclusion

Unlike prior studies that have found that students ignore the structure of proofs or think that they do not matter (Selden & Selden, 2003; Weber, 2009, 2010), we found that the students in our study attended to, and made sense of, the logical structure of the arguments for the existence of a mathematical object. However, how the students saw the structure as working or not working to prove the existence claim was sometimes inconsistent with mathematical norms.

In our view, mathematicians would not see the Invalid Proof as proving the existence of the mathematical object since the argument starts with the desired property (e.g., $a * x = b$) and as such, it assumes the existence of the desired object. However, students in our study saw this as acceptable for two reasons. First, some of the students viewed the Invalid Proof's framework as fitting the logical structure of the existence statement. For them, the statement required that one show how to find the desired object and thus needed to start with the property and end with the object itself. Second, other students viewed it as a way to add transparency when showing the existence of the desired object by explaining how one would find it. We saw these students as valuing the explanatory nature of Invalid Proof while acknowledging it was not the only way to prove the existence of the object. These ways of reasoning can explain why students may think that a proof that argues the existence of a mathematical object should be structured like the Invalid Proof.

We see mathematicians as viewing the Valid Proof's framework as introducing the variables, introducing an object as a candidate, and then showing that it fits the desired criteria. This structure functions to prove the existence of the desired mathematical object in the views of the mathematics community. However, the students in our study did not always share this view. In particular, some students in our study saw the line that introduced the object (e.g., Choose $x = a^{-1} * b$) as assuming the existence of the mathematical object. In other words, the argument did not prove the existence of the object since it assumed it from the start. Such a view explains why students may not think that proofs of existence claims should be structured in this way.

We see our study as important steps in supporting students in engaging in proof-activity related to existence claims by providing different ways in which students might think about proofs of existence claims and their frameworks. We hope that future work will investigate how instruction can build on these ways of thinking to support students in constructing, comprehending, and validating constructive existence proofs.

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Discursive Transgressive Actions
Exhibited in a History of Calculus Course

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Abstract: *Existing studies that explore the use of primary source materials in the teaching and learning of mathematics largely document the benefits of such an experience. However, there is little research which explores how those benefits are manifested through the study of primary source materials. To explore the progression of encountering a barrier, overcoming that barrier, and gaining a new understanding or perspective of mathematics via study with primary source materials, I use the transgressive actions lens of psychological theorist, Jozef Kozielecki (1986). This framing allows for the investigation of barriers which hinder learning and actions overcoming those barriers. Although many actions and barriers were discovered through this study, I focus specifically on actions of engaging in discursive communities and the barriers those actions overcome.*

Keywords: Transgressive actions, Discourse, Primary Source Projects

In recent decades, mathematics educators increasingly advocate for the use of history in the teaching and learning of mathematics and report many benefits for its place in the mathematics curriculum (Furinghetti, 2020; Tzanakis & Thomaidis, 2011). It has been speculated that studying mathematics through a historical lens can lead students to perceive mathematics as a more humanized discipline (Fried, 2001). Students can also become more cognizant of mathematics as a collection of contributions across many cultures, a constantly changing discipline which influences our scientific and societal development, and a basis for perpetual dialogue with other disciplines instead of viewing mathematics as a polished and pristine discipline to be admired from afar (Clark et al., 2019). Can and colleagues found that the study of abstract algebra via curriculum materials which incorporate primary source material, author commentary, and student tasks, also known as Primary Source Projects (PSPs), can provide learning experiences in which students position themselves as professionals in the field of mathematics (Can et al., in preparation). These types of outcomes are centered around what Jankvist (2009) calls *metaperspective issues of mathematics* (*meta-issues*). As opposed to issues concerning theories, concepts, and procedures, *meta-issues* are more focused on how mathematics has evolved and the human and cultural influences in its development.

Although these benefits to studying mathematics history—and importantly, studying mathematics via primary sources—are widely discussed in the literature, much less is known about how students come to these realizations and what barriers students face which lead them to have alternate conceptions about mathematics. I have used Kozielecki's theory in which he conceptualizes transgressive actions to explore the progression of how students of mathematics history who encounter barriers overcome those barriers and reach a new understanding or perspective.

Theoretical Framework

Kozielecki's (1986) seminal piece, "A Transgressive Model of Man," offers a unique lens which maps a person's previous state of having/being to a new having/being through the crossing of a barrier and thus allows for an exploration of how students overcome obstacles when learning mathematics. The term *transgression* may carry a negative connotation describing an action which violates social norms; however, Kozielecki (1986), described a *transgressive action* as a "purposeful action that leads to an outcome exceeding the boundaries of the individual's past achievements" (p. 89). In this sense, there are two identifiable objects which accompany a transgressive action. To transgress, an individual must traverse a *boundary* that leads to some *outcome* which exceeds their past achievements. Boundaries are characterized by Kozielecki as a "demarcation line that marks out the scope and type of positive or negative value scored by a person so far" (p. 90). A more appropriate definition related to mathematics learning is offered by Semadeni (2015), who describes a boundary as a limit to one's "own mathematical knowledge" or "deep rooted convictions" (p. 27). The outcome which results from having crossed such a boundary corresponds to the individual's new "having or being" (Kozielecki, 1986, p. 89). This new having or being is operationalized as an outcome that resulted from crossing a boundary. Figure 1 displays a model of the transgressive process.

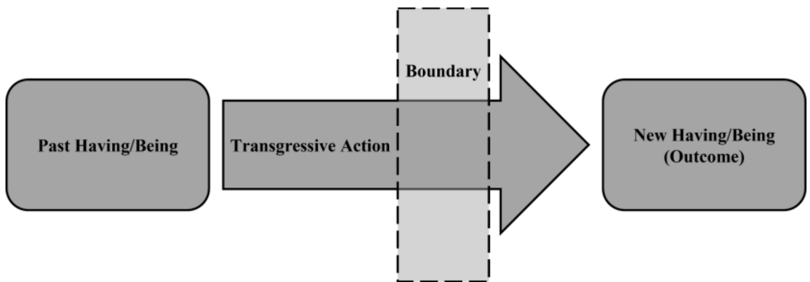

Figure 1. A model demonstrating the relationship among boundary, transgressive action, and students' past having/being and new having/being

Previous theoretical works have described transgressive behaviors of mathematics students and barriers to their growth and development in the realms of probability and statistics (Lakoma, 2015) and geometry and arithmetic (Semadeni, 2015). Findings from students who transgressed in abstract algebra courses provide evidence that the theory has merit in exploring mathematical learning experiences in which students take actions such as engaging in discourse with different individuals and groups of individuals to overcome certain barriers (Can et al., in preparation). Unfortunately, a major limitation of their study was that data were not collected with the transgressions framework in mind. Therefore, they were limited in their ability to identify, much less expand on, barriers and transgressive actions present in the existing data.

The goal of the current research study was to develop an interview instrument that could be used to investigate the highlights from the resulting codes of the project described in Can et al. (in preparation). Particularly, the interview protocol was constructed to explore the various ways in which students engaged in discourse to overcome barriers they faced. Drawing upon aspects of discourse defined by Sfard (2008), I am interested in the role of "overlapping communities of discourse" (p. 91). Students participated in discourse through whole class discussion, working in small groups, and with partners both in class and outside of class time. I claim that engaging in discourse is a transgressive action because students do so purposefully, and all the discursive actions referenced here do lead to an outcome which exceeds a barrier. Although some outcomes and transgressive action-barrier pairs unrelated to discourse were illuminated as a result of this study, my primary aim in this paper is to address the following question: How do students overcome barriers via discursive transgressive actions in a history of calculus course?

Methods

In Spring of 2021, four undergraduate students in a History of Calculus course were interviewed mid-semester. Students were chosen to represent diverse backgrounds, learning experiences, and educational trajectories. The course was delivered as a synchronous hybrid style during the COVID-19 pandemic in which students were required to participate via Zoom if they were enrolled as a "remote" student or participate in person otherwise (Table 1).[1] History of Calculus is unique because its audience comprises mathematics students who have already received instruction in calculus. The goal of the course is to examine the historical development of calculus but not to learn calculus content. Therefore, students primarily engaged in Jankvist's (2009) meta-issues of calculus through the extensive study of primary sources contained within PSPs, as well as those as stand-alone primary historical sources.

Since barriers and transgressive actions were the main focus in the design of the interview protocol, few outcomes were revealed in the analysis. From the author's previous work on a related project, it was apparent that transgressive actions and barriers required a more nuanced exploration. This can be attributed to the novelty of Kozielecki's theory in a mathematics education context. Any outcomes that were identified will not be included here; however, they will be considered in a redesign of the interview protocol to capture all three components for future investigation: *barriers, transgressive actions*, and *outcomes*.

Table 1. Student demographics

Name	Gender	Class Setting	Major	Class Standing
Anna	Female	Online (Zoom)	Actuarial Science	Junior
Sandra	Female	In-Person	Applied Mathematics	Senior
Brent	Male	Online (Zoom)	Actuarial Science	Senior
Esther	Female	Online (Zoom)	Pure Mathematics	Junior

At the end of the semester, the four interview transcripts were cleaned and coded using an inductive coding method as described by Miles, Huberman, and Saldaña (2014). Once an interview was coded, I returned to the codebook to combine redundant codes and delete codes that were irrelevant to the present research project. Each cycle of coding ended with this reflective cleaning process. By the end of the fourth cycle of coding, and thus analysis of the fourth interview, I had categorized codes based on *barrier* and *transgressive action*. Within each of those categories, some codes were further categorized by similarities. The focus of this article is the category, *discourse*.

Coding Process

Coding the data required looking at each interview transcript for evidence that a transgressive action had occurred. Kozielecki (1986) stated that transgressive actions overcome barriers, so I had to search for instances when a participant mentioned that they could not do something or were not able to do something before engaging in some type of transgressive action. They also could have stated that there was some limiting aspect they experienced in the learning process. For example, Brent wrote the historical text was "referencing something that we know now, and

[1] There were two course instructors in Spring 2021 to help facilitate the connection of the remote and in-person sections.

using a different word for it because of the time differential." I coded this response as an *unfamiliar language* barrier because the student indicated he had trouble comprehending mathematics in a form that he did not consider to be normal. It should be noted that PSPs often use an English translation of the source's original language which most closely resembles the spirit of the original text. An antiquated form of English (in the case of, for example, a 17th century translation of a Newton text from Latin) is also a source of contention among students and was included in the *unfamiliar language* category.

Brent revealed being limited by a barrier of unfamiliar language, but he also indicated that he overcame the barrier by saying "it was important to have like [instructor name] in this scenario to be able to bounce off questions, because otherwise I would just hit a wall, and you feel like, not something that I could ever really figure out on my own." This transgressive action was coded *expert discourse*, meaning the student communicated with someone they perceived as an expert in the work they were doing. Brent's exchange will be further explored in the results.

Some excerpts of the transcripts were not as transparent as the previous example. There were instances in the transcript where either a barrier or transgressive action was identified without being accompanied by its counterpart. These were still of interest and could benefit the research to identify its existence. Consider Anna who said, "I was kind of disconnected, to the mathematicians, initially, because I'm… not a math major… so I didn't really have a connection to historical mathematics as I do now." Anna clearly transgressed through a barrier of being disconnected from mathematicians due to her feeling like an outsider to her mathematical community, but she did not specifically state what led to this transgression. It is evident that she transgressed over the barrier because there is a clear barrier, previous having/being, and a new having/being. In this instance, the three mentioned components are closely related. Anna stated that she was disconnected from the mathematician because her major is not pure mathematics. The disconnect was her previous having/being and the subsequent connection is the new having/being. Her barrier was *being disconnected from mathematics* because she was not a pure mathematics major. It is logical to think that something occurred within the class or while interacting with the PSP to initiate this connection, but that is only speculation based on her interview. When revising the interview protocol, consideration will be given to constructing additional probing questions which enable the participant to explicitly connect barriers and transgressive actions.

Pairing Transgressive Actions with Barriers

Once all data were coded to separate transgressive action and barrier codes, I mapped each barrier to a transgressive action based on the context of that relevant portion of the interview. After listing all the barriers, I reviewed each instance where the barrier code was matched to a place in the data. I would then search through the proximal data for a matching transgressive action. Some of these action-barrier pairs had only one instance while others occurred multiple times.

Results

The resulting analysis led to some transgressive actions and barriers being grouped together based on similar characteristics. Discourse was a prevalent theme throughout the data and was partitioned into smaller subsets. Students mentioned that communicating with different people in various settings helped shape their understanding of mathematics and historical mathematics.

Instructor/Expert Discourse

The *instructor/expert discourse* surfaced when students spoke with someone that they considered knowledgeable about the field of mathematics or to the specific barrier which limited the learning ability.[2] Experts included the course instructor/facilitator, other students in the course, and other mathematics instructors they knew outside of class. However, it was evident students often perceived the instructor/facilitator as an expert in the field of mathematics or mathematical history. Anna recalls a time during the semester when she was being limited by her *background mathematical knowledge* (barrier). Although she was in an upper-division writing course in the major, she admitted that she had not "seen algebra or geometry since middle school" and was having trouble solving a quadratic equation. One of the course instructors advised her against dividing by a variable (because of the possibility of dividing by zero) and guided her in a more productive direction. Anna was at an impasse because she could not proceed without having found two roots of the equation. The instructor's interjection allowed Anna and her group mates to make progress on the task.

In contrast, Brent sought the instructor's guidance. Brent admitted that he "left a question mark next to [problematic text] and was like, 'I don't know.' My roommates don't know… just ask the professor." Of course, Brent asking the professor for the correct answer would represent a transgressive action, but, like most pedagogically minded professors, the professor did not respond with this simple option. She was available to "bounce off questions" as Brent stated because he would otherwise "just hit a wall." Brent's words represent an intersection of *instructor/facilitator discourse* transgressive action and *unfamiliar language* barrier. He gave an example that he remembered from a historical text, "calling 'velocities' 'evanescent increments.'" To Brent, this was something that one could not simply "look at a textbook" or "Google and find it." He later stated, "I definitely need the interaction of someone who knows the material [so] that I can ask these questions" referring to questions regarding the unfamiliar language.

Group Discourse

Working in small groups was a major component of the course implementation. *Group discourse* occurred when students were in breakout rooms on Zoom or in person and when students participated in a group chat outside of the class. Groups typically consisted of three to five people, and the course instructor occasionally joined each group to listen and help guide conversation when necessary. To distinguish between group and expert discourse, only accounts of students receiving help from other members of their groups were included in this code. Any mention of the instructor intervening in a small group session was coded to *instructor/facilitator discourse*. Students described their group experiences as peers talking about the problems and sharing their written work. Anna stated, "we're really all contributing and really all trying to decipher what Fermat was getting at… we were able to grasp what he was doing and work through it completely." She referred to her breakout group and how communicating with her group members helped her understand the writings of Fermat and overcome the barrier of *navigating a new experience*. Anna said that working with Fermat was different from other PSPs because this project required "getting to an answer" which she valued as "a lot more mathematics" as opposed to other PSPs which focused on "going about the proof and working through the proof and making ourselves understand what they [the primary source author] were doing." Working with a PSP that demanded more instrumental mathematics (Skemp, 1987) was

[2] I will italicize text to signify codes from my analysis.

a new experience for her and the rest of the group. By her own admission, the entire group was able to transgress this barrier and understand Fermat's mathematical writing through a productive group discourse.

Esther recalls working in a small group and using that space to help her overcome a barrier of following a *non-rigorous proof*. Esther referred to an encounter when their group was navigating Fermat's instructions to divide by what Kleiner (2001) calls "Fermat's mysterious e." (The "e" to which Kleiner refers is not Euler's constant but functioned much like "*h*" in the much-later developed concept of the definition of derivative, for example, in the term ($x + h$).) She does not "understand why the author is doing it this way." The disconnect from following a proof (or, as was the case in historical sources, a "demonstration") which many modern mathematicians would not consider rigorous confines Esther to draw from modern mathematical techniques and standards. To make that connection, Esther asks questions in her small group, and "someone will either answer, or … that question will then make us talk about another topic relating to the next question." She explained the benefit of collaborating with her group members stating, "it helps me to think about the problem in a different way." In both Anna and Esther's situation, their discourse with group members benefited not only them but also others in the group. Esther claimed that in her group member's attempt to explain Fermat's adequality concept, the group member was able to clarify the concept for himself as well. She said, "I think that clarified it for the both of us, because that was something that he was trying to explain." Here, our perspective on discourse allows us to justify the progress made on the Fermat reading as a product of communication with the self and another person. Both Esther and her peer overcame a barrier through discursive action.

Whole Class Discourse

Recall that discourse is more than just speaking. It is participating in communication which includes listening, talking, writing, and any other exchange of information that draws people together into a discursive community. Those that are drawn together in *whole class discourse* are students, instructors, and even the historical mathematicians in whose writing the class is engaged. This discourse is a transgressive action because participating in classroom discourse is a purposeful action, and students could make the choice to tune out the instructor and their classmates in a whole class setting. *Whole class discourse* occurred when the class was interacting together to make progress on PSP tasks and primary source excerpts. This was not simply a lecture delivery from the professor but an exchange of ideas among the professor and students in the class (both on Zoom and in-person). Anna stated the course instructor's expertise "in the main session is really helpful to steer the conversation and steer us working through the PSP together." Notice that Anna talked about interacting with the expert instructor. This is an instance where there can be more than one transgressive action at play. The class receives expert interaction, but the interaction exists to guide the whole class conversation as they make progress on the PSP together. Sandra indicated she encountered the barrier of *navigating a new experience* of reading mathematics as words without symbols. To her, the discourse among the whole class is " a bit more interactive than like most math classes, but it's still similar." The similarity lies in how she is "already used to… learning math which is like the teacher is up there… guiding us." She claims that in the whole class setting, there are "a few people who… , want to answer or like to answer, and they can… carry the rest of us. So, it's less pressure." Notice the affordance of a whole class discourse. Sandra, who disclosed in the interview that she had issues with anxiety, feels more comfortable when there is communication among the whole class. The way the instructor facilitated discourse among the whole class was handled delicately.

Sandra enjoyed the interaction because she could participate in modes of discourse other than constant speech. She compared this course to her Spanish course saying, “I really liked that this one is interactive but not, like oppressive.”

Discussion

Although Kozielecki’s (1986) theory is primarily cognitive in nature, when paired with Sfard’s (2008) commognitive theory, the findings suggest that transgressive actions may also be sociocultural. The limited research that has been conducted in this area focused primarily on transgressive actions in which the transgressor acted alone. However, the purposeful action of engaging in discourse with others has theoretical implications of unveiling more sociocultural transgressive actions in future projects.

Additionally, the discourse theme highlighted in this paper provides evidence that learning mathematics can be a collaborative process just as situated learning theorists, such as Lave and Wenger (1991), have suggested regarding entrance into any community of practice. Students often referenced the transgressive action of communicating with an expert or a peer in order to understand the material that was presented to them. This is an interesting finding which hints at denouncing a more traditional, lecture-style course delivery. Much of the History of Calculus course was rooted in the promotion of discussion among peers. Although students may feel uncomfortable in this non-traditional sense, it ultimately helped them overcome many barriers that one would expect students to face in any mathematics course. I have highlighted that expert discourse can help students overcome barriers of *unfamiliar language* and *background mathematical knowledge*. The expert probably has knowledge which aids them in identifying gaps in mathematical knowledge and allows guidance in discovery of the knowledge needed to be successful in the mathematical task. Group discourse can help students overcome barriers of *navigating a new experience* and following a *non-rigorous proof*. The collaborative nature of a group discussion could help students form connections with each other in which they build trust. That bond may allow group members to feel safe when making mistakes and asking for help when they have trouble reasoning about mathematics. Whole class discourse also leads students in *navigating a new experience*. Students converse with knowledgeable individuals who can actively guide their thinking. This dynamic guidance may not be present in a textbook or lecture notes which is more typical to a mathematics class according to students who were interviewed.

In essence, students may require a more personal collaborative transgressive action when reading mathematical text, especially when that text is from primary historical (mathematical) sources. In these cases, it was not just the unfamiliar language, but students stated they did not know where to begin or how to think about the text. The text may have been written in English, but the writing style and language conventions from several centuries ago proved to be an obstacle. It is apparent that group, expert, and whole class discourse seemed to bridge the gap and allow students to make meaning of the mathematical text. As Sandra stated, “some of the stuff is just… it gets a little confusing, and I don’t know how to… translate it properly. But I’ve just been… asking some other people that are in the class… for… a little bit of help.” Sandra is aware of the power of discourse. She and her classmates took advantage of its supportive nature and were ultimately victorious over barriers they faced.

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Understanding Conceptualizations of Active Learning in Mathematics Departments

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Several mathematics departments have increased their use of active learning to address low student success rates. However, it is unclear whether those involved in active learning have a consistent conceptualization of it. Like other educational terms, the phrase "active learning" is in danger of becoming overused and misunderstood, which puts the utility of active learning into question. This study examines 116 conceptualizations of active learning across six institutions in the process of change to use active learning. Analysis included three different comparisons: by stakeholder, by institution, and by the roles of students, teacher, content, and equity. Findings show that many participants conceptualize active learning as student engagement and activities other than lecture. Only eight participants mentioned issues of equity. Comparison within individual institutions shows that departments may hold common understandings of active learning. Implications of these findings include a need and rationale for conceptualizations to evolve through professional development.

Keywords: active learning, definitions, departmental change, undergraduate mathematics

There have been numerous calls for mathematics departments to increase their use of active learning (AL) in undergraduate mathematics courses to address low success rates (e.g., Conference Board of the Mathematical Sciences, 2016). Decades of evidence point to AL as promising in increasing student success in postsecondary education (Freeman et al., 2014; Theobald et al., 2020). Although AL has the potential to positively impact student learning, instructors may not know what AL means or be familiar with the research about its benefits (PCAST, 2012). Decades ago, Bonwell and Eison (1991) recognized the pitfalls in having no common definition of AL. At the time, national calls for faculty "to actively involve and engage students in the process of learning" were becoming prominent (Bonwell & Eison, 1991, p.iii). Yet, such terms were not consistently interpreted by faculty in ways that higher education researchers intended. In one study, faculty were asked how they determine if students are involved in learning; responses made it clear that faculty considered "involved" to be synonymous with "paying attention" or "being alert" in lectures rather than engagement with material (Stark et al.,1988, p. 95). Whereas AL is now a widely used phrase in higher education, the lack of a universally accepted, nuanced definition allows for a wide interpretation of what AL is. Therefore, one might expect that mathematics departments aiming to incorporate AL in their courses may choose a variety of instructional methods and pedagogical approaches, with varying effect. With over three decades of use as a phrase in education, "active learning" sometimes is treated as a buzzword. As a result, people may hold multiple, contradictory definitions of AL. To

[1] All authors contributed equally to this work with Williams starting and providing ongoing leadership.

successfully improve student outcomes, it is critical that departments develop a shared vision for what effective instruction looks like and have the support to carry out this vision (Smith et al., 2021). This study analyzes the conceptualizations of AL of different stakeholders in several mathematics departments across the United States. Based on this analysis, we argue there is a need for continued professional development focused on AL.

Literature Review

One definition of AL is a teaching approach used during class that engages or involves students in the learning process through methods that are not lecturing (Prince, 2004). Many scholars expand on this definition, emphasizing the importance of students engaging in activities which require higher-order thinking (analysis, synthesis, and evaluation), support the development of skills, and allow for exploration of attitudes and values (Bonwell & Eison, 1991; Freeman et al., 2014). Many of these characteristics are in the working definition that Freeman et al. (2014) developed after analyzing definitions provided by over 300 biology faculty:

> Active learning engages students in the process of learning through activities and/or discussion in class, as opposed to passively listening to an expert. It emphasizes higher-order thinking and often involves group work. (pp. 8413-8414)

Note that this definition includes an emphasis on group work. Although not always included in definitions of AL, methods that support collaboration between students (e.g., pausing lectures to let students reflect with a partner, small group work on problems) are generally considered a key part of AL (e.g., Laursen & Rasmussen, 2019).

AL is sometimes considered to encompass other approaches such as inquiry-based learning, problem-based learning, cooperative-based learning, and collaborative based learning (Laursen & Rasmussen, 2019; Prince, 2004). For example, Laursen and Rasmussen (2019) consider inquiry instruction to be a branch of AL that distinguishes itself by emphasizing student learning through a sequence of coherent, challenging tasks that allow students to create and reinvent mathematics. Laursen and Rasmussen distilled inquiry instruction into a set of four pillars: *(1) students engage deeply with coherent and meaningful mathematical tasks, (2) students collaboratively process mathematical ideas, (3) instructors inquire into student thinking, (4) instructors foster equity in their design and facilitation choices.*

Although focused on inquiry, we consider these pillars to be a useful working definition for AL. Unlike many descriptions of AL, these pillars bring attention to the significance of instructors' roles in shaping learning opportunities within the classroom. Furthermore, the fourth pillar calls attention to the importance of attending to issues of equity in the classroom, a characteristic that is notably absent in many scholarly definitions of AL. This inclusion is appropriate given recent research that calls into question whether all AL strategies equitably promote positive outcomes for students (Brown, 2018; Johnson et al., 2020). For example, Johnson et al. (2020) found significant differences in how men and women performed in classes following an inquiry-oriented curriculum. Their analysis suggests that men received more benefits (measured by learning gains) from the curriculum compared to women. Such research points to the importance of an explicit focus on equity in AL.

Despite clear evidence that AL strategies can improve student outcomes (e.g., Theobald et al., 2020), there is a lag in widespread adoption of AL in undergraduate education (e.g., Lane et al., 2020; Stains et al., 2018). It is exceedingly complex and time-intensive to help instructors learn how to implement new teaching techniques. To be sustainable, such efforts must be systemic and accompanied by cultural change at the department and institution level (Smith et al., 2021; White et al., 2020). Instructors must believe that improving their teaching (and associated student learning) is valued at the institutional and departmental levels (through internal grants, annual review, and promotion/tenure guidelines); otherwise, the substantive efforts involved are not worth instructors' time (Smith et al., 2021). Resources such as the MAA's *Instructional Practices* Guide (2018) and the EQUIP observation protocol (Reinholz & Shah, 2018) provide useful tools for departments and instructors seeking to adopt AL and improve equitable outcomes for mathematics students. At the core of improvement efforts, however, there must be a common vision of improved or effective mathematics teaching and learning (Elrod & Kezar, 2016).

Purpose and Research Questions

This research is a part of a larger mixed-methods study investigating institutions seeking to infuse AL into their precalculus and calculus courses. In this proposal, we examine how relevant stakeholders in these institutions conceptualize AL. To address this purpose, we consider three research questions: **RQ1:** How do conceptualizations of AL involve roles of the student, teacher, content, and equity? **RQ2:** How do conceptualizations of AL compare among different types of stakeholders across several institutions? **RQ3:** How do conceptualizations of AL compare among different stakeholders within one institution?

Methods

The SEMINAL project takes a multi-case study approach to understand what conditions, strategies, interventions, and actions at the departmental and classroom levels contribute to the institutionalization of AL in the undergraduate calculus sequence across varied institutions. The SEMINAL Project collected data from nine incentivized case study sites: mathematics departments provided with resources to institutionalize AL in their precalculus and calculus courses. Data from six of those sites comprise the foundation for this paper. All six medium-sized institutions have graduate programs in mathematics; five are minority-serving institutions and the sixth is approaching that classification. The SEMINAL research team conducted 2-3 day site visits at each of these sites during 2018-2019 and 2019-2020 which included 20-30 interviews of a wide range of stakeholders, including students, faculty, and administrators. For full details on SEMINAL methodology, see (Smith et al., 2021). Participants were typically asked to define, describe, or characterize AL. On occasion, depending upon the participant's role in the change process, they were asked what makes a good AL task, whether the department had a shared vision for AL, or about their level of commitment and experience with AL. Responses to this part of the interview were extracted and analyzed for this study. For validity purposes, our research team randomly selected a small subset of the interviews based on their variance in roles and institutions. We individually analyzed these interview excerpts using an initial codebook informed by our review of literature on AL and our field notes from the interviews. We then met to reconcile codes and revise our codebook, which was used to code the rest of the interview excerpts (see Table 1 for a list of our codes). There were a total of 116 excerpts from participants across six institutions and six stakeholder roles.

To analyze interview excerpts by stakeholder role, we first examined the frequency counts of particular codes and code combinations (e.g., *student* + *not active learning*) and looked for initial patterns. The prevalence or rarity of certain codes led to a further analysis of the first four codes in Table 1: student, content, teacher, and equity. We completed rounds of *in vivo* and open qualitative coding (Saldaña, 2016) for each of the four codes and sorted excerpts into data-driven themes. Later, we constructed pivot tables to examine the frequency of each code combination (16 possibilities) for each stakeholder role and to compare across roles. Finally, we re-introduced the code *not active learning* to our analysis since it was a commonly coded descriptor across all stakeholder roles. To analyze interview excerpts by institution, pairs of researchers analyzed the data by each institution, using an open-coding process to identify patterns in how stakeholders at each institution discussed AL. Throughout this process, we met as a large group to discuss our findings across institutions and stakeholder roles.

Table 1. Codebook

Stakeholder attended to in their definition	Description
Student	Describes what students are doing or expected to do
Content	Describes what math is being worked on for AL
Teacher	Describes what the teachers are doing or expected to do
Equity	Mentions anything about equity, equal opportunities for students, etc.
Not active learning	Defines AL by what it's not, such as “not lecture”
Specific strategies	Gives an example(s) of specific strategies for implementing AL
Citing a definition	References an "established" or "scholarly" definition or theory; e.g. the four-pillars, discovery-based learning, inquiry-based learning
Problematic	Researcher interpretation; problematic way of describing AL

Findings

We had hypothesized that we would find differences in definitions of AL based on stakeholder role; for example, we expected instructors might have more nuanced definitions than administrators. However, while we found many differences in definitions, none of those differences were captured when separating the responses by stakeholder role. Thus, our conclusion for RQ2 is the variations in conceptualizations of AL are not explained by the differing roles of the stakeholders we interviewed. For the remainder of this section, we present our findings for RQ1and RQ3 separately and in this order.

RQ1: Comparing Definitions to Students, Teacher, Content, and Equity

The most common code used for all stakeholders was *students* (100 of the 116 excerpts); stakeholders were most likely to describe what the students were doing in their definition of AL. Further analysis of the *student* code revealed that each type of stakeholder was describing what students are doing for AL in similar ways, but ranged from being specific to very vague. Common expressions used were “engaged,” “think and do something,” “working with each other,” and “teaching the problem to someone else.” One instructor said that students needed “to futz around with the distance between separating the intervals and see if they can arrive at a visual intuition at what a derivative is” to give a specific example of what they meant by students building knowledge. In contrast, only eight excerpts were coded as addressing *equity*; typical responses referenced strategies that were good for “all” students.

The second most common code across all stakeholders was *not active learning*. Many of the descriptions involved definitively stating that AL is not lecturing: “It’s the kids doing the math, not me;” “Math is something you do and not watch;” “definitely a more interactive experience, not lecturing;” and “Students have to do more than just sit and listen to a lecture. They have to do mathematics in order to learn it.”

Analysis of the *teacher* (57 excerpts) code revealed three themes that present opportunities for instructional improvement with AL. First, many of the participants' descriptions of AL implied some role of the teacher but did not make explicit the teachers' actions during instruction. One participant said AL meant "engaging back and forth with the students" while another said AL meant "having an engaging environment for students." In both of these teacher roles, it is not clear what exactly the teacher is doing to engage students. Second, many participants mentioned "facilitating group work;" however, few mentioned specific facilitation moves, such as asking questions or eliciting student thinking. The difference in the grain size of these responses makes it difficult to know what teacher moves the participants consider to be part of facilitation of AL. Finally, many descriptions of the teacher's actions were reliant upon student actions (e.g., "students have to struggle through the problems by themselves"), as opposed to describing how "the teacher provides opportunities for students to reflect" and engage during class.

Of the 31 excerpts that addressed mathematical *content*, 22 of these were from participants in the instructor role. While *content* was not commonly included in descriptions of AL, those participants who addressed content mentioned key components of the first pillar of inquiry-instruction (Laursen & Rasmussen, 2019); for instance, participant's definitions included "not just memorizing," "multiple ways to solve," appropriate ability levels, low floor-high ceiling tasks, "tactile" hands-on activities, drawing on relevant topics, and relating to prior knowledge. Thus, all together, the participants who addressed content created a robust understanding of appropriate mathematical tasks for AL classrooms.

RQ3: Institutional Definitions of AL

Each of these institutions were engaged in change efforts to implement or infuse "active learning" in their precalculus and calculus courses. From prior research, we know that successful change efforts have a shared vision and aim for the initiative. Therefore, we wanted to examine the extent to which each institution had a locally shared definition of AL. While not all individuals at the same sites shared the same definition, we did identify common themes in how they were defining AL. Examining responses across each institution, there were distinct ways in which the individuals were conceptualizing AL in their local contexts that further illustrate the codes for *students* and *strategies* (summarized in Table 2).

At institution 1, the shared definition of AL was primarily focused on students not listening to lecture. The most prevalent strategy to accomplish this broad definition was the use of metacognitive strategies to have students reflect on their own learning process. At institution 2, AL was defined as having students engaged in the learning process or having the teaching and learning being student-centered. This institution was also the one with the most references to equity in their definition of AL. Strategies used to foster this definition were the use of technology such as polling systems, videos, and flipped classrooms. Institution 3 is rather unique since they have multiple campus locations that are geographically separated. Many of the stakeholders from institution 3 discussed how their definition of AL may be different from the other campus location. The definitions at this institution were too varied to identify an emerging theme for students, but many stakeholders discussed the use of group work as a strategy for AL.

At institution 4, stakeholders emphasized the importance of students discovering new knowledge and being involved in the learning process. The strategies discussed to facilitate discovery-based learning included tactile activities (e.g., derivative domino train) and board-work. As one instructor stated, AL, "allows students to kind of generate the knowledge that you're trying to teach them on their own." At institution 5, stakeholders viewed AL from a largely cognitive perspective in which students must do their own “thinking." As the director of the student success center stated, AL features “engagement that the *mind* is active while learning is occurring.” Institution 5 had the most varied responses in terms of strategies referenced across stakeholders and within each individual stakeholder’s definition. Some of the strategies cited included flipped classrooms, think-pair-share, low-entry point but challenging tasks, and board work. At institution 6, AL largely focused on students communicating and engaging with mathematics. This was accomplished through group work, worksheets and reading the textbook.

Table 2*. Local Conceptions of Students’ Role in Active Learning and Strategies to Support Active Learning, Summarized by Institution*

Site	Prevalent definition of students’ role in active learning	Prevalent instructional strategies for active learning
1	Not listening to lecture	Metacognitive reflection activities to help students learn how to learn
2	Engaged in the learning process	Using technology
3	N/A (too varied)	Group Work
4	Discovering and communicating new knowledge	Tactile activities/hands-on learning, color-coded response cards, group work, and board work
5	Participatory cognitive thinking	Group work, flipped classrooms, think-pair-share, low-entry point but challenging tasks, board work, and classroom discussions.
6	Communicating and engaging with course topics	Group Work; worksheets; reading the textbook

Discussion

There was a wide variation in conceptions of AL, yet most stakeholders emphasized student engagement and *not active learning*, which suggests that stakeholders recognize AL requires giving students a role in the classroom which is, in many ways, nontraditional. Furthermore, the emphasis on students engaging *with one another* suggests that the second of Laursen and Rasmussen’s (2019) pillars: *students collaboratively process mathematical ideas* is a common component of stakeholders’ conceptions of AL. While *content* was less frequently coded, those who used it showed a sophisticated understanding that parallels pillar one: *students engage deeply with coherent and meaningful mathematical tasks*. At the same time, stakeholders sometimes neglected to consider the role of the teacher in their definition of AL. Teachers play a pivotal role in designing and facilitating good AL tasks and setting up norms that support an inclusive space. Moreover, our research suggests that equity is rarely made explicit in definitions of AL (it was the least used code of *students, teacher, content,* and *equity*), despite common assumptions that AL is more equitable. Together, our analysis demonstrates that the third and fourth pillars: *instructors inquire into student thinking* and *instructors foster equity in their design and facilitation choices* are often not explicit in definitions of AL.

Although we did not notice significant differences in how different stakeholders conceptualize AL, there were clear themes in how stakeholders within particular institutions conceptualized AL. For example, *not active learning* was a common code for definitions from institution 1. This is not too surprising considering that local change efforts focused on meeting instructors where they were at, and the phrase "not lecture" was intentionally used to encourage hesitant instructors to take steps toward AL. Similarly, institution 5 has incorporated several professional development activities to feature and promote various AL strategies within the mathematics department, and they promote AL as a spectrum of strategies. We posit that this approach to change led to the variety of strategies that appeared in conceptualizations of AL provided by stakeholders within this institution. At institution 6, AL largely focused on students communicating and engaging with mathematics through a structured approach to class time that featured mini-lectures and small group discussions. These course elements are reflected in the definitions of AL provided by participants which emphasized group work, worksheets and reading the textbook.

Limitations, Implications, and Future Work

This research was conducted within a broader grant-funded research project that narrowed the population of interest to mathematics departments with graduate programs. At every site, a representative sample of stakeholders were interviewed. However, the focus of this project was precalculus and calculus courses, which are broadly applicable across two and four year colleges. The interviews were conducted by a number of different research teams across the six sites whose data are featured in this study; at times, different interview questions were asked of people in particular roles. Thus, the data about AL tasks are drawn only from the subset of stakeholders who were asked that question. Despite this variation, interviewees were consistently asked their definition of AL. Finally, this study focused on people's reported definitions, not their observed classroom practices.

Our analysis only focuses on reported definitions; we know departmental norms and conversations around teaching influence what instructional practices are used (Author et al., 2021). We see nuanced understanding of AL strategies as a necessary but not sufficient condition for the effective use of AL strategies in mathematics classes. The importance of nuance is apparent in the following definition from one instructor in this study,

> So, to be *engaged* and *involved* in what you learn, not to be just a member of the audience, but to process the information that you get to practice it, and if possible, which is not easy in mathematics, to try to push the knowledge by yourself a few steps away without the help of the professor, based on what you got from the lecture in class.

This quotation corroborates findings from Stark et al. (1988) that faculty may not interpret the terms engagement and involvement in ways that educational researchers do. A number of participants had similar definitions that focused more on the mental experiences of learners rather than on the types of in-class tasks that are likely to support such mental experiences. Such cases demonstrate the importance of understanding stakeholders' conceptions of AL as a foundation for future work that ties beliefs and practices.

Finally, our findings indicate that departments who regularly discuss AL develop common understandings of what AL entails. This suggests that departments seeking to improve their instruction via AL should provide ongoing professional development and conversations about AL. Such professional development should build on a common understanding of AL as student engagement to include other aspects of AL, such as explicit guidance on the instructor's role, development of good tasks, and how to build equity while using AL methods.

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One Preservice Teacher's Refined Understanding of Compactness in Support of Her Technological Innovations Used in Planning a Lesson on Gerrymandering

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This report describes how one preservice elementary mathematics teacher refined her understanding of compactness of polygons over a two-week intervention unit dedicated to introducing preservice (PSTs) to the idea of teaching mathematics to support the development of a sociopolitical disposition (Bartell et al., 2017). The study took place in a university course designed to help PSTs learn mathematics content and pedagogy as well as how to use mathematical action technology (Dick & Hollebrands, 2011) in their classrooms. Utilizing Brown's (2009) Pedagogical Design Capacity Framework, I determined instances of adaptation which I refer to as "technological repurposing" in this context. The results highlight the progression of one PST's understanding of compactness and how this contributed to making an innovative technological adaptation to a set of lesson resources developed to help middle school students make sense of Gerrymandering.

Keywords: technology, conceptual understanding, preservice teachers

Teaching mathematics with technology and teaching mathematics to develop a sociopolitical disposition are complex practices that mathematics teachers can engage in to better support student learning. In order to better support teachers in engaging in these practices, it's important to understand more about what enables teachers to adapt their curriculum resources to meet these goals. Few studies have examined how teachers utilize external (e.g., curriculum) and cognitive (e.g., content knowledge) resources in making curricular decisions to support the multifaceted goals of teaching with mathematical action technology (MAT) (Dick & Hollebrands, 2011), teaching mathematics to develop a sociopolitical disposition (Bartell et al., 2017) and teaching for conceptual understanding (Simon, 2018). In the pages that follow, I highlight one prospective elementary mathematics teacher's refinement of her understanding of *compactness* of polygons (as measured by the Polsby Popper Index) and how this enabled her curricular decision to make an innovative technological adaptation to a lesson activity related to how compactness can be utilized to understand Gerrymandering.

The Course

The course where the intervention unit took place can best be described as part mathematics content, part methods, and part technology. In years past, the course was developed with the goal of developing Preservice Mathematics Teachers' (PSTs) *technological pedagogical content knowledge* (TPACK) (Koehler & Mishra, 2009). The semester in which this study took place (Spring 2021) occurred in the midst of the COVID-19 pandemic which means that every meeting session was held virtually via a Web conferencing application. As a regular practice in the course, we analyzed curriculum and technological applications and collaboratively planned lessons that met the dual goals of integrating mathematical action technology (MAT) (Dick & Hollebrands, 2011) and teaching for *conceptual understanding* (Simon, 2018). Though what constitutes conceptual understanding is a frequently debated topic (Baroody, 2007; Star, 2007), in this course I chose to include Simon's (2018) *What is a mathematical concept?* as one of our assigned readings. This reading assignment was treated as a starting point for our conversations about writing *conceptually oriented goal statements* (COGS). It was chosen because of its focus on justification and goal-directed planning for instruction.

For this semester, an additional two-week long unit engaged PSTs in a lesson design experience based on the curriculum resource, *High School Mathematics Lesson to Explore, Understand, and Respond to Social Injustice* (Berry III et al., 2020). In this unit PSTs were assigned the task of adapting the curriculum so that it integrated MAT. The section that PSTs adapted was related to understanding the social injustices surrounding Gerrymandering. Despite the relevance to national political conversations, I also found that the visual nature of this content to be particularly conducive to integrating a more dynamic version of the lesson with help of Desmos Activity Builder components.

Desmos Activity Builder (DAB) is a free online software that allows lesson designers to integrate Desmos's graphing calculator software with other "components" to create a module of slides for students to work through independently or for an instructor to facilitate with in a classroom. The "components" that we utilized the most in the course were text boxes, dynamic geometry environments, graphing calculators, and text input boxes where students can submit responses to prompts. DAB played a major role in both the facilitation of the learning experience for the PSTs and the Lesson Revision Assignment. During the facilitation of the lesson, I built a DAB lesson experience[1] adapted from the written materials from Berry III et al. (2020) and PSTs were instructed to revise these DAB slides for their Lesson Revision Assignment.

Theoretical Perspectives

As part of my analysis, I adapted Brown's (2009) *pedagogical design capacity framework* (PDC) to gain insights into PSTs decision making when designing a lesson that integrates both MAT (Dick & Hollebrands, 2011) and a *critical mathematics context* (Rubel & McCloskey, 2021). Both MAT and critical mathematics contexts played a secondary role in this analysis, but were relevant to my planning the intervention sequence. In particular, Dick & Hollebrands's, "A Guide for Choosing and Using Interactive Technology Scenarios," (2011, pp. xvi-xvii) was discussed and utilized frequently throughout the course and Rubel and McCloskey's (2021) framing of the *contextualization of mathematics* helped me to better understand the PSTs beliefs around the role of contexts in mathematics tasks.

Framing the analysis around the PDC framework allowed for PSTs level of agency in relation to the curriculum to be made more transparent. Conceptualizing this level of agency through the constructs of *offloading, adapting,* and *improvising* allowed for a glimpse at PSTs knowledge, values, beliefs, and goals related to designing tasks that support conceptual understanding, integration of technology, and incorporating critical mathematics contexts. Brown (2009) describes *offloading* as utilizing the curriculum as it is written and *improvising* as when teacher's personal resources are playing a more dominant role in the design of curriculum. These constructs did appear in the data gathered, but in this paper I focus on *adapted* content since it allowed for a more transparent look into PSTs decision making. Given that the curricular materials came in a digital format, I described certain acts of *adaptation* as acts of *technological repurposing* due to the way that much of the original technological components remained intact. In common usage, repurposing refers to attempting to use (perhaps with slight modification) objects in a way that differs from its original intent. For example, a person may utilize an object that is difficult to create on their own (for example a vehicle tire) for a purpose that it was not intended for (swinging from a tree). Since the digital curricular materials involved technology

[1] Link to the activity will be made available in unblinded paper

that the PSTs had varying levels of experience with, it may be the case that that this “difficulty to create” factor contributed to keeping much of the original content intact, but utilized to meet a new goal.

Methods

I define the instructional unit of focus as the sequence of classes in which PSTs engaged in the learning the lesson materials as students (Session 1) then brainstormed ideas for how they would implement the same or similar content with middle school students (Session 2). This instructional unit also includes the work of reflecting on the lesson content (Post-Session 1 Reflection Assignment), and creating a revised set of DAB slides to submit for their assignment (Post-Session 2 Assignments). These sessions served as the contexts where the data sources for this study came from. The analysis of classroom video data, discussion transcriptions, and the PSTs documentational work submitted to me after Sessions was conducted in support of the research question:

> What resources do PSTs draw from as they adapt curricular resources that integrate a critical mathematics context to meet the dual goals of teaching mathematics for conceptual understanding and for developing an understanding of matters of social injustice?

As the instructor of record for the course, I obtained IRB approval to analyze my students' submitted assignments and video recorded class meeting sessions. After my students were assigned grades for all of their work in this sequence of activities, I emailed a request to each student to obtain their consent to analyze their submitted assignments and the video recorded interactions that took place during our regularly scheduled class meeting time. Of the 15 PSTs enrolled in the course, eight of them agreed to participate in the study. Once granted permission, I compared my original implemented DAB slides with the revised DAB slides submitted by each participant. I noted how many slides were in their set, what order their slides were in, and to what degree they made any changes to an “original slide” (i.e., a slide that I provided to them). I found that most participants kept many of the original slides as is. When this occurred, I coded this as an act of *offloading* (Brown, 2009). Some participants added new slides to supplement the original content. In this case, I coded this as an act of *improvisation* (Brown, 2009). Some participants kept much of the content on an individual slide intact, but either edited the information, question prompts, or the MAT on the slide to highlight a new idea. These were coded as acts of *adaptation* (Brown, 2009). Once the PDC coding concluded, I defined a subset of adapted slides as “repurposed slides” where the technological capabilities along with the questions prompts and informative content were kept mostly intact with some modifications which were made to support a new goal. In the results to follow, I focus on how one student's engagement in class discussions and submitted reflection assignment helped to better understand what enabled her to engage in an innovative approach to repurposing a technological resource.

Results

Of the eight participants, *technological repurposing* was observed in the three participants' revised DAB slides. Here I focus on one student's development of a refined understanding of *compactness* and how this affected her adaptation of the curricular resources.

Summary of Megan's Technological Repurposing

Megan (pseudonym) adapted a DAB slide where the original goal was to have students explore how changing the boundaries of a polygon affects the value of the Polsby Popper Index[2]. In the original slide (see Figure 1) that I implemented with my students, I wanted them to discover how the polygon's "P-index" increases as it becomes more regular. I did define "P-index" later in Session 1, but did not provide the formula until after Session 1. In Megan's repurposed slide (see Figure 2), she animated the point "p3" (green, top middle) to constantly vary over a given set of y-coordinates while fixing the coordinates of every other vertex. In doing so the animation displays a lower "P-index" when the shape becomes concave. In Megan's rationale for her adaptations, she states that her goal for her students would be to "*...look at the difference in the [Polsby Popper Index] while considering how the area and perimeter changes...*"

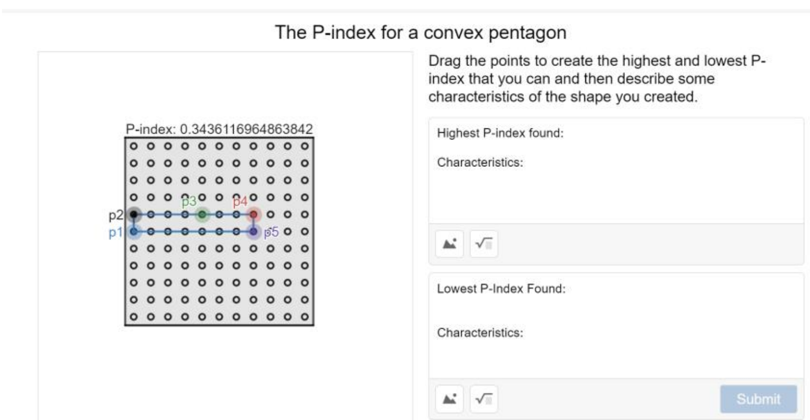

Figure 1. Original Slide Megan repurposed

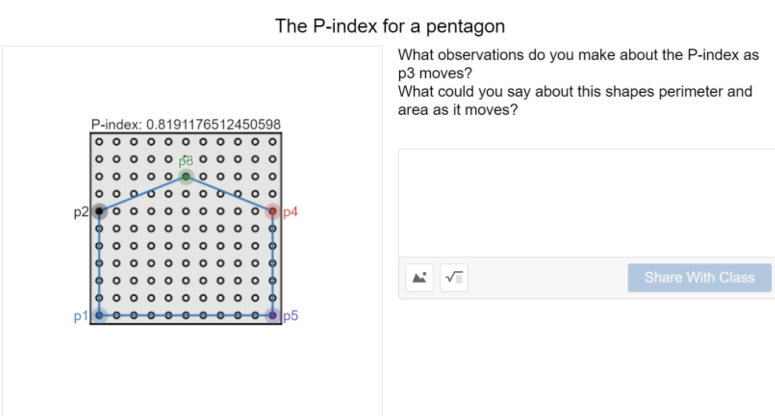

Figure 2. Megan's Technological Repurposing

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[2] The Polsby Popper index is commonly used as a measure of *compactness* of a shape which has been used to identify instances of Gerrymandering in legislative districts and is given by the formula $\mathrm{PP(D)} = 4\pi \cdot \frac{\mathrm{Area(D)}}{(\mathrm{Perimeter(D)})^2}$ where D is the given shape. In this activity, we focused on polygons and I introduced it as "p-index" to avoid students searching the web for more information prior to their exploration.

In Session 2 of the intervention, I concluded the class with an instructor-led demonstration with the slide shown in Figure 1 to highlight how the Polsby Popper Index is determined solely by the area of a shape when the perimeter is held constant. This proved to be a notable part of the intervention for Megan's refinement of understanding since her technological repurposing was done to highlight this idea in a very similar way to how I demonstrated to the class. I take this to be evidence that Megan found this to be a valuable demonstration to her own understanding and that it could be valuable to middle school students as well. It is not clear how familiar Megan was with the idea of compactness prior to the start of this intervention unit, but in her post-lesson reflection assignment she states how she, "was unfamiliar with the pp index." [*Megan's Post Session 1 Homework Reflection*]

Below I describe the progression of Megan's understanding of compactness and how this contributed to her technological repurposing. In Session 1, Megan engaged in the content as a learner and claimed that she experienced computer problems during some of the lesson, but was still able to formulate some initial ideas about the characteristics of compact and not compact polygons. After Session 1, the class was assigned a news article[3] to read about Gerrymandering where they were asked to reflect on how what they read about Gerrymandering connected to their exploration in class. In Session 2, Megan and her classmates were assigned to small groups to discuss what they liked and disliked about the activity they engaged in and then as a whole group they shared out some of the small group discussion main points. At the end of this session, a student had a question about the connection between the area, perimeter and Polsby-Popper Index that I addressed with a demonstration. Megan and her classmates then had one week after this session to ask other questions about the assignment or content and submit a revised version of the activity that they felt could be implemented with middle school students.

Megan's Refinement of Her Understanding of Compactness

Session 1. During Megan's first interactions with the slides geared towards understanding the compactness of polygons, she observed that compactness had to do with how, "the angles and sides relate to each other like how close they are" [*Session 1, In-class Activity, Slide 9*]. Here I have inferred that she meant "all of the angle measures were close to being equal" and all of the side lengths were close to being equal." This is supported in responses recorded on future slides. In particular, on the slide where she worked with quadrilaterals, she accurately observed that any square results in the same highest Polsby Popper value possible regardless of its area. She also states for the pentagon, that the highest Polsby-Popper (referred to as P-Index in class), results when a shape "looks fairly even like they have the same or similar side lengths and angles" [*Session 1, In-class Activity, Slide 4*]. It seems reasonable to me that she was able to generalize her engagement with the quadrilaterals to her responses given above related to the shape needing to be close to regular.

Although Megan appears to be making the connection between regular polygons and the value of the Polsby-Popper Index, when asked about the characteristics of polygons that have a low Polsby Popper value during Session 1, she consistently attends to observations about the shape containing obtuse angles. She states, "obtuse angles made a smaller p index." I take this to mean that within this context, she realized that all of the shapes that yield low Polsby Popper values contain obtuse angles, but she does not state how they also have at least one acute angle and that it is when there are large differences in the angles that the Polsby Popper values tend to be lower.

[3] https://truthout.org/articles/35-states-at-risk-of-rigged-districts-due-to-gerrymandering-report-finds/

Post Session 1. After Session 1 when prompted on her reflection assignment to respond to the following, "Using [the Polsby-Popper formula] and what you know what Gerrymandering, please explain why a Gerrymandered district might give a smaller [Polsby-Popper value]". She responded,

> "Gerrymandered districts are not as compact. When they manipulate districts, the area is smaller in comparison to the perimeter. Since the area is smaller while still having a larger perimeter, it has a smaller pp index." [*Post Session 1*, *Homework Reflection*]

I took this response to be evidence that she was beginning to refine what she knows about compactness to make it more mathematically formal.

Session 2. In Session 2, PSTs had the opportunity to collaboratively think about how they might implement a similar lesson (with middle school students) to the one they experienced in Session 1. In the passages below, we see how Megan seemed to maintain her previous understanding of compactness and its role in Gerrymandering. She appears to be trying to think about how she can make the connection clearer to students. In her small group planning conversation she states,

> "I had a little bit of an issue going from the p-index to the Gerrymandering part of it, so I'm trying to figure out how to make that transition a little bit more smooth to where like the connections can be made easier." [*Session 2*, *Small Group Planning Discussion*]

Also, in this discussion, when asked by a classmate if Gerrymandered districts have a high or low Polsby Popper Index she states,

> "I think it's low because the way that they are manipulated, in like weird shapes and get stretched so then their perimeter's larger than the area that they make up. But I think that influences the P index from my understanding." [*Session 2, Small Group Planning Discussion*]

Minutes later when explaining to the whole class, she states,

> "Since the perimeter was manipulated to be a little bit larger in comparison to the area, the p value was really low…when we go from the p-index to the Gerrymandering, it'd be important to talk about why the lower [Polsby Popper Value] matters, because the districts that are a little bit manipulated more for gerrymandering will have a lower p-index." [*Session 2, Whole Group Discussion*]

At this point in the intervention, it appears that she knows there is a relationship between the area and perimeter and that the perimeter being "stretched" decreases the Polsby Popper Index, but as we will see below she later refines this to explicitly point out how that when the perimeter is *fixed* while the area is decreased, this results in a less compact shape.

Post Session 2. In her submitted lesson activity, she states her Conceptually Oriented Goal Statement as,

> "Students will be able to use their prior knowledge of area, perimeter, ratios, and proportions to understand the Polsby-popper index. They will manipulate shapes using Desmos grids and observe the changes in the P-index. Students will be asked to create shapes with similar perimeters but different areas, and make observations. Based on the observations students can conclude that if the area is smaller given the same perimeter, then so will the P-index. [Post-Session 2, Submitted Revised Activity]

In her submitted revised activity, she provides a slide (figure 2) that animates a point of a pentagon (similar to how I demonstrated in class) that fluctuates to make the pentagon change from convex to concave and asks students to consider the area and perimeter as the Polsby

Popper Index goes from a larger to a smaller value. She also adds in a note to me on a slide that states,

> "[I don't know] how to do it but I kind of feel like I want to help [students] create the connection of area and perimeter sooner because [I] didn't make the connection. Is it possible to show the area and perimeter as they change the shape???" [Post-Session 2, Submitted Revised Activity].

On the another slide she asks, "What influences or determines how compact something is?" and then her rationale in her submitted assignment states and asks,

> "…I added the pentagon with the slider because I wanted students to narrow down their thinking and observations. I didn't feel like I made the connections between the area and perimeter and compactness and pp index very well. I thought by adding this and asking students to look at the difference in the pp index while considering how their area and perimeter changes, they might grasp that better…Is it possible to show the area and perimeter as they change the shape? I think this could help if we showed it on some of the last slides." [*Post Session 2, Submitted Revised Activity*]

Discussion

In the results above, we can see how Megan began her understanding of compactness by attending to the regularity of polygons. Though she does not initially identify regularity of the shapes as an important feature, she does attend to the sides and angles being the same or close to same. In later utterances, we see Megan accurately describe how the compactness (as measured by the Polsby Popper Index) depends on the ratio of the area to the perimeter of the polygons. Near the end of the intervention lesson, we see how she articulates that when the perimeter is held constant and the area is smaller, this leads to a smaller Polsby-Popper Index.

The overall results of my analysis including Megan's peers brought out multiple approaches to thinking about the what the "P-index" was measuring. Some PSTs described connections to area, some described connections to angles, some described the ratio of area to perimeter, some described it as "close to circle," and some described it as "close to regular." These results may suggest the task students engaged in was conducive to eliciting multiple ways of thinking about a relatively novel concept. Compactness of polygons is typically not taught in K-12 schools and it would appear that this idea was new to Megan at the start of the intervention.

In this paper, I focused on Megan's *adaptation* (Brown, 2009) (or *technological repurposing*, in this case) because it served as an instructive example of how one student refined her understanding of compactness to aide her pedagogical decision with technology. This highlights the complexity knowledge resources involved in teaching with technology and teaching about *critical mathematics contexts* (Rubel & McCloskey, 2021) in mathematics classes. Megan had to use her prior technological knowledge, and her refined mathematical knowledge in order to make the animation that highlighted an important feature relevant for students to make a key insight about the content. This was an example of how her technological innovation was supported by her refined content knowledge which helped to provide what she believed to be a good pedagogical decision. In other words, Megan displayed her technological pedagogical content knowledge (TPACK) (Koehler & Mishra, 2009) in repurposing the provided curricular materials.

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Modeling Student Definitions of Equivalence:
Operational vs. Structural Views and Extracted vs. Stipulated Definitions

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This paper describes a model of student thinking around equivalence (conceptualized as any type of equivalence relation), presenting vignettes from student conceptions from various college courses ranging from developmental to linear algebra, and courses in between (e.g., calculus). In this model, we conceptualize student definitions along a continuous plane with two-dimensions: the extent to which definitions are extracted vs. stipulated; and the extent to which conceptions of equivalence are operational or structural. We present examples to illustrate how this model may help us to recognize ill-defined or limited thinking on the part of students even when they appear to be able to provide "standard" definitions of equivalence, as well as to highlight cases in which students are providing mathematically valid, if non-standard, definitions of equivalence. We hope that this framework will serve as a useful tool for analyzing student work, as well as exploring instructional and curricular handling of equivalence.

Keywords: Equivalence, Equation, Solution Set, Operational Thinking, Structural Thinking, Definitions

Equivalence is central to mathematics at all levels, and across all domains. In mathematics education, much research has focused on studying how students think about the equals sign in primary school (Knuth et al., 2006) through post-secondary (Fyfe et al., 2020), because students' conceptions of the equals sign have been shown to be related to their ability to perform arithmetic and algebraic calculations. However, equality is just one example of the larger concept of equivalence—other types of equivalence occur extensively throughout the K-16 curriculum, but are rarely, if ever, taught under one unifying idea called equivalence (Wladis et al., 2020). On the other hand, multiple types of equivalence (e.g., similar/congruent figures, function types, expressions or equations with the "same form") are contained in the *Common Core Mathematics Standards* but are never explicitly labeled as a type of equivalence.

When equivalence is not explicitly defined, students may extract their own non-standard, ill-defined, or unstable definitions, or they may inappropriately use the definition of equivalence from one area (e.g., expressions) in another area where it cannot be directly applied to obtain the "standard" definition expected of them (e.g., equations). In this paper we will illustrate this problem by presenting examples of students' definitions around equivalence and a model for analyzing student definitions, focusing on college students' definitions of equivalent equations. Student examples will be used as vignettes to illustrate the model. Our aim in presenting this model is to start a conversation about student definitions of equivalence and to present an initial framework that can then be further tested, refined, and revised by future empirical work.

Theoretical Framework

Formally, we define equivalence through the notion of an equivalence relation. The formal definition of an equivalence relation most often given in advanced mathematics classes is that of a binary relation that follows the identity, symmetry and transitive properties. However, another equivalent but more accessible definition of an ***equivalence relation*** is that of a partition on a set, or more informally: If we have a set of objects, and a rule for sorting objects into sets so that each goes into one and only one set (and this rule is mathematically well-defined), then this "sorting" is an equivalence relation, and two objects are equivalent if they belong to the same set.

We do not advocate at this time for teaching any particular group of students this generalized definition of an equivalence relation; we simply note that if we did want to discuss this more generalized definition with students, that the definition of a partition on a set is accessible to students at many different developmental levels (in fact, it bears a striking similarity to preschool sorting tasks in the mathematics curriculum). Our primary motivation for introducing this definition is to define equivalence rigorously—this includes not just definitions of equality, or insertionally equivalent equations (i.e., equations that have the same solution set), but also anything in the curriculum which meets the definition of an equivalence relation.

This will also help us to more precisely discuss student definitions. "Experts" often point out when students use "incorrect" definitions, but we note that in existing curricula and classroom practice the word equivalence is often ill-defined (or never explicitly defined), even though it takes on different definitions in different contexts. When students have no explicit definitions of equivalence, this presents several potential problems: students may incorrectly apply one definition to another context where it fails to produce the standard definition (e.g., definition of equivalent expressions to equations); they may have only ill-defined or operational definitions of equivalence which inhibit their ability to reason through problems; or they may use valid but non-standard definitions of equivalence, in which case they are being penalized for not knowing certain socio-mathematical norms even when they are reasoning correctly. We hope that the model presented here will allow us to better understand student thinking about equivalence, and to better recognize when these three situations (as well as others) might be occurring.

Model of Equivalence

Our model of student thinking about equivalence conceptualizes student definitions as existing on a two-dimensional plane with two axes: operational vs. structural conceptions of equivalence (Sfard, 1991, 1992, 1995), and extracted vs. stipulated definitions of equivalence (Edwards & Ward, 2004, 2008). In operational thinking, a student thinks of mathematical entities as a process of computation; in structural thinking, they think of them as abstract objects in and of themselves which can then been seen as objects for even higher-order processes; objects are seen as reified processes (e.g., $6x$ is seen as an object itself, and not just as the process of multiplying x by 6), however when students view something as an object which is not the reification of any process, this is called a *pseudostructural* conception (p.75, Sfard, 1992)[1].

Extracted definitions are created to describe actual observed usage (e.g., a student may extract a meaning for equivalence their instructional experiences, whether or not they have encountered an explicit definition). In contrast, *stipulated* definitions are those definitions that are stated explicitly—to determine if something fits the definition one must consult the definition directly (Edwards & Ward, 2008)[2]. We note that in our model, a stipulated definition may be

[1] We note that process and object dichotomy is also related to other theories such as APOS theory (Arnon et al., 2014) and the notion of a procept (Gray & Tall, 2011), but we have insufficient space to discuss these distinctions.
[2] Mathematical definitions are typically seen as stipulated rather than extracted, although there may be many (both correct and incorrect) features of students' concept images that stem from extracted rather than stipulated knowledge around the concept definition (see e.g., Edwards and Ward, 2004 for examples).

stipulated by the student or an authority—the key features we use to determine if a definition is stipulated in our framework is whether or it appears to be explicit, well-defined, and stable. We note that while we have displayed our model in Table 1 as a two-by-two grid for the sake of simplicity, these categories are not necessarily binary, but conceptualized as more of a spectrum. In that sense, Table 1 could perhaps better be represented by a 2D coordinate plane.

Table 1: Model of Student Thinking About Equivalence.

	Extracted Definition	Stipulated Definition
Operational Conception of Equivalence	**Pseudo-Process View**: Students see equivalence as a computational process, and their approaches to those processes are dictated by prior experience in ways that are extracted rather than stipulated. Definitions of equivalence are typically non-standard, ill-defined, and/or unstable.	**Process-View**: Students see equivalence as a process, but do process computations by referring to stipulated rules or properties. Students with this view may be able to perform calculations correctly but this does not necessarily translate to being able to use stipulated definitions to recognize equivalent objects.
Structural Conception of Equivalence	**Pseudo-object view:** The student is able to consider whether two objects are equivalent without reverting to an explicit computation, perhaps by considering the structure of the objects; but definitions of equivalence are typically extracted in some way from experience rather than based on stipulated definitions of equivalence, and as a result are typically non-standard, ill-defined, and/or unstable	**Object view:** The student is able to consider whether two objects are equivalent without reverting to an explicit computation, perhaps by considering the structure of the objects; definitions of equivalence used to determine equivalence are stipulated. The student conceptualizes equivalence classes (or solution sets) as objects, although they need not do this formally.

Method

Data for this study were collected from 124 students at an urban community college through open-ended questions in 18 different courses, from developmental elementary algebra (similar to Algebra I in high school) to linear algebra. Student responses were analyzed using thematic analysis (Braun & Clarke, 2006). Responses coded as indicative of an operational-view of equivalence provided evidence of thinking of equivalence as an algorithm; those coded as indicative of a structural-view of equivalence provided evidence of thinking of equivalence as a fixed trait of an object, or reasoning about equivalence via its general properties.

In coding student work, students often struggled provide definitions of equivalent equations for several different reasons. One issue appeared to be that students attempted to apply the definition of equivalent expressions to equivalent equations. For example, in Figure 1, we see the work of two students-- one in elementary algebra and one in linear algebra-- both of whom give somewhat similar definitions of equivalent equations. The elementary algebra student gives a more ill-defined definition ("same answer") but we see from the examples that they provide that they appear to be thinking about equivalent arithmetic expressions. We would classify this response as a pseudo-process view, as the definition is not well-defined, and because it appears to center around arithmetic calculation.

We see similar work by the linear algebra student in Figure 1, with some differences; they give broader examples of equivalence (describing also when two vectors are equivalent) and their definition is a bit more detailed ("when two quantities are the same on both sides of an equation"). But like the elementary algebra student in Figure 1, they conflate the definition of equivalent expressions with equations (they include an algebra example, but only show identical expressions as equal). Their definition of equivalent equations is also not fully well-defined ("check if both sides are the same"), because the word "same" here is not well-defined. While their answer does show signs of having been exposed to more examples of mathematical equivalence, this does not appear to have positively impacted their definition of equivalent equations; we would still classify their definitions as extracted, because they are ill-defined.

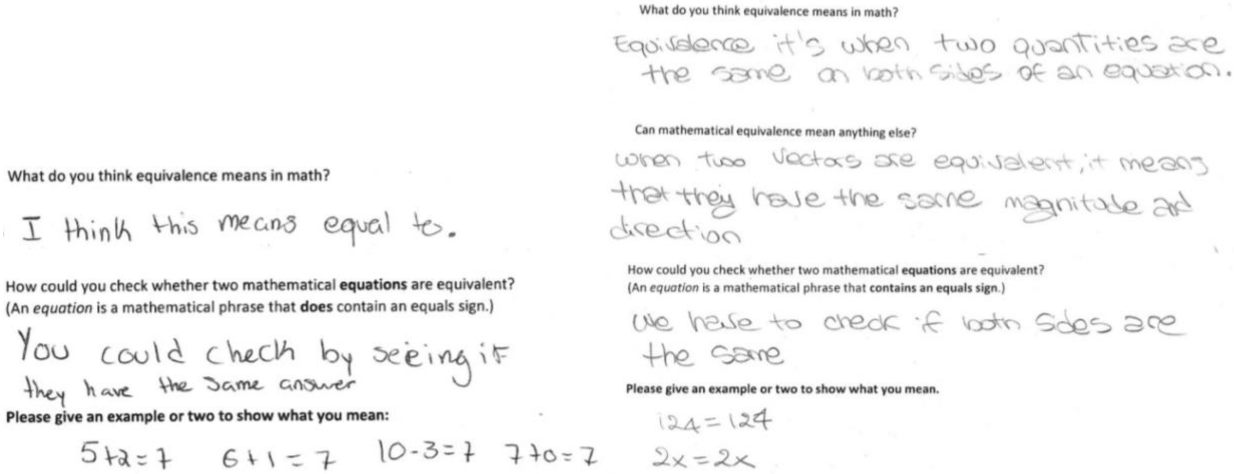


Figure 1: *Definitions from an elementary algebra student (on left) and a linear algebra student (on right), conflating the definition of equivalent expressions with equivalent equations*

Students who applied the definition of equivalent expressions to equations may even do this in a way that is mathematically valid (i.e., fits the definition of an equivalence relation), even though it is not one of the "standard" definitions of equivalent equations (e.g., same solution set).

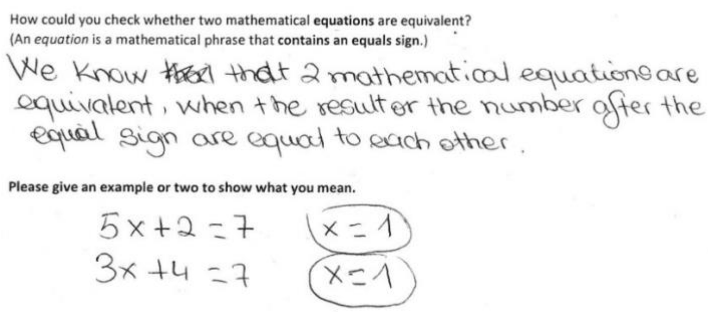
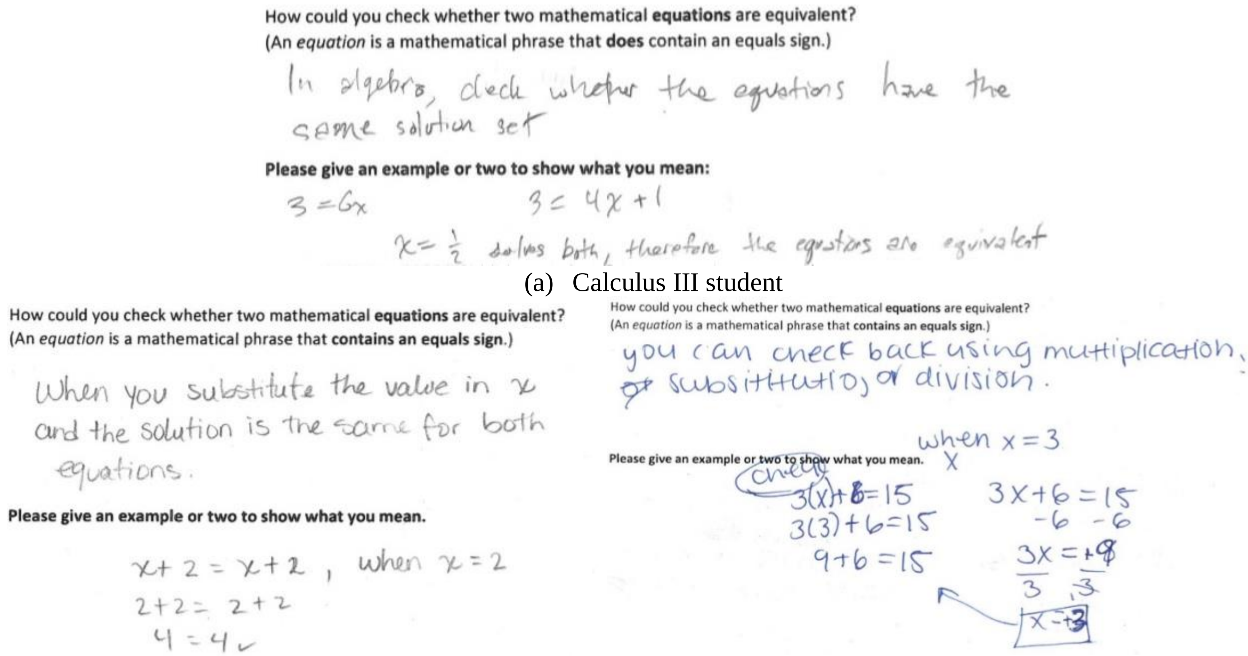


Figure 2. *Precalculus student's non-standard structural definition of equivalent equations*

Consider Figure 2, where a precalculus student has defined equivalent equations as two equations where "the result or the number after the equal sign are equivalent", and based on their examples, this seems to suggest that any equations of the form $expression = n$ for fixed n would be equivalent to one another. This is similar to definitions given by other students in other research (Wladis et al., 2020). This student is particularly interesting, because the two equations that they have given also happen to have the same solution set, so it is unclear if this is an implied part of their definition as well. Whether it includes this feature or not, we would classify this definition as a structural view even though it is a "non-standard" definition, because the student has given what could be a well-defined but alternate definition of equivalence (whether or not their definition is fully well-defined is unclear, as they haven't filled out all the details)[3].


How could you check whether two mathematical **equations** are equivalent?
(An *equation* is a mathematical phrase that **does** contain an equals sign.)

In algebra, check whether the equations have the same solution set

Please give an example or two to show what you mean:

$3 = 6x$ $\quad$ $3 = 4x + 1$

$x = \frac{1}{2}$ solves both, therefore the equations are equivalent

(a) Calculus III student

How could you check whether two mathematical **equations** are equivalent?
(An *equation* is a mathematical phrase that **contains an equals sign**.)

When you substitute the value in x and the solution is the same for both equations.

Please give an example or two to show what you mean.

$x + 2 = x + 2$, when $x = 2$

$2 + 2 = 2 + 2$

$4 = 4$

How could you check whether two mathematical **equations** are equivalent?
(An *equation* is a mathematical phrase that **contains an equals sign**.)

you can check back using multiplication, or subsitutio, or division.

Please give an example or two to show what you mean. when $x = 3$

check

$3(x) + 6 = 15$

$3(3) + 6 = 15$

$9 + 6 = 15$

$3x + 6 = 15$

$-6 \quad -6$

$3x = 9$

$\frac{3x}{3} = \frac{9}{3}$

$\boxed{x = 3}$

(b) introductory stats student (c) intermediate algebra/precalculus student

Figure 3*: Examples of different ways that students used notion of "solving" in defining equivalent equations*

In contrast to the previous examples, some students did draw in some way on the notion of "solving" equations or the solution sets of equations when defining equivalence. However, the ways in which students drew on notions of "solving" also fell into different areas of our framework. Simply talking about the "solution" of an equation was not sufficient to classify work as either stipulated or structural even though it sounds like it is related to the standard insertional equivalence definition of equations (i.e., same solution set). In Figure 3(a), we see the work of a Calculus III student, who appears to have a well-defined and structural view of equivalent equations: they define equivalent equations as having the same solution set (seeming to conceptualize the solution set as a fixed object); and their definition appears to be well-defined, not just because of their definitions, but also because they have provided an example which shows that their interpretation of "same solution" appears to be the "standard" one. We note that this is critical, as many students used the language of "same solution" but actually meant it to describe equivalent sides of an equation (equivalent expressions) rather than equivalent equations. See, for example, the work of an introductory statistics student in Figure 3(b). This student wrote that two equations are equivalent if you "substitute the value in for x and the solution is the same for both equations": this sounds like the standard definition of equivalent equations (if an incomplete one that does not account for the possibility that x may have more than one value), however, looking at the example this student has provided, we see that to them "solution" actually denotes the quantity which results from simplifying each side of an equation (not the solution set of two different equations). In this sense, the students' definition is ill-defined, because the vocabulary that they are using appears to be ill-defined and has

[3] This student may be drawing on notions of equations with the "same form" (e.g., $y = mx + b$, $ax^2 + bx + c = 0$) which is another type of equivalence that is commonly used in the algebra curriculum, even if it is not called equivalence in the curriculum (however, "same form" could in fact be codified as a formal equivalence relation, and students may be noticing this when they draw on it in their equivalence definitions (Wladis et al., 2020).

multiple, perhaps vague, meanings. For these reasons, we would classify this work in (b) as a pseudo-process view, even though on the surface the definition initially looked very similar to the one given in (a). The third example of student work in Figure 3(c) shows another common approach that students used, in which they drew on notions of solving when asked about equivalent equations, but struggled to relate these notions to any well-defined definition of equivalence. This student has solved an equation and checked the solution by substituting it back into the original equation; however, it is unclear what the definition of equivalent equations is, or even which two objects the student is claiming are equivalent (perhaps equivalence for them is not about the relationship between two objects, but is instead names a process of checking the solution of an equation). Because of this, we classify this as a pseudo-process view—there is no well-defined stated definition, and the student's focus is on computation.

Students also gave a variety of other non-standard definitions of equivalence that might possibly have been well-defined definitions of equivalence relations (e.g., equivalent arithmetic equations as ones that express the same additive relationship; equivalent algebraic equations which express the same relationship between the variables), which for the sake of space we do not share here. However, we note that by de-coupling our categorization of student definitions of equivalence from notions of what is "standard" and thinking more carefully about the extent to which student definitions of equivalence are stipulated definitions which meet the criteria of an equivalence relation; and the extent to which student conceptions of equivalence are structural or operational, we may be able to achieve two critical goals more effectively: 1) we may be able to better identify student thinking which "sounds right", but is actually ill-defined; and 2) we may be able to identify valid student thinking that simply does not adhere to "standard" definitions. Both of these goals may better help us to tailor instruction to students.

We now briefly describe some overall trends we found in coding open-ended questions on definitions of equivalence (Table 2). Students primarily associated equivalence with equality, and rarely cited other forms (e.g., equivalent equations), although the incidence of non-equality examples rose somewhat with course level. Similarly, students at all levels were extremely likely to give ill-defined or vague definitions of equivalence when asked. In terms of student definitions of equivalent equations, most students conflated this with the definition of equivalent expressions; this did not appear to improve with course level, suggesting that the lack of explicit definitions of equivalent equations in textbooks and curricula (Wladis et al., 2020) may well be contributing to student difficulty in understanding the how definitions of equivalence vary in different contexts. Some of these definitions, while non-standard, may have qualified as formal equivalence relations, and therefore mathematically valid reasoning—the prevalence of this was not correlated with course level, suggesting that students at all levels may sometimes be generating valid but non-standard definitions. Many students associated equivalent equations with solving, but this was rarely done in a well-defined way: roughly one quarter of all students at all course levels solved an equation but did not relate this in any well-defined way to the definition of equivalent equations (most commonly this involved solving a single equation, and then checking the answer, with no clear mention of which two things were actually equivalent); fewer students did this at levels of precalculus and above, but the differences by course level were not large. Small numbers of students did interpret equivalent equations to mean equations which have the same solution set, and did so in a well-defined way; this was slightly more common as course levels went up; however, the vast majority of these students did so in a operational way (i.e., solved two equations and said they were equivalent, without discussing the solution set in a more general or structural way); this is perhaps to be expected, given the operational way in which the question itself was phrased, however, this does follow patterns observed in questions without this more operational wording, such as the more general question about the definition of equivalence given on this set of questions (although student tendencies to use structural rather than operational definitions did increase with course level). However, we note that overall, structural and well-defined definitions were rare among all students, suggesting that instruction which specifically includes explicit stipulated definitions, and which encourages structural reasoning is needed at all levels.

Table 2. Summary of student definitions of equivalence

	elem. alg. or below	**inter. alg. or 100-level**	**200-level or above**
general definition of equivalence			
ill-defined or vague	67%	71%	60%
cited equality	94%	87%	80%
other valid definition	0%	3%	16%
operational definition	41%	18%	17%
structural definition	0%	2%	17%
how to tell if two equations are equivalent			
conflated w/ equiv. expressions	44%	48%	44%
of these, possible WD defn.	19%	6%	16%
finding solution set, operational	0%	3%	8%
related to "solving" but ill-defined	22%	29%	16%
solution set, structural	0%	2%	4%
total *n*	36	62	25

Discussion and Conclusion

The model of student thinking around definitions of equivalence that is presented here aims to refocus our attention from whether definitions look like a “standard” definition so that we consider more carefully the extent to which student definitions are explicit and well-defined as well as the extent to which students are able to think structurally rather than just operationally. Using this lens allows us to pinpoint places where students appear to understand a standard definition but upon further reflection we find that this definition is not well-defined or is wholly operational, limiting the student’s ability to use it. On the other hand, it also allows us to recognize when students’ reasoning is mathematically valid, and when students are recognizing more generalized instances of equivalence relations, even when they are not able to define them fully. Evidence from student examples here suggests that students do notice many kinds of “sameness”, yet struggle to articulate this in mathematically well-defined ways, just as they struggle to articulate “standard” definitions of equivalence in well-defined ways. This suggests that students are capable of noticing and assimilating more generalized notions of equivalence, but need more explicit definitions and language in order to be able to do this rigorously. Future research is necessary to better understand what kinds of explicit definitions of equivalence work best for students in different contexts, and the extent to which discussions of the more general notion of an equivalence relation might be helpful in instruction. This framework may also be able to serve as a measure of instruction and curricula, to assess how the concept of equivalence is presented to students as they are learning at various levels in the curriculum.

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Concept Usage of the Definition of a Limit of a Sequence in Proof Constructions

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In this report, we examine a case study of one student's proof constructions concerning limits of sequences. This student's case is interesting because he demonstrates that he has a robust understanding of the concept of a limit of a sequence and its definition, both of which the mathematics education research literature establishes to be notoriously difficult for students. However, in failing to coordinate the construction of his proofs properly with the limit definition, he ultimately fails at his proof construction tasks. We examine how his proof constructions went wrong, and then present some implications that this case could have for the instruction of real analysis.

Keywords: proof, limits, real analysis, concept usage, concept definition

This report explores a case study of a student's proof constructions about limits of sequences. The concept of a limit is notoriously difficult for mathematics students to understand (Williams, 1991). The student featured in this report is a notable exception to this observation. He has a fairly strong understanding of both the concept of a limit of a sequence and its formal definition. However, as we will see, these understandings are not sufficient for him to construct valid proofs about limits of sequences.

By using Moore's (1994) concept-understanding scheme, we will demonstrate how this student's failure to construct proofs about limits of sequences can be explained in terms of his inability to connect his understanding of the definition to the strategies he uses to construct his proofs. We will then briefly discuss why this may arise from real analysis instruction, and what implications this could have on the instruction of proof construction in real analysis.

Literature Review

Math education researchers have found that students often leave calculus courses with nonstandard understandings about limits. Among these misconceptions are the ideas that limits can be determined by plugging a finite amount of numbers into the formula for a function (Williams, 1991), that a sequence or function is not allowed to reach its limit (Williams, 1991; Roh, 2008), that infinity can be treated as a number with which to do limit calculations (Oehrtman, 2009; Ely, 2010), and that a limit serves as an upper or lower bound for a sequence or a function (Sierpińska, 1987; Williams, 1991; Szydlik, 2000). These difficulties have sparked research into ways that students can better understand the formal definition of a limit (Cottrill et al., 1996; Swinyard. 2011; Swinyard and Larsen, 2012; Oehrtman et al., 2014). For example, Oehrtman and his colleagues (2014) found that encouraging students to think in terms of error bounds and acceptable ranges around the limit of a sequence helped them to reinvent the definition of a limit of a sequence.

Much of the research on limits has been devoted to students' understanding of a limit as a standalone concept, and not to how it is used in their proof constructions. A notable exception is Roh and Lee (2017), who described how the use of a graphical epsilon-strip intervention helped a group of students to prove that every convergent sequence is Cauchy. Given the established difficulty that students have with learning real analysis (e.g. Alcock & Simpson, 2004, 2005), however, more research of this kind is needed. This report aims to further contribute to our knowledge about how the limit concept is used by students to construct proofs.

Theoretical Perspective

In this report, we examine a student's understanding of limits of sequences through Moore's (1994) "concept-understanding scheme."

Tall and Vinner (1981) introduced the notions of the "concept image" and "concept definition" of a mathematical concept. The concept image is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). The concept definition is the formal, mathematically sound definition of the concept. Through his analysis of students' struggles in an introduction to proof course, Moore introduced the complementary notion of "concept usage." Concept usage is the way in which an individual uses a concept to produce examples or proofs. With respect to proofs, Moore explained that students with well-developed concept usage should be able to apply definitions of the concept in a proof and to structure their proof in consonance with the definition of the concept.

Moore grouped the concept image, concept definition, and concept usage into a theoretical frame he called the "concept-understanding scheme" of a concept. He found that students' concept usages sometimes drew on both their concept definitions and concept images, and that their concept definitions could draw on their concept images. Importantly, though, he found that students often had three separate schemas for their concept images, concept definitions, and concept usages, and that a failure to fluently connect these three components of understanding was a key factor in their difficulties with proof constructions. In other words, well-developed concept images, concept definitions, and concept usages are not enough to guarantee success at proof construction if they are not adequately related. In this report, we show how an inadequate relationship between a student's concept image and concept definition of the limit concept and his concept usage of that concept led him to struggle with two proof construction tasks.

Methods

This case study is drawn from a larger study taking place at a large public university in the northeastern US. The larger study aimed to understand how students who have completed a real analysis course use their formal and informal conceptions of limits of sequences to create proof constructions for statements about limits. Each of the seven participants took part in four to five interviews, loosely structured according to a teaching experiment methodology (Steffe & Thompson, 2000). The first interview dealt with the participant's conceptions about limits of sequences. The second and third interviews had participants verbally construct proofs for various statements about limits, with the use of scratch work to aid their creation and presentation of their proofs. In the second interview, the proof tasks were about sequences defined by a particular formula (e.g., prove that $\left(\frac{1}{n}\right)$ converges to 0). In this report, we call these "explicit sequences." In the third interview, the proof tasks involved proving the convergence of sequences that were not defined by a formula, but rather by some property or by relation to another convergent sequence (e.g., prove that if (x_n) converges to 1, then $(2x_n)$ converges to 2). We call these "abstract sequences." After these three interviews, the participants responses were analyzed to create hypotheses about the strategies and conceptions of limits each participant used in their proof constructions. The fourth and fifth interviews were created to test these hypotheses using carefully chosen proof construction tasks, and in some case proof reading and comprehension tasks. Each interview was 60 to 90 minutes long, was held on Zoom, and was video-recorded and transcribed.

We examine the case of Greg, who was a junior mathematics major at the time of this study. He had completed an introductory real analysis course in the fall semester before his interviews began. In this report, we analyze three proof construction tasks completed by Greg according to the concept-understanding scheme. We looked for verbalizations in Greg's proofs that indicated mental pictures or informal ways of talking about limits (concept image) and use of the definition of a limit (concept definition). We then sought evidence for or against the proposition that Greg's concept image and especially his concept definition were a factor in the strategies he used to structure his proofs (concept usage).

Results

Concept Image and Concept Definition of a Limit of a Sequence

The first interview with Greg was largely devoted to understanding his concept image and concept definition for a limit of a sequence. Greg's concept image of limits conformed highly to conventional conceptions of limits. He described convergence of a sequence as meaning that if "a finite number of terms at the beginning of the sequence are ignored, then the later terms can be made arbitrarily close" to the limit. When prompted to consider non-normative conceptions of limits, Greg disagreed with them. He did not believe that a limit must serve as an upper or lower bound for its sequence terms, he rejected the idea that a sequence must not reach its limit within a finite number of terms, and he differentiated between the concept of a limit and a cluster point by pointing out that a sequence could only have one limit.

Greg was also able to provide a concept definition for a limit when asked. He defined L to be the limit of the sequence (x_n) if for all ε > 0 there exists a natural number N such that for all $n \geq N$, $|x_n - L| < \varepsilon$. Furthermore, he was able to explain this definition as it related to his informal understandings of a limit. For example, he explained that $N(\varepsilon)$ could be considered to be the number of terms at the beginning of the sequence that can be ignored so that the remaining terms fall within an ε-neighborhood of the limit. In this sense, his concept image and concept definition of a limit of a sequence appeared to be tightly related.

First Proof: An Explicit Sequence

During the second interview, Greg was asked to prove that the sequence defined by the formula $x_n = \frac{1}{\sqrt[3]{n}}$ converges to 0. He began by writing down the definition of a limit of a sequence (see Figure 1 for reference). Then he informed the interviewer that "what we want to do in this proof is choose a suitable capital N - that is what are our main goal should be in this proof…We're choosing the appropriate number of initial terms to ignore and beyond which the sequence terms lie in a certain neighborhood."

Greg proceeded to explain that he wanted to show that the absolute value of the difference between the expression $\frac{1}{\sqrt[3]{N}}$ and the proposed limit, 0, could be made smaller than an arbitrary positive ε. His strategy was to "manipulate this expression so that…N is some expression of ε." Using algebra, he determined that this inequality implied the inequality $N > \frac{1}{\varepsilon^3}$. He then stated that this work was not "technically" part of the proof. He reiterated that choosing N to be greater than $\frac{1}{\varepsilon^3}$ would establish the "initial finite number of terms [of the sequence] we ignore." Finally, Greg established how choosing $n > N$ implied that $\left|\frac{1}{\sqrt[3]{n}} - 0\right| < \varepsilon$, completing the proof.

In this task, we see Greg demonstrate the successful construction of a limit proof. He once again gives a correct concept definition of a limit, implying that he believes it to be relevant to his proof construction. He also relates his concept image of a limit to his proof when he identifies his choice of N as the number of initial terms of the sequence to be ignored. However, when the interviewer asked Greg how confident he was that his proof was correct, Greg stated he was only "fairly confident." He explained, "the whole approach of sort of reverse engineering the choice of N…I think that part I'm rather sure of. I've encountered enough proofs of sequences and limits of sequences in my real analysis course to be slightly confident about that." It appears that Greg's confidence in his proof stems not from a sense that his proof aligns with the concept definition, but from a sense that he has constructed the proof in ways that agree with his prior experiences with similar proofs. Though Greg gave a normatively valid proof of this statement, his comment suggests that there are at least two explanations for how he structured his proof: by use of the definition, or by the employment of a familiar procedure.

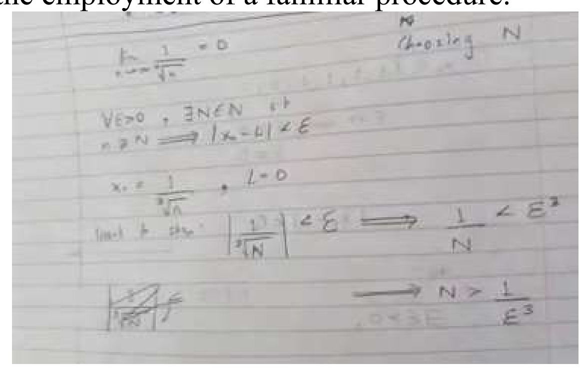

Figure 1. Greg's scratch work for the first proof.

Second Proof: An Abstract Sequence

During the third interview, Greg was asked to prove that a sequence (x_n) converges to 0 if and only if $(|x_n|)$ converges to 0. He began, again, by writing down the definition of a limit. He then noted that since this statement was a biconditional, he would have to prove two "directions" of the proof. He began by assuming that the sequence $(|x_n|)$ converges to 0, with the goal of proving that (x_n) converges to 0. By applying the definition of convergence to the hypothesis, Greg said that "for all epsilon larger than 0, there's a capital N such that lowercase n greater than or equal to capital N implies that…the absolute value of the absolute value of x_n less than 0 [i.e., $||x_n - 0||$], which is pretty much the same thing as $|x_n|$." Greg concludes that we "pretty much automatically" arrive at the convergence of (x_n) to 0. His work for this part is shown in Figure 2. Very quickly, he explains that the reverse conditional is proved in a similar way, relying on the fact that $||x_n|| = |x_n|$.

The interviewer was interested in how Greg understood the role of N in his conclusions of the two subproofs. Restricting his attention to the first subproof, the interviewer asked Greg to explain what value of N would satisfy the definition of a limit for (x_n) when the value of ε is 0.1. Greg asked if this would depend on the specific sequence that (x_n) represented, and the interviewer asked Greg to think about this. He focused on finding an example, the constant sequence $x_n = 0$, and eventually explained that any natural number would suffice for the value of N for this sequence. Hoping to see if Greg could provide a general rule for the value of N that satisfied the definition of the limit for (x_n), the interviewer began this exchange:

Interviewer: So is there anything in your proof that would help you find such a capital N, so if I asked you for a specific epsilon like 1. Is there anything in your proof that addresses [the choice of N], or is that kind of a thing that comes on a case-by-case basis, depending on what the sequence actually is?

Greg: I honestly think that it might be on a case-by-case basis, like in this one [the example of the constant sequence 0].

From the chain of implications at the bottom of his scratch work and from his verbalization of the proof, one might expect that Greg understood that the same value of N(ε) will work for both sequences, but Greg did not make any reference to the definition of the convergence of $(|x_n|)$ in explaining the choice of N in these follow-up questions. However, Greg also did not explicitly state that the choice of N(ε) was independent for these sequences. Moreover, the interviewer may have primed him to think of choosing fixed numbers by asking about fixed values of ε like 0.1 and 1. Of course, for an exact value of ε, the exact choice of N will depend on the specific sequences involved. In short, the data collected in this episode was not enough to infer whether Greg's concept usage in structuring his proof adequately took the concept definition of the convergence of (x_n) into consideration. So, in the next interview, the interviewer attempted to elicit a similar proof construction to further investigate how Greg conceived of the role N plays in proofs about abstract sequences.

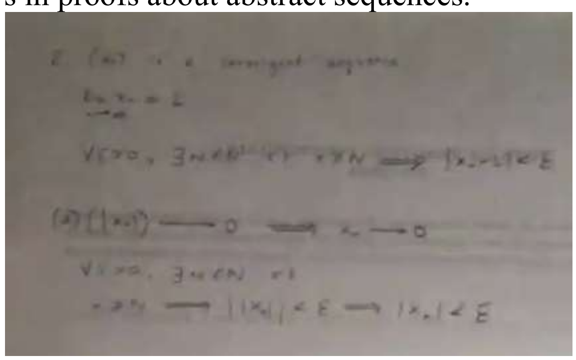

Figure 2. Greg's scratch work for the first conditional of the second proof.

Third Proof: Another Abstract Sequence

In the fourth interview, the interviewer asked Greg to prove that if the sequence (x_n) converges to 0, then the sequence $\left(\frac{x_n}{2}\right)$ converges to 0. Note that, just like in the second proof presented above, the choice of N(ε) for the sequence (x_n) will also satisfy the definition of convergence for the sequence $\left(\frac{x_n}{2}\right)$. Greg's immediate reaction was that the algebraic limit theorem may apply to this task. However, he had trouble recalling what the algebraic limit theorem stated. When the interviewer suggested that he try the proof by using the definition of a limit, Greg agreed that this was the next thing he wanted to do.

Greg first stated what it would mean for the sequence (x_n) to converge to 0. Then, he wrote down what he wanted to show to conclude that $\left(\frac{x_n}{2}\right)$ converges to 0 (the first two lines in Figure 3). Greg states that "I feel like we're almost finished the proof right there [the first two lines in his scratch work]." He goes on to say:

"$|x_n|$ is less than epsilon, that would automatically imply that $x_n$over 2
is less than epsilon…for all n. Now we know that given epsilon greater than 0…there

exists a capital N belonging to the natural numbers, such that lowercase n greater than or equal to capital N implies that x_n over 2 is less than epsilon, which means that x_n over 2 converges to 0."

Once again, Greg's verbalization of his proof construction was vague about how he established the existence of this N(ε). The exchange following immediately after clarified this:

Interviewer: You mentioned that we've now shown that given epsilon greater than 0, there's this capital N in the natural numbers, such that all this stuff implies x_n over 2 will be less than epsilon. What is that capital N?

Greg: So the capital N is the point beyond which the sequence x_n or the sequence x_n over 2, um, well, would it be the same? Could I choose the same? …It's not necessarily the same capital N, but what's important is that there exists a capital N for both x_n and x_n over 2. It's not necessarily the same- [he is interrupted by the interviewer, who speaks at the same time.]

Interviewer: Ok, and what allows you to assert that?

Greg: What allows me to assert that is that we know both sequences converge.

This appears to be the first time Greg is considering whether these two parameters are related. He concludes that they need not be identical, and when further asked to explain what N(ε) is for the sequence $\left(\frac{x_n}{2}\right)$ he resorts to circular reasoning. Greg's concept usage of a limit is distinctly different in this proof construction compared to his first proof construction. Whereas in the first proof construction Greg attended very carefully to identifying N(ε), in the third proof he does not. His conclusion appears to be based on the fact that the inequality $\left|\frac{x_n}{2}\right| < \varepsilon$ can be derived at all, without paying attention to the indices for which it holds true. Hence, this is an episode in which Greg's concept usage of a limit is clearly divorced from the concept definition.

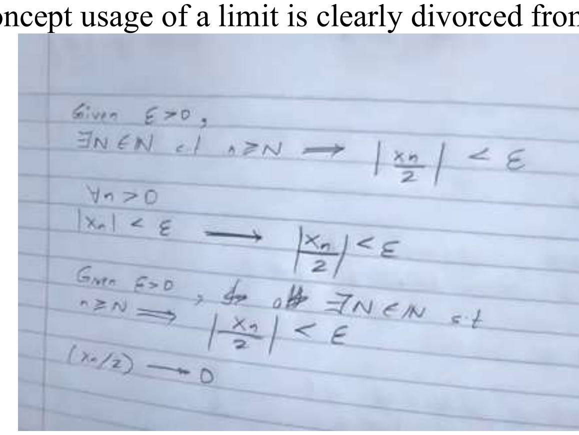

Figure 3. Greg's scratch work for the third proof.

Discussion

In the case of Greg, we see a student who has a normative concept image and concept definition of the limit of a sequence. He is able to accurately recall and explain the definition of a limit. He can also construct valid proofs that explicit sequences converge to a particular limit and explain these proofs using his concept image and definition of a limit. In these proof constructions, he explicitly states that his goal is to find a value of N that corresponds to an arbitrary positive ε for which the conditions of the definition hold. From his first proof construction, then, we might believe that Greg has a strong concept understanding of a limit of a sequence. However, when presented with statements about the convergence of abstract sequences, Greg's proof constructions neglect the role of N in establishing the limit of the sequence in question. Despite beginning these proofs by writing down the definition of a limit, Greg appears to be satisfied that his proof is completed if he can conclude that $|x_n - L| < \varepsilon$, without being able to describe the value of N(ε) that makes this true. In cases like the third proof presented in the results, we see that Greg's concept usage of the definition of a limit of a sequence could be improved by structuring his proof more carefully and explicitly to attend to the values of N that satisfy the definition of a limit.

We see two possible interpretations for this data within the concept-understanding scheme. The first is that Greg really did use the definition of a limit to structure his first proof but failed to do so completely for his third (and perhaps second) proof. The second and more likely interpretation of our results is that Greg did not substantially use the definition of a limit to structure any of these proofs. Although Greg repeatedly said that his goal in the first proof was to determine a value of N, this goal may not have been supplied entirely by the definition of a limit. His remarks about his confidence in the correctness of this proof show that he was thinking about previous examples of proofs that explicit sequences converge. Therefore, it is likely the case that he was able to explain his proof construction in relation to the concept definition of a limit, but that his concept usage in the proof amounted to the application of a familiar procedure. That is, Greg considered that his goal was to find N by algebraically manipulating the inequality $|x_n - L| < \varepsilon$, and then to reverse this process. In proof tasks involving abstract sequences, it is often not possible to find an explicit formula or bound for N through algebra alone. For example, in the second and third proofs presented here, it is necessary to logically relate the value of N(ε) for the sequence in the conclusion to some N value (implied by the definition of convergence) corresponding to the sequence in the hypothesis.

The case of Greg has implications for instruction. Instructors should take care when inferring a student's concept understanding from their written proofs. It has already been recognized in the literature (e.g., Moore, 2016) that mathematicians must use judgment to reconstruct what a student was thinking based on their written proofs. However, this is discussed in the context of minor mistakes or gaps left in the written proofs. With Greg, we see that even his first correct proof is not sufficient to conclude that he has a complete concept understanding, and specifically concept usage, of the definition of a limit. Furthermore, it was unclear from Greg's second and third proof that he understood the nature of the N value that implied convergence of the sequences in question. An instructor may be tempted to give the benefit of the doubt to Greg that he understood that the same value of N satisfied the definition of convergence for both sequences, but speaking to him about his third proof clarified that this was not the case.

Instructors may also be able to help students build a concept usage of limits that are linked to their concept definitions and concept images. It is common to acquaint real analysis students with the definition of a limit by presenting and assigning proof construction tasks involving explicit sequences. This may have the unintended effect of fostering a concept usage that relies on procedures like the one used in Greg's first proof, which may be more salient to students than how the structure of these proofs mimics the concept definition. By giving students a mix of explicit and abstract sequence tasks soon after they learn the definition of a limit, instructors may be able to help students realize the inadequacy of these procedures and focus more on the way that these proofs are driven by the concept definition.

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Promoting Quantitative Reasoning in Calculus: Developing productive understandings of Rate of Change with an adapted Calculus 1 curriculum

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The purpose of this study is to explore the benefits of an adapted Calculus 1 curriculum designed to support students in reasoning about quantities covarying. Since researchers indicate that students often have unproductive meanings for rate of change (Byerley et al., 2012; Simon & Blume, 1994; Castillo-Garsow, 2010), this study deliberately addressed how to support students in developing productive understandings for rate of change that could be leveraged into a productive understanding for instantaneous rate of change.

Keywords: Rate of Change, Calculus, Instructional Intervention

Calculus is the mathematics of how quantities change. The main idea of Calculus 1 (in the US curriculum) can be summed up as "You know how much of a quantity you have at all times and want to know how fast that quantity is changing at all times" (derivatives). Understanding how quantities vary and utilizing Calculus to model these quantities is an essential skill for STEM students. However, typical Calculus courses in the US are heavy on procedural fluency with little focus on conceptual understanding (Bressoud et al., 2016). Additionally, research has shown that even high-performing students demonstrate impoverished understandings of key Calculus concepts (Selden et al., 2001; Carlson et al., 2002) and that the learning of the derivative concept is complex (Tall & Vinner, 1981; Park, 2013; Zandieh, 2000; Oehrtman, 2002; Monk, 1994; Ubuz, 2007; Yu, 2020). Since derivatives represent something that we call "instantaneous rate of change," then students' understandings of rate of change are pertinent to their understanding of derivatives. However, researchers (Byerley et al., 2012; Simon & Blume, 1994; Castillo-Garsow, 2010) indicate that students have unconventional meanings for rate of change. Due to these issues, the purpose of this study is to provide an example of an instructional intervention in a Calculus 1 course designed to support students in developing productive understandings of rate of change. The research question this study explores is:

How does an instructional intervention designed to support students in using quantitative and covariational reasoning aid them in understanding the idea of rate of change?

Literature Review

Researchers in Post-Calculus education and engineering education support the finding that many students have unproductive meanings for rate of change. Rasmussen and King (2000) reported that students in a differential equations course conflated the number of fish in a pond with the rate of change of the fish in the pond with respect to time elapsed. Prince et al. (2012) reported that their engineering students struggled to distinguish a rate of heat transfer from an amount of heat transfer. Ibrahim and Robello (2012) indicated that even students who demonstrated understandings of rate of change in motion contexts failed to transfer these understandings to non-motion contexts such as work. It should be clear then that our Calculus courses need to better support students in developing robust understandings of rate of change in order for them to leverage derivatives (rate of change functions) in their respective fields.

One example of the benefits of revising curriculum to support students in developing rich meanings for rate of change functions is Thompson and Ashbrook's (2019) Calculus course. This course was designed to support students in overcoming difficulties, such as students thinking that variables do not vary, believing that Calculus is a set of rules and procedures, and a derivative is a slope of a tangent rather than concerning a rate of change (Thompson et al., 2013). Thompson (2019) reported that students in his course performed higher on average than students in a traditional Calculus course on an 11-item Calculus concept inventory that focused on variation, covariation, and rate of change understandings. This study leverages Thompson's work on a conceptual approach to Calculus by adapting a standard Calculus 1 curriculum to include a unit on the meaning of rate of change.

Theoretical Background

The Calculus classes in this study employ a standard Calculus 1 curriculum that has been adapted to focus on students' reasoning about quantities covarying. The instructional interventions' design is influenced by Smith and Thompson's (2007) theory of *quantitative reasoning* and researchers' findings on *covariational reasoning* (Carlson et al., 2002, Thompson & Carlson, 2017).

Quantitative Reasoning

Thompson (1990) defines *quantitative reasoning* as analyzing a situation in terms of quantities and their relationships. A *quantity* is a conceived attribute of an object that an individual envisions having a measurement. Thompson (2011) defines *quantification* as the process in which one assigns numerical values to an attribute they have conceptualized. Smith and Thompson's theory of quantitative reasoning influenced the design of the in-class activities that support students in imagining quantities and how students may represent them using mathematical expressions.

Conceptual Analysis

Thompson and Thompson (1994a) provided a conceptual curriculum for speed that I leverage to articulate the productive ways of thinking for rate of change (Figure 1). Put together in one statement, a rate of change quantifies a multiplicative relationship between 2 varying quantities.

1. Rate of change is a **quantification of variations**
2. Rate of change relates variations in **two varying quantities**
3. Rate as a quantification of variations in two quantities is made by a **multiplicative comparison** of these variations
4. To say that rate of change of quantity Y with respect to quantity X is "m" is to mean that the variation in quantity Y (Δy) is m times as large as the variation in quantity X (Δx), i.e., $\Delta y = m\Delta x$

Figure 1: Productive Ways of Thinking for Rate of Change

In a typical Calculus 1 course, instantaneous rate of change is introduced to students via the limit definition of derivative, $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{(x+\Delta x)-(x)}$ (Stewart, 2013; Larson et al., 2006). One productive interpretation of the limit is the multiplicative relationship (the value of $f'(x)$) between a variation in a function's output ($f(x + \Delta x) - f(x)$) and a variation in the input ($(x + \Delta x) - (x)$) so long as the variation from the input value x is arbitrarily small ($\lim_{\Delta x \to 0}$). The convergence of this limit is what we call "instantaneous rate of change," which represents the relationship between two varying quantities with respect to one another's relative size of variation. To use a derivative value as a rate of change having some value m, one must imagine the input quantity varying while simultaneously imagining the output quantity varying m times as much as the input quantity's variation. This is the same meaning we might attribute to an average rate of change over a small interval, where someone imagines the necessary constant rate of change to achieve the same accrual in one quantity with respect to the accrual size of the other quantity. The question then becomes what is needed to support students to construct such a meaning. This study describes such an activity that demonstrates the potential for helping students reason about rate of change robustly.

Methodology/Results

This study includes two Calculus 1 courses from the Fall 2020 and Spring 2021 semesters through a southwestern university via Zoom. These classes initially contained 40 engineering or computer science students (a few students withdrew from the class as the semester progressed). The instructor designed activities through Desmos to serve as didactic objects (Thompson, 2002) to facilitate a conversation about what rate of change entails. Additional data points include a pre-test at the start of the semester and exams that included questions focused on variation, covariation, and rate of change. Questions and answer choices on the Pre-Test and course exams were all designed with the research in mind. For example, answer choices were created to capture the various ways students might have reasoned about a rate of change.

Pre-Test

The pre-test involved a set of 10 questions that assessed students' understanding of function, variation, and rate of change. Four items were from the Precalculus Concept Assessment (Carlson et al., 2010). Each student took the pre-test on the first day of class, and items were scored immediately. This study will discuss the results of 2 of the 10 items.

Pre-Test Results

Figure 2 displays the results of students' responses to a question about constant rate of change. The results support previous literature that many students interpret the value of a rate of change as an amount to add (Byerley et al., 2012, Yu, 2020). A majority of the students seemed to have interpreted the value of the constant rate of change as the change in the weight of the fish for a 1-unit change in the input quantity.

5) The weight of a fish is modeled by the formula $w = 1.24x + 0.31$ where w is the weight of the fish in pounds in terms of the number of years x since the fish was born. Which of the following describes what 1.24 conveys in the context of this situation?

I. For any change in the age of the fish, Δx, the change in the weight of the fish is $(1.24) \cdot (\Delta x)$

II. The weight of the fish increases by 124% each year

III. The fish gains 1.24 pounds every year

Answer Choice	Number of Responses	Percentage of Responses
a) I only	5	6.25%
b) II only	2	2.5%
c) III only	52	65%
d) I and II only	8	10%
e) I and III only	13	16.25%

Figure 2: Results on Interpreting a Constant Rate of Change (n = 80)

Figure 3 displays students' responses to a question about what information they would use to estimate an instantaneous rate of change. The two most chosen responses were designed to capture the thinking of what would happen in the next hour (72.8-23) or how long it took to drive 1 hour ($\frac{23}{1}$). The responses support the idea that students were likely reasoning about the value of a rate of change as the amount of change for a 1-unit change in the input quantity.

8) Suppose you are driving your car on the freeway. Below is a table that has some data points on the distance you have traveled and the associated times.

Time since start of trip (in hours)	Distance Traveled on Trip (in miles)
1	23
1.03	25.2
1.5	53.1
2	72.8

Which of the following is the best estimate of your car's speed at one hour since you started driving?

Answer Choice	Number of Responses	Percentage of Responses
a) $\frac{25.2-23}{1.03-1}$	6	7.5%
b) $72.8-23$	15	18.75%
c) $\frac{23+25.2+53.1+72.8}{4}$	1	1.25%
d) $\frac{23}{1}$	57	71.25%
e) Cannot be determined	1	1.25%

Figure 3:Results on Estimating an Instantaneous Rate of Change ($n = 80$)

Due to the Pre-Test results, the instructor designed an instructional intervention to perturb students' understanding of rate of change to set them up for developing a productive meaning for instantaneous rate of change.

The Desmos Activity (a summary of the activity can be seen here: Desmos Activity Summary)

The following is one example of a series of lessons designed to support students in reasoning about quantities covarying with each other.

Students were initially presented with a situation of "The teacher is walking away from the wall at a constant rate of 1.7 meters per second. What does it mean to have a constant speed of 1.7 meters per second?" The instructor used this to elicit students' current ways of understanding constant rate of change with the anticipation that most students are thinking about 1.7 meters and 1 second (instead of 1.7 describing the multiplicative relationship between variations in meters traveled and time elapsed). True to this anticipation, most students had a response similar to "your position changes by 1.7 meters for every second that passes".

The instructor then utilized the Desmos activity to push and perturb this way of thinking by presenting scenarios where thinking about 1-unit changes in the input would not work. Then the activity allowed students to explore what 1.7 measures in the context and that there must be 1.7 of something in this situation. Additionally, scaffolded questions were provided to aid students in thinking about the quantities and what was being measured with a constant rate of change. For example, the graphing part of the activity [Figure 4] included questions that focused on what quantities were being represented and how the quantities varied. The instructor continually asked students to think about what 1.7 represented in this context and where 1.7 shows up in the graph.

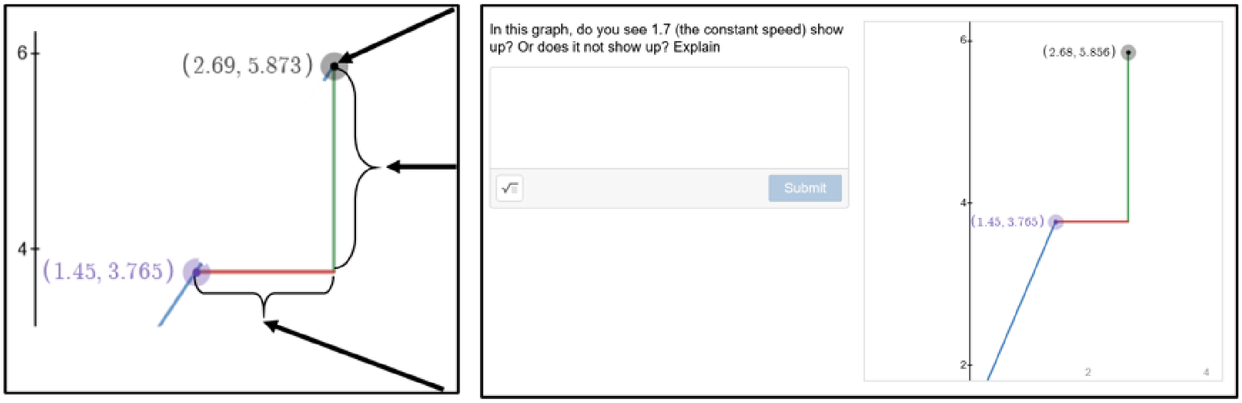


Figure 4: Examples of the Desmos Activity

As the activity continued, students were given a Desmos page where they could interact and use the change in the input to measure out the associated change in the output quantity. Figure 5 displays one screenshot of a student's work where they measured the value of change in the output quantity with the value of the change in the input (One large tick mark indicated 1-unit of the change in the input quantity, and the smaller tick marks indicated one-tenth of that unit).

At the end of the lesson, students were once again asked what 1.7 measured about the situation. A majority of the students noted that "1.7 represented how many times larger the change in distance traveled would be with respect to the change in time". It would be too hasty here to claim that students understanding completely changed; however, it is apparent that students were at least cognizant of the difference between their initial statement and how the activity helped them alter their meaning.

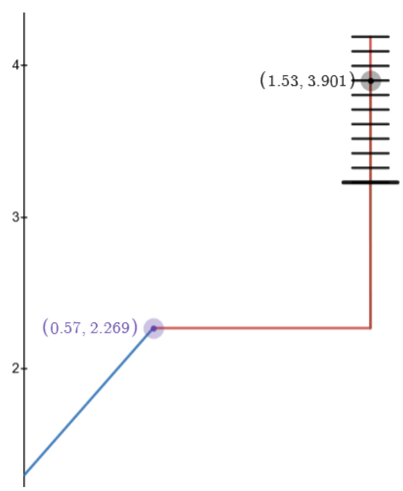


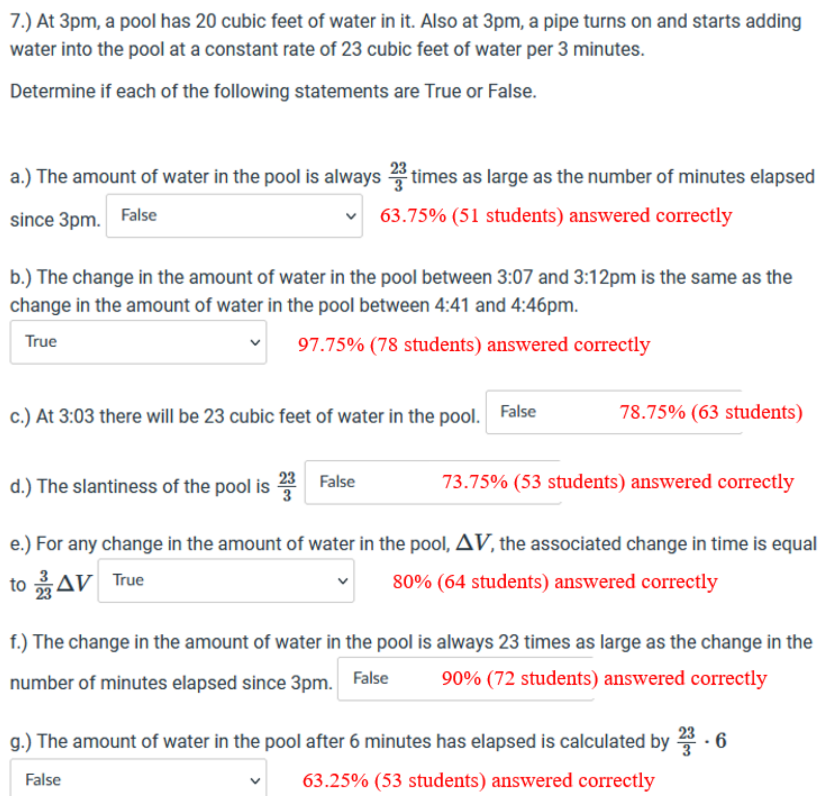
Figure 5: Measuring the Change in Output using the Change in Input as a Unit

Following this Desmos module, students were prompted to describe what they remembered from the previous class. Forty-eight students explained something akin to "We found how many times we could fit the change in time into the change in the distance," and 13 students replied, "the constant rate (of 1.7) would be maintained regardless of how much we changed the input". Overall it seemed that students recalled the animation where students measured out the change in distance using the change in time as a unit of measure and that this multiplicative relationship was captured by the value of the rate of change.

Post-Tests

The post-tests included the course exams and one post-course survey. These assessments were a mixture of multiple choice and short free-response questions. Since measurements were taken throughout each semester and each students' responses were tracked, this mitigates the possibility that students guessed correctly on these questions. Additionally, while the contexts of each question differ (e.g., graphical, algebraic, tabular), I argue that the consistent scoring across these contexts support the notion that students were leveraging their new meaning for rate of change, instead of viewing these contexts as disconnected from one another (Zandieh, 2000).

Figure 6 displays one of the questions on a course exam that assessed students' interpretation of a value of a constant rate of change. Compared to the Pre-Test, where most students did not choose the option that related the changes between 2 quantities, on this exam, 80% of students identified the reciprocal relationship that that constant rate of change implied. While there were still students who continued to not distinguish between changes in the quantity (ΔV) with the amount of the quantity (V) [Parts a and g], it appeared that many students' meanings for rate of change had shifted from thinking of 1-unit changes in the input towards one that described the multiplicative relationship between the changes between 2 varying quantities.



7.) At 3pm, a pool has 20 cubic feet of water in it. Also at 3pm, a pipe turns on and starts adding water into the pool at a constant rate of 23 cubic feet of water per 3 minutes.

Determine if each of the following statements are True or False.

a.) The amount of water in the pool is always $\frac{23}{3}$ times as large as the number of minutes elapsed since 3pm. False 63.75% (51 students) answered correctly

b.) The change in the amount of water in the pool between 3:07 and 3:12pm is the same as the change in the amount of water in the pool between 4:41 and 4:46pm. True 97.75% (78 students) answered correctly

c.) At 3:03 there will be 23 cubic feet of water in the pool. False 78.75% (63 students)

d.) The slantiness of the pool is $\frac{23}{3}$ False 73.75% (53 students) answered correctly

e.) For any change in the amount of water in the pool, ΔV, the associated change in time is equal to $\frac{3}{23}\Delta V$ True 80% (64 students) answered correctly

f.) The change in the amount of water in the pool is always 23 times as large as the change in the number of minutes elapsed since 3pm. False 90% (72 students) answered correctly

g.) The amount of water in the pool after 6 minutes has elapsed is calculated by $\frac{23}{3} \cdot 6$ False 63.25% (53 students) answered correctly

Figure 6: Post-Test on Constant Rate of Change

Figure 7 displays a question on the final exam about interpreting the derivative at an input value (interpreting the value of an instantaneous rate of change). The aggregated results indicate that most students considered the value of the instantaneous rate of change outside of just a 1-unit change in the input value due to their choice in selecting choice two as part of their answer (Note that by the end of the course, three students had withdrawn from the course so this data is out of 77 instead of 80 students). Overall, 54 of the students (roughly 70%) chose both correct answers (and none of the incorrect ones), indicating an overall shift in how the students initially conceived the value of a rate of change since the beginning of the course.

Jaime is baking a cake and has a thermometer that keeps track of his cakes' temperature as it bakes in the oven. Let $c(t)$ represent the temperature of the cake in degrees Fahrenheit after it has been put into the oven for t minutes.

Assume that $c'(23) = 3.8$, which of the following would be true? (Select all that apply)

- [ ] The slope of the time is 3.8 when the temperature is 23 degrees Fahrenheit 7.79% (6 students)
- [x] In a small enough interval around 23 minutes, the cake is essentially changing temperatures at a constant rate of 3.8 degrees Fahrenheit per minute. 84.42% (65 students)
- [x] If the cake were to keep changing at the rate of this derivative value between times 23 minutes and 24 minutes, then the temperature of the cake would change by 3.8 degrees Fahrenheit. 87.01% (67 students)
- [ ] The temperature of the cake at time 22 minutes is 3.8 degrees Fahrenheit less than the temperature of the cake at 23 minutes. 11.68% (9 students)
- [ ] Every 23 minutes, the cake will increase in temperature by 3.8 degrees Fahrenheit 2.6% (2 students)
- [ ] The temperature of the cake changed by a total of 3.8 degrees Fahrenheit since it was put into the oven. 6.5% (5 students)
- [ ] The temperature of the cake will change by 3.8 degrees Fahrenheit between times 23 minutes and 24 minutes. 9.09% (7 students)
- [ ] When the cake reached 23 minutes in the oven, it gained 3.8 degrees Fahrenheit in temperature. 5.19% (4 students)

Figure 7: Final Exam Question on Interpreting Instantaneous Rate of Change

Discussion

From the results of this adaptation to a standard Calculus 1 curriculum, there is evidence that a curriculum focused on aiding students in reasoning about how quantities covary and the meaning of the value of a rate can help students build productive understandings of derivative as instantaneous rate of change that they may need for their future STEM courses. This finding aligns with other previous works on alternative curriculums that support students in reasoning covariationally (Thompson & Ashbrook, 2019; Carlson et al., 2001; Ely & Samuels, 2019). Additionally, this study serves as an example of possible changes teachers can employ to support their students in developing productive meanings for Calculus topics.

However, several limitations should also be considered. For example, due to the lack of student interviews, the data is only based on assessments that have not been officially validated and do not describe what features of the Desmos module accounted for the possible changes in student thinking. The questions (except for those drawn from PCA) have not gone through extensive testing and thus are limited in providing definitive evidence of student reasoning. Despite this, the results appear optimistic in preventing students from confusing rate of change functions with amount functions as indicated by Post-Calculus education researchers (Rasmussen & King, 2000; Prince et al., 2012; Ibrahim & Robello, 2012).

Future studies will investigate individual student thinking as students work through the Desmos module and how to improve the tasks.

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Networking Multiple Reasoning Perspectives to Characterize Students' Thinking about Quantities and Quantitative Relationships

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We aim to respond to the enduring challenge of characterizing students' reasoning about functions by networking Thompson's theory of quantitative reasoning and Lithner's theory of mathematical reasoning. We situate this article as an inquiry into how coordinating the two theories can provide insight into how students reason about quantities and quantitative relationships. We provide data to show how combining the two reasoning perspectives provides a rich lens to characterize students' reasoning. Our findings indicate that students generated knowledge by shifting from less to more sophisticated reasoning about quantitative relationships. We also note that students' quantitative and covariational reasoning enabled them to flexibly create multiple representations for quantitative relationships.

Keywords: Mathematical Reasoning, Quantitative Reasoning, Networking Theories, Quadratic Functions, Covariational Reasoning

Networking theories are defined as a diversity of approaches or ways of making theories interact (Kidron et al., 2018). For us, networking theories are creating a rich web of multiple theories to shed light on students' meaningful understanding. Networking theoretical perspectives has been popular for several decades, where mathematics education community invites scholars for communication, collaboration, and cooperation (Kidron et al., 2018) in particular among Congresses of the Europe Society for Research in Mathematics Education (CERME) 4–6 working groups in Europe (Bikner-Ahsbash & Prediger, 2010). Another example of this is the *International Group for the Psychology of Mathematics Education* (PME), which encourages mathematics education scholars to join the critical conversation to create diverse theories (2002). In response to these calls, we aim to explore how networking theories of different grain sizes might shed light on students' ways of thinking about functions by communicating, collaborating, and cooperating within multiple theoretical perspectives. We network Thompson's quantitative reasoning and Lithner's mathematical reasoning.

Purpose of Networking

The National Council of Teachers of Mathematics (NCTM, 2014) emphasizes the need to promote reasoning as students learn mathematics. Scholars have reported students' difficulty in reasoning about functions when the functional relationship represents quantities and quantitative relationships (Altindis, 2021; Altindis & Fonger, 2019; Carlson et al., 2002; Fonger & Altindis, 2019; Moore & Carlson, 2013). One way to develop foundational reasoning abilities is to promote students' ways of thinking (Oehrtman et al., 2008). While scholars encourage students to employ authentic reasoning that is specific to each individual, in most cases students' reasoning about functions is characterized with a single theoretical lens (e.g, Moore & Carlson, 2012; Thompson, 2011). Hence, merging multiple theoretical perspectives to characterize students' ways of thinking is an important area of inquiry. In response to this need, we aim to merge Thompson's theory of quantitative reasoning and Lithner's theory of mathematical reasoning to characterize students' thinking.

Theoretical Orientation

Quantitative Reasoning

Thompson situated the theory of quantitative reasoning is based on Piaget's (2001) work on mental images that students create. Creation of the mental images is a cognitively demanding process for students during conceptualization of quantities, quantification, and relationships among quantities (Thompson, 2011). Students can develop conceptualizations of function by engaging with and reasoning about quantities that covary simultaneously. Their ability to build an image of changing quantities involves several layers: the first being perceiving a change in one quantity, the second being shifting into conceiving the two quantities as coordinated, and the final layer being the construction of an image of the two changing quantities as they covary simultaneously. In this study, we employ Thompson's definition of quantity- a quality of an object which is measurable.

There are several central tenets of quantitative reasoning: quantity in mind (not real world), quantification, and quantitative operations. *Quantity in mind* are mental constructions of quantities (Thompson, 2011). *Quantification* is "the process of conceptualizing an object and attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, or multi-linear) with its unit." (Thompson, 2011, p. 37). *Quantitative operations* are the relationships among quantities. Quantitative operations involve operating within quantities (Thompson, 2011).

Quantitative and covariational reasoning

Quantitative Reasoning is a foundation for covariational reasoning; covariational reasoning empowers students to see invariant relationships between covarying quantities (Thompson, 2011). For Thompson and his colleagues, covariational reasoning is being able to think about "two quantities' values varying" and the two quantities "varying simultaneously" (Thompson & Carlson, 2017, p. 425). Thompson's perspective of "covariation" is students' understanding of the relationships between quantities that vary continuously. We adopt Thompson and Carlson's (2017) major levels of covariational reasoning as indicated in Table 1.

Mathematical Reasoning

Lithner's theory of mathematical reasoning seeks to characterize the nature of students' thought processes during task solving in mathematics. Tasks entail exercises, tests, group work and other work that are requested from students. Just like Thompson's theory, Lithner's theory is based on Piaget's theory of mental images and seeks to explain how these images come to be. In the theory, reasoning is defined as "the line of thought adopted to produce assertions and reach conclusions in task solving" (Lithner, 2008, p. 257). The components involved in reasoning consist of *objects*, *transformations*, and *concepts*. Objects are the fundamental entity that one is working with in task solving. These may be real world objects or abstract [mathematical] objects. Examples of [mathematical] objects include functions, numbers, matrices, equations, graphs, etc. Real-world objects may include things like marbles, dice, apples, etc. A transformation is an action taken on an object or set of objects to produce another object(s). Transformations can occur between real world and abstract objects. For example, counting the number of marbles on the desk is a transformation on a real-world object to an abstract object (number). A concept, on the other hand, is a coherent set of objects, their properties, and transformations between the objects. An example of a concept is the notion of a function, or infinity. Most concepts usually have several properties, and one needs to be able to identify the relevant properties in a given problematic situation involving the concepts (Geteregechi, 2020). Such relevant properties are called intrinsic properties while irrelevant ones are called surface properties. As an example, a student who claims that the function $y = 5x$ has no y-intercept because there is no constant is said to anchor their argument on surface properties while a student who recognizes that the y-intercept is 0 because the solution of the equation at $x = 0$ is 0 is said to anchor their reasoning on intrinsic properties.

Table 1. Major Levels of Covariational Reasoning (Thompson & Carlson, 2017, p. 442)

Level	Definition	Verbal Reasoning about Function in the Growing Rectangle Context for This Study
Chunky Continuous Covariation	"The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with a chunky continuous variation."	The student thinks and describes that the area of the rectangle is growing because the height is growing, conceiving that both length and height are varying at an interval. E.g., each time the area increases, the length also increases.
Coordination of Values	The person coordinates the values of one variable (x) with the values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y)."	The student thinks and describes that change in height and change in area as discrete points. E.g., when the height is two, the area is 12; when the height is 3, the area is 27, which would then create a graph by lining up (2, 12), (3, 27) for height.
Gross Coordination of Values	"The person forms a gross image of quantities' values varying together, such as 'this quantity increases while that quantity decreases.' The person does not envision that individual values of quantities go together. Instead, they envision a loose, non multiplicative link between the overall changes in two quantities' values."	The student thinks and describes that the area is increasing while the height is increasing, and they do not conceive that values of height and area are changing together.

Main components of Lithner's theory of mathematical reasoning. The main components of Lithner's theory of MR include flexibility, argumentation, novelty, and object anchoring. *Argumentation* refers to any actions aimed at convincing the solver or someone else of the truth of stated assertions during task solving. During argumentation, one points to various objects and their properties. Argumentation may be based on (anchored) *intrinsic mathematical properties* or *surface properties*. An argument is said to be plausible if it is anchored on intrinsic properties. *Flexibility* refers to the ability to use different approaches and accommodate adaptations to a given situation. *Novelty* occurs when a solver creates a new strategy or recreates a forgotten strategy in a task solving situation. A necessary precursor for novelty to occur is a problematic situation in task solving. A problematic situation happens when a solver does not know the solution or parts of the solution prior to task solving (Geteregechi 2020; Lithner, 2008). If a solver meets a problematic situation and is able to overcome it by posing plausible arguments, then, we say that the solver has engaged in knowledge generation.

Methodology

In order to network the two theories to characterize students' reasoning about functions when they are given quantitatively rich tasks, we adopt Kidron et al.'s (2008) framework for networking theories. We begin by (a) finding a common aspect addressed by the theories, (b) identifying ideas shared in both theories, (c) comparing and contrasting the approaches according to the aspect under investigation, and (d) connecting the results in a complimentary manner that may provide deeper insights into the aspect under consideration.

The aspect under consideration for our case was the kinds of students' reasoning about functions. While Lithner's theory provides a well-specified definition of the term reasoning, Thompson's theory of quantitative reasoning is more silent about the term. The lack of characterization of reasoning is not a surprising observation and has been documented in other studies (e.g, Jeannotte & Kieran, 2017; Authors). Nevertheless, Thompson's theory provides a detailed explanation of reasoning in the context of quantities. Since the concept of function deals with quantities and how they vary, we believe that Thompson's theory and Lithner's theory can be used in a complementary way and provide a strong foundation for characterizing reasoning in the context of functions.

In order to conduct our networking effectively, we start by examining a common aspect in both theories, which is the kinds of students' reasoning about functions. We now identify some of the ideas that are shared by both theories and compare and contrast the approaches based on the shared ideas. To begin with, we start with the idea of flexibility. The notion of flexibility is articulated in Lithner's theory as the ability to see a situation from multiple perspectives and switch between different approaches as the situation may demand. A student who does not show this kind of flexibility is said to experience "fixation" (Lithner, 2008, p. 267). We find this component of Lithner's theory to be very general in nature when applied to the case of functions. Although Thompson's theory does not use the term flexibility, the descriptions in the theory make it clear that a student needs to be able to see quantities in an unconventional way. For example, if a student perceives the height and the length by mapping height of 1 cm with length of 4 cm, then the student shifts into perceiving that for every 1 cm change in height there is 4 cm change in the length.

Our second consideration was argumentation. In both Lithner and Thompson's theories, argumentation is a key component in determining students' reasoning. Since both theories emanated from Piaget's theory of mental images, justifying one's claims is an important consideration in characterizing how they are reasoning. The difference between the two theories is that Lithner's theory provides a way of characterizing the quality of arguments based on whether they are founded on mathematical properties of the involved objects or not. While this argumentation is subtly presented in Thompson's theory, Lithner's theory provides us with a way of bringing it forth and making sure it is at the center of any efforts of characterizing reasoning in the context of functions.

Data Sources and Task

Data originated from the first author's work with secondary school students. This work is from a design-based research methodology (Cobb, Jackson, & Sharpe, 2017). It was a teaching experiment (Steffe & Thompson, 2000) with eight secondary school students. The teaching experiment took place at a community center, for eight consecutive teaching episodes for approximately two weeks. The data is enhanced transcription of small and whole group interactions when they engage in quantitatively rich tasks at after school settings. The growing rectangle task (Figure 1) is modified from the "Gamma tasks" (Ellis, 2011). The enacted task characteristics includes dynamic growing rectangles and their videos (Altindis & Raja, 2021). Students were asked to investigate the relationship between height, length, and area of the growing rectangle.

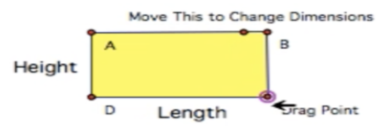


Figure 1: The represents a growing rectangle task

Network Theories in Action: Method of Networking

The first author analyzed an excerpt of data using constructs from Thompson's theory while the second and the third authors analyzed the same excerpt using Lithner's perspective. For Lithner's perspective, our goal was to determine whether the students met a problematic situation and how they reasoned their way out of it. We did this by searching for incidents where the students argued in support of the various assertions that they made and determined the quality of these arguments by assessing their mathematical foundations. Finally, we characterized the forms of reasoning as creative or imitative. From Thompson's perspective, the first author coded the data for incidents that reflected major levels of covariational reasoning (Thompson & Carlson, 2017).

Following these analyses, the three of us met to discuss our individual coding processes and to identify parts of the excerpt that each of us highlighted as important in characterizing students' reasoning about functions. We then compared these parts of the excerpt and how each perspective characterized them. In doing this, our aim was to gain deeper understanding of how using these two theoretical lenses on the same excerpt can be used to offer more insights into the students' reasoning about functions. In the discussion section, we highlight issues that were important for each perspective and how both can be leveraged simultaneously to promote meaningful learning of functions.

Results

By analyzing the data from the two theoretical perspectives, we concluded that, (1) Students *generate knowledge* when they shift among major levels of covariational reasoning. (2) Students showed *flexibility* in creating and connecting representations when reasoning about quantities and quantitative relationships.

(1) *Generation of Knowledge when Moving Across Levels of Covariational Reasoning.* We found that when students moved across levels of covariational reasoning (see Table 1), they benefited from the generated knowledge. The following vignette is taken from Mert and Tarik's small-group interactions when they were exploring the relationship between the height and length in the growing rectangle task.

Mert: So like the height is one. The length is something. Like one's going to be two. So like one three we can tell like tell how much it is. one is a billion the other like 0.001. I am going to make exactly 1. So every one, like, like, wait Is that true? When height is 2,

Tarik: Oh When it is 2, that is going to work. For every one. For height increases length increases by 2cm [Figure 2]

Mert: When the length is by 2, height increases by 1. As the height increases by 1, the length increases by 2, which is making the area larger.

Tarik : Oh and when it's two of one. it is a one to two. For every, When the height is increasing by 1 cm, the length is increasing by 2 cm.

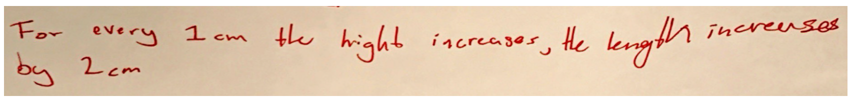


Figure 2: Represents Tarik's written rtifacts explaining the relationship between height and length of the growing rectangle.

From the above excerpt, we noted that Mert mapped the values of the height with the value of length. Then Tarik noticed that the change in length affected the change in height. We noticed that Mert encountered a problematic situation expressed as an uncertainty. Their initial move of addressing the uncertainty was by Mert and Tarik engaging in gross coordination of height and length without addressing the magnitude of the changes in the height and length. Then, they *generated knowledge* by noticing that for every one cm increase in height, the length increased by 2 cm. This was an instance of the students engaging in chunky continuous covariational reasoning. We concluded from this data that students engage in knowledge generation, when they shift from major levels of covariational reasoning, in this case a shift from gross coordination to chunky continuous covariational reasoning. We argue that Lithner' theory of mathematical reasoning can characterize the reasoning entailed in the shifts between the two major levels of covariation reasoning described above (Table 1), which originated from Thompson and Carlson.

(2) Students' Flexibility when Reasoning about Quantities and Quantitative Relationships. We found that students who were able to reason about quantities and quantitative relationships gained flexibility to create multiple representations. The following data excerpt is taken from Mert and Tarik small group interactions, when they explored the relationship between height and length of a growing rectangle.

Tarik: So, it is for every 1. So, it's a one to two ratio. So for every one the height, one centimeter, the height increases the length increases by 2cm. .

NA: Say, one more time. Or can you write that much? what you just said for me, okay, you can use this.

Tarik: as the height increases 1, and length increases by 2 therefore they are making the area larger.

Mert: I think it is going to go straight.

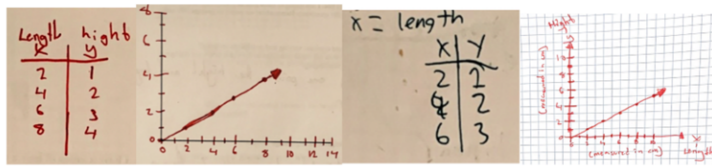
Tarik: No look, don't start the graph yet. We make this [Figure 3b] first.

Mert: Sketch.

Tarik: x and y axis. X is length, and Y is height. You can write on the side, it does not matter.

Mert: Wait. This goes on linearly like that (Figure 3a).

Tarik: No, we have to sketch it (Figure 3b).



(a) (b)

Figure 3: (a); Mert's table and graph. (b) Tarik's table and graph representing height and length of growing rectangle.

From the above data excerpt, we concluded that students were able to translate the table into a graph which gave them a geometric perspective of their function. The ability to create these multiple representations (verbal description, table, algebraic equation, and graph) of the function is an indication of flexibility. In this case Mert and Tarik were not fixated on a single perspective of the problem. Tarik noticed that the relationship between height and length is covarying in the interval of 1 cm (line 11). With that reasoning, Tarik and Mert were able to create table and graph (Figure 3) to represent height and length of the growing rectangle. Mert and Tarik's robust covariational reasoning enabled them to create a table and a graph. This flexibility allows them to see several quantities and quantitative relationships associated with the function.

Discussion and Conclusions

We networked Thompson's theory of quantitative reasoning and Lithner's theory of mathematical reasoning to characterize students' reasoning about quantities and quantitative relationships. Our choice of these two theories was informed by the following reasons. First, we found no differences in terms of how an object is defined by Thompson and Lithner' theories. Second, we agreed that the theories describe "reasoning" in similar ways. Thompson's theory doesn't define reasoning explicitly but describes how functions can be conceptualized using covariation reasoning by perceiving change in one quantity, shifting to view two quantities as coordinated, and building images of two covarying quantities simultaneously. From the description of what is entailed in covariation reasoning, one could characterize aspects of focus that would qualify a certain kind of reasoning as covariation. This kind of description, in our opinion, when assumed as a way of reasoning about covarying object, is limited to the concept of covariation, whereas Lithner's description of reasoning is more general. So they both speak to a kind of reasoning but are different in terms of the scope of the applicability of the kind of reasoning.

Our focus on student reasoning was in support of NCTM's (2014) and other researchers' emphasis to encourage student reasoning in learning functions. Our analysis of data from both theories revealed important aspects of student reasoning including their generation of knowledge and their flexibility in thinking as they engaged in quantitatively rich tasks and showed evidence of covariational reasoning. Knowledge generation is not only a key aspect of learning but also a daily aim for teaching. It is however not a trivial accomplishment in any teaching and learning setup. An approach that emerged from our analysis that enhanced knowledge generation was affording a problematic situation and assessing the nature of argumentation that aided knowledge generation. In particular, the argumentation needed to be anchored in the intrinsic mathematical properties of the objects. In a broader sense, our findings can speak to not only characterizing student reasoning but also supporting their reasoning by designing mathematical tasks that will improve and challenge student reasoning to higher levels.

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Reframing Relearning

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We propose an expanded conceptualization of "relearning", a construct that has a long history in the field of cognitive psychology and has more recently been applied to the field of education, specifically to teacher training. We illustrate how this broader conceptualization applies to other contexts in mathematics education (e.g. developmental mathematics courses and repeated courses), aiming to provide opportunities for researchers of relearning experiences to reconceptualize their work as describing part of a larger body of research in which findings from one section could suggest new avenues for others. By freeing constraints to which studies of relearning have previously been subjected, we provide researchers with theoretical tools to investigate relearning in their own context.

Keywords: relearning, teacher education, developmental mathematics, cognitive psychology

Despite mathematics being known for its hierarchical structure, it is often the subject of courses that are revisited, repeated, or otherwise redundant for students in undergraduate institutions. In some instances, as with math content courses for future teachers, this revisiting of content studied before is an expected and encouraged part of student experiences. In other instances-as with courses that are failed and then retaken, courses that fail to transfer from other institutions and are thus retaken, or developmental (or remedial) math courses that are required due to insufficient performance on a placement test-the repetition is unexpected and often undesirable. All of these cases involve to a significant degree the phenomenon of *relearning*, or learning about content that one has tried to learn before in a previous math course. As a term, relearning has been used for over a century to describe specific studies of memory in cognitive psychology. Beyond these studies in cognitive psychology, Zazkis (2011) has used the term *relearning* to describe the experience of preservice elementary teachers in their math content courses and to distinguish this experience from those that can be well-described by traditional theories of learning. We contend that these current conceptualizations of relearning are subject to unnecessary constraints that have resulted in their theoretical underdevelopment and have limited the potential applicability of the phenomenon of relearning in undergraduate mathematics education at large. Although this broader notion of relearning is not limited to the domain of mathematics, we believe it is of particular interest to mathematics education given that the hierarchical structure of mathematics and higher frequency of course repetition (particularly at the college level) leads to an increased number of opportunities for students to experience this phenomenon in mathematics compared to other subjects.

Existing Conceptualizations of *Relearning*

In cognitive psychology, the term *relearning* is used to describe scenarios in which an individual is studying some content (typically declarative knowledge such as lists of symbols, word-definition pairings, or sequences) they have previously memorized, to once again produce that content from memory in an experimental session. This use of the term *relearning* is attributed to German psychologist Hermann Ebbinghaus who in 1885 documented the number of verbal rehearsals necessary for him to memorize strings of randomly-ordered nonsense syllables as the lengths of the strings varied. He then recorded the number of rehearsals necessary to recite those same strings of syllables again from memory after varying intervals of time. Ebbinghaus labeled his experience of trying to memorize the same strings of syllables through verbal rehearsal a second time as "relearning". This research design involving observation of the amount of time participants needed to relearn to recite lists of terms multiple times was greatly influential on future studies of memory. Of particular significance was the 'savings in relearning' result (Nelson, 1985; Murre & Dros, 2015), or the observed inverse logarithmic relationship between the amount of time elapsed from the first learning trial to the relearning trial and the number of rehearsals required in the relearning trial for the individual to reproduce the material perfectly from memory. Ebbinghaus hypothesized that this trend was directly related to features of memory itself, as something that would fade rapidly at first, then more slowly over time. In this way, the 'savings in relearning' effect provided a means by which one could estimate retention, and thus provide information about the rate at which content was forgotten. Such an estimation is unmistakably valuable in educational contexts, in particular when accompanied by teaching and studying techniques that could anticipate such a curve to encourage maximum levels of retention (Abbot, 1909; Hill, 1914).

This "traditional" use of *relearning* in the study of memory would later come to be critiqued by more contemporary cognitive psychologists (Bahrick, 1979; Kintsch, 1974; Neisser, 1976). Critics noted that the restrictions on the content to be memorized and relearned also restricted the extent to which researchers could explore meaningful aspects of retention and relearning in authentic contexts. In particular, by only focusing on disconnected items in a series that could be memorized and reproduced in totality in one experimental session, researchers limited their applicability to contexts in which particular pieces of content are interrelated within a larger knowledge system such as a hierarchical structure (Nelson & Smith, 1972) or the structure of a language (Hansen, Umeda, & McKinney, 2002). Additionally, by limiting the amount of time between learning and relearning trials (sometimes to as little as a few seconds), traditional memory studies had little predictive value in an educational context in which students would be expected to retain information over the course of several weeks or months. Despite these critiques, studies of relearning in naturalistic educational contexts have largely retained the same scope in content and methodology that was critiqued over 40 years ago.

An example of this can be seen in more recent work investigating the use of the traditional successive relearning technique in "authentic educational contexts" (Rawson, Dunlosky, & Sciartelli, 2013, p. 524). These content/context pairings have included a list of definitions in an introductory psychology course (Rawson, Dunlosky & Sciartelli, 2013); a list of definitions, names of structures in the brain, and information about the steps in a mental process in a biopsychology course (Janes et al., 2020); and instructions for how to solve four types of probability problems in a laboratory setting (Rawson, Dunlosky & Janes, 2020). These studies share the same restrictions of focusing on content that is considered to be learned when it is memorized (with content retention being demonstrated in an experimental session), and thus fail to capture the range of authentic experiences with relearning in educational contexts (e.g. content courses for future teachers, all forms of developmental education, courses that students retake for any number of reasons, and preparatory courses for standardized tests such as the SAT).

The traditional focus on memorization may explain why a separate theory of relearning has more recently been developed by Zazkis (2011) in the field of content courses for preservice teachers (Zazkis did not reference the idea of relearning as it has been treated in cognitive psychology). While relearning as a term was used colloquially to describe the learning experience of preservice teachers learning mathematics prior to Zazkis (2011) (e.g. Nicol, 2006;

Klein, 2008; Hough et al., 2007), Zazkis' work marked the first acknowledgement of the experience of relearning as a phenomenon of theoretical significance in undergraduate math education. Zazkis argued that "contemporary" understandings of how people learn such as constructivism (Ernest, 1996) or situated cognition (Greeno, 1991) are insufficient for describing the experience of relearning for prospective elementary teachers. Specifically, "since prior cognitive structures have been constructed in the learner's mind some time ago, the reconstruction and reorganization processes involved [in relearning] are more challenging for the learner as well as for the instructor" (p. 13). Zazkis' notion of relearning allows mathematics teacher educators to more clearly focus on unique aspects of the learning experience noted to be fraught with resistance from preservice teachers (e.g. Nicol, 2006; Hough et al., 2007; Zazkis, 2011; Barlow et al., 2013). However, as is the case with cognitive psychology, we see potential for an even broader notion of relearning.

Zazkis' (2011) notion of relearning honed in the notion of "restructuring knowledge." In her work, this is the result of reconstructing previously-held knowledge and reorganizing it in a particular way seen as better-suited for the purposes of teaching. This definition is restrictive in that it only considers relearning to occur if two conditions are met: first, the learner began with an insufficient understanding of content from K-12 experiences; second, the outcome of that learning is reconstruction of previously-learned material. This is made clear by Zazkis and Rouleau (2018), who claimed: "It is unavoidable that some ideas of elementary mathematics have to be relearned as their domain of applicability was limited to early experiences. Those are unavoidable met-befores. However, in other cases, such as BEMDAS [the order of operations], relearning would not be required if there was no prior misleading learning," (p. 161). It is suggested that students are only considered to be relearning when they are expanding the "domain of applicability" of their content understanding or correcting "prior misleading learning". Hence, if a student had acquired what the researchers would consider to be a desirable understanding of the order of operations in K-12, then relearning would not necessarily take place when that same content was studied again in the teacher education course. A broader perspective would be to consider that relearning may involve outcomes other than significant restructuring of knowledge. Using this broader perspective, all preservice with the task of learning about content seen before would be considered *relearning*, but the outcome of that task may vary according to a multitude of factors (e.g., the learning that occurred in the K-12 context, the perceived quality of understanding of the content before relearning it). It may also be the case that multiple learning outcomes could exist simultaneously for one individual such that he or she may be reconstructing their understanding of some mathematical topics while achieving different outcomes for others.

Despite their surface differences, we argue that the inherent phenomenon being described as 'relearning' in cognitive psychology and teacher education is inherently the same. Their ostensible dissimilarity stems from the fact that they both describe different types of relearning, subject to restrictions that are relevant to the foci of their respective fields. By viewing them as separate instantiations of the same general phenomenon, both fields would increase the likelihood of theoretical advancements as problems from one field are reconceptualized using the lens of the other, and as explanatory mechanisms and moderating variables are shared. Furthermore, divorcing the term *relearning* from the norms of a particular context allows for the focus to shift from answering the question: 'what outcome *should* students get as a result of this experience?' (e.g., memorization, knowledge reorganization) to 'what outcomes *are* occurring and how do the circumstances of this particular context determine which outcomes are possible?'

Proposal of Theoretical Perspective

At the most basic level, we contend that relearning requires three things: some content (in our case mathematical), a "time 1" (T1) representing a past occurrence in which an individual has tried to learn about that content, and a "time 2" (T2) representing the most recent time an individual has tried to learn about that same content again. In general, it should be compared to the phrase "learning for the first time" in that it does not suggest a particular way in which ideas are cognitively organized or how that organization takes place. Instead, our proposal is to define a context along with defining variables of interest to structure explorations into student experiences in that context. Although the name *relearning* appears to suggest some degree of mastery of content at T1, we make no such assumption in our treatment of this construct. That is, T1 learning need not cross any threshold or meet any criteria for relearning to be said to occur at T2. This is not to say that different levels of proficiency do not matter, but instead that a certain level of proficiency at T1 is not *required* for the phenomenon to take place. By placing additional restrictions on these three components, we can recognize various types of relearning as they are currently conceptualized in the fields of cognitive psychology, preservice teacher education, and other contexts. Each of these fields focuses on a particular type of relearning by requiring certain values of the variables: motivation and possible learning outcomes.

By 'motivation' we mean the main rationale that justifies the beginning of the learning (or relearning) experience for the individual. In the relearning scenario, it is the answer to the question: why is the individual relearning material at this particular time? (e.g., was learning at T1 deemed insufficient, if so why?) Importantly, this question is asked of the *relearning context* rather than the individual. For instance, an individual required to participate in a psych study of memory for course credit and an individual required to take a math content course for future teachers might both list 'academic requirement' as their motivation for beginning the relearning experience. The motivation behind the design of the two scenarios, however, is very different. Whereas the motivation of relearning in memory studies in cognitive psychology is for the individual to meet or exceed proficiency in recalling content from T1, the motivation in math teacher education is for students to acquire a new type of proficiency of content at T1 for the purposes of teaching.

By 'learning outcomes' we mean the resultant relationship between a student and material they have seen before at the end of a relearning experience. This is *not* a grade or an indication of passing/failing. For a scenario in which one is learning for the first time, we ask what was learned. This may be determined by examining a student's answers to a very well-designed exam. The same is not true for a relearning scenario. In asking what was learned, we mean to answer the question: what was the value of their learning experience in terms of their understanding of content this time around? The answer to this question requires one to reference, in some way, the content that was learned before. This is simply not possible in a course where that content is being learned for the first time. The realm of possibilities for what the value of this variable might be is constrained by both the context and the individual. For instance, the restructuring outcome addressed by Zazkis earlier may be one of several possible outcomes for preservice elementary teachers in math content courses. Determining the range of learning outcomes that exist in a relearning experience and comparing it to the desired or range of desirable outcomes would be one of the first ways in which one could begin to determine which contextual elements are or are not supporting students in meeting course expectations.

While the content at T1 and T2 need not be identical, it does need to cross a particular threshold of similarity such that the content learning goals at T2 are essentially the same as those at T1. For some studies of memory in cognitive psychology this criterion is more clearly filled as the materials to be memorized are completely identical at T1 and T2. In the field of preservice teacher mathematics education, the issue of determining content similarity is more complex. This is because math content courses for future teachers often have additional learning goals that would not be considered in the K-12 context. Zazkis (2011) describes the experience of preservice elementary teachers relearning the concept of divisibility. There are several instances in which the preservice teachers "abuse" divisibility rules to make erroneous conclusions. For example, inappropriately extending the idea that if the last digit of a number is even, then it is divisible by 2 leads to the conclusion that the number 359 must be divisible by 3 since 9 is divisible by 3 (p. 60). The learning goal of the lesson occurring with the preservice teachers in this instance is the same as the learning goal that would occur with 4th grade students learning about divisibility for the first time. Interestingly, it is precisely because these content learning goals are the same in the relearning situation that the most relevant questions regarding student experiences at T2 can be asked. While it wouldn't be wrong to ask 'can a preservice teacher produce the prime factorization of the number 359?' a question that would be more relevant in the context would be: 'Given that this student has *already* learned about prime factorization, why are they approaching the task of producing the prime factorization of 359 in this way *now*?'.

Alternative Relearning Contexts

If one considers the basic components to be the only criteria necessary, then several other more common college math experiences are capable of being classified as experiences with relearning. Three types of course experiences that heavily involve the experience of relearning but do not currently utilize the concept for the purposes of theoretical advancement are: college math courses which are retaken, traditional developmental math courses, and non-traditional forms of developmental courses featuring co-requisite sections or supplemental instruction. For the sake of space, we briefly consider developmental courses in more depth. Traditional developmental math courses are semester-long courses taught in college settings whose content mirrors that of pre-algebra and algebra courses offered in the middle and high school settings. These are typically non-credit courses that are required for students to take after placement in order to graduate. Given the high failure rates of these courses combined with their status as degree requirements, students often need to retake developmental courses, sometimes multiple times (Ngo & Velasquez, 2020; Fay, 2020). Thus, it is highly likely that students are spending significant time learning about content they have seen before, either from a previous K-12 math course or from a previously attempted developmental math course in college. In fact, this similarity to content learned at a time T1 is explicitly acknowledged as a common feature and a point of *concern* for developmental math educators. For example, Stigler, Givvin, and Thompson (2010) provide the following summary of a typical developmental math experience:

> Thus, students who failed to learn how to divide fractions in elementary school, and who also probably did not benefit from attempts to reteach the algorithm in middle and high school, are basically presented the same material in the same way yet again. It should be no surprise that the methods that failed to work the first time also don't work in community college. And yet that is the best we have been able to do thus far. (p. 4)

Although the authors' comments focused on the community college context, similar sentiments have been made across developmental math courses offered at 2- and 4-year institutions (see Ngo, 2020). In this case, not only is the mathematical content itself considered similar enough to be "retaught" but the teaching methods used to do so are similar as well. The use of the word 'reteach' here is colloquial, and not suggestive of an underlying theory of relearning as the primary activity in developmental math classrooms. The present framework would serve as a guide into the type of theoretically rigorous investigation of student experiences in these courses that has been called for numerous times before (Grubb & Cox, 2005; Sitomer et al., 2012; Mesa, Wladis & Watkins, 2014).

The developmental math relearning experience differs from an experience with relearning in preservice teacher mathematics education and studies of memory in cognitive psychology due to the motivation behind the T2 experience and the range of possible learning outcomes considered. For example, content courses for future teachers are a mandatory component of the normal curriculum for the major. Relearning in this context is motivated by the career requirement for a different relationship to mathematical content. In contrast, a developmental math experience with relearning is motivated by demonstrated insufficient proficiency with the content learned at T1, either according to the institution (e.g. failing the course previously, poor placement exam results) or the student. Preservice teachers may also have developed an insufficient proficiency with the content in their K-12 math experiences, but T1 proficiency does not dictate whether a preservice teacher would be required to take the course. Likewise, whereas the learning outcome of reconstruction through the reconsideration of previous understandings and restructuring of content may be desirable in both teacher education and developmental math, other learning outcomes are possible. For instance, it is a reasonable learning outcome in a developmental math course for an individual to acquire the content proficiency that they should have acquired the first time they took the course. Learning outcomes in this domain are much more bound to the learning outcomes that were possible at T1, much like studies of memory in cognitive psychology. While this outcome is also possible in math teacher contexts, it would be far from desirable. Learning outcomes in these contexts are made to intentionally expand beyond those possible at T1 due to students' differences in age and mathematical experience.

Comparison to Alternative Relearning Conceptualizations

Due to the hierarchical structure of mathematics, one could argue that you would be hard-pressed to find any college math course that didn't include learning about at least *some* content that a student had seen before. Thus, one might argue that instances of relearning are really simply special cases of students building on prior knowledge. Recall that in order for a scenario to be labeled as relearning, the content learning goals at T2 are essentially the same as the content learning goals at T1. This would exclude cases, for instance, in which calculus instructors reference common *algebraic* errors when teaching students how to find critical values of functions whose derivatives involve fractions (Stewart & Reeder, 2017). The content learning goals at hand are focused on novel Calculus concepts of derivatives and local maxima and minima, not the algebra that might be involved in solving a problem related to these concepts. Like the title of the book Stewart and Reeder's chapter comes from, *And the Rest is Just Algebra*, learning of algebra in this context is considered trivial, or at best secondary.

While we agree that it would be possible to view relearning scenarios like any other learning scenario in which prior knowledge is used, we contend that this level of generality would be less advantageous for understanding student experiences. Consider the comparison between the above examples from calculus with the educational scenarios described in Cox (2015). In her analysis of curriculum and instructional activities across six developmental math courses, Cox describes instructional strategies for teaching students about fraction representations. One strategy involved walking students through "a review of division more broadly" in order to contextualize the idea of fraction division within a larger domain of part-whole relationships between numbers (p. 274). The instructor asked students to produce mathematical problems whose solutions would be represented by various fractions. For instance, 3/.25 could be a representation for the solution of the problem "How many quarters do I need to make $3?" (p. 274). One *could* think of these students as building on prior knowledge of division to produce a new type of understanding of what previously may have been only a mathematical "rule". However, considering this lesson to be an example of relearning allows the primary area of focus to be more specific to the purpose and impact of the repeated content. Using this lens, we might wonder: what kinds of problems would these students be able to solve more easily using this type of understanding, than they could in their previous algebra course? To what extent did students perceive this lesson as a "review" and how did that impact the value they gained from this additional lesson in fraction division?

In a similar way, relearning can be distinguished from McGowen and Tall's notion of a met-before (McGowen & Tall, 2010). A met-before is defined as "a mental structure that we have now as a result of experiences we have met-before," (p. 171). McGowen and Tall use met-befores to construct mental models of students' understanding of content by considering how students employ mental structures formed by previous experiences with mathematical content to learn new things. The notion of a met-before is not incompatible with the notion of relearning, but the two terms represent different types of entities. Met-befores are mental structures of previously seen content, whereas relearning is an experience that takes place when a student is learning about the same content at a different timepoint. However, met-befores may be a particularly useful concept for examining how the particularities of a relearning context restrict the kinds of learning outcomes that are possible for students given that they are capable of being both supportive and unsupportive according to the context in which they are encountered.

Conclusions

One of the most difficult and yet most useful aspects of defining a phenomenon is determining the boundary conditions. Deciding whether or not an educational scenario can be described as relearning is not always obvious. While some scenarios (like the cognitive psychology experiments described above) more clearly fit the definition of relearning, deciding to view experiences in naturalistic educational contexts as relearning requires a bit more thought. In the contexts discussed above of teacher education, students repeating courses, and developmental math courses, the question is not so much 'is this an instance of relearning?' but rather 'to what extent can I describe student experiences in this context as relearning?' Exploring the boundaries around what is and is not best described as relearning in a given context is a method by which instructors can work to better understand the scope of experiences students are having with content in their course. In the realm of teacher education, a consequence of considering the role of relearning within a math methods course is that it requires one to also carefully consider the role of pedagogical content knowledge. The extent to which the two can or should be thought of separately is a point of debate for teacher educators. One benefit of considering the role of relearning in such a course is that the phenomenon is not inextricably tied to the teacher education context like pedagogical content knowledge is. This enables teacher educators to compare the experiences of their students with those students in other relearning contexts in which the motivation of teaching the content is not present. This, in turn, may help shed light on the influence of the expectation of teaching over student experiences, even in courses that are "purely" subject-matter based. How similar are the experiences of students in a developmental algebra course and those in a math course for future teachers aimed at that same mathematical content? Which learning outcomes might only be possible in either context due to the difference in motivation for relearning?

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The Staging of Proof by Contradiction in Texts: An Exploration of Disciplinary Artifacts

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Drawing on Vygotskian perspectives of cultural artifacts and the work of Plut and Pesic (2003) this theoretical report argues that, as culturally supportive tools (CSTs), texts play a role both in students' development and in the reproduction of culture. The extent to which these roles are fulfilled, however, can vary. Indeed, as argued by Plut and Pesic (2003), texts can be functional or dysfunction with respect to either development ot culture. Taking these possibilities into consideration this theoretical report explores: (1) the staging of proof by contradiction in Introduction to Proof (ItP) texts, and (2) the extent to which these ItP texts (cultural artifacts) align and/or fail to align with historical /philosophical treatments of proof by contradiction; that is, the extent to which these CSTS may be either functional or dysfunctional.

Keywords: Proof by Contradiction, Contradiction, Cultural Artifacts, Textbook Analysis

Introduction

Research on students' difficulties with proof by contradiction has tended to cast students' difficulties as an *attribute of students*. Specifically, researchers have argued that students experience difficulties because: (i) students' dislike this form of proof (Harel and Sowder, 1998); (ii) prefer to reason constructively (Harel & Sowder, 1998; Leron, 1985;) or from known objects as opposed to the absurd (Antonin & Mariotti, 2006), and/or (iii) experience difficulties accepting the logical theories (metatheorems) required for proof by contradiction and, therefore, the results of such proofs (Antonini & Mariotti, 2008). In other words, students supposed difficulties with proof by contradiction are essentialized (in the sense of Gutierrez, 2008): they are portrayed as characteristics of students, their ways of reasoning and believing, seeing and knowing. One issue with this tendency is that essentialization results in students' ways of reasoning and knowing being treated as separate from rather than linked to mathematicians' didactical practices and curricular resources. But what if students' ways of reasoning about proof by contradiction are not, in fact, that which arise irrespective of the instructional and cultural milieus students experience? The focus of this theoretical report is to begin to explore this question in the context of ItP texts. Specifically, this report explores the questions: (1) How might we characterize the staging of proof by contradiction in ItP texts, i.e., didactic CSTs; and (2) Does this staging align with discourses found in historical/philosophical artifacts?

Why Investigate Texts?

Plut and Pesic (2003) argue that by taking a Vygotskian perspective of texts we can see texts as cultural products with a cultural mission. Specifically, textbooks – those written, symbolic artifacts used for didactical purposes and henceforth referred to as *texts* – not only function as a "formative influence on the individual development" but also play a critical role in "the cultural reproduction of society" (p. 502). One of the ways texts do this is by providing "samples of knowledge that competent adults in certain cultures have selected, classified and didactically shaped as organized systems of knowledge" (Plut & Pesic, 2003, p.502). Beyond this role, texts also mediate various psychological and social practices. Consequently, texts can and should be thought of as disciplinary artifacts, which function as culturally supportive tools (CSTs):

"independent symbolic entities of an external world (whose function is) cultural mediation … (and which) become constitutive elements of an emerging (psychological) structure" (p. 503).

Texts as CSTs are of interest because of the many roles CSTs play. CSTs both represent and are representative of cultures (including disciplinary cultures such as the mathematics community); they mediate the internalization of knowledge and, therefore, can be a constitutive of ways of reasoning and knowing, and CSTs "enable or support … the very self-reproduction and development of culture" (Plut & Pesic, 2003). Indeed, returning to the focus of this theoretical report, it can be argued that *Introduction to Proof* (ItP) texts are the very artifacts that the mathematics community (i.e., the discipline of mathematics) produces to play these roles: they provide a sampling of expert knowledge which both represents and is representative of the discipline's cultural practice of proving, they are designed to mediate students' ways of knowing and reasoning about the practice of proving, and are developed to support the reproduction of not only cultural practices but also the culture of the discipline of mathematics itself.

On Examining and Operationalizing the CST Concept

One of the key questions asked by Plut and Pesic (2003) in their discussion of texts as CSTs is: "Is it possible to classify CST according to their validity" (p. 509). Turning to ItP texts, *what would it mean for such texts to be valid CSTs for proof by contradiction*? One criterion might be that a text's stance towards the practice of proof by contradiction aligns with that found in non-didactical disciplinary artifacts, such as the historical-philosophical artifacts of the discipline where meta-mathematical discussions are found. Such criteria would necessarily be seeking to determine if the texts are acting as a means for cultural reproduction. Another criterion might be that derived from the work of Brousseau (1997); namely, that the text's staging of proof by contradiction not only mediates development but also hampers the emergence of didactical obstacles (i.e., obstacles which are neither epistemological nor ontological but due to didactical aspects), such as those that arise from the encoding of didactical transpositions (i.e., shifts in meanings which arise from efforts to make concepts teachable). And, while the first criterion – that of alignment to disciplinary artifacts - explores the question of cultural reproduction, this second criterion would enable examinations of questions of internalization: Can the text support the internalization of valid practices and ways of knowing? Indeed, there are many criteria that might fruitfully inform questions of the validity of ItP texts as CSTs for proof by contradiction. Due to space limitations, this theoretical report explores the first criteria: *To what extent do ItP texts meet the criteria of cultural reproduction*? The selection of this criteria is due in part to the timing of the research (a pandemic) and in part to the belief that such research is foundational.

Is alignment enough? Theoretical discussions of CSTs tend to characterize CSTs as playing a "positive" or functional role (Plut & Pesic, 2003). Yet, there is cause to consider alternative possibilities. Specifically, "the concept of cultural tools has inherent positive connotation (sic)" and ignores the possibility of a "dark side" of cultural influence (p. 510). Two examples of negative ("dark side") influences include CSTs that play a *restrictive function* and those that have a *destructive function* (Plut & Pesic, 2003). In a restrictive function, a CST acts in a "controlling and directing" manner which "narrows the wide range of developmental possibilities" (p. 510). Whereas, when playing a destructive function, CSTs are "not orienting but disorienting, not regulative but disintegrative, bringing chaos instead of order and structure, restricting or even destroying the developmental potentials" (p. 510). These negative functions or dysfunctions lie in contrast to the positive functional roles CST might play, such as: supporting the coding of cultural material; fostering productive internalization; mediating the emergence of shared ways of knowing, enabling participation in cultural practices, etc. Thus, questions about alignment between historical/philosophical and didactic texts are important, for they may inform explorations of the extent to which CSTs range from functional to dysfunctional.

The Staging of Proof by Contradiction in Historical/Philosophical Cultural Artifacts

In this theoretical report, the phrase "historical/philosophical cultural artifacts" refers to the written artifacts produced by members of a discipline (or cultural group) with the aim of describing an object or practice of the discipline to other disciplinary experts and/or with the aim of placing the object/construct within a broader theoretical framework related to the discipline. Artifacts that belong to this category include but are not limited to manuscripts written by members of the discipline and those produced by researchers, be they educational, historical or philosophical. In this report, commentaries on proof by contradiction produced independently by David Hilbert, G. H. Hardy, and G. Polya are considered, as are writings on proof by contradiction produced by the logician and philosopher Ludwig Wittgenstein and further described by Alfred Nordmann (2010), and historical analyses of the emergence of proof by Hans Neils Jahnke (2010); in particular, his remarks regarding the origins of indirect proof.

The Commentaries of Mathematicians in Historical/Philosophical Artifacts

If one turns to mathematicians' writings on proof, in general, it is not difficult to find remarks on proof by contradiction. In this report, analyses were restricted to the three most well-known commentaries on proof by contradiction, those of Hilbert, Hardy, and Polya. In the early 1920s, David Hilbert put forth what became known as the formalist program. He is said to have sought to rid mathematics of the many paradoxes and inconsistencies emerging during the development of set theory and other areas of advanced mathematics. His practices and goals put him at odds with some in the mathematics community; in particular, those who are often referred to as the intuitionists and/or constructivist. Debates ensued. One comment from these debates often attributed to Hilbert is the following: "without proof by contradiction a mathematician is a pugilist with his hands tied behind his back." A pugilist is a boxer. The image Hilbert's remark conjures is of a fighter without access to his fists. Thus, without contradiction we have nothing to fight with. Proof by contradiction is a fighter's tool, one with which we can rid mathematics not only of the paradoxes that plagued a generation but also any inconsistencies.

In the famous essay, *A Mathematician's Apology* (1940/2005), G. H. Hardy offered a slightly different characterization of proof by contradiction (i.e., reductio ad absurdum):

> "Reductio ad absurdum, which Euclid loved so much, is one of a mathematician's finest weapons[5]. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game." (p.19)

A gambit is a risky move, one which often entails some level of loss while at the same time can potentially lead to significant gain. Why does Hardy refer to proof by contradiction as a gambit? Why does he say, "a mathematician offers the game"?-One possibility is that, as a mathematician of the 1940s, Hardy was well aware of the controversies of the foundations crisis and the impact of the non-Euclidean geometries of Gauss, Bolyai, Lobachevsky, Reimann and others. In regard to the latter, many of these mathematicians (e.g., Gauss) had sought to establish that the fifth postulate of Euclidean geometry was actually a proposition by assuming the postulate was false and attempting to derive a contradiction. In the case of Gauss, Bolyai, and Lobachevsky, this approach, along with the assumption that all lines are infinite, led to their (independent) development of hyperbolic geometry. In the case of Reimann, he took the same approach but with an assumption that lines are finite and developed elliptic and then Reimannian geometries. Returning to the work of Gauss, we find an example of that which Hardy describes. As a gambit, Gauss negated the fifth postulate. He then sought out a contradiction (hoping to preserve the "game" of Euclidean geometry). Instead, what he found was a new game. Indeed, writing to his colleague Taurinus, Gauss remarked he'd developed "a geometry quite different from Euclid's, logically coherent, and one that I am entirely satisfied with … The theorems are paradoxical but not self-contradictory or illogical … All my efforts to find a contradiction have failed" (Gauss's letter to Taurinus, quoted in Gray, 2006, p. 63). Thus, it was as Lockhart warned in *A Mathematicians Lament,* (2002), in mathematics "your creations do what they do, whether you like it or not." Looking across these commentaries we can see an emerging pattern: Proof by Contradiction is a means for establishing the consistency of a mathematical theory. It is a weapon against paradoxes and inconsistencies; a mechanism by which we test the limits of a mathematical theory or even from which new theories emerge.

But what of Polya? In Polya's famous 1957 problem solving text, *How to Solve It*, he discussed proof by contradiction and then there is a section titled *Objections* where he notes:

> … it would be foolish to repudiate "reductio ad absurdum" as a tool of discovery. It may present itself naturally and bring a decision when all other means seem to be exhausted ... Experience shows that usually there is little difficulty in converting an indirect proof into a direct proof, or in rearranging a proof found by a long "reductio ad absurdum" into a more pleasant form … In short, if we wish to make full use of our capacities, we should be familiar …with "reductio ad absurdum"... When, however, we have succeeded in deriving a result … we should not fail to look back at the solution and ask: *Can you derive the result differently*? (p.169).

Unlike Hilbert and Hardy, Polya does not refer to proof by contradiction as a "fine" weapon, nor does he describe it as a primary fighting tool like the fists of a boxer. Instead, it is characterized as a necessary method "when all other means seem to be exhausted." From Polya's perspective, proof by contradiction is a last resort, which should be avoided if other techniques are possible.

Researchers' explorations of the discipline as Historical/Cultural Artifacts

In addition to commentaries by mathematicians, others have written about mathematics at a metamathematical level. Among these scholars, two stand out due to the attention paid to indirect proof: the philosopher and logician Ludwig Wittgenstein and the mathematician and historian Hans Niels Jahnke. Regarding Wittgenstein, his work is of interest because Wittgenstein argued that proofs can be viewed as pictures or as experiments. This is noteworthy because it is the later that Wittgenstein argued was best exemplified by proof by contradiction. A proof is a picture when symbols have been employed to produce a surveyable, reproducible representation of an activity. Though the activity itself was an experiment – the experiment is rendered in symbols. As Nordhamm (2010) notes, the proof is "something to be surveyed and seen." In contrast, a proof by contradiction is an experiment. We do not survey such proofs but rather reenact them. "The reductio argument …exemplifies the proof as an experiment that probes commitments and establishes the connection between inference and decision" (Nordhamm, 2010, p. 195). Indeed, it is this aspect of indirect proof Wittgenstein focused on when arguing, whereas pictures show us what is, the reductio argument tests our commitments and reveals the "domain of the imaginable;" it demonstrates not only what can but also what cannot be within our theory.

Turning to Jahnke's (2010) writings on the origins of proof, two points are important. First, Jahnke notes that during the era of Euclid's Elements a great deal of discussion occurred among Greek mathematicians on the topic of axioms, postulates, and hypotheses. Though often equated in modern mathematics, Greek mathematicians made subtle distinctions. Moreover, these components of mathematical theories were themselves examined. Indeed, when engaging in dialectic practices participants negotiated hypotheses prior to their use. Specifically, participants would examine hypotheses in terms of the desirable and undesirable consequences. Speaking to this practice, Jahnke remarks "The extreme case of an undesired consequence would be a logical contradiction, which would necessarily lead to the rejection of a hypothesis. … the procedure of indirect proof in mathematics can be considered as directly related" (p. 19) Indeed, as Jahnke (2010) argues, that this practice may explain the "frequent occurrence of indirect proof in the mathematics of the early Greek period" (p. 19). The second important point concerns not indirect proof, itself, but rather mathematics:

> "From the times of Plato and Artistotle to the nineteenth century, mathematics was considered as a body of absolute truths resting on intuitively safe foundations … In contrast, modern mathematics and its philosophy would consider the axioms of mathematics simply as statements on which mathematicians agree; the epistemological qualification of the axioms as true or safe is ignored" (Jahnke, 2010, p. 27).

To be sure, after the foundations crisis a modern view emerged where: "mathematics deals exclusively with hypothetical states of things and asserts no matter of fact whatever" (Peirce 1935, p.191 cited in Jahnke, 2010, p. 27). These remarks are of interest for they speak to the idea that, from a modern point of view, proofs are tools not for determining "absolute truths" but for building theories and establishing their consistency. Something is "true" in the sense that it is valid in a mathematical system. And, proofs by contradiction are the means by which we fetter out undesired consequences or, to echo sentiments expressed by Hilbert, determine consistency.

The Historical-Philosophical Staging of Proof by Contradiction: A Summary

As illustrated by the mathematicians' commentaries, views on proof by contradiction vary. A prevailing theme among many is that proof by contradiction is a powerful tool (or weapon) for riding theories of inconsistencies and for understanding what one's theory can and cannot do. Standing out as a counter-perspective, we find Polya's view: among problem solving methods it is a last resort, one which should be replaced with other methods, when possible. Interestingly, it is this view that we find in texts for novices, as will be seen in the next section.

The Staging of Proof by Contradiction in Didactic Artifacts

Didactic artifacts are artifacts produced by a cultural group with the intent of teaching the recipients concepts or practices employed within a culture. In this case, the cultural group is the mathematics community and their "culture" is the discipline of mathematics. Of interest are those texts designed to enculturate novices into the discipline's practice of proving, generally, and their staging of proof by contradiction, in particular. Thus, for this analysis 11 *Introduction to Proof* (ItP) texts were selected. The majority were published during the past two decades as ItP courses became more prevalent. Two earlier texts were included to see if significant differences in the staging of proof by contradiction were evident. Analyses were limited to the ItP texts' sections or chapters on proof by contradiction.

Drawing on grounded theory (Strauss & Corbin, 1994) traditions, a process of iterative categorization was employed to identify the staging practices in didactic artifacts. This analytic approach involved cycles of description, categorization, and comparison followed by cycles of recategorization and comparison until stable categories emerged. Specifically, this approach produced both a general categorization and three subcategories. Before describing distinctions, we explore commonalities. All of the ItP texts, including those of the early period, focused on proof by contradiction as a method with specific logical underpinnings. For instance, nearly all of the texts included symbolic logic justifications of the method (e.g., $(\sim P \Rightarrow R \wedge \sim R) \Rightarrow P$) before getting into details about the distinct steps involved. The texts also tended to provide a specific set of steps to follow, which ended with remarks such as, "find a contradiction." Seldom did these texts include an answer to the question "a contradiction to what body of knowledge?" or specify what was to be taken as known and not known, as illustrated below.

> PROOF TECHNIQUE *Proof by Contradiction*. To prove an implication, it is enough to assume that the hypothesis is true and that the negation of the conclusion is true and then deduce any contradiction. Proof by contradiction is based on the rule from proof by contradiction stated in Section 2.1. (Barnier & Feldman, 2000, p.41).[1]

Indeed, the approaches in ItP texts tended to align with the sentiments expressed by Polya: the method is a problem-solving approach; the problem is to find a proof and the solution is a "proof." Proofs are not constituent components of mathematical theories, they are a means to an end. In some cases, the intellectual need for proof by contradiction was discussed but these discussions were infrequent and focused on verification rather than on the other functions of proof. Moreover, many texts included commentaries describing the method as "odd," "strange" or to be "avoided." Thus, due to the general lack of discussion and/or inclusion of mathematical theories, the staging of proof by contradiction in didactical artifacts was categorized as both *atheoretical* and *methodological*, with methodological meaning: focused on the procedures employed by a discipline. We now turn to the three subcategories.

Subcategory 1 ItP texts were those in which front matter or appendices included axioms or stated "givens" that were referenced in the sections on proof by contradiction. The texts did not present the concept of a mathematical theory nor did they discuss the method in relation to mathematical theories but did employ parts of a *reference theory* when exemplifying proof by contradiction. The focus in these texts was on the methodology, its role in the verification of results and underlying logic. Descriptors for these texts include: methodological (MT), reference theory (RT), intellectual need of validation (INV) and problem-solving approach (PS). Only two texts in the sample were in subcategory 1: Bittenger (1970) and Lankins (2016).

Subcategory 2 ItP texts met all of the criteria of Category 1, except they did not include a reference theory. Instead, the staging was *atheoretical* (AT): no theoretical components were explicitly included in relation to proof by contradiction or were withheld until subsequent, unreferenced chapters. These texts also alluded to but did not specifically claim universal truths when deriving a contradiction. And, in some cases, the idea of a contradiction was described in relation to a "known fact." Descriptors for this category include: MT, PS, INV, and AT. In the

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[1] The reference rule is $(P \wedge \sim Q \Rightarrow O) \equiv (P \Rightarrow Q)$ (See p.36).

sample, 6 texts were in Category 2: Lay (2001), Babich and Person (2005), Cunningham (2012), Chartrand et al (2013), Cullinane (2013), and Meier and Smith (2017).

Subcategory 3 ItP texts met the criteria of Categories 2, except rather than being atheoretical these texts tended to reference *universal truths* (UT). In other words, not only was no theory provided (or referenced) but the exemplifying proofs refrenced that which is known universally. For instance, in Solow's (1982) text, he remarks "Thus, in a proof by contradiction, you assume that A is true and that NOT B is true, and, somehow, you must use this information to reach a contradiction to something that *you absolutely know to be true*." (*emphasis added*, p.65). In Hammack (2013) it is argued that to recognize a contradiction on must merely recognize "*nonsense*" (p.111). And in Barnier and Feldman, students are told that they should simply "*assume the usual facts*" when the authors discuss proofs involving the reals. Codes for this category include: MT, PS, INV, and UT. In the sample, 3 texts were in Category 3: Solow (1982), Hammack (2013) and Barnier and Feldman (2000).

In summary, the analyses of the didactic artifacts revealed that when enculturating novices into the discipline, these CSTs tended to focus on proof by contradiction as a methodological, problem-solving technique focused on validation. Theories were either employed as reference theories or not employed at all, with the exception of a few cases in which mathematics was treated as body of knowledge with taken-as-shared universal truths. Across the categories the distinguishing feature was their treatment of mathematical theories and the notion of "truth."

Discussion

The primary aim of this theoretical report was to explore the questions: (1) How might we characterize the staging of proof by contradiction in ItP texts; and, (2) Does this staging align with that found in historical/philosophical artifacts? As described in the findings of the text analysis, the sample ItP texts staged proof by contradiction as a problem-solving methodology in a manner which was either *atheoretical*, or employed a *reference theory* without a discussion of mathematical theories, or approached the method as relying on *universal truths.* Thus, the didactic artifacts differed in significant ways from most of the historical-philosophical artifacts examined in the study. To be sure, with the exception of Polya (1945), historical /philosophical artifacts tended to describe proof by contradiction as a means for testing theories, hypotheses, and axioms – their coherency, consistency and role in a broader mathematical system rather than restrict their function to verification and problem solving. Returning to the idea of texts as CSTs and the work of Plut and Pesic (2003), the question of whether or not misalignment is indicative of dysfunction seems all the more salient. Indeed, this work raises several questions: Is the lack of alignment between historic/philosophical and didactic artifacts problematic? In particular, if one criterion for validating CSTs is that of cultural reproduction, is a problem-solving methodological approach focused on validation sufficient? Or are such artifacts dysfunctional by being restrictive or destructive? Certainly, the didactic artifacts cast this form of proof in a narrow manner, suggesting at the least a restrictive dysfunction or the need for research that examines the impacts on students as they progress in their enculturation to the discipline. Moreover, there is cause to question: is the lack of alignment (this dysfunction at the enculturative level) destructive: "not orienting but disorienting, not regulative but disintegrative, bringing chaos instead of order and structure, restricting or even destroying the developmental potentials" (Plut & Pesic, 2003, p. 510). Indeed, it is not clear when students are asked to negate a statement and produce "nonsense" that the students will recognize they are building the *domain of the imaginable;* not only revealing what the theories can do, but also what they cannot.

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Theorizing Proof as Becoming Using the Post-Structural Philosophy of Gilles Deleuze and Félix Guattari

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In much of the literature in mathematics education and beyond, proof is often seen as consisting of both subjective and objective aspects where mental processes and meaning-making provide pathways to some form of objective statement or theorem. In this theoretical report, I explore an alternative approach to the conceptualization of proof that is rooted in the post-structural philosophy of Gilles Deleuze and Félix Guattari. Proceeding in this manner allows us to see proof as a becoming, as a material that is always entangled with the world, in flux, and never stable. I illustrate a particular manifestation of Deleuzio-Guattarian becoming, that of faciality, with regards to proof by utilizing story excerpts from the AMS text Living Proof: Stories of Resilience Along the Mathematical Journey. I argue why viewing proof as becoming is useful with regard to expert mathematicians and conclude with a brief discussion about research implications.

Keywords: Proof, Deleuze and Guattari, Post-Structural Philosophy, Becoming, Faciality

In Rota (1997), the author states that "Everyone knows what a mathematical proof is. A proof of a mathematical theorem is a sequence of steps which leads to the desired conclusion" (p. 183). While this account of proof would certainly be familiar to mathematicians, students, and researchers alike, the philosophical and mathematics education literature has proposed various ways to further nuance this conceptualization. For example, Kitcher (1981) states that an argument is rigorous "if and only if the sequence of statements has the conclusion as its last member, and every statement which occurs in it is either a premise or a statement obtainable from previous statements by means of an elementary logical inference" (p. 469). Brown (1997) argues for the usefulness of picture proofs, providing a platonic argument that "*some 'pictures' are not really pictures, but rather are windows to Plato's heaven.…*As telescopes help the unaided eye, so some diagrams are instruments (rather than representations) which help the unaided mind's eye" (p. 174). While these accounts appear to express an objectivist view of proof in that logical statements, transcendent conclusions, and truth are emphasized, other scholars have conceptualized proof or aspects of proof by also elaborating on the subjective qualities (e.g., CadwalladerOlsker, 2011; Harel and Sowder, 1998; Raman, 2003). For example, Harel and Sowder's (1998) notion of proving involves that of a *proof scheme* which "*consists of what constitutes ascertaining and persuading for that person*" (p. 244). This sort of definition assumes that there is a degree of subjectivity involved in the proof process. On the other hand, Harel and Sowder's (1998) proof scheme notion also takes into account an objectivity in that ascertaining and persuading involves mitigating doubt to obtain truth.

In a recent theoretical investigation, Czocher and Weber (2020) produce an interesting *proof cluster definition* that is derivative of Wittgenstein's notion of *family resemblance*. The idea is that proof should be defined using a variety of criteria but that an argument only has to meet any one of those criterion in order to be considered a proof in a minimal respect. For example, Czocher and Weber (2020), as part of the cluster definition, indicate that arguments are convincing to mathematicians and "*sanctioned* by the mathematical community" (p. 61) which both involve the subjectivities of people. Another property is that a proof is an "*a priori* *justification* that shows that a theorem is a logically necessary consequence (i.e., a deductive consequence) of axioms, assumptions, or previously established claims" (p. 61) which is a more objective aspect. This subject/object binary also seems to manifest in constructivist-inspired frameworks such as Kidron and Dreyfus' (2014) notion of *proof image* where "the interplay between the learner's intuitive and logical thinking as well as the construction of knowledge that results from and enables progress of this interplay" (p. 299) are emphasized.

Focus and Motivation

While the above ideas certainly add to our understanding of what proof is, I argue that these accounts center fairly straightforward subject/object relations, often involving the subjectivities of individuals in relation to, and perhaps resembling or mimicking, that of objective truth or rigor. That is, the proof literature often seems to emphasize *intentionality* which refers to "referentiality, relatedness, directedness or 'aboutness'" (Crotty, 1998, p. 44). If we want to further understand what proof is in relation to professional and student experience, I suggest also seeing proof *as a becoming* as opposed to foregrounding its subject and object. Such an approach does not fully dismiss subjectivity and objectivity, but it does not center it either. That is, becoming is often a *post-intentional* project that "is not meant to suggest a departure from or an opposition to intentionality. Rather, the conceptual move is meant to experiment along the edges and margins of phenomenology using some post-structural ideas" (Vagle, 2018, p. 128). In this report, I utilize the philosophy of Gilles Deleuze and Félix Guattari (D&G), which views material not as stable entities, but as a becoming that is "neither imitation nor resemblance" and "that can no longer be attributed to or subjugated by anything signifying" (Deleuze and Guattari, 1987, p. 10). Rather, becoming can be seen as a formation of something new, an irreducible entanglement known as a *rhizome*, that cannot be separated from the world and that always functions with the political, artistic, experiential, and experimental while also intersecting with logical, scientific, and cognitive aspects. Such a post-intentional program is better carried out through philosophical inquiry rather than with cognitive or social frameworks and methodologies that often disentangle phenomena along disciplinary lines. While D&G's ideas have rarely, if ever, been utilized explicitly in proof research, post-structural notions (including those of Deleuze) have been used in the study of mathematical materiality and embodiment in school mathematics (e.g., de Freitas, 2013; de Freitas & Sinclair, 2013; Roth & Maheux, 2015) of which the work represented in this report certainly finds connection. I argue that studying proof as becoming is important since it helps us to understand the activity that underlies the subjectivities and norms regarding the intentional proof conceptions found in the literature while also exploring the ways that these can be overcome. Together, these processes *constitute proof itself* and studying them will further our thinking regarding what proof can be. Finally, since this work has some focus on doctoral student proof-related experiences associated with their dissertations, it may also help inform our understanding of more "expert-level" proof involvement called for by Weber and Mejia-Ramos (2011).

In the next section, I provide further overview of some of D&G's ideas followed by a juxtaposition of a specific type of becoming discussed by Deleuze and Guattari (1987), that of *faciality*, with the notion of proof. I utilize excerpts from the AMS text *Living Proof: Stories of Resilience Along the Mathematical Journey*[1] to help demonstrate such concepts and end the report with a brief discussion concerning research implications.

[1] Excerpts reprinted with permission from the American Mathematical Society and their respective authors.

Deleuze and Guattari and their Relationship to Mathematics

Gilles Deleuze and Félix Guattari were both French philosophers in the post-structural tradition which consisted of theorists such as Michel Foucault, Jacques Derrida, Jean-François Lyotard and others. Post-structuralism is a response to the enlightenment ideals of structuralism which often centers logic, rationalism, positivism, humanism, and stable identities. Post-structuralism, on the other hand, embraces the exploration of the metaphysical, the irrational, and the instable. In the case of D&G, their interest lies not in the centering of the scientific nor the linguistic but rather in the material. For example, in *A Thousand Plateaus* (Deleuze & Guattari, 1987), D&G write about the notion of a book: "We will never ask what a book means, as signified or signifier; we will not look for anything to understand in it" (p. 4). Rather, Deleuze and Guattari (1987) state, "we will ask what it functions with" (p. 4). In reading D&G, it becomes clear that a "book" could just as well stand in for any other notion such as "human," "capitalism," "thought," "fascism," "music," or "proof." D&G are interested in the conditions which stable identities arise without asking what they actually are, what they mean, or what they represent. For example, regarding a hurricane, Smith and Protevi (2020) discuss D&G's notion of the *virtual* and how this creates the material conditions for such a phenomenon to *actualize*:

> Here it should be intuitively clear that there is no central command, but a self-organization of multiple processes of air and water movement propelled by temperature and pressure differences. All hurricanes form when intensive processes of wind and ocean currents reach singular points. These singular points, however, are not unique to any one hurricane, but are virtual for each actual hurricane, just as the boiling point of water is virtual for each actual pot of tea on the stove. In other words, all hurricanes share the same virtual structure even as they are singular individuations or actualizations of that structure. (para. 36)

Smith (2006) indicates that Deleuze's view of mathematics is somewhat similar to that of the hurricane in that it involves movement between the *axiomatic* and the *problematic* and that

> The fundamental difference between these two modes of formalisation can be seen in their differing methods of deduction: in axiomatics, a deduction moves from axioms to the theorems that are derived from it, whereas in problematics a deduction moves from the problem to the ideal accidents and events that condition the problem and form the cases that resolve it. (p. 145)

In describing an example of problematics which Deleuze refers to concerning the solvability of the quintic, Smith (2006) states that:

> In 1824, Abel proved the startling result that the quintic was in fact *unsolvable*, but the method he used was as important as the result: Able recognized that there was a pattern to the solutions of the first four cases, and that it was this pattern that held the key to understanding the recalcitrance of the fifth. Abel showed that the question of 'solvability' had to be determined internally by the *intrinsic* conditions of the problem itself, which then progressively specifies its own 'fields' of solvability. (p. 160)

This example highlights the wandering aspect (or "method") of the problematic side of mathematics, one in which solutions are not centered but are considered unforced events. Smith (2003) states that "In Deleuzian terms, one might say that while 'progress' can be made at the level of theorematics and axiomatics, all 'becoming' occurs at the level of problematics" (p. 424). Problematics corresponds closely to the virtuality of the hurricane in the example discussed previously. Smith (2003) also explains that "problematic concepts often (though not always) have their source in what Deleuze terms the 'ambulatory' sciences, which includes sciences such as metallurgy, surveying, stone-cutting, and perspective" (p. 423). This seems similar in spirit to Rota's (1997) phenomenological analysis of proof where he suggests that proof is "an opening up of possibilities" (p. 191) as opposed to being "devised for the explicit purpose of proving what it purports to prove" (p.190). Therefore, in taking up D&G's concepts and applying them to the notion of proof, the focus often moves away from solutions and the cognitive strategies for attaining them to the chaotic processes of becoming that often move outside of mathematics itself. In what ways, then, can we theorize about proof as becoming without resorting to intentional relations of the subject/object binary? This is taken up in the next section regarding Deleuze and Guattari's (1987) concept of the *face* and how it is formed through *faciality* which is a particular kind of virtual process that allows us to understand how more traditional notions of proof develop and form a kind of "face" that blocks further becoming. Utilizing story excerpts from the AMS text *Living Proof: Stories of Resilience Along the Mathematical Journey*, I demonstrate this notion of faciality and how we may escape the face of proof via another process given by Deleuze and Guattari (1987), that of *dismantlement*, which does not render the face (and thus traditional notions of proof) as useless but as a conduit to further becoming. By dismantling the face, proof becomes "an opening up of possibilities" (Rota, 1997, p. 191) as opposed to a "regime" of capture and confinement discussed in Deleuze and Guattari (1987).

The Face of Proof

In *A Thousand Plateaus*, Deleuze and Guattari (1987) discuss semiotic systems and propose the idea that there are indeed several such systems or "regimes." Two in particular, the *signifying* and *post-signifying* regimes, together constitute a mixed system composed of a "white wall" or "screen" with that of a pattern of "black holes." The idea is that the white wall defines a sort of enclosed space in which all expressions and communications project but that such signifiers merely serve to point to a transcendent totality. Wasser (2018) asks "What does the signifying regime signify? Ultimately, it signifies itself: it asserts its own structure as meaningful" (p. 89). Concerning the black holes that populate the white wall (that is, the post-signifying regime), Wasser (2018) states that it "bears witness to a redundancy of subjects, to a redoubling of the interpellated subject with his point of subjectification" (p. 93). The black hole system consists of processes that originate from "points of subjectification" generated by conflicts (D&G call these "betrayals") that then induce *proceedings* or *lines of flight* from these points. These lines of flight are then captured by black holes resulting in the formation of a subject.

The processes associated with the white wall and the black holes form the face through a becoming known as faciality that contains us within the norms and expectations of a facial landscape (e.g., laws, policies, mathematical axioms, definitions and theorems) while also prompting us to identify, feel, and believe in certain ways (e.g., "This is my favorite movie," "I believe in God," "My proof is correct!" "I can't find the answer!"). The result is a face: points of subjectivity against a "normed" landscape.

Proof also has a face that is formed by a faciality process. I demonstrate the facialization of proof by utilizing the following story excerpt from *Living Proof: Stories of Resilience Along the Mathematical Journey* in which mathematician Dr. Robert Allen recounts a story from his time as a doctoral student in the process of trying to obtain a particularly difficult result related to his dissertation work. While this story can likely be interpreted in a variety of ways using the theory just described, I mostly focus on the black hole or subjectivity "capturing" aspect of faciality.

> This was causing me to lay awake at night thinking about how this result was going to keep me from earning my PhD. In times like these, a long hot shower usually relaxes me enough to fall asleep.…As I am standing under the scalding water trying to wash away

> the anxiety and frustration of my situation, a statement/question pops in my head. "I must know something about compact operators. What do I know about compact operators?'' The answer was three words: The Spectral Theorem. That's right, the glory of every functional analysis class on the planet. By this time, the steam had covered the shower door, creating a wonderful writing surface. So, I started writing the Spectral Theorem. I then feverishly began to connect the dots, and, in the fog of a steaming shower, I wrote the cutest result in my dissertation. I immediately jumped out of the shower to grab a piece of paper and a pencil to jot this idea down. Yes, I was running around my apartment in my birthday suit, dripping wet, giggling like a preteen watching *Twilight* for the 18th time (Team Jacob). Once the adrenaline worked its way through my system and I had put some clothes on, I sat down to write the results and send them to my advisor, proclaiming victory! Then all the self-doubt flooded back in the single thought: "What if there is something wrong with my proof that I can't see?"…Fortunately, this story has a happy ending. The proof was correct. (Allen, 2019, p. 80)

This excerpt serves to illustrate the facial characteristics of proof which includes both the white wall (the body of relevant mathematics related to compact operators, such as the Spectral Theorem) as well as the black hole which captures Dr. Allen's line of flight. In this case, one might see the line of flight as his experimentation and celebration that is generated when Dr. Allen seeks a new direction through the Spectral Theorem. Deleuze and Guattari (1987) indicate that this line of flight is constituted by a "split" subject: one who enunciates a *mental reality* and one who bears a *dominant reality* "of which the mental reality just mentioned is a part, even when it seems to oppose it" (p. 129). One might suggest that there is the Dr. Allen who enunciates his success and celebrates a victory while there is the Dr. Allen that takes on the very different dominant reality of *verification uncertainty*. These subjects are in a virtual entanglement with each other, propelling the line of flight toward the black hole. Deleuze and Guattari (1987) indicate that the subject who enunciates a mental reality "*recoils into the subject*" (p. 129) who bears the dominant reality. That is, the Dr. Allen who enunciates victory eventually "submits" to the Dr. Allen who bears uncertainty, thus completing the subjectification process through the actualization of a subjectivity (the disposition of verification uncertainty) that pauses Dr. Allen's line of flight. This black hole, or "eye," highlights the axiomatic face of proof that privileges solution (fully obtained or not) as end goal over further becoming.

Through this discussion of the "split" subject, Deleuze and Guattari (1987) conclude that "there is no subject, only collective assemblages of enunciation" (p. 130). Individuals are not subjects and do not express subjectivity. Rather, the "subject", and thus to an extent proof itself, is constituted by and subjectified through a chaotic and virtual process of faciality which includes the line of flight that is ultimately captured via the formation of a black hole that severs becoming. It is here that the face of proof, that is, the more traditional subject/object conceptions found in the literature, begins to actualize. How can proof escape these black holes of subjectivity and the limitations imposed by its face? This question is taken up in the next section.

Dismantling the Face

The white wall confines meaning and ultimately points all signifiers toward a singular formation. In the case of proof, we might consider the signifiers as all the relevant definitions, axioms, lemmas, and theorems that lead and are in redundancy to the statement in need of proof. Black holes that seem to correspond to *axiomatic subjectivities* of "correctness," "incorrectness," "conviction," and "uncertainty" capture lines of flight, keeping them from attaining further freedom. Through this process of capture and confinement, this becoming which Deleuze and

Guattari (1987) call faciality, proof expresses itself as a face much in the same way a human face expresses itself through facial traits such as the eyes set against a sort of "screen." Indeed, Deleuze and Guattari (1987) comment on how faciality produces the actual human face (whose norm is the face of Christ) through a selection and "deviance detection" process that occurs via a virtual "overcoding" of the body through complex historical, psychological, artistic, and cultural developments that emphasize face over body. Through this particular becoming of faciality, D&G offer an interesting metaphysical account of racism. The face and its regimes are something to overcome. However, Deleuze and Guattari (1987) state that "We can't turn back.…we are born into them, and it is there we must stand battle. Not in the sense of a necessary stage, but in the sense of a tool for which a new use must be invented" (p. 189). How do we reinvent the face of proof? Deleuze and Guattari (1987) suggest through dismantlement:

> Dismantling the face is the same as breaking through the white wall of the signifier and getting out of the black hole of subjectivity.…Find your black holes and white walls, know them, know your faces; it is the only way you will be able to dismantle them and draw your lines of flight.…With what joy the painters used the face of Christ himself, taking it in every sense and direction; and it was not simply the joy of a desire to paint, but the joy of all desires. (Deleuze and Guattari, 1987, p. 188-189)

Here, D&G seem to suggest that the face must not be completely erased but escaped from as quickly as it is approached. This process of dismantling is itself a becoming, one that is not purely cognitive or social in nature but rather multiplicitous and rhizomatically entangled with the world. While it is certainly possible to argue that Dr. Allen had escaped his initial blockage in his struggles via the line of flight generated by his turn to the Spectral Theorem, this flight is also recaptured when he encounters verification uncertainty. How does one escape the black hole while avoiding recapture and therefore engaging in a real dismantling of proof's face? To illustrate such a possible scenario, I end this section with a brief discussion concerning the story written by Dr. David Neel from the book *Living Proof: Stories of Resilience Along the Mathematical Journey*, focusing specifically on the escape from the face. Here, like the previous story, Neel (2019) recounts a difficult problem related to his dissertation experience in combinatorics and, like Dr. Allen, gets caught in a black hole where he tries, but seems unable, to obtain the solution. Dr. Neel is invited to his supervisor's house for a week to work on the problem where they continue to struggle. However, after a few days, there is a potential escape:

> we would pack a picnic in the morning and hike up into the state park which bordered his backyard. He knew a nice clearing for some lunch, only a half-hour's walk. Talk through the problem, aloud, feet in motion, see if we could sketch out the missing pieces. (Neel, 2019, p. 85)

This is a critical moment: Dr. Neel and his supervisor leave the house where they had been struggling to work through these results. A new line of flight is drawn from the interiority of the house (and perhaps of the mathematician) where they had been working and falling into a black hole. Dr. Neel describes the adventure, no longer that of proof but rather a "hike-proof:"

> We reached another split in the path. We may have made a joke about graph theory. Two paths, but the shorter one was marked: "Path Closed for Maintenance." Still two options, really: forward or back. We do not have a map. Ken is pretty sure he remembers where this forward path emerges. It should be fine. Keep walking. Keep talking about combinatorics.…It feels as if this has become some elemental struggle. And we're still, somehow, talking about combinatorics. But now, we have it. It snapped into place. Now, we are sketching it out more fully, outlining it for each other, repeating, so that if only at

> least one of us can somehow walk or crawl free once more into that world outside this state park we can explain it to someone. (Neel, 2019, p. 86)

It is here, amidst Dr. Neel's virtual entanglement with the mathematical material, his supervisor, and the hike where he witnesses and is a part of a new rhizomatic formation (a "hike-proof") where solutions begin to lose their centeredness and are, in a sense, "accidentally" stumbled upon. Thus, proof becomes problematics as similarly discussed in Smith (2006). Eventually, the pair are able to find their way out of the state park and the story ends with the following:

> The moral is clear: care for one another, keep walking, do not despair. One other moral: Ken's example, his kindness and generosity. He was a model mentor and a good man. I had hoped and expected more years and many more chances to thank him. (Neel, 2019, p. 86)

Unlike the previous story, Dr. Neel's line of flight does not break or pause with a black hole of axiomatic subjectivity but rather with what Roquet (2014) calls *cosmic subjectivity* which is "a form of self-understanding drawn not through social frames, but by reflecting the self against the backdrop of the larger galaxy" (p. 124). A literal galaxy may not be involved in this story, but the rhizomatic formation of the "hike-proof" serves as a kind of stand in for a such a universe. Dr. Neel has escaped the black hole experienced with his initial struggles and has seen proof as "an opening up of possibilities" (Rota, 1997, p. 191) which is, in this case, an entanglement of mathematical material with the world while avoiding recapture. The result is, I believe, a partial dismantling of proof as face. A face that contains proof within the realm of intentionality and blocks further becoming.

Becoming-Proof

Applying post-structural philosophy to the study of proof is an interesting challenge since proof is often seen as constituting the very foundations of mathematics and connected to traditional notions of intentionality when studied in traditional cognitive and social manners. With a post-structural approach, proof (and thus mathematics itself) becomes a rhizomatic complex that cannot be separated from the world. This is important since both students and mathematicians are also entangled with the reality of mathematical materiality, how it is produced, and what is produced in turn. Studying proof's materiality will help us to understand how proof itself is constituted virtually by relations of power and desire that limit or encourage its becoming and therefore allows us to see proof as immanent (as opposed to transcendent) or as "*knowing in being*" (Jackson & Mazzei, 2012, p. 9). Therefore, understanding the materiality of proof will also help us to better grasp the broader lived experiences of students and mathematicians as they experience these becomings which ultimately fuel or impede student and expert desire to pursue advanced mathematical study. However, as described earlier in the report, frameworks, theories, and ideas that are situated along tight disciplinary lines (strict cognitive and social theories, grounded theory, straight-forward forms of thematic analyses) will not allow us to readily witness such complicated becomings since there is often too much focus on stability and intentionality. Rather, the theories of Deleuze and Guattari, Foucault, Derrida, and others where issues of power and the problems of intentionality are explored provide a stronger grounding for the study of becoming. Additionally, there exists qualitative, post-structural research methodologies such as the "thinking with theory" approach of Jackson and Mazzei's (2012) which appears to have connections with the efforts made in this report to understand the presented stories. Thus, both post-structural theory and methodology can help us to better understand the material openings and becomings that begin to close and disentangle with more traditional modes of inquiry.

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A Framework for a ‘Set-Oriented Perspective’ in Combinatorics Using the Theory of Register of Semiotic Representation

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Combinatorics education research has repeatedly affirmed that attending to and reasoning about the set of outcomes in a counting problem is productive for students. A ‘set-oriented perspective’ (Lockwood, 2014) is a way of thinking about counting so that sets of outcomes are intrinsic to the counting process. However, Lockwood’s model of student reasoning in combinatorics does not capture necessary nuances when reasoning about sets of outcomes. Specifically, many counting arguments use properties of a specific representation of the set of outcomes, instead of the set of outcomes themselves. I draw from Duval’s (1995) theory of register of semiotic representations to present a complementary framework that distinguishes between Events, Encoded Events, and Numerical Solutions, which occur in different registers but are each fundamental to a set-oriented perspective. I then analyze theoretical examples and examples from the literature to demonstrate the utility of the framework.

Keywords: Combinatorics, Encoding, Semiotic Representations, Student Thinking

Counting problems are combinatorial tasks that ask the solver to count the number of ways an event can occur, or to count the size of a set. As combinatorics has become increasingly important due to its applications in fields such as computer science and data science, there is a need to understand better how students reason about counting problems. Despite its relevance, combinatorics is also a field that requires critical thinking and ingenuity (Kapur, 1970; Tucker, 2002), and research has repeatedly demonstrated that students at all levels have difficulty correctly solving counting problems (e.g., Batanero et al, 1997) and recognizing errors in solutions (e.g., Eizenberg & Zaslavsky, 2004). There is much research that has investigated how students reason about counting problems (e.g. English, 1991; English 1993; Halani, 2012; Montenegro et al., 2021), and in particular Lockwood’s (2013) model has been frequently used to analyze and understand student thinking. Lockwood has also repeatedly concluded that attending to sets of outcomes is an important part of producing and understanding counting arguments. By a ‘set-oriented perspective’ (Lockwood, 2014), she means “a way of thinking about counting that involves attending to sets of outcomes as an intrinsic component of solving counting problems” (pg. 31).

Despite the attention to sets of outcomes, Lockwood’s model does not describe some of the nuances of how set of outcomes are created, represented, or used. For example, some counting arguments rely on properties of specific representations of the outcomes, and these properties may not appear in other representations. In this theoretical report, I draw from Duval’s (1995) notion of semiotic representations to propose a complementary framework that focuses on the relationship between how sets of outcomes are represented and interpreted in the resolution of counting problems. This framework distinguishes between Events (what is being counted in a counting problem), Encoded Events (events inscribed according to a decided semiotic representation), and Numerical Solutions (algebraic or arithmetic expressions), which are all different registers of representation of the same mathematical object. Specifically, I will argue that counting processes (as described by Lockwood) are often dependent on how events are encoded, and that different methods of encoding can lead to different solution processes.

Literature Review and a Set-Oriented Perspective

Student difficulties in combinatorics are well known, and they include distinguishing between order when it doesn't matter, or conversely failing to distinguish between order when it does matter (Batanero et al, 1997). Solutions to counting problems are also difficult to verify and may contain subtle errors (Eizenberg & Zaslavsky, 2004). Attention has been increasingly placed on student reasoning (e.g. English, 1991; English, 1993; Halani, 2012; Lockwood, 2014; Lockwood & De Chenne, 2021), and of importance to this report has been Lockwood's repeated call to adopt a set-oriented perspective towards counting (e.g. Lockwood, 2014; Lockwood & Gibson, 2016; Lockwood & De Chenne, 2020; Lockwood et al., 2015). By a set-oriented perspective, Lockwood is describing attending to sets of outcomes as a fundamental aspect of solving a counting problem.

Lockwood's (2013) model for student reasoning in combinatorics describes students as reasoning between Mathematical Formulas/Expressions, Counting Processes, and Sets of Outcomes. Formulas and expressions are algebraic or arithmetic expressions that yield a number, and these are typically thought of as the solution to a counting problem. Counting processes are the real or imagined enumerative processes one uses as they solve a counting problem. Sets of Outcomes are the set of elements being counted in a counting problem. While the term set was used intentionally to invoke that the elements being counted are unordered but with a fixed (and, in our case, finite) cardinality, a marked aspect of a set-oriented perspective is reasoning about the structure of a set of outcomes. In doing so, Lockwood and colleagues have argued that a counting process might impose a specific order on a set of outcomes (Lockwood & De Chenne, 2020), and that reasoning about lists of outcomes is productive (Lockwood & Gibson, 2016). Lockwood & De Chenne have also argued that reasoning about encoding outcomes—where students reason about how to inscribe sets of outcomes in a way that aligns with a counting process—can be a useful way to engage in a set-oriented perspective. I will argue that what Lockwood & De Chenne (2021) describe as encoding sheds light on a larger combinatorics issue: many counting arguments proceed by arguing about a specific representation of the outcomes. In particular, I will argue that sets of outcomes are often too far removed from a numerical solution for students to reason between the two without an intermediate representation of the set of outcomes. Further, it is often this intermediate representation that students reason about a counting process, and having access to a larger number of representations can give a more robust understanding of counting.

Other additions to Lockwood's model have been proposed, and specifically Modabbernia (2021) argued that Lockwood's model would benefit from the inclusion of "detecting choices." This addition to the model specifically argues that as students reason about the number of possible choices at each stage in a counting argument, it is important for them to recognize all choices. While this is a necessary part of counting, my framework folds in those concerns. To demonstrate the utility of my framework, I will analyze data from Modabbernia's (2021) paper. Moreover, I will argue that what Modabbernia described as the student not detecting possible choices may have been a case of the student not realizing a possible representation of the outcomes. Hence, the issue goes beyond the ability to detect choices and speaks to a larger aspect of counting: many counting processes reflect the content of a representation of the outcomes, and vice versa. When the content of the representation is not reflected in a numerical expression, there is an understandable difficulty in converting between the two.

Theory of Register of Semiotic Representation

To develop this framework, I draw from Duval's (1995) theory of register of semiotic representation. For the purpose of this report, semiotic representations are representations of mathematical objects and ideas that utilize symbols. For example, an algebraic expression such as $y = 3x + 4$ and the graph of the same line are two different semiotic representations of the same object. Register can refer to the medium or mode of representation, and as such an algebraic representation and a graphical representation occur in different registers; but register can also distinguish between two distinct types of representations in the same medium. For example, $y = 3x + 4$ and $\{(x, y) \in \mathbb{R}^2 : y - 3x = 4\}$ are two representations of the same mathematical object that use symbols common to algebra, but they occur in different registers. Duval (1995) points out the importance of not confusing a semiotic representation with the mathematical object itself, and in fact the content of the representation depends on the register. That is, different representations of the same mathematical object do not state the same properties of the object, but what is explicitly stated is the content of the representation (Pino-Fan et al., 2015). The possible treatments of the representation depend on the content of the representation, and so different representations can fulfill different purposes.

One of the paramount purposes of semiotic representations in mathematics is the ability to substitute some signs for other signs (Duval, 2006). For example, the representation $1 + 2$ may be transformed into 3. The two types of transformations, treatments and conversions, are differentiated by the beginning and ending registers. Treatments are transformations of a representation to a representation in the same register; transforming $1 + 2$ into 3 is an example of a treatment. For treatments, the possible transformations depend on the specific register used (Duval, 2006). Conversions are transformations of a representation into a representation of the same object in a different register; transforming $1 + 2$ into the representation "the sum of the numbers one and two" is a conversion. This requires recognition of two representations for the same object, even when the contents of the two representations are different (Duval, 2006). Lastly, transitional auxiliary representations are those that are introduced so as to enable conversions between representations. Typically, transitional auxiliary representations are only used temporarily, and are discarded after conversion is complete.

Previous works in combinatorics education that have examined registers of semiotic representations largely used the theory to examine how students use registers (such as trees and systematic lists) to convert between natural language counting problems and numerical expressions as a solution (e.g., Montenegro et al., 2021). Here, I will use the theory to create a framework for a more elementary aspect of counting problems: representing the objects that a problem counts. In doing so, I will point out that the objects being counted can be represented in different and flexible ways. Some of these ways are more conducive to counting, different representations may lead to different solutions, and students may be unaware of other ways to represent the same object. Hence, part of the complexity in reasoning about sets of outcomes is determining a register of representation that is useful for a given counting problem, and some 'clever' solutions are nothing more than determining a novel representation that is useful in a counting problem.

Proposed Framework

By applying Duval's theory of register of semiotic representation to solving counting problems, the goal of a counting problem is to convert the departure representation into the arrival register, where typically the departure register is natural language and the arrival register is a numerical expression (Montenegro et al., 2021). Yet, counting problems can frequently be solved in different ways, resulting in mathematical expressions that are numerically the same although the representations are different. In particular, I propose that a natural language departure representation often requires in intermediate inscribed register from which a counting argument is formed, before converting between the intermediate register and the arrival register. The purpose of this proposed framework is to create a way to distinguish the set of outcomes as described in the problem, and the encoded set of outcomes that students reason about. When considering Lockwood's model of student reasoning and her set-oriented perspective, Duval's theory warrants a more nuanced approach to a *set of outcomes* and a *counting process*. The proposed framework here introduces the terms *Events*, *Encoded Events*, and *Numerical Solution* as a way to describe better the different registers students may be reasoning about while solving a counting process. Figure 1 illustrates this framework.

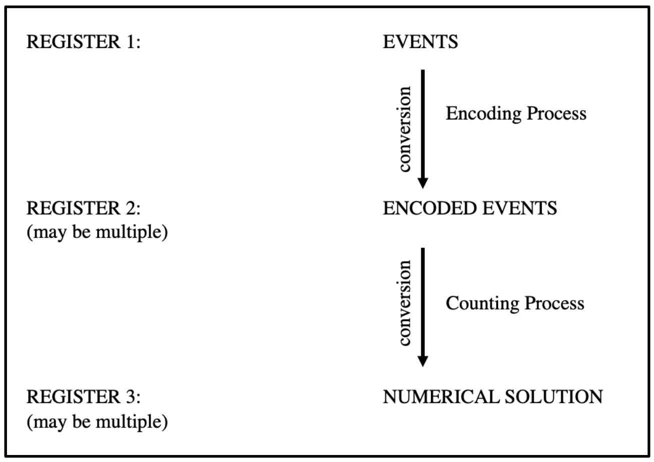

Figure 1: Framework for a Set-Oriented Perspective

Events

A classic counting problem is "how many ways are there to flip a coin three times in a row?" This problem asks the counter to determine the number of ways a physical phenomenon might take place, and hence I adopt the term Event to indicate this. Events are described in a natural language register, and they are the objects being counted in a counting problem. The Events in the aforementioned problem are ways to flip a coin three times in a row.

Encoded Events

Encoded Events are an inscribed representation of Events that occur in a different register. Conversion between Events and Encoded Events occurs by deciding a register for the Encoded Events so that every event can be represented and each event has a distinct representation. I refer to this as the 'encoding process.' For example, if the events are all ways to flip a coin three times in a row, encoded events might be concatenations of three letters, where each letter is either an H or a T. That is, HHT would represent flipping a coin on the first and second toss, and flipping a tail on the third toss. Important to this framework is the fact that there is no single way to encode events, and that other ways of encoding the same events might have different content. An alternative register would be to encode the events as subsets of $\{1,2,3\}$, where the number in each subset indicates which coin flips landed heads. For example, $\{1,2\}$ would represent a coin landing on heads on the first and second flips, and landing on tails on the third flip. While HHT and $\{1,2\}$ represent the same event, they are in different registers and have different contents.

When solving counting problems, textbooks and other literature might refer to making a counting argument that would justify a numerical solution based on set equality. Yet, some of those arguments are based on specific representations of the set, which is different from the set itself. This does not acknowledge that the contents of two different representations are not the same, which can lead to an attempt to justify a counting argument from a representation without the necessary content. This mismatching of representation with counting argument will be illustrated in the examples of the following section, and it can lead to understandable difficulties for students. Further, not all representations are conducive to counting, and some conducive representations are hard to realize (e.g. the 'stars and bars' representation). By being familiar with more possible representations, I hypothesize that the counter would have a more robust understanding of counting.

Numerical Solution

A Numerical Solution is the arrival register of a counting problem, and it is an algebraic or arithmetic representation that uses symbols common to combinatorics, such as nPk or $\binom{n}{k}$. In terms of Lockwood's model, the conversion between Encoded Events and Numerical Solutions is the counting process, and I will adopt the term counting process in this framework. The Numerical Solution depends on the Encoded Events used, and the counting process makes use of the content of the representation in the Encoded Events. For example, a numerical solution that counts coin flips represented like HHT could be 2^3. The counting process would be to use positional reasoning (where we reason about the number of characters that can occupy each of three positions) and to apply the multiplication principle. In contrast, a numerical solution that counts coin flips represented as subsets of $\{1,2,3\}$ could be $\sum_{k=0}^{3}\binom{3}{k}$, where the counting process makes use of the content of the representation by partitioning the subsets by cardinality, and then counting the number of *k*-subsets using $\binom{3}{k}$. A counting argument that uses positional reasoning for this representation would fail because the representation does not have positions. Because the contents of the two representations are different, the content of the numerical solutions are different. While a counter may reason about a numerical solution directly from events, I hypothesize that doing so necessitates reasoning previously about encoded events for a counting problem that is reasonably isomorphic.

Two Hypothetical Examples and One Example from Literature

Here, I analyze two hypothetical mathematical examples, and one example from Modabbernia (2021) using my proposed framework. I chose the hypothetical examples to illustrate cases where standard counting arguments do not use expected representations. The example from Modabbernia is used to show how two representations can result in two counting arguments, and that students can have difficulty verifying a counting argument if the content of a representation does not reflect the content used in the argument.

The Two Coin Flips Problem

The Two Coin Flips Problem asks "How many ways are there to flip a coin five times in a row, where exactly two of the coin flips are heads?" Here, the order of the coin flips matter, but rather than counting every arrangement of coin flips we are only counting those where a head was flipped exactly twice. The Events in this problem are all ways to flip a coin three times where a head was flipped exactly twice. Because the Events in this problem are sequential coin flips, which are typically encoded as strings of sequential Hs and Ts, one might expect that this method of encoding is productive in this problem. While we can produce a solution using this method of encoding (a case breakdown by the first coin flip to be heads would result in 4+3+2+1), this solution does not scale well as the number of coin flips increases or as the required number of heads increases. Hence, what is required for this problem is a different method of encoding the outcomes, which might be a sticking point for students.

Instead, the relevant properties of the Events in this case are the locations of coin flips that are heads (or, conversely, the locations of the coin flips that are tails). We can encode the Events as 2-element subsets of the set $\{1, 2, 3, 4, 5\}$, where the elements in the subset correspond to which coin flips landed on heads. For example, $\{2, 3\}$ would encode that flips 2 and 3 landed on heads, while the rest landed on tails. Thus, the Encoded Events in this problem would be all 2-element subsets of the set $\{1, 2, 3, 4, 5\}$, which is a departure from the expected way of encoding the events, and students may encounter some difficulty in the conversion between the Events and the Encoded Events. At the point where students were solving this problem, they would most likely have already encountered combination problems, so their counting process (the conversion between Encoded Events and Numerical Solutions) might be that $\binom{5}{2}$ is a numerical representation of the number of 2-element subsets from a set of cardinality 5.

The Hotdog Problem

The Hotdog Problem asks "How many ways are there to buy three hotdogs from a hotdog vendor that sells five varieties of hotdogs?" Here, the order in which the hotdogs are purchased does not matter, and one can buy multiple of the same variety of hotdog. The Events in this problem are all ways to buy three hotdogs, where each hotdog is one of five varieties. A reasonable way to encode the events are as multisets of cardinality three with elements from $\{1,2,3,4,5\}$. For example, the encoded event $\{2, 3, 3\}$ would indicate purchasing one variety 2 hotdog, and two variety 3 hotdog. However, this method of encoding the events is not particularly conducive to counting, in the sense that it is difficult to convert between the encoded events and a mathematical expression without already knowing a formula that counts multisets (a solution that uses nested sums can be found, by indexing the multisets by the lowest element, however the feasibility of using this solution technique scales poorly as the parameter values increase).

Instead, a common way of encoding these events is to use the 'stars and bars' argument. For the sake of brevity, suffice it to say that bars represent separations between varieties, and the number of stars represent the number of hotdogs purchased in each variety. For example, |*|**| | and $\{2, 3, 3\}$ represent the same outcome. However, with the events encoded as arrangements of three stars and four bars, an isomorphic counting process to the Two Coin Flips Problem reveals that there $\binom{3+5-1}{3}$ ways to purchase three hotdogs from five varieties.

Coffee Problem (Adapted from Modabbernia)

Modabbernia (2021) describes a student's (named Henry) work as he demonstrates that two different numerical solutions both correctly solve the same counting problem. I have changed the wording of the problem and the parameter values in the problem for space concerns. The Coffee Problem asks "A group of six people have a meeting. After the meeting, three of the people decide to go to a coffee shop, and they invite the remaining three people. The coffee shop sells five different types of coffee. If the remaining three people may each decide to get coffee or not to get coffee, how many possible coffee orders are there?" Two mathematical expressions that solve this problem are $5^3 \times \left(\binom{3}{0} \cdot 5^0 + \binom{3}{1} \cdot 5^1 + \binom{3}{2} \cdot 5^2 + \binom{3}{3} \cdot 5^3\right)$ and $5^3 \times 6^3$.

The first numerical expression breaks the events into cases based on how many of the remaining three people get coffee. Because the content of the encoded outcomes must reflect the content of the counting argument, it might include the orders of the three people going to the coffee shop, which of the remaining people go to the coffee shop, and the orders of the remaining people. For example, a representation of the form $(1, 5, 4)\{4, 5\}(2, 2)$ might be used, where (1, 5, 4) indicates the orders of the three people who must get coffee, $\{3, 4\}$ indicates which of the remaining people decide to get coffee (where the remaining people are labeled as 4, 5, and 6), and (2, 2) indicates the orders of those remaining people. Another encoded event using this encoding method might be $(2, 3, 4)\{6\}(1)$. Of note in this representation is that the number of remaining people getting coffee might change, which is reflected by the size of the middle subset and the size of the right-most ordered values.

The second numerical expression, $5^3 \times 6^3$, does not break the events into cases. Rather, the number of possibilities for each remaining person is increased by 1 because they have the option of not getting coffee. In Modabbernia (2021), the author discussed the student's difficulties of detecting this remaining option. The content of these representations might include the coffee orders of the three people who must get coffee, and the "coffee orders" of the three people who might get coffee (quotations used to signify that one possibility is not to get coffee). For example, a representation (1, 5, 4)(2, 2, 0) might encode the same event as $(1, 5, 4)\{4, 5\}(2, 2)$, where the ordered triple (2, 2, 0) indicates that people 4 and 5 get coffee number 2, and person 6 does not get coffee.

Both of these ways of encoding the events create unique representations for every event, and they can both be used to create a reasonable counting argument; yet, the contents of the two representations are different, and the counting process for each representation reflects the contents. In Modabbernia (2021), the student Harry was able to explain the first solution, yet he had difficulty explaining the second solution. However, Modabbernia also notes that "Interestingly, considering one element of the set of outcomes helped [Harry] to convince himself that the option of not choosing is an option" (pg. 194). By this, Modabbernia is stating that Harry was able to justify that the solution $5^3 \times 6^3$ was numerically equal to the previous solution after he wrote down how one of the events could be represented (using the above method of encoding). One possible explanation for this is that Harry recognized that all of the events could be represented in this way, and that $5^3 \times 6^3$ counted the number of representations. Therefore, 'Detecting Choices' fits into this proposed framework, and it is an example of why Lockwood's model of student thinking in combinatorics needs a complementary model that specifically examines representations of sets of outcomes.

Conclusion

In this theoretical report, I have proposed a framework for student thinking in combinatorics that is particularly well-suited for analyzing student thinking that displays a 'set-oriented perspective' (Lockwood, 2014). This framework draws from Duval's (1995) theory of register of semiotic representations, arguing that the content of the representation of the set of outcomes informs the counting process, and that different representations may lead to different solutions. One conclusion of this framework is that discussion of counting should explicitly include how sets of outcomes can be represented, the utility of different representations, and the conversion process between encoded outcomes and numerical solutions. Hence, this framework offers additional aspects of counting that are not characterized in Lockwood's (2013) model of student thinking in combinatorics, yet are fundamental to counting using a set-oriented perspective.

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Combining Sealey, Von Korff & Rebello, Jones, and Swidan & Yerushalmy into a Comprehensive Decomposition of the "Integral with Bounds" Concept

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Previous calculus education work on integrals, including definite integrals and accumulation functions, has created useful theoretical frameworks that decompose the "integrals with bounds" concept into constituent parts. Yet, each framework focuses on distinct aspects of the integral and leaves certain parts implicit. Further, definitions and operationalizations are absent in many of these frameworks. This theoretical paper contributes by: (a) pulling together the various pieces in these frameworks into a comprehensive decomposition of the integral concept, (b) explicitly defining the processes and objects within it, and (c) operationalizing the processes and objects within each of the numeric, graphical, and symbolic representations. This comprehensive framework is useful for researchers, curriculum or task writers, and instructors alike to have a more complete picture of the elements that make up the integral concept.

Keywords: calculus, definite integrals, accumulation functions, theoretical framework

There has been a steady focus recently on integration in calculus education (e.g., Ely, 2017; Hall, 2010; Hu & Rebello, 2013a, 2013b; Jones, 2015a, 2015b; Kouropatov & Dreyfus, 2014; Sealey, 2006, 2014; Swidan & Yerushalmy, 2014; Von Korff & Rebello, 2012). An important part of this work has been to theoretically describe the constituent parts of the integral concept to provide the field with a shared understanding of the ideas that make it up (e.g., Sealey, 2014; Von Korff & Rebello, 2012). Such frameworks are crucial, as they provide a lens that influences what researchers may see, what curriculum writers or task designers may attend to, and what instructors might guide their students toward during learning. However, the existing frameworks in the literature each appear to speak to different aspects of the broader *integrals with bounds* (IBs) concept, including parts of definite integrals, $\int_a^b f(x)dx$, or of accumulation functions, $g(X) = \int_a^X f(x)dx$. In our own prior work that led to this theoretical paper, we had been engaged in designing a hypothetical learning trajectory for definite integrals and accumulation functions and found the real need to pull this work together into a single comprehensive framework. Thus, this theoretical paper is meant to contribute to calculus education research by (a) combining existing frameworks on integrals into a more complete decomposition of IBs and (b) making explicit certain parts of these frameworks that were previously implicit. The outcome is a comprehensive framework that can provide researchers, curriculum/task writers, and instructors with an enhanced view of what makes up the IB concept.

Definitions

A definite integral is one with fixed bounds, $\int_a^b f(x)dx$, and an accumulation function has a variable upper bound, $g(X) = \int_a^X f(x)dx$. We refer to these collectively under the label *integrals with bounds* (IBs), to distinguish them from unbounded indefinite integrals, $\int f(x)dx$. Because common approaches to integration rely on Riemman sums ($\sum f(x)\Delta x$) (e.g., Hughes-Hallett et al., 2012; Stewart et al., 2021), we take this "sum" meaning as the foundation for IBs, as opposed to "area under a curve" or "antiderivative" meanings. This view is strongly compatible with a large portion of the research literature to date that builds on sum-based meanings (e.g., Chhetri & Oehrtman, 2015; Ely, 2017; Hu & Rebello, 2013b; Jones, 2015a; Jones et al., 2017; McGee & Martinez-Planell, 2014; Nguyen & Rebello, 2011; Sealey, 2006, 2014; Simmons & Oehrtman, 2017; Swidan & Yerushalmy, 2016; Von Korff & Rebello, 2012). However, we wish to be clear we are aware of other approaches that exist as alternatives to sum-based interpretations (Kouropatov & Dreyfus, 2014; Thompson et al., 2013).

Literature Review, with a Focus on Previous Frameworks

To start, much of the work on definite integrals or accumulation functions contains the theme that they are fundamentally quantitative structures (e.g., Blomhøj & Kjeldsen, 2007; Chhetri & Oehrtman, 2015; Ely, 2017; Hu & Rebello, 2013b; Jones, 2015a, 2015b; Kouropatov & Dreyfus, 2014; Nguyen & Rebello, 2011; Sealey, 2014; Simmons & Oehrtman, 2017; Swidan & Yerushalmy, 2016; Thompson, 1994; Thompson & Silverman, 2008; Von Korff & Rebello, 2012). We explicitly incorporate this stance into our framework and speak of the integral in terms of quantitative relationships. We see numeric, graphical, or symbolic representations as literal *representations* of this more fundamental quantitative structure.

Previous Integral Frameworks

Sealey's Riemann Integral Framework. Sealey's (2014) framework was proposed as a "decomposition of the Riemann integral into its mathematical components" (p. 230), consisting of five layers: *orienting, product, sum, limit, function* (see Figure 1). *Orienting* deals with attending to the individual parts of the integral, $f(x)$ and Δx. *Product* involves multiplication between them, $f(x) \cdot \Delta x$. *Sum* is adding the products $f(x)\Delta x$ over many subintervals, $\sum f(x)\Delta x$. *Limit* considers a Riemann sum, $\sum_{i=1}^{n} f(x)\Delta x$, for every *n*, with a limit as $n \to \infty$, $\lim_{n\to\infty} \sum_{i=1}^{n} f(x)\Delta x$ Finally, *function* switches a basic definite integral with fixed bounds, $\int_a^b$, to having a variable upper bound, $F(b)$. We note that this final layer has the feel more of a variable bound than as a true accumulation function (e.g., Swidan & Yerushalmy, 2014; Thompson & Silverman, 2008). Sealey's (2014) framework was presented mostly within the symbolic representation, with less attention to numeric or graphical representations (Figure 1).

Layer 0: Orienting	$f(x)$ and Δx
Layer 1: Product	$f(x)\Delta x$
Layer 2: Summation	$\sum f(x)\Delta x$
Layer 3: Limit	$\lim_{n\to\infty} \sum_{i=1}^{n} f(x)\Delta x$
Layer 4: Function	$F(b) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x)\Delta x$

Figure 1. Sealey's framework (adapted from 2014, p. 242).

Von Korff & Rebello's Integral Framework. Von Korff and Rebello's (2012) framework was specifically meant for contextualized integrals, but works well as a general framework. Their framework is similar to Sealey's (2014), with three key differences (Figure 2). First, they use a specific *quantity* layer in place of the general *orienting* layer, focused on making sense of the quantities in the context. Second, and a much more important difference, von Korff and Rebello (2012) used *infinitesimals*, which represent "microscopic" changes in or amounts of a quantity (see also Von Korff & Rebello, 2014). Symbolically, Δx represents a macroscopic change or amount, while *dx* represents an infinitesimal change or amount. Infinitesimals can formalized either with a "limit," where an increasingly tiny Δx receives the special notation *dx*

(Jones & Dorko, 2015), or with the hyperreal numbers (Ely, 2017, 2020). However, the informal idea of infinitesimals itself has been shown to be quite powerful for first-year calculus and for science applications (Amos & Heckler, 2015; Ely, 2017, 2020; Jones, 2015a; Nguyen & Rebello, 2011; Schermerhorn & Thompson, 2019a, 2019b). Thus, Von Korff and Rebello (2012) give each layer the dual nature of a macroscopic version and an infinitesimal version (Figure 2). Consequently, there is no separate "limit" layer, but rather a switch at *any* stage from macroscopic to infinitesimal such as an infinitesimal product, $f(x)dx$. The arrows show many possible paths at starting with the basic quantities and building toward a definite integral or accumulation function. The third difference is that Von Korff and Rebello's (2012) framework is explicit in using the integral symbol, $\int$, as a literal "sum" symbol.

Macroscopic quantity: Δt	Macroscopic product: $v \cdot \Delta t$	Macroscopic sum: $\sum v \, \Delta t$	Macroscopic function: $\Delta x(T) = \sum_0^{T/\Delta t} v(t)\Delta t$
Infinitesimal quantity: dt	Infinitesimal product: $v \cdot dt$	Infinitesimal sum: $\int_{t=a}^{t=b} v \, dt$	Infinitesimal function: $\Delta x(T) = \int_a^T v(t)dt$

Figure 2. von Korff and Rebello's framework (adapted from 2012, p. 3).

Jones' AUP structure. Jones (2013) documented an empirical student understanding structure called *adding up pieces* (AUP). Its usefulness as a reasoning structure (e.g., Chhetri & Oehrtman, 2015; Jones, 2015a) led it to becoming a theoretical framework in its own right (e.g., Ely, 2017; Jones, in press). While AUP has resemblance to the other two frameworks, it adds key elements as well. AUP consists of a *partition*, a *target quantity*, and a *sum* (Figure 3). *Partition* means chopping the domain into tiny (infinitesimal) pieces. Thus, this framework considers the partitioning as its own layer, as a precursor to the other layers. *Target quantity* is the construction of some quantity of interest within each of the partition pieces. This is akin to the *product* layer in Sealey (2014) and Von Korff and Rebello (2012), except that the target quantity does *not* have to be constructed via a product, $Q = f(x)dx$ (Chhetri & Oehrtman, 2015; Ely, 2017; Simmons & Oehrtman, 2017). Rather, other quantitative structures are permitted, such as $L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ or $V = \pi r^2 h$. *Sum* is then adding the target quantity across the partition pieces to get a total amount, where the integral symbol, $\int$, is again used with a literal "sum" meaning. Also within AUP research, Simmons and Oehrtman (2017) helped push past a linear ordering, by suggesting that the layers can be traced in different orders, such as considering the target quantity first, and then partitioning.

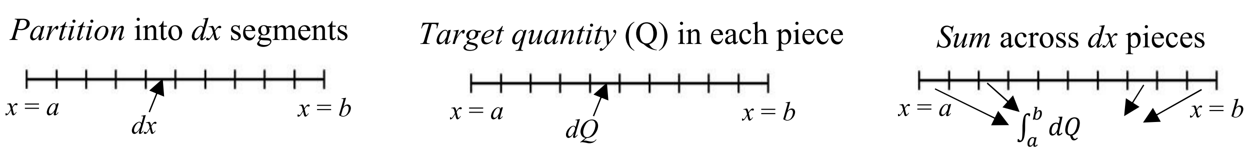

Figure 3. Jones' AUP structure (adapted from Jones, in press).

Swidan & Yerushalmy's Accumultion Function Framework. The frameworks presented so far focus on definite integrals, with the "function" layer being much less developed (Sealey, 2014; Von Korff & Rebello, 2012). On the other hand, Swidan and Yerushalmy (2016) provided a framework (Figure 4) explicitly addressing accumulation functions, $g(X) = \int_a^X f(x)dx$. Their first three categories – *delta x*, *product* and *sum* – are essentially the same as *partition*, *product* (or *target quantity*), and *sum* from the others. However, the lower bound is a more salient construct in these, as a *zero accumulation* point. That is, the integral is a *net amount* and some previous amount of the quantity may have existed, in a sense, *before* the start of the integral's accumulation (i.e., lower bound). The integral finds the additional amount past $x = a$. Swidan and Yerushalmy (2016) end with a function layer that is much more developed. It includes function properties, like where the intercept is positioned, whether the function is increasing or decreasing, and its concavity (see also Kouropatov & Dreyfus, 2014; Thompson et al., 2013).

(A) Delta x →	(B) Product →	(C) Sum of products →	(D) Accumulation function properties
Ideas in A: Partition	Ideas in B: Lower bound and initial value	Ideas in C: Zero-point accumulation, relation between bounds	Ideas in D: Accumulation function, position, tendency, and concavity

Figure 4. Swidan's framework (adapted from 2016, p. 42).

Summary of the State of the Literature and the Need for a Comprehensive Framework

The hope is that by now, the reader can see that there are several frameworks that theoretically decompose parts of the IB concept, including for definite integrals and accumulation functions. However, each framework focuses on certain components, leaving others implicit. Further, the layers of these frameworks ostensibly rely on mathematical processes that culminate into objects used in subsequent layers, such as the product process becoming an object (the resulting quantity) that is then needed in the process of summing (Sealey, 2014; Swidan & Yerushalmy, 2016; Von Korff & Rebello, 2012). These processes and objects have not been explicitly defined, and previous frameworks lack operationalizations for them within different modes of representation (i.e., graphical, numeric, and symbolic). These issues become important when one attempts to use these frameworks to construct learning arcs for IBs over the course of an entire integration unit. In fact, the problems we ran into while creating a hypothetical learning trajectory is exactly what prompted this theoretical paper. We needed to first organize the different layers, define the processes and objects, and consider them within distinct representations before we could proceed. The field of calculus education will benefit from synthesizing these prior frameworks into a single, comprehensive decomposition of the IB concept. This theoretical paper contributes by providing this needed framework.

Processes and Objects

The previous frameworks all implicitly use the idea of *processes* and *objects* (Sfard, 1991, 1992). That is, each layer must transition from a process to an object that is then available for use in the next layer. Here, a *process* refers to some operation that might be done, such as adding numbers together or multiplying two quantities – though the operation can be imagined without necessarily being carried out (called *interiorization*, Sfard, 1991). Much of mathematics involves taking operations and working with them in their own right as objects, such as thinking of a summed total as an entity that can be manipulated, or a product as an actual result that can be used in another process. Sfard called this *reifying* the process into an *object*. As an example, the process of finding the target quantity through some quantitative structure must result in the conceptualization of a little bit of that target quantity in each partition piece, before those little bits can be thought of as being added together (Ely, 2017; Jones, 2013, 2015a). A key part of the contribution of the framework presented in this paper is to explicitly define these processes and objects and to operationalize them in terms of different representations.

The Framework: A Comprehensive Decomposition of the IB Concept

In this section, we present our IB framework (Figure 5). The framework is *not* meant to be restrictive in terms of possible understanding or usage, because the larger notion of integration can include other ideas, such as computing integrals with antiderivatives, constructing an integral for a specific context, or extending integrals to improper integrals. Rather, our framework is meant as a decomposition of the central IB structure (as in Sealey, 2014; Zandieh, 2000).

The framework consists of seven layers, which are listed on the left in Figure 5, along with their process (P) and object (O) definitions. The first layer deals with *orienting to quantities* (Sealey, 2014; Von Korff & Rebello, 2012), the next three deal specifically with *definite integrals*, $\int_a^b f(x)dx$ (Jones, 2013, 2015a; Sealey, 2014; Von Korff & Rebello, 2012), and the last three layers deal with extending to *accumulation functions*, $g(X) = \int_a^X f(x)dx$ (Swidan & Yerushalmy, 2016). We adopt Von Korff and Rebello's (2012) use of infinitesimals, as a powerful construct for first-year calculus and its applications (Amos & Heckler, 2015; Ely, 2020; Schermerhorn & Thompson, 2019a, 2019b). Thus, we do not use a separate "limit" layer (as in Sealey, 2014) and instead define each layer at both the macroscopic and infinitesimals (I) levels, where applicable. We also take the broader *target quantity* view, which is not limited to product-only structures (Sealey, 2014; Swidan & Yerushalmy, 2016; Von Korff & Rebello, 2012), but permits non-product structure as well (Chhetri & Oehrtman, 2015; Simmons & Oehrtman, 2017). Thus, *target quantity* is operationalized in terms beyond only products. Finally, we think of these layers fairly hierarchically in *theoretical* terms, because they build on each other, but we note that a person does not necessarily have to think through them in this linear order, in line with what Simmons and Oehrtman (2017; 2019) have discussed.

Within each cell, we provide an operationalization of what "understanding" that layer's *process* (P) and *object* (O) would look like within each of the numeric, graphical, and symbolic representations. As operationalizations, these definitions are given in terms of cognitive actions. Finally, because of the stance that integrals are fundamentally quantitative structures, each definition is given in terms of *quantities*, though these definitions could also be applied to generic mathematical variables as well.

Layer Definitions	**Numeric Operationalizations**	**Graphical Operationalizations**	**Symbolic Operationalizations**
Orienting to the context (layer 1)			
Basic quantities: P: Identify quantities O: A set of quantities	P: Pair a number with an associated quantity O: A set of quantities associated with the numbers	P: Pair a visual length/size with an associated quantity O: A set of quantities associated with lengths/sizes	P: Pair a symbol with an associated quantity O: A set of quantities and associated symbols
Definite integrals (layers 2–4)			
Partition: P: Chop the domain quantity into partitioned pieces O: A set of partitioned pieces	P: Partition a numeric interval into smaller numeric subintervals O: Recognize the subintervals as a set that can be manipulated (e.g., made smaller) I: Subintervals are numerically very small	P: Partition a portion of the horizontal axis (or $\mathbb{R}^n$ region) into smaller segments/pieces O: Recognize those segments as a collection that can be manipulated (e.g., made smaller) I: Segments along axis are drawn very small	P: Use "Δ" to mean a change in or relatively small amount of a quantity O: See "Δ[symbol]" as one piece among a set of pieces I: "*d*" is used instead of "Δ" to indicate infinitesimal scale
Target quantity:	P: Perform a numeric operation relating the	P: Create a graphical object (e.g., rectangle, prism, right	P: Use or interpret a symbolic expression as a quantitative

P: Use a quantitative relationship within a partition piece (e.g., product or other structure) O: The conceptualization of the resulting target quantity	quantities (e.g., $40N\times0.01m$ or $\pi(3\,ft)^2(0.1\,ft)$) O: Recognize the result as the target quantity (e.g., $0.4J$ or $0.9\pi\,ft^3$) I: Use a small numeric amount of one quantity, giving a small amount of the resulting target quantity	triangle) where the parts are individual quantities related together through the object O: Recognize that the "size" (area, volume, length) signifies the target quantity I: Object is drawn with small components (e.g., rectangles with very thin widths)	structure (e.g., $y\Delta x$ or $\sqrt{(\Delta x)^2+(\Delta y)^2}$). O: Recognize that the expression signifies a resulting quantity (e.g., $y\Delta x = area$) I: Use "d" to indicate the infinitesimal scale (e.g., $\sqrt{(dx)^2+(dy)^2}$)
Summation: P: Add the target quantity across the partitioned pieces O: The aggregated total amount of the target quantity	P: Adding target quantity's numeric values across the partitioned pieces O: Recognize the sum signifies a total amount of the target quantity. I: Sum is imagined over as small of numeric intervals as possible, creating many pieces the addition happens over	P: Imagine the objects filling up a graphical space (e.g., an area, volume, or length) O: Recognize the combined area/volume/length as the total amount of the target quantity I: Objects are drawn as small as possible (e.g., a set of rectangles with thin widths)	P: Use "$\sum$" to denote a sum of the symbolic expression. O: Recognize that "$\sum[expression]$" signifies the total amount of the target quantity I: Use the integral symbol "$\int$" to mean a "sum" over infinitesimal bits
Extending to Accumulation Function (layers 5–7)			
Net Amount: P: Combine integral with a "previous" amount (at $x = a$) O: The integral's value is seen as an *additional* amount	P: Add the integral's numeric result to a previously existing amount of the target quantity O: Recognize the integral's numeric result as an inherent "additional/net" amount (I: n/a)	P: Attach some previously existing amount to the "left" edge of the area/volume/length O: See the area/volume/length inherently as an "additional/net" amount (I: n/a)	P: Notate adding the integral's value with an existing amount (e.g., "A+I") O: Conceive of the integral symbol, "$\int$", as an inherent "additional/net" amount. (I: n/a)
Variable upper bound: P: Change the upper bound to produce different amounts O: The upper bound is a true input variable	P: Use different numeric values for the upper bound, giving different results O: Identify the upper bound as a truly variable value I: The numeric amounts by which the upper bound increases are very small	P: Add on new objects past where the original partition ended to get different amounts O: Conceive of the graphical boundary edge as fluid and changeable I: The new objects are drawn as small as possible	P: Use a non-number symbol to notate a changing upper bound value (e.g., "#" or "X") O: Perceive the symbol used as denoting a true variable I: x is seen as increasing by continuous amounts
Accum. function: P: Track upper-bound inputs with accumulated-total outputs. O: The bound-amount relationship is an actual function, with function properties	P: Steadily increase the upper bound value and track the progress of accumulation O: Identify accumulated numeric value as a function of the upper bound, with usual function properties (e.g., increasing value) I: The accumulation is tracked over upper bounds increasing by very small amounts	P: Steadily add new objects and track the aggregated amount up to those ending locations O: Recognize the graph of the accumulated amount as a function of the ending location, with graphical properties (e.g., concave up) I: The graph is drawn continuously	P: Use symbols (e.g., $A(X)$) to notate the accumulated amount for a given X upper bound O: Recognize $A(X)$ as a function of X, and the expression $\int_a^X f(x)dx$ as its symbolic "rule" I: The domain for X is continuous

Figure 5. A comprehensive framework, including a refined set of layers, the inclusion of different representations, and an operationalization of the process-object and infinitesimal understandings for each layer.

A Discussion of This Framework

The purpose of creating a framework for the IB concept is to aid researchers, curriculum writers or task designers, and instructors. In our own work, we found this framework necessary before we could embark on a hypothetical learning trajectory spanning an entire integration unit on definite integrals, accumulation functions, and the Fundamental Theorem. We see the contributions of this framework as follows. (1) It brings together distinct frameworks that each focused on different parts of the IB structure into a single, coherent whole (Jones, 2013, 2015a; Sealey, 2014; Swidan & Yerushalmy, 2016; Von Korff & Rebello, 2012). (2) It provides clear definitions for the processes and objects (Sfard, 1991, 1992) that make up each layer, which has hitherto been implicit. (3) It treats each layer in all three numeric, graphical, and symbolic representations, with operationalizations for the processes and objects in each. (4) It incorporates key ideas from recent research on: (a) infinitesimals (Amos & Heckler, 2015; Ely, 2020; Jones, 2015a, 2019; Von Korff & Rebello, 2014), (b) thinking of integrals quantitatively (Blomhøj & Kjeldsen, 2007; Hu & Rebello, 2013b; Pina & Loverude, 2019; Thompson & Silverman, 2008), (c) and using various quantitative structures to create or interpret integrals (Chhetri & Oehrtman, 2015; Ely, 2017; Simmons & Oehrtman, 2017).

We see many ways this framework can be useful. We believe it provides researchers with a clearer view of the different components of IBs at one glance. For example, it could easily serve as a tool of analysis for identifying the specific layers and representations in use in student thinking, learning, or reasoning. We believe the framework provides curriculum writers and task designers with clearer goals for their curriculum or tasks to hit. For example, one could attend to how students would progress through this framework in a systematic manner. Task writers could identify which parts of the layers/representations are being elicited by their tasks. Holes in current learning progressions can be seen more easily, allowing the inclusion of tasks that can address those holes. We believe the framework provides instructors with a tool to ensure that their students understand each layer explicitly, and whether they can flexibly use the different representational operationalizations to describe the structure for each layer. Instructors could use this framework, and its operationalizations, to design homework questions or assessment questions targeting specific aspects of integrals.

Of course, this framework has limitations as well. As stated earlier, we were primarily focused on the *structure* of the IB concept. We did not focus on the computation of integrals, proving integral properties, or general families of antiderivatives (see Black & Wittmann, 2007; Christensen & Thompson, 2010; González-Martín, 2005; Grundmeier et al., 2006). Additional work may wish to identify how such ideas relate to this framework. Further, we wish to be clear that despite the hierarchical presentation of the layers, people might be able to think or reason in non-linear ways across these layers, especially once the integral concept is understood (Simmons & Oehrtman, 2017, 2019). However, despite these limitations, we strongly believe this framework captures the essential conceptual components of the IB structure well. We believe it extends easily to integrals of various types, including second semester integrals, like $\int_a^b \pi f(x)^2 dx$ and $\int_{x=a}^{x=b} \sqrt{(dx)^2 + (dy)^2}$, double or triple integrals, $\iint_R f(x,y)dA$ and $\iiint_R f(x,y,z)dV$, or path integrals, $\int_C f(x,y)ds$ and $\int_C \mathbf{F} \cdot d\mathbf{r}$ (Ely, 2017; Jones, 2020; Jones & Dorko, 2015). Each of these types of integrals consist of quantities (e.g., dx, f, or $\mathbf{F}$), quantitative structures (e.g., $\pi f^2 dx$, $\sqrt{(dx)^2 + (dy)^2}$, or $\mathbf{F} \cdot d\mathbf{r}$), and a summation of the target quantity across some domain. If useful, these can be extended to accumulation functions by allowing the summation to continue on with a variable upper bound, and by coordinating the upper bound inputs with different accumulated totals, such as $\int_a^X \pi f(x)^2 dx$ (Kouropatov & Dreyfus, 2014; Swidan & Yerushalmy, 2016; Thompson & Silverman, 2008). Thus, we propose this framework as a useful organization for future work on integrals with bounds.

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Meanings, Reasoning, and Modeling with Definite Integrals: Comparing Adding Up Pieces and Accumulation from Rate

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Abstract. Approaches to integration based on quantitative reasoning have largely developed along two parallel lines. One focuses on continuous accumulation from rate, with accumulation functions as the primary object. The other focuses on summing infinitesimal bits of a quantity, with definite integrals as the primary object. No work has put these two approaches in direct conversation with each other, which is the purpose and contribution of this theoretical paper. In this paper, we unpack both approaches in terms of meanings and reasoning. Because modeling is a key motive for using quantitatively-grounded approaches in the first place, we then analyze and discuss each approach's method of modeling two example contexts.

Keywords: calculus, definite integrals, quantitative reasoning, adding up pieces, accumulation

Introduction

One crucial theme in research on integration is that the common area-under-a-curve and antiderivatives meanings are insufficient, and that *quantitatively-grounded* meanings are needed (see Jones, 2015; Sealey, 2006). Yet, quantity-based approaches to integrals have largely developed along two parallel lines. One approach, which we call *accumulation from rate* (AR), focuses on dynamic rate-of-change integrands where the integral is a continuous accumulation (Kouropatov & Dreyfus, 2014; Swidan & Yerushalmy, 2016; Thompson, 1994; Thompson & Ashbrook, 2019; Thompson & Silverman, 2008). In AR, accumulation functions, $f(x) = \int_a^x r_f(t)dt$, are the primary construct. The other approach, *adding up pieces* (AUP), focuses on zooming into tiny (infinitesimal) pieces of a domain to conceptualize tiny (infinitesimal) amounts of a quantity associated with each piece, which are added up (Chhetri & Oehrtman, 2015; Ely, 2017; Jones, 2013, 2015; Simmons & Oehrtman, 2017). In AUP, definite integrals, $\int_a^b f(x)dx$, are the primary construct. While, AR and AUP are both quantitative approaches, they have important differences. No literature puts these two approaches in direct conversation with each other, and this theoretical analysis paper partly fills this need by examining the meanings, reasoning, and modeling practices within the two approaches. We do this by explaining the *meaning* contained within each approach, as well as the entailed *reasoning* in each approach, and then examining how *modeling* might be done based within each approach.

Adding Up Pieces

AUP is a structure comprised of three elements: a *partition,* a *target quantity,* and a *sum* (Jones, 2013, 2021). To illustrate, consider $c = \int_{b_1}^{b_2} a\, db$, where *a, b,* and *c* are quantities related by $c = a \cdot b$. *Partition* means dividing the quantity "*b*" (e.g., a length or a time interval) into small pieces. These pieces can begin at a "macroscopic" scale (Δb), but can then be scaled to infinitesimal (*db*). In *target quantity,* each of these *db* pieces now has a tiny "infinitesimal" amount of the quantity of interest, *c*, constructed by the relationship $a \cdot db$. The infinitesimal bit of *c* can be notated "*dc*." Importantly, in AUP, the quantitative relationship does *not* have to be a product and can be based on other relationships (Chhetri & Oehrtman, 2015; Ely, 2017), leading to a more general format, $\int_{b_1}^{b_2} dc$, where *dc* is the result of whatever the quantitative operation is. *Sum* then means that the target quantity is added together across all of the partition pieces to give a total amount. The integral symbol itself, $\int$, denotes the summation, with the bounds [b_1,b_2] representing the portion of b that is summed over. This structure is summarized in Figure 1.

Here, "infinitesimal" essentially means "extremely tiny," though it can be formalized either through the limit ($\lim_{\Delta x \to 0}$) or the hyperreals (Dray & Manogue, 2010; Ely, 2017). Much research has shown the reasoning power across math and science in allowing differentials to retain their quantitative meaning (Amos & Heckler, 2015; Dray et al., 2008; Hu & Rebello, 2013; Jones, 2015; Nguyen & Rebello, 2011; Pina & Loverude, 2019; Schermerhorn & Thompson, 2019a, 2019b; Von Korff & Rebello, 2014).

Partition: Quantity b is partitioned into infinitesimal pieces, represented by db

$$c = \int_{b_1}^{b_2} a\,db$$

Sum: $\int$ is the sum of infinitesimal bits of c between b_1 and b_2

Target quantity: $a \cdot db$ is a quantitative structure giving an infinitesimal amount of c (dc) in each db piece

Partition/Target quantity: dc represents infinitesimal pieces of c over *implicit* partition of b (from the bounds)

$$c = \int_{b=b_1}^{b=b_2} dc$$

Sum: $\int$ is the sum of infinitesimal amounts of C across the pieces between b_1 and b_2

Figure 1. Correspondence between AUP and the definite integral notation

Reasoning Involved in Adding Up Pieces

Each part of the *partition-quantity-sum* structure of AUP cognitively involves a process that leads to an object that is then needed in the next part (Sealey, 2014; Sfard, 1991, 1992). *Partition* is the process of dividing into infinitesimal pieces, leading to a set of infinitesimally-sized pieces. *Target quantity* is the process of relating quantities together, leading to an envisioned target quantity associated with each infinitesimal piece. For example, relating pressure and an infinitesimal area, $P \cdot dA$, produces an infinitesimal amount of force, dF, in that piece. *Sum* is the process of adding the target quantity, leading to the total amount of the target quantity.

Because these processes and objects deal with *infinitesimal* pieces, *scaling-continuous covariational reasoning* (Ellis et al., 2020; Ely & Ellis, 2018) is a foundational image. This type of covariation imagines a static continuum that is infinitely zoomable. For example, one could scale time to arbitrarily small increments and coordinate an object's displacement as becoming increasingly small within those increments as well. In scaling-continuous reasoning, no matter the scale factor, the new increment is always a continuum, and always has the same dimension and quantitative character as the original interval. This avoids the problematic collapse metaphor (Oehrtman, 2009), which treats the differential quantity as having disappeared entirely.

AUP is fundamentally built on quantitative reasoning. Simmons and Oehrtman (2019) used the term *basic model* to refer to the overarching quantitative structure that would apply if all quantities had constant values, such as $Mass = \rho V$, or $Force = GmM/r^2$. They explain that in non-constant cases, partitioning into infinitesimal pieces (e.g., dV) allows one to reason that the basic model essentially holds within each piece (e.g., $dM = \rho \cdot dV$), called a *local model*.

While quantitative reasoning is the basis for interpreting or setting up definite integrals to model a situation, Ely (2017) explained that within AUP there is a distinction between *setting up* an integral versus *computing* the integral via the Fundamental Theorem of Calculus (FTC). That is, once the integral has been set up, one may need to rearrange the integral expression to ensure it matches the $\int_a^b f(x)dx$ structure required for computing integrals via the FTC. In order for the FTC to become available in AUP, definite integrals must be extended to accumulation functions (see Sealey, 2014; Swidan & Yerushalmy, 2016; Von Korff & Rebello, 2012). Recently, Stevens (2021) provided a trajectory for doing so, by extending *partition* in AUP to adding on new infinitesimal pieces past the original partition. Generalizing to an arbitrary stopping point for the partition, $\int_a^X$, leads to the input-output relationship given in an accumulation function. Stevens (2021) found that students fairly naturally extended from definite integrals to accumulation functions, and even made their own connections that led to the FTC.

Accumulation from Rate

AR comes from the work of Thompson and colleagues (Thompson, 1994; Thompson et al., 2013; Thompson & Silverman, 2008) and also contains a structure, which we summarize as *rate function*, *bits of variation*, and *accumulation*. While this is based on Thompson and Ashbrook (2019) we wish to be clear that this is our own breakdown and language. To illustrate, consider an accumulation function, $f(x) = \int_a^x r_f(t)dt$. *Rate function* entails conceptualizing $f(x)$ and x in dynamic covariation (Saldanha & Thompson, 1998), with $r_f(t)$ indicating the rate at which $f(x)$ increases or decreases at any moment between a and x. *Bits of variation* means that as the independent variable, x, varies by tiny "infinitesimal" amounts, the rate of change function determines how much the dependent variable, $f(x)$, increases or decreases at each moment. Because the integrand must always be a rate in AR, other types of quantitative relationships are required to first be transformed into a rate-of-change relationship, *variation in* $f(x)$ = *rate* × *variation in* x. To construct *bits of variation* in $f(x)$, Δt intervals are made that start at $x = a$, with each interval having its own rate, r. As x varies by infinitesimal dt bits *within* these Δt intervals, there is an accompanying bit of variation in $f(x)$ given by rate × dt. Finally, *accumulation* means that the increases/decreases in $f(x)$ are tracked open-endedly as x changes, producing a net accumulation from the starting point at $x = a$. Once this image is established, if Δt also becomes infinitesimal in size, the rates become exact ($r_f(t)$) and the accruals in $f(x)$ become exact ($r_f(t)dt$). This structure is summarized in Figure 2.

$$\underbrace{f(x)}_{\text{$f(x)$ is the net variation of the quantity from $x = a$ to the current value of x}} = \int_a^{\overbrace{x}^{\text{\textit{Accumulate} bits of $f(x)$ open-endedly from $x = a$ to the current value of x}}} \underbrace{\overbrace{r_f(t)}^{\text{$r_f(t)$ is the \textit{rate function} for $f(x)$ at any moment t.}}dt}_{\text{$r_f(t)dt$ represents a \textit{bit of variation} in f as t varies slightly by dt.}}$$

Figure 2. AR structural elements in integral notation

With AR, accumulation is formalized by first approximating using macroscopic Δx intervals with constant rates, and considering *dx*-sized variations within the intervals. It switches from approximate to exact by then allowing the Δx intervals to *also* shrink to infinitesimal in size. The approximation part of the process produces a (piecewise linear) accumulation function, given by $A(x) = \left[\sum_{k=1}^{\left\lfloor\frac{x-a}{\Delta x}\right\rfloor} r(a + (k-1)\Delta x)\Delta x\right] + r(x)\big(x - \text{left}(x)\big)$ (Kouropatov & Dreyfus, 2014; Thompson & Ashbrook, 2019). As with AUP, this transition can be formalized through either the limit concept, $\lim\limits_{\Delta x \to 0} A(x)$, or by taking Δx to be hyperreal infinitesimal.

Reasoning Involved in Accumulation from Rate

In AR, one must reason about accumulation functions to make sense of *any* integral. One crucial part is perceiving that $f(x)$ is a *function*, and that it is a function of x and not of the "dummy variable" t that is inside the integral (Kourapatov & Dreyfus, 2014). Another important idea is that the bounds of integration, a and x, differ significantly in role and in variable type. The lower bound, a, is a "zero accumulation" point (Swidan & Yerushalmy, 2016), whereas x represents a continual, open-ended accumulation with no fixed terminating location.

A robust conception of accumulation functions from rate relies on smooth, continuous covariation, the highest level of covariational reasoning (Castillo-Garsow et al., 2013; Thompson & Carlson, 2017). It also entails reasoning in terms of something moving fluidly (Thompson & Carlson, 2017); the varying quantities are imagined as being tacitly parametrized by conceptual time (Oehrtman et al., 2008; Paoletti & Moore, 2017; Thompson, 2011). Accumulation functions rely on the objectification of a rate-of-change function that quantifies this covariation. This begins with an image of constant rate of change (Thompson, 1994), from which the idea of a nonconstant rate of change is abstracted by imagining that a function has constant rates of change over infinitesimal intervals of its independent variable (Thompson et al., 2013). Thompson and Carlson (2017) describe a moment as a small interval around x over which $r_f(t)$ is "essentially constant," implying the accumulation is "essentially linear" over that moment.

To develop exact accumulation functions, the reasoner must imagine variation and accumulation at the infinitesimal scale. In AUP, the transition from finite to infinitesimal scale is accompanied by a change in notation from Δx to dx. In contrast, with AR, Δx and dx can actually both refer to macroscopic or infinitesimal increments (Thompson & Ashbrook, 2019). The difference between these notations is not one of *scale*, but of *role*. If x starts varying at some point a up to some "current location" $x = X$, Δx is used to denote a partition interval that begins at a. The current location X varies smoothly by dx amounts within any such Δx interval as the accumulation progresses. The approximation is then made exact when the learner imagines this same phenomena for infinitesimal Δx: X varies smoothly through infinitesimal $(x, x + \Delta x]$. The function $f(x)$ then aggregates bits of accumulation while this variation occurs (Thompson & Ashbrook, 2019, §5.3). The integral $f(x) = \int_a^x r_f(t)dt$ is notation that means the accumulation over an interval from a to x of the function f that comes from this rate-of-change function r_f.

With the AR approach, a definite integral is simply a specific value of an accumulation function, $f(b) = \int_a^b r_f(t)dt$. The way accumulation functions are developed and the way definite integrals are defined means that the FTC is essentially directly built into the AR approach. This could be considered one of AR's big advantages. Part 2 of the FTC states that $\int_a^b r_f(t)dt = f(b) - f(a)$, which is essentially the *definition* of AR's definite integral, or an immediate corollary of it. Part 1 of the FTC, states that if f is an accumulation function $f(x) = \int_a^x r_f(t)dt$, then f is an antiderivative of r_f. Using the same "rate-of-change function" rephrasing, this part also follows immediately from the way accumulation functions are defined in AR.

Modeling in AUP and in AR: Two Example Contexts

Example Context #1: Fluid flow

Consider the problem: a fluid runs through a pipe into a tank, and the pipe has a device on it that records the flow rate (R) of the fluid in liters per minute. If the flow rate is non-constant, find the amount of fluid (A) that passed through the pipe over the time interval, $t_1 \leq t \leq t_2$ minutes.

Modeling fluid flow in AUP. Modeling in AUP could begin with the basic model $A = R \times t$, and recognizing it only applies to constant rates. By *partitioning* the time interval into infinitesimal time intervals, dt, the basic model scales down to become an appropriate local model: $dA = R \times dt$ (see Figure 3). Summing these infinitesimal amounts of fluid over the time segments from $t = t_1$ to $t = t_2$ produces the total amount of fluid that flowed through the pipe over $t_1 \leq t \leq t_2$: $A = \int_{t_1}^{t_2} R(t)dt$ (Figure 3). This structure already fits the format needed for the FTC and an antiderivative of R could be found and evaluated at $t = t_1$ and $t = t_2$.

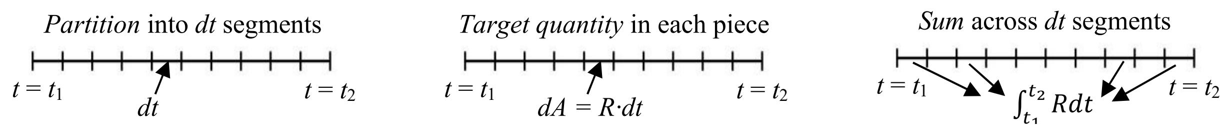

Figure 3. Modeling fluid amount through AUP

Modeling fluid flow in AR. Modeling in AR could begin by imagining the amount A covarying with elapsed time x starting at some $x = t_1$. The rate of change function here is directly provided in the context, $R_A(t)$, in liters per minute. The current time x begins at t_1 and increases open-endedly in infinitesimal dt amounts within Δt time intervals (Figure 4). The structure *rate × time = amount* provides variational bits of A. An accumulation of the variations in A leads to the integral structure $A(x) = \int_{t_1}^{x} R_A(t)dt$. The total amount up to $x = t_2$ is this accumulation function evaluated at t_2, $\int_{t_1}^{t_2} R_A(t)dt$, determined by $A(t_2) - A(t_1)$ (Figure 4).

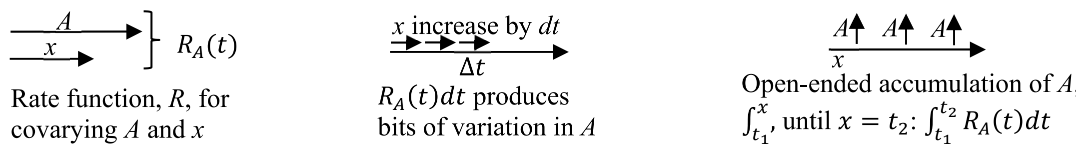

Figure 4. Modeling fluid amount through AR

Comparison of AUP and AR in fluid flow. AUP uses *scaling* to imagine an infinitesimal partition of time. Each piece of time has an associated amount of fluid, which are added to capture the total amount of fluid over that time interval. AR uses *smooth covariation* to imagine the fluid amount varying as elapsed time changes continuously. The variation of fluid over infinitesimal times is tracked to conceptualize an ongoing net accumulation, which is then truncated at the desired time value. While both approaches use infinitesimal time segments, they construct "fluid amount" in quite different ways (static partition versus dynamic covariation).

Example Context #2: Arc Length

Consider a curve defined by $y = f(x)$ between $x = a$ and $x = b$, with x and y in units of centimeters. Find the length of this curve.

Modeling arc length in AUP. One could begin by recognizing the distance formula (based on the Pythagorean Theorem) as a basic model: $s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$. However, this basic model only works for straight lines. But one can scale to the infinitesimal level by partitioning the curve into infinitesimal segments with infinitesimal lengths ds (Figure 5). At this scale, each segment is essentially linear. If one imagines an infinitesimal right triangle for one such segment, with legs dx and dy and hypotenuse ds, the local model $ds = \sqrt{(dx)^2 + (dy)^2}$ can be applied for the length of that segment. Summing these ds lengths produces the total arc length, $s = \int_{x=a}^{x=b} ds$, or $s = \int_{x=a}^{x=b} \sqrt{(dx)^2 + (dy)^2}$ (Figure 5). While this integral is a completely legitimate model in AUP, evaluating this integral via the FTC would now require altering into the $\int_a^b f(x)dx$ structure (Ely, 2017). To do so, one can "factor out" $(dx)^2$ from inside the square root to create $\int_{x=a}^{x=b} \sqrt{1 + (dy/dx)^2}\ dx$. Equally, one could factor out a $(dy)^2$ to create the expression $\int_{y=f(a)}^{y=f(b)} \sqrt{(dx/dy)^2 + 1}\ dy$ instead. Here, dy/dx (or dx/dy) is the ratio of infinitesimal changes in *x* and *y*, which precisely defines the derivative, y' (or x'): $s = \int_a^b \sqrt{1 + (y')^2}\ dx$.

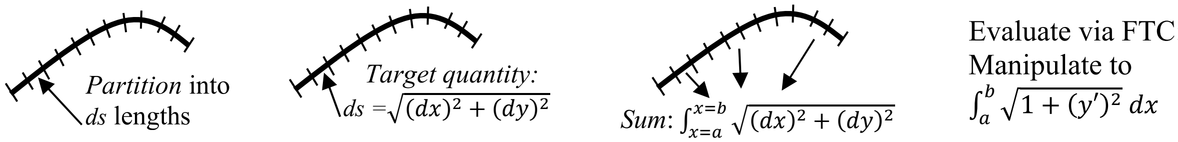

Figure 5. Modeling arc length through AUP

Modeling arc length in AR. This explanation is based on Thompson and Ashbrook's approach (2019, §8.4). The first step would be to start at $x = a$ and imagine covariation between some current position along the *x*-axis, $x = X$, and the net arc-length-so-far, $s(X)$, with the arc length's growth rate at each moment given by some rate function $r_s(x)$. Constructing straight line segments over Δx intervals, *X* is seen an varying smoothly by *dx* amounts within each Δx interval (Figure 6). As *X* varies, the *y* value also varies by $dy = r_f(\text{left}(x))dx$, where "left(*x*)" denotes the *x*-value of the left end of the Δx-interval in which *x* is currently varying (Figure 6). Note that r_f is *not* the arc-length rate function, r_s, but is the rate at which $y = f(x)$ increases within the Δx interval. This is needed because it is the rate r_f at this point that is extrapolated across the entire Δx interval as *dx* grows. Having constructed *dy*, one can relate a small variation in arc length, *ds*, with *dx* and *dy* at the infinitesimal scale, $ds = \sqrt{(dx)^2 + r_f(\text{left}(x))^2 (dx)^2}$. Factoring out $(dx)^2$ and dividing gives $ds/dx = \sqrt{1 + \left(r_f(\text{left}(x))\right)^2}$. At this point, the reasoner also imagines Δx taking on an infinitesimal value, so that *ds*/*dx* becomes essentially equal to$r_s(x)$, and $r_f(left(x))$ becomes essentially equal to $r_f(x)$. This finally gives the rate function r_s: $r_s(x) = \sqrt{1 + \left(r_f(x)\right)^2}$. One can accumulate bits of increase in arc length $r_s(x)dx$ open-endendly: $s(X) = \int_a^X \sqrt{1 + (r_f(x))^2} dx$. Once it is known that the rate function is a "derivative",

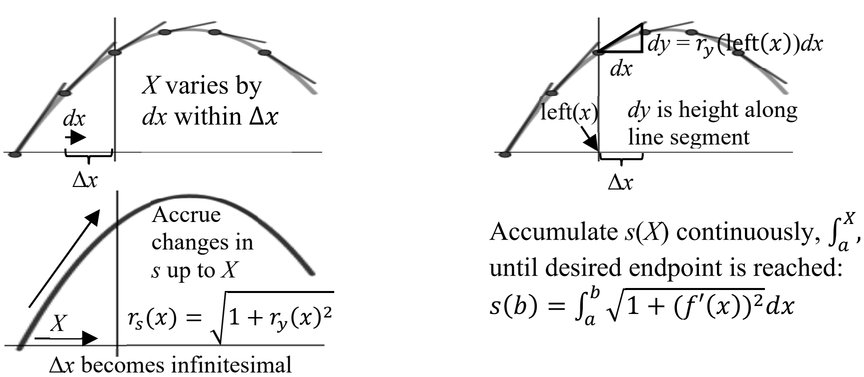

Figure 6. Modeling arc length through AR

this is also equal to $\int_a^X \sqrt{1+(f'(x))^2}dx$. To produce an arc length up to a specific location $X = b$, one evaluates this exact accumulation function at that value, $s(b) = \int_a^b \sqrt{1+(f'(x))^2}dx$.

Comparison of AUP and AR in arc length. The difference between AUP and AR is more pronounced in this context. In AR, the steps involved include all those used in the AUP modeling process, but there are a number of additional steps as well. The reasoner imagines both a Δx partition interval *and* a dx varying inside it. This Δx also becomes infinitesimal, making it subtle to distinguish from dx. While in AUP $ds = \sqrt{(dx)^2 + (dy)^2}$ is the direct quantitative relationship used to construct the integral, in AR it only serves as an intermediate step toward deriving the rate function $r_s(x) = \frac{ds}{dx}$. Further, in AR, the reasoner conceptualizes all of this before the integral expression is written. AUP allows the reasoner to represent sums along the way before the entire process is complete, such as $s = \int_{x=a}^{x=b} ds$ or $\int_{x=a}^{x=b} \sqrt{(dx)^2 + (dy)^2}$. These expressions would be meaningless in AR. Lastly, by focusing on the rate at the moment of accumulation as a function of x, the reasoner works with $r_f(\text{left}(x))dx$ instead of dy. If one wanted to do the integral with respect to y, one would have to make a different construction for $dx = r_y\big(\text{bottom}(y)\big)dy$, whereas AUP already has both dx and dy available in the quantitative relationship governed by the infinitesimal right triangle.

Discussion

This paper contributes by putting AUP and AR in direct conversation with each other in terms of meaning, reasoning, and modeling. While both are quantitatively-based approaches, AR and AUP use very distinct meanings for integrals. AR creates an image of variables in dynamic, continuous covariation (Thompson & Carlson, 2017) based on an explicit rate function (Thompson, 1994). AUP creates an image of a static infinitesimal partition, with bits of the target quantity in each piece, based on the idea of zooming (Ely & Ellis, 2018; Jones, 2013; Jones & Dorko, 2015). These meanings lead to distinct types of reasoning. AR requires smooth-continuous covariational reasoning, and the coordination of elements of a dynamic system into an encapsulated rate function (Thompson & Carlson, 2017). AUP is based on scaling covariation reasoning (Ellis et al., 2020), involving a zoomable continuum where basic models for quantitative relationships can become appropriate local models. Where AR requires all integrals conceptualized through *amount = rate × variation*, AUP allows any quantitative relationship to serve as a basic or local model (Chhetri & Oehrtman, 2015; Simmons & Oehrtman, 2019).

One benefit to AR is that the FTC is an immediate consequence of the definitions of accumulation functions and definite integrals. In AUP, further reasoning is needed beyond the definition of the definite integral to develop the FTC. However, a key benefit to AUP is its power and flexibility in modeling, by allowing different quantitative relationships to be used (Chhetri & Oehrtman, 2015; Ely, 2017; Jones, 2015, 2020; Simmons & Oehrtman, 2017). AUP handles rate contexts, non-rate contexts, and even non-product contexts (such as arc length) equally. The exact same partition-quantity-sum structure functions the same in all cases. Further, progress in modeling with AUP can be inscribed by intermediate integral expressions, such as $\int_{x=a}^{x=b} dF$, prior to the FTC-ready $\int_a^b f(x)dx$ format. On the other hand, AR is constrained by the requirement to perceive all integrals as rates. Non-rate or non-product contexts must first be reconceptualized as rates. The full rate function, $r_f(t)$, must be described before an integral expression that is meaningful in AR can be written. In conclusion, while AR and AUP are both valid quantitative approaches, the meanings, reasoning, and modeling is different in each.

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On the Theory of Conceptualizing an Animation as a Didactic Object

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In this paper, I propose a theorization of the mental operations involved in conceptualizing an animation as a didactic object. I begin by motivating the idea of an animation as a didactic object. Then, I leverage the foundations of radical constructivism, conceptual analysis, instructional conversation, intersubjectivity, and reflective discourse to make explicit the theoretical relationships that underlie the mental operations of conceptualizing an animation as a didactic object. I offer a visualization and an explanation of the interaction between the mental operations that an instructor or researcher may engage in to conceive of an animation as a didactic object. Lastly, I discuss limitations and future directions.

Keywords: Didactic Object, Didactic Model, Animations, Conceptual Analysis

Introduction

Instructors have long since struggled to help students visualize mathematical ideas. Until recently, attempts to support students' visualizations were limited to static diagrams. Now, advances in technology have allowed instructors and curriculum writers to display static and dynamic images in the hopes that students understand visually what relationships are being expressed symbolically and to help students conceptualize problem contexts. The mathematics education community has called for the inclusion of visualization in the teaching of calculus to help support students' understandings (e.g. Arcavi, 2003). Recent calculus curriculum reform efforts have often sought to leverage the power of technology to illustrate the coherence of calculus concepts using graphs of functions (e.g., Swidan, 2019; Thompson, Byerley, & Hatfield, 2013). With these tools, teachers of calculus might use graphs in their efforts to support students in moving beyond procedure-oriented mathematics and shift their students' understandings to a more conceptual one.

This paper focuses on animations because they are widely used in curriculum (e.g Thompson et al., 2013). In mathematics education, animations are researched in two different ways. The first employs animated cartoon characters which researchers have found useful in representing classroom scenarios for supporting preservice teachers reflections on their teaching (e.g., Chazan & Herbst, 2012; Chieu & Herbst; 2016; Herbst & Kosko, 2014). The second employs an animation as a visual representation that "generates a series of frames, so that each frame appears as an alteration of the previous one" (Bétrancourt & Tversky, 2000, p. 313). However, utilizing this definition for a mathematical animation would be problematic because then any compilation of images formatted as a video could be considered an animation. Hence, I extend Bétrancourt and Tversky's (2000) definition of animation to define a mathematical animation as an animation that depicts variation or continuous motion.

High school and college mathematics instructors offer animations to help students understand situations involving varying quantities that otherwise are described in the text, perhaps accompanied by a diagram. For example, Figure 1 is replicated from a calculus textbook's section on related rates using GeoGebra. It is accompanied by the diagram below the problem statement. The purpose of the diagram is to assist students in understanding the situation, as described in the text.

A rubber band connects the hour hand and minute hand on a 12-hour clock. What is the rate of change of the band's length with respect to time at each moment between 3:00 pm and 7:00 pm?

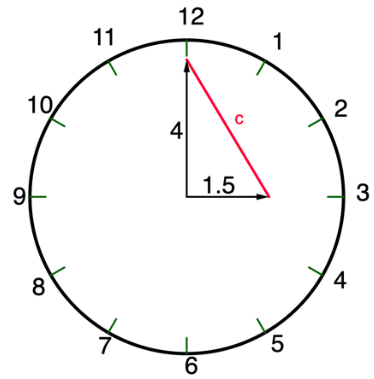

Figure 1. Static diagram of clock scenario.

For students to understand the diagram in a meaningful way, they must envision how the hands of the clock vary in relation to time, how the hands vary in relation to each other, and that the rubber band's length also varies systematically with regard to the clock hands' positions. However, it is well documented (e.g., Carlson, Jacobs, Coe, Larson & Hsu, 2002; Moore & Carlson, 2012; Oehrtman, Carlson & Thompson, 2008; Thompson, 1994) that students have difficulty imagining situations dynamically when presented in text, even with a static diagram. Even in an elementary education setting, mathematics educators have found that diagrams are not self-evident. Rather, they are ambiguous and have to be interpreted actively by the students (Steinbring 2005; Söbbeke 2005).

Now imagine that the static diagram was animated for a student to view while solving the problem. The intention behind the animation is that students understand that the minute hand makes a complete revolution each time the hour hand makes 1/12 complete rotation. Meaning the minute hand rotates 12 times as fast as the hour hand. Even with the animation, it is up to students to decide the hands' angular displacements, the directed angle between the hands, and the hands' rates of change of angular velocity are salient quantities. They must also decide that the relationship between hands' rates, the relationship between the rubber band's length, and the directed angle between the hands are significant to the question. Consequently, animations can be helpful aids to students' envisioning a situation, but there are many aspects of situations students must still provide for themselves to complete their understanding productively. In particular, students must conceptualize relevant quantities within the animation and conceptualize relationships among them.

My discussion of the animation for the students, suggests the animation might be used best in supporting students' understanding were they to experience it in the context of a teacher holding a thoughtful discussion about interpreting it—quantities they can see in it, which of them might be relevant to problem's question, and how to think about relationships among them. The animation would support such a conversation more productively than would a static diagram. With the diagram alone, students would need to envision for themselves all that varies and envision the variations and appropriate relationships to each other. Finally, without a well-crafted teacher-guided reflective conversation, students could easily flounder because of their unawareness of the many decisions and interpretations they must make to understand the animation productively. In describing the animation in this context, I mean to utilize an animation as a *didactic object*.

Didactic Objects and Model

Thompson (2002) offered the terms “didactic objects” and “didactic models” as a means by which a teacher might create a dynamical space in which students have the opportunity to construct the understandings the teacher intends. Thompson defines *didactic object* as “a thing to talk about’ that is designed with the intention of supporting reflective mathematical discourse” (Thompson, 2002, p. 198). Thompson went on to say, “I hasten to point out that objects cannot be didactic in and of themselves. Rather, they are didactic because of the conversations that are enabled by someone having conceptualized them as such” (ibid, p. 198). Didactic objects entail images of conversations amongst varying participants about the object that will be propitious for the kinds of engagement out of which advanced understandings might emerge. It is important to note that an object with no discourse is simply an object and is not inherently didactic, the object only becomes didactic when conceptualized as defined previously (Thompson, 2002).

This wide interpretation of “object” gives the creator a limitless perspective on what type of artifact can be implemented. However, the instructor must be aware of why she has selected a specific object and how the object will assist students in developing conceptual understandings of mathematics. Therefore, the design for guiding a conceptual understanding using an animation as a didactic object should include questions that prompt *reflective mathematical discourse*. The questions that are included in the instructional conversation should be conceptual rather than calculational (Thompson, Phillip, Thompson, & Boyd, 1994). Questions that are calculational prompt and solidify answer getting behaviors and the idea that only one process or answer exists. Conceptual questions that elicit deeper thinking and multiple answers will support stronger meanings, as well as a conceptual understanding. In this way, the goal of didactic objects is to support ways of thinking that go beyond what is actually present in the discussion surrounding the object.

A *didactic model* is “a scheme of meanings, actions, and interpretations that constitute the instructor’s or instructional designer’s image of all that needs to be understood for someone to make sense of the didactic object in the way he or she intends” (Thompson, 2002, p. 212). Didactic models are not models of students' experiences; rather, they should encapsulate how a student's understandings might evolve into sophisticated, advanced, and coherent understandings. By focusing on the development of advanced understandings of didactic objects, one also addresses the instructional actions to support such development. This is true because the conversations an instructor facilitates regarding objects aim to support students’ abstraction of mental operations and operative mathematical structures by creating a context wherein students might participate communally in mathematical reasoning the instructor intends each to develop personally (Thompson, 2002).

Thompson discussed three issues surrounding the design and use of didactic objects: (a) employing didactic objects to manage reflective conversations aimed at supporting students’ understanding of fractions as conveying relative size, (b) ways teachers might overestimate the impact of using animations on students’ understandings, and (c) issues of designing artifacts with the intention of using them as didactic objects. A central theme in all examples was the importance of teachers’ ability to decenter from his or her mathematical meanings in the quest to hear students’ meanings.

Other researchers have found the construct *didactic object* useful in their research to design curriculum (Bowers, Bezuk, & Aguilar, 2011; Counrtney, 2010), to support students in constructing formulas (Guy, 2020), and to support students' imagery of varying quantities in calculus (Mirin, Yu & Kahn, 2020). However, in prior literature when researchers describe the process of conceiving an artifact as a didactic objects they are not explicit in describing the mental process(es) that the researcher(s) engaged in to conceive of the artifact as a didactic object. Moreover, the implementation of a didactic object is most likely to differ across individuals since the conceptualization is dependent on the instructor or researcher's *conceptual analysis*, *didactic model*, and *instructional conversations*. Thus, this paper seeks to make explicit the theoretical foundations for conceiving of an animation as a didactic object and seeks to explore the following question:

RQ: What mental operations must an instructor/researcher engage in to conceive of an animation as a didactic object?

Underlying Theory and Theoretical Constructs

In this section, I will describe the other theoretical constructs of *conceptual analysis*, *instructional conversation*, *reflective discourse*, and *intersubjectivity* before then utilizing these constructs to describe the mental operations entailed in conceiving of an animation as a didactic object.

Conceptual Analysis and Instructional Conversation

Stemming from *radical constructivism* (Glasersfeld, 1995) the method of *conceptual analysis* (Glasersfeld, 1995; Thompson, 2008) plays an integral part in conceptualizing an animation a didactic object. For a curriculum developer or researcher to orient themselves to radical constructivism means they are interested in designing mathematical tasks to engage learners in the types of actions that promote their construction of desired understandings and place them is such situations that foster abstractions from those actions. The focus of designing such tasks aims to answer the question, "What mental operations must be carried out in to see the presented situation in the particular way one is seeing it?" (Glasersfeld, 1995, p.78). Glasersfeld called the method for proposing answers to the question a *conceptual analysis* and noted there are two ways to employ a conceptual analysis. Glasersfeld's first method of conceptual analysis involves the creation of models of knowing that might help one think about how others might know a specific idea. The second, was to determine the actions that might be most propitious for achieving desired understandings and actions that would help students build more powerful ways to deal mathematically with their environment. Thompson (2008) expanded on Glasersfeld's descriptions of conceptual analysis by offering two additional uses for a conceptual analysis. A conceptual analysis can also be used to describe ways of knowing that might be unproductive or problematic in specific situations and in analyzing the coherence of various ways of understanding a body of ideas.

The four ways of using conceptual analysis have implications for instructional designs as conceptual analyses of mathematical ideas cannot be carried out abstractly. Rather, one doing a conceptual analysis imagines students doing *something* in the context of *discussing it* (Thompson, 2002). In this regard we can leverage conceptual analyses to "anticipate the conceptual operations that underlie a particular way of thinking and therefore the design of conversations that might support students developing them" (Thompson, 2002, p. 197). Thompson calls such conversations *instructional conversations* and notes that while a teacher is aware of an intended instructional conversation, students do not need to be aware of it in the same way or at all.

Reflective Discourse and Intersubjectivity

Taking a radical constructivist stance means that I interpret discourse involving two or more entities as a collective activity in which individuals participate. Each individual has schemes that suggest to them how to participate. It is these schemes through which the individual interprets the actions and utterances of other individuals (Thompson, 2000). Hence, *reflective discourse* "is characterized by repeated shifts such that what the students and teacher do in action subsequently becomes an explicit object of discussion" (Cobb, Boufi, McClain, & Whitenack, 1997). This means that the teacher shifts the discourse such that the student's actions and meanings become the object of the discussion; implying that the student first has to contribute. One should be careful to note that student participation alone does not mean students will engage in reflective discourse, rather participation supports students to reflect and reorganize prior activities and does not cause, determine, or generate it (ibid). In this sense, guiding and developing a reflective discourse requires careful consideration and judgment on the instructor's part in which a conceptual analysis will help guide the instructor to implement the discourse.

For two people to communicate successfully in a reflective discourse does not mean that they have come to the same understandings and meanings, rather, there is no reason for each person to believe they have misinterpreted each other. *Intersubjectivity* is the state in which each person in the interaction feels confident that the other(s) involved thinks or anticipates as they do. More eloquently stated by Steffe and Thompson (2000) and Thompson (2000), "people have reached a state of reciprocal assimilations where further assimilations are unproblematic". The notion of intersubjectivity therefore implies the focus of the interaction is not about agreement, but about the understandings and meanings in the moment of each individual. It would be impossible to claim in any instance that two individuals have the same meanings because by doing so would imply that the two individuals have identical schemes and have had the exact same experiences. In this sense, intersubjectivity can be used as a lens in which to view reflective discourse in the moment of the discourse between the student and the instructor or researcher when using an animation as a didactic object.

Putting it All Together

In theorizing the mental operations an instructor/researcher may engage in when conceptualizing an animation as a didactic object, I offer Figure 2 as a visual representation of the relationships between all the constructs defined and explained previously. I will also contrast the conceptualization of an instructional conversation without a didactic object to highlight an important distinction that an instructor/researcher must engage in when conceptualizing an animation as a didactic object.

The Red Path

A conceptual analysis imagines students doing *something* in the context of *discussing it* (Thompson, 2002). Instructional conversations support students developing the conceptual operations that underlie a particular way of thinking as a result of a conceptual analysis, but do not inherently include didactic objects. Hence, a researcher or instructor could choose to have an instructional conversation with a student or class (implementing the instructional conversation) and then reflect on how productive the conversation was in supporting the students' understandings (analysis of the instructional conversation). In turn, the analysis of the instructional conversation might inform the researcher or instructor of improvements to be made on the initial conceptual analysis or the instructional conversation. In Figure 2, it should be noted that the shaded purple box depicts the mental operations that an instructor may engage in when conceiving of an instructional conversation with or without a didactic object and that outside of the shaded purple box is the enactment and analysis of the instructional conversation.

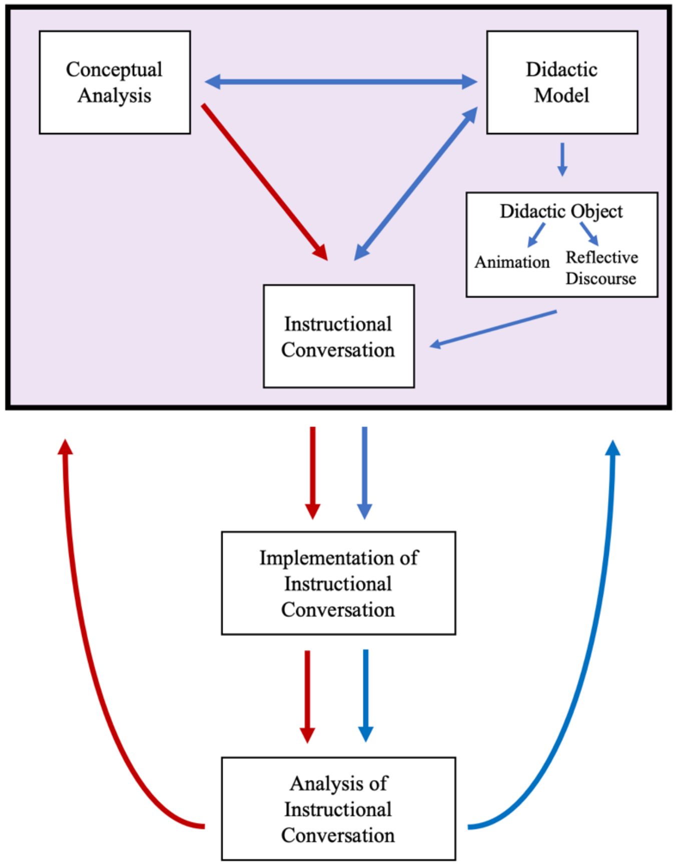

Figure 2. A visualization of the theoretical relationships.

While engaging in The Red Path researchers or instructors need not engage in a *second-order model*, but they must at least engage in a *first-order model.* First-order models are "models the observed subject constructs to order, comprehend, and control his or her experience" (Steffe, Glasersfeld, Richards, & Cobb, 1983, p. xvi), which no one else can have access to. When creating a conceptual analysis, the researcher/instructor creates a first-order model of an epistemic student's thinking. Namely, a model of mental actions that govern the student's mathematical perception, activity, and anticipation of results of the activity Many instructors may think it is impossible to teach without didactic objects and may find the red path impractical; however, it is quite possible that an instructor intends to use an animation but does not take careful consideration to conceptualize it as a didactic object. Meaning, that upon implementation they may focus students' attention to irrelevant information and not think about how the animation can be used to advance students' mathematics.

The Blue Path

Since a conceptual analysis supports the design of a didactic object from students' understandings, then, within the method of conceptual analysis one thinks about describing things students might reperceive and things about which a teacher might hold productive discussions with them (Thompson, 2002). Specifically, that the discussions that the instructor envisions with an artifact support reflective mathematical discourse focused on mathematical topics, where the discourse becomes an explicit object of student reflection. Students participating in reflective discourse have the opportunity to construct deeper understandings and cognitive connections.

A didactic object also entails the images of conversations to support students developing an advanced way of thinking. Thus, from an instructional design perspective it reasons that the instructional conversation an instructor intends to implement from a conceptual analysis plays a role in the designing and implementing the didactic object in the scope of the conceptual analysis. Moreover, the relationship between conceptual analyses and instructional conversations ensures that both student learning and instruction is thought of when creating a didactic object through the means of the didactic model. Thompson (2002) notes that a didactic model differs from Simon's idea of a learning trajectory because didactic models make clear the separation between descriptions of instruction and descriptions of learning (p. 213). In essence, didactic models allow the possibility of multiple approaches to the same goal whereas as learning trajectories are specific instructional sequences.

In designing reflective discourse using the researcher/instructor's didactic model, they must engage in creating a *second-order model*, whereas in The Red Path they need not necessarily engage in a *second-order* model. *Second-order models* are those models the researcher/instructor construct of the subject's knowledge in order to explain their observations or experience of the subject's states and activities (Steffe et al., 1983). Specifically, a researcher/instructor would have to think about how students might interpret the animation using their schemes and how the researcher/instructor would respond to such interpretations. A researcher/instructor must take careful consideration to think about the ways in which students may interpret the animation as a didact object because we cannot take for granted what is the conversation in which students will actually participate in when implementing an instructional conversation.

Similarly, to The Red Path, in The Blue Path the researcher/instructor implements the instructional conversation with a student or class, reflects on how productive the conversation was in supporting the students' understandings, and then the analysis of the instructional conversation informs the researcher/instructor of improvements to be made on the initial conceptual analysis, instructional conversation, and/or didactic model.

Limitations and Future Research

I recognize that animations implemented as didactic objects might differ depending on the instructor or researcher's conceptual analysis, didactic model, and instructional conversations. This places a necessity to further investigate to what extent the role of the researcher/instructor's mathematical meanings plays in conceptualizing an animation as a didactic object and how does that conceptualization impact the instructors' implementation of the didactic object. Moreover, what appears to be an important implementation of animations as didactic objects is encouraging that students anticipate how the values of quantities would vary before watching the animation (Hegarty, Kriz, & Cate, 2003; Schnotz & Rasch, 2005). In this sense, it may be pertinent to explore the lens that *quantitative reasoning* (Thompson, 1990, 1993, 1994, 2011) and *covariational reasoning* (Carlson et al., 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017) may offer in the mental operations that a researcher/instructor must engage in when conceiving of an animation as a didactic object outlined in this paper. Lastly, although this paper focuses on animations as didactic objects, future research should also investigate the viability of the model in relation to empirical studies and in relation to other artifacts (e.g. static diagrams and applets) being conceived of as didactic objects.

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An Analysis of Multi-Step Questions in a Calculus II Course

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This paper describes some of the psychometric properties of a new type of online exam question called multi-step questions. The multi-step questions in this study are given through computer assisted assessment on two Calculus II exams. Multi-step questions are of a form in which the question is broken up into its constituent parts, and a student must work and submit their answer for a part before moving onto the next part. These questions are analyzed both qualitatively and quantitively through a categorization on Anderson and Krathwohl's (2001) Taxonomy Table and by using Rasch analysis with the Partial Credit Model (Masters, 1982).

Keywords: Calculus, Computer assisted assessment, Exams, Multi-step questions, Psychometrics

A crucial facet of undergraduate calculus courses is assessment. Making conclusions based on exam results assumes that the exams have properties in which conclusions can be made. Exams are often being altered by changing current questions, adding new questions, and removing old questions. Sometimes creating new questions requires using different means to ask questions. As questions are altered, analyzing the psychometric properties of the exams is important to ensure that the exam questions are working as intended.

This study aims to analyze the psychometric properties of multi-step questions in Calculus II. Multi-step questions are questions given through computer assisted assessment (CAA) in which a question is broken down into parts, and students must work on and submit their answer for one part before continuing onto the next part. Giving multi-step questions on exams allows for students to earn partial credit even if they make a mistake somewhere in their work. Once a preceding part is submitted on its final submission, the correct answer for that part is shown so that the student can use it in the following part. This continues until the student has completed the question in its entirety. In this study, these questions are given through an online education platform. Automatic item generation (AIG) is used to change numbers for question variation on exams. In some cases, these questions are given through question pools in which the online platform randomly selects a question out of the pool to give to the student to work.

Literature Review

Computer assisted assessment allows for students to complete homework assignments and exams through a computer based platform. It enhances exams by increasing the quality of assessment (Draaijer, 2019). With questions involving partial credit, it allows for the use of a consistent grading scale to allow for partial credit points. It also allows for the use of various types of questions, such as placing questions within question pools and multi-step questions.

Automatic item generation is used to create test items in an automated manner under an established item model (Lai, 2009). Embretson and Kingston (2018) attribute the use of AIG to decreasing item familiarity and cheating. By using AIG, scores are able to be tracked automatically and students are able to earn partial credit (Singley & Bennett, 2002). A great benefit of AIG is the ability to generate more questions by reusing item models (Gierl & Lai, 2012).

The exact process of multi-step questions that occur in this study was not found in the literature. Similar occurrences were found in literature, such as computer algebra systems trying to automatically give partial credit (Erabadda et al., 2016; Kadupitiua, 2016). In this case, multi-step questions look something such as Figure 1, where a student would first see everything in the first block. Once the student either used all of their submissions or got the answer correct, they can view the next part to work. This continues until the student is notified that they have completed the problem. By breaking down the questions and having blanks within the parts, students can earn consistent partial credit when taking these exams.

Question 2

This question has several parts that must be completed sequentially. If you skip a part of the question, you will not receive any points for the skipped part, and you will not be able to come back to the skipped part.

<u>Integration By Parts Exercise</u>

Part 1 of 4
Use Integration By Parts to evaluate the integral

$$\int (5x+6)e^{4x}dx.$$

First, decide on appropriate u and dv.

$$u = \_\_\_\_\_\_\_\_\_\_ \qquad dv = \_\_\_\_\_\_\_\_\_\_ dx$$

Part 2 of 4
Since $u = 5x + 6$ and $dv = e^{4x}dx$, find du and v.

$$du = \_\_\_\_\_\_\_\_\_\_ \, dx \qquad v = \_\_\_\_\_\_\_\_\_\_$$

Part 3 of 4
We have obtained $du = 5dx$ and $v = \frac{e^{4x}}{4}$. Next, apply Integration By Parts formula.

$$\int (5x+6)e^{4x}dx = \_\_\_\_\_\_\_\_\_\_ - \int \_\_\_\_\_\_\_\_\_\_ dx$$

Part 4 of 4
Complete the integration and write the final answer. (Use C for the constant of integration.)

$$\int (5x+6)e^{4x}dx = \frac{5x+6}{4}e^{4x} - \int \frac{5}{4}e^{4x}dx = \_\_\_\_\_\_\_\_\_\_$$

$$\int (5x+6)e^{4x}dx = \frac{5x+6}{4}e^{4x} - \int \frac{5}{4}e^{4x}dx = C - \frac{5e^{4x}}{16} + \frac{1}{4}e^{4x}(5x+6)$$

You have completed the problem.

Figure 1. Example of a Multi-Step Question

Theoretical Framework

Anderson and Krathwohl's (2001) revision to Bloom's (1956) Taxonomy created a two-dimensional Taxonomy Table with the Knowledge Dimension and the Cognitive Process

Dimension. The Knowledge Dimension is comprised of four types of knowledge: factual, conceptual, procedural, and metacognitive, while the Cognitive Process Dimension is comprised of six cognitive processes: remember, understand, apply, analyze, evaluate, and create. Combined, these dimensions form the Taxonomy Table used to categorize exam questions in this study as seen in Table 1. Along with the Taxonomy Table, the theoretical framework requires the Partial Credit Model (Masters, 1982), otherwise known as PCM. This model was chosen because it is a latent trait model that transforms ordinal ratings to interval measures. It also provides clear requirements for evaluating item properties. The combination of the Taxonomy Table and the PCM form a foundation in which psychometric properties can be analyzed.

Table 1. Calculus II Questions Categorized on the Taxonomy Table (Krathwhol, 2002, p.16)

Knowledge Dimension	Cognitive Process Dimension					
	Remember	Understand	Apply	Analyze	Evaluate	Create
Factual						
Conceptual						
Procedural						
Metacognitive						

Methods

De-identified student data containing students' scores on each question for two exams for Calculus II were obtained and considered in this study. The two exams considered in this study were the second and third exams during the semester, labelled as Exam 2 and Exam 3, respectively. There were 395 students who took Exam 2 and 362 students who took Exam 3. These exams were taken using CAA on an online education platform in a proctored mathematics laboratory setting. The students were already familiar with the online platform before taking these exams since they had an exam prior to the considered exams and were to complete their homework through the same platform. When taking an exam, a student is given three submissions on each question that is not multiple choice. The student can use the first two submissions with no penalties, but there is a 50% penalty on points earned on the third submission. If a question is multiple choice, the student is only given one submission.

In addition to the multi-step questions, the exams had other types of questions on them as well. The first action was to remove the other types of questions to get a better understanding of the properties of the multi-step questions. Partial credit categories were then combined to meet the requirements to use the Partial Credit Model (Masters, 1982) provided by Linacre (2002), such as having at least ten items in a category. This was done through a process of taking into account the exam questions as well as trying to reach a minimum of ten items per partial credit category. After this, each exam question was individually examined and categorized by quantitatively placing the items on the Taxonomy Table. Once each item was categorized, a final inspection was done to verify that the questions were categorized into the correct category.

After question categorization, the psych package (Revelle, 2021) was used on R software to perform exploratory factor analysis to evaluate dimensionality. A scree plot was produced to determine if the items were unidimensional. Having the items be unidimensional is critical because, in addition to being a requirement of the PCM, it also facilitates clear interpretations of item order and person order. Once the determination was made that the items were unidimensional, the principal axis was checked to make sure all the questions considered were providing information.

Once the exploratory factor analysis was complete, it was determined that the PCM (Masters 1982) could be used to analyze the exam data. The eRm (Mair et al., 2020) package was used on R software to perform the Rasch analysis using the PCM function.

Results

Table 2 shows the results of the qualitative categorization of the exam questions on Anderson and Krathwohl's (2001) Taxonomy Table.

Table 2. Calculus II Questions Categorized on the Taxonomy Table

Knowledge Dimension	Cognitive Process Dimension					
	Remember	Understand	Apply	Analyze	Evaluate	Create
Factual	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Conceptual	0.00%	0.00%	0.00%	16.67%	16.67%	0.00%
Procedural	0.00%	0.00%	41.67%	8.33%	16.67%	0.00%
Metacognitive	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

The majority of questions were classified as 'apply procedure'. No multi-step questions from the exams were classified in the 'remember', 'understand', or 'create' columns or in the 'factual knowledge' and 'metacognitive knowledge' rows. Seeing that the category with the majority percentage of exam questions was categorized as 'apply procedure', one may infer that the multi-step questions for Calculus II are more procedural. It can be argued that this is the case because the questions are broken down into their constituent parts rather than asking a student to work a problem from start to finish using their own knowledge to follow steps.

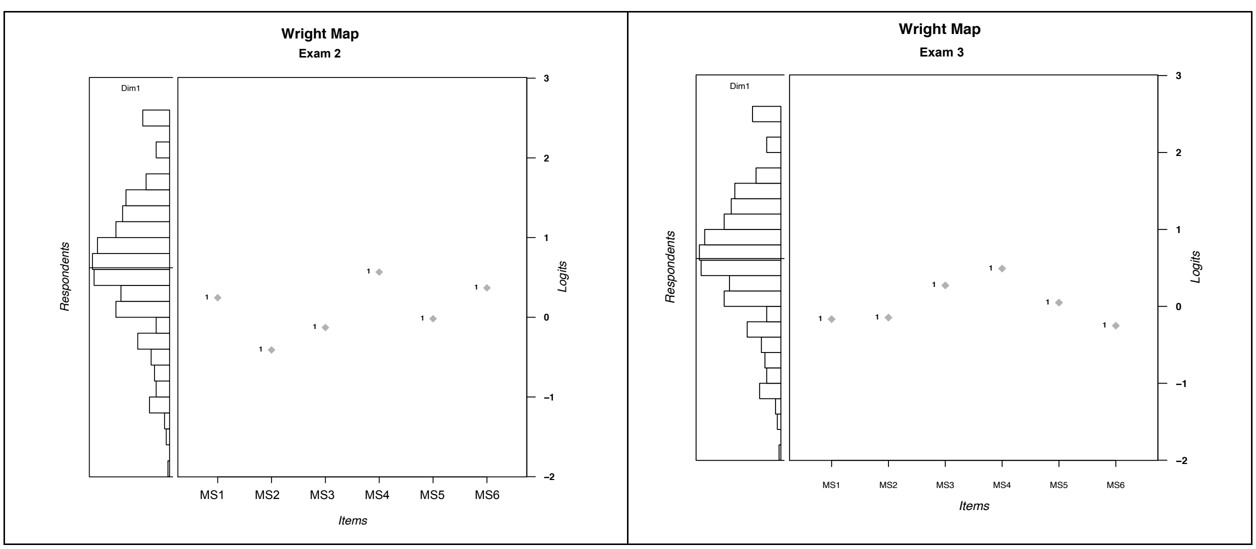

Figure 2: Wright Maps for Exam 2 and Exam 3

Above, in Figure 2, are the Wright Maps for the multi-step questions on the two exams. Looking into Exam 2 a bit further, it was found that of the 395 students that took Exam 2, person ability was lower than the lowest item difficulty (-0.41) for 44 students or about 11.14%. On the other hand, it was found that person ability was higher than the highest item difficulty (0.57) for 213 students or about 53.92%. Only about 34.94% of students' abilities were in the item difficulty range for this exam. Similar results were found for Exam 3. Of the 362 students that took Exam 3, 27 students' person ability was lower than the lowest item difficulty (-0.25), which is about 7.46%. It was found that for 258 students, or about 71.27% of students taking Exam 3, person ability scores were higher than the most difficult item (0.49).

The visual representations of the person locations shown in Figure 3 for Exam 2 shed light onto how little information is able to be drawn from the analysis of these questions. Figure 3 expresses the relationships between percentage correct and person ability for each exam question. Consider MS1. Students with person ability scores over two standard deviations above the average had scores ranging from the minimum to the maximum. Similar cases occur in items MS3, MS4 and MS6. With scores close to the maximum having so much congestion, variation amongst students with average to high person ability are not being differentiated. The ability of the students goes much beyond what is asked on these questions.

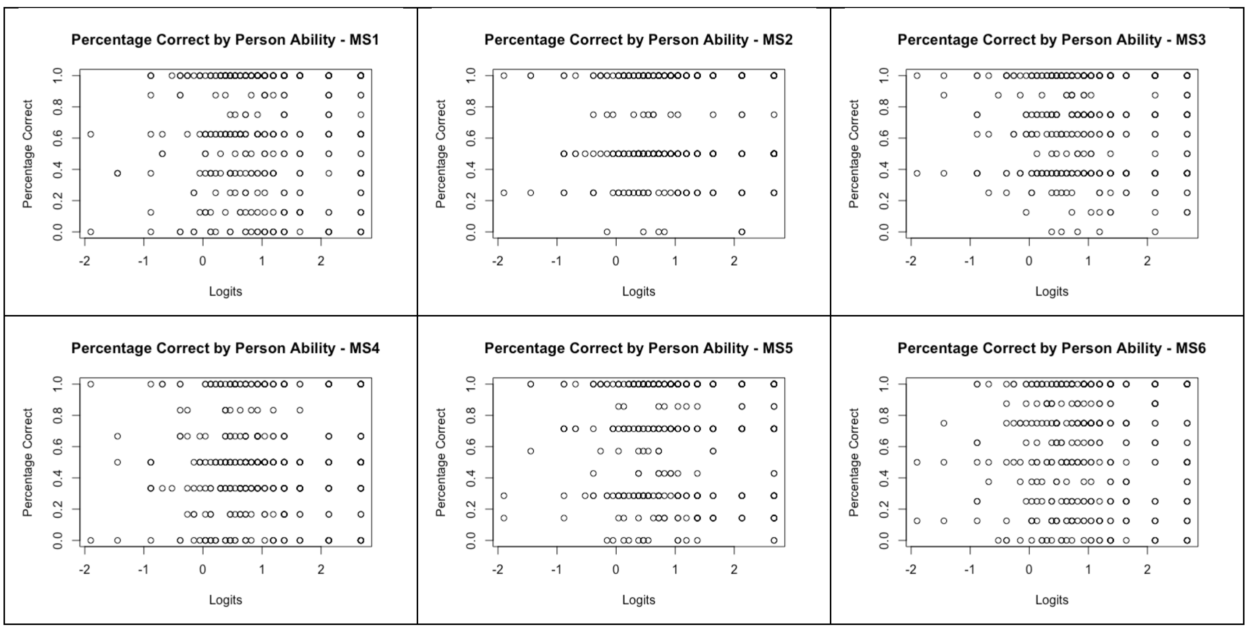

Figure 3: Percentage Correct by Person Ability for Exam 2

The lack of information being given by Exam 2 is consistent with Exam 3. Consider Figure 4, which shows the person-item map for Exam 3. When the location of person abilities and item difficulties are placed along the same latent dimension, it is clear that half of the locations of item difficulty fall below the average; hence, the exam questions being given are too easy and cannot accurately measure person ability because they are not differentiating between students in the upper echelon of person ability.

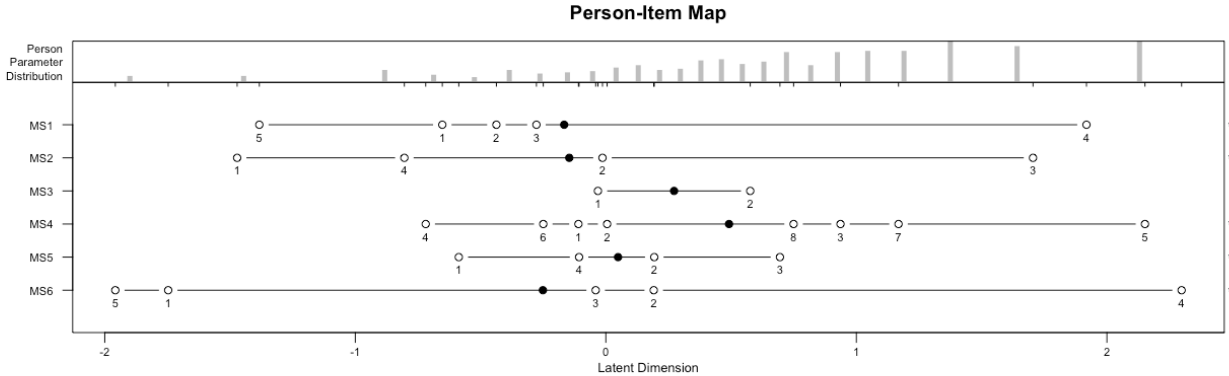

Figure 4: Person-Item Map for Exam 3

Discussion

Overall, the questions considered in this study are not beneficial to measuring student understanding and cognitive skills because they are too easy for the students taking them. There is a lack of student differentiation due to the lack of difficulty on these exam questions. Although multi-step questions allow for consistent partial credit, they could be making exam questions too easy for the average calculus student. It would be beneficial to analyze the questions with various difficulty levels. This would allow strong conclusions to be made about whether or not giving multi-step questions on exams is beneficial for calculus students. Five of the twelve questions considered in this study were categorized as ‘apply procedure’. More research needs to be done to determine if the exam questions were too easy because students were mostly asked to apply a procedure or if they were too easy because of the breakdown of the questions.

As computer assisted assessment is incorporated into more and more classrooms, it is important to find methods in which partial credit can be given, especially for courses such as calculus in which there are many steps to reach the correct answer. Calculus courses demand deep cognitive reasoning skills, and instructors are aiming to measure student understanding and cognitive skills in these classrooms but continue to run into constraints, such as giving consistent partial credit. Multi-step questions might have the potential to successfully differentiate students and give consistent partial credit if different types of questions are asked. Increasing the difficulty level of multi-step questions could better differentiate person abilities. Although this may result in lower exam averages, grading scales could be changed to ensure that students are still passing even though they are scoring lower due to the higher difficulty of the exams.

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Theoretical Development of the Constant Rate of Change Assessment (CRCA) Instrument

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This theoretical report presents the constant rate of change assessment (CRCA) taxonomy that is designed based on the theory of quantitative reasoning, proportional reasoning, and covariational reasoning. The paper also presents the development of four novel assessment items informed by the CRCA Taxonomy. I also share the contribution of clinical interview data in the development of the CRCA instrument.

Keywords: assessment items, constant rate of change, proportional reasoning, precalculus

Introduction

The ideas students study in Precalculus are foundational to build meanings for ideas in higher- level mathematics courses (Calculus, Linear Algebra, etc.) and to persist in continuing a STEM degree. Sonnet & Sadler (2014) reported that most Precalculus curricula in college reintroduce the procedural techniques students learn in high school, and therefore, students struggle in their college Calculus courses. In an effort to improve students success and understanding of ideas in Calculus courses, researchers have identified the idea of constant rate of change as foundational to understanding linear functions, the idea of average rate of change, proportionality, and slope (Thompson, 2008; Byerley, 2016 & Coe, 2007). The studies also reported the disconnections students and teachers have as they conceive these ideas as separate sets of actions and associated with unrelated contexts (Lobato, 2006; Lobato & Siebart, 2002; Lobato & Thanheiser, 2002; & Coe, 2007). If students build meanings for the idea of constant rate of change that entails thinking about two quantities co-accumulating so that their increments are in constant proportion regardless of their size, they will be better prepared to conceptualize functional relationships in Calculus (Thompson & Carlson, 2017). Therefore, it is an academic interest to develop assessment items that are designed to assess students understanding and ways of thinking of the ideas of a quantity, change in quantity, variable, formula, ratio, rate, linear function, slope, an average rate of change, etc. that are related to learning and understanding the idea of a constant rate of change. In this report, I share a theoretical development of four assessment items from my current study of developing the Constant Rate of Change Assessment (CRCA) instrument. The CRCA instrument is informed by a theoretical report (Khan, 2021a) on conceptual analysis (Glaserfeld, 1995 & Thompson, 2008) of the idea of a constant rate of change and a proposed hypothetical learning trajectory (Simon, 1995; Simon & Tzur, 2004) of ideas to be foundational for understanding the constant rate of change. I will present a Constant Rate of Change Assessment (CRCA) Taxonomy that details the specific ideas and reasoning to be assessed by the CRCA instrument. I conclude by sharing clinical interview results and discuss how the clinical interview plays a role in developing the assessment items.

Background: Foundational Reasoning for Understanding the Constant Rate of Change

Mathematics educators and researchers have investigated students' understanding of proportionality, rate, and ratio at various mathematical grade levels, with findings that repeatedly manifest student difficulties in applying proportional reasoning when interpreting the rate of change in a modeling context (Tourniaire & Pulos, 1985; Doerr & O'Neil, 2011; Orton, 1983 & Yoon, Byerley & Thompson, 2015). In an exploratory study (Khan, 2020) with calculus students, I found that thinking about constant rate of change requires proportional reasoning (Lamon, 2007; Lesh, Post& Behr, 1988; Thompson, 1994), covariational reasoning (Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Thompson & Carlson, 2017), and quantitative reasoning (Thompson, 1994 & 2011). This section provides a brief description of the reasoning abilities and understanding students need to develop to understand the idea of constant rate of change and guide our effort to design items to assess students' understanding of the idea of constant rate of change.

A student engages in quantitative reasoning (Thompson, 1988, 1990, 1993, 1994 & 2011) as she conceives quantity as a measurable attribute of an object and conceives measuring it as a multiplicative comparison of two fixed quantities. A quantity is a student's conceptualization of an attribute of an object that can be measured (Thompson, 1990, 1993, 1994 & 2011). When a student is presented with a candle burning context where a 14-inch candle is lit and steadily burns until it is burned out, the student imagines measuring the original length (attribute) of the candle (object) or measuring the remaining length (attribute) of the candle at any elapsed time (attribute) since it started burning, she is conceptualizing quantities. Therefore, Thompson (2011) claimed that quantities exist in an individual's mind when she conceptualizes measuring the quality of an object that can assume a measure.

Proportional reasoning as a theory interacts with quantitative reasoning (Thompson, 1994 & 2011) as one conceives quantity as a measurable attribute of an object; she conceives a multiplicative comparison of two fixed quantities. She conceives ratio as a result of the multiplicative comparison of the quantities. She conceives rate as a proportional relationship between the measure of two varying quantities. Researchers have reported that student's ability to engage in proportional reasoning revolves around the understanding of rational numbers, fractions, the idea of ratio and rate, and over the years, educators have developed multiple definitions and distinctions of ratio and rate (Lesh, Post & Behr, 1988; Kieren, 1976; Lamon, 2006; 2007 Kaput & West, 1994 & Verganaud, 1983; 1988). Thompson (1994), Thompson & Thompson (1994 & 1996) suggested that the distinction between rate and ratio depends on an individual's mental operations on how she comprehends the given rate and ratio within a context. Thompson (1994) defined ratio as a result of comparing two quantities multiplicatively and defined rate as a reflectively abstracted constant ratio. When one conceives the idea of rate, she thinks about the characteristic of rate as two quantities are covarying. A person reconceives ratio as a rate when she applies the ratio to a different situation and thinks about the ratio as a rate that characterizes covariation between quantities. In other words, a ratio is a multiplicative comparison of the measures of two non-varying quantities, and rate is the proportional relationship between two varying quantities' measures (Thompson & Thompson, 1994). Researchers vaguely adopted the term proportional reasoning, and Lamon (2007) said that 'anything and everything related to ratio and proportion' (p. 637) is referred to as proportional reasoning.

When two quantities vary in relation to each other, the mental operation that supports the dynamic images in students' thinking is referred to as covariational reasoning (Carlson et al., 2002; Thompson & Carlson, 2017). When a student engages in proportional reasoning, she simultaneously engages in covariational and quantitative reasoning. A student engages in proportional reasoning when she conceives the invariant relationship of quantities in a dynamic situation or applies her understanding of proportionality in mathematical modeling situations. Constructing the idea of rate involves envisioning two quantities in a situation vary smoothly and continuously, and the changes (increases or decreases) in one quantity or variable's value is a simultaneous result of changes in another quantity or variable's value; and as the two quantities covary, the multiplicative comparisons of their measures remain proportional.

A student thinks about the idea of constant rate of change as relating two covarying quantities when changes in one quantity's value are proportional to the corresponding changes in the other quantity's values. She thinks about using variables x and y to represent the values of two quantities that change together, if the quantities are related by a constant rate of change, then $\Delta y = m. \Delta x$ where Δy represents the changes in y values and Δx represents the changes in x values and m is proportionality constant. The changes in y's values are m times as large as changes in x's values. The theory mentioned above informs the role of the related ideas- quantity (varying and fixed), covariation, changes in quantity's varying values, representation of quantities using variables, expressions, formula, variable (delta notation) to express any change in quantity's values, graphical representation of a dynamic situation, ratio, rate, proportionality, linear graph, functional relationship, etc. in understanding the idea of constant rate of change. The ideas involved in understanding the idea of constant rate of change are mentioned in the CRCA Taxonomy in the next section of the report.

The CRCA Taxonomy

The CRCA Taxonomy (Figure 1) includes three reasoning abilities that are foundational for learning the idea of constant rate of change. The taxonomy includes understanding various ideas related to the concept of quantity, proportionality, constant rate of change, and constant rate of change in a functional relationship. The CRCA is a 20-item multiple-choice exam with each question having five answer choices. The CRCA also includes three already validated items from PCA (Carlson, Oehrtman & Engelke, 2010) instrument and two items drawn and modified from Lobato & Ellis (2010). Each item assesses at least more than three understandings of concepts mentioned in the CRCA Taxonomy. In the next section, I will introduce four novel items that I designed based on the Taxonomy and provide brief rationale of how each item can assess multiple understandings of the CRCA Taxonomy.

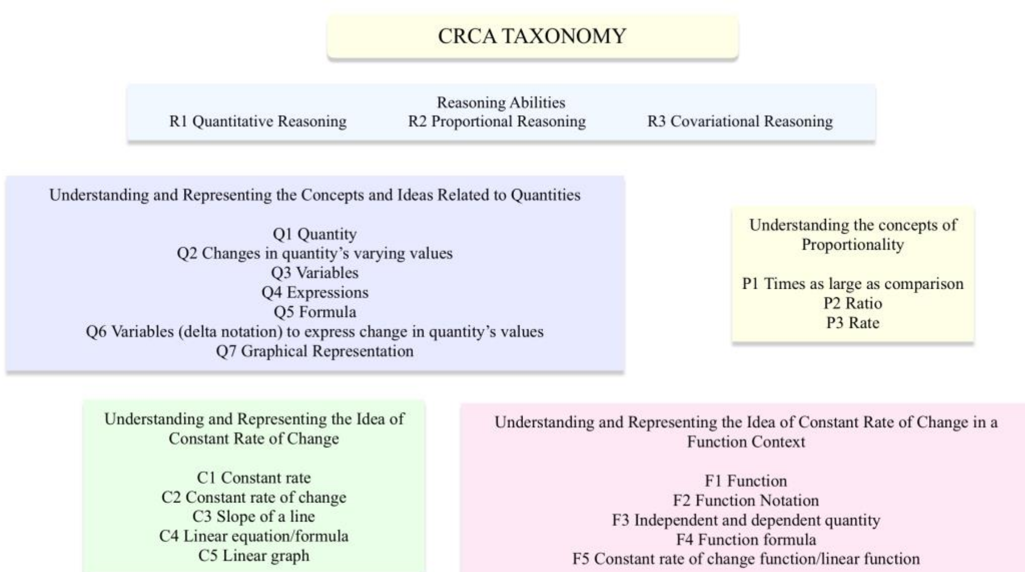

Figure 1. The CRCA Taxonomy

The Process of Developing the CRCA

I am using a four-phase technique to develop and validate the CRCA, suggested by Lissitz and Samuelsen (2007). Developing and validating an instrument is a time-worthy process. In this report, I only intend to set forth my ongoing effort of the first two phases in developing the CRCA. According to Lissitz and Samuelsen (2007), developing a valid instrument should always begin by identifying the concepts or ideas worthy of assessment. I worded questions and chose item distractors based on the observation of students thinking during nine semesters of consecutive precalculus teaching and analyzing exploratory teaching interviews (Steffe & Thompson, 2000) data on student thinking of the idea of constant rate of change (Khan, 2020 & 2021b). Next, I conducted interviews with students using open-ended items to identify and refine distractors based on students' thinking. I used the CRCA as a precalculus course assignment quiz after introducing the concept of constant rate of change and proportionality. The quantitative data from the quiz provides insights in favor of some chosen distractors and suggests further iteration. I plan to repeat clinical interviews (Clement, 2000) until each item is validated to clear interpretation, assess the concepts mentioned in the CRCA Taxonomy, and the distractors reflect students' thinking as suggested during the iterative interviews.

Here, I briefly mention the intended phases of the CRCA development-

Phase I. I began my investigation (Khan, 2020 & 2021b) by interviewing calculus and precalculus students, to answer the research question-how do students think about the idea of constant rate of change? The literature and results of my prior investigations provided insights that students' understanding of the idea of constant rate of change has a key role in improving students' understanding of other precalculus and calculus topics. This led me to develop the first collection of assessment items that will inform the foundational reasoning abilities and the foundational ideas to understand the idea of constant rate of change. The first pass of the development included identifying already validated items from PCA and other instruments relevant to the CRCA Taxonomy. I designed other items based on the data from the tasks I used in previous studies (Khan, 2020 & 2021b). I engaged with students in clinical interviews with the open-ended form of the items to ensure that students interpreted the wordings of each item as I intended. The students' interpretations were analyzed carefully to address the required changes to improve the wording of the items.

Phase II. To gain a sense of the quantitative performance of the assessment items and the use of the distractors, I used the items as a class quiz in the Spring 2020 semester in one section of the precalculus course. The students took the quiz after they completed class discussions and assignments based on the idea of constant rate of change. I noticed the variation of choosing different distractors among students with an average score of 59%. Based on the phases I and II insights, I went through another iteration of updating the assessment items.

Phase III. The next step includes circulating the CRCA instrument among precalculus sections taught by different instructors using different curricula in the beginning of the semester. Then based on the data, I will choose participants for another round of clinical interviews (Clement, 2000). I will select interview participants who receive less than the mean score. Their thinking in justification of the incorrect choices will provide crucial arguments to accept or reject the distractors before another iteration. I will remove the distractors chosen by fewer than 5% of students. In this phase, I will establish the internal content validity of the CRCA.

Phase IV. This phase will be essential to examine external measures to establish CRCA's validity as a tool for determining students' understanding of the idea of constant rate of change. During this phase, I will administer the CRCA as a pre-post-assessment to precalculus-level students. The pre-post-CRCA mean scores will be important to analyze the role of instructional intervention to the instrument CRCA.

Table 1. below presents the most recent version of four-novel items from the CRCA-

Table 1. Four novel items from the CRCA

Items	Choices
1. Jim tosses a pebble into a pond, and it produces a circular ripple that travels outward. Jim notices that the circumference of the circle increases at a constant rate as the radius of the circle expands. Which of the following formula represents a change in the circle's circumference with respect to a change in the circle's radius? Here, C represents the circle's circumference in cm and r represents the circle's radius in cm.	a. $\frac{C}{r} = \pi$ b. $C = 4\pi r$ c. $\Delta C = 4\pi\Delta r$ d. $\Delta C = 2\pi\Delta r$ e. $C = 4\pi r^2$
2.On Saturday Lee drove 200-miles on a road trip at a constant speed of 75 miles per hour. On Sunday he drove 2/3 as far as he drove on Saturday, while maintaining the same constant speed of 75 miles per hour. How long did Lee drive on Sunday?	a. 200/75 hours b. 2/3*200 hours c. 3/2*(200/75) hours d. 2/3*(200/75) hours e. 2/3*75 hours
3. What is the constant rate of change of y with respect to z, when $y = 525 - \frac{1}{3}z$?	a. 525 b. 1/3 c. 524.66667 d. -1/3 e. 3
A function f has a constant rate of change of 7. $f(10) = 132$. What is the value of $f(0)$?	a. 0 b. 132/7 c. 132-7/10 d. 62 e. 70

All the items mentioned above require students to conceptualize quantities as a first response. The items are followed by students' thinking about the relation between two or more quantities and covariation in the values of the quantities. Item 1 requires all three reasoning abilities from students, including the understanding of Q1, Q2, Q3, Q5, Q6, C2, C3, P1, and P2 from the CRCA Taxonomy. To answer item 2 with correct choice, students will think about the situation with reasonings R1 and R2, including the understanding of Q1, P1, P2, P3, and C1. Item 3 requires R1 and R3 reasonings and understanding of Q1, Q2, Q3, Q4, C1, C2, C3, and C4. Item 4 inquires students thinking about the idea of constant rate of change in a functional relationship. Students will think about quantities and their changing values by interpreting the function formula and meaning for a function to have a constant rate of change of 7. This item requires students' R1, R2, and R3 reasoning abilities and understanding of C1, C2, F1, F2, F3, F4, and F5.

Clinical Interview Results in Developing the CRCA

CC and Alexi (pseudonyms) were enrolled in Precalculus courses in a large southwest university in the US during the clinical interviews (Clement, 2000). They participated in 1 hour and 30 minutes long sessions with open-ended CRCA items at the end of the Fall 2020 term. The data was collected using zoom video/audio recording during the COVID-19 pandemic. Both participants were already exposed to the idea of constant rate of change as part of their coursework and used the annotation feature of zoom to share their scratch work. The qualitative data analysis method in this study is supported by grounded theory (Strauss & Corbin, 1994). The open-ended version of item 1 (Table 1) initially did not mention the respective variables to express the circle's circumference (in cm) and the circle's radius (in cm). CC exhibited difficulties interpreting the problem without given variable names. I encouraged her to use any variable names of her choice to write the formula. Alexi, on the other hand, was prompt in thinking about using variables of her choice, and she justified choosing $\Delta C = 2\pi \Delta r$ as her answer because "delta represents change, and the question is asking for to represent the change in the circle's circumference with respect to the change in the circle's radius."

While thinking about item 2 (Table 1), CC's first instinct was to correlate the problem with the distance formula, $d = st$ where d represents the distance (in miles), s represents the speed (in mile/hours) and t represents the time (in hours). She thinks that the distance and time in both days are proportional as Lee maintained a constant speed to drive during both road trips. She used t as a variable to express how long it took Lee to drive on both road trips but explicitly mentioned that t does not express the same quantity on both days. Alexi took a different approach; she made a drawing to make sense of the item. She then thinks that the distance and time for both road trips are proportional. She thinks that if the distance of the second road trip is 2/3's of 200 miles, then the time to complete the second road trip is 2/3's of the time to complete the first road trip. Figure 2 and Figure 3 below show their work (item 2 was numbered 4 during the interviews)-

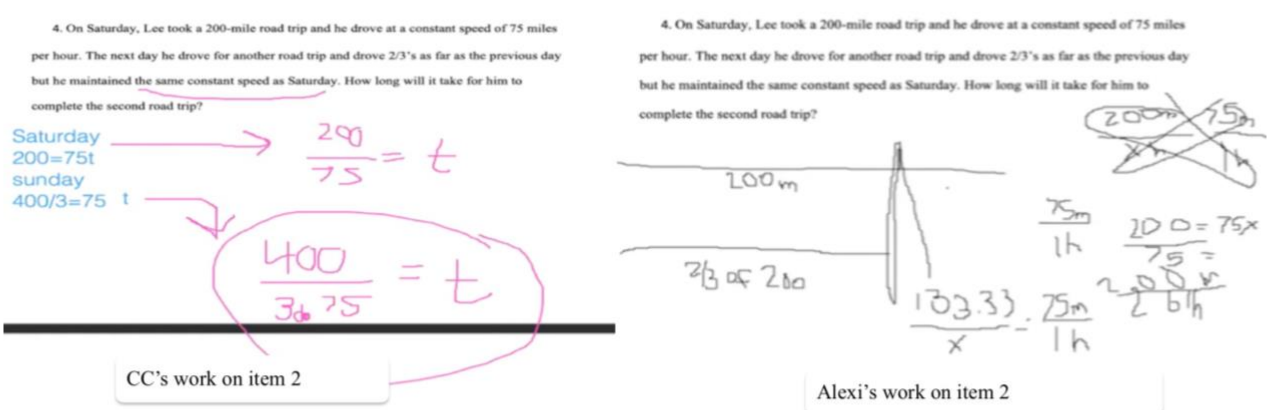

Figure 2. CC and Alexi's work on item 2.

CC and Alexi were initially thinking about finding pairs of values of (z, y) while working on item 3 (Table 1) and thinking about 'rise over run' formula to find the constant rate of change of y with respect to z in this context. They figured the slope of this equation would be -1/3 from the calculations and later noticed that -1/3 is the constant multiplier of z in the given equation that represents the slope. They reflected in their thinking and realized that the role of the constant multiplier in a linear equation is to express the constant rate of change when y and z covary and changes at a constant rate.

For item 4 (Table 1), CC thinks about an equation or formula she could work with, and she thinks about eventually solving for 0 as the item asked for the value of $f(0)$. Alexi, alternatively, thinks about the role of the constant rate of change of 7 in a function formula. She thinks that for every change in the independent variable's value of f, the dependent value will be "7*10 subtracted from 132". CC and Alexi both compared the imaginary function formula in this context with the general equation $y = mx + b$. However, CC first solved for b and gets 62 as the value using the point (10, 132), then she used the equation $y = 7x + 62$ to solve for y when x is 0. The interviewer then probed CC to think about what b represents in a linear equation. After some probing, CC realizes that the corresponding y-value when x is 0 represents the y-intercept, and the general variable b is used in the equation $y = mx + b$ to represent the y-intercept of the context.

Discussion and Future Work

The data from the clinical interviews (Clement, 2000) suggest that it is important to introduce variables to represent the varying quantities in an assessment item for students to make sense of the problem context. Therefore, I revised the CRCA items to define appropriate variables in all problem contexts. The quantitative data from the quiz suggest that 57% of students chose the correct answer for item 1. CC and Alexi's thinking for item 2 indicates that students are likely to use proportional reasoning to compare road trips. I have chosen the distractors as expressions for item 2 so that the students focus on the quantitative and proportional relationship between quantities' values rather than thinking about calculating values. 78% of the students chose the correct answer for item 2 in their quiz. For item 3, 525 and 524.66667 were popular answer choices among 30% of the students, suggesting the students might identify 525-1/3 as a multiplier of z or they might think 525 can be replaced with mx in a general linear equation $y = mx + b$. In either possible case, students need to think about each element of a linear equation to understand the idea of constant rate of change in a linear equation. CC and Alexi showed ways of thinking that led them to the correct value of $f(0)$ while working on item 4. However, 40% of the students chose 0 as the correct answer in the quiz and 38% chose 62. The students who chose the answer choice 0 might think there is no vertical intercept when the independent quantity's value is 0.

The data provided insights to revise the CRCA items with distractors that might reflect students' thinking. However, the development of an instrument is a process that will require more iterations to make sufficient comments on the content validity. The development of the CRCA is still in progress with more iterations and will proceed with phases III and IV in due time. The theoretical framework in designing each assessment item is easily overlooked when we discuss the overall development of an assessment instrument and the validity of the instrument to measure students' performance. Therefore, this report features the role of theory in building a taxonomy for an assessment instrument and how the theoretical framework and taxonomy inform each assessment item. I have presented brief data to support the process of theoretical development of four CRCA assessment items. The items are not limited to use only to assess students' thinking about the idea of constant rate of change. Researchers can use and modify the items as teaching materials or tasks to investigate how students think about the idea of constant rate of change.

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The Story of Circulating Conversations Methodology towards RUME Research Questions

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The goal of this paper is to convey an Indigenous research paradigm to the RUME (Research in Undergraduate Math Education) community in a way as authentic as possible. This paper stories my PhD research journey of applying an Indigenous research paradigm to research in undergraduate math education at Sitting Bull College (SBC). For this study, Circulating Conversations Methodology (CCM) was named as the theoretical framework with one of its key features as co-connecting knowledge. This paper shares the process of developing Circulating Conversations Methodology (CCM) within an Indigenous research paradigm and shares its results of four research questions. Within an Indigenous research paradigm, the process is the product of my research (Wilson, 2008).

Keywords: Indigenous research paradigm, relationality, co-connecting knowledge, tribal college, math curriculum and pedagogy

In seeking to follow Indigenous ways of knowing and being throughout every aspect of this research, how knowledge is transferred is equally significant with the knowledge itself (Kimmerer, 2013; Kovach, 2009; Wilson, 2008). The goal of this paper is to discuss my process of applying an Indigenous research paradigm which in turn led to the development of research questions. Through this experiential process, I am continually learning from Shawn Wilson (Opaskwayak Cree) that "the process is the product" (Wilson, 2008, p. 103). Within my understanding of an Indigenous research paradigm, the process to arrive at the research questions is equally as significant as the answer to the research questions. This process-centric and relational way of writing and view of knowledge will likely feel striking to a Western trained reader. In this paper, the research questions are part of the product, that is developed in process through CCM, and so will be shared near the end. I will begin this paper with the cultural protocol of introductions.

Introductions

Hau mitakuyepi. Ċhaƞte waṡteya nape ċhiyuzapi. Danny Luecke emaċiyapi. Fargo, North Dakota el wathi na Fargo emataƞhaƞ. Ina Kathy Jo Dahlgren eċiyapi. Ate Lenny Luecke eċiyapi.

In Lakhol'iyapi (the Lakota language) I said, hello my relatives. With a good heart I shake your hand. My name is Danny Luecke, and I am from and currently live in Fargo, North Dakota. I shared my parent's names in my desire to honor all my ancestors. I am from multiple European nations as well as Choctaw Nation and reflect upon the bind of embracing or neglecting my Choctaw heritage because of my predominantly white background, privileges, and experiences. I am honored by your interest in reading my work and learning with me. I pray that this proposal would strengthen the relationship between us and strengthen your relationships with Indigenous knowledges and Indigenous Peoples. There are so many connections I do not know. I strongly dismiss any notions of being an 'expert' (Kovach, 2009; Wilson, 2008; Windchief & Pedro, Ch.2, 2019). All I share with you are some of the connections I have made. I acknowledge the land I am from as the land of Oceti Sakowin, Anishinaabe, and multiple more Nations. I honor and thank them for their millennia of sustainably partnering with the Land as a living relative. In humility, I introduce Land as one who has been here long before any of us and will be here long after any of us (L. T. Smith et al., 2018, Ch.1).

While Western research demands the notion of objectivity, an Indigenous research paradigm embraces the clear articulation of subjectivity (Archibald, 2008; Grande, 2004; Kovach, 2009; Meyer, 2014; Wilson, 2008; Windchief & Pedro, 2019). A first-person introduction like this and a story writing style may seem unusual to you. It certainly was to me when I began learning about Indigenous research paradigms. Shawn Wilson wrote a seminal work that likely has the most influence on me and this research titled "Research Is Ceremony: Indigenous Research Methods" (2008). Wilson and his co-researchers developed a saying that I have embraced also. "If research doesn't change you as a person, then you haven't done it right" (Wilson, 2008, p. 135). I know I have changed dramatically through this process, personally as well as my professional views towards research and writing. I did not grow up participating in spiritual ceremony. I am learner to Indigenous ways of knowing and being. My continual greatest fear is not respecting the Indigenous knowledges and Indigenous Peoples that I am connecting with and learning from. Today, I am trusting Creator, my academic elders, and the relationships being developed through the research process to guide me.

Another seminal work towards an Indigenous research paradigm written by Jo-Ann Archibald, also known as Q'um Q'um Xiiem, (Stó:lo Nation) is titled "Indigenous Storywork: Educating the Heart, Mind, Body, and Spirit" (2008). In doing any research with Indigenous communities, she shares pivotal self-reflection questions addressing issues from past and ongoing colonialism within research. "Was I doing anything different from earlier 'outsider' academics who created a legacy of mistrust among First Nations concerning academic research? How was my research going to benefit the education and wellbeing of Indigenous peoples and their communities? How would I address ethical issues related to respect and ownership of Indigenous intellectual property?" (Archibald, 2008, p. 36). As I seek to do RUME at a tribal college, these questions help guide my work by guiding my heart, mind, body, and spirit.

Wilson elaborates in sharing that "As we Indigenous scholars have begun to assert our power, we are no longer allowing others to speak in our stead. We are beginning to articulate our own research paradigms and to demand that research conducted in our communities follows our codes of conduct and honors our systems of knowledge and worldviews" (Wilson, 2008, p. 8). Circulating Conversations Methodology (CCM) seeks to follow this demand in every possible way. Thank you for joining in unraveling this CCM journey towards RUME research questions.

Indigenous Research Paradigm

While reading "Research is Ceremony" (2008) the first time, I wrote down in my notebook "Relationality is the sum of the whole Indigenous research paradigm." Going through the book for a third time months later, the actual quote reads, "Relationality *seems to* sum up the whole Indigenous research paradigm *to me*"{emphasis added} (Wilson, 2008, p. 70). This epiphany moment struck my heart and mind. In my first reading, I had removed the subjectivity and opted for a more definitive way of knowing. It was not until the third reading, and after a discussion with my mentor, professor of education, Dr. Hollie Mackey (Northern Cheyenne) about my absolutist writing style at that time, did the revelation come that my reading and writing patterns were not matching the subjectivity inherent within relationality (H. Mackey, personal communication, September 21, 2020). I was reading the seminal pieces with an eye for the single precise definition for an Indigenous research paradigm so I could extract that out of context into my work (L. T. Smith et al., 2018). The single definition for an Indigenous research paradigm is **not** written in any of the seminal works, which fully aligns with the paradigm itself. There is no one way to apply an Indigenous research paradigm! There **cannot** be one way because it is dependent on all relations. This may include spirituality, a specific place, a specific language and culture, and certainly a dependence on the researcher and participants themselves (Archibald, 2008; Kovach, 2009; Wilson, 2008).

Relationality to me is the idea that everything is in relationship (Wilson, 2008), that everything [including knowledge] is alive and connected (Meyer, 2014). Wilson taught me that it goes beyond the idea that I have a web of relationships to I **am** the web of relationships. This is not for humans only, knowledge as a living entity does not have relationships, but knowledge **is** relationships. This reality of nature and knowledge is distinct from constructivism that centers human knowing (Hatch, 2002; Kovach, 2009). Wilson says relationality to him is that "relationships form reality" (Wilson, 2008, p. 137). I laughingly remember the essence of relationality via seeing relationality as a contraction of the two words relationship and reality. Mathematically, it may be seen as emphasizing the study of the edges instead of the vertices. This assumption about the nature of reality, that is ontology, impacts not only research but perspectives about science and math as living entities themselves (Kimmerer, 2013). Greg Cajete (Santa Clara Pueblo), a well-known Native scientist, is quoted by Manulani Aluli Meyer ('Ōiwi Hawai'i) by saying "The perspective of Native science goes beyond objective measurement honoring the primacy of direct experience, interconnectedness, relationship, holism, quality and values, and they are specific to tribe, context, and cultural tradition" (Meyer, 2014, p. 98). Wilson (2008) brought me to tears as he shared a metaphor describing relationality applied to knowledge, and therefore my responsibility to the knowledge in its relational context.

> So the way I see it, gaining knowledge is more like being married to someone – you don't own your spouse or children but you do share a special relationship. It is a relationship that you are accountable to. And therefore it becomes cultural appropriation when someone comes and uses that knowledge out of its context, out of the special relationships that went into forming it. You have to build a relationship with an idea or with knowledge, just like you have to with anything or anyone else… For someone else to come along and use this knowledge in an inappropriate manner is like raping that relationship. You know that sexual exploitation and total denigration of our humanity was a big part of colonialism. Now that is taking place with our ideas and knowledge. Our knowledge is being stripped of its relationships and being used without accountability. (p. 114)

This metaphor hits the heart, body, and spirit. I can feel the knowledge and perhaps you may too. This metaphor not only helped me crystallize knowledge as relational (and therefore personal, subjective, experiential and holistic) but also demonstrated the obligation of responsibility and accountability I have towards the Indigenous knowledges and Indigenous Peoples I learn from. Linda Tuhiwai Smith (Ngāti Awa, Ngāti Porou) addresses the specific connection between research and Indigenous Peoples in her high impact book "Decolonizing Methodologies" (1999, 2012). Here are the first words of her introduction.

> From the vantage point of the colonized, a position which I write, and choose to privilege, the term 'research' is inextricably linked to European imperialism and colonialism. The word itself, 'research', is probably one of the dirtiest words in the Indigenous world's vocabulary. When mentioned in many Indigenous contexts, it stirs up

> silence, it conjures up bad memories, it raises a smile that is knowing and distrustful… At a commonsense level research was talked about both in terms of its absolute worthlessness to us, the indigenous world, and its absolute usefulness to those who wielded it as an instrument. It told us things already known, suggested things that would not work, and made careers for people who already had jobs. (p. 1-2)

As Indigenous communities/nations are asserting their sovereignty, there is a growing demand for research by, for, and with the community towards an indigenizing or decolonizing outcome (Kovach, 2009; L. T. Smith, 1999; Tuck, 2009; Wilson, 2008; Windchief & Pedro, 2019). This demand fit with my experiences. When I first read the quote above, I recalled an experience I had a couple months previous with a tribal college administrator who strongly warned me of parasite research. With an adamant tone, the administrator declared 'We are stopping it here!' Not fully understanding what was meant by the declaration I sheepishly asked what was meant by parasite research. The administrator continued that parasite research(ers) take, take, take, and give nothing back. They show up for a short period of time to extract data solely for their own benefit and then disappear, giving nothing back to us or the community.

An Indigenous research paradigm, grounded in Indigenous knowledges, moreover a tribal-specific knowledge and language, emphasis giving back to the community and strengthening all relationships in the process (Archibald, 2008; Kovach, 2009; Wilson, 2008). From my viewpoint, themes of relationality, subjective knowledge, holism, and story seem to circulate with values of responsibility, respect, and reciprocity to form the dynamic and place-based research paradigm (Archibald, 2008; Kovach, 2009; L. T. Smith et al., 2018; Wilson, 2008; Windchief & Pedro, 2019). However, my understanding of an Indigenous research paradigm is only my understanding. Each person, including you, will connect with it in their own way and join in the joint responsibility of being in relationship with an Indigenous research paradigm (Archibald, 2008; Kovach, 2009; Wilson, 2008).

Circulating Conversations Methodology (CCM)

With a context of hurtful research with Indigenous Peoples and with the core of an Indigenous research paradigm centered around relationality and relational accountability, I began the research for my PhD with Sitting Bull College (SBC), where multiple personal and professional friendships had already been established. SBC is a tribal college chartered by Standing Rock Nation guided by Dakota/Lakota culture, values, and language. I was confident that I could not come in with my research questions, framework, or agenda. I was confident that I wanted to do research that was beneficial and actionable for the SBC math instructors and that outside of directly talking with them I had no aspiration of thinking I could determine that on my own. Two statements rang in my ears after a discussion with Dr. Josh Mattes, engineering/math instructor at SBC. From his perspective, looking at the intersection of math and Dakota/Lakota language and culture was an "excellent idea" and that "even minimal results here would be beneficial [for SBC math instructors]" (J. Mattes, personal communication, September 25, 2020).

Within my literature review, I found precisely one article about collegiate math and Indigenous languages. It excited me showing a potential path but also warned of difficulties with the delicate relationship between math education research and fluent elders (Ruef et al., 2020). Due to the implications of the COVID-19 pandemic, a time of waiting, struggle, research roadblocks, prayer, and Choctaw identity development became the norm. I saw no clear path forward. Then at the end of January 2021, a spiritual moment of connection brought Sunshine and I together for our first meeting. Sunshine Carlow (Lakota) is an instructor and the financial manager for Lakȟól'iyapi Wahóȟpi Wičhákini Owáyawa (Lakota Language Immersion Nest) located at SBC. This new relationship made a way to learn about the intersection of undergraduate math education and Dakota/Lakota language and culture.

After initial introductions and first interviews with Hollie, Sunshine, Josh, and my advisors, more focused conversations happened within the next week and a half about potential research directions at the intersection of undergraduate math and Dakota/Lakota language and culture. With all the conversation notes in front of me, drawing upon what I remembered hearing, I sought to holistically (heart, mind, body, spirit) connect all the ideas together. This time of synthesis formed an initial one-pager with four first draft research topics looking at content, development methodology, student affect, and faculty experiences. I circulated amongst Josh, Sunshine, Hollie, and my advisors to connect with each of them and listen to their feedback on the initial four topics. Again, seeking to holistically connect each of their responses altogether illuminated two topics. Content and development methodology becoming research question 2-4 and 1, respectively, through a final round of conversations with each key person.

In a spiritual moment of epiphany midway through, I came to realize the pattern of relationships I was enacting **was** literally a web. My experiential journey of conversations with each key person was my theoretical framework! I named it **Circulating Conversations Methodology (CCM).** Although time moved forward as CCM happened, the connecting of themes and ideas was anything but linear as shown in Figure 1 below.

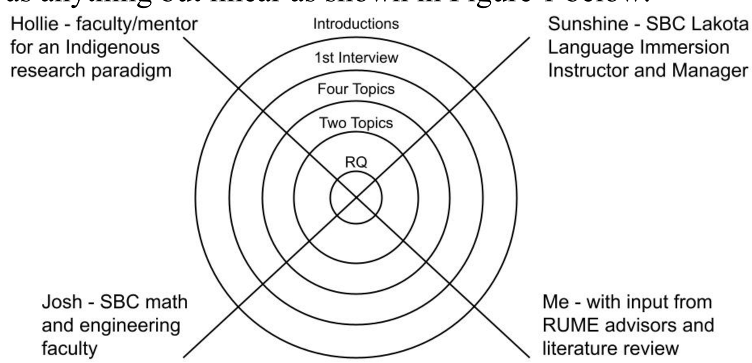

Figure 1: A diagram showing that Circulating Conversations Methodology is a circular web.

Like a spider web that glistens and waves with the wind, each intersection point between circles and strands is unique. The circles represent different stages, moving from one to the next through moments of synthesis. The strands represent central people in the development of the research questions. Each intersection point is an essential conversation in the web. Conversations followed Kovach's conversational interview protocol (2010). The RUME research questions were the end goal of this particular CCM that brought together multiple people, ideas, value systems, and institutions. Hollie connected her knowledge of an Indigenous research paradigm and Indigenous Knowledges. Sunshine connected her knowledge of Dakota/Lakota language and culture. Josh connected his knowledge of teaching math and pre-engineering courses at Sitting Bull College. I and my advisors connected with an impetus for my PhD research and our PhD level understanding of mathematics. Through circulating conversations, the research questions developed through an iterative, circular, and collaborative process.

Co-Connecting Knowledge: Relationships Form Reality

Circulating Conversations Methodology (CCM) is based in an Indigenous research paradigm. Within this paradigm, theoretical frameworks and/or methodologies are as diverse as in Western research paradigms. Until we can articulate the vast array of these theoretical frameworks, they may be seen as vague or fuzzy (H. Mackey, personal communication, February 19, 2021). To attempt further clarity for CCM, one critical component is co-connecting knowledge, a term like CCM that was developed through the process. CCM is not haphazardly talking to a few different people. It is specifically based in the ontology (nature of reality) and epistemology (nature of thinking and knowing) of relationality. As Hollie and I discussed the initial four research topics, we recognized our word choice of "co-constructing knowledge" was a Western term that was distinct from the activity we were doing. We found ourselves stuck in Western terminology "to describe something that's far more nuanced" and desired to "come up with something that actually catches what it is" (H. Mackey, personal communication, February 19, 2021). Co-connecting knowledge became that term. None of our conversations constructed, created, found or discovered new knowledge. Rather it was the collaborative connecting via conversation and story that new relationships/knowledge developed.

Co-connecting knowledge describes the space where theory from the literature can connect with personal experiential knowledge in practice. It describes the space where intellectual knowledge can connect with spiritual knowledge (Meyer, 2014). It allows a relational worldview to connect with the neuroscience that says learning is new connections in the brain. Plus, co-connecting knowledge gives the space to connect all of these connections together holistically. Co-connecting knowledge aligns with an ontology and epistemology of relationality where knowledge is not owned, discovered, created or constructed but rather "knowledge is shared with all creation… the idea belongs to the cosmos, to all of the relations that it has formed, not to the individual who happens to be the first to write about it" (Wilson, 2008, p.56, 114).

Co-connecting knowledge instead of co-constructing knowledge is one example of how an ontology (nature of reality) and epistemology (nature of thinking and knowing) based in relationality is distinct from Western research paradigms (Grande, 2004; Kovach, 2009; Wilson, 2008; Windchief & Pedro, 2019). Some Western frameworks/methods are popular in Indigenous communities such as participatory action research, critical/feminist paradigms, and constructivist paradigm because the expansive intersection in seeing knowledge as personal, subjective, and political, recognizing a larger meaning to the mantra 'knowledge is power' (Grande, 2004; Gutiérrez, 2012; Kovach, 2009; Sfard, 1998). However, these Western frameworks are still based in Western constructs such as human-centrism and progressivism (Grande, 2004; Kovach, 2009). For example, Gutiérrez's work (2012) in equity recognizes math education as going well beyond individual intellectual capacity. However, math is still viewed through a human-centric lens. Similar can be said for Sfard's work on using multiple metaphors for learning (1998). The ontology and epistemology of these frameworks are not based in relationality and Indigenous knowledges. An Indigenous research paradigm is distinct in its decolonizing aim, tribal-specific knowledges, and knowledge being bound to place through ancestors, language, and land. Further, an Indigenous research paradigm can certainly include quantitative methods as well (Grande, 2004; Kovach, 2009; Windchief & Pedro, 2019).

Scientific/Academic Rigor

A distinct ontology and epistemology of relationality and a distinct set of values of being accountable to all these relationships through respect, responsibility, and reciprocity leads to distinct validity measures. Scientific/academic rigor in my understanding is the alignment of ontology (what is real?), epistemology (how do I know what is real?), methodology (how do I find out more and explore this reality?), and axiology (what moral beliefs will guide this search for reality?) (Wilson, 2001, 2008). Even if a research project meets the Western standards of judgement, like validity and reliability, but does not show respect to the relationships between researcher, participants, topic, Land, and community it would be considered inauthentic or non-credible within Indigenous research paradigms. Wilson (2008) explains,

> We don't need externally imposed measures or tests of whether or not something is 'true,' we have our own ways of ensuring this. We have our own ways or questions to ask, so that we know that what we are saying is strong enough to say, 'Yes, we can go ahead and design a program for our children or our community based on what we have learned from this research." And we have trust or faith enough so that we are willing to use this in our communities, for our own people. (p. 102)

I have been conditioned and trained into a specific ontology through my Western education. In contrast, to remind myself of the heart and core of an Indigenous research paradigm I often look back to the words of Margaret Kovach (Plains Cree/Saulteaux). In part she shares, "The sacredness of Indigenous research [and knowledge] is bound in ceremony, spirit, land, place, nature, relationships, language, dreams, humor, purpose, and stories in an explicable, holistic, non-fragmented way" (Kovach, 2009, p. 140).

I attempted to follow this holistic way through Circulating Conversations Methodology (CCM). In that attempt I learned that the process of determining the research questions is equivalent in significance as the research questions and results themselves. Without this CCM process, the four research questions for the next phase of this research would not exist in this way whatsoever. Looking back in my reflective journal I see how much I and my attitudes towards my PhD research have changed. I am indeed continually experiencing and learning that "the process is the product" (Wilson, 2008, p. 103).

Research Questions

The four research questions that were co-connected via CCM are:

1. In what ways can an Indigenous research paradigm lead an individual researcher towards more ethical and impactful (beneficial and actionable) RUME at TCUs?
2. In what ways can Western higher order math concepts (HOMC) be *identified within* Dakota/Lakota space, place, and language, to inform possible SBC math curricular/pedagogical adjustments for TCU math courses?
3. In what ways can Dakota/Lakota culture and language be *identified within* Western HOMC, to inform possible Lakota Language Immersion Nest curricular adjustments?
4. In what ways can Dakota/Lakota space, place, and language *represent* non-Western HOMC?

I did not come up with these research questions myself. I did not choose a topic or gap in the literature. Rather I chose some values and a process, that is an Indigenous research paradigm, and it guided me throughout. I did not come into SBC with my agenda for research to be done on Indigenous communities. Instead, every word of the research questions has a specific moment of co-connecting knowledge through CCM that brought that wording or idea about.

The goal of this paper was to introduce an Indigenous research paradigm through my journey of CCM. Future RUME at tribal colleges could potentially use this research paradigm and theoretical framework. My work at SBC has built from these research questions and will hopefully be presented in the future. Thank you for joining me in unraveling this journey. I pray that you were able to connect holistically with some of this writing and it can be beneficial to you and your work. In Choctaw, Yakoke. In Lakota, Pilamayayelo. In English, Thank you.

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Strengthening the RUME field: substantive research questions and suitable claims

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In this report, we discuss a common type of methods found in RUME research: qualitative coding techniques. Claims stemming from such an approach depend on warrants for the appropriateness of the methods and data collected. Yet, we commonly find methodological misalignments between the types of research questions asked and the methods used to investigate these questions. When the result of such research is often a framework with relative counts. Thus, researchers are making implicit claims that rely on statistical generalizability that is unwarranted by common sampling techniques. We advocate for future researchers to examine the alignment between their research questions and methods and move beyond producing frameworks as results to using these frameworks to explore deeper relational questions.

Keywords: research questions, methodologies, grounded theory, qualitative coding

Qualitative coding techniques, like those used when conducting research "in the spirit of" (or "inspired by") grounded theory (iGT) or thematic analysis, are commonly applied in undergraduate mathematics education research. In these studies, researchers usually collect data by recording classroom lessons or conducting task-based interviews. They transcribe the recording, and then perform inductive or deductive coding techniques (or sometimes use content analysis) to reduce the data to sets of categories. In RUME, the category set is often referred to as a framework. The framework is reported with (relative) frequencies, and the authors attempt to interpret descriptive statistics for the categorical variables, which usually involves comparing frequencies across categories or perhaps across demographic variables. These frameworks tend to be paired with questions like:

(Conception-RQ) How do undergraduate students conceive of topic X? Or

(Belief Comparison –RQ) To what extent do instructors hold different types of beliefs $\{X_i\}$?

Such questions are often methodologically misaligned with the types of samples and types of analysis techniques applied. If we turn to the commonly used Toulmin's argumentation scheme (1958), we can point explicitly to the role of flawed *warrants.* The strength of *claims* that answer these questions rely on the type of *data* collected. To make a claim that the set of student conceptions is exhaustive, or to claim one type of belief is more prevalent than another, researchers would need to move beyond convenience samples and counts. In Figure 1, we outline a Toulmin's argumentation diagram from a common type of claim (and rebuttal), found in manuscripts that report iGT methods. The data tends to reflect convenience samples and the claims rely on exhaustive/representative samples to make assurances of fully capturing all conceptions. In this sense, the typical *warrant* is insufficient because iGT methods do not allow for connecting between a convenience sample and such a claim. Further, such results are often paired with a limitation around sample-to-population generalizability that at best weakens the tie between data and claim and at worst completely invalidates the inference.

In our experiences reviewing for journals and for this conference, we often find these mismatched questions and claims. This leaves such manuscripts and proposals unable to make a compelling case for their contribution. In this report, our goal is reflect on the types of research questions that *are* answerable with iGT methods to advocate for a shift away from questions such as those above (*singular focus*) into deeper explorations of relationships, causality, and change (*relational focus*) that can be addressed productively with such methods. We will discuss and synthesize methodological essays on conducting high quality qualitative research, focusing on formulating research questions, and matching them to iGT methods contextualizing in undergraduate mathematics.

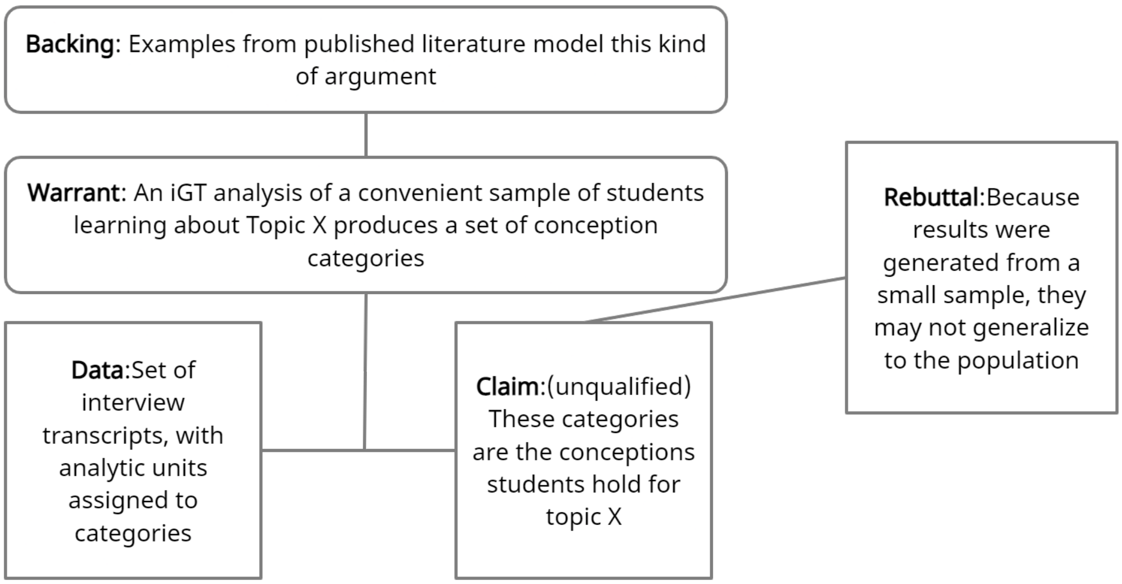

Figure 1. A Toulmin's Argumentation Diagram for Typical Claims from a "Framework Paper"

Analyzing Qualitative Research Questions

Formulating research questions means attending not only to their content but also toward foreshadowing their methods and their answers. Maxwell (1996) outlined four types of research questions that are relevant to our considerations: *generalizing*, *particularizing*, *variance*, and *process*. Generalizing questions are about a population. *Conception-RQ* is phrased as a generalizing question about students. Answering them requires selecting a sample from the population that will allow for generalization of any results about the sample back to the population. Particularizing questions are about populations *within a particular social context*. Particularizing questions frame studies as cases *of* some larger phenomenon. The focus is on "developing an adequate description, interpretation, and theory" of that case (p.89-90) rather than generalizing findings from the sample to a broader population. Variance questions are about differences and correlation. They lead with *does*, *how much*, *to what extent*, and *is there* and call for analyses that explain differences in some outcome within the phenomenon of interest. *Belief Comparison –RQ* is a variance question, seeking a univariate distribution of individuals across beliefs. Finally, *process* questions focus on unpacking how outcomes come about, rather than on whether a pair of explanatory variables is related or the extent to which an outcome is explained by a given variable.

We see that *Conception-RQ* and *Belief Comparison –RQ* are actually not robustly qualitative questions that would be well-matched to iGT methods. Answering *Conception-RQ* suggests a survey of students' responses to mathematics tasks that draw on topic X while *Belief Comparison –RQ* suggests a survey of instructors'' beliefs. We do not suggest that authors formulating these questions have confused qualitative approach with survey methods or that these questions cannot be answered qualitatively, but rather that though the answers to these questions could be found, through iGT procedures the *type of answer* would be similar to what one would expect as when conducting a survey, namely, a list of categories with (relative) frequencies for each category. What is the contribution of a list of categories, without knowing that the list is (likely to be) exhaustive? What is the contribution of a set of (relative) frequencies for those categories without knowing that they were generated from a representative sample? The issue with addressing generalizing questions with iGT methods is that they call for the researchers and the audience to make statistical inferences without assuring the validity of such an inference. The issue with addressing variance questions is that qualitative methods are not capable of doing so. The choice to focus on differences between groups defined along dimensions arising from the analytic constructs can lead to shallow analyses and weak conclusions because, in quantitative terms, the cell counts are often not sufficient in size to warrant claims bout comparisons between groups. Thus, whether an outcome of interest is related to another variable across analytic units and the extent of the relationship within or across analytic units is best treated by quantitative methods.

The *How?* Question

How? is a common start to a qualitative research question. Indeed, a popular heuristic for formulating qualitative research questions is to begin them with *How…?*. This leads to trouble when the research design for the study is still vaguely quantitative, as with variance questions. For example, *How does professors' mathematical philosophy influence their grading?* Such a question, phrased as a *how?* question suggests looking at variation in grading policies associated with variations in mathematical philosophies. Aside from being potentially deceitful, *how* questions are often imprecise. There are at least three operationalizations of a *how?* question that determine the kind of answer it should receive:

1. *in what ways* is answered by a descriptive list,
2. *what are the steps* is answered by a sequence or procedure, and
3. *under what combinations of conditions will the outcomes occur* calls for an explanation or prediction.

Judging by the answer/question pairings in RUME papers, the *how* in *Conception-RQ* is typically unpacked as *What are the different ways students conceive of topic X*? Its answer is a comprehensive list. Unpacking *Conception-RQ* in the second sense might involve conducting a set of cognitive interviews with a diverse array of students to generate a hypothetical learning trajectory for topic X. Unpacking *Conception-RQ* in the third sense might suggest using a teaching experiment or variation theory to generate multiple adjacent learning environments to provide evidence for the kinds of pedagogical materials and supports that result in an adequate conception of topic X. Neither of the second two senses are amenable to iGT methods, and an answer to the first sense might be difficult to interpret because of the implicit claim that such a list exhaustive, but rarely developed using a sample to warrant such a claim.

Multi-Dimensional Research Questions

In contrast to the examples above, we suggest iGT methods are best aligned with relating two or more dimensions of interest, within and across participants. We use Spradley's (1980) social dimensions as a tool to operationalize such questions. These dimensions can be found in Table 1, and for the purposes of mathematics education, we expand upon the object category to not just consist of physical objects, but also mental entities. Although Spradley introduced these dimensions in the context of ethnography, we find them equally amendable to addressing research questions that are answerable by iGT methods. We use them to add structure to research questions. A research question can have either a *singular focus* or a *relational focus*. Singular focus questions have a form like "Can you describe all the ways…?", such as *Concept-RQ* or "can you compare counts of all the ways?" Questions with a relational focus include at least two dimensions and are answered via claims that attend to the nature of the relationships among the dimensions rather than through numerical comparisons of variation within the dimensions. iGT procedures enable deep analysis of those relations and are suited to address research questions with foci like the meanings participants attribute to events or the influence of the social context on the events that unfold. These can include descriptive questions, which ask about "what happened" in terms of (potentially) observable (or inferable) behavior or events, interpretive questions, which ask about the meanings of these things ascribed by the people involved (involving thoughts, feelings, and intentions), or theoretical questions, which ask about *why* these things happened or how they can be explained (Maxwell, 1992). Because there are many explanatory variables in qualitative research, a good heuristic is "to assume that different combinations of causes might land a set of analytic units in the same cell" and then seek to elaborate the set of causes leading to that outcome[1] (Ragin, 2004, p. 137). To concretize an example in the RUME setting, consider the research question:

- How does the questions instructors ask while orchestrating discussion in a calculus lesson support student engagement in authentic mathematical activity?

This question includes a number of dimensions: an activity (orchestrating discussion), an event (a calculus lesson), acts (questions), and activity (student mathematical activity.) Taking a relation lens, we can further subdivide this question into context components (orchestrating discussion; calculus lesson) and focal relationship under investigation (questions; student mathematical activity). An adequate framework operationalizes question types and a series of acts or description of student mathematical activity. Then applying the framework can provide insight into how particular teacher questions may relate/promote/constrain students' mathematical activity.

Table 1 Spradley's (1980) dimensions of social contexts with examples from RUME contexts

Dimension	Definition	RUME Example
Space	The physical space or places	Classroom and its configuration
Actor	The people involved	Undergraduate Students
Activity	A set of related acts people do	Orchestrating a Discussion About Derivatives
Object/Knowledge	The physical or mental entities that are present	Conceptions of Derivatives
Act	Single actions that people do	Questions an Instructor Asks
Event	A set of related activities that people carry out	A Calculus Lesson
Time	The sequencing that takes place over time	Stages of Problem-Solving

Goal	The things people are trying to accomplish	Engender Students in Authentic Mathematical Activity

Reporting Framework Studies with Warranted Claims and Substantial Contributions

In this section, we provide some general reflection on ways that we have seen iGT methods reported with attention to making warranted claims and substantial contributions.

Distinguishing between making framework and using a framework and reporting appropriately

A simple heuristic is that inductive coding procedures create frameworks while deductive coding procedures apply them. The former is a set of inferences; the latter is a set of starting assumptions. The main distinguishing feature between inductive and deductive coding is that inductive procedures develop the codes *from the data* whereas deductive coding procedures are decided *a priori*, usually with an already-existing codebook, a set of codes developed from a thorough literature review, or using non-interpretive labels (e.g., correct/incorrect). In either case, the decision to code inductively or deductively should be justified in relation to prior literature and the focal research question. Inductive coding procedures are used to generate explanations of phenomena that are 'close' to the data, in the sense of Glaser and Strauss's (1967) use of the term grounded theory, because the codes are generated from the text, the researcher's descriptions of observations, or the participants' own words. Open (or *in vivo*) coding is only the first step of this bottom-up approach. Codes then usually need to be refined, combined, deleted, or supplemented. In Grounded Theory, and sometimes in iGT, open coding is followed by axial coding (developing a hierarchy, for example) and selective coding (choosing a category representative of the core or essence of the research topic). Reporting on an inductive coding approach should be accompanied with a rigorous description of how the codes and themes were developed. While axial and selective coding are not strictly necessary to generate interesting results (especially when using iGT procedures, rather than fully implementing a Grounded Theory methodology), we note that typically it not sufficient for publication to create a categorization scheme. In contrast, deductive coding procedures use an existing framework, or one developed from the literature, to analyze data. The deductive approach is appropriate when there is already active research in an area. For example, there are ample categories describing levels of sophistication of the mental actions essential to covariational reasoning (see Thompson & Carlson, 2017, for example). The definitions for the existing codebook can be compared to the data collected and applied in a systematic way. Given that previous work exists on the topic, it is likely neither necessary nor desirable to step back all the way to open coding when conducting a study about students' covariational reasoning. More concretely, if the authors are using an existing framework, they are not open coding and so the methods should not be reported as such. The authors should be clear about the manner(s) that their coding procedure diverges from prior use of a framework, including amending definitions or grain-sizes, as well as tender a rationale for why the pre-existing theories are appropriate for their setting.

Using (Relative) Frequencies as context, not claims

Framework studies often report frequencies of observations for each of the categories in the framework. While they can provide context, comparison claims are unwarranted. For example, suppose that a set of differential equation lessons are coded with an observational protocol with several types of teaching moves categorized as high, medium, or low. Now, suppose a researcher records a single lesson from a sample of 10 instructors. In one lesson, they observe m instances low teaching moves with a total of N coded teaching moves. Reporting that "m/N percent of the time, the teacher did m" would only be true if *all times in the classroom* were coded. That is, only if there were full coverage of classroom time by the codebook. Similarly, one must take care when comparing m/N for different classrooms or even across multiple lessons, if those environments did not afford opportunities to exhibit all N codes. Even less suitable are claims that teachers or classrooms or students could be assigned to a single category. If the grain size of analysis is "moves" (or "utterances") it should be carefully considered how to transform the unit of inference to "teacher," "student," or "classroom."

Additionally, evidence for the existence of a code is not a strong argument for its centrality to a phenomenon since the code may exist only as an artefact of the researcher's lens on the observations. Thus, we advocate for using frequencies to contextualize the data rather than as results. Without deploying a representative or exhaustive sample, frequencies are may be artefacts of the study rather than serve as data for claims. However, frequencies can serve an important role in exploring relationships between categories via co-occurrences of categories under investigation or identification of non-confirming cases.

Sampling Purposefully to Answer an RQ

The field often uses convenience samples because it is difficult to gain access to natural educational settings and because working in RUME, our population sizes are rather small to begin with. Yet, the power of an inference is limited by the sample and the educational environment that produced it. If a sample is not representative, the researcher should be transparent about sample selection and tender a rationale to warrant why *this* sample provides the appropriate insight. For example, purposeful sampling can be a very strong approach when the participants (or lessons or classroom sessions or tasks) are selected along dimensions important to the research question or because they are central to the theories or constructs under consideration. A purposeful sample can also serve to assure that any inferences or findings "adequately represent the entire *range* of variation" (Maxwell, 1996, p. 72) (p. 72). Shadish, Cook, and Campbell (2002, p. 350) advocate for articulating the prototypical features of constructs, how the features relate to each other, and how the constructs of interest relate to other constructs similar and dissimilar to them. From such an analysis, a purposeful sample can be selected. Another approach to sampling is theoretical saturation which does not rely on a number of participants set in advance; data are collected from new participants (or additional task environments or additional classroom observations or additional scholastic contexts, depending on the analytic dimensions relevant to constituting the data set) until the researcher is confident that new information would produce few or no changes to the developing codebook (see Guest, Bunce, & Johnson, 2006, p. 65). Theoretical saturation can support claims that categories are exhaustive because additional data is not producing distinct categories.

In any case, to produce a claim of any strength in answer to a research question that relies on counting occurrences of categories, the authors must provide some sense that the methods produced an accurate list of the categories *and* that the distribution of (relative) frequencies observed for those categories can reasonably be interpreted as usual. For counts to lead to meaningful comparisons, such as for variance or generalizing questions, an argument needs to be made that the sample is representative or exhaustive. This is pairing variance or generalizing questions with iGT methods and is rarely methodologically sound.

Reporting "which people were included and why?" is not the only relevant consideration. Tasks, classrooms, and lessons are *also* sampled in any study. These are analytic choices which impact the strength of any resulting claims and therefore should be justified in the methods and included in the discussion. Attending to these analytic choices is especially important when the research questions are about reasoning or activities or beliefs, because the learning environment or interview protocol will occasion the behavior, conceptions, or perceptions that can be observed.

Making claims that *can* be generalized

Finally, we briefly address the complex issues surrounding delimiting the study and characterizing its contribution. As we mentioned, in small-scale qualitative research, statements about generality are not derived from sample-to-population inference. We see a lot of papers, many conference proceedings, some manuscripts for review, and some even published, that conduct a thorough analysis, make important claims, and then apologize that "the sample wasn't big enough to be able to generalize." This kind of admission negates the entire study and betrays a misunderstanding of the relationship between sampling and generalization. Sample size is not what determines sample-to-population generalizability; representativeness of the sample determines the validity of *statistical inference*. When conducting an iGT study, statistical inference is rarely the purpose. The goal of a qualitative study is not to generalize to the population but to account for "as many of the temporal/contextual variables as may be necessary so that the generalization *will* hold" (Lincoln & Guba, 1985, p. 116) in other settings. Another approach to generating, testing, and generalizing the explanations and relationships observed in an iGT study is to base claims in arguments about necessary conditions to achieve a given outcome or on arguments about sufficient conditions for causal factors to lead to the outcomes (Ragin, 2004). A third approach, "analytic generalization" (Firestone, 1993) seeks to elaborate threats to generalizability or to identify critical or deviant cases that may extend or challenge existing explanations. In this approach, it is the *theory* (an explanation) that is being tested rather than relationships among variables.

Conclusions

Cai, et al. (2019a) recently reflected on methodological alignment suggesting that researchers begin with hypotheses and "claims that they want to make" in order to work backward to identify the appropriate methods and data needed. We suspect such an exercise would be generally useful to many of us in the field of undergraduate mathematics education. We hope in this theoretical piece we made a substantial case for increasing our attention to the alignment of research questions, methods, and claims. In a recent discussion with a colleague, he mentioned that "[paraphrasing] RUME is a young field. We are still just making categories of things." While there is merit in considering the relative age of our field and certainly calling for additional foundational research, there is little need for more category papers. There is also risk of building the field upon methodologically shaky foundations. Rather, we advocate for research that operationalizes *how* questions not as "*in what ways,*" but rather pose questions that can provide theoretical insight into relations. iGT methods are at their best when serving to explore relationships between focal dimensions. Rather than asking, *How do undergraduate students conceive of topic X?* a researcher might ask: *How do undergraduate students' conceptions of topic X relate to their epistemological beliefs? How do undergraduate students' conceptions of topic X develop over time? How do undergraduate students' conceptions of topic X interrelate? How do undergraduate students conceptions' of topic X relate to their mathematical activity Y?*

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Four Ways Students Interpret and Reason with Points and Portions of Graphs of Functions: An Intersection of Two Theoretical Frameworks

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In this theoretical report, we examine the intersection of two previously-recognized dimensions of students' reasoning about how symbolic notations represent elements of graphs of functions. One dimension distinguishes location-thinking, where notations refer only to a point's location on a graph, from value-thinking, where such a point is treated as a multiplicative object. The other dimension distinguishes a nominal interpretation of expressions, where expressions refer to positions in the plane, from a magnitude interpretation, where expressions describe a measure of length. Taken together these dimensions provide four distinct ways students reason about expressions, especially those involving function notation, on graphs. Each case reveals new meanings and affordances indicated by the interplay between the two dimensions. We provide both a theoretical account and empirical example of each case.

Keywords: Mathematical Representations, Graphical Interpretations, Graphs of Functions

The use of multiple representations, including visualizations, to illustrate concepts is a hallmark of mathematical thought. Indeed, the ability to use and connect multiple representations, including algebraic symbols and graphs, is central to the teaching and learning of mathematics (NCTM, 2000, 2014). However, the use of graphical representations, in particular, may pose challenges for students (Leinhardt et al., 1990), despite their widespread use in secondary and undergraduate mathematics. In extreme instances, students may avoid reasoning with graphs altogether, even when their use would afford more efficient solutions (Dawkins & Epperson, 2014). In order to better support student learning, researchers have begun to propose frameworks to characterize various distinctions in students' understanding of graphical representations. Such frameworks include those that characterize how students create and reason with graphs, in terms of how they conceptualize coordinate systems (Lee et al., 2019), the trace of a graph (Moore & Thompson, 2015), or even the intersection of these two frameworks (Paoletti et al., 2018). Although these frameworks tease apart important details of students' reasoning with graphs, they do not account for how students may connect symbols to graphs and what such symbols represent, which may be significant. In fact, research has shown that students may not always make key connections among symbols and graphs of functions (e.g., Knuth, 2000). In this theoretical report, we consider two frameworks that account for two dimensions of students' interpretations of symbols on graphs of functions, related to their conceptions of points and positions on graphs: (1) David et al.'s (2019) value-thinking and location-thinking framework which relates students' interpretation of points and (2) Parr's (2021) description of nominal and magnitude interpretations which distinguishes students' interpretations of expressions to signify positions in graphs. By examining these dimensions at their intersections, we uncover nuances in students' graphical interpretations that may contribute to their difficulty in making sense of graphical representations or the concepts they illustrate.

The Intersection of Two Theoretical Frameworks

Inherent in both theoretical frameworks are notions of notation and interpretation. We frame these concepts using the language of semiotics (Barthes, 1957). A notation such as $f(a)$ is a sign, comprised of the symbols (signifier) and that which they indicate or represent (signified). The signified can be a mental object, a mark, or collection of marks (each of which could be a signifier). An interpretation is then the association between the symbol and the signified.

Value-Thinking & Location-Thinking: Two Ways to Reason about Points

We draw on David et al.'s (2019) constructs of value-thinking and location-thinking to distinguish students' reasoning about points on curves in the Cartesian plane. These terms refer to distinctions in students' attention when reasoning about points along curves in the Cartesian plane. These constructs emerged in the context of students evaluating statements about functions from Calculus (real-valued functions of one variable). The symbolic elements related to functions, such as $f(a)$, thus serve as signifiers. The two types of thinking are distinguished based on what the student treats these symbols as signifying on the graph.

Value-thinking. A student engaged in value-thinking views points along a graph as representing a pair of values simultaneously, typically an input and output value of a function. This way of thinking entails conceiving of inputs and outputs as distinct from each other, and distinct from the point in space representing them. Students thinking in this way often represent input values on the input axis and output values on the output axis. Value-thinking is consistent with the notion of interpreting a point as a multiplicative object, as described by Saldanha and Thompson (1998) and Thompson and Carlson (2017). Evidence from several studies suggests that conceiving of points as multiplicative objects affords reasoning quantitatively and covariationally when working with graphs (Moore et al., 2019; Thompson et al., 2017). When a student uses value-thinking in the Cartesian plane, $f(a)$ refers to the vertical component of the associated point on the graph from the horizontal axis, which can be denoted on the vertical axis.

Location-thinking. A student using location-thinking refers to and focuses on the location of the point in the plane, rather than reasoning about it as a multiplicative object. While value-thinking emphasizes the pair of values represented by a point, location-thinking emphasizes the location of the point in space. Because of this, students engaged in location-thinking often label outputs of a function at points along the curve, rather than along an output axis. Furthermore, these students will reason about the output as referring to the location of the point along the curve. In other words, they treat the signifier $f(a)$ as referring to a point on the graph.

When a student is location-thinking, this referent, the point, is a singular entity in the moment, not decomposable into components. When they are value-thinking, the point is multifarious—it represents a multiplicative object, a coordination of two components. This signification involved in value-thinking coordinates more meanings with the point on a graph, which more readily affords further analysis (David et al., 2019; Sencindiver, 2020). Figure 1 (left) summarizes the signification involved in value-thinking and location-thinking.

Magnitude & Nominal Interpretations: Two Ways to Interpret Expressions in Graphs

A second framing of students' understanding of graphs offered by Parr (2021) describes how students relate expressions, typically involving input or output variables (e.g., $f(b)–f(a)$), with graphs. We focus here on two of the four distinct ways Parr (2021) observed students interpreting symbolic expressions on graphs. These two ways of thinking are distinguished based on what the student treats these symbolic elements as signifying in the coordinate plane.

Magnitude interpretation. Parr (2021) describes a magnitude interpretation as treating an expression as a measurement of a quantity, one that is based on particular positions represented in the plane. A magnitude interpretation of an expression often involves representing an amount of a quantity as a length of a segment on a graph from a reference point on an axis or curve.

Nominal interpretation. In contrast, Parr (2021) describes a *nominal* interpretation of expressions as referring to or used as labels without quantitative significance, much like the use of labels in an anatomical diagram. A student who interprets expressions nominally may place an expression on a graph (on an axis or on a curve) to label a particular position in the Cartesian coordinate system. The nominal interpretation of an expression aligns with the use of the term *in name only*. That is, a student may place an expression at a particular position in the plane without reference to why the expression is placed where it is. When reasoning about the expression, though, the student reasons about the position labeled using the expression. Thus, a nominal interpretation may be limited to a comparison of equality between two expressions based solely on their spatial positions.

When students use a nominal interpretation of a symbolic expression, its referent, a position in the plane, is a singular, non-decomposable entity. When they use a magnitude interpretation, this reference is multifarious. The symbolic expression signifies a position, which itself indicates a relevant endpoint for a measurement. A student using a magnitude interpretation mentally constructs a portion of a graph (e.g., a segment, an arc length) from a reference point to the relevant position and uses the expression to also signify the measurement of the length of this portion of the graph. This signification involved in a magnitude interpretation coordinates additional meanings for positions in the plane. Figure 1 (right) summarizes the signification involved in the magnitude and nominal interpretations of expressions in the plane.

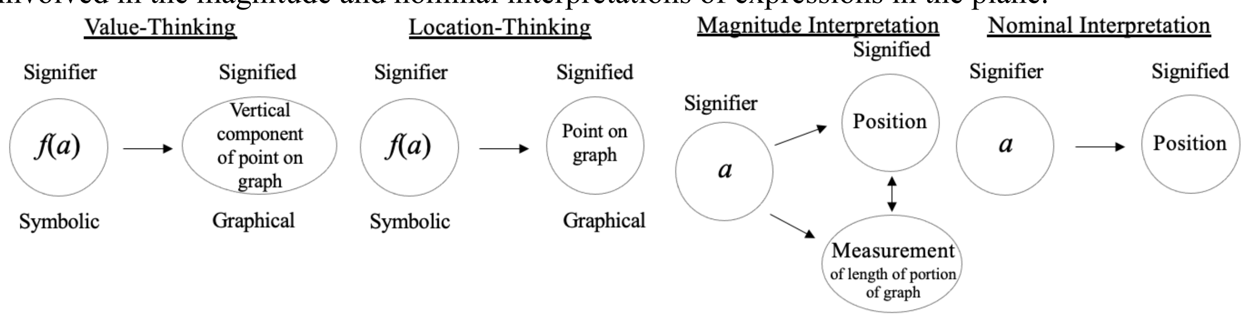

Figure 1. Signification in value and location-thinking (left) and magnitude and nominal interpretations (right).

The Intersection of Interpretations of Expressions and Interpretations of Points

Each of these frameworks describes distinct aspects of students' interpretations of graphs. The value-thinking and location-thinking framework describes whether students uncouple a point as a multiplicative object, or treat a point's location as synonymous with the output of a function. In contrast, the distinction between nominal and magnitude interpretations recognizes the differences in students' meanings for the placement of an expression on a graph, and their subsequent reasoning with it. A nominal interpretation of an expression on a graph involves treating the expression as a label, indicating a position somewhere in the plane. A student with a magnitude interpretation of an expression on a graph uses the expression to indicate an amount of a quantity represented with distance.

Table 1 shows how each of these two dimensions intersect to create four ways of thinking and uses function notation as an example in each case. To be clear, the four categories created are meant to characterize a student's thinking with a particular task or in a particular instance, rather than characterize a student and all of the ways she is capable of thinking. In fact, we suspect that students may demonstrate thinking indicative of different ways of thinking within the same task, including having a nominal interpretation of one expression, such as *a*, and magnitude interpretation of another expression, such as $f(a)$. We provide a description of each of the four cases and illustrative empirical examples.

Table 1. Four ways of interpreting function notation on graphs.

		Ways of thinking about points (David et al., 2019)	
		Value-thinking (point as ordered pair)	Location-thinking (point as output)
Ways of interpreting expressions on graphs (Parr, 2021)	Magnitude (Measures a length)	*Case 1* $f(a)$ means the measure of the vertical distance from the horizontal axis to the point	*Case 2* $f(a)$ means the measure of a distance to the point along the graph from a reference point
	Nominal (Focus solely on position)	*Case 3* $f(a)$ means the vertical position of a point on the graph	*Case 4* $f(a)$ means the position of a point on the graph

Case 1 (Magnitude + Value-Thinking):

Case 1 involves using both a magnitude interpretation of expressions and value-thinking about points. In this way of thinking, for the point $(a, f(a))$, a can mean the horizontal distance from the origin to the point, and $f(a)$ can mean the vertical distance. (These distances may not be along the respective axis, but could be parallel to the axis.) A student reasoning this way would coordinate two distances, forming a multiplicative object to comprise a position.

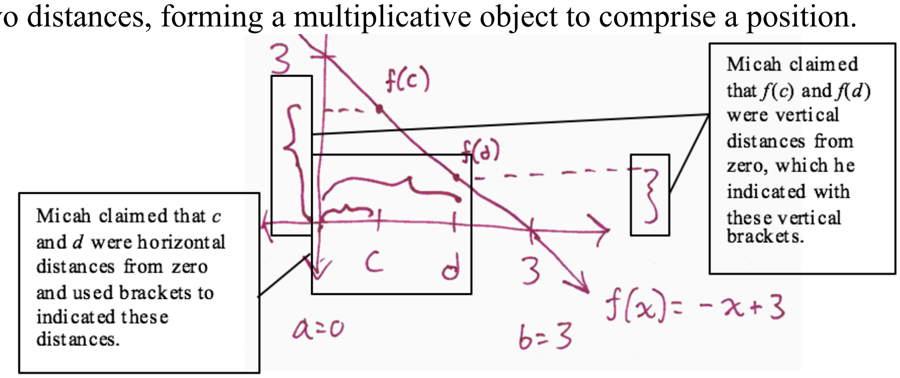

Figure 2. Micah's labels of magnitudes c, d, f(c) and f(d) while value-thinking (Parr, 2021, p. 22).

As an example, we turn to an episode with Micah from Parr (2021). Micah explained from a graph of a linear function he drew (Figure 2) that, "c is less than d but $f(c)$ is not less than $f(d)$." To justify this claim, he continued, "so…the distance here from 0 to c [*draws in horizontal curly bracket from the origin to c on x-axis*] is less than the distance from 0 to d [*draws in horizontal curly bracket from the origin to d on x-axis*]" (as quoted in Parr, 2021, p. 22). Micah reasoned similarly with $f(c)$ and $f(d)$.

Although Micah labeled $f(c)$ and $f(d)$ at positions along the graph, Micah engaged in value-thinking, decomposing these points into vertical and horizontal components. In other words, Micah did not conceive of $f(c)$ and $f(d)$ as points along the graph, as illustrated when he drew the vertical brackets and compared the distances. To reason about the expressions in this instance, Micah compared horizontal and vertical distances, which he measured from a reference point (the origin) to the positions he labeled. Thus, Micah interpreted c, d, $f(c)$ and $f(d)$ as magnitudes, distances from the origin along an axis, or parallel to an axis.

Case 2 (Magnitude + Location-Thinking):

Case 2 involves using a magnitude interpretation of expressions and location-thinking about points. In the moment, a student thinking in this way would interpret expressions as signifying lengths or distances in the graph, yet would not be thinking of points as multiplicative objects. After having identified a point in the plane, such a student would create labels and reason in ways that do not acknowledge the point as a multiplicative object ($a, f(a)$). For example, a student may associate magnitudes with the arc length of a curve between two points, or may measure in reference to other perceivable features presented in the graph.

To illustrate this case, we provide the example of Lisa from Sencindiver (2020). Lisa conveyed a meaning of $f(a)$ consistent with a measure of an arc length. Lisa represented $f(a)$ by first marking the lowest point depicted on the curve with an 'x' (Figure 3), measuring a distance of a along the arc, and marking the end of the distance with another 'x'. She then labeled the length along the curve '$f(a)$', saying "yeah, it'd be like the whole distance from here [*gesturing to the lowest blue 'x' in Figure 3*] to here [*gesturing to the other blue 'x' in the Figure 3*]" (as quoted in Sencindiver, 2020, p. 121). In the moment, Lisa did not seem to be thinking of points on the curve as multiplicative objects, but rather as spatial locations in the plane, with $f(a)$ representing the arc length between these markings. Further, Lisa continued describing $f(a+h)$ as an arc length through a similar construction, and $f(a+h)– f(a)$ as the difference of the two lengths.

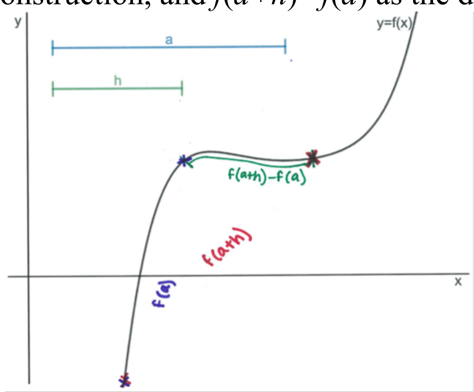

Figure 3. Lisa's labels of magnitudes f(a), f(a+h), and f(a+h)-f(a) while location-thinking (Sencindiver, 2020, p. 121).

Case 3 (Nominal + Value-Thinking):

Case 3 involves using both a nominal interpretation of expressions and value-thinking about points. Thus, a student using these ways of thinking interprets inputs and outputs, such as a and $f(a)$, as labels for particular positions, and as horizontal and vertical components of the ordered pair of the point $(a, f(a))$. This student may reason with a point similar to one reporting battleship coordinates. To do so, the student coordinates two positions to give a third position, still forming a multiplicative object. However, this is *not* a multiplicative object of multiple distances. In other words, a and $f(a)$ can be thought of as positions on the axes, without thinking about the distance that variable represents between the origin and the location on the axis. Likewise, a student can decouple a point into two positions by projecting vertically and horizontally to positions on the x-axis and y-axis, respectively.

One example of a student using these ways of thinking comes from Martha in Parr (2021). She claimed that the $f(c)$ and $f(d)$ she labeled on a monotone decreasing graph were not equal and could never be equal (Figure 4). She explained this claim by saying,

> because $f(c)$ and $f(d)$ are separate values. So I know I can't, they could be… maybe,?... No, I don't think so. Yeah, I think f of yeah… c and d are gonna be, if they're [c and d]

separate values, like they're [*c* and *d*] labeled separately I think they're [*f*(*c*) and *f*(*d*)] gonna be separate (Parr, 2021, p. 15).

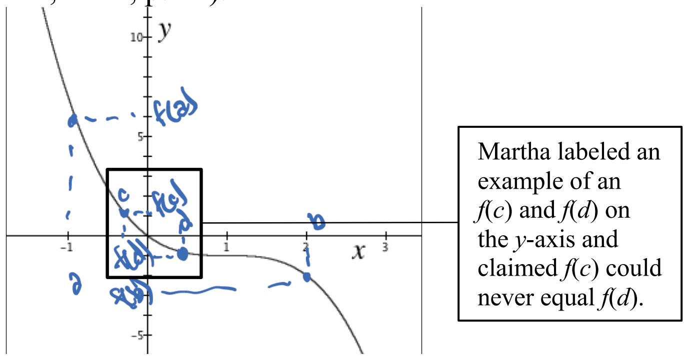

Figure 4. Martha's labels of c, d, f(c) and f(d) interpreted nominally while value-thinking (Parr, 2021, p. 15).

In this instance, Martha engaged in value-thinking and conceived of points as coordinates of two components, labeled on the appropriate axis (*c* appears to be labeled on the graph, but this was due to space constraints). Additionally, Martha was reasoning about *c* and *d*, (and *f*(*c*) and *f*(*d*)) as labels for positions on the graph. Her language of "labeled separately" indicates that she was thinking of *c* and *d* being labeled in two different places on the graph, presumably on the *x*-axis, as she did in Figure 4. Martha then considered *c* and *d* to be unequal ("separate values") because of how she was interpreting these expressions nominally.

Case 4 (Nominal + Location-Thinking):

Case 4 involves using both a nominal interpretation of expressions and location-thinking about points. Thus, a student using these ways of thinking interprets expressions as labels for particular positions, and points as outputs along the curve. Such a student may use an output label, such as *f*(*a*), for a position along the curve. This position may correspond with the input *a*, but is not an indication of a measurement along the curve. In reasoning this way, the student may coordinate an input *a* with an output label *f*(*a*) (thought of as a point), but in the moment, the student does not conceive of the point on the curve itself as a multiplicative object.

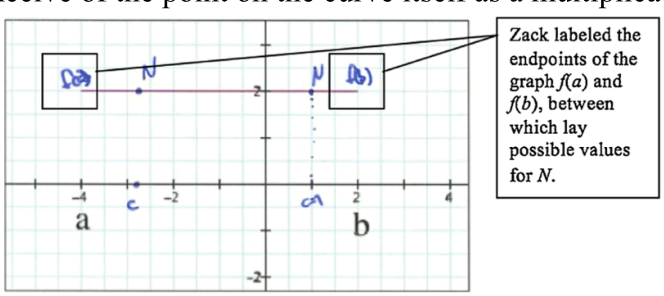

Figure 5. Zack's labels of f(a) and f(b) interpreted nominally while location-thinking, for which he claimed f(a)≠f(b) (David et al., 2019, p. 10).

To illustrate this case, we provide the example of Zack from David et al. (2019). In this instance, Zack pointed to the endpoints of the graph, which he had labeled as *f*(*a*) and *f*(*b*) respectively (Figure 5), and claimed that "when I input a (points to left endpoint of the graph) I know that's not going to be *f*(*b*) (points to right endpoint of the graph), so *f*(*a*) does not equal *f*(*b*)." As explained by David et al. (2019), Zack engaged in location-thinking, conceiving of the points he labeled on the graph solely as outputs. Further, by justifying that *f*(*a*) does not equal *f*(*b*) by indicating the ends of the graph, Zack interpreted *f*(*a*) and *f*(*b*) nominally. He considered them as labels for positions and reasoned about these positions, rather than any measurements associated with these positions, such as the height of the points, to reason about $f(a)$ and $f(b)$.

Discussion

Our theoretical findings shed light on students' ways of interpreting expressions related to points and portions of graphs. By analyzing the intersection of two recent frameworks, we see four distinct modes of interpreting symbolic expressions on graphs, allowing us to tease apart details of the mental actions involved in each of the four cases. Further, applying a semiotic lens to each framework and considering them in light of each other advances our understanding of the frameworks independently. For instance, we offer an elaboration on the notion of "value" in value-thinking by contrasting Case 1 and Case 3. Although value-thinking as described in David et al. (2019) may be interpreted as only referring to Case 1 (i.e., students conceptualizing a point as a multiplicative object of a pair of values of measurements from the point to the axes or along the axes), this work highlights the reality that students may conceptualize a point as a multiplicative object of positions on the axes, without reference to measurements as in Case 3.

The coordination of these two frameworks allows us to see important parallels and interplays between them. Location-thinking's view of *point* is singular, while value-thinking's view is multifarious. A nominal interpretation of *position* is singular, while a magnitude interpretation of it is multifarious. By overlaying these, multiple shades of meaning and signification become apparent. For instance, in Case 1 (magnitude+value), the meaning of point and position are both multifarious for the student, which allows us to see that the student has formed *a multiplicative object of coordinated the distances* from the axes to the point on the graph.

Of the four cases, Case 1 provides the most robust and flexible combination, but all four cases reveal affordances and constraints for the mathematical activity potentially available to the students, and suggest ways instructors can support this activity. For instance, Case 4 (nominal+location) may afford reasoning in geometric contexts where horizontal and vertical components are not privileged. Yet, a student using this reasoning will need to coordinate input and output components of the graph of a function before they can interpret more complex symbolic statements, such as the Mean Value Theorem. Likewise, aspects of reasoning with Case 2 (magnitude+location) afford conceptualizing quantities such as arc length along the curve as measurable, which is critical for multiple topics in Multivariable Calculus (e.g., line integrals of vector-valued functions), as well as reasoning about quantities within spatial coordinate systems (Lee et al., 2020). However, this sort of thinking may constrain students' productive activity when a graph is representing information along orthogonal axes. Case 3 (nominal+value) may be sufficient for students in finding numerical values from graphs, if they do not need to reason within the graph further.

The intersection of the two frameworks we described may help instructors account for differences in students' reasoning about points and portions of graphs. In contexts that utilize graphs of functions in the Cartesian plane, such as Calculus, supporting students in conceptualizing points as multiplicative objects of coordinated distances (Case 1) may be considered along the two dimensions of the frameworks we described. If a student is not using a Case 1 (magnitude+value) conception of points and positions, distinguishing whether they are failing to conceptualize a point as a multiplicative object, or a position as indicating an endpoint of a length to be measured may be helpful. Further research in this area may include teaching experiments to study the extent to which students may be constrained to certain cases of thinking. Such studies may also shed light on what factors support student in transitioning from one way of thinking to another, such as those that are more productive in a given context.

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Importance of a Shared Coherent Language for Mathematics Learning

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Learning Progressions is usually connected with conceptual development and suggested as a way to bring coherence to how we think about learning and the curriculum. In this session, we present our view that Learning Progressions is not sufficient for successful learning of mathematics without attendance to concept development through a shared coherent language. We draw on the conceptual change learning theory to show that the difficulties students experience in learning mathematics are also due to the inconsistent representation of a concept from lower to higher levels of mathematics education. We argue that investigation is needed into how treatment of a concept in a restricted context in high school CCSSM-aligned textbook may lead to a potential conflict and difficulty of concept development in the broader context, later in college.

Keywords: mathematics learning, language, conceptual change, learning progressions

Various studies have shown that about 59% of students in two-year colleges and 33% of students in four-year colleges are taking some type of developmental mathematics courses, with an average student taking two to three successive courses. Even more worrisome is that about 50% of two-year and 58% of four-year college students enrolled in these courses do not complete all of their required developmental math courses. Furthermore, only about 20% of students who complete all of their developmental math courses successfully complete college level math courses (Brock et al., 2016). Such situation is mostly explained by the procedural nature of mathematics instruction in high schools (Zenati, 2019) rather than a conceptual development.

Conceptual development is often linked with Learning Progressions (LP), and LP is suggested as a way to bring coherence to how we think about learning and the curriculum (Siemon, 2017). The LP is described as "successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic" (The National Research Council [NRC], 2007, p. 214). Teachers rely heavily on standards and curricular materials in their instruction and assessment (Davis, 2009; Remillard, 2000; Schneider & Krajcik, 2002; Van Zoest & Bohl, 2002). In particular, as stated in Van Zoest and Bohl (2002), "historically mathematics textbooks have played the role of a mathematical authority and reference for students and teachers in mathematics classrooms" (p. 268). In this respect, one of the attempts to improve students' learning of mathematics at K-12 levels—with a focus on LP, and by bringing coherence—was the development and implementation of Common Core State Standards for Mathematics (CCSSM) (2010). The developers of the CCSSM noted that "the development of the standards began with research-based learning progressions detailing what is known today about how students' mathematical knowledge, skill, and understanding develop over time. The knowledge and skills students need to be prepared for mathematics in college, career, and life are woven throughout the mathematics standards" (p. 4).

The development and implementation of LPs requires attending to students' conceptual progress in order to promote intended learning (Tzur, 2008), i.e., building on their previously constructed conceptions. Therefore, an important focus of LP is suggested to be a shared language around a set of topics that point to the underlying conceptual structure of the mathematics that is the focus of the LP (Siemon, 2017), so that students can develop a conceptual understanding and cognitive structure by building up concepts over the years through LP. Tall and Vinner (1981) also argue that the development of concepts includes all mental pictures and attributes and associated properties and processes of a concept, i.e., concept image, as well as a form of words that is used to specify that concept, i.e., concept definition. A conflict and/or incoherency within and between concept images and concept definitions that are formed over the years of LP can yield a cognitive conflict and difficulty of forming an appropriate concept image, which can seriously impede the development of the formal theory in the mind of the individual student. Therefore, our view is that LP is not sufficient for successful learning of mathematics without attendance to concept development through a shared coherent language. We draw on the conceptual change learning theory and attempt to show that the difficulties students experience in learning mathematics are also due to the inconsistent representation of a concept from lower to higher levels of mathematics education.

In this paper, we, first, describe the conceptual change theoretical perspectives. Then, we present how conceptual change theory is used to identify mathematical concepts that require conceptual change as students progress through elementary, middle and high school. Last, we explain the shared coherent language theoretical perspective.

Conceptual Change Theoretical Perspectives

Conceptual change ideas have been used for explaining students' difficulties with learning certain science concepts and for developing teaching strategies to initiate conceptual change of students' understanding of science concepts. Since the 1980s many studies have shown that students come to classrooms with knowledge that is not consistent with formal views of science (Duit & Treagust, 2003). Conceptual change researchers thus focused on investigating the development of students' previous knowledge toward intended science concepts. Consequently, their perspective of students' knowledge structures was fundamental to this research.

In general, two theoretical perspectives regarding knowledge structure could be recognized in literature (Ozdemir & Clark, 2007). According to one perspective, students' knowledge structure is an ecology of quasi-independent elements (e.g., diSessa & Sherin, 1998). These researchers argue that students' knowledge structure consists of multiple conceptual elements, which are spontaneously connected and activated according to the situation. Conceptual change process then involves revising and refining elements and their interactions by addition, elimination, and reorganization in order to strengthen knowledge structure. Another perspective assumes that students' knowledge is best represented as a coherent unified framework of theory-like character (Chi & Roscoe, 2005; Posner, Strike, Hewson, & Gertzog, 1982; Vosniadou & Vamvakoussi, 2006). These naïve theories develop through every day experience and require revolutionary change so that students acquire knowledge consistent with formal views of science. For the purpose of this paper, students' prior knowledge is assumed to be a coherent framework of theory-like structure (Vosniadou & Vamvakoussi, 2006).

While the conceptual change teaching approach has been used in science teaching since 1980 (Duit & Treagust, 2003), mathematics education researchers have attempted to apply conceptual change ideas to the teaching and learning of mathematics as recently as 2002 (e.g., Merenluoto & Lehtinen, 2002). One of the arguments for not using conceptual change in mathematics was that there are no revolutions in mathematics like in science. The revolutionary change in science results in discarding the older theories in favor of the new one, which is not a characteristic of conceptual change in mathematics where an old structure is retained as a substructure of the new one (e.g., enlarging number sets). However, Vamvakoussi and Vosniadou (2004) suggested that the answer to question “Are there revolutions in mathematics?” depends on the definition of “revolution”. They suggest that the conceptual change approach can be applied to mathematics learning if the focus shifts from the change of mathematical theories, to the development of mathematical concepts. For example, the change from understanding ratios of integers as operations to understanding ratios of integers as numbers can be interpreted as an ontological shift from the category of processes to the category of objects (Vamvakoussi & Vosniadou, 2004).

Greer (2004) also claims that there are revolutions in mathematics. For example, formal mathematics had not accepted negative numbers until the 20th century, even though their functional counterparts existed in the real world (e.g., debit and credit transactions). He quotes De Morgan (1910 [originally 1831], p. 103-104): “3 – 8 is an impossibility, it requires to take from 3 more than there is in 3, which is absurd” (p. 542), to illustrate his struggle with conceptual change. Other examples include rejection of the belief that the Euclidean geometry is a unique way to describe the space and the introduction of non-Euclidean geometries (e.g., hyperbolic and spherical geometry) in the 19th -20th century; or Descartes notation of exponents $2x^3$ rather than in terms of geometrical roots $2A$ cubus, which allowed considerations of expressions such as x^0, x^{-1}, $x^{-1/2}$ and of dimensions beyond three. All of these show “that there have been changes in mathematics at the meta-level whereby earlier views have indeed been displaced” (Greer, 2004, p. 542).

Mathematical Concepts that Require Conceptual Change

Mathematical concepts identified in research that utilized conceptual change include the density of rational and real numbers, the use of the minus sign in algebra, exponents, limits, continuity, and tangency. Most researchers examined students’ prior knowledge to identify ideas that are inadequate for learning the new information. Students’ experience with natural numbers in everyday life along with the early mathematics instruction supports the development of the notion of the discreteness of numbers. Thus, understanding that there are infinitely many numbers between any two real numbers is difficult for students to comprehend (Merenluoto & Lehtinen, 2004; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2005; Vosniadou & Vamvakoussi, 2004). The knowledge about the minus sign in arithmetic, which usually indicates an action ‘subtract two numbers’, appears to present an obstacle in learning algebra concepts, where the minus sign can assume more than one meaning at the same time (Vlassis, 2004; Christou & Vosniadou, 2005; Christou et al., 2007). The notion of exponents is another challenging algebra concept for students (Pitta-Pantazi et al., 2007). Students’ original understanding of powers with positive exponents in terms of ‘multiply *a* by itself as many times as the number in the exponent’, is not adequate in solving problems involving negative and rational exponents. Merenluoto and Lehtinen (2002) suggested that students come to calculus courses with an everyday understanding of the term ‘limit’ as ‘limiter’ or ‘something that ends’ and the term ‘continuity’ as ‘something that never ends’, which are not compatible with mathematical definitions of the terms ‘limit’ and ‘continuity’.

Biza et al. (2008), however, used the conceptual change theoretical framework to analyze a sequence in which students learn about a tangent. They argued that the initial understanding of the concept of a tangent of a curve as ‘a line that touches the curve at one point, and divides a plane into two parts, one of which contains the whole curve’ is mismatched with the formal definition of the tangent of a curve at the given point as ‘a line that has the slope equal to the derivative of the function at that point’.

Shared Coherent Language Theoretical Perspective

We conceptualize the *shared coherent language* as *the use of consistent and accurate mathematical language through LP over the grade levels and mathematical content areas*. Mathematical language includes precise mathematical phrases, mathematics terminology, definitions, properties, and rules as appropriate for the developmental level of the students. Siemon et al. (2017) explain that, although it is not appropriate to teach negative numbers to children in kindergarten, a basic understanding that there is no *first* number should be communicated. They argue that the phrase “the first number is 1” when counting or presenting numbers on the number line is problematic because students may develop an understanding that there are no numbers less than one. Clearly such an understanding may present an obstacle in students’ learning of negative, rational, and irrational numbers. Siemon et al. (2017) provide a list of mathematical phrases that are commonly used and corresponding phrases that should be used to support special education students to develop conceptual understanding. Additionally, Karp et al. (2015) described 12 mathematics rules (e.g., PEMDAS, FOIL, factor rainbow, etc.) that are commonly used by elementary school teachers that expire in the middle school. In our opinion, these are not mathematics rules, but mnemonics and "rules of thumb” that will interfere with students’ learning of advanced mathematical concepts.

We believe that using appropriate mathematical phrases as well as accurate mathematical terminology, definitions, properties and notation is essential for learning of mathematics of all students. As described in the previous section, the conceptual change theoretical perspective is used to identify mathematical concepts that require conceptual change. We then propose to examine mathematical language used to present each identified concept in curricula and associated resources for teachers and students across grade levels and mathematical content areas.

Conclusion

In conclusion, although students can operate quite happily with the restricted notion of a concept initially in a restricted context, this can lead to an inappropriate concept development, and hence, they may be unable to cope with the concept in a broader context. As argued by Tall and Vinner, the teaching programme (i.e., curriculum) itself is responsible for this unhappy situation. Therefore, it is our view that LP is not sufficient for successful learning of mathematics without attendance to concept development through a shared coherent language. Given the number of mathematical concepts that require conceptual change coupled with teachers’ reliance on textbooks for content, we believe it is essential that researchers need to look at LP in textbooks for particular topics that students struggle with. Specifically, investigation is needed into how treatment of a concept in a restricted context in high school CCSSM-aligned textbook may lead to a potential conflict and difficulty of concept development in the broader context, later in the college.

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A framework for analyzing students' reasoning about equivalence across undergraduate mathematics

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Establishing and leveraging equivalence is a central practice in mathematics. Though there have been many studies of students' uses of equivalence, much of the research thus far has been domain-specific, and the literature generally lacks coherence within and across mathematical domains. In this theoretical paper, we propose an initial unifying framework for capturing the different ways that students might establish equivalence. Using constructs born out of the K-12 literature, we discuss how this framework can be applied to student reasoning in undergraduate settings. We do so by presenting the results of conceptual analyses of students' possible uses of equivalence when thinking about vectors, isomorphisms and homeomorphisms, and single-variable limits. We then conclude with a detailed analysis of student data from combinatorics that identifies productive aspects of their uses of equivalence when constructing permutations.

Keywords: Equivalence, Conceptual Analysis, Student Thinking

Equivalence is a pervasive mathematical concept that is fundamental to constructing relationships between mathematical objects at all levels (Carpenter, Franke, & Levi, 2003; Cook, 2018; Hamdan, 2006; Kieran & Sfard, 1999; Knuth et al., 2006; Lockwood & Reed, 2020; Moore, 2013; Ni, 2001; Steffe, 2004; Stylianides et al., 2004). In postsecondary mathematics, equivalence is fundamental to students' thinking about topics such as angle measure (Moore, 2013), logic (Stylianides et al., 2004), combinatorics (Lockwood & Reed, 2020), and abstract algebra (Cook, 2012, 2018; Larsen, 2013). There is, however, evidence that students throughout K-16 mathematics face difficulties in reasoning about equivalence (Chesney et al., 2013; Godfrey & Thomas, 2008; Kieran, 1981; McNeil et al., 2006; Weinberg, 2009). We propose that one reason for these difficulties is that equivalence is often treated in compartmentalized, context-specific ways that emphasize its utility within a context but not its common, overarching structure. This is significant because, as noted by Asghari (2019), "equivalence has had many different faces and […] many different names" (p. 4675).

We note that very little has been done to develop a clear, unifying image of what is involved in productively reasoning with equivalence across domains in undergraduate mathematics. In this theoretical report, we seek to begin to address this need by presenting an initial theoretical framework that characterizes key aspects students' reasoning with equivalence. Specifically, we first present theoretical analyses of the ways that students might operationalize equivalence when reasoning about (1) vectors, (2) isomorphisms and homeomorphisms, and (3) limits in single-variable calculus. Then, we present an analysis of students' mathematical activity in combinatorics that highlights how they conceived of various sets of outcomes as equivalent. In doing so, we demonstrate the utility of the framework for highlighting key aspects of students' productive engagement with equivalence across multiple mathematical domains.

Background Literature

As there is much more literature on equivalence at the K-12 level than at the postsecondary level, we draw on K-12 literature in situating our paper. Because of spatial restrictions, we only discuss the works that largely informed our framework. The K-12 equivalence literature holds two key implications for our theory-building objectives. First, there are a plethora of *explicit* calls for instruction to attend to equivalence (McNeil & Alibali, 2005; McNeil et. al., 2006; Ni, 2001; Smith; 1995; Solares & Kieran, 2013; Stephens, 2006). While this has been somewhat achieved at the K-12 level, we have observed that equivalence in postsecondary domains often remains backgrounded.

Second, the K-12 literature contains descriptions of various in ways in which students might interpret equivalence, specifically in the context of the equals sign. These descriptions provided an initial foundation for our framework. A fundamental distinction in K-12 involves students viewing the equal sign *operationally* (as a indicator to "do something") or *relationally* (as an indicator that the objects in question are in some way the same) (Kieran, 1981; Knuth et al., 2005). But what does a *relational* understanding of the equal sign entail? As an example, we consider the equivalent algebraic expressions $2x + 2y$ and $x + y + x + y$: What does it mean to say that two objects are *in some way* the same? Our framework stems from three possible ways to interpret the equivalence of these two expressions that appear in the K-12 literature (Liebenberg et al., 1999; Saldana & Kieran, 2005; Solares & Kieran, 2013; Zwetzschler & Prediger, 2013):

1. *Numerical*: these two expressions are equivalent because, for any real numbers x and y, the expressions $2x + 2y$ and $x + y + x + y$ have the same numerical value.
2. *Transformational:* these two expressions are equivalent because one can be transformed into the other using algebraic rules (e.g. associativity and commutativity of addition).
3. *Descriptive:* these two expressions are equivalence because they both describe the perimeter of a rectangle with sides x and y.

We propose more general, refined versions of these interpretations in the next section and illustrate how they capture key aspects of students' reasoning across different domains.

Theoretical Framework

Our framework takes the form of a *conceptual analysis*, an explicit description of "what students might understand when they know a particular idea in various ways" (Thompson, 2008, p. 43). We find conceptual analyses to be useful for our theory-building objectives in three ways. First, conceptual analyses offer means to identify desirable interpretations of equivalence that can inspire targets of instruction (Thompson, 2008). Conceptual analyses can form unifying threads within and across courses and curricula (O'Bryan, 2018). Finally, conceptual analyses can also enable researchers to create *models of students' thinking* (Clement, 2000; Steffe & Thompson, 2000). These models are useful for both researchers and instructors because they can be employed to explain students' mathematical activity and render it sensible in some way.

In this report, we shall illustrate how results of the cross-domain conceptual analysis of equivalence that we present serves these purposes. The conceptual analysis was informed by our analyses of (1) the K-16 literature on equivalence, and (2) data collected for teaching experiments that had been previously conducted in abstract algebra (Cook, 2018) and combinatorics (Lockwood & Reed, 2020; Reed & Lockwood, 2020). In this framework, we describe three interpretations of equivalence that we hypothesize are useful for reasoning about equivalence across mathematical domains (these are featured in Table 1).

Table 1. A framework for analyzing students' reasoning about equivalence.

Interpretation of equivalence	Description	Example from undergraduate mathematics
Common characteristic	Interpreting or determining equivalence based upon a perceived attribute that the objects in question have in common.	Interpreting that parallel lines are equivalent because "the common property will be the slope" (Hamdan, 2006, p. 143).
Descriptive	Interpreting or determining that objects are equivalent because they describe the same quantity or serve the same purpose with respect to a given situation.	Determining that -3 and 9 are equivalent modulo 12 because they both function as the additive inverse of 3 (Cook, 2012).
Transformational	Interpreting or determining the relationship between equivalent objects in terms of the actions by which one object has been or might be transformed into another.	Interpreting that two matrices are row-equivalent because one can be obtained by applying a sequence of elementary row operations to the other (Berman, Koichu, & Shvartsman, 2013).

In the next two sections, we illustrate the utility of this framework by (a) elaborating theoretical analyses of how these constructs might capture relevant aspects of students' reasoning about equivalence in the context of vectors and magnitudes, isomorphisms and homeomorphisms, and single-variable limits, and (b) using the framework to conduct a detailed analysis of students' reasoning from a teaching experiment in enumerative combinatorics. Together, these will demonstrate ways in which the framework can contribute to a broader, unifying perspective on equivalence that may be applicable across domains.

Using the Framework to Gain Insight into Equivalence Across Domains

We now illustrate how the interpretations detailed above capture productive aspects of reasoning about equivalence in the contexts of vectors, isomorphisms and homeomorphisms, and single-variable limits.

Vectors and Magnitudes

Vector equations provide an example that extends work done at the K-12 level to the undergraduate curriculum. For example, consider the equation $\|5v\| = 5\|v\|$, where $\|\cdot\|$ denotes a vector norm. First, a student might employ *transformational* equivalence to consider that the equality $\|5v\| = 5\|v\|$ follows from allowable operations on vector norms. This transformation might be described as "pulling the 5 out." More formally, the definition of a norm requires that the norm function satisfy the property $\|cv\| = |c| \cdot \|v\|$ for any real constant c and vector v. In a *common characteristic* interpretation of the equation, a student might appeal to the fact that given any vector v, $\|5v\|$ and $5\|v\|$ give the same numerical value[1]. Finally, a descriptive equivalence interpretation could involve reasoning with magnitudes. Following Thompson and colleagues (2014), the magnitude of a quantity A is the size of that quantity measured with respect to a unit[2]. From this perspective, $\|5v\| = 5\|v\|$ could be interpreted *descriptively* as a statement that the measure of the length of $5v$ (when using the length of v as a unit) is 5.

Isomorphic and Homeomorphic Spaces

Significant identifications commonly made in advanced mathematics establish spaces as equivalent in the sense of possessing the same essential features. The standard method of determining such an equivalence entails the finding of a (usually bijective) map between the spaces such that the map satisfies certain topological, analytic, or algebraic properties. A homeomorphism, for instance, is a bijective map, f, such that both f and its inverse, f^{-1}, are continuous. A group isomorphism, ϕ, is a bijective map such that ϕ preserves the group operation: $\phi(a * b) = \phi(a) \cdot \phi(b)$, where $*$ and $\cdot$ are the binary operations of the two groups. A student using such mappings to change one space into another would employ *transformational equivalence*, as the maps are the means by which elements of one space are transformed into elements of another. The analytic, algebraic, and topological qualifications of the bijective maps afford other implications, however, that also constitute interpretations of equivalence. Given two isomorphic groups G and H, G is abelian if and only if H is abelian. If M and N are homeomorphic metric spaces, then they share convergent sequences. Put another way, viewing the equivalence between spaces this way focuses on their *common characteristics.* One benefit of this interpretation is that such fundamental results as those we have given above become intuitive (if not obvious). Another is that it can be leveraged to justify that certain spaces are *not* the same: an abelian group cannot be equivalent (isomorphic) to a non-abelian group, and a connected topological space cannot be equivalent (homeomorphic) to one that is disconnected.

Single-Variable Limits

Limits underlie most curricular treatments of fundamental operations in single-variable calculus: derivatives, integrals, and series. One formulation of limits answers the question: *At a given domain value, a, of a function, f, is there a single real number, L, that f approximates to any desired error bound via domain restrictions of f around a?* The mathematical necessity of such a question can be seen by examining $\frac{e^x-1}{x^2}$, which does not admit a readily available output for all domain values. While numerical and graphical methods might allow determination of rather obvious limiting values, L, for certain functions, f, the most efficient way to determine the limits of functions - such as $\frac{e^x-1}{x^2}$ - at points of discontinuity is to find an alternate, continuous function f^* that has the same limit as f at a.

For simpler functions, f^* can be determined algebraically. For instance, $x + 1$ can be used to determine the limit of $\frac{x^2-1}{x-1}$ at $x = 1$ by noting that $\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x + 1$, so that $\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x + 1 = 2$. We consider the determining f^* in this way to be an example of transformational equivalence, specifically by obtaining $x + 1$ from $\frac{x^2-1}{x-1}$ through a series of algebraic

[1] Notice that numerical equivalence from the K-12 literature is subsumed in common characteristic equivalence.
[2] Symbolically, $|A| = m(A) \cdot |u|$ where $|A|$ is the magnitude, $m(A)$ is the measure of A in unit u, and $|u|$ is the magnitude of the unit.

transformations. These operations by themselves, however, do not constitute the utility of interpreting f and f^* as equivalent for the purpose of limit calculations. Rather, f and f^* are also equivalent because of a *common characteristic*: they share the same output values in their common domain (that is, all real numbers except 1). Because of this common characteristic, the output $f^*(1) = 2$ is approximated by values of f for any error bound given a sufficiently small domain interval around $x = 1$, thus constituting the limit of f at $x = 1$. As such, the limit of f is determined because of the *common characteristic* equivalence of f and f^*, yet f^* is likely to be originally determined *transformationally.*

As functions, f, vary in complexity, engaging in algebraic transformations becomes increasingly insufficient, requiring new ways to determine suitable f^*. For instance, while many functions share the same limiting value as $\frac{e^x-1}{x^2}$ at $x = 0, \frac{e^x-1}{x^2}$ admits no readily available algebraic transformations. From this perspective, limit theorems - such as L'Hopital's rule or the squeeze theorem - can be viewed as providing the means of generating useful equivalent functions, f^*. While desired functions f^* have a common characteristic with f that their limits evaluate to the same number, there are many such functions, g, that have this same common characteristic. We consider that students might productively generate more robust understandings of limit theorems as ways to establish equivalence between f^* and f via applications of ideas fundamental to calculus, those of locality and approximation, for instance.

An Analysis of Students' Reasoning in Combinatorics

We now demonstrate the utility of the framework for capturing key aspects of students' reasoning with equivalence in combinatorics. Lockwood & Reed (2020) characterized an *equivalence way of thinking* to describe a general approach that students in teaching experiments (Steffe & Thompson, 2000) used to solve enumerative combinatorics problems successfully. Broadly, their *equivalence way of thinking* involved identifying outcomes of counting processes as the same and then using division to account for such 'duplicate' outcomes. Our analysis here furthers this work by explicating *how* the students employed equivalence in multiple ways. Specifically, we discuss the counting activity exhibited by novice counters (pseudonyms Carson, Anne-Marie, and Aaron) when solving the Horse Race Problem, which states: "There are 10 horses in a race. In how many different ways can the horses finish in first, second, and third place?"

The students first answered $10 \cdot 9 \cdot 8$, enumerating the sequence of events in which 10 horses finish the race, but only 9 horses remain after the first horse finishes, followed by 8 horses that compete for a third-place spot. The interviewer then introduced the notation $\frac{10!}{7!}$ as another way to express the solution and asked the students to justify why $\frac{10!}{7!}$ was also a solution. The students first argued that $\frac{10!}{7!}$ gave another way of writing $10 \cdot 9 \cdot 8$ as $\frac{10 \cdot 9 \cdot 8 \cdot 7!}{7!}$, yielding cancellation of 7!. This first response employed *transformational equivalence*, as the students enacted algebraic transformations in which $\frac{10!}{7!}$ transformed into $10 \cdot 9 \cdot 8$. Wanting to give the students opportunities to make other – combinatorially based - connections, the interviewer asked, "can you explain why this answer might make sense aside from the fact that its numerically equivalent to 10 times 9 times 8?" The following conversation ensued:

Carson: So, the way I'm thinking about it, is that we know kind of the method to get the number of ways that 10 horses can finish a race, and that's 10!. … So, there's 10! total outcomes, and then we know for any given first 3 there's gonna be 7!, because that's saying we know the first 3 horses have finished. How can the last 7 horses finish? So that's gonna be 7!. But all we care about is how many given first 3s there are. So, if we divide the total number of outcomes by the number of potential of outcomes for the last 7 horses - that will give us the potential number of outcomes for the first 3. If that makes sense?

Interviewer: It makes sense to me. Are you guys following what he's saying?

Anne-Marie: I see why, like 10! would be looking at all 10 positions for each 10 horses. I just feel like it'd be more intuitive to subtract the 7! than it would be divide but I see why dividing works better.

We note two complementary interpretations of equivalence that Carson engaged in for the Horse Race Problem. First, Carson employed descriptive equivalence to establish similitude of the expressions $10 \cdot 9 \cdot 8$ and $\frac{10!}{7!}$. By establishing that $10 \cdot 9 \cdot 8$ and $\frac{10!}{7!}$ counted the same total collection of the first three race finishers, Carson argued that the expressions described the same outcome set. This use of descriptive equivalence is commonly employed in combinatorial proof. Second, in the underlined portions, Carson argued that there were 7! orderings of 10 horses that represented each single desired ordering of the first three horses. This representation of the single outcome in 7! ways was the first time that the students identified what they would later call "duplicate" outcomes and set the foundation for what Lockwood & Reed (2020) called an *equivalence way of thinking*. For Carson, the assumption that there were 7! representations of the same desired outcome provided the impetus for the division of 10! by 7!, and constitutes another use of descriptive equivalence, as the 7! duplicates represent the same desired quantity.

This discussion of the utility in dividing versus subtracting, initiated above by Anne-Marie, became a prevalent distinction for these students. While the students could articulate that there were 7! arrangements of the 10 horses for any specific arrangement of gold, silver and bronze medalists, at this point in the experiment only Carson could articulate why division meaningfully accounted for those 7! extraneous arrangements to produce a single desired outcome.

Figure 1: Arrangements of A-E

ABCDE	ACBDE	BACDE	BCADE	CABDE	CBADE	DABCE	DBACE	EABCD	EBACD
ABCED	ACBED	BACED	BCAED	CABED	CBADE	DABED	DBACE	EABDC	EBACD
ABDCE	ACDBE	BADCE	BCDAE	CADBE	CBDAE	DACBE	DBCAE	EACBD	EBCAD
ABDEC	ACDEB	BADEC	BCDEA	CADEB	CBDEA	DACEB	DBCEA	EACDB	EBCDA
ABECD	ACEBD	BAECD	BCEAD	CAEBD	CBEAD	DAEBC	DBEAC	EADBC	EBDAC
ABEDC	ACEDB	BAEDC	BCEDA	CAEDB	CBEDA	DAECB	DBECA	EADCB	EBDCA
ADBCE	AEBCD	BDACE	BEACD	CDABE	CEABD	DCABE	DEABC	ECABD	EDABC
ADBEC	AEBDC	BDAEC	BEADC	CDAEB	CEBAD	DCAEB	DEACB	ECADB	EDABC
ADCBE	AECBD	BDCAE	BECAD	CDBAE	CEBAD	DCBAE	DEBAC	ECBAD	EDBAC
ADEBC	AEDBC	BDEAC	BEDAC	CDEAB	CEDAB	DCEAB	DECAB	ECDAB	EDCAB
ADECB	AEDCB	BDECA	BEDCA	CDEBA	CEDBA	DCEBA	DECBA	ECDBA	EDCBA

To elicit further reflection, the interviewer provided a printed list of the 5! arrangements of the letters A through E (Figure 1) and asked the students to find the 20 groups of letters that could represent first and second place finishers. Notice that the entries in the list were spaced apart according to the fixed first two letters. The students noticed this arrangement, and subsequently circled the 20 groups that reflected this spacing. Seeing that there were six elements to each grouping, Aaron asked why division by 3! made more sense than "getting rid of the other 5". The following exchange occurred after the interviewer pointed out that 3! was 6:

Aaron: Well, since there are 6 options for each AB, then dividing by 6 would just mean you would get 1, because that's all you're looking for. But then 5! would give you the number of groups (i.e. arrangements of A-E) and 3! would give you the number of combinations in each group (i.e. arrangements of the 3^{rd}-5^{th} letters).

Carson: Well, 3! gives you the number of ways you can arrange the last 3 letters given the first 2 letters.

Anne-Marie similarly explained that she understood why division by 6 created the single desired outcome, and that the 6 was achieved by 3!. As with Carson in the Horse Race Problem, the students' generation of a desired outcome from a collection of representative outcomes constitutes employment of descriptive equivalence. Accordingly, the students' motivations for division were rooted in considering each of the 6 outcomes as a version of the desired singular outcome from which generation of the 1 desired from the 6 duplicates could follow.

Following this activity, the students expressed solutions to permutation problems through division, and explained their process as "getting rid of unwanted" outcomes. In general, the students throughout the rest of the teaching experiment *explicitly* attended to whether certain outcomes generated by a counting process could be seen as duplicates of other outcomes under the constraints of the problem, thus continuing to employ descriptive equivalence. This was a notable component of students' determination of when multiplication was appropriate and when addition was appropriate. As determining the operations appropriate for the constraints of a particular counting problem is an area of difficulty for students (e.g., Batanero et al., 1997), the students' use of descriptive equivalence was productive for their overall counting.

Conclusion

In this report, we have presented and discussed an initial framework for analyzing students' reasoning about equivalence across undergraduate mathematics. We exemplified the utility of this framework by demonstrating its constructs through a discussion of three different mathematical concepts, and by presenting student data from a combinatorial context. We are motivated by the fact that despite the fundamental nature of equivalence in K-16 mathematics, few frameworks offer constructs and language that span domains and levels of mathematics.

As exemplified in our analyses, students might interpret established equivalences between objects and spaces in myriad ways, each of which might have implications for the ways students carry out goal-oriented activity with the objects. In addition to providing unifying accounts of the associations that students make between various mathematical objects, this framework also offers tools for identifying productive aspects of students' engagements with equivalence, such as the productivity of the combinatorics students' uses of *descriptive* equivalence to determine whether subtraction or division was appropriate in a permutation calculation.

Our hope is that we and other researchers can refine this framework by applying it to empirical data in a variety of domains and topics. Moreover, conceptual analyses such as those in this report can serve as a foundation for design research that targets these concepts. We offer these theoretical analyses as inspiration for future conceptual analyses and empirical studies in which equivalence is considered to serve a key role in students' reasoning.

Acknowledgments

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Calculus Reconceptualized Through Quantities

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We delineate the historical evolution of Calculus as a subject. We reframe Calculus as a study of quantities, and reclaim for differentials the interpretation given by Leibniz, infinitesimal quantities. With a conceptual analysis, we create a framework in Calculus for relevant types of quantities and the relationships between them. We demonstrate how this framework is a coherent reconceptualization for Calculus and can be a powerful tool for instruction and understanding.

Keywords: Calculus, Quantitative Reasoning, Infinitesimals, Differentials

Introduction

Calculus has been an area of extensive educational research. Due to its role in the U.S. STEM curriculum, student success in Calculus has many stakeholders: in high school and in college, in mathematics and in other STEM fields (Bressoud et al., 2013).

What is Calculus? It has been summarized, particularly for Single Variable Calculus (SVC), as limits, derivatives, integrals, and the Fundamental Theorem of Calculus (FTC), although some exclude limits (Thompson, Byerly, Hatfield, 2013). Calculus as invented by Leibniz does not match Calculus as it is taught today. It was originally a study of varying quantities (Boyer, 1949). One of the central notions of Calculus is quantities which are arbitrarily small. Leibniz's approach was to use infinitesimal quantities, such as *dx*, referred to as differentials, with which he constructed ratios such as $\frac{dy}{dx}$ and infinite sums such as $\int f(x)dx$. An alternate approach employed by Cauchy was to make *dx* arbitrarily small by being any value so that $0<dx<\Delta x$ for some specified Δx. Weierstrass used limits, and subsequently provided the first rigorous proofs of Calculus techniques (Boyer, 1949). In this formulation, which is the dominant pedagogy: *dx* is not a quantity but an indicator of a variable and its role; the derivative $\frac{dy}{dx}$ is not a ratio of quantities but shorthand for a limit of a difference quotient; the integral $\int f(x)dx$ is not a sum of quantities but shorthand for a limit of Riemann sums.

Presently for Calculus, in instruction and research, the overwhelmingly dominant perspective is to place limits early in the development of the material. The topic occurs first in most textbooks, and it is presented in the research as an essential step in understanding derivatives (Zandieh, 2000) and integrals (Sealey, 2014). Problematically, student difficulty with limits is extensive (Oehrtman, 2009).

Nearly a century after Weierstrass, Robinson (1966) provided the mathematical proofs for infinitesimals. Calculus textbooks by Keisler (1976) and Henle & Kleinberg (1979) adapted those proofs for an undergraduate audience. Instances of teaching Calculus interpreting differentials as infinitesimals (Sullivan, 1976; Ely, 2017; Thompson & Dreyfus, 2016) often include pleas to adopt the given method and seem to be isolated instances rather than the establishment of a new standard.

Recently, some researchers have called for an alternate approach for Calculus which is separated from Analysis and is centered on applications involving variable quantities with a central role for differentials (Augusto-Milner & Jimenez-Rodriguez, 2021). Some research has found that students can develop robust conceptions of the three main ideas of SVC without limits

(Tall, 1986; Samuels, 2012). In non-mathematical STEM fields, where practical considerations are paramount, differentials as infinitesimal quantities are used extensively (Fraser, 2015).

For this theoretical report, we revisit the literature on quantitative reasoning, and observe some theoretical gaps before filling them in. We perform a conceptual analysis of Single Variable Calculus, using the lens of quantitative reasoning, which produces a framework for relevant quantities and the relationships between them. We demonstrate how this framework is a coherent reconceptualization for Calculus and can be a powerful tool for instruction and understanding.

Theoretical Lens

One stage of this work is a *conceptual analysis* of Single Variable Calculus. Conceptual analysis "can be employed to describe ways of understanding ideas that have the potential of becoming goals of instruction or of being guides for curricular development... [It is for] describing ways of knowing that might be propitious for students mathematical learning" (Thompson, 2008, p46). In this conceptual analysis, we particularly attend to *quantitative reasoning*. This mode of reasoning entails "conceptualizing a situation in terms of quantities and relationships among quantities" (Thompson & Carlson, 2017, p425), where a quantity is a measurable attribute combined with a way to measure that attribute (ibid.).

Quantitative Reasoning

Schwartz (1988) identified two types of quantities: extensive and intensive. An *extensive quantity* can be measured directly and exists independently (i.e. it does not stand in relation to another quantity). An *intensive quantity* cannot be measured directly and expresses a multiplicative relationship between two quantities. Nunes et al. (2003) noted that extensive quantities can be combined in the original units. However, they did not address what the outcome of a combination could be. To resolve that question, we contribute two additional quantity types.

A *disjunctive quantity* is a subtractive combination of two quantities of like units which has the units of the original quantities. As with an intensive quantity, it is important to maintain the awareness that a disjunctive quantity is comprised of two extensive quantities while also forming a consolidated single quantity. Similar to an intensive quantity, it increases with increases in one or decreases in the other of its constituents. A *conjunctive quantity* is an additive combination of multiple quantities in the same units which operates in the units of the original quantities. It increases with an increase in any constituent quantity. The last three types are more complex to form than extensive quantities, as they require the coordination of multiple quantities, and collectively we call them *compound quantities*.

Differentials as Infinitesimals

We interpret differentials as infinitesimals consistent with the conceptions of Leibniz and the proofs of Robinson. Here we present a brief explanation of some key ideas; for a full explanation see (Keisler, 1976). A positive infinitesimal is a positive number ε which is less than every positive real number: $0<\varepsilon<r$, $\forall r \in\mathbb{R}^+$. One can visualize infinitesimals graphically, as in Figure 1. Given the real number line, infinite magnification at a point reveals only one real value *a* and an infinite number of values of the form $a+\varepsilon$ (called *hyperreals*), each one infinitely close to the real number *a*. Every finite hyperreal number can be written with a unique real part (called the standard part), meaning it is infinitely close to a unique real number, which can be expressed as

$a+\varepsilon \approx a$ or $st(a+\varepsilon)=a$. One can do arithmetic with hyperreal numbers, just as one would do with real numbers. Typically, at the end of a calculation, one gets a final real answer by taking the real (standard) part. Note that $st(\varepsilon)=0$. Limit statements have equivalent hyperreal statements.

Infinitesimal rules provide procedures to calculate derivatives. Define $f'(x) = st\left[\frac{dy}{dx}\right]$.

Sample exercise: for $y = x^2$, find $f'(x)$

$$\frac{dy}{dx} = \frac{f(x+dx)-f(x)}{dx} = \frac{(x+dx)^2-x^2}{dx} = \frac{x^2+2x\,dx+(dx)^2-x^2}{dx} = 2x + dx$$

$$f'(x) = st\left[\frac{dy}{dx}\right] = st[2x + dx] = 2x$$

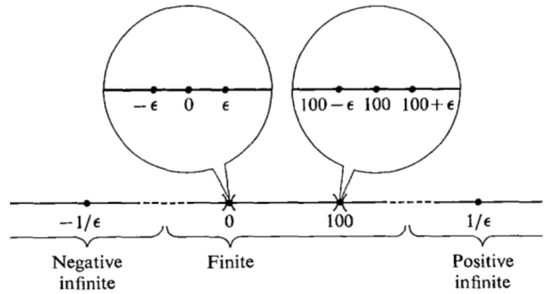

Fig 1. Infinite magnification reveals values infinitely close to a real number (from Keisler, 1976).

Conceptual Analysis of Calculus

To this point, we have argued for the primary role in Calculus of quantities, and briefly described how differentials operate as meaningful infinitesimal quantities. Next, we seek to develop a categorization of the quantities we encounter in (Single Variable) Calculus, and describe the ways they interact with each other. This opens the way for a quantitatively oriented reconsideration of the Calculus curriculum.

To begin, we consider one version of the central statement of Calculus, the FTC: $f(b) - f(a) = \int_a^b f'(x)dx$. We investigate which quantities are present in that formula. On the left side of the equation, there are two extensive quantities, which might be called amounts. They are combined into a disjunctive quantity, often referred to as the change in $f(x)$. On the right side is an integral, which in Leibniz' formulation was an infinite sum. This is a conjunctive quantity, commonly referred to as an accumulation. The terms being summed are each composed of an (infinitesimal) change in x, and the rate of change for $f(x)$, an intensive quantity. The four types of quantities previously described in the section on Quantitative Reasoning are all present in the FTC. We state now, and will continue to support in the rest of the report, that the four essential quantities of Calculus are: Amount, Change, Rate, Accumulation. Those four quantities and the relationships between them form the ACRA Framework for Quantities in Calculus.

An *amount* is an extensive quantity which could take on any real finite value. In an applied situation, it would be the measured value of a quantified property (including units) (Thompson & Carlson, 2017). A variable is used to represent an amount, and Thompson & Carlson noted that a variable might be varying, a parameter, or constant. We can make statements about single amounts, or relationships between two or more amounts. The former can be represented graphically with a point on a coordinate axis, the latter with a point on a coordinate graph.

A *change* is a disjunctive quantity which is the difference between two amounts of the same quantity. It can be written for real quantities as $\Delta y \equiv y_2 - y_1$; if $y=f(x)$, then $\Delta y \equiv f(x+\Delta x) - f(x)$. Including infinitesimal quantities, we might have dx or dy; if $y=f(x)$, then $dy \equiv f(x+dx) - f(x)$. Graphically, it can be represented with a (directed) segment on (or parallel to) an axis. Real and infinitesimal versions are depicted in Figure 2a-b.

Table 1. The ACRA Framework for Quantities in Calculus.

QUANTITY	**DESCRIPTION AND GRAPH**	**VALUES INVOLVED**	**FORMULA**
Amount	A magnitude or extensive quantity Graphically, a point on a coordinate axis or graph	real	x, y
		infinitesimal	
Change	A difference between two amounts (of the same quantity)	real	$\Delta x = x_2 - x_1$ $\Delta y = f(x+\Delta x) - f(x)$
	Graphically, a directed line segment on (or parallel to) a coordinate axis	infinitesimal	dx, dy $dy = f(x+dx) - f(x)$
Rate	A quotient of two changes (of different quantities)	real	$m = \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1}$
	Graphically, the slope of a straight line	infinitesimal	$\dfrac{dy}{dx}$
Accumulation	A sum of consecutive changes	real	$x_n - x_o = \Sigma_i (x_i - x_{i-1}) = \Sigma_i \Delta x_i$ $f(b) - f(a) = \Sigma_i (y_i - y_{i-1})$ $= \Sigma_i \Delta y_i = \Sigma_i m_i \cdot \Delta x_i$
	Graphically, a directed line segment on (or parallel to) a coordinate axis, composed of subsegments	infinitesimal	$b - a = \int_a^b dx$ $f(b) - f(a) = \int_{x=a}^{x=b} dy$
FURTHER RELATIONSHIPS			
Rate Equation	The product of two rates is another rate (focus on the case in which one denominator variable and one numerator variable match)	real	$\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$
		infinitesimal	$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
Change Equation	The product of a rate (of one varying quantity with respect to another varying quantity) with a change in the other quantity is a change in the first quantity	real	$\Delta y = \dfrac{\Delta y}{\Delta x} \cdot \Delta x$
		infinitesimal	$dy = \dfrac{dy}{dx} \cdot dx$

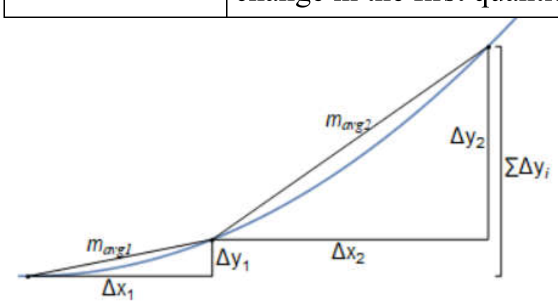

Fig 2a. A graphical representation of a real change (Δx_1), rate (m_{avg1}), and accumulation ($\sum \Delta y_i$)

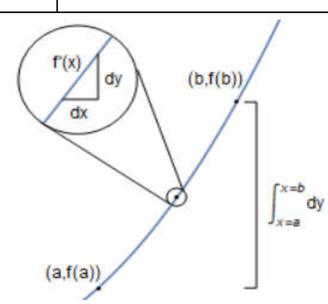

Fig 2b. A graphical representation of an infinitesimal change (dx), rate (f'(x)), and accumulation ($\int dy$)

A *rate* is an intensive quantity which is the ratio of two changes (of different quantities) (Schwartz, 1988). For real changes, the rate is $\frac{\Delta y}{\Delta x}$. For infinitesimal changes, the rate is $\frac{dy}{dx}$. Graphically, it is a line's slope. Real and infinitesimal versions are depicted in Figure 2a-b.

An *accumulation* is a conjunctive quantity which is the sum of consecutive changes. Note that an accumulation is itself a change, but specifically one for which we also consider all constitutive changes. It may involve only one amount or multiple related amounts, and only real changes or including infinitesimal changes. An accumulation may occur in multiple variations:

- A finite sum of real numbers: $x_n - x_o = \Sigma_i (x_i - x_{i-1}) = \Sigma_i (\Delta x_i)$
- A finite sum of reals from an amount relationship $y = f(x)$: $y_n - y_o = \Sigma_i (\Delta y_i) = \Sigma_i m_i \cdot (\Delta x_i)$
- An infinite sum of infinitesimal changes: $b - a = \int_a^b dx$
- An infinite sum of infinitesimals from an amount relationship $y=f(x)$: $f(b) - f(a) = \int_{x=a}^{x=b} dy$

Graphically, the accumulation can be represented as a directed segment on (or parallel to) a coordinate axis, with marked subsegments which represent the constituent consecutive changes.

As it stands, we have elucidated the categories of quantities in Calculus. Now we describe the ways in which they interrelate. The first three are extracted from the definitions, the last two are additional relationships.

- A difference of two amounts (of the same quantity) is a change
- A quotient of two changes (of different quantities) is a rate
- A sum of consecutive changes is an accumulation.
- The product of two rates is another rate (we focus on the case where one denominator variable and one numerator variable match)
- The product of a rate (of one varying quantity with respect to another varying quantity) with a change in the other varying quantity is a change in the first varying quantity.

Note that the quantities and their defining relationships are hierarchical, in the sense that *change* is defined in terms of *amount*, and *rate* and *accumulation* are each defined in terms of *change*.

Here we make several observations. As noted by Johnson (2011), a situation involving rate can be understood two ways, either associating extensive quantities or constructing an intensive quantity. Within the ACRA framework, the latter aligns with the definition of rate, while the former matches a change equation. Thus we can expect that both interpretations are necessary and useful.

One can take a real accumulation equation and substitute using the real change equation to get: $f(b) - f(a) = \Sigma_i m_i \cdot \Delta x_i$, where m_i is $\frac{\Delta y_i}{\Delta x_i}$, the average rate on the interval $[x_{i-1}, x_i]$. One can proceed similarly for an infinitesimal version: $f(b) - f(a) = \int_a^b f'(x)dx$, where $f'(x)$ is the infinitesimal rate. This suggests that, if we want students to conceive of an integral as an accumulation, we can support that prior to Calculus courses by fostering a conception of accumulation with real values on which they can build.

If students meaningfully understand ACRA, the four quantities and the relationships between them, they can immediately construct the FTC using those meanings, as in Table 2. There is potential for students to conceive of the upper boundary as variable, which is conventionally indicated using the letter t, to form the FTC in function form: $f(t) - f(a) = \int_a^t f'(x)dx$. We do not underestimate the difficulty of students coming to possess this conception robustly; research shows extensive student difficulty (Radmehr & Drake, 2017). However, here lies a promising path to this key concept which builds on conceptions of quantities coherently connected throughout Calculus.

Table 2. A meaningful construction of the FTC using infinitesimals and ACRA.

$f(b) - f(a)$	=	$\int_{x=a}^{x=b} df$	=	$\int_a^b \frac{df}{dx} dx$	=	$\int_a^b f'(x)dx$
The total change (accumulation)	is	The integral (infinite sum) of every infinitesimal change	is	The integral (infinite sum) of every infinitesimal change divided by infinitesimal input change times infinitesimal input change	is	The integral (infinite sum) of infinitesimal rate (as a function) times infinitesimal input change

Examples of pedagogical implementation

We now give some examples of typical problems from a wide range of SVC topics to show how the ACRA framework can motivate meaningful student solution processes.

Chain rule. Some research has shown that student knowledge and application of the chain rule is weak (Clark et al., 1997), and that function notation interferes with student performance (Dunmyre & Fortune, 2018). In the ACRA framework, a chain rule calculation is simply variable substitutions (which represent relationships between quantities with amount equations) and a rate equation. Further, using arithmetic to cancel the differentials is justified.

Sample exercise: for $y = \ln(\tan(x^2))$, find $\frac{dy}{dx}$.

Make *amount equations* for substitutions: $y = \ln u$ … $u = \tan v$ … $v = x^2$

Calculate *infinitesimal rates*: $\frac{dy}{du} = u^{-1}$ … $\frac{du}{dv} = \sec^2 v$ … $\frac{dv}{dx} = 2x$

Substitute in the *rate equation*: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = (u^{-1})(\sec^2 v)(2x) = [\tan(x^2)]^{-1} \cdot \sec^2(x^2) \cdot 2x$

Related rates. Notoriously challenging for students are related rates questions (Engelke-Infante, 2007). Using the ACRA framework, they can be solved by identifying the relevant quantities, then implementing an amount equation and infinitesimal change and rate equations.

Sample exercise: Suppose you are filling a spherical balloon with water. The water is flowing at 20 cm^3/min. When the radius is 6cm, how fast is the radius increasing?

Given: $r = 6$, an *amount*; $\frac{dV}{dt} = 20$, an *infinitesimal rate* … Requested: $\frac{dr}{dt}$, an *infinitesimal rate*

First construct an *amount equation* for r & V, a necessary precursor: $V = {}^4/_3\, \pi r^3$

From this, generate an *infinitesimal change equation*: $dV = 4\pi r^2\, dr$

Divide by dt, creating an *infinitesimal rate equation*: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

Substitute known values and solve.

Integration applications. There is extensive documentation that applications of integration are difficult for students (Wagner, 2018; Jones, 2013, 2015). The most common conception of integral is antiderivative or area under a curve (Jones, 2015; Fisher & Samuels, 2016), which is of limited help for modeling. There are several frameworks for productive quantitative reasoning about integrals. Quantitative-Based Summation (Simmons & Oehrtman, 2019) and Adding Up Pieces (Jones, 2013) detail how Riemann sums and integrals can be conceived as a sum of quantities. Multiplicative-Based Summation (Jones, 2015) notes the importance of recognizing the multiplicative and additive structure in Riemann sums and integrals. ACRA builds on these because its relationship rules prescribe *how* to form the needed expressions.

Sample exercise: For density function $p(x)$ (kg/m), find the mass m (kg) between $x=a$ and $x=b$.

Recognize that density is an *infinitesimal rate* of mass per length: $p = \frac{dm}{dx}$

The desired mass is an *accumulation*: $m = \int_{x=a}^{x=b} dm = \int_a^b \frac{dm}{dx} dx = \int_a^b p\, dx$

Integration applications in which the rate function given is *not* a derivative with the boundary variable are particularly challenging for students (Simmons & Oehrtman, 2017). Unlike the previous problem, it is not a straightforward application of the FTC. The ACRA framework suggests the solution method: use an infinitesimal change equation to match the variable of integration to the boundary variable while also aligning with the given information.

<u>Sample exercise</u>: A hill has its mass *m* spread unevenly in a field, density p (kg/m^2) is related to *r* (m), the distance from the center: $p = 8e^{-r}$. How much mass is between 1m and 2m from center? [Note that $m'(r)$ is not given, so it is impossible to use $m(2) - m(1) = \int_1^2 m'(r)dr$.]

Let *A* = area. Note that density is an *infinitesimal rate* of mass per area: $p = \frac{dm}{dA}$

The requested mass is an *accumulation*: $m = \int_{r=1}^{r=2} dm$

Use an *infinitesimal change equation* … twice: $= \int_{r=1}^{r=2} \frac{dm}{dA} dA = \int_1^2 \frac{dm}{dA} \cdot \frac{dA}{dr} dr$

Form an *amount equation* for *A* and *r* to find that *infinitesimal rate*: $A = \pi r^2 \rightarrow \frac{dA}{dr} = 2\pi r$

Substitute and solve $= \int_1^2 8e^{-r} \cdot 2\pi r\, dr$

Discussion & Conclusion

We have described the interpretation of differentials as meaningful infinitesimal quantities, used a conceptual analysis to construct the ACRA Framework for Quantities in Calculus, and demonstrated the productive usage of that framework in a wide variety of problems from SVC. In the sample solutions, every step is drawn from ACRA, so it is accompanied by a foundation in a real equation and a meaningful interpretation. This approach can provide a robust, consistent backbone of meaning for students across topics and contexts in SVC. Future research could consist of devising a hypothetical learning trajectory (Larson et al., 2008) for each topic in SVC, enumerating goals, learning evolution, teacher role, and tasks. Future work could also seek to extend this conceptual analysis and framework to multivariable calculus.

This conception of Calculus is a striking departure from usual practice. However, in the standard curriculum, the hardest material (limits) is presented first as the foundation, and students often fail to connect it meaningfully to their subsequent Calculus conceptions. Conceptions of infinitesimal values occur spontaneously for students within their math classes without intentional instruction, or even with lessons to the contrary (Tall, 1992; Ely, 2010). From that, one could conclude that it is an achievable and desirable goal to build on these notions.

Wagner (2018) documented difficulties with integration applications for students without a quantitative conception of differentials. In fact, he found that the more successful students were those who utilized quantitative meanings learned in non-mathematics classes. Jones (2017) noted that student understandings of derivative were most robust in a narrow range of applications, and a wider variety should be introduced. More broadly, Ferguson (2012) documented the mismatch between Calculus taught in Mathematics Department courses and the use of the content in STE courses. The standard Calculus pedagogy discourages quantitative reasoning (Augusto-Milner, Jimenez-Rodriguez, 2021), and it eschews infinitesimals even though they are at the heart of the STE disciplines and have a rigorous mathematical foundation. This theoretical report suggests that one solution may be a Calculus curriculum organized by the ACRA framework and utilizing differentials as infinitesimal quantities.

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Instructions and constructions in mathematical proof

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In mathematics education, proofs are often conceptualized as series of assertions connected by logical deduction. We argue that this perspective has had a considerable influence on how mathematics educators analyze proofs, conceptualize proof comprehension, and teach proof. However, in both mathematical practice and in undergraduate instruction, the sentences in proofs are not always assertions. Some sentences are instructions and some proofs are best viewed as constructions that explain how to build a mathematical object. As instructions have a different truth semantics from assertions, models of proof as a series of assertions are incomplete. We present a model of how we believe construction-based proofs should be validated and understood. We conclude by using process-object theories to explain why some proofs are intrinsically difficult for students to understand.

Keywords: Proof; Proof comprehension; Proof reading

Proof plays a central role in advanced mathematics classes. University mathematics students are expected to be able to write proofs (e.g., Karunakaran, 2018), check proofs for correctness (e.g., Selden & Selden, 2003), and learn from the proofs that they read (e.g., Conradie & Firth, 2000). Unfortunately, as the previous citations illustrate, a large body of research demonstrates that many university mathematics students struggle with each of these tasks. Naturally, there has been considerable effort to understand why university students struggle with proof and how mathematics educations might ameliorate the situation.

Any mathematics education research on proof presupposes (perhaps tentative working) answers to difficult epistemological questions: What is a proof? What does it mean for a proof to be correct? What does it mean to understand a proof? As Balacheff (2008) documented, mathematics educators notoriously disagree on what the answers to these questions should be. Nonetheless, we have found that many mathematics educators have conceptualized proof as a series of mathematical assertions, where each new assertion in a proof is either an acceptable starting point or a deductive consequence of previous assertions. We believe this perspective is limiting; we contend that many proofs in both mathematical practice and undergraduate mathematical textbooks are actually not of this type.

We have four goals for this theoretical report. First, we hope to persuade our audience that many proofs, both in mathematical practice and in undergraduate mathematics, are not series of assertions. Instead, many proofs contain *constructions*, which are series of instructions that the reader is asked to apply to build mathematical objects. Second, because instructions have different truth semantics than assertions, we argue that the standards for evaluating construction-based proofs differs from evaluating sequences of assertions. We describe what those standards are. Third, we provide a model of what it means to comprehend construction-based proofs. Finally, we use process-object theories of education (Tall et al., 1999) to explain why some construction proofs will be intrinsically difficult for students to understand.

Assertions and instructions in mathematical proof

Assertions in proofs

In this proposal, a *mathematical assertion*[1] is a declarative statement that expresses a relationship between mathematical objects and their properties. Mathematical assertions may be assigned epistemological values. For instance, "5 is a prime number" is a true assertion and "*p* is a prime number" may be true, false, or indeterminate, depending upon how *p* is defined. There may be a logical relationship between assertions. In particular, one assertion may be a logical consequence of another.

Many mathematics educators' conceptualization and analysis of proofs treat proofs as a series of assertions. For instance, in a series of influential articles, Duval (1991, 2007) claimed that proofs are series of assertions in which specific attention is paid to the operational status of each assertion. Stylianides' (2007) widely used characterization of proof begins with the premise that a proof is a special type of "connected series of assertions for or against a claim" (p. 291).

To analyze the validity of a proof, mathematics educators often conceptualize proofs as a sequence of claims $A \rightarrow B \rightarrow C \rightarrow D\ldots$ where the validator is expected to understand how each new claim follows from previous assertions in the proof (e.g., Inglis & Alcock, 2012). This assumption is the basis for using Toulmin's (2003) scheme to analyze proofs (e.g., Reid & Knipping, 2010), where a new assertion of a proof is viewed as the "conclusion", previous assertions in the proof are viewed as "data", and there is a (sometimes implicit) "warrant" that is itself an assertion for why the data necessitates the conclusion. Toulmin's scheme has been commonly used to analyze the structure and correctness of student-generated arguments (e.g., Pedemonte, 2007), proofs in lectures (e.g., Fukawa-Connelly, 2014), and textbook proofs (Weber & Alcock, 2005).

The most widely used models of proof comprehension also explicitly characterize proof as a sequence of assertions. For instance, Yang and Lin's (2008) Reading Comprehension for Geometry Proof (RCGP) model was heavily influenced by Duval's (1991, 2007) theoretical analysis of proof, and is comprised of four levels: Understanding the meaning of assertions in a proof, recognizing the operational status of each assertion (is it a definition, hypothesis, deduction, or conclusion?), justifying how a new assertion can be deduced from previous assertions, and encapsulating a string of assertions into a general method for deducing a conclusion from a set of premises. Finally, proof is often introduced to students as an argument that presents "statements" and "reasons", as is encapsulated in a rather extremely literal way with the two-column proof format commonly used in high school geometry (e.g. Herbst, 2002).

To avoid misinterpration, it would be inaccurate to say that *all* mathematics educators characterize proof as a series of mathematical assertions. Simon's (1996) transformational reasoning and Harel and Sowder's (1998) transformational proof schemes are two important alternative accounts of proof. We also do not claim that the authors that we cited here would claim that only assertions may appear in a proof. Our point is simply that the presumption that proofs are series of assertions undergirds much of the research on proof in mathematics education.

Instructions in proofs

[1] Throughout this proposal, we will treat "mathematical assertion" and "mathematical statement" synonymously.

In this proposal, we define an *instruction* in a proof as a prompt for the reader to engage in a mathematical action. Instructions are typically provided in two forms: as an imperative command or as a declarative sentence with "we" as the subject. A corpus analysis has found that instructions are commonly used in the proofs that mathematicians write for one another (Tanswell & Inglis, in press). We also claim that instructions are common in the proofs that undergraduate students read. To illustrate, consider the passage below from an undergraduate real analysis textbook. (We number the sentences in the proof to facilitate discussion, but otherwise the proof is copied verbatim).

Theorem 3.57 (Bolzano-Weierstrass). Every bounded sequence of real numbers has a convergent subsequence.

[1] Proof. Suppose that (x_n) is a bounded sequence of real numbers.

[2] Let $M = \sup_{n\in\mathbf{N}} x_n$, $m = \inf_{\in\mathbf{N}} x_n$,

[3] and define the closed interval $I_0 = [m, M]$.

[4] Divide $I_0 = L_0 \cup R_0$ in half into two closed intervals, where $L_0 = [m, (m + M)/2]$, $R_0 = [(m + M)/2, M]$.

[5] At least one of the intervals L_0, R_0 contains infinitely many terms of the sequence, meaning that $x_n \in L_0$ or $x_n \in R_0$ for infinitely many $n \in N$ (even if the terms themselves are repeated).

[6] Choose I_1 to be one of the intervals L_0, R_0 that contains infinitely many terms and choose $n_1 \in \mathbf{N}$ such that $x_{n1} \in I_1$.

[7] Divide $I_1 = L_1 \cup R_1$ in half into two closed intervals.

[8] One or both of the intervals L_1, R_1 contains infinitely many terms of the sequence.

[9] Choose I_2 to be one of these intervals and choose $n_2 > n_1$ such that $x_{\mathrm{n}2} \in I_2$.

[10] This is always possible because I_2 contains infinitely many terms of the sequence.

[11] Divide I_2 in half, pick a closed half-interval I_3 that contains infinitely many terms, and choose $n_3 > n_2$ such that $x_{n3} \in I_3$.

[12] Continuing in this way, we get a nested sequence of intervals $I_1 \supset I_2 \supset I_3 \supset \ldots I_k \supset \ldots$ of length $|I_k| = 2^{-k} (M - m)$, together with a subsequence (x_{nk}) such that $x_{nk} \in I_k$.

[13] Let $\varepsilon > 0$ be given.

[14] Since $|I_k| \to 0$ as $k \to \infty$, there exists $\mathrm{K} \in \mathrm{N}$ such that $|I_k| < \varepsilon$ for all $k > K$.

[15] Furthermore, since $x_{nk} \in I_K$ for all $k > K$ we have $|x_{nj} - x_{nk}| < \varepsilon$ for all $j, k > K$.

[16] This proves that (x_{nk}) is a Cauchy sequence, and therefore it converges by Theorem 3.46.

(Hunter, 2014, p. 89)

Note that the reader is given a large number of imperatives where the reader is asked to build a sequence of terms and a sequence of intervals by acting upon mathematical objects that have been defined earlier in the proof. The reader is asked to *divide* intervals (see lines [3], [7], and [11]), *choose* points ([3], [9], [11]), *pick* intervals ([11]), and *continue* a process ([12]). We have found this is common in introductory textbooks. In set theory textbooks, the reader is asked to build models, well-order sets, extend partial orderings, and pick and choose objects (e.g., Kunen, 1980). In graph theory textbooks, the reader is asked to partition vertices and color edges (e.g., Anderson, 2002).

Some mathematics educators have noted the prevalence of imperatives in mathematical writing. Pimm (1987) remarked on the "common use of the imperative in mathematical discourse" and said this is a "topic worthy of considerable attention" (p. 72), an observation also made by Ernest (1998). However, Pimm and Ernest do not elaborate on the role of imperatives further. Rotman (1988) claimed that "mathematics is so permeated by instructions for actions to be carried out, orders, commands, injunctions to be obeyed … that mathematical text seems at times to be little more than sequences of instructions written entirely in an operational exhortory language" (p. 8). However, there is little work on how imperatives (or instructions more generally) work in proof in terms of validity and comprehension.

Constructions in proofs

Many proofs in undergraduate education are of the form, "if you give me an object X with property P, I will show you how to construct an object Y with property Q". The Bolzano-Weierstrauss proof presented earlier is an example of a construction proof. Line [1] in the proof stipulates that (x_n) is a bounded sequence Lines [2]-[12] of the proof provide a construction that the reader is to execute (or imagine executing) to produce a subsequence. Lines [12]-[16] constitute a verification that the subsequence that the reader has produced is actually convergent. (Note [12] contains an implicit instruction "*continuing* in this way" and then makes observations about the objects that result). The construction is composed of imperative commands: the reader is asked to *divide* intervals in half, *pick* sub-intervals, *choose* an index of a sequence term in that subinterval, and *continue* doing this an infinite number of times. We use this proof to make three points about constructions.

First, as the reader is reading the construction, we suggest she should be asking whether it is possible to carry out the steps in the construction. If it is not obvious to a mathematically knowledgeable reader why the instruction can be followed, the author is obligated to justify this in the proof. Note step [9] in the proof directs the reader to do two things: (i) "choose I_2 to be one of these intervals", and (ii) "choose $n_2 > n_1$ such that $x_{n2} \in I_2$". The author justifies that (i) is possible in line [8] and that (ii) is possible in line [10].

Second, the author may ask the reader to obey instructions that a human being may not be able to actually carry out in practice. In our example proof, the reader is asked to execute an infinite number of steps, which no human being can do during their finite lifetime. In some cases, the reader will not be able to carry out even a single step in the proof. For instance, consider the sequence (x_n) that is defined by:

$x_n = 1$ if $n = 1$ or there is an $m \leq n$ such that m is an odd perfect number,

$x_n = 0$ otherwise.

The terms of this sequence will have an infimum of 0 and a supremum of 1. The sequence will have infinitely many terms in the interval [0, ½] if and only if there are no odd perfect numbers, and infinitely many terms in the interval [½, 1] if and only if there is an odd perfect number. It is presently unknown whether there are any odd perfect numbers, so someone following the construction in the proof could not be expected to decide whether to choose L_0 = [0, ½] or R_0 = [½, 1] as I_1. The main point here is that executing an instruction may sometimes require a type of mathematical omniscience that no human mathematician has.

Finally, in construction-based proofs, claims about the objects that were generated are justified by how the objects that were constructed—that is, from the construction itself. When we follow the construction up through line [12], we generate a sequence of intervals and a sequence of indices. Line [12] then asserts three claims about these objects: (i) "we get a nested sequence of intervals $I_1 \supset I_2 \supset I_3 \supset \ldots I_k \supset$", (ii) "length $|I_k| = 2^{-k}(M - m)$", and (iii) "a subsequence (x_{nk}) such that $x_{nk} \in I_k$". None of these claims are justified in the proof. The implicit justifications for these claims comes from the construction itself. (i) follows because we are continually picking sub-intervals. (ii) follows because each sub-interval is chosen to be half the length of the original interval. (iii) follows because we are choosing our n_k's to be increasing and in $I_{k.}$ Lines [13]-[16]

constitute a straightforward proof as a sequence of mathematical assertions that if the sequence of intervals and indices had the properties given in [12], then (x_{nk}) would be a Cauchy subsequence.

How instructions and constructions differ from assertions

The use of instructions poses challenges for accounts of how proofs should be understood and how we can decide whether a proof is correct. In some mathematics educators' accounting, a step in a proof is judged to be correct by the reader if she can see how the assertion expressed in this step is a valid deductive consequence from previous assertions. Instructions differ from assertions in that they cannot be true or false. "p is a prime number" can sometimes be assigned a truth value; "double p" or "choose a divisor of p" cannot. Similarly, instructions cannot be deductive consequences of previous statements, which poses a challenge in performing Toulmin-style analyses on construction-based proofs.

One potential objection is that construction-based proofs do not differ significantly from assertion-based proofs because one can easily translate instructions to assertions. For instance, saying "Choose a Z with property P" can be viewed as two assertions: "There exists an object with property P" and "Z is assumed to be an object with property P" (with the second assertion justified by the first). We are not persuaded by this objection. We believe it is more parsimonious to take mathematicians' language at face value and believe that they mean what they say. And even if every instruction could be translated into assertions, this implies that more attention in the classroom and the mathematics education literature needs to be paid to how one translates instructions into assertions. We now go further and argue that not all instructions can be translated into assertions in a straightforward manner.

The proof that we presented uses the Axiom of Choice twice—both in picking the interval sequence and in choosing the subsequence term. The use of the Axiom of Choice was masked in line [12] by "continuing in this way". (The Axiom of Choice is not necessary to prove the theorem; one can avoid it by saying "If the left interval contains infinitely many terms, choose that one. Otherwise choose the right interval" and "Let n_k be the least index greater than n_{k-1} that is in the interval I_k". But the Axiom of Choice was used in this proof). Mathematicians are often surprised to find that the Axiom of Choice is invoked in this proof. Indeed, some early 20th century mathematicians who *opposed* the Axiom of Choice published this proof in their textbooks (Moore, 1982)! Hence, we have the situation where mathematicians who oppose the Axiom of Choice and undergraduates who have not heard of the Axiom of Choice can be convinced by this proof. Clearly these individuals are not translating the proof into assertions to obtain this conviction.

A model of validity and comprehension for construction-based proofs

We distinguish between four levels of understanding in which the construction in a proof be understood:

1. Understanding the literal meaning of each instruction in the construction;
2. Being able to justify why it will always be possible to execute each instruction in the construction;
3. Being able to justify why the output of the construction has specified mathematical properties;
4. Interpreting the construction as being goal-directed.

Levels (1), (2), and (3) relate to checking that the claims about the construction are valid. Level (4) speaks to a broader understanding. Regarding level (1), it is worth noting that we cannot expect the reader to actually apply the instruction in any situation, for as we noted, this might involve the reader having knowledge that no current human being has. In these cases, it may be better to say a student understands a construction if she could apply each step if the knowledge needed to do so is either obvious or otherwise given to the student.

Regarding Level 2, as we noted earlier, instructions do not have the same truth semantics as declarative mathematical statements. Ordering the reader to "choose $n_1 \in \mathbf{N}$ such that $x_{n1} \in I_1$" (line [6]) can neither be true or false. For an instruction to be permissible in a proof, it must always be theoretically possible for the reader to implement it, regardless of the previous actions she had taken in following the construction. This suggests two ways that the presence of an instruction may invalidate a proof. First, the reader may find herself in a situation in which it is impossible to follow the instruction. For instance, if the reader was asked to "pick a prime $p > n$ that is even", this instruction would be impossible to follow if $n \geq 2$. Second, a proof might be invalid if the reader could implement an instruction, but it is unclear why this is so. In this case, the proof would have a gap in it. The gap could be bridged by giving more detailed instructions on how the broader instruction could be carried out.

Regarding Level 3, the student needs to use the instructions in the construction to justify why the output of the construction has the properties that it does. We believe this is quite difficult, and we elaborate on this understanding shortly. Regarding Level 4, some scholars have remarked that part of understanding a proof is understanding the general method (e.g., Hanna & Barbreau, 2008; Leron, 1983; Rav, 1999). Students who fail to understand the motivation behind a method may be perplexed and view steps in the proof as "pulling a rabbit out of a hat" (Leron, 1983). What we suggest is that this understanding can be achieved in construction-based proofs by viewing the constructions as *goal directed*. Each step in the construction is designed to ensure that future steps in the construction will be possible to execute or to ensure that the object that is constructed has a desirable property. In this sense, each instruction in the construction permits two types of justification. The first, which we discussed in Level 2, is that it is possible for the reader to carry out the instruction. The second is the motivation for including that instruction in the first place. How does this instruction facilitate the construction of an object with desirable properties? For instance, in line [9] of the proof above, the reader is asked to "choose $n_2 > n_1$ such that $x_{n2} \in I_2$". The reader can ask why this is possible—a justification is given in step [10]. Or the reader can ask why this instruction was included. The answer to this question is that sequence of indices can form a subsequence ($n_2 > n_1$ ensures the indices are increasing) whose tails are within a fixed distance of one another.

Why some construction-based proofs are hard to understand

For the sake of brevity, this section will assume that the reader has some familiarity with the process-object theories of concept understanding in mathematics education (e.g., Tall et al., 1999). We briefly highlight the main points. To fix our terminology, we will frame our discussion in terms of APOS theory (Cottrill et al., 1996). However, as Tall et al. (1999) discussed, the themes that we discuss are equally well supported by the panoply of other process-object theories in mathematics education as well (e.g., Davis, 1984; Sfard, 1991).

An individual may understand a procedure as an *action* if she can apply a set of mathematical transformations in response to an external stimulus. The individual may understand the same procedure as a *process* if she views the procedure as transforming inputs into outputs. She "can describe and reflect upon the steps of the action without necessarily performing them" (Cottrill et al., 1996, p. 171). The key point here is that an individual who only holds an action understanding of a procedure cannot reason about the output of the procedure without actually performing its steps. An individual with a process understanding can anticipate the result of the procedure without actually performing each of the steps.

In a construction-based proof, a Level 1 understanding of a construction requires an action level understanding of the construction. The reader only needs to have a literal understanding of each of the instructions of the construction. However, Level 2 and Level 3 understandings require having a process understanding of the construction. The reader needs to see why steps in the construction are *necessarily* possible to carry out and that the output will *necessarily* have some properties, for arbitrary inputs.

Process-object theories posit that one gains a process understanding by applying a procedure and reflecting upon one's actions. However, proofs simply state what the constructions are, but does not ask the reader to actually carry out the instruction. Furhter, some constructions, such as the one in the proof of the Bolzano-Weierstrauss Theorem presented in this proposal, may be impossible to actually carry out, either because they involve applying infinitely many steps or because some instructions require mathematical omniscience to carry out that no human being possesses. This suggests that some construction-based proofs may be extremely difficult for students to understand.

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A Framework for Designing Intellectual Need-Provoking Tasks

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Intellectual need (IN) is a powerful way to support learning by engaging students and helping them view mathematics as less arbitrary. While IN has been developed theoretically, much less has been done to build frameworks for how to actually create IN provoking tasks – both in terms of what a task designer might attend to and how to attend to those things. In this theoretical paper, we review key premises in IN, from which we extract several components that should be taken up in IN task design. We then describe a process one can use to address these components systematically in constructing a task specifically meant to provoke IN.

Keywords: Task Design, Intellectual Need, Learning Environments

Introduction and Literature Review

As instructors, we routinely look for ways to support our students' learning. In particular, we want our students to "engage with mathematical content in ways that go beyond using known facts in standard procedural ways" (Mesa, Burn, & White, 2015, p. 84). Harel's (1998) idea of intellectual necessity, as part of his DNR framework (Harel, 2008a; 2008b), is a powerful framework for conceptualizing and planning instruction. The theory of intellectual need (IN) has been widely used to design instructional tasks. For example, Harel (e.g., 2013b) has conducted numerous teaching experiments for various mathematical topics based on the necessity principle. Leatham, Peterson, Stockero, and Van Zoest (2015) used the idea to design and analyze effective "openings" for lessons, and Abrahamson, Trninic, Gutiérrez, Huth, and Lee (2011) used the idea to design "hooks" for their curriculum of mediated discovery. Other teachers and researchers have used the idea to conduct professional development workshops (e.g., Meyer, 2015), design instructional tasks (e.g., Koichu, 2012; Caglayan, 2015; Foster & de Villers, 2015) and analyze classroom instruction (e.g., Rabin, Fuller, & Harel, 2013; Zazkis & Kontorovich, 2016). Despite the numerous examples of IN-provoking tasks, there have not yet been descriptions of principles for designing such tasks. In this theoretical paper, our contribution is to synthesize ideas within IN theory and to organize them into a systematic framework for designing IN-provoking tasks.

Theoretical Background

Intellectual Need

The instructional principle of intellectual need (IN) was originally suggested by Harel and Tall (1991), who proposed, "if students do not see the rationale for an idea... the idea would seem to them as being evoked arbitrarily; it does not become a concept of the students" (p. 41). This was formalized by Harel (1998) as the necessity principle: "For students to learn what we intend to teach them, they must have a need for it, where 'need' refers to intellectual need" (p. 501).

Harel (2013b) defined the necessity principle as the perceived need to resolve "a perturbational state resulting from an individual's encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, his or her current knowledge" (p. 122). The IN of the individual is defined to be the aforementioned problematic situation. This perturbation is rooted in the individual's experience within the discipline—in this case, of mathematics—and is based on the learner's epistemological justification for the mathematical concept, where an epistemological justification is “the learner’s discernment of how and why a particular piece of knowledge came to be” (Harel, 2013a, p. 8).

IN is grounded in Piaget’s (1985) notion of disequilibrium and situated within Harel’s DNR framework for mathematics curriculum and instruction (e.g., Harel, 2008a). The DNR perspective is based on eight premises about learning mathematics, which we summarize and comment on here (the premises are italicized). Within the premises, we bold certain terms relating to learning environments and we underline certain terms relating to designing a task.

- The *epistemophilia* principle asserts that all learners are capable of developing a desire to be puzzled and to solve puzzles. These puzzles can be embodied in the creation of tasks, and the process of task development is the central focus of our paper.
- The *knowing* premise describes learning as a developmental process of resolving disequilibrium and the *knowing-knowledge linkage* asserts that all knowledge is a result of such a resolution. Thus, the goal of these tasks is to help students <u>enter</u> into a perturbational state that, when <u>resolved</u>, will foster construction of the target knowledge. This perturbation and its resolution are mediated by the **tools and artifacts** that determine whether a scenario is viewed as problematic, and avenues for resolving the perturbation.
- The *knowledge of mathematics* premise asserts that mathematical knowledge consists of institutionalized ways of understanding (WoUs) and ways of thinking (WoTs), which are, respectively, products and characteristics of mental acts (see, e.g., Harel, 2008b). From a complementary perspective, the *subjectivity* premise asserts that students’ knowledge is personally constructed as they engage in various tasks and the *interdependency* premise describes the reflexive relationship between students’ actions and their views of the world. Thus, when designing tasks, it is essential to attend to both the institutionalized WoUs and WoTs, which we refer to as the “**Math**” (with a capital M), as well as the WoUs and WoTs of the **students** who will be engaging with the task.
- The *context dependency* of learning premise proposes that the context in which the Math is learned influences what and how they are learned. Thus, when designing tasks, it is important to consider both the **curriculum** in which the learning is taking place and the **students’** relationship with the context as part of the <u>educational setting</u>.
- The *teaching* premise highlights the role of expert guidance and collaboration as part of the learning process. Thus, it is essential to consider the interactions between the **teacher** and **students** as the task is <u>presented</u> and the disequilibrium is <u>resolved</u>.

The subjective nature of IN is of particular importance, because the situations that “constitute IN for one person may not be so for another” (Harel, 2010, p. 365).

Harel (2008b, 2013b) also explained the relationship between *psychological* need and IN. Psychological need is the <u>interest or willingness to engage</u> in the problem in the first place (Harel, 2008b) and thus pertain to one’s motivation with respect to a task or context (Harel, 2013b). This willingness is crucial before the student can ever personally interact with the problematic situation that provides the IN for the Math. For example, a student with interest in cars might find a task about cars engaging and be motivated to investigate it. Upon investigating it, they may be set up to encounter the problem within the task that is unsolvable by their current knowledge. Alternatively, a student might enjoy puzzles and be willing to work on a provided puzzle, leading to the mathematical problem within it. We view psychological needs as a key factor in designing IN-provoking tasks and incorporate them into our framework.

Closely related to psychological needs are *affective* needs (Harel, 2013b), which also relate to whether one desires to engage with a problem or not. However, affective needs deal more with social factors, such as desiring to get a good grade, to please the teacher, or to be seen as successful. While these affective needs can serve to help students engage in a problem, we consider *psychological* needs to be more powerful in creating motivation. Further, desiring a good grade or wanting to please the teacher are difficult to build into a task, and consequently we do not consider them in our task design framework.

The theory of intellectual need has counterparts in other theorizations of teaching and learning. For example, it shares some ideas with Brousseau's Theory of Didactical Situations (TDS) (Brousseau, 1997). TDS incorporates the idea of a "fundamental situation" in which a mathematical concept constitutes an "a priori optimal solution" for a problem (Artigue et al., 2014, p. 49), which is very similar to the idea of an intellectual need-provoking task. However, in contrast to TDS and other theorizations, Harel's DNR framework is unique in its particular focus on student cognition that is actively mediated by a teacher within an instructional context and with a specific relationship to the discipline.

Task Design

There are numerous theories and projects that have been developed to address issues related to the design and implementation of tasks. To organize the vast body of work, Kieran, Doorman, and Ohtani (2015) draw on Cobb et al's (2003) perspective to distinguish between grand, intermediate, and domain-specific frames for task design. For example, constructivism (e.g., von Glaserfeld, 1987), in which Harel's DNR framework is situated, is a grand frame in that it presents a general theory of learning that can be applied both in and outside of educational settings. A domain-specific frame would describe methods for task design in a specific content area, such as calculus or algebra. In contrast, intermediate frames, such as the DNR framework, "present the complex interactions between task, teacher, teaching methods, educational environment, mathematical knowledge, and learning" (Watson & Ohtani, 2015 p. 5). Although the principles of the DNR framework provide a method for thinking about and identifying these interactions, it does not operationalize a theorization of learning environments in a way that would guide an instructor through the process of creating intellectual need-provoking tasks. Thus, we see the creation of a framework for designing intellectual need-provoking tasks as a component of an intermediate frame of task design that can be used to connect that frame to various domain-specific frames.

Theorization of Learning Environments

Weinberg and Jones (2020) proposed a theorization of learning environments that can facilitate task design for IN-provoking tasks. The model (Figure 1) identified five nodes that correspond to the key aspects of the DNR premises as they relate to task design: The Math (grounded in the knowledge of mathematics premise), the curriculum (part of the context dependency premise), the students (related to the subjectivity, interdependency, context dependency, and teaching premises), the available tools and artifacts (grounded in the knowing-knowledge linkage premise), and the teacher (part of the teaching premise). By making these nodes explicit, the model facilitates the task design process by structuring thinking about individual nodes and relationships between collections of nodes, which are illustrated by the lines, faces, and tetrahedra in the model.

However, while this framework extracted key aspects of learning environments based on Harel's work, there is still an important shortcoming. We believe it desirable for an IN task design framework to do more than just state *what* should be attended to, but to also provide a structure for *how* one might attend to the five nodes in the hexahedron (Figure 1). Harel's work only has general suggestions for how to design an IN-provoking task, including (a) recognizing what INs might be for one's population of students, (b) transforming this need into a set of questions, (c) constructing a sequence of problems whose solutions make progress on the questions, and (d) helping students elicit the Math from the solutions (Harel, 2013b, p. 149). While these are useful recommendations, we find that they lack the detail that might be necessary in a guide for creating IN-provoking tasks. In the next section, we provide this needed detail. The individual aspects of our framework can be seen across Harel's work (e.g., Harel, 2008a; 2008b; 2013b), and our main contribution is to take the ideas woven within Harel's premises and to organize the pieces into a systematic structure for designing IN tasks.

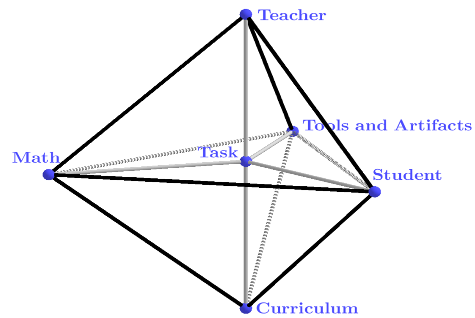

Figure 1. Previous framework of learning environments.

Beyond the "What": The "How" of IN Task Design Process

In this section we provide our main contribution by addressing the *how* of IN task design. We see designing an IN task as consisting of two main "stages," which we call the *development* stage and the *anticipation* stage. The development stage corresponds to the bottom half of the hexahedron, including the *curriculum, student, Math,* and *tools/artifacts* nodes. It is the stage of design dealing with actually putting together potential task(s). The anticipation stage corresponds to the top half of the hexahedron, including the *teacher, student, math,* and *tools/artifacts* nodes. It is the stage of design that imagines how the task would play out, how the IN might be created, how it might be resolved, and how the knowledge can be seen by the students as the solution. In the remainder of this section, we unpack these two stages, explaining how they build from the pieces within Harel's premises and how they attend to all of the nodes in the hexahedron.

The Development Stage

The development stage, which corresponds to the bottom half of the hexahedron (Figure 2) contains three components: *considering the setting* of the lesson, *constructing a question context* (or several candidate contexts), and *identifying the mathematical problem* (within the contexts).

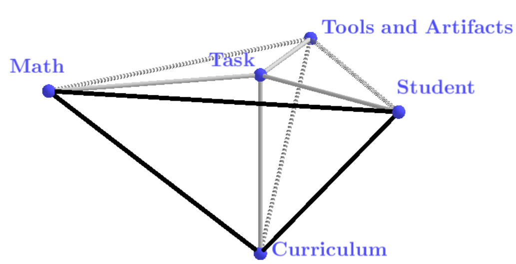

Figure 2. Development stage corresponding to the bottom half of the hexahedron.

Considering the setting. The *setting* is the broader educational setting of the lesson, including the type of class (e.g., a remedial class, a class for math education majors, a proof-based class, etc.); the overall curriculum used and where the target Math fits into that curriculum; the students in the class and their interests, background, and identities; the course goals; and the class norms. This component comes out of Harel's *context dependency* premise, though our description here of what constitutes the "setting" helps elaborate different aspects of this premise. The setting involves the curriculum node in terms of what ideas are available, how the Math connects to previous ideas, and what the goals are for the lesson. It also involves the student node by considering what kinds of interests, mathematical identities, and expectations the students have. It also considers the norms that have been established for the students.

Constructing a question context. The *question context* is the scenario or context presented to the class that is used as a vehicle for exploring some idea that will lead to the relevant Math. The context can be about some real-world phenomenon like motion, money, or health, or about a purely mathematical context like different types of functions. The context would include some initial question that is posed to get the class going, such as "Which is the best business option?" or "How fast is the object going when it hits the ground?" This component originates in Harel's psychological needs, which is crucial in deciding what context is ultimately used by the teacher in the classroom. The *question context* involves the Math node, because the context must "contain" the target Math knowledge. The initial question must lead toward this knowledge. The context also relates to the tools and artifacts that would be available. That is, does the context need to be introduced via a video or image? Does it require data? What computational tools are available? Do students need access to particular symbols or terminology to express or understand the question? Importantly, the question context also directly relates to the student node because it considers whether the students would find the context interesting or engaging, or comprehensible. Note that at this point in the process, a teacher might find several candidate contexts, and the remaining steps in task design might help determine which context is best.

Identifying the mathematical problem. The mathematical problem is the moment the students in the class enter into a perturbational state—that is, the moment the students experience IN. It is how and when the mathematical issue arises that makes students feel the lack in their current knowledge and the need to create new knowledge to address the issue. Continuing the previous examples (in "Question Context"), an instructor might assume that students will not know how to guarantee an option is the best, or how to find a speed at a single instant in time. This component is the crux of an IN task, so it is critical for a teacher developing candidate tasks to identify where the mathematical problem exists within the context. Because the mathematical problem creates the IN for the intended knowledge, it is centrally based on the Math node, though it relates to the tools/artifacts and student nodes, as well. It relates to tools/artifacts in terms of the means that are needed to perceive the mathematical problem, such as a resulting expression or image that does not make sense. It relates to students in terms of whether they have the background to be able to reach and identify the problem. By identifying the problem within possible candidate contexts, a task designer might decide that some problems are more apparent or easily understood, thereby helping decide which context to use.

The Anticipation Stage

The anticipation stage, corresponding to the top half of the hexahedron (Figure 3) consists of taking candidate tasks created during the development stage and imagining how they might play out in the classroom and how the teacher might guide the class through the task. If one wishes to create an IN-provoking task, it is imperative to envision how the class will bump into the mathematical problem and whether they have the tools to resolve it. This stage contains two components: *presenting the context/problem* and *resolving the problem*.

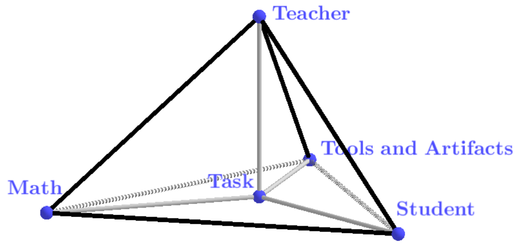

Figure 3. Anticipation stage corresponding to the top half of the hexahedron.

Presenting the context in order to reach the problem. The *presentation* includes both the content of the task (e.g., the background information that will be provided to the students and the phrasing of the questions) as well as the actions the teacher will take, how they expect students to proceed, and how they plan on encountering the *problem*. It also includes whether the context and initial question are understandable to the students, and if they're presented in a way that might create psychological need (i.e., engagement in the question). This component is connected to the *knowing* premise and the *teaching* premise, in that one imagines the steps leading to perturbation and how the teacher will guide the students toward that perturbation. It also connects to psychological needs, in terms of whether the teacher presents the initial questions and highlights the problem in ways that might make the students more curious and interested in solving the question/problem. If one is considering multiple possible tasks, thinking through the presentation could illuminate if a certain context or task would be more understandable or if the problem would be more easily encountered, helping in deciding which context and task to use.

Resolving the problem. Once the problem has been encountered and the IN created, the *resolution* is the teacher's plan for helping the class navigate the solution to the problem in a way that develops the target knowledge. It involves thinking through the students' and teacher's potential actions to resolve the perturbational state. It also involves thinking about how the Math will be developed and used *as* the solution, as well as how competing ideas might be addressed. It must also anticipate whether the students, by the end, would be able to identify how the knowledge resolved the situation, building on the *knowing-knowledge* premise and the idea of *epistemological justification*. Thinking through the resolution might also help one decide if certain contexts and tasks might be better to use than others.

Connection to nodes. Both the *presentation* and *resolution* components involve the teacher, student, tools/artifacts, and Math nodes. They both obviously relate to the teacher, because the teacher is the one that presents the context and question and guides the students through the problem and its resolution. The Math node is also a major part, as the whole purpose of the presentation is to guide to the mathematical problem, and the whole purpose of the resolution is to create the mathematical knowledge that resolves it. This stage also relates to the students, because the way the context might be made engaging depends on the students' interests and knowledge, and the way the students can work up to and from the problem depends on their background and their expectations of their role in the classroom. Tools/artifacts are a key part of this stage because the tools available (e.g., calculator, ruler) and artifacts created (e.g., charts, figures, expressions) are needed components of the classwork leading to and from the problem.

Note about the Two Stages

Lastly, we note that our description of these two stages, and their components, followed a "linear" order. While it would generally make sense for the *development* stage to precede the *anticipation* stage, because the development stage creates a candidate task and the anticipation stage imagines how it would play out, we wish to be clear that the components within each stage could be done in different orders.

Discussion

Harel's work (2008a, 2008b, 2010, 2013a, 2013b) has provided the field of mathematics education with the important construct of intellectual need, which provides a way for us to think about how mathematics can be less arbitrary and more meaningful for our students. Our framework extends Harel's theory by providing an explicit guide for creating tasks specifically meant to provoke IN in students. To do this, we took the ideas woven throughout Harel's theory and systematically organized them into an approach for designing IN-provoking tasks.

This framework has both theoretical and practical uses. Theoretically, our framework helps identify components that play a significant role in a task intended to induce IN. It separates out the question context from the mathematical problem, acknowledges the tools/artifacts necessary to present or make progress on the task, expects that notions of *resolution* be an explicit part of the task itself, and so on. For example, if a task is presented in the literature with a description only of how the mathematical problem can be reached but without a description of how it is to be resolved, that task might require elaboration or revision. Our framework also provides researchers with a tool for analyzing tasks intended to provoke IN. In fact, we believe empirical work examining tasks in light of this/framework is a key next step in this work.

Practically, our framework gives instructors tools to be able to craft IN-provoking tasks. It helps them first develop candidate tasks by thinking through the setting, question context, and where the problem lies within the question context. It then walks through anticipating how it might play out and whether certain tasks might work better than others at either leading *to* the mathematical problem or *from* the problem to the resolution.

While we believe our/framework makes a significant contribution to the area of task design, we also believe there is additional work needed in this area. One issue we have considered is cases when students think they already know the solution to a task and therefore do not honestly engage in it. For example, many students in college calculus courses have previously taken a calculus class. If a teacher creates a task for, say, optimization in calculus based on using derivatives, when the task is first presented a student might think, "Oh, I've seen this. You just take a derivative and set it to zero." This belief might prevent the student from identifying and confronting their own incomplete understanding and from entering a state of disequilibrium. Another issue we have considered is distinguishing between curiosity and confusion, and how either relates to perturbation. That is, when a student experiences disequilibrium, they might interpret this as confusion—which can carry a negative connotation—rather than curiosity or wonderment. A final issue we have considered is the role of "good performance" from a teacher during the setup of a question context. We believe that the same question could be presented in a way that stokes psychological need in students, who become interested and vested in the question. However, the same question could be presented in a way that is not found interesting at all by students, limiting the psychological need that might propel them toward the mathematical problem.

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Conceptualizing Mathematical Transformation as Substitution Equivalence: The Critical Role of Student Definitions

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This theoretical paper explores student conceptions of transformation as substitution equivalence by linking it to their definitions of substitution and equivalence. This work draws on the work of Sfard (1995) to conceptualize substitution equivalence and its components, equivalence and substitution, as a spectrum from computational to structural. We provide examples of students' work to illustrate how student notions of substitution, equivalence, and substitution equivalence as an approach to justifying transformation may related to one another.

Keywords: Equivalence, Substitution, Substitution Equivalence, Structural Thinking, Definitions

Transformation has often been framed as a core mathematical activity (Kieran, 2004), and all mathematical calculation, whether arithmetic, simplifying expressions, or finding the solution sets of equations, can be viewed as a process of transformation. Thus, with the goal of exploring the core mathematical ideas that justify why particular transformations are mathematically valid, we view mathematical transformation through the lens of substitution equivalence, conceptualizing it as a process of replacing one symbolic object with an equivalent one, and naming this process substitution (Wladis et al., 2020). This also includes the process of identifying sub-objects and replacing them with equivalent ones in order to generate a new equivalent object. This process is non-trivial for many students, and we hypothesize that substitution equivalence may be intimately connected to many of the struggles that students have with symbolic mathematics at various levels and domains. Little attention has been paid formally to students' notions of substitution equivalence, even though these notions may be intricately linked to the ways in which students think about and execute various types of mathematical transformation. In this paper we attempt to address that gap, by providing a model of student thinking around substitution equivalence. First we describe the model, including the theories and body of research literature which have informed its creation, and then we proceed to use the model to analyze a few vignettes of student work, in order to illustrate its potential affordances.

Substitution Equivalence as a Lens for Mathematical Transformation

In this paper, we focus specifically on student thinking around *substitution equivalence*, or the notion that two expressions, equations, or other mathematical objects are equivalent if one can be generated from the other through a sequence of substitutions carried out through a combination of correct interpretation of syntactic structure and appropriate use of mathematical properties (Wladis et al., 2020).

Definition of Substitution: In order to see clearly how mathematical activity could be viewed through the lens of substitution equivalence, we define substitution more broadly than has been done explicitly in much existing research and curricula, as the process of replacing any mathematical object (or any unified subpart of an object) with an equivalent object, regardless of complexity. This includes the replacement of x in $2x^2 - 2x + 1$ with -3, but also, e.g., the replacement of $x^2 - 6x = 1$ with the equivalent equation $x^2 - 6x - 1 = 0$ during solving.

Definition of Equivalence: We note that the idea of substitution equivalence is wholly dependent upon an underlying equivalence relation of some kind and depends upon a specific stipulated definition of equivalence. This may be a particular context-specific definition of equivalence (e.g., two equations are equivalent if they have the same solution set), or a more generalized concept of equivalence (e.g., an equivalence relation); however, any definition of equivalence that satisfies the definition of an equivalence relation could be used.

Definition of Substitution Equivalence: We define the domain of substitution equivalence as composed of two main ideas, which we illustrate in more detail in subsequent sections. According to our model, students who have a notion of substitution equivalence recognize:

1. **The *general* notion of substitution equivalence:** They understand that we can replace an object with *any* other equivalent object when problem-solving.
2. **That substitution of unified sub-objects can be used to generate equivalent objects:** They understand that objects can be broken into unified sub-objects, and that we can replace any unified sub-object with any equivalent unified sub-object (and the process of substitution leaves the rest of the structure of that object unchanged).

The second notion leads us to another core definition: We use the term *subexpression* (or sub-object, more generally) to denote a substring of an expression (or other object) that can be treated as a unified object without changing the syntactic meaning of the original expression (or object). E.g., $a - b$ is a subexpression of $a - b - c$, but $b - c$ is not (because putting parentheses around $b - c$ would change the syntactic meaning of the expression).

Model of Operational and Structural Thinking about Substitution Equivalence

Wladis et al (2020) described key features of student thinking around substitution equivalence on a spectrum from structural versus operational approaches. This paper aims to take this further by describing explicitly how student conceptions of substitution equivalence may be dependent upon student definitions of substitution and equivalence (see Figure 1).

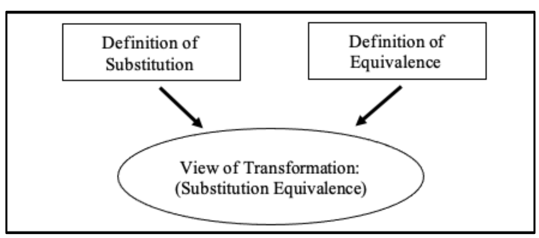


Figure 1: *Model of Student Thinking about Substitution Equivalence*

In the model in Figure 1, holding well-defined and standard definitions of both substitution and equivalence are necessary but not sufficient conditions for students to develop a view of transformation justified by substitution equivalence. A student may have trouble thinking of transformation as substitution equivalence because (a) their definitions of substitution are too narrow; (b) their definitions of equivalence are ill-defined, unstable, or invalid; (c) they do not draw on their knowledge of substitution and/or equivalence when performing transformation; or a combination of all of these. We conceptualize student views of substitution, equivalence, and transformation as being on a continuum from operational to structural (Table 1). This model is based on the notion that the *ability* to conceptualize transformation as a process of substitution equivalence may be useful for students in developing deeper understanding of the justification behind their transformation work (and a way of checking the validity of that work).

Table 1: *Components of substitution equivalence model*

	Operational Thinking	Structural Thinking
View of Transformation	Students see transformations of expressions and equations (or other objects) as a process of “operating on” the original object itself. They may or may not see this as linked to any notion of equivalence.	Students see each step in a transformation as the process of replacing one object with an equivalent one through substitution, using properties and existing syntactic structure. They appear to have some notion of an equivalence class as an object (which need not be formally defined).
Definition of Equivalence	Students either ignore the notion of equivalence entirely, or appear to have only vague, ill-defined, or unstable notions of equivalence, or try to apply one definition of equivalence that works only in one context to another context.	Students have a well-defined and relatively stable definition of equivalence, and recognize that it is context-dependent. They recognize that equivalence is a fixed trait (two objects are either equivalent under a particular definition or not—they do not “become” equivalent).
Definition of Substitution	Students see substitution only as plugging a number in for a variable (and then computing the result). They see variables as representing only numbers.	Students see replacement of any object (or sub-object) with an equivalent one as substitution. They see variables as representing any valid mathematical object, including numbers or (potentially complex) expressions.

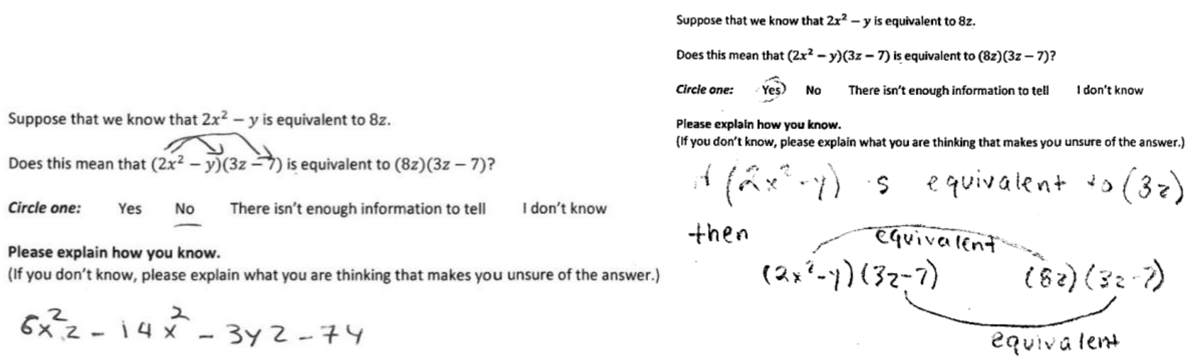
Development of the Model

This work draws on data collected from multiple classes across six years at a northeastern community college, including classroom observations, cognitive interviews, and open-ended questionnaires. These data were analyzed using conceptual analysis (Thompson, 2008) to generate and refine models of students’ thinking to explain their written work and utterances. We note that these models of students’ thinking are based on what the students communicate in the moment and are situated within the given task. Further, their strategies and responses may be impacted a myriad of factors, including but not limited to the wording of the question, the environment they responded in, or the established sociomathematical norms of the classrooms they participate (Yackel et al., 2000).

This analysis was heavily influenced by the work of Sfard (1995), and existing literature about the students’ definitions of mathematical concepts (Edwards & Ward, 2004) and their understanding of equals sign (e.g., Knuth et al., 2006). Sfard (1995) describes that students can conceive of a mathematical concept as a combination of two ways: operationally (as a process, often of computation) or structurally (abstract entities in and of themselves; Sfard, 1995). In terms of equality, similar language and ideas are used in the literature to describe the students’ conceptions of the equals sign, often either operationally (as a ‘do something symbol’; Kieran, 1981), or relationally (as a relationship between two entities; Knuth et al., 2006), though further refining these categories (Rittle-Johnson et al., 2011; Stephens et al., 2013) has been the focus of other research. Though research on equality is plentiful, research on substitution and substitution equivalence as a broader concept is comparatively minimal. For example, substitutive aspects of equivalence have been investigated in the context of arithmetic (Jones & Pratt, 2012), and Musgrave, Hatfield, and Thompson (2015) have found that secondary teachers had particular difficulty correctly applying a given substitution property to expressions when they found operations to be unfamiliar or had difficulty thinking of symbols simultaneously as both a process and an object. They argue that if teachers are having difficulties with these ideas, then these are likely stumbling blocks for students as well.

Vignettes: A Model in Action

We now provide examples of students' written work to illustrate how one might use the model we present here. These are intended to highlight the continuum of the operational and structural views. To see how students' views of transformation as substitution equivalence can vary along this spectrum, we present two developmental elementary algebra students' responses about assessing whether or not two expressions are equivalent (Figure 2), where the first response (Figure 2a) exemplifies an operational view and the second response (Figure 2b) exemplifies a structural view. The first student's response (Figure 2a) appears to foreground computation and symbolic manipulation. In cognitive interviews (not included here because of lack of space), students on similar problems have explained similar work by stating that they can only tell if two expressions are equivalent if they both simplify to the same final "answer", so this approach may happen when students have an internal computational definition of equivalence as "expressions that simplify to the same thing". Regardless, this student's response foregrounds computation, and hence would be considered as an operational view of transformation. In contrast, the response in Figure 2b illustrates exactly how the two equivalent subexpressions are substituted into the larger expressions using arrows to indicate the relationship between each piece and to highlight the structure of the two expressions. They map each unified subexpression in the first expression to an equivalent unified-subexpression in the same place in the second expression, in order to illustrate how they know that the two expressions are equivalent. Though the student doesn't explicitly use the substitution, we do see evidence that they are looking at the underlying structure and visualizing a replacement or exchange of one equivalent sub-part with another.



(a) (b)

Figure 2: *Examples of students' responses rooted in an operational view (a) and structural view (b) of equivalence*

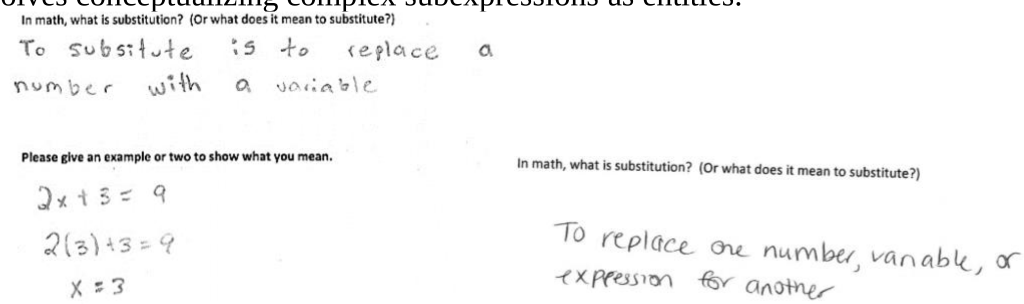
Students' Definitions of Equivalence

To see how students' definitions of equivalence can vary along this spectrum, we refer to the previous two examples and consider the definitions of equivalence the students seem to be evoking. These responses exemplify operational and structural definitions of equivalence, respectively. In the first response (Figure 2a), the student attempted to simplify the expressions to determine whether they are equivalent, and then appeared to decide that they are not equivalent after they could not immediately simplify them both to the same expression. This definition of equivalence appeared to be computational (e.g., “two expressions are equivalent only if they simplify to the same thing”), and their work doesn’t seem acknowledge the equivalence within their work. Because the student abandoned the attempt after this did not work, this suggests that they did not see a way to use the structure of the given expressions to determine equivalence beyond simplifying both sides to see if the results are the same.

In contrast, the response in Figure 2b that the student may have a structural definition of equivalence. In this example, they are drawing on the structure of two complex expressions to show how they map to one another in such a way that each subexpression is either the same or equivalent, and leverage that equivalence to show that the final result will be equivalent. This apparent definition of equivalence appears to be well-defined and potentially could be a fixed trait of a set of objects.

Students’ Definitions of Substitution

To exemplify the differences along this spectrum, we look at two students’ definitions of substitution (Figure 3). Throughout data collection, the response in Figure 3a (“putting a number in for a letter”) is one of the most common given by students at all levels, from elementary algebra through linear algebra. This narrow definition of substitution would be considered operational, while the response in Figure 3b would be considered structural view of substitution. This is because their definition affords a greater variety of terms to be replaced for one another, which involves conceptualizing complex subexpressions as entities.



(a) *(b)*

Figure 3: *Examples of an operational (a) and structural (b) definition of substitution.*

In order to see how student views of substitution may impact their view of transformation of expressions, we further examined students’ responses to a task to identify instances of substitution, and found that their responses were typically consistent with their definitions (e.g., only recognizing transformation as substitution when it involved a number being substituted in for a letter if that was their stated definition); we include one example of this in the next section.

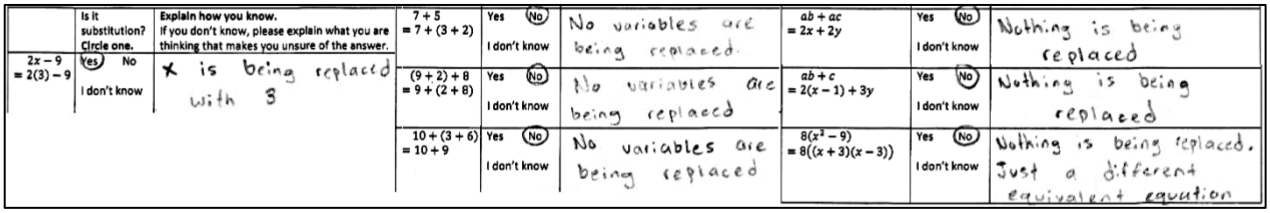
Using the Framework to Analyze Student work longitudinally

In order to illustrate the potential of this model for deeper analysis, we consider responses from an Algebra I student (whom we call Epsilon, like ε) across multiple tasks and points in time.

Substitution: We first consider Epsilon’s definition of substitution (Figure 3a), where they have given an operational definition, rather than a structural one. This correlates with the extent to which they identify different computations as substitution in the following work (Figure 4).

We can see in Figure 4 that Epsilon rarely identified computation as substitution when it was more complex or generalized. They notice, for example, that the expressions in the last example in Figure 4 are equivalent, but they do not see replacement of the subexpression $x^2 - 9$ with

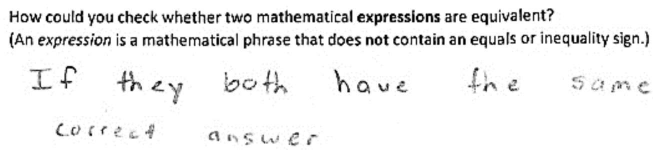
$(x+3)(x-3)$ as an instance of substitution ("nothing is being replaced"), which is consistent with the more limited operational definition of substitution that they gave in Figure 3a.



	Is it substitution? Circle one.	Explain how you know. If you don't know, please explain what you are thinking that makes you unsure of the answer.
$2x-9 = 2(3)-9$	Yes No I don't know	x is being replaced with 3
$7+5 = 7+(3+2)$	Yes No I don't know	No variables are being replaced.
$(9+2)+8 = 9+(2+8)$	Yes No I don't know	No variables are being replaced
$10+(3+6) = 10+9$	Yes No I don't know	No variables are being replaced
$ab+ac = 2x+2y$	Yes No I don't know	Nothing is being replaced
$ab+c = 2(x-1)+3y$	Yes No I don't know	Nothing is being replaced
$8(x^2-9) = 8((x+3)(x-3))$	Yes No I don't know	Nothing is being replaced. Just a different equivalent equation

Figure 4: *Epsilon's interpretations of substitution in specific contexts*

Equivalence: Now we consider Epsilon's definition of equivalent expressions (Figure 5).



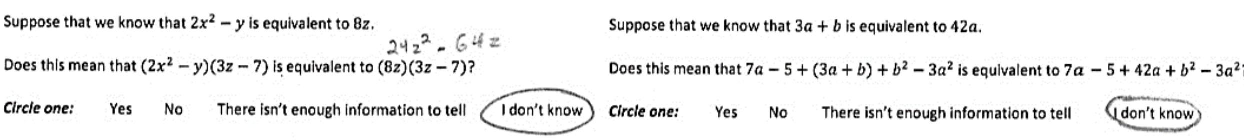
How could you check whether two mathematical **expressions** are equivalent?
(An *expression* is a mathematical phrase that does **not** contain an equals or inequality sign.)

If they both have the same correct answer

Figure 5: *Epsilon's definitions of equivalent expressions*

Epsilon provided a seemingly correct (if perhaps incomplete or ill-defined) definition of equivalent expressions. We cannot be sure the extent to which they understand that expressions have to have the same value for every possible combination of variable values or that this applies to algebraic and not just arithmetic expressions, and the word "answer" is also ill-defined; however, this definition is in line with the standard definition used in algebra, and they have been able to correctly identify equivalent algebraic expressions in last example in Figure 4 (as well as other questions not shown here), suggesting that their definition is at least somewhat standard. Their definition also appears to be operational, as it is rooted in computations with expressions.

Substitution equivalence: Now we consider the extent to which Epsilon recognizes instances of substitution equivalence in certain algebra examples (see Figure 6).



Suppose that we know that $2x^2 - y$ is equivalent to $8z$.

$24z^2 - 64z$

Does this mean that $(2x^2 - y)(3z - 7)$ is equivalent to $(8z)(3z - 7)$?

Circle one: Yes No There isn't enough information to tell I don't know

Suppose that we know that $3a + b$ is equivalent to $42a$.

Does this mean that $7a - 5 + (3a + b) + b^2 - 3a^2$ is equivalent to $7a - 5 + 42a + b^2 - 3a^2$?

Circle one: Yes No There isn't enough information to tell I don't know

Figure 6. *Epsilon's recognition of substitution equivalence in some examples*

In Figure 6, Epsilon does not recognize either example as substitution equivalence. On the left in Figure 6, they attempt to simplify one of the expressions, but this does not help them to identify whether the two expressions are equivalent. They do not appear to draw on the given fact that $2x^2 - y$ is equivalent to $8z$ when attempting to determine if the two expressions are equivalent. This suggests that they may not have a notion of substitution equivalence or are unable to draw on it in this problem context. Epsilon's operational approach to determining if the two expressions are equivalent suggests that their operational conception of equivalence may be limiting Epsilon's ability to recognize and use substitution equivalence when performing mathematical transformations. Another barrier to Epsilon developing a robust notion of substitution equivalence and linking this to their transformation work may be their narrow notion of substitution itself. Just as they do not recognize most of the transformations in Figure 4 as substitution, they likely do not recognize the transformations in Figure 6 as substitution either.

Potential impacts of instruction: Epsilon was actually part of a cohort that took part in a semester-long classroom intervention in which students were taught broader structural definitions of substitution, equivalence, and how to view transformation as substitution equivalence explicitly (as well as other concepts). One sample of Epsilon's work after the intervention can be seen in Figure 7.

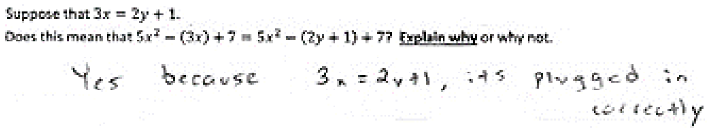


Figure 7: *Epsilon's identification of substitution equivalence after an intervention that addressed it explicitly*

After the intervention, Epsilon was not able to identify substitution equivalence in all cases, but they were able to recognize it in cases similar to questions where they had not recognized it at the start of the term. In Figure 7 we see how they are able to see a complex equation as an equivalence relationship between two structurally identical expressions where one equivalent subexpression could be conceptualized as having been substituted for another. Epsilon's use of the words "plugged in" are a common phrase often used by students to indicate substitute. We do note, however, that this language still suggests a computational approach. However, Epsilon is drawing on structural features of equivalent algebraic expressions through the lens of substitution equivalence, even if their approach still contains some computational elements. We have insufficient space to discuss the intervention at length here—we simply include this short example as a demonstration that more structural and well-defined definitions of substitution, equivalence, and substitution equivalence approaches to transformation can all be learned, even by students in developmental mathematics courses in college, given the right supports.

Conclusion

We have presented a model which describes how student definitions of substitution and equivalence may related to their ability to justify computational work through the lens of substitution equivalence. Using student examples, we have illustrated some of the affordances of this lens. We have demonstrated how students may struggle with substitution equivalence for different reasons, which may then require different instructional approaches. For example, if a student's definition of equivalence is ill-defined, it mays be important to find ways for them to correct their internal definition; whereas if a student has broad and well-defined definitions of substitution and equivalence, a more effective intervention may be one which helps them to see the connections between this existing knowledge and the work that they do when they perform transformations. These are very different approaches to solving what might on the surface look like similar errors, but which actually stem from very different underlying patterns of student thinking about the mathematics. Thus, we hope that this model may aid us to better tailor instruction to respond to student thinking, and to better think about how definitions of substitution and equivalence are presented in instruction. We have also shown through one particular student example that students are able to learn to think about transformation through a substitution equivalence lens with the right kind of instructional approaches, even when they are in developmental math courses. Further research is needed to better understand what approaches may be most effective, as well as to investigate which ways of thinking may be most productive for students in different contexts.

Acknowledgments

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Two Initial Schemes for Enumerating Permutations: A Preliminary Report

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In prior research, we proposed an initial learning trajectory resulting from analyses of two undergraduate students' schemes for enumerating permutations. We explained the levels of this learning trajectory using an elaborated theory of levels of abstraction for both operations on combinatorial composites and, in more advanced levels, symbolic representations of computational reasoning about combinatorial composites. In this preliminary report, we elaborate on this initial learning trajectory by incorporating two additional permutation enumeration schemes, identified from subsequent data that were collected in a recent teaching-experiment research study of students' developing combinatorial reasoning.

Keywords: Permutations, Combinatorics, Learning Trajectories, Schemes, Teaching Experiment

The importance of combinatorics in K-16 curricula has been well documented in the research literature (cf. DeBellis & Rosenstein, 2004; Hart & Martin, 2018; Kapur, 1970; Lockwood et al., 2020). Permutations and combinations—archetypical examples of "ordered" and "unordered" structures—are ubiquitous within combinatorics. Thus, supporting students to develop powerful means to enumerate permutations and combinations is a worthwhile instructional goal. In this preliminary report, we provide an initial analysis of data from a recent teaching experiment (Steffe & Thompson, 2000) with undergraduate students. This report elaborates an initial learning trajectory suggested in prior research (Antonides & Battista, under review a).

Literature Review

Drawing on the Realistic Mathematics Education principle of guided reinvention (Gravemeijer, 1999), Lockwood et al. (2015) conducted a teaching experiment with a pair of undergraduate students. Their goal was to guide the students to develop, or "reinvent," an algebraically generalized formula for counting certain combinatorial structures, including permutations. Their students successfully came to suggest $n!$ as the appropriate counting formula for counting permutations, but their suggestion came from looking at patterns in empirically-derived numerical results rather than by appealing to structural or multiplicative reasoning.

To investigate how students might construct combinatorial counting formulas *with* supporting conceptualizations-based reasoning, we conducted two one-on-one teaching experiments (Steffe & Thompson, 2000) with preservice middle-school teachers (Antonides & Battista, under review a). Our study also contained elements of design experiments (Cobb et al., 2003) in that we aimed to develop both a domain-specific theory of student learning and an instructional means to support this learning, conceptualized as a learning trajectory (LT). In our view, a LT is "a detailed description of the sequence of thoughts, ways of reasoning, and strategies that a student employs while involved in learning a topic," conceptualized as levels of sophistication, "including specification of how the student deals with all instructional tasks and social interactions during this sequence" (Battista, 2011, p. 510).

Transitions from one level of sophistication to the next were enabled by specific cognitive processes. Among these, abstraction and generalization were critical, and we described our students' conceptual progress using a theory of levels of abstraction (Battista, 2007), which we elaborated and extended to the realm of combinatorics. This theory of abstraction is outlined in the Theoretical Framework section. Instructionally, our study represented permutations as vertically-oriented "towers" of multiple colors of connecting cubes, without repeating colors in each tower—an adaptation of the instructional approach utilized in Maher and colleagues' longitudinal research (cf. Maher et al., 2011). We found that starting with small numbers of cubes in which all permutations can be modeled, then progressing to larger numbers in which modeling all permutations is infeasible, and finally progressing to variables served as a viable means for students to progressively interiorize the relevant combinatorial structures and enumeration schemes.

Due to space constraints, we provide only a brief overview of the levels of sophistication suggested in our initial learning trajectory for permutations. Levels 1, 2, and 3 of the learning trajectory are characterized by operating on spatial representations to generate permutation composites with no distinguishable system, a partial system, and a complete system (respectively). Levels 1 and 2 were hypothetical since they were not exhibited explicitly by the two students in the study, but these levels are supported by prior empirical research (e.g., English, 1993; Maher et al., 2011).

The remaining levels involve progressively sophisticated schemes for enumerating permutations via operations on numerical/algebraic symbols. At Level 4, permutations were enumerated recursively using a two-factor multiplicative structure; for example, if a student knew that there were 6 possible 3-cube towers that could be made with 3 different colors of cubes, then they might multiply 6×4 to enumerate towers 4-cubes-high each containing 4 different colors of cubes. At Level 5, multiple two-factor multiplicative structures can be unified into a single recursive multiplicative structure; for instance, knowing that there are 120 possible 5-cube towers each containing 5 different colors of cubes, a student might enumerate 9-cube towers each containing 9 different colors of cubes by multiplying $(((120 \times 6) \times 7) \times 8) \times 9$. At Level 6, these multiplicative enumeration structures have been reflected on, decomposed, and recombined to form a non-recursive multiplicative structure, such as $9 \times 8 \times \ldots \times 1$, which may be symbolized as 9!. At Level 7, such structures are algebraically generalized. At Level 8, conceptual operations can be performed on factorial-like structures; for example, a student at this level could make sense of and reason about the formula $P(n,k) = \dfrac{n!}{(n-k)!}$.

Theoretical Framework

Following Battista (1999), we define a *scheme* to be an organized sequence of actions or operations that has been abstracted and, to some degree, generalized so that it can be applied in response to similar or analogous circumstances. A scheme includes a mechanism for recognizing a situation, a mental model that is activated to interpret actions/operations within the situation, and a set of anticipated or expected results of one's actions.

Abstraction is taken to be the mental process of selecting, coordinating, unifying, and registering in memory a collection of mental items (objects or actions) that an individual perceives or conceives (Battista, 1999). Moreover, abstraction occurs at multiple levels (Antonides & Battista, under review a; Battista, 2007; Steffe & Cobb, 1988). At the *perceptual level*, abstraction isolates an item from the experiential flow and grasps it as a unit, entering it into working memory. At the *internalized level*, an item can be re-presented (mentally visualized or re-enacted) in the absence of relevant perceptual material, enabling the item to be recurrently used in reasoning; it is at this level that a *concept* has been formed (von Glasersfeld, 1982). At the *interiorized level*, an item can be reflected upon, operated on, and analyzed. A more general structure is abstracted, and the item can be recognized and used in novel situations. The *second interiorized level* is characterized by the construction of symbols that act as "pointers" to, and substitutes for, the abstracted material in reasoning. At the *third interiorized level*, these symbols can be used in novel and more complex operations.

Context and Methods

Five undergraduate students each participated in a one-on-one teaching experiment. The data discussed in this report relate to KC, AR, and EM (not the students' real initials). KC (she/her/hers) was a first-year student majoring in psychology enrolled in a second-semester developmental mathematics course. AR (she/her/hers) was a third-year preservice elementary teacher, and EM (she/her/hers) was a first-year preservice elementary teacher. Both students were enrolled in a mathematics content course for future elementary teachers focusing on number and operations. EM had studied some combinatorics in a high school probability course.

All data were collected remotely using Zoom. Each student used two electronic devices—a laptop and an iPad—so that the teacher-researcher (the first author) could see their inscriptions (on the iPad) and most of their gestures and expressions (on the laptop). The teacher-researcher also developed multiple digital learning environments using Geometer's Sketchpad, and he gave students remote access to his computer when these programs were to be used. Drawing on the instructional findings of our prior study, we used multi-colored digital squares in Geometer's Sketchpad as an alternative to physical multilink cubes to develop student reasoning about permutations.

Findings

The findings reported here are an elaboration of the preliminary learning trajectory proposed by Antonides and Battista (under review a) discussed in the Literature Review section. Specifically, we interpret our findings as an elaboration of the levels of sophistication prior to Level 4, and as an exposition of some of the initial ways students can enumerate permutations prior to abstracting relevant recursive spatial and numerical structures.

Options-Focused Enumeration Scheme

The form of reasoning that we have come to call an *options-focused enumeration scheme* was prevalent throughout this study. This scheme is activated by assimilating a situation as calling for the enumeration of permutations; their mental model includes abstractions of the composite of objects that are to be arranged and of the composite of positions into which these objects are to be placed. The activity of the scheme involves counting the *options* for the objects that can be placed in each position, then finding the sum of these values. The anticipated result of this activity is an enumeration of potential permutation composites. When a student uses an options-focused enumeration scheme, they do not conceptualize the construction of a given composite as consisting of multiple *dependent* steps—for instance, if a red square is placed in the bottom position of a 3-square tower, then it cannot reoccur in another position of the same tower.

For example, to enumerate 3-cube towers each containing 3 colors of cubes, KC said she would multiply 3×3. She reasoned that there are 3 colors that can go in the first position, there are 3 colors that can go in the second position, and there are 3 colors that can go in the third position, so there are 9 possible towers. In fact, all five students of this study initially said there were 9 possible towers—including the two students who had learned about permutations in a prior high-school course. EM, however, quickly modified her answer to 3!, which she explained by linking this multiplicative structure to the decreasing number of options available in each successive step.

Notably, one of the two students in our prior study solved an equivalent task (with physical multilink cubes rather than digital squares) by constructing each tower one-by-one, finding there to be 6 potential 3-cube towers. However, she said she thought the total number of towers would be found by multiplying 3×3. She resolved her uncertainty by repeating her construction process and abstracting the spatial property that each color will appear in each position twice (e.g., in red-green-blue and red-blue-green towers, the red cube appears on top twice). This alternative spatial structuring enabled her to justify for herself that 6 is the correct number of potential 3-cube towers, each containing 3 different colors of cubes.

Partial Spatial-Temporal-Enactive Enumeration Scheme

A *partial spatial-temporal-enactive enumeration scheme* is activated by assimilating a situation as involving the enumeration of permutations, with a mental model that includes abstractions of composites of the objects to be arranged, the positions into which they are to be placed, *and* the permutation composites that are to be enumerated. The activity of the scheme involves (1) a sequence of actions on perceptual material oriented toward determining the number of potential positions into which one object may be placed, (2) performing a similar sequence of actions for each remaining object, (3) finding the sum of these values (or keeping a running total while counting), and (4) multiplying this sum by the number of objects being arranged. Students using this scheme, as with the previous scheme, do not conceptualize the construction of a given composite as consisting of multiple dependent steps, which often leads to inaccurate enumerations. However, the partial spatial-temporal-enactive enumeration *is* a conceptual step forward from the options-focused enumeration scheme in that students conceptualize and coordinate their actions around counting combinatorial composites, not options.

For example, AR exhibited an instance of this scheme during her initial enumeration of 4-square towers each containing 4 different colors of squares. AR initially predicted the number of towers would be either 16 or 20, using an options-focused scheme to produce the former, but she expressed stronger confidence in the latter. An excerpt of her explanation is given below; of note, while the task asked about the number of "towers" that could be made, AR chose to use a horizontal spatial orientation.

AR: So if this was one, um, tower, then you would change—So this would be 1 possible way, and then the blue could go here and here. So that's 2, 3. And then the green can go 4, 5. And I guess I forgot to count the black, because then I could just multiply the 5 different ways that this tower can be made, and then there's still the other squares. So I would multiply that by 4.

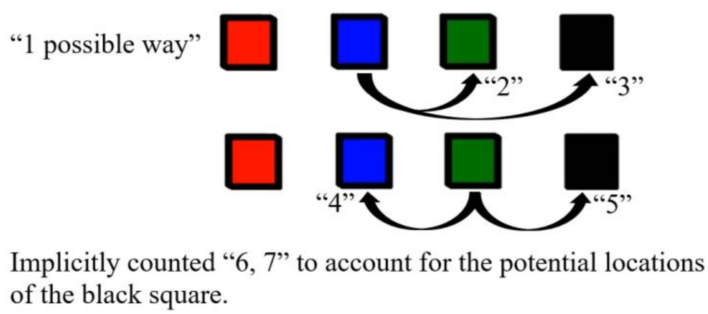

Figure 1. AR's actions on squares to enumerate 4-square permutations

We hypothesize that when AR moved a square to the position of another square, such as moving the blue square to the position of the green square when she uttered “2”, AR mentally imagined the blue square taking the place of the green square, and the green square moving to the left. Similarly, when she uttered “3”, we hypothesize AR imagined the blue square taking the position of the black square, and the blue and black squares both shifted to the left.

With this interpretation, we infer that AR’s enumeration consisted of the following 4-square composites: (1) red-blue-green-black; (2) red-green-blue-black and (3) red-green-black-blue; (4) red-green-blue-black [duplicate] and (5) red-blue-black-green; (6) red-black-blue-green and (7) red-blue-black-green [duplicate]. While she did not explicitly act on the black square, we infer from her statement, “and then there’s still the other squares,” that she would have performed the analogous set of actions using the black square, resulting in composites (6) and (7). Her enumeration would not have included red-black-green-blue. However, as indicated by AR’s final statement, “So I would multiply that [the number of 3-square permutations] by 4,” her enumeration of 4-square permutations was a form of spatially-linked multiplicative reasoning.

Conclusions

In this preliminary report, we have outlined two schemes that students have used along their trajectories toward developing increasingly sophisticated schemes and concepts for reasoning about permutations. The two schemes outlined here—the options-focused enumeration scheme and the partial spatial-temporal-enactive enumeration scheme—provide an elaboration of the learning trajectory outlined in prior research (Antonides and Battista, under review a). However, in light of our findings and given the preliminary nature of this report, it is difficult to conceptualize the specific way in which these schemes fit into the initial learning trajectory as originally conceived.

To resolve this issue, we reconceptualize the first three levels of sophistication to the following reformulations. Level 1 reasoning consists of an attempt to enumerate permutation composites, either symbolically or by using perceptual material for the units in the composites, but with no discernable structuring informing this enumeration. For instance, generating permutation composites seemingly at random, without an apparent system in place, would constitute Level 1 reasoning, consistent with findings from the youngest student populations included in studies by English (1991) and Piaget and Inhelder (1951/1975). Level 2 consists of an attempt to enumerate permutation composites, either symbolically or perceptually, with a discernable structuring but one that does not enable an accurate enumeration. The two schemes introduced in our Findings are both instances of Level 2 reasoning, since for both schemes students have a partial spatial-temporal-enactive structuring (Antonides & Battista, under review b) and thus their conceptualization leads to omitting certain composites while double-counting others. Finally, Level 3 reasoning is characterized by enumerating permutation composites, using perceptual material for the units in each composite, informed by an inferred structuring that enables systematic enumeration.

Upon further analysis, additional insights regarding our students’ combinatorial schemes and concepts may emerge. We anticipate being able to provide a more fully elaborated learning trajectory for permutations, with descriptions of how to instructionally support this development, from our research findings. However, one potential avenue for future research would be to investigate additional combinatorial schemes that would fall under our reconceptualized version of Level 2 reasoning.

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The Effect of Inquiry-Based Versus Lecture-Based Instruction on Calculus I Students' Math Anxiety

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Math anxiety affects student learning and academic performance. Highly math-anxious individuals exhibit physical, mental, and emotional symptoms. These symptoms often have a short-term and long-term impact on students' mathematics learning and their performance both inside and outside of school. Hoping to identify the possible measures to reduce math anxiety, this study investigated the effects of inquiry-based learning (IBL) on Calculus I students' math anxiety, compared to lecture-based instruction (LBI). We used a short version of the Mathematics Anxiety Rating Scale (MARS-S) as a pre- and post-test to collect the data from Calculus I students. A total of 15 participants from the IBL group and 20 from the LBI group responded to both the pre- and post-tests. The results show that the IBL group's math anxiety slightly decreased and the LBI group's slightly increased; however, there was not enough evidence to conclude that both of these changes were statistically significant.

Keywords: inquiry-based learning, lecture-based learning, math anxiety, collaboration

Math anxiety affects student learning and academic performance. Highly math-anxious students exhibit physical, mental, and emotional symptoms. Physical symptoms include nausea, sweaty palms, and increased cardiovascular activity (Ashcraft, 2002; Chang & Beilock, 2016). Mental symptoms include an inability to concentrate and mind blanking (Plaisance, 2009; Ruffins, 2007). Emotional symptoms include extreme nervousness and apprehension (Mattarella-Micke et al., 2011). These symptoms often have a short-term and long-term impact on students' mathematics learning and their performance both inside and outside of school. In a short term, students may begin to dislike mathematics and take fewer mathematics courses, and in the long term, they tend to avoid mathematics and mathematics-related courses (Godbey, 1997; Hembree, 1990).

Due to the substantial impact of math anxiety on mathematics learning and mathematics performance, it is essential to diagnose the causes of math anxiety and to determine some potential interventions to reduce such anxiety. Therefore, we investigated the effects of inquiry-based learning (IBL) instruction on Calculus I students' math anxiety, with lecture-based instruction (LBI) used for comparison.

Literature Review

Math anxiety has been a part of the human experience for centuries. The verse, "Multiplication is vexation ... and practice drives me mad" goes back at least to the 16th century (Dowker et al., 2016). In 1957, Dreger and Aiken introduced the concept "number anxiety," and math anxiety received increasing attention thereafter. Richardson and Suinn (1972) conducted the first formal study of math anxiety, who characterize math anxiety as "feelings of tension and anxiety that interface with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (p. 551). Since then, studies on math anxiety have been substantially investigated.

Traditional lecturing, which is a predominant mode of instruction in college mathematics courses across the United States (Stains et. al., 2018) and is ineffective in helping students learn mathematics (Boaler, 2008), could be one of the possible reasons for evoking math anxiety among students. The LBI does not offer substantial opportunities for students to share each other's ideas and experiences with their teachers and peers. On the other hand, IBL, which is an active learning pedagogy, provides extensive opportunities for students where they can work in pairs or groups to make conjectures, gather information for problem-solving, and present their work to groups and to the whole class (Kogan & Laursen, 2014). Through a comparative study, Laursen et al. (2014) reported that students in IBL math-track courses achieved greater learning gains than their non-IBL peers in cognitive, affective, and collaborative areas. Similarly, Laursen et al. (2011) found that the IBL students were involved more in interacting with each other, with the instructor, and they were more involved in setting the course pace and direction. It is also reported that IBL enhances students' conceptual understanding (Jensen, 2006), communication skills, confidence, and self-efficacy (Laursen et al., 2011). Considering the benefits of IBL as a ground, this study sought to examine the relative changes in the scores of Calculus I students' math anxiety, using a short version of the Mathematics Anxiety Rating Scale (MARS-S).

Method

Research Context and Participants

The students, who were enrolled in Calculus I courses via IBL and LBI during Spring 2021 at a university located in the Midwestern United States were the sample for this study. Students, who received the IBL instruction were IBL group and those who received the LBI were LBI group. In this study, about 65% (n = 15) of the students off the 23 from the two IBL sections and about 41% (n = 20) students off the 49 from one of the LBI sections responded to both the pre- and post-MARS survey. Table 1 shows the distribution of IBL and LBI participants by their gender and academic standing.

Table 1. Distribution of IBL and LBI students by their major and academic standing.

Characteristics	IBL Group		Lecture-Based Group	
	Frequency	Percentage	Frequency	Percentage
Gender				
Male/Man	5	33.3%	8	40.0%
Female/Woman	10	66.7%	11	55.0%
Non-Binary	0	0.0%	1	5.0%
Academic Standing				
Freshman	11	73.3%	15	75.0%
Sophomore	2	13.3%	1	5.0%
Junior	2	13.3%	4	20.0%
Senior	0	0.0%	0	0.0%

Regarding the instructors, the IBL instructor has 15 years of experience in teaching Calculus I at the university and high school level via active learning, including an IBL. The LBI instructor has 2 and a half years of experience in teaching Calculus I at the university level and 14 years of experience in teaching undergraduate-level mathematics courses via a lecture-based approach. Both the instructors taught remotely using audio-visual conferencing; the IBL instructor taught using Microsoft Teams, whereas the LBI instructor taught using Zoom. Throughout the semester, both IBL and LBI classes had class meetings every Monday, Wednesday, and Friday.

The IBL instructor engaged students collaboratively in sequentially organized pre-tasks and tasks in and out of the class. Students were supposed to practice the pre-tasks before the class for a better understanding of the material during the next day's class meeting. The instructor usually began the class by welcoming each student and briefly describing the tasks and activities for that day. Then, the students were sent to Teams breakout rooms, where they shared each other's ideas, asked questions, made conjectures, and solved problems while they were working with their small group members. The instructor visited each group at least once, or as needed and prompted students if they had any questions or concerns. In the end, the students were returned to the main room, where the instructor facilitated whole class discussion. On the other hand, the LBI instructor began the class by asking students whether they had any questions or concerns from the previous class. If they had, then, the instructor solved the examples or explained the concepts as needed. After that, the instructor usually began the lecture by solving preselected examples using the *Notability* app from his iPad. Occasionally, the instructor paused during the lecture and asked some questions to the whole class. Students were never sent to breakout rooms and never provided opportunities for group discussions.

Data Collection and Analysis

Qualtrics online survey was used to collect the pre- and post-MARS data from both IBL and LBI groups after receiving an institutional review board (IRB) approval. The MARS-S survey is the 30-item anxiety measure instrument that was developed by Suinn and Winston (2003). It is a 5-point Likert-type scale survey in which each item on the scale represents a situation that may arouse anxiety to the respondent. A score of 1 indicates that the respondent is not anxious at all by that situation, whereas a score of 5 indicates that the respondent is anxious very much. The pre-MARS survey, in conjunction with a demographic questionnaire, was administered during the second week and the post-MARS was administered during the eleventh week of the class. A link to the survey and consent form were posted on Blackboard and made available for students on the day of MARS administration soon after the class. On the very day, one of the researchers had joined each of the IBL and LBI classes remotely through audio-visual conferencing platform and read the consent form for students at the beginning of the class and requested them to respond to the survey if they elect to participate voluntarily. The data thus collected were analyzed using the SPSS 27.0 and Microsoft Excel spreadsheet. Initial data screening for the pre- and post-MARS scores of the IBL and LBI groups were conducted and assumptions for the *t* tests were checked, before running the full analysis

Results

First, the response to the 30 items of the pre- and post-MARS survey was analyzed to see the mean and standard deviations for each of the 30 items. The average of the anxiety scores and the standard deviations for all the 30 items in the MARS for the IBL pretest were 2.43 and 1.00 and posttest were 2.40 and 1.03; LBI pretest were 2.63 and 1.08 and posttest were 2.69 and 1.01, respectively. Table 2 shows the participants' extreme math anxiety levels among the 30 items of the MARS. For the IBL group, item 3 received the highest level of math anxiety in both the pretest and posttest, whereas item 29 and item 27 received the lowest level of math anxiety in the pretest and posttest respectively. For the LBI group, item 4 and item 27 received the highest and lowest level of math anxiety in the pretest, and items 1 and 17 received, respectively, the highest and lowest level of math anxiety in the posttest.

Table 2. Participants' extreme math anxiety levels on the 30-item MARS.

Items that received the highest and lowest levels of math anxiety from IBL and LBI groups				
		Item		M*(SD)*
IBL				
Pretest	Highest Anxiety	3	Thinking about an upcoming mathematics test one day before.	4.07*(1.16)*
	Lowest Anxiety	29	Being given a set of subtraction problems to solve.	1.00*(0.00)*
Posttest	Highest Anxiety	3	Thinking about an upcoming mathematics test one day before.	4.20*(1.08)*
	Lowest Anxiety	27	Watching someone work with a calculator.	1.07*(0.26)*
LBI				
Pretest	Highest Anxiety	4	Thinking about an upcoming mathematics test one hour before.	4.50*(0.83)*
	Lowest Anxiety	27	Watching someone work with a calculator.	1.05*(0.22*
Posttest	Highest Anxiety	1	Taking an examination (final) in a mathematics course.	4.45*(0.89)*
	Lowest Anxiety	17	Adding up 976 + 777 on paper.	1.25*(0.55)*

Second, paired samples *t* tests were conducted to examine the changes in the math anxiety levels within and between the IBL and LBI groups. All the *t* test results were obtained at the 0.05 level of significance. Although there are some changes in the math anxiety scores between and within the IBL and LBI groups, none of these changes were found to be statistically significant. Table 3 shows that the math anxiety score of the IBL students in the pretest was about 6 points lower and in the posttest was about 7 points lower than that of the LBI students, however, neither of these differences were statistically significant. These results suggest that the math anxiety levels of the students from both groups were about the same at the beginning of the semester and also at the end of the semester.

Table 3. Paired samples t tests between IBL and LBI groups.

	Paired Differences								
	IBL			LBI					
	Mean	SD	N	Mean	SD	N	*t*	*df*	*p*
Pretest	73.00	19.56	15	79.00	15.46	20	–0.92	14	0.37
Posttest	71.87	18.79	15	80.55	22.20	20	–1.03	14	0.32

Table 4 shows the changes in the math anxiety scores of both IBL and LBI groups from pre- to post-test. Although the IBL students' math anxiety scores slightly decreased by 1.13 points and LBI students' math anxiety scores slightly increased by 1.55 points from the pretest to the posttest, neither of these changes were statistically significant. These results suggest that the math anxiety levels of the students from both groups did not change significantly from the beginning to the end of the semester.

Table 4. Paired samples t tests within IBL and LBI groups.

	Paired Differences							
	Pretest		Posttest					
	Mean	SD	Mean	SD	N	*t*	*df*	*p*
IBL	73	19.56	71.87	18.79	15	0.38	14	0.71
LBI	79	15.46	80.55	22.20	20	–0.35	19	0.73

Discussions and Conclusions

Math anxiety is a prominent issue in the United States and across the world. Research studies have found that it can begin at least as early in children attending first and second grade (Raver, 2014) and peaks in middle school and high school (Jackson & Leffingwell, 1999; Oxford & Vordick, 2006; Scarpello, 2007). Once established, it can impact peoples' everyday activities involving numeracy and higher-level mathematics learning throughout their lives (Oxford & Vordick, 2006). Ultimately, it can have a significant impact on students' overall performance in and out of the class.

Therefore, it is essential to diagnose the possible causes of math anxiety and to determine some potential interventions to reduce such anxiety before students get into the tornado of math anxiety. As such, this study sought to investigate whether inquiry-based instruction significantly decreases the math anxiety among Calculus I students, compared to lecture-based instruction. Comparison of both the IBL and LBI groups' pre- and post-MARS survey data after a semester-long instruction revealed that there is not a significant difference in the math anxiety scores within and between these two groups. The IBL students were found to be a little bit less anxious than the LBI students at the beginning of the semester. IBL students' mean score was slightly decreased from 73 to 71.87 from the beginning to the end of the semester, however, during the same period of time, the LBI students' mean score was slightly increased from 79 to 80.55. Although there is some level of change in the math anxiety scores of the IBL students from pre- to posttest, we do not have enough evidence to conclude that inquiry-based learning reduces students' math anxiety compared to lecture-based instruction.

This unprecedented outcome could be the consequence of the small sample size. Although a fairly high percentage of IBL students, 65% (n = 15) responded to both the pre- and post-MARS survey compared to LBI students, 45% (n = 20), these numbers were not large enough to provide a desirable level of statistical power.

Suggestion

Due to the COVID-19 pandemic during Spring 2021, most of the Calculus I classes were taught online. Due to the online settings, it was difficult to find a reasonably large number of participants for this study from both the IBL and LBI classes. Thus, we suggest replicating this research when the classes resume in-person with a reasonably large sample size over a span of at least a year. Nonetheless, the results of this study can be useful for researchers, Calculus I students and instructors, and professional development organizers to obtain a general picture of the relationship between IBL and math anxiety.

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How 2020 (Didn't) Change Calculus Instructors' DEI Engagement

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Increasing diversity and advancing equity in postsecondary mathematics education is garnering much-needed attention. Efforts to diversify STEM fields, and in the longer term to dismantle systemic barriers, requires awareness and engagement on the part of mathematics faculty, particularly those who teach introductory courses to large numbers of students. A survey of 1064 instructors of introductory STEM courses (305 mathematicians) captured data on their awareness and engagement with various diversity, equity, and inclusion (DEI) issues and initiatives. This preliminary report provides insight into how those instructors' beliefs and behaviors regarding DEI shifted during 2020 – a year in which the status quo was interrupted by a global pandemic, and which saw nearly unprecedented national conversations about social (particularly racial) justice. We find changing beliefs were common, while changes in activities were uncommon. Ongoing qualitative analysis will reveal much about those beliefs and how and why they changed.

Keywords: Diversity Equity Inclusion, Undergraduate STEM, Professional Practice

Attention to diversity, equity, and inclusion (DEI) in STEM has steadily risen over the last few decades. Investigating and redressing systemic inequities in postsecondary mathematics are now widely (though not unanimously) acknowledged as a fundamental and urgent charge for researchers of undergraduate mathematics education. In this preliminary report, we describe responses from approximately 300 calculus instructors to multiple-choice questions about changes in equity-related activities (uncommon) and views about DEI (common). We provide sample natural-language text responses explaining these changes in views, and our plans for completing a directed content analysis of the full set of 199 written responses. The results of that analysis will be completed by the end of this semester (Fall 2021) and presented at the research conference in Spring 2022. Our research goals are to (a) document calculus instructors' beliefs and activities as they relate to DEI issues; and (b) identify mechanisms by which views might be shifted toward higher engagement (or the reverse). While 2020 was full of events which cannot be recreated for the purposes of professional development or institutional transformation, we anticipate that many of the lessons learned from these queries will inform targets and processes for future work aimed at creating a more inclusive STEM environment.

Methods

Data collection occurred as a follow-up to a larger study of introductory chemistry, mathematics, and physics instructors (across ranks and titles) at postsecondary institutions across the United States. The larger study included a survey of instructional practices, related beliefs, contextual factors, and individual characteristics. The final question on an initial survey sent to thousands of instructors asked if we could invite them to participate in a follow-up survey focused on DEI issues in STEM. Of the 3,769 who took the initial survey, 2,229 agreed and were invited to participate in this study; 1,064 completed it. What began as a small pilot investigation gathered enough data to warrant meticulous analysis and has real potential to increase our understanding of the DEI climate in postsecondary STEM. Of the respondents, we focus only on the 305 mathematicians, all of whom taught a single-variable calculus course in the 2017-18 and/or 2018-19 academic years.

The survey asked briefly about a range of beliefs about different race-gender groups, factors which might be used to explain existing gaps in representation within STEM, engagement with DEI initiatives, and personally experiencing and witnessing discriminatory behavior in professional settings. This paper reports on only the questions which asked about changes in behaviors and views of DEI during 2020 (see Preliminary Results for question text). This survey was completed electronically via the online Qualtrics platform (2018), and was approved by the Western Michigan University Human Subjects Institutional Review Board (#17-06-10).

We employ a mixed-methods triangulation approach to understanding these data (Creswell & Plano Clark, 2007). In this study, we use quantitative methods to analyze responses to multiple-choice survey questions and to explore relationships between variables. We use qualitative methods to analyze and understand the natural language data provided by respondents in an open-ended free response survey question asking them to explain their answer to a previous multiple-choice question. These analyses will be triangulated to provide more robust understandings of the underlying phenomena of interest: calculus instructors' DEI views and related activities, and what (if anything) about those views and activities changed in 2020 (and why). Text data will be analyzed using qualitative content analysis, "a research method for the subjective interpretation of the content of text data through the systematic classification process of coding and identifying themes or patterns" (Hsieh & Shannon, 2005, p. 1278). More specifically, we engage in directed content analysis, in which initial coding categories are identified from existing theoretical framings and empirical research (relevant literature is briefly reviewed in the next section). The initial codes are not entirely prescriptive; new codes and subcodes are generated through iteration and constant comparison (Hsieh & Shannon, 2005; Miles & Huberman, 1994). At the later stages of the overall analysis, instructors' personal and professional contexts will be incorporated. We omit these markers from the initial phases to minimize the impact of our own biases on the subjective interpretation of participants' responses.

Theoretical Framing & Relevant Literature

We omit a review of literature indicating that instructor beliefs and activities about dimensions of identity and DEI issues/initiatives impact their students directly in favor of describing theories and research which inform our conceptual framing and initial coding scheme. Briefly, instructors are key actors in the development and evolution of STEM culture, in students' introduction to that culture, and have the power to influence DEI through their professional activities (e.g., Busch et al., 2021; Canning et al., 2019; Gandhi-Lee et al., 2017; Rainey et al., 2019; Reinholz & Apkarian, 2018; Schein, 2010). Thus, understanding their engagement and views will support efforts (in the short-term) to create more inclusive educational experiences and (in the longer term) to engage in institutional transformation.

When coding natural language data, we will attend to what (if any) dimensions of diversity instructors specify. In job applications, faculty have been documented as referring to unspecified diversity, federally defined categories (e.g., race, gender, disability status), and economic class with some variation depending on context and discipline (Lee Baker et al., 2016; Schmaling et al., 2015). The prominence of race-neutral rhetoric in American culture also contributes to an avoidance to name race at all and to the use of euphemisms for race (Carter et al., 2017; Vaught & Castagno, 2008).

Additionally, we will attend to what (if any) aspects of their views are mentioned as changing, and to what those changes are ascribed (if at all). For example, awareness of racial disparities does not necessarily lead to understanding, empathy, or behavioral changes (Vaught & Castagno, 2008). Support can also diminish rapidly, as evidenced by the massive decrease in support for the *Black Lives Matter* movement following a historic high. With regard to instructor behaviors, there is evidence that STEM instructors are heavily influenced by personal experience, and less influenced by empirical studies, when making decisions about pedagogy (Andrews & Lemons, 2015; Oleson & Hora, 2014).

Finally, we will attend to broader ideologies, prose, and rhetoric related to dismantling or upholding systems of oppression in higher education. This includes a myth of meritocracy which masks differential accessibility and experience (Liu, 2011; Taylor & Shallish, 2019; Yosso et al., 2009). Views of racism (and by extension, other discriminatory -isms) as an issue of individual pathology, as opposed to a systemic condition, impact how people view evidence of differential treatment as well as the solutions they might support (Carter et al., 2017; Patton, 2016; Vaught & Castagno, 2008). We will also, of course, monitor for explicitly discriminatory statements.

Preliminary Results

One set of survey questions asked participants whether they had engaged in four specific activities centering equity issues in 2020 and, separately, in 2019. The options were Yes; No, but I want(ed) to; No, and I do/did not plan to. Most respondents answered both items, and these are cross-tabulated in Table 1.

Table 1. Cross-tabulated responses to "have you engaged in any of the following activities" in 2019 and in 2020. Responses reported only for those who responded to both 2019 and 2020 questions. Entries list counts and (proportion), calculated from the number of respondents who answered both questions.

Attended nonmandatory talks/workshops focused on equity issues. (1) [N=294]			
	2019: No, no plan	2019: No, but wanted to	2019: Yes
2020: No, no plan	45 (0.15)	0 (0)	15 (0.05)
2020: No, but want to	8 (0.03)	25 (0.09)	24 (0.08)
2020: Yes	14 (0.05)	14 (0.05)	139 (0.47)
Organized a talk/workshop focused on equity issues. (2) [N=295]			
	2019: No, no plan	2019: No, but wanted to	2019: Yes
2020: No, no plan	181 (0.61)	2 (0.01)	3 (0.01)
2020: No, but want to	23 (0.08)	28 (0.09)	13 (0.04)
2020: Yes	9 (0.03)	5 (0.02)	31 (0.11)
Read books/journal articles, etc. focused on equity issues. (3) [N=295]			
	2019: No, no plan	2019: No, but wanted to	2019: Yes
2020: No, no plan	32 (0.11)	0 (0)	6 (0.02)
2020: No, but want to	8 (0.03)	13 (0.04)	7 (0.02)
2020: Yes	12 (0.04)	15 (0.05)	202 (0.68)
Was a member of a group or committee (including as an advisor) dedicated to taking action around equity issues. (4) [N=298]			
	2019: No, no plan	2019: No, but wanted to	2019: Yes
2020: No, no plan	91 (0.31)	2 (0.01)	2 (0.01)
2020: No, but want to	27 (0.09)	42 (0.14)	14 (0.05)
2020: Yes	9 (0.03)	19 (0.06)	92 (0.31)

While the American consciousness about social justice was raised in 2020, our participants did not report much change in their behavior regarding these four activities (71%, 81%, 83%, and 76% reported no change regarding activities 1-4). The majority of our participants (68%) reported reading literature focused on equity issues in both 2019 and 2020; only 9% reported doing so in 2020 while not having done so in 2019, and 4% reported the opposite change in behavior. While 10% reported attending non-mandatory talks/workshops focused on equity in 2020 without having done so in 2019, 13% reported a shift in the opposite direction; however, 47% reported attending such events in both 2019 and 2020. Activities associated with a larger commitment, and perhaps being seen as having some expertise (committee work, workshop organizing) were less common in both 2019 and 2020, again with little reported change.

Of the 305 survey respondents, 300 answered a yes/no question about changed views of DEI issues during 2020, and 199 of these answered an associated free response item asking them to explain their response. Of the 300 respondents, 128 (43%) reported that their views of DEI changed over the course of 2020, while 172 (57%) reported that their views did *not* change. Participants were asked to explain their response regardless of whether their views had changed, this was taken up more by those who reported a change (118, or 92% of changers) than those who reported no change (81, or 47% of non-changers). The 199 free responses range in length from 1 to 366 words, (M = 42, SD = 52). These open-ended responses are currently being coded by the research team, according to the process outlined in the methods section and informed by the reviewed literature. This analysis is far from complete, but we offer a few sample responses (Table 2) and early interpretations, based on documented phenomena and theories.

Table 2. Sample quotes from recent/current calculus instructors explaining their (lack of) change in views of DEI issues and initiatives in 2020. These are reproduced in their entirety, with no corrections or edits.

Explanations for changed views of DEI	
P1	I became aware of the systemic problems in education.
P2	For many years now, I have felt that we need to actively battle equity gaps. This year, with the pandemic, I feel that I have made more exceptions personally for my students. Somehow, teaching during the pandemic has allowed me to reduce the professional distance between me and my students, and I am able to empathize more with their struggles.
P3	After reading articles about the difficulties faced by Black and Latinx people in earning math degrees, I am more in favor of affirmative action admission policies for students and hiring policies for faculty.
Explanations for unchanged views of DEI	
P4	I just would like to teach mathematics to my students, treating each person as infinitely precious in the eyes of God. I was hired to teach math, and I realize the extent of my influence, which is not going to solve the problems of the world. I treat everyone with love and respect where I can, on a very small scale, in my classroom, and that is all I can do. It is up to me to be an expert in my subject, not in politics.
P5	It is actually annoying to see how this agenda gets pushed down people's throats. You can't get a PhD just because you are black or Hispanic, you have to be good at the subject. If one wants to be good at math then it all starts in grade 1 and has nothing to do with race. It is a cultural thing.
P6	I am a member of a minority in mathematics, and I still believe that there is systemic racism and sexism in STEM and more diversity, equity, and inclusion is beneficial to everyone.

Participant 1 (P1) states simply that they have become aware of systemic problems in education, without specifying the dimension(s) of oppression to which they refer. This may reflect a discomfort naming racism, sexism, classism, ableism, and so on, which is a common avoidance mechanism. Awareness is valuable, but it does not necessarily lead to action (Vaught & Castagno, 2008). P2 reports how personal experiences connecting with their students, and making accommodations related to the pandemic, has brought them closer to their students and helped them empathize with students' struggles. This echoes previous research on teacher decision-making, which suggests that personal experiences are a powerful driver for change. P2 also does not name the specific dimensions of struggle or challenge. In contrast, P3 reports changing views brought about by reading articles about Black and Latinx experiences rather than personal interactions. We note that they mention only two (large and non-homogenous) racial groups in their response, and do not mention other characteristics. On a promising note, they indicate that their changed mindset incorporates a change in support for policies which were intended to redress historic inequities. While affirmative action has only been directly linked to increased diversity (Murrell & Jones, 1996), disrupting racialized power structures requires shifts in policy as well as mindset.

There is also a range of ideas presented by the three whose views did not change. P4 acknowledges "the problems of the world," though they do not mention specific issues, and appears unaware of actions they could take within and outside their classroom; they also draw an inaccurate distinction between mathematics and politics (Barany, 2020). Interestingly, they seem to employ contradictory thinking. First, that "being good at math" starts in elementary school and has nothing to do with race. Of course, elementary school experiences have a lot to do with race (e.g., Carter et al., 2017; Ladson-Billings & Tate, 1995). And while strong, positive, early experiences with mathematics are likely beneficial for students regardless of their eventual interests, to suggest that someone who has not pursued math since the age of 6 cannot succeed is, frankly, a ridiculous statement and an insult to educators and students everywhere. This is followed by "it is a cultural thing," which is often code for race – and in fact P5 references two racial groups (Vaught & Castagno, 2008). We expect such contradictions to arise elsewhere and will leverage extant literature to untangle their meanings. Finally, we note that individual identities and experiences will have an impact on participants' views and responses. Though their identity is unspecified, P6 notes that, as a member of a minority group in mathematics, they were aware of systemic racism and sexism in STEM before it was trending. As we complete our qualitative content analysis, we will consider participants' own identities and contexts, as volunteered on the survey, and examine patterns in responses. Rather than make sweeping statements about individuals based on a single short response to an online survey, we will consider broad patterns and the prevalence of particular sentiments and views.

We hope that the RUME community will help us think about how these lessons can support future DEI initiatives in ways that support meaningful change, and in considering what lessons from an unprecedented year might be transferable to more precedented times.

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Undergraduate's Covariational Reasoning Across Function Representations

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Covariational Reasoning is the mental actions, constructions, and processes used to coordinate two or more quantities and interpret the relation between them. While research has shown that covariational reasoning is critical in a variety of fields, there has been a lack of studies on three-dimensional covariational reasoning. This study utilizes the Action-Process-Object-Schema (APOS) Theory framework to analyze how a student applies covariational reasoning to a parametric representation to model a real-life three-dimensional scenario. Preliminary results suggest that students' focus on experiential time may inhibit their ability to reason about two or three quantities relating to each other irrespective to time.

Keywords: Covariational Reasoning, APOS Theory, Calculus

Introduction

Covariational reasoning is the mental actions, constructions, and processes used to coordinate two quantities and interpret the relation between them (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Carlson et al. (2002) showed students with strong covariational reasoning ability, but no calculus background, were able to complete the same limits and differentiation tasks that a group of second semester calculus students struggled with. While research has shown that covariational reasoning is critical in a variety of fields, there are still unknowns such as: (1) the mental processes students enact to understand covariational reasoning, (2) the foundations students need to develop their reasoning abilities, and (3) how students are able to apply covariational reasoning in different environments.

The goal of this study is to expand the existing literature by focusing on how students apply covariational reasoning in a different environment in comparison to previous studies. To accomplish this end, this study will focus on analyzing how students are able to interact and understand a three-dimensional model which utilizes a mixture of linear and nonlinear functions. The overarching question the study aims to answer is as follows:

How are students able to use covariational reasoning to create a parametric representation to model a real-life 3-dimensional problem?

Literature Review

Over the past several decades, numerous articles and studies focusing on the role covariational reasoning plays in various contexts have been published, such as:

- The initial study from Carlson et al. (2002) which focused on coordinating quantity changes related to instantaneous rate of change;
- Using trigonometric functions to relate radian measures to arc lengths in a circle context (Moore & LaForest, 2014);
- Reasoning about two quantities through time as a third parameter (Paoletti & Moore, 2017); and
- Examining the role of reasoning about magnitudes when graphically representing covarying quantities (Moore, Stevens, Paoletti, Hobson, & Liang, 2019).

As our study focuses on a three-dimensional model, we considered two forms of reasoning associated with covariational reasoning: simultaneous-independent reasoning and change-dependent reasoning. *Simultaneous-independent reasoning* focuses on how two quantities vary with respect to a third quantity, which frequently is time (Stalvey & Vidakovic, 2015). This definition is particularly relevant to the discussion on parametric representations since the focus is describing how two or more functions vary relative to one another expressed through coordinates of the points. While these functions normally have the same input, the changes that occur in one function do not directly cause the changes in another function. *Change-dependent reasoning* focuses on how a quantity directly causes changes in a different quantity (Stalvey & Vidakovic, 2015), such as how the height of an object may change over time.

Theoretical Framework

This study will utilize the Action-Process-Object-Schema (APOS) Theory framework to analyze and interpret our results. APOS Theory describes the mental structures (Actions, Process, Objects, and Schemas) that individuals construct to learn a mathematical concept. Developing these structures are considered stages in the learning process (Arnon, et al., 2013). We briefly describe each construct below using definitions from Arnon et al. (2013).

When an individual first learns a new concept they start at the Action stage, which is described as when an individual can take a mathematical object and perform an explicit transformation based on external cues. These actions can be simple or complex, depending on the objects they are acting upon. After repeating an Action, individuals move away from relying on external cues and can control the procedure internally. At the Process stage, individuals can implicitly carry out the transformation and even deviate from the external cues they previously relied on. Students who can then act on this dynamic, internal procedure as a static object are said to be at the Object stage. These now-static objects can then be acted on by new external cues to continue developing the concept. Finally, a Schema is an ever-changing mental structure that an individual constructs and reconstructs. Schemas include Actions, Processes, Objects, and other Schemas about a single mathematical concept. Schema development occurs both through the stages an individual may take through a concept as well as through the connections between other mental structures related to the concept.

Methodology

To examine students' covariational reasoning while modeling a real-life problem, we developed a virtual model of a bird flying in a helix pattern around a tower. The student could change their view of the tower without interrupting the bird flight by rotating horizontally and vertically around the tower. The student could pause the bird's flight or leave it to loop. A flag was presented on the ground parallel to the tower to provide an additional landmark the student may use to reason through the variations in the bird's horizontal, vertical, and height displacement. After being showed how to change the views of the model, a student was prompted to answer questions along two separate goals: (task 1) graph an individual quantity with respect to time and (task 2) graph two or three quantities irrespective to time.

Prior to completing the task, students were asked to supply their personal definition for *function* and *derivative*. After completing the task and discussing their answers with the interviewer, students were again able to present their definitions for *function* and *derivative* to see if their definition had developed.

Volunteers to complete the interview were solicited from a Calculus 1 course at a large southeastern university during Spring 2021. One student volunteered to participate: pseudonym

Jane. Jane was a first-year university student, who (at the time of the study) was taking calculus for the third time. This was her second time taking it on the university level, prior to which they had taken a first semester calculus course through the International Baccalaureate program at their secondary institution.

Data Analysis

At the action level understanding of a *function*, in the first task we would expect a student to be reliant on selecting specific moments in time and their corresponding height to create their graph and recognize a linear relationship. Whereas students possessing a process level understanding would abstract this point process to create a smooth representation and a continuous line.

For the second task, students who understand covariational reasoning at the action level would start graphing individual points and connecting them to create their parametric representation. Students with a strong definition of derivative could recognize that as they plot more points, if they were to plot infinitely many points, then a smooth representation could emerge, which students could internalize to recognize how changes in one function coordinate with another function and thus attain a process level understanding of covariation reasoning.

After transcribing the interview, the authors analyzed Jane's responses to the task to identify evidence for simultaneous-independent and/or change-dependent reasoning in terms of APOS Theory. We present preliminary results from this analysis.

Results

Based upon the student's personal definition of *function* and their response to question 1, it was evident the student possessed an action level understanding of *function*. This caused them difficulties in coordinating each function to create a polar representation, which in conjunction with their definition of *derivative*, showed they also possessed an action level understanding of covariational reasoning. We present evidence for her level of conception through how she created a linear representation and a parametric representation of the bird's flight.

Creating a Linear Representation

A feature of the instrument that was not implemented was an explicit measurement tool, whether for measuring time or for determining numerical values for the bird's position. This meant if students depended on having specific inputs for determining a function representation or constructing a parametric representation, students would need to create their own measurement tool, which is what Jane did. As shown in the following image, Jane not only measured the period of the bird's flight around the tower, but also constructed ratios between the height of the bird to the flag to the tower. These ratios were indicated in her scratchwork and during the interview where she detailed using a piece of paper to measure the differences in height to create her ratio.

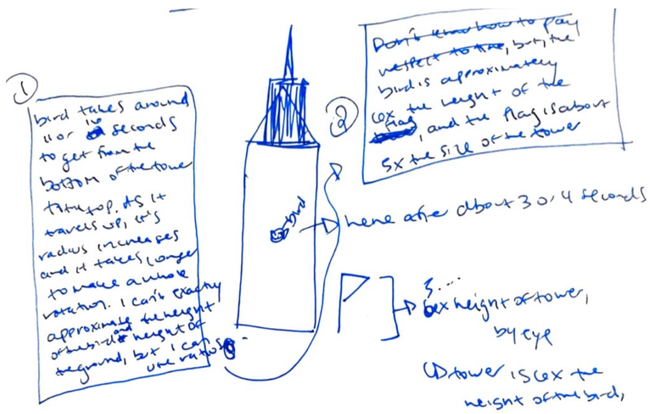

Figure 1: Scanned image showing Jane's Scratchwork for task 1. Transcribed writing:
1. Bird takes around 11 or 16 seconds to get from the bottom of the tower to the top
*2. Flag is approximately 6x height of the tower… tower is 5x height of flag**
3. Tower is 156x height of the bird

Jane used the flag, the tower, and the bird to create a measurement system based on the ratios between each. The flag specifically Jane used as a reference so that they could track the period of each cycle in the bird's movement and identify points to construct her representation. As shown in the next section, Jane specifically used the flag to indicate the side of the tower the bird was on and create "snapshots" of the bird's motion. Between finding a numerical value for the period of the bird's flight and using the flag to create a measurement system of ratios, Jane needed a system of points to create a linear representation.

Creating a Polar Representation

Something to note about Jane's solution to creating a parametric representation for the polar representation is that they misinterpreted the quantities to be coordinated and created a representation that showed the bird's height with the bird's horizontal translation.

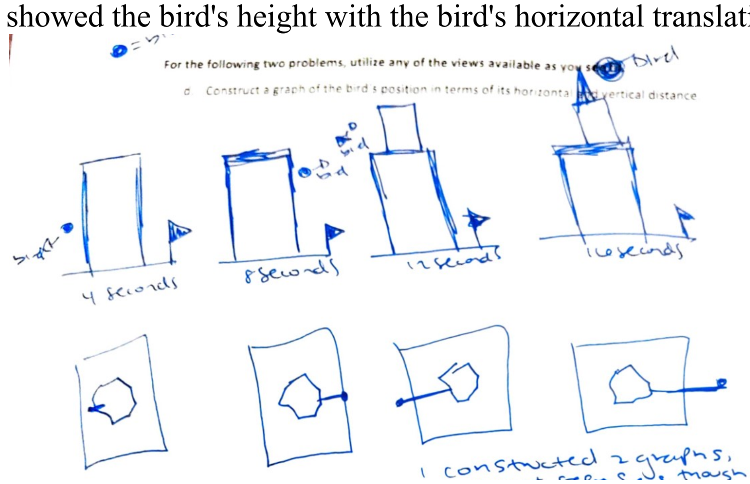

Figure 2: Scanned image showing Jane's Scratchwork for task 2.

This was the closest Jane got to creating a parametric representation, which she accomplished by creating "snapshots". Jane took her pre-existing relations between the bird's height and horizontal position and connected them through time to describe the bird's position. This is where students with a strong process level understanding of covariational reasoning would be able to begin interacting with simultaneous-independent reasoning. Specifically, they could track and understand how the height and the horizontal position relate to each other, with time being implicit in their representation.

In Jane's response there was a clear change-dependent thought process being applied. In the interview Jane explained that she would start with the bird's height at a given time, then determine its horizontal position at that time. In other words, Jane used time to move between each component of the position but could not separate time in the representation. This is where students' definition of *derivative* is a factor since derivatives describe the relationship between different values. Jane described derivatives as, "rates of change that… pay respect to time". Jane's understanding of derivative is tied to how a value relates to time.

Implications

These results show that students with an undeveloped understanding of *function* and *derivative* face challenges applying covariational reasoning. Students with an action level understanding of *function* cannot continually interpret relationships between quantities over extended periods. More research should be directed into understanding the constructions students use to produce continuous parametric representations. This will in turn help students' covariational reasoning abilities because it will prepare students to interact with continuous representations, rather than the "snapshots" we saw in our results.

On the other hand, the role of time in teaching derivatives may need to be de-emphasized. While authors such as Keene (2007) illustrated that students often incorporate time as they consider different attributes of a physical example changing, this may prohibit reasoning about quantities changing irrespective to time. The numerous examples of derivatives with time may encourage students to overgeneralize derivatives as a quantity changing over time. A stronger understanding of derivative would have helped Jane coordinate how the height and the horizontal translation changed relative to each other.

Limitations and Future Work

Despite the results the study was able to produce, there were multiple constraints that appeared. The most immediate was that as a pilot study there was a single participant in the study. This study was also conducted during the COVID-19 pandemic, which meant the study was conducted virtually. This meant that some of the physical actions that students produce when interacting with the instrument were difficult to observe.

After observations during the study and feedback during the interview, the instrument and directions could use further development. For instance, different tools were built in for students to interact with the model which were largely unused. Some of the tools were specifically implemented to determine how students would reason with an invariant relationship, namely whether they would be able to recognize the presence of an invariant relationship and whether they would represent it in their graph. Emphasizing these tools could provide valuable data.

Asides from technical corrections, there are other directions this research could go in future iterations. This target audience for this study were first semester calculus students, which is what led to focusing on students' reasoning capabilities between linear and trigonometric representations since these are some of the first representations students interact with. While could be investigated further, beyond changing the target audience, future iterations could focus on how students are able to use covariational reasonings to interpret relationships with other functions such as exponentials, logarithmic, or even investigating how students recognize and analyze piecewise functions.

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Mathematical Modeling as a Way of Expressing the Flexibility of Solution Strategies

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Mathematical modeling tasks connect authentic situations to classroom mathematics and allow for a variety of solution strategies. To this end, we designed a hybrid project that centered on a mathematical modeling task for elementary preservice teachers within an integrated content and methods module. Participants engaged in the Soccer Task as both students and teachers, individually and in groups, using multiple platforms and modalities. Using artifact analysis, we explored how the preservice teachers used mathematical strategies of varying complexities, drew on their funds of knowledge, and reflected on the meaning of fairness in different contexts. We present preliminary findings and suggest prompts for discussion.

Keywords: mathematical modeling, preservice teachers, funds of knowledge, hybrid teaching

Purpose and Background

Teaching mathematical modeling requires a level of openness typically not found in traditional curriculum, resulting in a different, oftentimes more demanding, way of teaching mathematics (Bennett, 2017; Doerr, 2006). While there are several affordances of modeling in the teaching and learning of mathematics, this practice can be challenging to incorporate into preservice teacher (PST) preparation programs, particularly at the elementary level (Bartell et al., 2010; Kaiser et al., 2017). To address this issue, we created the Hybrid Teaching Project (HTP), during which undergraduate PSTs experienced mathematical modeling from the perspective of both a learner and a teacher. Although the HTP was initially created in response to the COVID-19 pandemic and the need for multiple teaching modalities, it provided PSTs the opportunity to engage in authentic mathematics situations via mathematical modeling in a hybrid structure. The research question guiding this project is:

1. How do PSTs draw on their own funds of knowledge via multiple solution strategies when solving a modeling task?

Conceptual Framework

In the Common Core State Standards (CCSSI, 2010), the standard for mathematical practice "Model with Mathematics" states that "mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (p.7). Mathematics education researchers have defined mathematical modeling as a process for connecting the real world to the world of mathematics (Blum & Borromeo Ferri; 2009; Blum & Leiss, 2007) or as "a process in which students consider and make sense of an everyday situation that will be analyzed using mathematics for the purpose of understanding, explaining, or predicting something" (Anhalt et al., 2018, p. 202). A modeling diagram, often in the form of a cycle, accompanies most definitions to illustrate the nonlinear, complex, iterative nature of modeling (see Figure 1).

Mathematical modeling tasks can connect multiple content areas and, in this way, can augment existing K-12 curriculum as a replacement or extension of problem-solving tasks (Mousoulides et al., 2008). Indeed, the work of Doerr and English (2003) demonstrated how students could not only find a solution to an authentic mathematics problem, but could also create a generalized solution, a procedure or model, that would be transferable to many similar situations. The research on mathematical modeling in the early grades (K-5) is limited (Lyn English's (2006; 2008; 2012) work in Australia is an exception) but has gained prominence in recent years in the U.S. (e.g., Turner et al., 2021; Carlson et al., 2016; Suh et al., 2017).

Examples of mathematical modeling tasks at the elementary grades level include tasks that introduce complex systems via tables of data, such as the Olympic Swimming Team Selection task (English, 2008). Similar tasks related to selecting sports teams (e.g., M2C3, 2018), have the goal of connecting to students' prior experiences and funds of knowledge. The notion of *funds of knowledge* views households as wellsprings of cultural resources, skills, and knowledge that children can leverage to make connections between their school and home learning (Civil, 2002; Moll et al., 1992). Recent research has explored how mathematical modeling tasks can draw on elementary students' funds of knowledge to help them connect classroom mathematics to authentic situations in their school, family, home, and community (e.g., Civil et al., 2021; Turner et al., 2021; Wickstrom et al., 2017). However, few studies examine the mathematical thinking and strategies of elementary PSTs engaging in modeling in their undergraduate courses.

Methods

Context and Participants

The project reported in this paper spanned two courses, a mathematics content course and a mathematics teaching methods course for elementary PSTs. At this institution, the content and teaching methods courses are integrated as part of a STEM block for elementary PSTs (see Heaton & Lewis, 2011; Homp & Lewis, 2021), meaning they have a shared syllabus and some shared projects. Additionally, during typical semesters, PSTs taking both courses participate in a practicum experience in elementary schools two days a week. Participants of the first iteration of this project were 62 elementary PSTs across two sections in the first semester of their preparation program. In this paper, we focus on the work of one section of 30 students, of which the first two authors were the instructors for the content course and the teaching methods course, respectively.

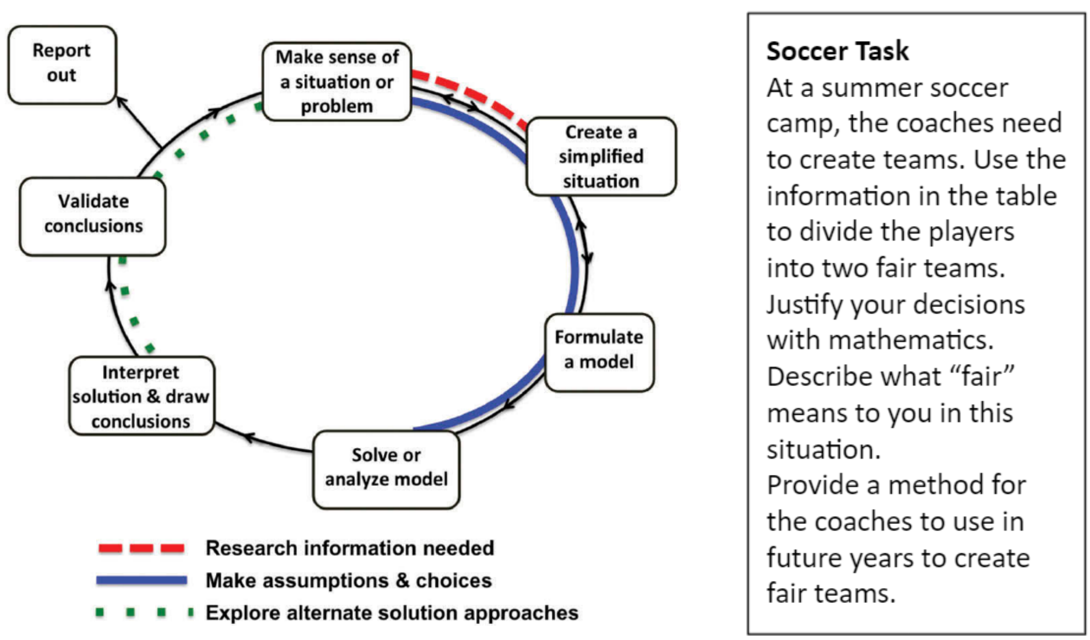

Figure 1. The mathematical modeling cycle and the Soccer Task, as presented to PSTs

Structure of the Hybrid Teaching Project

The HTP consisted of multiple scaffolded parts. As mentioned, we re-designed this project since PSTs participated in class via a hybrid format and were not able to interact with elementary students in their classrooms. We initially provided PSTs with the modeling cycle diagram shown in Figure 1 (from Anhalt et al., 2018), as well as the Soccer task prompt (M2C3, 2018). In Figure 2, we share a small portion of the player information table from the task. To see the complete task, including the table with soccer player skills, visit the M2C3 project site.

Player Name	100 meter sprint time	Shots on Goal (out of 5 tries)	Defensive blocks out of 10
Annette	14.41 sec	score score score blocked blocked	2
Brianna	13.80 sec	score score wide blocked blocked	7
Chandra	14.95 sec	score score blocked wide wide	8

Figure 2. Portion of soccer player information table, as presented to PSTs

The first component of the HTP asked PSTs to individually solve the Soccer task, in which they were given data on 12 soccer players and told to create two fair teams (M2C3, 2018). The information table in the task included three skills: speed, shots on goal (scoring), and defensive blocks (see Figure 2); thus, PSTs first had to make sense of the task and decide what skills they deemed important. PSTs' assumptions changed based on their funds of knowledge and lived experiences. For instance, a PST familiar with playing soccer might make the assumption that a "blocked" shot indicates a higher skill level than a "wide" shot. Furthermore, in designing the HTP, we purposefully chose a modeling task and reflection assignments that emphasized PSTs multiple mathematical knowledge bases (Turner et al., 2012) and funds of knowledge.

The second component of the HTP was a task analysis which required PSTs to work in a group to compare each other's individual solutions, as well as compare their solutions to how they think an elementary student would solve the task. The third component of the HTP prompted PSTs to modify the original Soccer task; they had the flexibility to alter the task in multiple ways. PSTs were required to adapt the task to a specific grade level by connecting to appropriate mathematical content and possibly changing the complexity of the information table. For example, if the task was modified for first grade, then the data for sprint speed should be modified to display whole numbers rather than decimal representations up to the hundredths place. PSTs were also allowed to change the sports context (e.g., change soccer to volleyball) and change the skills and variables as necessary (e.g., change "speed" to "vertical jump").

Preliminary Findings

We present our preliminary findings from artifact analysis (Hatch, 2002) of the first component of the HTP project: the mathematical solutions and strategies from the Soccer Task. Additionally, we explored PSTs' notions of "fairness" and their use of funds of knowledge in their written work through Concept and Focused coding techniques (Saldaña, 2016).

Flexibility of Task Solutions

Overall, 28 PSTs explicitly stated their final two teams for the Soccer task, and 26 of these were unique solutions. In other words, there was a high level of flexibility of solutions and very little repetition of team configurations for the 12 players. Note that there are 924 (or "12 Choose

6") possible team configurations in this task. The variability of possible solutions indicates a variety of avenues for further analysis. For instance, we could analyze how PSTs placed pairs of players on the same or opposing teams: Annette (player A) and Brianna (player B) were two of the fastest soccer players with similar "shots on goal" outcomes (see Figure 2). Of the 26 unique solutions, 13 placed players A and B on the same team, and 13 placed A and B on different teams. This analysis of team configurations could be repeated for other pairs of players and emphasizes the ability of this task to generate multiple solutions and strategies.

Flexibility of Mathematical Strategies

A variety of team configurations for the Soccer task leads to questions about the strategies used to create the teams, (i.e., the models). We first analyzed the 30 solution strategies based on the mathematical concepts used (e.g., summing, ranking/comparing, calculating averages), tools and representations used (e.g., Excel spreadsheets, tables), and the soccer skills attended to (i.e., speed, shots on goal, defensive blocks). Then, we grouped and sorted the models based on their level of complexity, similar to the work of Anhalt et al., 2018. We created a continuum of complexity ranging from low complexity (one mathematical strategy and/or one skill focused on) to high complexity (two or more mathematical strategies and/or three skills focused on).

Models that had aspects of randomness, rather than specified methods or rationales, were generally considered less complex. Five models that were on the low complexity end of the continuum seemed to consider multiple soccer skills but did not use a mathematical strategy to rank or assign players to teams. For example, PST5 found the top two players in each category and divided them between the two teams, but then distributed the remaining players without a specified mathematical method. However, PST5 added authenticity to his model by leveraging his knowledge of soccer as a sport to guide some of his decisions, such as giving each team a "good defender", placing the fastest players "at midfield", and placing the "best scorers at forward", which created, in his opinion, fair teams. He noted, "As someone who plans on being a coach as well as a teacher, I tried to separate them by equal levels of talent." The evidence of PSTs' funds of knowledge is summarized in Table 1.

Table 1. Mathematical complexity of PST models paired with evidence of funds of knowledge

	Low Mathematical Complexity	Low-Moderate Complexity	Moderate-High Complexity	High Mathematical Complexity
PST	PST5	PST16	PST27	PST30
Evidence of Funds of Knowledge	knowledge of soccer positions and interest as future coach	opinion of most important skills for soccer	related "strength" in sports context mathematical average	notion of "fair" was validated by the modeling process

Moving toward more mathematically complex models, 10 PSTs created models that focused on only one soccer skill (e.g., "blocks"), then created a ranking to distribute players to each team. PST16 ranked and ordered players based on their number of "blocks" and considered the number of goals scored as a tiebreaker. She deliberately chose to focus on defensive blocks, stating, "my reason for this is that defense is the most important aspect...defense is most important because the better your team is at defense the less goals the other team scores." Then, PST16 matched players on opposite ends of the list (i.e., 1 and 12, 2 and 11, etc.) and placed pairs on opposing teams. Other PSTs used this distribution method and called it "the rainbow method" due to the concentric arc shapes formed during the pairing process.

Two of the high complexity solutions considered all three skill categories in their models, but in different ways. Both PSTs based their models on their knowledge and assumptions about what it means to be "strong" or a "good player" in soccer. PST27 calculated an average for each of the three skills and created three categories: strong runners, strong "shooters", and strong blockers, where "strong" was defined to be above the average. Then, she placed players that were strong in one category first, so that they "can participate in the game in the way they are strongest." Finally, she distributed players that were strong in two categories based on what skills she determined the teams were lacking.

PST30 ranked players across all three skills and considered the top and bottom two players in each category, assigning one of each pair to a different team. Then, PST30 placed the remaining players on teams in three different ways, calculating "average skill levels" for both teams on all three attempts. She ultimately chose the configuration for which the two teams had the closest averages to each other. Although this model incorporated a modified guess-and-check method, it utilized the validating and revising steps of the modeling process to verify that the teams created were as fair as possible.

Based on these findings related to the flexibility and complexity of mathematical strategies, we present some themes and suggest prompts for audience discussion.

Discussion Prompts

The HTP and the Soccer task provided opportunities for PSTs to explore the modeling process and the benefits of flexible, open-ended tasks. The Soccer task highlighted the often-ambiguous nature of authentic mathematics in real-world situations and how assumptions impact final answers. *What assumptions did PSTs make when solving the Soccer task and how did these assumptions influence their mathematical models?*

Within the modeling process, an important part of making sense of the situation and making assumptions is to simplify the situation so that a model is feasible. *While we explored the complexity of models, how could we also explore the "degrees of simplicity," in other words, the extent to which PSTs simplified the task to create a more manageable model?*

We deemed several models to be quite mathematically complex. However, while some PSTs suggested giving more importance to one skill over the others, no models explicitly mentioned "weighted averages" or used this mathematical concept in their solution strategies. This presents an opportunity to revisit the Soccer task later in the semester (or possibly in future semesters) and let PSTs revise their previous models to incorporate other mathematical tools and concepts. *How can the revision process be a powerful tool in undergraduate mathematics courses?*

The examples of high complexity solutions came from students who would not be placed at the top of the class in the traditional sense, both in terms of class participation and grades. However, they created sophisticated mathematical models and successfully engaged in this task. *How can mathematical modeling tasks promote a growth mindset for PSTs and provide space for them to showcase their diverse ways of understanding and doing mathematics? Furthermore, how can mathematics education researchers leverage modeling tasks to highlight anti-deficit narratives in research on student learning?*

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Linear Algebra as a Prerequisite: A Pilot Study

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It is well known to anyone who teaches introductory linear algebra that it is often populated by students from across the STEM disciplines. However, as a research community we don't have a more systematic understanding of where exactly these students are coming from. In this report, we present findings from a preliminary investigation aimed at identifying fields of study, often referred to as client disciplines, that depend on introductory linear algebra as an important component of their undergraduate curriculum. By conducting a direct survey of undergraduate course catalogs from thirty colleges and universities across the United States we created a list of undergraduate majors requiring some kind of introductory linear algebra course as a requirement for degree completion. This report details and explores the variation between and within various majors and fields of study. Initial analysis revealed 60 distinct majors, nested within 15 fields of study.

Keywords: Linear Algebra, Prerequisites, Client Disciplines, Applications

Introduction

Historically, linear algebra has played an important role in the undergraduate mathematics curriculum, providing students majoring in mathematics with an introduction to more abstract concepts and preparing them for higher-level mathematics courses. As the mathematical sciences have become increasingly integrated with other disciplinary fields (National Research Council, 2013) there has also been an expansion in the applications of linear algebra outside of mathematics and, in turn, an increase in the number of students from client disciplines taking Linear Algebra courses at the undergraduate level. Despite the changing student population, applications of linear algebra concepts are generally not a primary component in the curriculum (Bergman & Kirin, under review), potentially creating "significant difficulties for students who struggle to grasp the more theoretical aspects of the course (Stewart, Andrews-Larson, & Zandieh, 2019).

At the same time, recommendations have been put forth by the research community that recognize the changing role that linear algebra plays in today's society. For instance, the Committee for the Undergraduate Mathematics Program (CUMP) states that "Every linear algebra course should incorporate interesting applications, both to highlight the broad usefulness of linear algebra and to help students see the role of the theory in the subject as it is applied" (Diefenderfer, Hill, Axler, Neudauer, & Strong, 2015). Similarly, when the Linear Algebra Curriculum Study Group, an NSF funded working group composed of both mathematicians from across the country and consultants from a variety of client disciplines, came together to produce a set of recommendations for a first course in linear algebra, their very first recommendation was: "The syllabus and presentation of the first course in linear algebra must respond to the needs of client disciplines" (p.41, Carlson et al., 1993).

More recently, in an article surveying research in linear algebra, Stewart, Andrews-Larson, and Zandieh (2019) point out that future research needs to focus on *how* linear algebra is applied in other fields, suggesting that such research could inform the design and selection of applications and topics in the classroom. We agree with these researchers that more attention needs to be paid to identifying, designing, and incorporating realistic applications of linear algebra into the classroom. In response to this, we feel that a natural first step is to systematically document *what* other disciplines rely heavily on linear algebra. Thus, we ask:

1. What undergraduate majors and fields of study require introductory linear algebra as part of their curriculum?
2. How common are these majors and fields of study across institutions?

Methods

We would like to begin the methods by defining both a field of study and a major. A field of study is a branch of knowledge (e.g., art, mathematics, philosophy) and we use the term interchangeably with disciplines. Whereas a major is defined as a program that results in a bachelor's degree upon completion (e.g., Bachelors of Science in applied physics).

In order to investigate how introductory linear algebra (200-300 level) is serving as a prerequisite, we chose to look at undergraduate majors that listed introductory linear algebra as a requirement for degree completion. For this pilot study, our data set consists of a sample of 30 institutions, including doctoral, masters, and bachelors-granting colleges and universities from across the United States. A direct survey method was used to obtain degree requirements as listed in the most recently available course catalog from each institution. Direct survey methods make use of available material that exists online or in printed format (Stefanidis & Fitzgerald, 2014) and allow for data collection in a systematic way (Kung, Yang, & Zhang, 2006). In other words, a direct survey method can help to provide a comprehensive snapshot of specific undergraduate programs of interest in the United States (Bell, 2012).

For this study we utilized university web sites and course catalogs as the primary source of information for the survey data. We began by searching course catalogs for mathematics courses currently offered at each institution in order to determine the specific course numbers for introductory linear algebra courses. We then searched the entire undergraduate course catalog at each institution for both the term “linear algebra” and specific course numbers. This search produced a list of all majors at each of the institutions that noted linear algebra as a requirement for completion. Note we did not list majors that listed linear algebra as an elective although nor did we list specific courses that required linear algebra, although, these are both dimensions of data collection we feel are worth exploring in a follow-up study. Once we had a list of majors, we then grouped like majors into their more general fields of study. In doing this our hope was to highlight the commonalities among the various majors while also recording the variation in focuses and specializations.

Results

As analysis is ongoing, in this preliminary report we share findings with respect to our first research question: *What undergraduate majors and fields of study require introductory linear algebra as part of their curriculum?*

Our results organize the 175 instances where an introductory linear algebra course was listed as a requirement for degree completion across the 30 institutions. These 175 instances were then grouped into 15 fields of study and collapsed into 60 unique majors. Table 1 shows the variation in the types of fields of study. These fields of study provide an overview of the types of disciplines that are leveraging linear algebra. Note that if a single institution offered both a Bachelor of Arts, BA, in mathematics and a Bachelor of Science, BS, in mathematics only one “mathematics” major was counted. However, if a single institution offered both a BS in mathematics and a BS in applied mathematics, they were counted as two different majors.

Table 1. Fields of Study

Fields of study		
Mathematics Engineering Computer Science Statistics Physics	Data Science Dual Programs (i.e. math/physics) Economics Chemistry Neuroscience	Astrophysics Geoscience Cognitive Science Industrial Technology Sociology

Again there were 60 unique majors across the 15 fields of study. Table 2 highlights how the 60 majors were distributed across the 15 fields of study. If a single institution offered both a BS in mathematics and a BS in applied mathematics, they were counted as two different majors.

Table 2. Fields of Study and types of majors

Fields of study (and # of unique majors within)		
Mathematics (10) Engineering (15) Computer Science (3) Statistics (5) Physics (5)	Data Science (4) Dual Programs (4) (i.e. math/physics) Economics (4) Chemistry (3) Neuroscience (2)	Astrophysics (1) Geoscience (1) Cognitive Science (1) Industrial Technology (1) Sociology (1)

While the fields of study can provide us with a general overview of the client disciplines using linear algebra, it is also important to consider the variation of concentrations within each field. This offers a more specific understanding of the applications that require linear algebra. Of the 15 fields of study, 10 had variation in the types of majors offered. Table 3 further illustrates this variety within a single discipline, engineering.

Table 3. Unique Engineering Majors within the Field of Study: Engineering

Unique Engineering Majors		
Bioengineering Bioengineering: Pre-med Chemical Engineering Civil Engineering Computer Engineering Cyber Security Engineering	Electrical Engineering Electronics Engineering Financial Engineering Industrial Engineering and Management	Manufacturing Engineering Mechanical Engineering Nuclear Engineering Systems Engineering Software Engineering

For our presentation we will further describe other fields of study and the various majors within them. Additionally, we will share results related to how these fields of study and majors are distributed across our sample of institutions. Such findings can help provide insight into how common these fields of study and majors are as well as differences that may exist across institutional types (e.g., doctoral, masters, bachelors granting).

Discussion and Implications

While some of our results were expected, others we found quite surprising. For instance it seemed obvious that many mathematics majors would require introductory linear algebra. From the existing research literature, and our own experience with the course, fields of study beyond mathematics such as engineering, physics, and chemistry were also fully anticipated. However, we were not expecting such a wide variety of disciplines, and we found some of the fields of study such as sociology and astronomy to be particularly interesting. We were also surprised by just how many unique majors required linear algebra (60) across the 30 institutions. The diversity of fields identified in these preliminary results and the large number of majors requiring linear algebra further support the notion that this kind of systematic study of the applications of linear algebra is both informative and worthwhile.

A systematic understanding of how linear algebra is being used as a prerequisite can be utilized in at least three different ways. First, it can be particularly helpful for curriculum designers who want to heed the calls to incorporate interesting applications (Diefenderfer et al., 2015) and to respond to the needs of client disciplines (Carlson, et al., 1993). Second, a better understanding of the majors requiring linear algebra can help linear algebra instructors better understand and serve their students' needs. Carlson, et al. (1993) claim that since most students currently take only one course in linear algebra, it is imperative that the course syllabus contain the topics and concepts needed most by the majority of the students. Lastly, this research can be leveraged by educational researchers by starting to identify topics that might be appropriate for a second course in linear algebra, one which includes more abstract topics, as we continue to document, and serve, industry needs (Stewart et al., 2019).

Directions for Future Study and Questions for the Audience

Again the results presented in this preliminary report are from a pilot study of 30 institutions. We have already begun to outline a larger study looking at a much larger data set of institutions. We are sharing these preliminary findings with the research community in order to foster discussion that can provide us with valuable insight as we decide how to move forward with our study. Some of the questions we would like to discuss with the audience are:

- Q1: In terms of adding to the research knowledge base, what would be the benefits (drawbacks) of focusing on introductory linear algebra (300 or below) versus any undergraduate linear algebra course?
- Q2: In terms of populations, we're considering R1 and R2 universities. What advantages or disadvantages might this and/or other populations have?
- Q3: What about courses that require linear instead of just majors? This would be a much larger data set but would also give much more insight on how linear algebra is being used as a prerequisite. Would this be worth pursuing and, if so, how?

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Generalizing in the Context of a Generic Example

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Generalizing is a keystone of mathematics instruction at all levels. The goal of this preliminary report is to begin articulating the ways in which students generalize in the context of a generic example. The study takes place in a university introduction to proof classroom, and follows one group as they seek to first explain why a sequence converges and then come up with more general conjectures. Using transcript data and Ellis' (2007) generalizing taxonomy I describe student activity while generalizing, and frame the data according to the process of generalizing with a generic example.

Keywords: generalizing, generic example, real analysis

Generalizing is a keystone of mathematics instruction at all levels. Generalization can be thought of as the broadening the context of a particular argument or claim (Harel & Tall, 1991). Many scholars have studied generalization (e.g., Ellis, 2007; Park & Kim, 2017) and continue to express a need for better understanding how students generalize, and can be supported in this venture. One way that mathematicians generalize is through reasoning generically about a specific example, this is referred to as a generic example. Mason and Pimm (1984) define a generic example as a concrete example in which one can see the general. They give a number of examples in everyday life, such as Kleenex. Kleenex is facial tissue, but the name is synonymous with the object. It is a specific example of facial tissue, but one may be predisposed to consider all things called Kleenex as a facial tissue. Similarly, Harel and Tall (1991) define a generic example as a specific example seen by an educator as representing an abstract idea (pp. 41).

In the case of proof specifically, Balacheff (1987) describes a generic example as an attempt by a student to prove a general claim. Cassabut et al. (2012) describe Balacheff's conceptualization of generic example (since the original manuscript is in French), "the generic example makes the reasons for the truth of an assertion explicit by operations or transformations on an object that is a characteristic representative of its class" (pp. 173). In other words, generic examples are descriptions of students' attempts to prove a conjecture in general, but doing so in the context of a specific example, which is often seen as a naïve proof construction.

I view a generic example similarly to Mason and Pimm (1984), and Harel and Tall (1991), in that a generic example is something specific that one can use to reason more generally. I seek to expand on these definitions of generic examples by viewing them as processes through which one can reason generically about the specific. I do not believe that there are examples that exist that are purely generic, but rather it describes a large class of examples that can be reasoned about in a generic way to make sense of general behavior. This study examines the nature of generalizing in an introduction to proofs course taught using curriculum materials developed using the design heuristics of Realistic Mathematics Education. In particular, this preliminary report focuses on how students generalize in the context of generic examples, and the processes one goes through while reasoning generically about an example.

Theoretical Perspective

There is a growing body of literature on student thinking on generalizing in mathematics. Ellis (2007) reviews a large body of research and contributes a student-centered approach to generalizing. Ellis (2007) presents a taxonomy for student generalizing actions and the reflection generalizations that are produced. Ellis argues that the actions involved in generalizing include *relating*, *searching*, or *extending*. *Relating* involves a student making a connection between two scenarios. *Searching* involves reviewing a number of examples in order to find some common property, this action needs to be repeated in order to count as searching (pp. 238). *Extending* involves a student noticing a common property or relationship and then also goes on to expand the pattern to capture more cases (pp. 241). These then result in reflection generalizations which are *identifications or statements*, *definitions*, or *influences*. When a student makes an *identification or statement* they verbally or in writing express a general statement in the form of a rule, pattern, or property (pp. 245). In order for a reflection generalization to count as a *definition* the student's statement (verbal or written) must communicate the "fundamental character of a pattern, relation, class, or other phenomenon" (pp. 248). Lastly, *influence* applies a previously developed generalization to a new context (pp. 249).

Using Ellis' (2007) framework allows the researcher to have a student-oriented perspective when studying generalizing activity. However, Ellis does not explicitly attend to the notion of generic examples. This study seeks to contribute to the growing body of student thinking on generalizing by illustrating how students generalize in the context of a generic example. In particular this study seeks to address the question: What are the features of student activity when generalizing in the context of a generic example?

Methodology

Data for this study comes from a larger, ongoing NSF-funded project (ASPIRE in Math, DUE 1916490) that is developing Introduction to Proof curricula and accompanying instructor support materials for the guided reinvention of the foundations of real analysis. The ASPIRE in Math project curriculum is being implemented in several university and community college courses across the West Coast of the United States as well as other community colleges across the country. The data for this preliminary study is from an implementation at a university Introduction to Proofs course. This course was taught remotely by two instructors (one faculty member and one doctoral student) assisted by another doctoral student. The class had 14 students whose demographics were representative of the university at large. The class met synchronously over Zoom for 10 weeks, twice a week for 1 hour and 50 minutes. The class activities were captured using screen recording technology, zoom recording, and Google Docs. This particular study focuses on one class day and follows the assisting doctoral student (the teacher-researcher for this study or TR) and his small group of 4 students as they were working on a generalization task and culminates with the students coming up with general conjectures.

Due to space constraints only the relevant portion of the instructional sequence is presented here. For more details see Larsen, Alzaga Elizondo, et al. (submitted), Larsen, Strand et al. (in progress), and Strand, et al. (in progress). In this episode students were working with a sequence generated by a root approximation method called the Bisection Method. Students develop this method by determining an algorithm one might use to approximate a root of a continuous function that has a sign change. The method involves first determining an interval where there is a sign change, the left endpoints are labeled with a_n and the right endpoints are labeled with b_n (where $n \in \mathbb{N}$). Then, one finds the midpoint between a_1 and b_1, with $a_1 < b_1$, called c_1, test the output of c_1, and replacing a_1 or b_1 with c_1 depending on the output sign, and then repeating until desired accuracy of the root is obtained. The students are then reoriented by the instructors to no longer assume there is a root, but to prove that one exists. This method generates the left endpoint sequence denoted by a_n and this sequence serves as a generic example of a sequence that is increasing and bounded above. The episode analyzed below follows almost immediately, and involves two tasks: (a) why does *the* left endpoint sequence converge, and (b) come up with general conjectures to explain why *the* left endpoint sequence converges.

From the video data collected, a transcript of the day was created. This transcript was analyzed using Ellis' (2007) taxonomy for generalizing actions and reflection generalizations. In order to identify instances of generalizing, the transcripts were coded according to utterances by students that evidenced generalizing activity based on the framework. Once the coding was complete, an analytic memo was written to recreate the story of the classroom interactions and patterns describing how students reasoned with a generic example were identified.

Initial Results

The analysis of the data revealed that these two tasks represent three phases of student activity while generalizing with a generic example: (a) make sense of the example, (b) figuring out what makes the example tick, and (c) generalizing and conjecturing further. I will illustrate these phases below.

Make Sense of the Generic Example

In order to generalize from a generic example one first needs to make sense of the example itself. Below, I will give instances of two students offering generalizations about why *the* left endpoint sequence converges. The two instances seek to illustrate the ways students generalize when they have and have not yet identified the left endpoint sequence as a generic example.

The teacher researcher starts the task by asking the students to come up with a reason for why the left endpoint sequence must converge. Leah begins by offering the following generalization:

> Well, I, when I was doing the homework, I was just thinking about, if you start with a point, an "a" value that has a positive outcome, and a "b" value that has a negative outcome, there's going to be some point where that switches over and depending on the function, the first few values you have aren't going to be exactly where the switch is going to happen. So, you just have to keep narrowing it down.

Leah explains that one starts by determining where the sign change occurs, and that the first values aren't necessarily near the location of the root, so one continues to iterate the method – but she does not explicitly attend to how she knows the left endpoint sequences converge. She is relating different situations in which she has encountered the bisection method. In this way, Leah is still working to make sense of the example of *the* left endpoint sequence because she has described the bisection method but not the sequences that are generated. In contrast, Maya has already made sense of *the* left endpoint as an example:

> So, when you're doing this method, each next term of the left endpoint sequence that you're making is either going to be the same as the term you were on because the other one changed, or it's going to go halfway to the other term, right? So um, you're either adding zero to it, if it stayed [sic] the same, or you're adding the difference between [a_n] and $b_{same\ n}$.

Maya's contribution starts with her orienting herself with the bisection method as Leah did, except that she also *searches* across various examples (mentally) and identifies a common pattern that is specific to the left endpoint sequence. Maya says "when you're doing this method" demonstrating that she has identified the process from a number of examples she's worked with in the course. She's generalized that the left endpoint sequence either stays the same or the terms move to the right along the number line, as evidenced by her statement "each next term of the left endpoint sequence that you're making is either going to be the same as the term you were on … or it's going to go halfway to the other term."

These excerpts reveal two different ways students may work to identify the example they are asked to consider. While both of the students worked to come up with reasons why *the* left endpoint sequence converges, Leah's contribution really told the story of how she was trying to make sense of the task. Whereas Maya identified *the* left endpoint sequence and also reasoned about it generally to come up with a reason that it must converge. The next subsection explores the different reasons the small group came up with to explain why *the* left endpoint sequence converges.

Figuring Out What Makes the Example Tick

In this phase the students identify the salient properties of why a particular phenomenon occurs within the context of the generic example. As evidenced in the way Maya discusses why *the* left endpoint sequence converges (see above) one can see that Maya has already worked to understand what the general features of this sequence are that convince her it must converge. A few minutes later Maya offers another reason for why *the* left endpoint converges.

Maya: I suspect you might be looking for something like it's increasing, and it has an upper bound of b_n.

TR: So a_n is increasing. And it has an upper bound of b_n. And so, what is...

Maya: or b… or first b?

TR: Okay, b_1.

Maya: Yeah, b_1 works.

Maya generalizes by *extending* since she has removed all context except for the salient features, and results in the *identification* of a general rule (i.e., that the sequence is increasing and bounded above by b_1). What follows is Leah making sense of Maya's contribution.

Leah: I like [the increasing and bounded statement], I didn't really think about how - I mean, it's kind of implied when you do the bisection method, but I didn't really think about using that it has an upper bound of b_1 to prove why it converges. I found that interesting.

TR: Yeah, that is interesting, right? Because what does that actually mean? What does this mean? If we have an upper bound of b_1, what does it mean about our sequence?

Leah: That our a_n will never cross that point? And in a lot of cases, it's not even going to get very close to it, because the b's are going to get closer to that convergence point

TR: I see. Okay. All right, does it so what is this purpose here of this increasing component, then? Because if I'm bounded, right, why does it matter that I'm also increasing?

Leah: Because it's gonna approach that point, but never cross it?

Maya's second contribution was important for at least two reasons. First, Maya sees that *the* left endpoint sequence is both increasing and bounded above by b_1 which are both salient properties of *the* left endpoint sequence that can be used to develop more general conjectures. Second, Maya's contribution assisted Leah in identifying the general properties that she struggled to articulate earlier in the task. Leah is now making sense of these concepts as helpful to justify why *the* left endpoint sequence converges. The next phase proceeded almost immediately.

Generalizing and Conjecturing Further

In this phase the students are reflecting on the properties they found previously and generalizing further so that the properties are no longer tied to *the* left endpoint sequence and can now explain general phenomenon. This phase was signaled by a shift to a new task by the teacher-researcher. Namely, "Write a general conjecture of the form 'Let x_n be a sequence. If ____, then x_n converges."

The statement that *the* left endpoint sequence is increasing and bounded above by b_1 was easier to generalize for the group. They were able to capture the property in the following way: if a sequence is increasing and bounded above, then x_n converges. The first statement that Maya gave was harder to generalize. Recall that Maya's initial contribution was that "you're either adding zero to [a_n], if it stayed [sic] the same, or you're adding the difference between [a_n] and $b_{same\ n}$." Leah attempts to reframe this conjecture more generally "can we say that if $(b_1 - a_1)/2^n$ goes to zero, then we know that a_n converges?" Leah has identified that the distance between consecutive terms of *the* left endpoint sequence can be expressed by $(b_1 - a_1)/2^n$. This is not fully general yet, because Leah is still relying on the context of the bisection method. A few minutes later, Maya offers a generalization that builds on Leah's thoughts:

> Okay, I have something for this one. I was just waiting so people can [sic] think. If the limit of basically the difference between steps goes to zero. So, if the limit of $a_{n+1} - a_n$, because you're just dealing with the… or I guess x_n, so $x_{n+1} - x_n$ goes to zero, then the sequence converges.

Here, Maya's generalization is again *extending* by removing context to apply to broader cases, these include changing the symbol a_n to x_n, and she further generalizes the $(b_1 - a_1)/2^n$ goes to zero property to the property that the distance between consecutive terms is decreasing. At this point the students have used *the* left endpoint sequence as a generic example of a sequence that is increasing and bounded above, as evidenced by their final generalizations.

Discussion and Conclusion

In this study I sought to identify the features of student activity when generalizing in the context of a generic example. This study starts to articulate a process for generalizing from a generic example which includes three phases: (a) make sense of the example, (b) figuring out what makes the example tick, and (c) generalizing and conjecturing further. The utility of these phases requires more research, but initially there is promise for instructional heuristics to support students at all levels to generalize with a generic example.

Furthermore, the collective generalizing that takes place in these episodes indicate that providing students opportunities to work on this generic example in small groups has promise. For example, Maya and Leah collaborated to make sense of the example and to come up with the general conjectures in the end. Ellis (2011) refers to the actions taken by Leah and Maya as generalizing-promoting actions. Therefore, this suggests that collective generalizing takes place in the complex interactions in the classroom between students and instructors and between students and other students.

The presentation will include an expanded data set to further articulate the process by which one reasons generically with an example. The presentation will also further explore the notion collective generalizing.

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More Than Just the Math: Embedded Tutors May Provide More Than We Hoped For

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Developmental math (DM) courses historically and disproportionately affect the persistence and success of minoritized students. Embedded tutors (ET) have been utilized in many classrooms as a way to support student learning in such courses (Hayes, 2021). We analyze interviews and classroom observations of a DM course to understand how an ET is utilized in the classroom. The data show that the tutor was mainly used to support mathematics learning, yet he possessed a wealth of cultural capital that could support students holistically. We plan to discuss with the audience ways that tutoring centers and faculty can amplify such assets of ETs in the classroom.

Keywords: Embedded tutor, Equity, Developmental Mathematics

Community colleges serve a large number of historically minoritized, first-generation, and low-income students (Bahr, 2010; Baum et al., 2016). Many community colleges offer a pathway for students to develop foundational knowledge of mathematics through a series of developmental mathematics (DM) courses, which provide the "skills necessary to perform college-level work at the level required by the institution" (Parsad & Lewis, 2003, p. 1). Students enrolled in these courses may be required to complete multiple semesters before they can officially enroll in college-level mathematics. It has been found that Black and Latina/o students are more likely to be placed into DM class (Bahr, 2010) and less likely to pass the class compared to their White peers (Hayes, 2021), causing delays to completion as well as increased financial strain (Larnell, 2013). The mismatch between the purpose of DM courses and student outcomes in such courses brings critique of the overall effectiveness of the DM pathway.

To compensate for overall student performance in DM courses, some community colleges have heavily relied on tutoring with the hope that it will increase student passing rates. Hayes (2021) found that students are more likely to pass DM courses for every additional hour of tutoring as well as more likely to enroll in the next math course. Many mathematics departments have created well-structured tutoring centers to provide specialized content help for students in all courses. In this study, we discuss the use of embedded tutoring, in which an individual "works in the classroom under the instructor's guidance to help students understand course concepts and enhance student engagement" (Hayes, 2021, p. 27). Many institutions utilize embedded tutors (ETs) in classrooms to provide immediate support in the moment of learning.

Because DM courses have such low success rates (Bettinger, Boatman, & Long, 2013), many institutions have begun to place ETs into the classroom to increase student engagement, aiding the instructor by increasing one-on-one support for students during their learning. Oftentimes it is seen as an equitable answer to the challenges of high classroom enrollment and providing students with more sources of expertise. In this paper, we analyze the ways in which an ET was incorporated in a DM course, and other ways an ET can support student success.

Conceptual Framework

We use two frameworks to guide this study: the four constructs of mentoring framework (Nora & Crisp, 2007) and the asset-based community cultural wealth framework (Yosso, 2005). Nora and Crisp (2007) realized that mentorship was a key component to student persistence, creating four constructs to mentorship. Similar to Henry, Bruland, & Sano-Franchini (2011), we adopt these four constructs and use it to encompass the role of an ET. In the context of the study, we define an ET to be a more experienced student at the institution who has completed the course that they are supporting, attending and engaging in every class meeting.

The first construct outlines *academic subject knowledge*, what is traditionally considered as the main role of a tutor. *Psychological and emotional support* describes a relationship between the tutor and a student where the student feels that their emotions are being listened to, encouragement is being provided, or when a student feels more comfortable with the tutor. *Goal setting and career paths* is a construct that describes how a tutor may be able to assist a student and provide advice for academic or career goals. The final construct is to be a *role model*. A tutor has their own lived experience as a student at the institution, often different than an instructor, from which students can gain insight, guidance, and awareness about being a successful student.

Because the ETs are also current students at the institution, they may have accumulated a wealth of knowledge different than an instructor may be able to provide for their students. Yosso (2005) stresses the importance of considering the non-traditional forms of capital that underrepresented students bring to their learning including linguistic, familial, social, aspirational, resistant, and navigational capital. *Linguistic capital* refers to the ability and skill needed to speak in more than one language/dialect, which can help bridge the gap in learning that is not caused by the content knowledge itself. *Familial capital* can form cultures of community and bring with it lessons of caring and expanding definitions of family. *Social capital* is the knowledge and ability to draw on social content and resources to best navigate educational institutions. *Resistant capital* refers to skills gained through challenges to inequality. This type of knowledge flows from generation to generation and teaches students to understand their value in a system that often devalues them. *Aspirational capital* is the "ability to maintain hopes and dreams for the future, even in the face of real and perceived barriers" (p. 77). *Navigational capital* is the practical knowledge gained from having to interact with institutions that were not built to support communities of color, and to be able to utilize such knowledge within the institution in order to navigate successfully to completion of a college degree.

We consider the purpose of an ET to be more than just academic support. ETs spend as much instructional time with students as the instructor, positioning the tutor differently than if the tutor was just in a tutoring center. Therefore, we want to know how an ET can be leveraged to holistically provide support for students in a DM class. This paper focuses on the following two research questions: 1) How is an ET utilized during instruction in a developmental mathematics classroom? 2) In what ways is an ET able to support students, beyond content support?

Methods

Data for this paper were collected during Fall 2016 as part of a larger study that focused on the student instructional experiences of nine Latinx students in a developmental mathematics course (Cawley, 2018) supported by an equity grant at an HSI-designated community college in Southern California. The researcher observed 25% of all course meetings for one section of intermediate algebra, Math 5, and interviewed students, the instructor, and the ET using semi-structured interview protocols. Alberto was assigned as the course ET. His duties included attending every class meeting and supporting the instructor and students. Alberto was a 23-year-old Latino male, a first-generation college student in his fifth year at the institution, majoring in Civil Engineering with plans to transfer. Alberto described being trained by the tutoring center, being instructed to have the least amount of impact on the thought process of the students and was expected to avoid direct answers by responding to students' questions with a question. He was not provided with any other ways to interact with students.

This paper focuses on the use and implementation of an ET in a DM course, as well as the opportunities in which an ET could support students outside of mathematical content. The data analyzed and discussed in this paper include one 90-minute interview with Alberto, triangulated by classroom observations/fieldnotes and focal student interviews. Alberto was interviewed to discuss his background, his role as an ET, and his views as a Latino studying in a STEM field. Nine students (Nancy, Raquel, Santiago, Teresa, Layana, Chris, Guillermo, Adriana, and Marisa) were interviewed about their instructional experiences, and specifically about their interactions with the tutor.

The data were analyzed using two frameworks through deductive coding. To answer RQ1, we use Nora and Crisp's (2007) four constructs that comprise mentoring. First, we reviewed the classroom observations, fieldnotes, student interviews, and tutor interview to provide a description of how the tutor was utilized in the classroom. We then reviewed these data sources again and applied the four codes whenever we saw the tutor providing academic support, psychological/emotional support, goal setting/career support, or acting as a role model. To answer RQ2, we use Yosso's (2005) *community cultural wealth* framework to analyze the tutor's interview transcript. We coded the transcript for moments when specific capital is referenced. Authors met to discuss the findings and discussed disagreements we had of application of codes.

Findings

How was Alberto utilized in the classroom?

Alberto sat in the back of the class in a row of seats near the door, behind all of the students. On the first day, the instructor thought he was going to inform students on the tutoring center and then leave. When she asked him to give his pitch, he had to clarify that he was actually going to be an ET for the class and that he would stay for the entire semester. She quickly repeated what he said to the class, and moved on. During this exchange, the instructor did not ask the tutor come to the front of the class and none of the students turned to look at him. His role continued to be unclear, and the instructor did not intentionally incorporate him into the class. In fact, he was not used by students in the first few weeks of class; on the seventh class meeting the instructor asked him, again, what his role was and he replied that he was there to help students with questions. For the remainder of the class, he assisted students during short periods of individual practice time, and only if students in the back row beckoned him over. Throughout all of the observations, most students did not interact with him.

Because of this structure, Alberto was mainly utilized to support students' mathematical understanding. Alberto only worked with a few students in the class, mainly those who were in the back rows. During these interactions, he would provide mathematical steps for students to follow. Alberto felt this class was different than others, less interactive. "I'm stuck in a situation where I can't interfere unless they ask. They don't ask as many questions…other classes are much more open and it's much easier [for students] to ask without feeling pressure." Oftentimes he would spend the entire classtime working with one student. He often helped Raquel; she stated that she would call on him to help her begin a problem or ask him to check her work, which she found very helpful. Nancy said she avoided getting tutoring because many times tutors showed you a different way to attempt problems than what the teacher shows, which would confuse her. Given that Alberto sat in on every class, this was a missed opportunity for academic support. Both Nancy and Santiago did not engage with Alberto because he sat in the back of the room and they were positioned in the front of the room. Santiago acknowledged that an ET is a major resource, "The tutor knows more than me, obviously. A tutor will probably tell me how to do it and I'll remember it for sure. He's more of a hands-on. I'm more of a one-on-one person than a one-to-twenty person". Teresa would not call on the tutor because she did not want to appear to need help, but sat next to another student who regularly called Alberto for help. She would wait for Alberto to help her peer so she could overhear the his suggestion. Two other focal students mentioned often going to the tutoring center, but not engaging with Alberto in the class.

The remaining three mentoring constructs were not as obviously present. Alberto provided *psychological support* for Raquel. Raquel stated that she felt part of the classroom community as long as she did not feel like she was ignored. She indicated that she preferred receiving help from Alberto because she felt more comfortable around him, and he always helped her when she needed it. While Alberto did not feel like he was a *role model*, he did feel that he needed to share his story with students, but did not share his story with students in Math 5. He indicated that as an ET in other math classes he was better integrated into the course, which provided him opportunities to share his story with students. He would tell students that he used to be in their position, and that he never imagined himself tutoring math. "Just by saying something so simple, I want to let them know that they can do it." Alberto nor the students indicated giving/receiving support with *goal setting or advice on career paths*. Similar to the previous two constructs, this may not have occurred because of the limited amount of time he was able to engage with students or not being trained to provide such advice.

How can a tutor support students beyond the math content?

Of the six forms of cultural capital, Alberto described moments that strongly relate to resistant, social, aspirational, and navigational capital. Familial and linguistic capital were not as prevalent in his discussion of his educational experience and support as a tutor.

Alberto demonstrated a strong understanding of his position as a Latino male in a STEM field, in which we found many instances of resistant and aspirational capital. All throughout schooling, Alberto saw scientists and mathematicians who were White, which he felt translated to implicit messaging, making someone who is different feel like they "aren't part of this developed world" which could be "discouraging in math…it's a super hard subject". Alberto explained that Latinos are often portrayed in the media as a day laborer or some other low-level worker, and that these types of messages often made Latinos in his community feel like "why try?" because no matter what they did, they could not change society. This created two groups of Latinos in his eyes: those who tried, and those who did not. He wanted to try.

> It seeps into all aspects of society, work, school…going into the classroom, I kind of just realized that I was going to have a lot of adversity with that…So just being aware of what you're going to face is going to help you more so you won't be surprised or disappointed.

Alberto's resistance to falling into the stereotype amplified his aspiration to succeed.

Alberto was originally placed in a lower-level DM course, requiring him to complete eight courses to arrive at Differential Equations, rather than six. His aspirational capital supported his determination to complete these courses successfully, admitting the path he needed to traverse would be lengthy. At times he felt like giving up, "There were points in time where I just thought I was done with school. I would just think about how much more I had still left. I would get very discouraged." In some instances, he had to drop or repeat classes, extending his time to over five years at the two-year institution. He did not think he was good at math. "It wasn't until Trig or Precalc that I started to realize it's not only Asians or Whites or anything. It's other people…they say race is a social construct. It's just about determination." His experience bolsters strong navigational capital and self-awareness. He began to understand that he *could do math*, but did not realize this until later in his journey. Had he not continued on to college-level math, he would not have experienced that realization. He also came to understand when he could and could not complete a class successfully; he withdrew from Precalculus once and from Differential Equations twice and also recognized moments when he needed to stop out for a semester. He learned to understand the system in which he was positioned and how to best support his learning while not giving up.

Alberto had people in his life that provided avenues of support for a career in STEM, which strengthened his social capital, and positioned him to be a support for others. His new step-mother and other Latino family members were in the engineering field. He recalled a Latino high school teacher who had a degree in mechanical engineering who provided a lot of mentorship and emotional support for his Latino students. While his teacher did not point out race or ethnicity directly, "he [would say] 'you guys can do it'. He would just use implicit encouragement so it can just be instilled for us to later use and we were told it is in us as well." Alberto found this encouraging because his teacher also struggled in math throughout his education, yet was a successful high school math teacher. Alberto acknowledged that as a student, hearing his type of support from accomplished Latinos impacted him greatly, which is why he felt the need to also share his story with students.

We did not find evidence to support linguistic capital, though Alberto's mother learned English as he was growing up, which was similar to the experiences of the students' he supported in Math 5. He spoke Spanish, but did not discuss this as something he used when tutoring. While he stated that he did not really care if students learned the material or not (contradicting familial capital), he did discuss how he would stay after hours to help students, and even described his relationship with some students in Math 5. For example, he talked about Raquel; he felt that she was not putting in as much time into her studies as she should, yet he still worked with her regularly throughout the class to support her learning, even encouraging her to go up to present her work at the board. His concern for her indicated that he did feel an obligation to her learning.

Discussion and Questions

As can be expected, tutors are trained to provide academic content knowledge support, yet can also act as models of academic success through lived experience and encouraging social interaction and well-being (Gordon et al., 2006). We see that the implementation of an ET in one section of Math 5 was structured in such a way to minimally allow for math support, and did not provide the opportunity for other important supports that the tutor demonstrated as necessary to thrive in DM courses. We argue that an ET brings with them more than just mathematics knowledge. Alternative forms of capital (e.g., resistant, aspirational) may supplement other areas of mentorship. While we recognize that ETs' main goal is to provide academic support, students also need social support to succeed (Tinto, 1975). Studies show that students often establish a meaningful connection with tutors, which in turn helps students feel more integrated to the campus which affects their learning (Reinheimer & McKenzie, 2011). We recognize that a tutor may not be prepared to fully take on important topics such as psychological well-being or career advising (Henry et al., 2011). Trainings can incorporate tools and resources so that ETs are equipped with current resources to best direct/support a student. Further, we recognize that the introduction/integration of an ET in the classroom needs to be carefully planned and maintained as students need multiple and regular opportunities to work with their ET. We would like to learn from the audience ways to capitalize on these additional ET supports in the classroom.

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Why do students rely on online homework over lecture?

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We observed student exam responses that used an approach from online homework that differed from the lecture approach. In this preliminary report we focus on three interviews investigating why. One student found learning from homework faster. Another found examples more memorable than concepts. We argue these students approached the exam as a didactical situation, adopting the role of students in a formal education setting whose job is to demonstrate an ability to solve a problem in exchange for recognition of this ability. This differs from an adidactical frame in which students engage with the mathematics in ways that demonstrate little to no consideration of the formal education setting. The third student began in an adidactical frame but switched to memorizing the homework formula because he viewed his attempts to understand the lecture approach unsuccessful. More data collection will occur in fall 2021.

Keywords: online homework, instructional triangle, didactical situation, adidactical situation

Our study begins with an observation made by the second author (SA) in spring 2020 regarding the multivariable calculus exam problem shown in Figure 1:

Free response. Let $\vec{A} = \langle 1,0,-1 \rangle$ and $\vec{B} = \langle 1,3,-5 \rangle$.
(a) Compute the component of $\vec{A}$ in the direction of $\vec{B}$. Show all of your work.
(b) On the figure below, clearly label a length that corresponds to the component of $\vec{A}$ in the direction of $\vec{B}$.

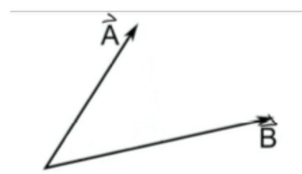

Figure 1. A problem from a multivariable calculus exam.

In class, SA had led students through a derivation of the formula for the component of $\vec{A}$ in the direction of $\vec{B}$ shown on the left in Figure 2. The derivation was quantitative in nature and emphasized that the component of $\vec{A}$ in the direction of $\vec{B}$ is a specific magnitude (specifically, a length) whose measure can be obtained by computing the dot product of $\vec{A}$ with the unit vector in the direction of $\vec{B}$. Part A was designed to assess students' computational fluency, whereas part B was designed to determine if students had developed notions of the aforementioned quantitative understanding. While some students solved part A of the exam problem using that formula, others used the formula shown on the right in Figure 2; this formula only appeared in the 'Practice Another Version' feature in the online homework (with no derivation) and was not presented or discussed in class.

Component of $\vec{A}$ along $\vec{B} = \vec{A} \cdot \frac{\vec{B}}{\lvert\vec{B}\rvert}$	The component of **u** along **v** is $\left\| \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\boldsymbol{v} \cdot \boldsymbol{v}} \right) \boldsymbol{v} \right\|$

Figure 2. Left: formula presented in class. Right: formula from online homework

This raised the question that motivated this study: why did some students use the formula from the online homework instead of the one presented in class? We see this question as an opportunity to gain insight into more general questions about students' engagement with lecture and online homework, which we discuss in the next section.

Literature on Student Learning from Homework and Lectures

Undergraduate students spend more time doing homework than they do in lecture (Ellis et al., 2015; Krause & Putnam, 2016; Lew & Zazkis, 2019). In a nationwide study of calculus classes, White and Mesa (2014) found that, on average, 78% of the mathematical tasks assigned in a term were homework tasks. Additionally, students report homework tasks are often more useful for their learning than lecture (Glass & Sue, 2008). However, few studies exist that provide insight into what students learn from homework. Lew et al. (2016) state the field also needs to understand more about what students learn from lectures. They found that students in an advanced mathematics class did not understand the ideas in a lecture that the professor intended for them to learn. Students' difficulty learning concepts from lectures is particularly problematic given that they are unlikely to experience significant conceptual development from homework problems (Dorko, 2021, 2020, 2019; White & Mesa, 2014). We addresses both of the gaps identified above by studying how students might learn from both homework and lecture.

Theoretical Perspective

We draw upon Dorko's (2021) adaptation of the instructional triangle (Cohen et al., 2003; Herbst & Chazan, 2012). Instruction is conceptualized as interactions between a teacher, the knowledge at stake (content), and the student. Students experience instruction in various milieu, or counterpart environments that provide resources for and feedback on work. Lectures are one milieu in which students interact with the teacher and knowledge at stake. Online homework is another and an exam is a third. The didactic contract, the set of mutual, implicit expectations about the roles and responsibilities the teacher and students have to one another, governs these interactions (Brousseau, 1997; Herbst & Chazan, 2012). Brousseau (1997) theorized that when students interact with teachers and/or with content, the situation can be characterized as didactical or adidactical. In a didactical situation, "the student acknowledges that while the teacher may assume they are asking a mathematical question, the student's response may be more determined by the obligations of the schooling environment than mathematical sense-making" (Dawkins, 2014, p.91). In an adidactical situation, the student engages with the mathematics "without apparent recourse to the schooling environment that enveloped it" (Dawkins, 2014, p. 91). We employ the instructional triangle part of the theory as a way to position lectures, homework, and exams as distinct (but connected) environments (milieu). This implies a need to research student learning in each environment and to consider how the environments and connections between them afford and constrain student learning. The language of didactical and adidactical situations offers a way to make sense of students' activity in various milieu.

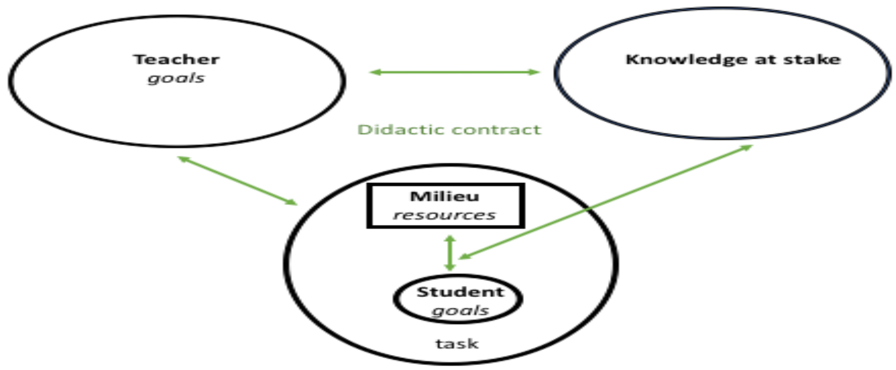

Figure 3. Dorko's (2021) version of the instructional triangle

Methods

The methods include two phases of data collection and three phases of analysis, as described below. The data come from three online sections of calculus III in fall 2020 and spring 2021, with more data collection planned in fall 2021.

Data Collection Phase 1: Written data

1. Video recording of the relevant class period: the online students watched a pre-recorded lecture video. Each relevant in-person lecture will be recorded in Fall 2021.
2. Students' lecture notes: all students, regardless of course format, uploaded lecture notes to the online course management site.
3. Students' scratchwork from relevant online homework: students uploaded their scratchwork from the relevant online homework.
4. Students' responses to exam problems: the first author photocopied responses to Question 3 (Figure 1) for all students who consented to participate in the research. The exam took 75 minutes and students were allowed a single 3 inch by 5 inch notecard.

Data Analysis Phase 1: Sort students' work from (3) and (4) above into categories:

(a) lecture method on both HW and exam, (b) online homework method on both HW and exam, (c) lecture method on HW, online homework method on exam, (d) online homework method on HW, lecture method on exam, and (e) other

Data Collection Phase 2: Interview[1] students from categories a, b, c, d above

i. Show students their work from exam Q3 and ask them to explain what they did and why
ii. Show students their lecture notes; ask what they understood of that formula and its derivation
iii. If students used the online formula on their homework, ask what they understood about it
iv. Ask why they used the online homework method (or vice versa as applicable)

Data Analysis Phase 2: Employ a constant comparative analysis (Strauss & Corbin, 1994) to seek themes in the data regarding why students employed one method over the other

Data Analysis Phase 3: Conduct a final phase of analysis in which the data (and the themes identified in Phase 2) are viewed through the lens of the constructs in the instructional triangle. This might include statements in which students express expectations, goals, or ideas about their relationships with the instructor, content, or milieu.

Results

[1] Interviews were conducted by the first author, who was not an instructor of the course. The interviews took place as soon as possible following the exam, which was the first exam in the semester.

We conducted interviews with John, Sarah, and Henry, all of whom watched the video recording of the lecture, submitted notes that included the "lecture formula" and its derivation, then used the online homework formula on their homework and on the exam (due to space constraints, we focus only on responses to exam question (a), Figure 1). John mentioned that he focused on the homework because it was faster than re-watching lecture:

John: Prior to this exam I just went and started looking over the homeworks…I just went and committed to memory the important equations, or whatever I felt was going to be most relevant on the exam… the homework's faster for me to go through. It's a lot easier for me to go and read down a whole bunch of homework problems than it is for me to go rewatch multiple 45 to hour minute long lectures. This exam actually went a little more rough because the review did not have everything that was on the exam on it.

We interpret John's statement that the exam was difficult because the review was not comprehensive as expressing an expectation (a "clause" in the didactic contract). Taken together, these statements suggest John viewed learning in this course as a didactical situation. John appeared to expect that the instructor was responsible for identifying the important knowledge by putting it in the exam review, and John's role as a student was to complete the review, then demonstrate the knowledge on the exam. John's activity (memorizing formulae) fulfills the obligations of the school environment, making it didactical activity. In terms of the research question, John relied on the online homework instead of the lectures because the homework was "faster". Speed was important to him because he saw mathematical learning as a didactic activity.

Henry[2] and Sarah contrast John in that they began in an adidactical frame. Henry described he wanted to understand the formula from lecture and asked for help. He was unable to understand it. When he got to the exam he was only able to solve the problem because he remembered the homework formula:

Henry: I try to understand what does, how does that [lecture formula] mean like in real life…. But I don't know why it's like this… I would like to know but I don't understand because every time I ask some other student like what is like this one… they say maybe if when you work or you have advanced courses then you will understand…why it is this way. Okay you have to know just how to solve it… It's confusing and annoying but yeah… I was really stuck honestly…so what I did here [on the exam], honestly, I remember one of the homework.

We take Henry's comments as evidence of an adidactic focus. In particular, Henry did not say he wanted to understand the formula so that he could do well on the exam, but rather he wanted to understand what it meant "in real life." However, Henry did not understand what had been presented in lecture. He used the formula from online homework because it stood out in his memory more. We take the comment about using that in hopes of obtaining partial credit as a brief shift to a didactical situation in which he hoped to exchange remembering a formula he did not understand for partial credit; like John, this activity met the obligations of the school environment. However, Henry did not find this satisfying: he commented that he still "wanted to know" the mathematics of the problem.

Like Henry, Sarah tried unsuccessfully to understand the formula from lecture and ultimately found that she remembered the online homework formula stuck because she did more examples with it than she did with the lecture formula:

[2] English is not Henry's first language.

Sarah: It took me awhile for, to take from the lesson notes to the homework I had to get some help and ask about it because I was, kept trying to find like, it just didn't click from the notes to that [lecture] formula, and finally I came to this [online homework] formula and was like oh that makes sense and I got the same thing with these numbers. I kind of just memorized [the online formula], I'm not going to lie. I learn math from examples. Like I have to do tons of examples. Just looking at lecture notes and having all of these like, not to be like words, but words, words and sentences kind of gets confusing sometimes. So the homeworks help me the most I think in like memorizing or like remembering.

We view Sarah's initial activity as adidactical because she tried to understand the formula from lecture, including asking for help. When she could not, she turned to the online formula. Her memorization of it indicates a didactic focus of trying to obtain good grades. We note that our interpretation hinges on interpreting Sarah's "it just didn't click" as indicating she was trying to understand and use the lecture formula (adidactical). An alternative explanation is that Sarah's activity was totally didactical in nature, and it is possible that she could not figure out how to use the lecture formula for all the problems but the online homework formula gave her the right answers. In support of this, it was important to Sarah that she "got the same thing" with both formulas, and she admitted to memorizing (didactical). Without knowing more about what Sarah meant by "it just didn't click" and the online formula "making sense", it is impossible to say whether her activity was adidactical then didactical, or complete didactical.

Discussion

Our provisional results align with and extend those of prior studies. Specifically, they affirm Dorko's (2021) assertion that students' use of examples during homework often supports procedural learning. Additionally, the students in this study turned toward memorizing the online formula because they did not understand the explanation in lecture, which echoes the Lew et al. (2016) findings. These results also suggest that students may rely on procedures from online homework because they consider them to be easier or more efficient. This finding is significant because it adds nuance to other findings about students' preference for procedures, such as those that students cling to procedures because they believe the nature of mathematics is procedural rather than conceptual (e.g., Spangler, 1992).

An alternative explanation for students' affinity for the online homework formula is that students obtain immediate feedback on online homework and try problems multiple times when given that option (Dorko, 2020). These students may have been more comfortable with the methods the homework suggests because they could use that method, then obtain feedback about it immediately. For instance, this could explain Sarah's preference for the online formula. Students may also have spent more time using the online homework formula than the one they used in class. Whether or not either of these factors influenced students' preference for the online homework formula is an area for future research.

One implication for instruction is that instructors realize online homework may provide students with different formulas than the instructor presented. Although these students were aware the two formulas served the same purpose, we hypothesize other students might be confused by seeing two different formulas. In particular for this study and the formula presented in class, the results suggest that some students may have gained little from the in-class lecture. Instructors might consider what assessment questions in their own classes might reveal about the way they taught particular topics, and use that data to alter their instruction in ways that are more understandable or memorable to students.

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Searching For The Math: Undergraduate Students' Strategies For Using the Internet to Learn About Novel Mathematical Concepts

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Students make extensive use of online resources to support their learning in college-level mathematics courses (Erickson, 2020) but little attention has been paid to the specific strategies that students employ when using search engines to discover the meaning of novel mathematical concepts. This preliminary report offers a series of case studies that illustrate contrasting strategies used by students to understand the meaning of an unfamiliar mathematical notation. The results of this analysis demonstrate that information-seeking strategies are an important mediating factor in mathematical learning particularly with respect to mathematical content that is not directly addressed by the instructor. I discuss implications for supporting students' study skills in undergraduate mathematics courses.

Keywords: Technology, Information-Seeking Strategies, Case Studies

Introduction

Recent examinations of undergraduate students' experiences with self-directed use of online resources reveal that students taking the same mathematics course may have radically different experiences as they study due to the different ways that they engage with the online environment outside of the classroom to support their learning (Erickson, 2019, 2020). While past research on mathematics education and the internet have been centered largely on interventions created by the institution (e.g. mobile tools, digital libraries, and collaborative learning tools) and online instruction for math educators (Borba, Askar, Engelbrecht, Gadanidis, Llinares, & Aguilar, 2016), there has been a recent push for mathematics education researchers to seriously investigate how students make use of online tools under their own initiative (Puustinen, Volckaert-Legrier, Coquin, & Bernicot, 2009; van de Sande, 2011, Anastasakis, Robinson, & Lerman, 2017; Erickson, 2019, 2020; Higgins & Minners, 2020). The present report contributes to this body of work by presenting an analysis of episodes in which students are asked to narrate their actions as they use the internet to learn about an unfamiliar mathematical notation, a process recorded with screen-capture technology. The preliminary analysis of this data provides examples of the different strategies that students can adopt with the same online tools and how these different approaches may help or hinder a student's understanding of newly-encountered mathematical concepts.

Online Information-Seeking Strategies

When confronted with an information problem-solving task (i.e., a problem that requires that the student seek out, evaluate, and make use of information) students have been shown to have difficulties choosing search terms, evaluating the credibility of information sources, and deciding when they have adequately brought their search to a close (Walraven, Brand-Gruwel, & Boshuizen, 2008). It is also the case that the difficulties that students encounter vary depending on their familiarity with the academic discipline in question and that it follows that these skills are domain-specific to some extent (MaKinster, Beghetto, & Plucker, 2002; Brand-Gruwel, Wopereis, & Vermetten, 2005). Thus it is important to study information-seeking practices in the context of the discipline in which they are employed.

Students' Use of Online Tools for Studying Mathematics

Students in lower-division mathematics courses extensively use online resources to support their mathematical studies (Erickson, 2020). Previous research has found that students' use of these resources primarily takes two forms: online answer engines like Symbolab and instructional videos such as those offered by Khan Academy. Crucially, though, these are both used to navigate homework assignments and study for tests, whereas there is little known about how students engage with online resources when they try to understand a new concept that is not introduced by a textbook or an instructor. Ironically, the latter situation is more characteristic of the types of information-seeking tasks that an individual may encounter in their everyday post-collegiate life and so a better understanding of students' use of online tools in those situations could help support the development of a more practical and empowering mathematics curriculum.

Context and Methodology

The present data comes from a larger explanatory mixed methods study (Creswell & Clark, 2017) in which over 250 students from over 25 US and Canadian colleges were surveyed about their use of online resources, particularly those that were not prescribed by the instructor, to support their study in lower-division mathematics classes. These surveys were supplemented with two sets of follow-up interviews, the first was a semi-structured interview in which students were asked to elaborate on their responses to the initial survey and the second was an extended interview in which students were asked to answer several mathematics question. The questions were chosen to be difficult enough to encourage the interviewees to make use of the internet while not being too difficult to solve. The interviewees were not required to make use of the internet, but if they did then they were asked to narrate the reasons for the actions they were taking with the computer as it happened.

The analysis presented below is based on the responses to the following question: *"Rewrite the following expression using exponential notation: 2↑↑3"*. This problem employs Knuth's up-arrow notation, a method of extending exponentiation that can be employed to represent numbers that would be unwieldy using exponential notation. A single up-arrow represents exponentiation, i.e., iterated multiplication. Analogously, two up-arrows represent iterated exponentiation which is called *tetration* and three up-arrows would represent iterated tetration. The notation was chosen for this task because it is not part of the curriculum in any typical undergraduate mathematics course. Thus, a student confronted with the task would be expected to make use of the internet in order to find out what the notation means.

As predicted, it proved to be the case that each of the interviewees (n = 7) were both unfamiliar with the notation and decided to use the internet in order to find out what the notation meant. The actions they took, however, varied dramatically and the analysis reported here is intended to unpack those differences. As a final note, the participants all used the Google search engine, so it can be assumed throughout that Google is the tool being used whenever there is a reference to an internet search.

Results

I will first present two contrasting case descriptions of the approach that a pair of students took in their exploration of the up-arrow problem. The case of Jason (all names are pseudonyms) is a student who was able to successfully and efficiently find the correct answer to the problem while the case of Ken presents a student who also found the correct answer but who took much longer to arrive at the solution. I will then briefly describe how the remaining participants in the extended interviews relate to these two cases and discuss important distinctions between students' information-seeking approaches.

Trust Exercise: The Case of Jason

After copying down the problem, Jason says "I don't know what that means." He does not seem chagrined by his lack of knowledge even though he has taken the calculus sequence and statistics. Rather, he immediately types *Two up arrows in math* into the search engine. The resulting search result screen highlights the Wikipedia entry on Knuth's Up-Arrow notation (see Figure 1) and Jason does not even bother to scroll downward.

> This looks pretty good, Wikipedia I look at a lot. Some people say it's not trustworthy […] but Wikipedia, I think it's fine.

He reads through the Wikipedia entry which conveniently uses the problem he is supposed to solve as an example. To J's credit, he takes a little time to confirm that he understands how tetration works, as described by Wikipedia, rather than simply repeating the provided answer. Once he does confirm the correct answer, about 5 minutes in total have elapsed.

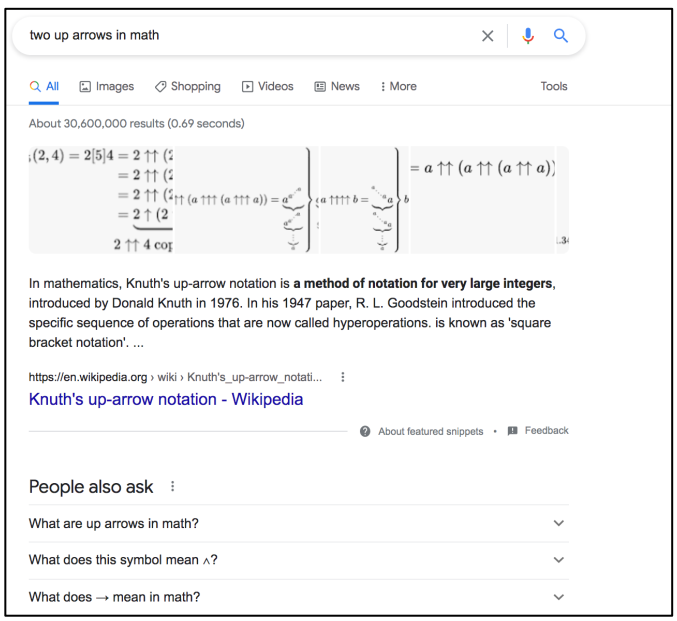

Figure 1. Google search result for "Two up arrows in Math" on February 2, 2021.

"I Don't Trust..": The Case of Ken

Ken also copies down the problem and confirms that he has not seen the notation before. He goes on to say, "usually I would ask the professor or a tutor however that is not a tool at the moment. Looking at this, I think it is saying 2 cubed however I'm not sure." He then searches for *Different math symbols* and clicks on the first link which states that it is an exhaustive list of math symbols. He spends a minute scrolling through the entire list, "right now I'm just scrolling through this random website to see if I can find that notation". He does not find anything that matches and goes back to the search engine to look up *double arrow up symbol math*. Despite the similar search terms, the result of this search is slightly different than in Jason's case (see Figure 2). Fortunately, the first excerpt is still information on Knuth's Up-Arrow notation. Rather than click on that first link, he scrolls down a little bit then scrolls back up to the top remarking that he doesn't know whether any of these entries are correct.

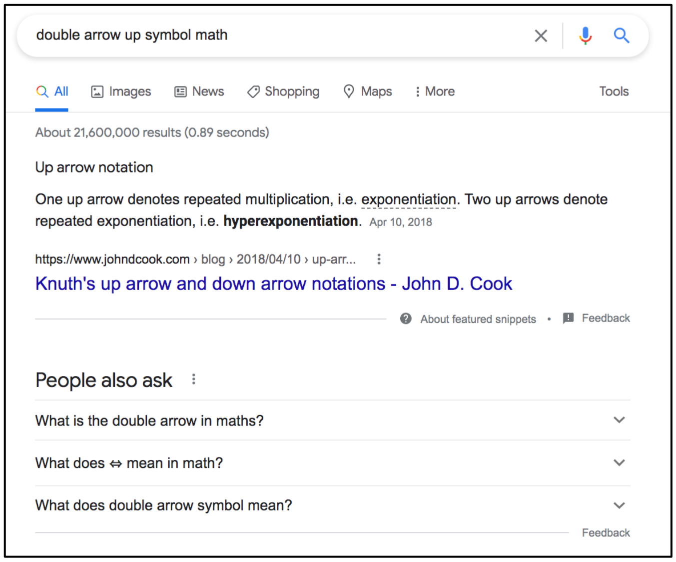

Figure 2. Google search result for "Double Arrow up Symbol Math" on February 8, 2021.

At this point, Ken might have clicked on the blog that was the first entry on the search results screen and found an explanation of Knuth's Up-Arrow notation, but instead he pressed on Image Search which brought up a number of lists of mathematical notation. He looked through these, commenting that they were difficult to see. He then went back to the excerpt seen in Figure 2, read it, and stated that "it still looks like 2 cubed but I haven't seen an example" while scrolling further down through the search results. He then clicked on a Youtube video but this was an explanation of the "if and only if" double arrow. Fortunately, Ken understood that this was not the operation in question and so he went back to the excerpt and finally clicked on the link. After looking through the example, he decides that is initial guess was correct that that the value of 2↑↑3 is 2^3 or 8. However, he notes that this is "just one website" and looks for a second website for corroboration. The first search result is Wikipedia, but Ken says "That's Wikipedia, I don't trust Wikipedia" and continues to scroll down to an .edu website with an explanation of the notation. After looking at another example on this website, he concludes that he was actually incorrect and that the correct answer is 16. This process took approximately 20 minutes.

Variations on the Themes: The Remaining Cases

The other five cases all contained elements of the two cases described above with two of the seven participants failing to generate the correct answer. Mary, much like Ken, produced the same blog entry with her initial internet search, chose not to click on the link because she did not feel certain that it was the notation in question and so clicked on image search instead. She also looked at lists of mathematical notation and was unable to locate the Up-Arrow notation. She then went further and searched for the term *math symbols* and looked through a glossary of math notation on Wikipedia. After failing to find the Up-Arrow notation there, she gave up. She never chose to return to that first search result which would have provided her with the definition she sought.

Another participant, George, used Google to search, came up with the Wikipedia article as the first search result, clicked on the article and was able to arrive at the correct solution. A fifth participant initially looked through a list of symbols but then proceeded to search for *mathematical arrow symbol up arrow*. This led them to the Wikipedia article as well and they were able to proceed to the correct answer – although in this case, the participant subsequently double-checked their understanding by looking up a video on the topic using YouTube. The sixth participant was initially unsure of how to proceed, stating that "since I can't put arrows in the search bar, I can't find it. The only way I could figure it out is by going to Google and to put arrow math symbols". However, she shortly searched for *math symbol two arrows pointing up* which led to the search result screen previously seen in Figure 2. She was then able to find the correct answer after looking at the blog. The final participant found the same blog by searching for *Two arrows pointing up between two numbers* but came to the incorrect answer of 2^16 or 65,536. Most remarkably, each of these subsequent cases relied on the same search engine and made use of or at least encountered the search results that were found by the first two participants.

Discussion

There is a tension between the information seeking strategies recommended by experts in the field (Scott & O'Sullivan, 2005) and the strategies described above. Generally, students are advised to look across various sources for corroboration rather than relying on a single source and they are have historically been told to distrust Wikipedia although there has been a shift toward greater reliance on that information source in recent years (Minguillón, Aibar, Lerga, Lladós, & Meseguer-Artola, 2018). However, in the cases recounted above, the most efficient students were those that were willing to immediately engage with Wikipedia as a source of relevant information. This suggests that an important aspect of information-seeking is the willingness of students to click on an information source and engage with the content contained within rather than skimming the search results in hopes of encountering a summary that more definitively signals that it contains the desired information. There was also an element of chance given that very similar search terms could return the Wikipedia excerpt as a first result which depicted a numerical example of the notation or they could return an excerpt of a blog entry that did not display an example of the notation. One strategy suggested by the frustrations encountered by some of these students could be for instructors to model information-seeking in their own classrooms. They could talk about how they seek out information online when they encounter an unfamiliar math term and which sources they judge to have reliable information.

Questions for the Audience

1. What is your information-seeking strategy for a task such as the one described above? When, if ever, would it be appropriate to discuss your own approach to this sort of information-seeking with students in an undergraduate mathematics course?
2. How might these students need to be making use of their mathematical knowledge? How might we analyze the way in which content knowledge interacts with information seeking in tasks like these?

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Potential Barriers to a Sustainable MGTA PD Program Focused on Equity and Inclusivity

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Mathematics Graduate Teaching Assistants (MGTAs) often have a significant role in teaching undergraduate mathematics courses, but typically have little prior teaching experience. Professional development (PD) for MGTAs is typically limited and often does not focus on evidence-based practices, but on departmental idiosyncrasies. This project responds to calls for implementation of equitable and inclusive practices by designing an MGTA professional development program to equip MGTAs with evidence-based teaching practices that are proven to support diverse groups of learners in engaging mathematical activities. Although department supports exist, this paper specifically reports on the barriers identified in one department's culture for developing such a professional development program.

Keywords: Graduate Teaching Assistants, Professional Development, Equity, Inclusivity

Introduction

Given mathematical graduate teaching assistants' (MGTAs') impact on undergraduate learners (Ellis, 2016; Miller et al., 2018; Selinski & Milbourne, 2015), it is critical that departments of mathematics attend to MGTAs' development as teachers. Developing MGTAs' teaching beliefs and practices is vital for improving student learning outcomes and fostering supportive learning environments. Instruction of post-secondary mathematics classes is often inequitable and harmful to underrepresented groups of students. Mulnix, Vandegrift, and Chaudhury (2016) contend that instituting active learning is necessary for "progress toward equity and inclusion" in STEM fields (p. 8). Students' positioning in their academic environments, such as having more opportunities to express their voice, can impact their learning and the development of their identities in significant ways (Adiredja & Andrews-Larson, 2017; Boaler, 2002). Professional organizations and national agencies, including the American Mathematical Society (AMS), the Mathematical Association of America (MAA), the National Science Foundation (NSF), and the National Research Council (NRC), have called for implementing teaching practices that actively engage students in post-secondary mathematics classrooms (Saxe et al., 2015).

The multi-institution, multi-stage study from which the data of this report originates responds to these calls for equity and inclusivity by designing and implementing a sustainable MGTA PD program that focuses on engagement, inclusivity, and equity. We focus our implementation at the department level for several reasons. First, mathematics departments are not a monolith, as each has local contexts that include serving populations with unique aspirations and needs, working within programs and networks with differing goals, and contending with varying cultural and institutional issues. It is necessary to examine this context to learn how innovation efforts can support or influence these local frameworks. Secondly, departments are relatively coherent units of culture (Reinholz & Apkarian, 2018); this coherence can support the influence of innovation efforts more broadly, leading to more sustainable and impactful implementations (Reinholz et al., 2020). Transformation within these cultures requires "exploring clashes of values that may lie at the heart of institutional resistance to change" (Reinholz et al., 2020, p. 3) in order to unite stakeholders in an ongoing and cyclical process to adapt to changing contextual factors (Reinholz & Apkarian, 2018). For these reasons, the initial stage of this larger study focuses on collecting baseline data to better understand the participating departments' cultures. In this report, we address one of the research questions that guide the project: *What identifiable aspects of departmental cultures inhibit extended MGTA PD that is focused on equity and inclusivity?*

Framework

This project explores, and is informed by, two dimensions of change: cultural and individual. For this report, we focus on the former. Cultural change refers to aspects of departmental or institutional cultures that inhibit or support the innovation efforts of implementing an MGTA PD program. It is necessary to study the local contextual factors that influence the sustainability of a PD program, as both the program and the participating MGTAs are embedded within the department culture.

Our project utilizes Reinholz and Apkarian's (2018) adaptation of Bolman and Deal's (2008) four-frame model of organizational change. This adaptation focuses on the context of higher education and allows us to understand how aspects of department cultures support or constrain improvement initiatives. The four-frame model views culture as a "historical and evolving set of structures and symbols and the resulting power relationships between people" (Reinholz & Apkarian, 2018, p. 3). This analytic tool specifies four frames for understanding systemic change: structures, symbols, power, and people. Reinholz and Apkarian (2018) further defined these constructs:

> *Structures* are the roles routines and practices of a department; their enactment and meaning are dependent on *symbols*, which are the norms, values, and ways of thinking in a department; changes are ultimately enacted by *people* whose individuality impacts their intentions and perceptions; and the distribution of *power* determines who makes certain decisions and influences interactions (p. 7)

Data Sources and Methods

We report on preliminary analysis of baseline interviews with mathematics department leadership at a large, PhD granting public university (LPU), which we studied because of the potential to implement MGTA PD at this site. The Mathematics Department employs MGTAs in class support or instructor roles. The LPU is located in a racially and ethnically diverse city, which the institution celebrates. At the time of data collection, the Mathematics Department was undergoing review and having ongoing conversations about its response to national discussions regarding equity and inclusivity.

We recruited department leaders who we identified as power-brokers, based on roles and hierarchies in the department, with abilities to influence department decisions for interviews. These are important figures in innovation efforts, as a "change effort requires sanction from the appropriate power holders to succeed" (Reinholz & Apkarian, 2018, p. 5). In particular, the department chair, the graduate coordinator, experienced instructors and coordinators, and those who contribute to existing MGTA PD within the department received invitations for interviews. Five department leaders accepted the invitation and were interviewed; interviews were audio recorded and transcripts were generated.

Our research team designed a protocol for semi-structured interviews that focused on learning about departmental culture, particularly regarding the roles, values, and perceptions of

MGTAs and equitable, inclusive, and engaging teaching practices in the department. We expected that the interviewees' responses to these questions would provide us with insights into potential barriers for developing a sustainable MGTA PD program focused on engaging, equitable, and inclusive teaching practices.

Analysis and Preliminary Results

Our team used an inductive open coding method that included sequences of summarizing interview transcripts to discuss and resolve codes (Miles & Huberman, 1994). During our second phase of analysis, we adopted an axial coding method to construct and connect emerging categories from our codes that related to aspects of the department's culture that may inhibit the implementation or sustainability of a potential MGTA PD program. We used the four-frame model as a lens to help illustrate some barriers that emerged. In our results below, we describe three themes and unpack their connection to and impact on MGTA PD using the four frames.

Instructors' Choice

We observed an unwritten policy of "instructors' choice" regarding faculty members' teaching that permeates the department's culture. Instructors' choice was construed as a *symbol* of the department, as it seems an integral and influential aspect of the department's identity. Besides a few explicitly pre-structured courses, instructors and coordinators have the freedom to determine their classroom pedagogy, what PD opportunities they want to pursue, and their own course *structures*. With respect to the *people* frame, instructors and coordinators have considerable agency within the department. This agency flows into *power*: instructors and coordinators are provided wide-ranging decision-making capacity, which includes the roles and responsibilities of MGTAs associated with their classes. We found that MGTAs have extremely different teaching experiences depending on the instructor or coordinator that they are assigned to.

MGTA PD Program. Ideally, an MGTA PD program would be a centralized *structure* within the department, which is antithetical to the notion of individual choice. Without having department *structures* to build from, such as a Diversity, Equity, and Inclusivity (DEI) committee or departmental incentives or responsibilities regarding use of engaging, equitable, and inclusive practices, developers and implementers of PD programs would need to create *structures* to unite stakeholders for this sustainable innovation effort. Thus, valuing individual choice may make collective efforts for MGTA PD difficult to enact.

Developers and implementers of PD programs would need to establish collective goals, a *structure*, for instructional staff that exercise their pedagogical freedom (agency recognized by the *people* frame) in a variety of ways. In particular, the interviews depicted instructors providing various roles and teaching experiences to MGTAs. Some MGTAs are incorporated as collaborators in various aspects of coordination planning while some interviewees expressed that other MGTAs are utilized as limited classroom support, "another set of eyes and ears," to a "sage on the stage." This represents a challenge for the PD program to be successful and sustainable, as the program's design would need to facilitate opportunities for MGTAs to enact practices that are still respectful of instructors' and coordinators' agency.

Lack of Internal Communication

One cost of instructors' choice is a lack of communication *structures* in the department. For example, it is not typical for instructors and coordinators to share their practices or work within the department outside of their course. Because of this, collective best practices do not emerge effectively - as even department leaders are working on shared problems in isolation, seemingly unaware of each other's work. This is also applicable to faculty PD, as one leader reflected, "our department doesn't know how many of us are doing this and whatnot."

MGTA PD Program. A sustainable PD program would need to be integrated with communication lines between department members, particularly those involved with MGTAs in their classrooms or under their coordination. Actively coordinating with participants of the PD and stakeholders of the innovation efforts is vital for the efficacy of the program. This is reflected in the *people* frame, which describes the usefulness of attending to shared visions and goals; communication would need to be a departmental responsibility that is properly incentivized.

The lack of internal communication *structures* appears to hinder a cultural consensus about the roles and responsibilities of MGTAs in the department. Even department leaders provided conflicting descriptions. This ambiguity and confusion would need to be addressed for a PD program to train and prepare MGTAs for their role. Without this shared vision, the uncertainty of MGTAs' roles represents a challenge to link the PD to all MGTAs' experiences, diminishing the effectiveness of the program. Again, inherently needed in the effort to clarify this confusion are communication *structures* which currently do not exist.

Another potential challenge that emerged from the interviews was the ability to coordinate availability of the PD. Many department leaders were unaware of PD for MGTAs, and it was not clear if anyone had the role of disseminating these opportunities. When PD is offered to graduate students within the department, it is not effectively communicated to other faculty members as one department leader recounted, "Our workshop was short this year just because we only actually had one graduate student and then the others were attending a workshop by [another faculty member] that overlapped … I was like, 'I wonder where that person is.'"

This challenge also impairs sustainability efforts; without means to disseminate information, a PD program would struggle with maintaining support. It is important to communicate and demonstrate progress early in the innovation process to participants and stakeholders (Reinholz & Apkarian, 2018). Otherwise, those with *power* to sanction change may waver in their support.

Addressing DEI Collectively as a Department is Not Currently a Priority

At the time we conducted interviews, the department was in the midst of internal conversations about their responsibility regarding DEI, for which there was broad disagreement. The formation and role of a centralized DEI-focused committee (a *structure*) was contentious, as some faculty members favored development of a collective mission statement (which can be construed as a *structure* or a *symbol*) to describe the department's values. This seemed to relate to the principle of instructors' choice, as one department leader described, "We don't need someone outside checking on whether or not we're promoting diversity in our classrooms because it's our mission." Even the need to respond to DEI was questioned, as one interviewee recounted, "Among the many things, some people ... felt that we should stay out of social justice issues, because it runs the risk of entering the political realm and that sort of stuff that should be separate from our academic responsibilities and our subject in mathematics." Another interviewee compared diversity statements to the anti-communism statements of the 1950s. However, some department leaders expressed concern about their lack of awareness regarding DEI, and expressed uncertainty about their individual and collective awareness of best practices.

Several interviewees described the incentives and faculty PD offered by the college; these values were not clearly reflected within the department. One department leader described the role of PD as "another one of those contentious questions that I've heard people bring up," as they shared skepticism regarding the nature of PD focusing on equity and inclusivity. They questioned PD providers' motives by referring to them as an "industry that's arisen with organizations and people making tons of money off of institutions." In general, the department leaders did not share a collective need for DEI practices. When asked about active DEI-related efforts within the department, instead of discussing specifics of collective efforts, all interviewees referred to a single untenured, tenure-track faculty member's work.

MGTA PD Program. Given the department's uneven stance on their role in responding to equity and inclusivity issues, it may not be an opportune time to institute a program focused on these issues. Similar to the formation of a DEI committee, an MGTA PD program focused on equitable and inclusive teaching practices could foster further discord. Additionally, since DEI is not collectively prioritized within the department, as indicated by missing *structures*, it is intuitive to anticipate that it will not become a priority, a centralized *structure*, for MGTAs.

Promotion protocols (another *structure*) in the department do not factor in equity and inclusivity. As one department leader stated, "I perceive that to be an area that's very sensitive. Because, again, it's not part of someone's job description," and shared, "there has been no conversation in the university whatsoever about that." Given that equitable and inclusive practices are not incentivized within the department, the potential lack of support for an MGTA PD program that focuses on these core principles and practices is a concern for the program's sustainability.

In conjunction with the developing stance on DEI, the department also identifies more with its research responsibilities. Given the increased expectations of faculty to conduct and publish research, it was generally accepted that they would focus less on teaching. Many graduate students and program goals were also focused on research or industry-oriented careers. Consequently, one department leader described that time is a barrier for MGTAs to pursue PD. PD would need to align with the goals and aspirations of those involved, faculty and MGTAs.

Conclusion and Questions for the Audience

We observe that instructors' choice and freedom may come at the cost of collective action, even in the case of prominent national and international issues. Additionally, our analysis showed that in departments with MGTA training, the providers and facilitators of the PD are not always able to communicate with other department members who have power over MGTAs' future teaching experiences. As a result, MGTAs do not necessarily have the opportunity, or the support, to perform the behaviors espoused by an MGTA PD program. And lastly, dissonant and developing value systems within the department make it difficult for an MGTA PD program to connect with the internal structure of the department that it is embedded within. Until a culture stabilizes, the innovation project would face a troubling task of uniting not just differing visions, but conflicting ones. We conclude with two questions for the RUME community: How can we leverage a department's existing *people, power*, *structures,* and *symbols* to foster more cultural support for a PD program focused on equity and inclusivity? What local barriers do you perceive towards implementing a sustainable PD program?

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Studying the Potential for Pedagogical Change Based on Perceptions and Proposed Initiatives Using the Four Frames

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It is well known that pedagogical transformations are difficult to sustain. To address this issue, more researchers are focusing on understanding change processes across a variety of contexts; however, to understand the complexities of change more research in this arena is needed, in addition to research that uses existing change theories to contribute to a collective understanding of the change process. This study addresses this need by examining the starting point of one "successful" institution's change efforts through the Four Frames, a theoretical framework for understanding change in STEM departments. Preliminary analysis identified key components within this institution's plan that may explain their success, including choosing to involve faculty perceived to be influential in the department, as evidenced by our analysis of their social network. These results support further analysis of change at this institution over time, as well as cross case comparisons with other institutions' change efforts.

Keywords: change, active learning, leadership, four frames model

Introduction

Mathematics departments are planning and executing changes that are altering teaching and learning practices. Many of these changes involve a focus on student engagement, such as active learning, which are responses to national calls for improving student success by moving away from passive teaching and learning practices (Conference Board of the Mathematical Sciences, 2016; PCAST, 2012). It has been substantiated that changing teaching and learning practices is difficult and that successful change for a department involves systemic thinking (Smith et al., 2021; White et al., 2020). There has been some focus on studying change within mathematics departments (e.g., Smith et al., 2021; numerous articles in three consecutive issues of 2021 PRIMUS), but a recent commentary from editors of IJRUME have declared an "urgent need for a systems approach that can account for this complex landscape of change" (Reinholz, Rasmussen, & Nardi, 2020, p. 155). They argue that research on change in a mathematics department needs to focus on four critical areas: 1) analyzing historical artifacts, 2) studying ongoing efforts, 3) creating new efforts, and 4) issues of equity and social justice (Reinholz, Rasmussen, & Nardi, 2020). With these foci, researchers can develop more contextualized and cohesive theories that will in turn help the mathematics education community enhance its ability to implement and sustain change (Reinholz, Rasmussen, & Nardi, 2020). Further, such research should draw from common change theories. In a review of 97 studies of change in STEM higher education settings from 1995 to 2019, Reinholz et al. (2021) found that 40 distinct change theories were used. This lack of a theoretical base makes it difficult for such studies to contribute to a growing understanding of change. Our study aims to support this need for more research into change by using an established change theory, the Four Frames, to analyze the starting point of change efforts at one institution.

*All authors contributed equally to this submission, and as such are listed in alphabetical order.

Theoretical Framing and Research Questions

Our work is guided by the Four Frames for organizational change originally conceptualized by Bolman and Deal (2008), and adapted by Reinholz and Apkarian (2018) for use in analyzing change in STEM departments. The four frames: structures, symbols, people, and power serve as lenses to analyze the cultures of STEM departments, and institutions more broadly. *Structures* are elements of a department that determine how individuals interact; this may include positions, incentives, and coordination systems. *Symbols* are the abstract cultural tools of the department that shape how the department gives meaning to structures. These include the values, beliefs, practices, and narratives that the department generates over time. The *people* frame emphasizes that departments are composed of individuals with their own needs, wants, identities, and sense of agency. The *power* frame emphasizes that differences in power influence how individuals interact within a department. The model can both be used as an analysis tool for understanding change and as a tool for enacting it. In this study we use the model to analyze the starting point of change efforts within one mathematics department.

To understand the potential for pedagogical change in a department, the research team used the Four Frames to generate the following research questions:

RQ1: What initiatives were proposed to drive pedagogical change?

RQ2: How did institutional and departmental efforts to improve student success align with these change initiatives?

RQ3: Which people were proposed to lead these initiatives?

RQ4: How were these people perceived by the department? In what ways were they positioned to create and sustain the change initiatives?

RQ1 and RQ2 deal with the structures and symbols involved in making departmental changes, while RQ3 and RQ4 deal with the people and power driving those changes. In our findings, we discuss RQ1 and RQ2 as a pair, followed by RQ3 and RQ4. As noted by Reinholz and Apkarian (2018), structures and symbols work together as a pair: "The structures are the visible signs of how a culture works, but the symbols determine how the structures are actually enacted" (p. 5). Similarly, the people and power frames work together as a pair since "[t]he people frame focuses on the importance of individuality, while power draws attention to the way that all of these individuals are linked in a political system" (pp. 5-6).

Methods

Data for this proposal come from SEMINAL (Association of Public and Land-Grant Universities, n.d.), a multi-case research project focusing on understanding how mathematics departments and institutions support the initiation, implementation, and sustainability of active learning in Precalculus through Calculus 2 courses. SEMINAL had multiple phases. During the second phase of the project, SEMINAL sent out a request for proposals from institutions looking to transform their Precalculus through Calculus 2 courses by institutionalizing active learning. Nine selected institutions received funds of approximately $50,000-100,000 to support their efforts. One of the participants of Phase 2 selected was a large, Southeastern, public university with two campuses, which we will call SEU. SEU has several part-time faculty, many of whom teach the lower division mathematics courses. A focus of SEU's efforts has been on building and expanding its coordination system in Precalculus through Calculus 2 courses as a way of supporting the use of active learning. As part of SEMINAL, we conducted two site visits during which we spoke with a wide range of SEU stakeholders: mathematics faculty and instructors, department chairs (of mathematics as well as client disciplines), administrators, learning assistants, and provosts. SEU also completed a climate and culture survey, which includes questions about their social network as it pertains to teaching. This preliminary study focuses only on SEU's plan of action to incorporate active learning into their mathematics courses, with the future goal of analyzing site visit data later to understand how their plan was enacted.

Analysis of Proposal

First, we created a summary sheet to identify evidence from the proposal that fit within each frame. This summary sheet included questions, based on the four frames, that guided our reading of the text of the proposal. For example, for the structures frame we asked questions such as "What roles/positions were discussed in the proposal? What routines are in place (e.g., monthly brown-bag lunches)? What practices does the department share (e.g., use of worksheets)? What new roles, routines, practices are being proposed?" At least two of the authors independently read through the proposal and filled in the summary sheet for each frame, before meeting to discuss what should be included in a reconciled version of the summary sheet. (Creswell & Poth, 2018). As a team this summary sheet helped identify pivotal structures, connect them to the people involved, and analyze evidence of symbols/power related to them. We augmented this analysis with social network survey data to better understand the role of the leadership team.

Social Network Analysis

We used in-degree centrality from social network analysis (Borgatti, Everett, & Johnson, 2013; Scott, 2000) to examine responses to three different items, that asked: 1) who they *discuss* instructional activities with, 2) who they go to for *advice*, and 3) who they go to for instructional *materials*. In-degree centrality is the number of people who report that they discuss instructional activities with a particular person. While other indices are available, we present only the in-degree centrality to highlight the roles of leaders in the change process as "the in-degrees of the vertices in the network indicate respective status by popularity, potential for influence or leadership, and so on" (Bandyopadhyay et al., 2011, p. 11).

Preliminary Findings

Change Initiatives

To address RQ1 and RQ2, we now consider the primary initiatives proposed by SEU and how they aligned with institutional and departmental efforts to support student success.

Coordination system. The primary initiative put forth in SEU's proposal was course coordination. The motivation behind this change initiative was to create a "stable cadre of faculty to regularly teach the [Precalculus through Calculus 2] courses who employ effective learning opportunities in the classroom" (SEU Proposal, p. 1). Previously, all general education mathematics courses were coordinated in the sense that they used the same textbook and learning outcomes. In addition, one person was designated to loosely oversee this coordination. Their goal was to develop a much more robust coordination system for their Precalculus and Calculus 1 courses by aligning the curriculum within and across the courses, creating additional coordinator positions, curating active learning resources into a common repository, setting up a system of peer-to-peer observations, using common assessment items, and holding pre-semester coordination meetings. To lead this effort, the PI and Co-PIs on the proposal planned to rotate as coordinators through Precalculus and Calculus 1.

Alignment with existing efforts. To support the development of a course coordination system for active learning, SEU proposed leveraging several existing departmental, college, and institutional programs. We discuss two particularly impactful college-level programs here: the Learning Assistant Program and Faculty Learning Communities.

Learning Assistant Program. The college had developed a learning assistant (LA) program, which was created to "support students in developing the knowledge, beliefs, and practices that will help them be successful in their first year courses" (SEU Proposal, p. 5). In particular, LAs were introduced in the classroom to support active learning. In 2017, the College of Science and Mathematics formalized its LA Program, which included a pedagogy seminar and regular instructor-LA meetings. The LA program was based on the University of Colorado, Boulder model. At the time of submitting the proposal, six math faculty were working with eight LAs, impacting approximately 300 students. Interest in using LAs in mathematics courses is relatively recent at SEU. After participating in a faculty learning community (FLC) involving chemistry, biology, and mathematics, a subset of mathematics faculty became interested in using LAs. Elements of the LA program are designed to support alignment in the course, and, with the instructor; "[t]he regular meetings with their faculty mentor help to ensure that LAs are supporting students in ways that align with the course learning objectives and the instructor's intentions" (SEU Proposal, p. 8).

Faculty Learning Communities. At the time of writing the proposal, SEU had already implemented a college-supported structure called faculty learning communities (FLCs). Twelve mathematics faculty had participated in a year-long FLC that was focused on expanding active learning in their courses; nine faculty members were slated to participate in the next round of FLCs, including four faculty teaching Precalculus and five faculty teaching Calculus 1. The three PI/Co-PIs involved in the instruction of mathematics courses at SEU were also part of this cohort of faculty. The purpose of these FLCs was to help faculty align course components, design meaningful assessment practices, create and implement active learning activities, and support students in adjusting to new learning expectations.

People Involved in Change Initiatives

In this section, we discuss how the leaders identified in the proposal were positioned within the department and aim to address RQ3 and RQ4. The PI and Co-PIs listed in the proposal consisted of three tenured Associate Professors of Mathematics, who regularly taught Precalculus and Calculus 1, and an Assistant Dean in the College of Science and Mathematics, who was involved in mathematics education research. We used in-degree centrality to analyze how these individuals were perceived by the department and to identify any other influential individuals in the department. We summarize these data in Table 1.

Table 1 displays the in-degree centrality of individuals for all three social network analysis items. We chose to include only the top nine individuals in the table, as the difference between the sums of the in-degrees values for the three items for 9th and 10th place individuals was substantial. These data suggest that the four leaders listed in the proposal were perceived to be influential in the teaching community at SEU. As these data were collected at the start of this project at SEU, this suggests that the PI and Co-PIs were well-positioned in the department to implement the proposed changes. This table also suggests that there might have been other influential people within the department, including people in power like the department chair and assistant department chair. From the analysis of the proposal, these particular people were in support of the plans. In addition, the most influential person, according to these data, appears to be the General Education Math Coordinator. Thus, there were other potential leaders within the department who were not among the four leaders in the proposal.

Table 1. In-degree centrality values for three items.

	Discuss Median = 4 Response rate = 58%	*Advice* Median = 2 Response rate = 49%	*Materials* Median = 1 Response rate = 39%
General Education Math Coordinator	18	12	13
Lecturer involved in College Algebra improvement efforts	20	11	6
PI and faculty member*	19	8	5
Department Chair	19	9	3
Co-PI and faculty member*	17	7	7
Co-PI and College Administrator*	16	9	1
Co-PI and faculty member*	14	5	5
Lecturer	16	5	2
Assistant Department Chair	14	5	0

*Leaders of the local SEMINAL project as identified in the proposal

Discussion and Questions for Audience

This analysis indicates that SEU developed a robust plan for changes that takes advantage of existing structures and values at the department, college, and institutional level, and emphasizes alignment across these structures to support their goals of incorporating active learning into mathematics courses. Furthermore, the social network analysis demonstrates that the people put forward as change agents in this process already had influence within the department. Future analysis will focus on the roles of these change agents as the plan was enacted, and whether others identified as influential in the department will become involved in change efforts. To support this work, we ask the audience the following questions:

- We selected SEU for analysis because they were able to "successfully" implement several of the changes they described in their original proposal. Our hope is to eventually expand this analysis to include more institutions, as well as analyze change across various points in time. As we engage in this analysis, what part of this analysis should we continue? How might we change our analysis for other institutional plans or for looking at how institutions enacted their plans?
- We are particularly interested in thinking about people as change agents. Is there additional data or analysis we could be doing to identify and track these change agents within these stories of change?

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Characterizing How Undergraduate STEM Instructors Do and Do Not Leverage Student Thinking

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At the K-12 level, there is evidence that instruction that leverages student thinking can lead to increased conceptual understanding and success for students. Yet the ways in which instructors leverage student thinking in undergraduate STEM context remains largely unexplored. To investigate our research question examining the extent to which, and how, STEM instructors leverage student thinking in their teaching, we collected data from faculty who taught courses in biology, physics, chemistry, and math. We interviewed participants before a lesson, filmed the lesson, and conducted a stimulated recall interview using video clips from the filmed lesson. Transcripts were analyzed using thematic analysis. Initial results highlight differences in how instructors access student thinking, their efforts to interpret student thinking, and how they leverage student thinking to inform instruction.

Keywords: Leveraging student thinking; Instructor practice; STEM education

There have been calls for increased attention to the teaching of introductory undergraduate science, technology, engineering, and mathematics (STEM) courses, and professional development (PD) for those who teach these courses, in an effort to improve enrollment and retention rates in STEM disciplines (Bok, 2013; Holdren & Lander, 2012). In K12 contexts, there is evidence that effective instruction is enabled by an instructor's attention to student thinking (e.g., Erickson, 2011). In particular, instruction that leverages student thinking can lead to increased conceptual understanding and more positive learning experiences for students (Carpenter et al., 1989; Thornton, 2006). However, few studies at the post-secondary level have examined how undergraduate instructors are leveraging student thinking in their teaching.

Teacher noticing is a useful framework from the K-12 level that has informed our research design, analysis, and interpretation of findings. This framework has two key premises: (1) the heart of teaching is action in the midst of the complex social environment of the classroom, and (2) student thinking is productive and resourceful to teaching (Sherin et al., 2011). It is important to note that the noticing required for effective teaching is specialized and goes beyond simply being observant (Ball, 2011). Most scholars agree that teacher noticing consists of attending to and making sense of particular events during instruction (Sherin et al., 2011). Jacobs, Lamb, and Philipp (2010) narrow this scope and describe professional noticing as three interrelated skills: attending, interpreting, and deciding how to respond to students' mathematical strategies.

Research on teacher noticing has been leveraged in both mathematics and science contexts. This points to the translatability of this work across disciplines. For example, there is evidence at the K-12 level that both science and mathematics teachers attend to students' process skills or errors as novices and can develop skills for better noticing the disciplinary substance of student thinking (e.g., Stockero, 2014; Barnhart & van Es, 2014). Examining teacher noticing across STEM disciplines at the post-secondary level affords the opportunity to consider disciplinary similarities or differences that may arise and would be beneficial to consider in designing and implementing PD to support instructor noticing and leveraging of student thinking.

The goal of this research was to investigate the extent to which, and how, instructors leverage student thinking in introductory STEM courses. That is, how the substance of student thinking informs instructors' teaching. We adopted this fairly broad definition of what it means to leverage student thinking in order to capture the variety of ways STEM instructors might draw on information about student thinking to inform their instruction. We asked the following research question: In what ways do college STEM instructors leverage student thinking in their teaching? Since this is a relatively unexplored area, we examined the thinking and practices of instructors as they planned, enacted, and reflected on a lesson, with the goal of richly characterizing instances of leveraging student thinking.

Methodology

Seven faculty from various STEM departments, including Mathematics, Biology, Physics, and Chemistry, who were recognized by their colleagues as individuals who made thoughtful decisions about their teaching, were invited and agreed to participate in this study. Participants were experienced instructors who regularly taught introductory STEM courses. For this proposal, we focus on two participants, Dr. Bio (full professor in Biology) and Dr. Chem (career-line instructor in Chemistry) who taught large enrollment (70 and 270, respectively) introductory STEM courses. In the talk we will discuss all seven participants.

To investigate the extent to which, and how, participants leveraged student thinking in their teaching, participants were interviewed before and after a class period which they selected to have filmed (the *target class*). The pre-instruction interview was a semi-structured interview designed to elicit the instructor's goals for class, knowledge of student understanding regarding the topic to be covered, and how this knowledge impacted their planning for class. Clips from the target class (2-5 clips) were selected for use in the post-instruction interview to stimulate discussion about student thinking and instructional decisions. The selected clips highlighted moments when the instructor had access to student thinking and the interviews prompted participants to discuss how student thinking informed their planning and real-time decision-making. Interviews were transcribed and analyzed using open coding and thematic analysis (Braun & Clarke, 2006). At least two researchers coded each transcript. We met regularly to discuss coding decisions, to come to consensus about what and how segments were coded, and to create and refine codes as necessary. This allowed us to identify emergent themes related to the ways in which instructors notice and leverage student thinking.

Results: STEM Instructors Leveraging of Student Thinking

This work has illuminated instructors' thinking and practice related to the ways that they leverage student thinking, highlighting some of the variation that exists across introductory college STEM classes and the dimensions on which they vary. Specifically, our highly contextualized data revealed how instructors got access to information about student thinking while teaching, how they attended to and interpreted student thinking, and how they leveraged student thinking to inform their instruction. The following sections describe these three main themes.

Accessing Student Thinking

Dr. Bio and Dr. Chem differed in the extent to which they had access to student thinking during the target class. They both incorporated times in their class when they stopped lecturing and provided an opportunity for students to work on questions individually or in small groups. However, Dr. Chem actively sought out information about student thinking using a variety of strategies, whereas Dr. Bio relied on very few strategies for accessing student thinking. Dr. Chem frequently accessed student thinking by eavesdropping on student conversations, looking at student written work, asking students to share their thinking, and by asking students to explain their reasoning during whole class discussions. In the following excerpt, Dr. Chem describes how she uses students' facial expressions as an indicator that students are confused. She said:

> Oh, it's easy, really easy [to tell by their faces if they're getting it]. You're going to get 200 people looking at you with this blank look on their face. Or the talking gets louder because they're talking back and forth because they work together on these problems. Their talking back and forth gets louder because they're not getting it, and they're not actually working the problem. They're talking to each other, trying to figure it out, and they're just not getting it. You can very easily tell by looking at your class … And so you get to know who's sitting where, and you get to learn to read their expressions. I rely a lot on that and just looking at them and seeing, you know, are they getting in or not?

This highlights that Dr. Chem can read her students' expressions, getting access to some information about student thinking, and then she uses that information to draw conclusions about whether or not students are "getting it".

In contrast, Dr. Bio accessed information about student thinking much less frequently and used fewer strategies to do so. Dr. Bio primarily accessed student thinking through clicker questions and through students volunteering their answer to a question during a whole class discussion. Dr. Bio expressed that he thought that clicker questions were the only way to access student thinking in a large class. He said:

> I mean, again, the kind of feedback that you get really is - in a class that size, can only be measured quickly on the responses to [clicker question]. And that's why I do use [clicker questions]. As not just as an engagement tool, but as a way of kind of measuring, 'okay, how are they doing with that?' … But it's hard to gauge where they are in their thinking. I think it's an iterative process where I really won't be able to gauge that until I re-engage them on a comparable question tomorrow and see if they've got it now.

Further, Dr. Bio acknowledged that he might not have sufficient access to information about student thinking. He commented that he often relied on end of semester surveys for feedback on which topics students felt were challenging in the class.

Attending to and Interpreting Student Thinking

Dr. Bio and Dr. Chem had different goals for when they encountered student thinking during the target class. Dr. Chem attended to student thinking and aimed to make sense of or interpret student reasoning, even when it was incorrect or incomplete. She carefully listened to what students were saying and worked to make sense of their thinking. Her goal for doing this was to diagnose where students were at in their progress toward the learning objectives, and then to use this information to help her decide how to respond. In the following excerpt, Dr. Chem described what she was thinking during an interaction with a student who was having trouble making progress on a problem. Dr. Chem asked the student to explain their work, and then said the following when reflecting on this interaction during the post-instruction interview:

> I figured out [what she was thinking] pretty quick. As soon as she said that's sodium, then I said, ‘okay, I see what you're doing and you're just completely screwed up’. So yeah, I understood perfectly what she needed and so I explained it to her.

In addition to making sense of individual student’s thinking, Dr. Chem also frequently attended to the whole class’s progress by interpreting the information she accessed from multiple students through eavesdropping and clicker questions. She used this information to gauge where the class was at with their understanding, and then adjusted her plans and pacing for class accordingly.

In contrast, Dr. Bio demonstrated that he had goals other than interpreting when he encountered student thinking. In particular, he often focused on using information about student thinking to gauge students’ level of participation instead of to gauge their level of understanding. Further, Dr. Bio frequently corrected or redirected student thinking instead of trying to make sense of it. In the following quote, Dr. Bio describes what was on his mind when he heard two sentences of a student sharing their thinking. Instead of reasoning through the student’s thinking, it seems as though Dr. Bio immediately concluded the student’s thinking was tangential and incorrect, and then moved on to explain the connection that he wanted students to make. In the interview, Dr. Bio said the following in response to this interaction:

> So that student … gave a kind of a tangential answer and instead of going to another student or another student, I went straight to the answer. But what I was thinking is that later on I want them to be thinking about gene selection, before and after passing on your genes. And so it was such an important point I guess that I wanted to make sure it was made, that I just went ahead and made it.

Leveraging Student Thinking

Dr. Bio and Dr. Chem differed in the immediacy and the extent to which they leveraged student thinking in the target class. Dr. Chem regularly leveraged student thinking during class, made changes from one class period to the next, and made changes each semester based on student thinking. Specifically, Dr. Chem planned for instruction based on students’ current understanding of the material, stating, “I won't move forward if they're not getting it. There's no reason to. I actually wanted to get farther on Tuesday then I did, but the class wasn't ready to go farther.” Dr. Chem frequently added clicker questions for the next class to provide students more opportunities to engage with key concepts. She made the following comment after discussing a topic her students were currently struggling with: “I have a new question that will address it. … I've posted the annotated slides so they can see how to do it, but … we're going to do another question that's similar to that [in class].” Notably, Dr. Chem also made changes from semester to semester. She said,

> I'll take notes of like - ‘I'll need to spend more time on this’, or ‘[students] really struggled with this’. Then when I get ready for class the next semester, I go back to that and that's what prompts me to make new slides.

In contrast, Dr. Bio did not leverage student thinking frequently, but instead tended to respond to information about student thinking by providing explanations about the content or slightly adjusting the pace of a single class period. Dr. Bio described how he adjusted the pacing of class, saying:

> For instance, with one of these [clicker questions] or something along those lines, that the majority of the class gets it, and gets it fairly quickly, then I feel that, okay, they did make that connection. They came along with me on that. … But also if a large number of students do put down an incorrect answer, then that tells me, okay, they didn't make that connection. They didn't make that conceptual leap. And so I need to either go back and address that differently the next time I try to teach it or I need to explain to them why.

This segment also illustrates that when Dr. Bio recognizes that students do not understand material, his response might be to give an explanation immediately or he might, instead, adjust instruction the next time he teaches. In fact, Dr. Bio never discussed making changes from one class to the next, but instead he focuses on making adjustments from semester to semester. This then would mean that his current students would not benefit from these changes. Further, it seems that since Dr. Bio did not take advantage of opportunities to access and interpret the substance of student thinking, he was constrained in his ability to make more immediate changes (besides adjusting the pacing of class) that were rooted in students' understanding.

Discussion

Both Dr. Bio and Dr. Chem were both experienced instructors, but they differed in the extent to which they accessed student thinking, attended to and interpreted student thinking, and how they leveraged student thinking. One reason for this might be their exposure to substantive student thinking. Dr. Chem capitalized on nearly every opportunity when she had access to information about student thinking. As such, Dr. Chem was in-tune with her students' thinking, demonstrating that she valued student thinking and saw it as central to her instruction. Dr. Chem discussed that this exposure to student thinking helped shape the knowledge that she relied on while teaching. Dr. Bio, on the other hand, focused on the sequencing and timing of the lesson and so he was not able to focus on making sense of student ideas or recognize that some student contributions could be leveraged to make a lesson point. It seemed as though Dr. Bio did not recognize student thinking as a valuable resource that could be leveraged to inform his teaching. As such, Dr. Bio did not have as many opportunities to learn and develop the knowledge that seems to be necessary for leveraging student thinking. This suggests that disciplinary knowledge alone is insufficient for leveraging student thinking since Dr. Bio was a content expert, a finding that has also been documented by other researchers (e.g., Speer & Wagner, 2009).

Dr. Chem's and Dr. Bio's perspective on how students learn seemed to impact their ability to access information about student thinking, which then, in turn, impacted the opportunities they had to interpret and leverage student thinking. Dr. Chem viewed her course as an opportunity to challenge common student errors and to support students in developing a more complete and normative way of thinking about the content. She facilitated student learning by giving them a chance to solve problems and get feedback from their peers and instructor. Dr. Bio, on the other hand, seemed to think that students learn best when they hear clearly articulated accurate ideas. Dr. Bio's goal for instruction was to facilitate learning by helping students recognize the connections between the content through the sharing of his knowledge. Together this highlights that it could be important to discuss how students learn and work to foster a responsive disposition that values in-progress student thinking when developing PD to support faculty in accessing, attending to and interpreting, and leveraging student thinking.

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Decisions, Decisions: Mathematics Instructors' Decision-Making about Content and Pedagogy when Teaching with IBL

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In this paper, I explore the kinds of decision-making situations undergraduate mathematics instructors finds themselves in and the decisions they make when teaching with inquiry-based learning (IBL) methods. I first provide a brief review of research on teacher decision-making, by which this exploration is guided. I then use survey responses from 20 instructors from 12 different universities across the United States, who reflected on decisions they had made or were anticipating making about content and pedagogy in lower-division mathematics courses they had taught or were going to teach using IBL. After classifying the decision-making situations that instructors reported, I discuss the findings and the next steps of this exploration.

Keywords: teacher decision-making, university mathematics teaching, inquiry-based learning

Inquiry-based learning (IBL) methods—a teaching approach that promotes 'active learning' for students—have become increasingly popular among practitioners in undergraduate mathematics education in the United States (Haberler et al., 2018; Laursen & Rasmussen, 2019). IBL is often used as an umbrella term to encompass various mathematics instructors' ways of engaging students (Ernst et al., 2017; Laursen & Rasmussen, 2019; Yoshinobu & Jones, 2012). Because IBL is not research-based and it has instead emerged from mathematicians' teaching practices, research in this area is still conceptualizing IBL, characterizing its different ways of implementation (e.g., Laursen & Rasmussen, 2019; Mesa et al., 2020; Shultz, 2020), theorizing its effects, and informing the ways it can be improved in a variety of environments, such as courses and institutions. My larger research goal is to conceptualize IBL through teacher decision-making in various situations. By investigating how and why instructors make decisions in specific situations, we can enhance our knowledge of IBL and the resources instructors teaching with IBL need when using these methods (e.g., knowledge, institutional and societal resources). As a preliminary step, I conducted a study on IBL instructors' decision-making, during which I asked instructors to describe decision-making situations in a course taught with IBL. The study was primarily designed to collect decision-making situations that instructors encounter while teaching an undergraduate lower-division course using IBL methods, for the purposes of generating situations in future studies of teacher decision-making in IBL. This paper serves the very early stages of this investigation. To explore IBL instructors' decision-making, I attempt to answer the following research questions in this report: What kinds of decision-making situations about content and pedagogy do instructors find themselves in when teaching with IBL, and what kinds of decisions do they make in these situations?

Background on Research on Teacher Decision-Making

Cognitive-based research on teacher decision-making has mostly focused on modeling teacher's routines and what teachers do when a routine is interrupted due to an unexpected event (i.e., modeling teacher's decision-making schemas for different scenarios), categorizing those scenarios and decisions, comparing novice versus expert teachers' teaching schemas and routines, and finding connections between decision-making and individual constructs, such as beliefs, and knowledge (Borko & Shavelson, 1990; Bishop, 1976; Bishop & Whitfield, 1972;

Schoenfeld, 2010; Shulman & Elstein, 1975). In particular, Schoenfeld's (2010) theory of teacher decision-making states that teacher's decisions are filtered through and are functions of their goals, orientations, and resources. These approaches assume that teachers make decisions based on personal and idiosyncratic schemas (e.g., what has worked in the past) and as such, focus on individual teachers rather than on teachers as a group.

Schoenfeld (2010) assumes that when placed in a contextualized situation with various options available to them, the teacher as an individual with a pre-existing set of goals, orientations, and resources, makes decisions (i.e., chooses an option) consistent with their goals. There are two types of situations: in familiar situations, making a decision is more automatic because existing routines and scripts are accessed and activated to reduce cognitive load; in unfamiliar situations, the subjective expected value of each available option is calculated and the option with maximum expected value is picked (see Gerami, 2021 for detailed example of such analysis). In this paper, I am interested in the situations instructors report finding themselves in and the decisions they make (or the options they choose) in these situations. Although the framework focuses on individual teacher decision-making, here I use the framework to study teacher decision-making across teachers by identifying similar decision-making situations that teachers find themselves in and comparing them.

Methods

The data for this study comes from a preliminary study that explored mathematics instructors' decision-making in IBL lower division courses via a survey and a follow-up interview. The data were collected from 20 instructors from 12 different institutions across the United States over Fall 2019 and Winter 2020. In this paper, I primarily focus on data from a section of the survey that elicited decision-making situations from the instructors. This section of the survey asked instructors to choose a lower-division course that they have taught or were going to teach with IBL methods and to describe five situations where they made or anticipated making a decision: about the content, about the course materials, about the assignments and assessments, about methods of teaching, and while teaching. For each situation, participants were asked to (1) list all options available to them when making the decision, (2) the desirable and undesirable outcomes of each option, (3) explain what they chose and why, and (4) identify the use and frequency of various teaching methods (e.g., lecturing, student small group work) and their learning objectives for their students. The remaining sections of the survey inquired about instructors' background and demographic information, their definition of IBL, reasons for using IBL, and personal gains and concerns about IBL. I followed up the instructors' responses to the survey questions via a semi-structured hour-long interview.

Here, I only use data from two decision-making situations—about the content and about the methods of teaching—because of lack of space and that the situations instructors provided about content and methods of teaching seemed complementary. Table 1 shows the instructors' characteristics organized by the lower-division courses they chose to describe the decision-making situations from. All names used in this report are pseudonyms. All instructors, except T5 and T7, had used IBL methods to teach their courses before participating in the study; T5 and T7 stated that they were going to use IBL methods in the future in their Calculus I course. On two 4-point Likert scale questions about being comfortable with IBL methods and being knowledgeable of IBL methods, all instructors indicated that: they were either fairly comfortable or slightly comfortable with IBL methods, and they were knowledgeable or slightly knowledgeable of IBL methods.

Table 1. Study participants by course

College Algebra/Pre-Calculus	Introduction to Proofs
T1: Greg	T12: Isabel
T2: Leah	T13: Bill
Calculus I/Differential Calculus	Introduction to Proofs and Linear Algebra
T3: Simon	T14: Emanuele
T4: Christina	Linear Algebra
T5: Kara	T15: Sebastian
T6: Kyle	T16: Michael
T7: Lynn	T17: Harvey
Calculus II/Integral Calculus	Abstract Algebra
T8: Matthew	T18: Ruth
T9: Luke	Mathematics for Elementary Teachers
T10: Sam	T19: James
Calculus III/Multivariable Calculus	Probability and Statistics
T11: Elaine	T20: Tim

To identify the types of situations and decisions instructors reported about content and pedagogy, I first summarized each scenario they described in a table. For example, Table 2 is the summary table of Leah's (T2) decision-making situation about the mathematics content in her pre-calculus course. To explain the situation, Leah wrote: "[W]hen we were studying trig[onometry] functions, I had to basically skip tan[gent] all together so that we could better cover cos[ine] and sin[e]. I showed one single example of a graph of the tan[gent] function." Leah listed four available options, ranging from skipping Tangent graphs altogether to covering them in depth in class, and described desirable and undesirable outcomes of each of her option. She chose option (1) in the end: "I gave a short lecture/example and didn't include it in any of their assessments ... I did this so that we could spend the extra time on graphs of sin[e] and cos[ine] instead. We needed more time on this." I labeled the situation as "whether and how to cover graph of Tangent functions," and characterized it as unfamiliar because this was Leah's first time teaching pre-calculus with IBL. As the survey elicited one content-related situation and one pedagogy-related situation from each instructor, this process resulted in total of 40 tables (two from each instructor) divided into two categories: 20 decision-making situations about the content, and 20 decision-making situations about methods of teaching (pedagogy). I then categorized the decision-making situations by open coding the labels. Within the same decision-making categories, I used the tables to find patterns among chosen options when the options were comparable.

Table 2. Summary table of T2's decision-making situation about content in her pre-calculus course

Decision-making situation label: whether and how to cover graph of Tangent functions		
Options	Desirable Outcomes	Undesirable Outcomes
(1) Cover Tan graphs via mini lecture [chosen]	students see some new examples, allows more time on Cos and Sin	short lecture likely results in the students not learning
(2) Skip Tan graphs altogether	more time to spend on Cos and Sin graphs	students do not see examples of the Tan graph, problematic for calculus series

(3) Ask students to learn Tan graphs on their own	more time to spend on Cos and Sin graphs	have to follow up (only a couple students would read the book), students not learning
(4) Cover Tan graphs in class	reinforce ideas from the Cos and Sin graphs	No more time to spend on to better learn Cos and Sin graphs

Preliminary Findings

When asked to describe a decision they made about the *content* in their IBL course, 15 of the 20 instructors provided situations where the content was mentioned in their decision-making, with eight of the situations coded as familiar. These situations were divided into three groups: whether and how to cover specific topics, how to introduce a new topic, and how to clear students' confusions or misconceptions about the content. The other five instructors mentioned no specific (e.g., Riemann sums) or generic (e.g., proofs) content (T8: "I had to decide what group projects to use. There are many published examples of possible projects, or I could have written my own, or adapted published projects"; T18: "I could have had the students carry out calculations using technology").

Twelve instructors described situations about whether and how to cover specific topics given the lack of time to cover all topics in the ways they desired (e.g., T15: "I usually have to make a choice between covering an application or two or instead picking another "core" topic, like Gram-Schmid and orthonormal bases."). These 12 instructors chose four different options: to cover the content via lecturing, to cover the content via student-led teaching methods (e.g., group work, class discussion, student presentations), ask students to learn the content on their own outside of class, and skip the content altogether.

Three instructors described situations where they had to decide how to introduce a new topic so that it could help students gain better understanding:

> *T9*: Deciding how to discuss using rectangles to approximate the area under curves (eg: using right-endpoint approximation with four rectangles). I had to make a decision how I wanted to introduce this topic to students so they would be able to grasp ideas like more rectangles produces a better approximation.

One of the instructors describing a situation involving introducing a new topic, also wanted to decide how to clear students' confusions or misconceptions about the content:

> *T1*: We were starting our section on compound interest (in the sense of banking). This topic can lead to some confusion, because interest compounded more than once leads to an overall interest rate greater than the nominal rate. I wanted to build understanding of this, and think some about why this happens.

The three instructors contemplating about how to introduce new topics to their students (T9, T11, and T1) all decided to have students work on problems that would make them think about the upcoming material and follow up their work by having class discussions.

When asked to describe a decision they made about *methods of teaching* in their IBL course, 17 instructors provided situations where methods of teaching were mentioned in their decision-making, with 14 of the situations coded as familiar. One instructor described a situation about assessment prior to a lesson without mentioning methods of teaching and two instructors did not provide a situation.

Most instructors who described a situation about their methods of teaching acknowledged making such decisions often and instead of picking one specific situation that they found themselves in once, they described a generic situation where they had to make a similar decision frequently. These situations were divided into two groups: choosing among various teaching methods, and deciding how to proceed when students are at an impasse.

Fourteen of the 17 instructors described situations where they had to choose among various teaching methods (e.g., group work, lecture, student presentation). Among these situations, the instructors were often choosing: between lecturing and other student-centered teaching methods (e.g., T8: “I had to decide how much time to devote to lecture vs. class discussion vs. group projects.”), or among various student-centered teaching methods (e.g., T10: “I had to decide how to balance student group work while efficiently being able to have students present their work to their peers.”). A few instructors also wondered how to arrange students to work: individually, in groups, or as a whole class (T7: “When having my students work on a problem in class I always have to decide if I want them to try it on their own, in a group, or as a whole class first.”). Most of these instructors explained that in these situations their decision-making is not straightforward, as they have experimented and chosen different options depending on the specific situation.

Three instructors described situations where students were at an impasse working on the materials on their own in class and the instructors had to decide how to proceed (e.g., T13: Essentially whole class is at an impasse and presentations are stalled out.”). Two of these instructors said that they often bring all students together for a whole class discussion to help students move forward; the other instructor explained she chooses among two options depending on the day (sending them home to think, show them the solution).

Discussion

The findings show that the instructors had a difficult time reflecting on decisions about content, as five of them provided situations where the content was not discussed, and when the content was mentioned, it was not always obvious how it played a role in making the decisions; most instructors ended up discussing *how* to teach instead. This shows that IBL instructors may not be fully aware of the decisions they make about the content (e.g., sequencing problems to teach a specific topic, choosing a problem with specific representations), or that they do not often make such decisions. Moreover, analyzing the decision-making situations about the methods of teaching showed that these decisions were more familiar and automatic and that teachers used complex routines to choose the right method in the moment. Future studies are needed to unpack these routines and examine other factors teachers consider, besides time economy, when they have to decide among the various teaching methods used in IBL. Concerns about coverage in IBL and the relationship between lecturing when under time pressure have been previously raised (Johnson et al., 2016; Yoshinobu & Jones, 2012). However, this study shows that instructors use other options when under time pressure, such as delegating the content to students to learn on their own through use of portfolios and homework and assigning some topics as extra credit. The analysis also shows that instructors in IBL may feel the need to introduce the content to students differently when they are not lecturing, and to think through how they would respond to student difficulties to the materials in class.

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Examining Elementary Prospective Teachers Use of Visuals in Fraction Addition and Subtraction Problems

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Mathematics learning is enhanced when students use visual and intuitive mathematical thinking. Visual representations create an integration between conceptual understanding and formulaic mathematics that allows the student to make sense of the underlying mathematics. In this study, we examined elementary prospective teachers' (PTs) use of visual models in making sense of fraction addition and subtraction concepts. We found that at the beginning of a math content course PTs relied on computational rules. As the course progressed, they incorporated more visuals in their solutions and improved their use of the visuals. We analyzed the types of visual models used and PTs' sense-making progression.

Keywords: fraction addition, fraction subtraction, prospective teachers, representations, visuals

Introduction

The role of visual representations in the problem-solving process is well studied (Boonen, Reed, Schoonenboom, & Jolles, 2016; van Essen & Hamaker, 1990; van Garderen, Scheuermann & Poch, 2014). Not only are visual representations teaching tools, but they also serve as a process developed by students to make sense of a problem (Stylianou, 2011) and hence are components of the problem-solving process. Visual representations provide a visual description and connect numerical, linguistic, and spatial quantities (Pape & Tchoshanov, 2001; Woodward et al., 2012). Once the problem structure is clear, visual representations can facilitate understanding of the problem, identification of a problem solution, self-monitoring the progress, and self-checking the accuracy of the solution (Boonen et al., 2014; Krawec, 2014; van Garderen, 2007). In addition, visual representations also help students explore multiple strategies and understand why mathematical rules work (Boaler, 2016).

A recent study has found pre-service teachers with high spatial skills can generate structured schematic representations showing a relationship between parts of the object correctly whereas those with low spatial skills tend to generate pictorial representations in terms of shape, color, or brightness (Özsoy, 2018). The use of appropriate visual representations to support the understanding of the problem and the improvement of problem-solving performance depends on how the teacher uses these visual representations. Incorrect or incomplete representation may confuse the students which may have an adverse affect on students' problem-solving process (Özsoy, 2018). Also, research (Thompson, 1985) suggests that beginning teachers' conceptions and practices are influenced by their schooling experience. Therefore, if teachers did not have experience with visual representation in their schooling, how can they be expected to teach the same to their students.

Given the importance of visual models in elementary curriculum and advanced mathematics thinking, mathematics teacher educators (MTEs) need to understand what knowledge in-coming prospective teachers (PTs) have and how they can support PTs to gain a conceptual understanding of mathematics through visual representations. Hence, we explore the question: What types of visual models do elementary PTs use to make sense of fraction addition and subtraction concepts, and how does their use of these models change throughout the semester? How does PTs' use of visual models contribute to their sense-making and problem-solving?

Theoretical Framework

Our study employs two theoretical frameworks: conceptualization of representation and conceptualization of mathematical pedagogical content knowledge.

Conceptualization of Representation

Representations are essential components used in describing problem solving processes in mathematical learning (Cifarelli, 1998). A representation is a configuration of signs, objects, or characters that stand for something other than itself (Goldin & Shteingold, 2001). The term representation refers to "the act of capturing a mathematical concept or relationship in some form and to the form itself" (NCTM, 2000, p. 67). In other words, representation is not just the final solution of a problem but is a process to capture student thinking in obtaining the solution (e.g., Ball 1993; Cai 2005). Pape and Tchoshanov (2001) argue that representations "refer to both the internal and external manifestations of mathematical concepts" (p. 118). External representations include conventional symbol systems of mathematics such as base-ten systems, numerals, diagrams, equations, etc. to structured learning environments such as manipulatives, computer-aided systems, etc. that support understanding of mathematical concepts (Goldin & Shteingold, 2001). Internal representations include mental images or abstractions of mathematical ideas constructed by the learner. These include students' own symbolization, language, visuals, and spatial representations that they have assigned to make sense of the mathematical constructs. These two representations interact and influence each other (Pape and Tchoshanov, 2001). For example, a PT can use a discrete model to internalize one and one-half of the whole. Similarly, a PT who internalizes 3/2 as 3 parts of size ½ can create a continuous model to externalize the mental image. Hence a conceptualization of representation is a two-way process; first internalizing external representations and second externalizing mental images (Figure 1). This interplay between the external and internal representations facilitates learning when PTs come to use various representations to understand abstract mathematical concepts.

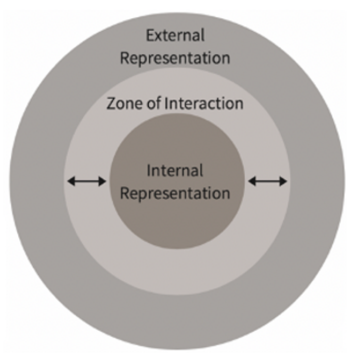


Figure 1. A picture showing the two different types of representations and their zone of interaction.

Pedagogical Content Knowledge

Our work is situated within Ball and colleagues' conceptualization of mathematical Pedagogical Content Knowledge (e.g., Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008), which is the knowledge of effective teaching including teachers' conceptual knowledge about the content, knowledge of curriculum, knowledge of teaching and instructional tools, and knowledge of students' learning. Examining PTs' visual representations can help to inform MTEs on how to incorporate multiple visual representations in their K-8 content courses, strengthening PTs in content as well as in their readiness to teach and address children's mathematical conceptions.

Methodology

For this study, our participants were prospective teachers enrolled in the second semester of a two-semester math content course for elementary teachers in a large public university. A total of 289 PTs participated in the study: 103 in Spring 2020, 58 in Fall 2020, and 128 in Spring 2021. Each semester of the study, a subset of the three researchers were the instructors of the course. All three semesters, the data collected from a pre-and post-assessment on fraction addition and a final exam question on fraction subtraction consisted of both the visuals used and the written explanations of the solution (Figure 2).

Pre-/ Post-assessment on Fraction Addition Spring 2021	**Final Exam Spring 2021**
Cory understands the meaning of fractions but has not yet learned any rules for adding fractions. Just using what Cory knows, how could you help Cory reason out a solution to the addition problem 3/4 + 4/3?	Below is Chase's solution: $\begin{array}{r} 3\frac{1}{7} \\ -\,2\frac{2}{7} \\ \hline 1\frac{1}{7} \end{array}$ Use a visual to help Chase see a correct solution of the subtraction problem 3 1/7 – 2 2/7.

Figure 2. Pre-and post-assessment on fraction addition and final exam on fraction subtraction.

For our study, visual models include schematic representations such as arrays, number lines, and diagrams. Also, we have used visuals, and visual models interchangeably in this paper. We used a descriptive coding process (Miles et al., 2014) and used descriptive statistics, namely frequency statistics to analyze the data.

Results

Here we present our results on our exploration of the research questions.

Research Question 1: What types of visual models do prospective elementary teachers use to make sense of fraction addition and subtraction concepts, and how does PTs' use of visual models change throughout the semester?

Participants utilized a variety of visual models. Table 1 shows the percentage of solutions using the listed visual model in each of the three assessments. We observed that PTs included a visual model in 47.3% of pre-assessment solutions and 69.6% of post-assessment solutions. Meanwhile, 63.1% of pre-assessment and 41.2% of post-assessment solutions included fraction rules or decimals. This percentage is heavily weighted in the Spring 2019 pre-assessment as PTs interpreted the problem as, "what would a child who is unaware of the rules do?", e.g., add straight across numerators and denominators. Directions were modified and a total of 3 PTs added straight across numerators and denominators over subsequent semesters. For the final assessment, which explicitly asked PTs to use a visual, visuals were included in 90.4% of solutions and 18.5% of solutions included fraction rules. We observed that discrete strip diagrams and number lines were used more frequently over the course of a semester.

Research Question 2: How does PTs' use of visual models contribute to their sense-making and problem-solving?

The fraction addition problem used in the pre- and post-assessments required PTs to create equal size pieces in the whole since the denominators in the two fractions were not equal. Table 2 lists the visual process of PTs in the pre- and post-assessments, and the percentage of each occurrence. Over the semester, the percentage of clear and complete solutions that used a visual model increased from 7.5% on the pre-assessment to 34.1% on the post-assessment. However, by the end of the semester, nearly a third (32.2%) of solutions in the post-assessment still had no visual component or no indication of how to use a suggested visual. Examples of some of the ways PTs use their visuals in making sense of the addition problem are shown in Figure 3.

Table 1. Percentage of participants using a particular type of visual model in three assessments.

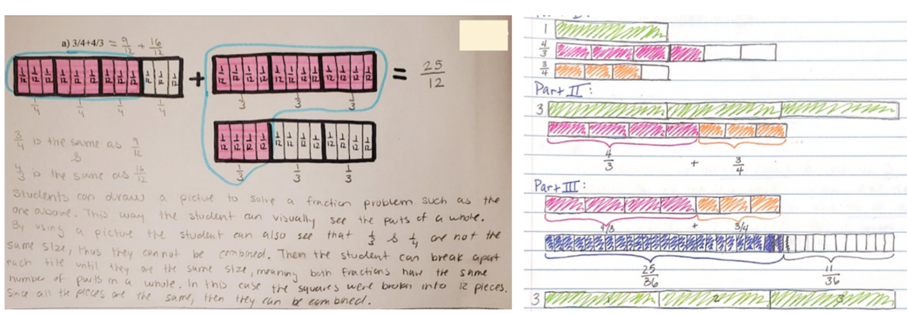
Visual used*	**Pre (%) N=281**	**Post (%) N=211**	**Final (%) N=276**
Discrete strip diagram	21	14.7	54.3
Continuous strip diagram	10	18	6.5
Number line	2.5	25.6	21.3
Pies	8.5	2.8	2.2
Array	4.6	8.5	0
Discrete objects and grouping	0.7	0	4.7
Place value blocks	0	0	1.4
Suggested a visual with words	7.5	8.1	0

*Some participants used more than one model in their explanation.

Table 2. Descriptive statistics on PTs' visual solutions to make sense of fraction addition.

PTs' visual process for sense making*	Pre (%) N=281	Post (%) N=211
a) Entirely visual solution with clear representation or explanation	7.5	34.1
b) Entirely visual solution with less clear representation or explanation	7.1	9.5
c) Initial visual set up to get equal pieces, but then added arithmetically	0	0.5
d) Visually attended to addition/acknowledged the need for equal size pieces visually, but did not show how to get an answer visually	6.4	10.4
e) Visually represented one or both fractions but did not alter the fraction visually to perform addition	17.4	10.4
f) Visually attended to addition; did not acknowledge the need for equal size pieces visually, and didn't complete solution visually	4.3	1.9
g) Explained what to do visually but didn't show	1.4	2.4
h) No indication of a visual or how to use suggested visual	56.6	32.2

*Some solutions contained more than one process.



(a) (b)

Figure 3. (a) An entire visual solution with a clear explanation. (b) A PT acknowledged the need for equal size pieces but did not show how to alter the visual representation to achieve equal size pieces.

The final assessment considered a problem with like denominators. PTs' visual process for the final assessment and the percentage of each occurrence is shown in Table 3. Figure 4 shows an example of a PT's process which is entirely visual with clear representation. In the final exam, 71% of PTs had an entirely visual solution with a clear representation or explanation.

Table 3. Descriptive statistics on PTs' visual solutions to make sense of fraction subtraction.

PTs' visual process for sense making*	Final (%) N=276
a) Entirely visual solution with clear representation or explanation	71
b) Entirely visual solution with less clear representation or explanation	7.2
c) Initial visual set up to make subtraction easy, then subtracted arithmetically	2.5
d) Converted given fractions to improper fractions, then subtracted visually	3.3
e) Visual set up but didn't show subtraction of 2 2/7 visually	0.7
f) Visually represented one or both fractions but did not perform subtraction	3.3
g) Attempt to show subtraction visually but didn't complete visually	1.4
h) No visual	10.9

*Some solutions contained more than one process.

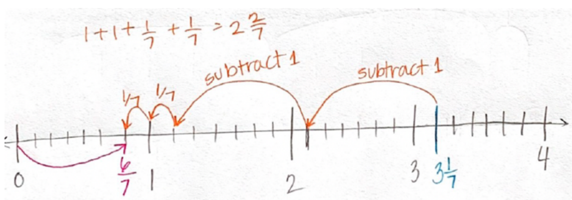


Figure 4. A PT's work showing solution entirely visual with clear representations.

Discussion

The results from the pre-assessments provided some immediate pedagogical implications. We found that very few PTs used a number line as their visual model on the pre-assessment. Studies have shown that a student's ability to work on a number line contributes to the development of numerical knowledge and correlates to improvements in estimation accuracy (Siegler & Booth, 2004; Schneider et al., 2009). With the conceptual gains that arise with the use of number lines, we hope that these future teachers will use them with their students. Therefore, we need to build up these future teachers' comfort level working with number lines. This is not to say that other models are not important, but rather the rarity of number line use requires specific attention.

The comparison of the pre-assessment to the post and final clearly shows that PTs get better at conceptualizing fractions and using visuals to communicate that understanding. Those results should not be surprising as that is one of the purposes of the course. The more interesting question is to find out the extent to which this learning persists. We plan to follow up with a group of these PTs while they are in their credential program and then again when they are in-service teachers to track how their conceptual understanding of fractions and their communication of that understanding changes over time. Our data showed much better performance on the final than on the post. This is not surprising as the final was high stakes whereas the post was only graded for completeness. When following up with groups of students, we will also be analyzing which of the culminating formats is a more accurate indication of future understanding.

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Undergraduate Calculus Students' Perceptions of the Characteristics of Good Responses

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We conducted preliminary research on effectiveness of contrasting examples in supporting students in introductory calculus to conceptualize explanation as a genre of mathematical writing. Drawing on principles from Variation Theory (Marton, 2014) we constructed sets of mathematical responses, each designed to contrast along a critical dimension. Participants compared responses, identified similarities and differences, and ranked responses from better to worse. Participants discerned dimensions of variation that were consistent with our intentions and positioned clarity and efficiency as particularly valuable features. However, we conclude the report by problematizing the limitations of variation theory as an instructional design based on the ways that such design is likely to perpetuate norms and expectations for communication in mathematics that risk marginalizing students.

Keywords: Learning to write in mathematics; Mathematical explanations; Variation theory; Mathematical communication

Learning to write in ways that reflect the norms and expectations of a mathematical community is fundamental for students' growth as members of that community. There is little research on how to support students in learning to write in mathematics. However, the few available studies suggest benefits of student's exposure to writing in the discipline (Bicer et al., 2011; Linhart, 2014). Due to the importance of learning to write in mathematics, Bicer et al. (2013) argued for a need to understand how to design instruction that helps to improve students' ability to write and construct meaningful mathematical responses.

This study uses Variation Theory (Marton, 2014) as a lens for investigating how students' experiences of contrast across examples of mathematical writing might support their development of an understanding of mathematical explanations as a genre of writing. By exposing students to sets of constructed examples we intended to support their development of a conception of good mathematical explanations.

Theoretical Framing

Writing in mathematics has not been well conceptualized (Quealy, 2014; Sumner, 2016; Wilcox & Monroe, 2011). Several scholars have advanced the argument, however, that mathematical writing is valuable as an instrument of learning and as an outcome of learning. For instance, Quealy (2014) argued that writing helps students build deeper connections to the subject by inviting students to reflect on their own reasoning and extend their understanding. Meaningful writing promotes positive attitude and understanding of mathematics and contributes to construction of new knowledge (Fung, 2010; Linhart, 2014; Seo, 2015). Writing promotes opportunities for students to practice mathematical inference, to communicate mathematical ideas, organize thinking, make connections, and interpret results (Fung, 2010).

Writing is a constructive activity that not only reinforces understanding but also allows for constructing new meanings (Seo, 2015). According to Seo (2015), engaging in writing is useful to students as a way of discussing assumptions, justifications, providing explanations, and criticizing mathematics ideas. Colonnese et al. (2018) found that including intensive writing in mathematics courses is one strategy for engaging students in critical thinking thus enhancing students' own critical thinking. Colonnese et al. (2018) posit that writing mathematical responses creates a higher level of cognitive demand for students by challenging them to explore mathematical relationships among ideas in their responses.

The argument for engaging students in writing is reflected in the Common Core, which calls for students to write about mathematical ideas to analyze and self-evaluate their thinking and that of their peers (National Governors Association et al., 2010). Using mathematical language to convey ideas precisely is essential to strengthening conceptual understanding. Students should learn to construct arguments, critique others' mathematical reasoning, explain the process of solving problems, and adopt clear definitions and vocabulary in mathematics. In addition, the writer's choice of text, structure, grammar and lexis is determined by the writer's role, purpose, context and the target audience (Nesi, 2012), including their disciplinary community. Therefore, every student can benefit from learning to write like members of their disciplinary community.

Using Variation Theory to Support Students' Learning to Write

Variation theorists argue that there are necessary conditions for meaningful learning to take place. Specifically, Marton (2014) argues that learning requires interacting with instances "that have . . . [some] meaning in common, though differing otherwise" (p. 45). In other words, learning happens when the learner is aware of both sameness and difference across experiences. A learner notices a critical feature in one situation when it differs from at least one other feature of a similar situation along some dimension of variability (Marton, 2014). Multiple features in the same dimension must be experienced simultaneously because discernment cannot occur without learners' awareness of other features on a common dimension. Thus, meaning is created based on how experiences differ from one another, and the difference is determined by the possible features in each dimension.

The implications for instruction are that an instructor must identify objects of learning (such as a specific concept or skill) (Marton & Pang, 2006) and critical features of those objects that learners should discern. The instructor must then construct opportunities for learners to contrast examples with the critical features against examples that do not share those features. In theory, this contrast will support learners to develop the intended object of learning.

Our goal in this pilot research was to respond to the following questions:

1. What dimensions of variability do learners discern among contrasting examples of mathematical responses to typical calculus problems and which features do they perceive as critical features of a "good" response?
2. How do the dimensions discerned by students relate to those that the designers intended?

Methods

We conducted this study in the context of an introductory Calculus I course. We developed a list of four critical features to define our object of learning: a mathematical response that includes exposition, that presents understanding and capacity related to a solution process even when the individual is stuck on a particular stage, that includes descriptions of erroneous solution paths, and that acknowledges in first-person language the human agent behind the solution. We designed sets of examples intended to make each dimension visible by varying constructed examples with respect to the intended dimension but otherwise holding features constant (see Figure 2 and Figure 3). We asked students to compare examples in each set, identify similarities and differences across the examples in each set, and state which example they preferred. We collected de-identified responses to in-class work, quizzes, exams, discussion posts, and recordings of class meetings as data sources.

Analysis

The second author coded inductively through open coding data (Emerson et al., 1995; Bogdan & Biklen, 2007) by noting the features and contrasts that students explicitly mentioned as they compared, contrasted, and ranked examples within each set. Codes were either direct phrases from the data or represented the researchers' interpretation of the data. We inferred, based on the features that participants explicitly mentioned, the corresponding dimension of variation that the student discerned. The process led to 13 distinct codes, seven of which we found across more than one participant's data. We validated codes by a second researcher applying the codes as an a priori scheme on the same data. We found consistency between the a priori coding and the inductive coding, indicating reliability for the code set.

Findings

The codes that emerged from the coding and data analysis are illustrated in Figure 2. We include only those codes that appeared in data from more than one participant. We developed four profiles detailing dimensions of variation, specific features along those dimensions, and the valorized features identified by four participants. In the following section, we illustrate those findings using Adam's profile. We compare the discerned dimensions identified by Adam to the intended dimensions on which we had based the examples.

Discerned Features	**Participants**				
	Adam	**Bobbi**	**Carl**	**Dina**	**Erin**
Clarity	/	//	//	/	//
Efficient use of words	/	/			/
Organization	/		/		///
Explanation	/		///	/	
Content knowledge/ understanding				/	//
Use of Labeling			//	/	
Accessibility		/			//

Figure 1. Dimensions of Variation discerned by participants based on examination of contrasting examples.

Participant 1: Adam

As shown in Figure 1, Adam discerned clarity of communication as a dimension of variation across the different examples, with some examples identified as "clearer" and others as "less clear". Other dimensions that Adam discerned included efficient use of words, organization, and length of the response. According to Adam, the features of a good response include providing just enough well-organized information to make communication clear. For example, in comparing the examples in Figure 2, Adam wrote that "the best response is response B because . . . response A has way too much. . .. Response B shows the same thing in a much more efficient way." Here we infer Adam discerning the amount of exposition as a dimension of variation and valorizing an economical response over a more descriptive one.

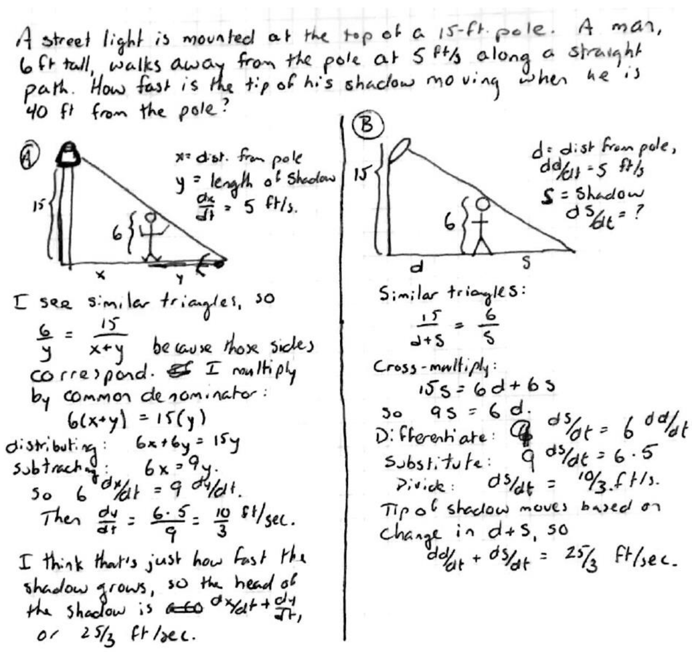

Figure 2. Responses that contrast with respect to representing the human agency responsible for the mathematics.

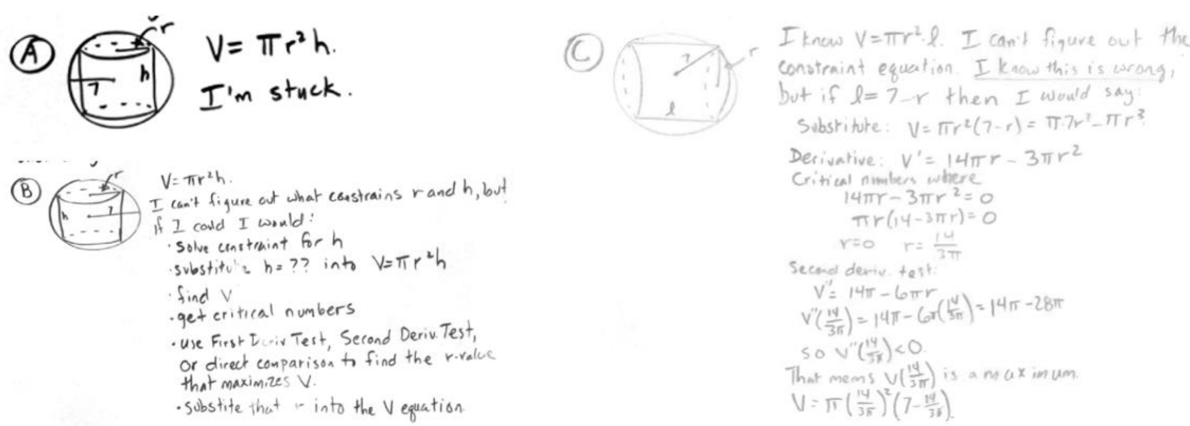

Figure 3. Example responses to "Find the largest possible volume of a cylinder inscribed in a sphere of radius 7".

Adam also responded to a set of examples designed to illuminate different ways of responding to a problem when one is unsure of a critical detail – in this case, as shown in Figure 3, a hypothetical situation in which respondents have not yet found a relationship between radius of the sphere and height of the cylinder in the inscribed cylinder problem. The responses differ in the amount of insight provided regarding how the respondent would use that relationship if they could express it. Adam clearly discerned the intended dimension of variation, and valorized the response that most fully illuminated the scope of the respondent's knowledge of the process and capacity to carry it out:

> C was the best response because it showed some work, that they knew was wrong, but that was right with the equation they used. Response B was the next best because they

illustrated what they would do, but they didn't actually show themself doing it. And, finally, response A showed nothing besides saying that they were stuck.

We note that in this instance, the valorized feature that Adam identified – *showing that one could carry out a process in spite of knowing that a component is incorrect* – was consistent with our intended critical feature.

Discussion

Writing in mathematics benefits students' development of mathematical understanding and is a useful skill for students who pursue mathematics as an academic or professional path. In this research, we sought to determine whether comparing examples designed to draw attention to particular features of mathematical writing could be effective at supporting students to develop understanding of qualities related to effective writing in mathematics. Our preliminary analysis found that participants who examined our sets of responses identified dimensions of variations that were consistent with those that we had intended, which according to Variation Theory is a necessary first condition for developing a systematic approach to supporting students in developing understandings of mathematical writing that is descriptive, narrative, and that creates opportunities to demonstrate knowledge even in contexts when the student has not yet fully solved the problem about which they are writing. For example, the dimensions of variation and critical features that Adam valorized, including clarity, efficiency, and demonstrating capacity in spite of a missing component, were consistent with our intended object of learning. However, we note that no students identified the dimension of human agency or the feature of responding in a way that identifies the first-person agent in the response, which was one of our intended dimensions and features for the set shown in Figure 2. It seems that example sets, when designed appropriately, can be effective at making visible the dimensions of variation and critical features that the designer perceives as characteristic of good mathematical writing.

However, a problematic aspect of this instructional design is that the object of learning, the dimensions of variation embedded in the examples, and the critical features valorized by instruction represent the norms and expectations that the instructor holds, as a representative of an existing mathematical community. For examples designed to support the development of mathematical concepts, this does not seem problematic. For examples designed to support more subjective concepts – such as the nature of good mathematical writing – this approach risks perpetuating the status quo by pushing students to write and communicate in ways that are associated with those groups that are most centrally represented in mathematics. We know that the status quo marginalizes students whose forms of expression -- grounded in their historical, social, and cultural backgrounds – differ from the forms of expression that are valorized by the culture of mathematics, which is itself dominated by white men. Knowing that students in this research discerned the dimensions of variation intended by instructors suggests that such examples could be effective at helping students develop a concept of mathematical writing that would be aligned with the instructor's values. However, if we want to broaden access to mathematical meaning and mathematical success, and if we intend to honor and respect the ways of writing and communicating that incorporate students' personal historical and cultural assets, then it will be necessary to find ways to support instructors to critically examine their assumptions about the nature of effective writing in mathematics and the critical features that differentiate effective writing from less effective writing.

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Female Students' Increased Belonging in Active Learning Calculus

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Women continue to be underrepresented in undergraduate science, technology, engineering, and mathematics (STEM) majors, presenting a lack of diversity that limits the field of scientific inquiry. Prior studies identify low sense of belonging as a key contributor to women's decisions to leave STEM. Incorporating active learning into introductory STEM courses offers promise in increasing female students' sense of belonging, and in turn, their persistence in STEM. The purpose of the current study is to explore to what female students in an active learning Calculus course attribute an increased sense of belonging. Findings suggest that while most female students did not experience increased feelings of belonging, 8% (N=3) did. Ongoing analysis is being conducted to investigate the experience of these three students more deeply.

Keywords: sense of belonging, active learning, undergraduate calculus, gender

Science, technology, engineering, and mathematics (STEM) fields continue to be male-dominated. Females are especially outnumbered in math-intensive fields such as computer science, engineering, and the physical sciences (Bui, 2014; Wang & Degol, 2017). This STEM gender gap is due at least in part to students' college major decisions (Carmichael, 2017; Chamberlain, 2017). Fewer females than males enter undergraduate STEM majors as freshmen (Eagan et al., 2016), and females are 1.5 times as likely as males to switch out of STEM majors (Chen, 2013; Ellis et al., 2016). Therefore, far more males are persisting and graduating with STEM degrees than females, resulting in a STEM workforce that lacks diversity.

Diversity benefits any field for a number of reasons. First, with more perspectives comes more and different approaches to solving problems (Gibbs, 2014). A diverse workforce opens up the possibility of identifying and tackling new types of problems that may not have been noticed by a more homogenous workforce. As Blickenstaff (2005) states, with a homogenous STEM workforce, "the field of scientific inquiry will be narrow and inbred" (p. 383). Additionally, research indicates that diverse groups are better at problem solving and hold more complex discussions than homogenous groups (Antonio et al., 2004; Hong & Page, 2004). STEM fields are missing perspective and ideas from talented females, and thus the problems and approaches to solving those problems remain male-centric.

One main reason females are fleeing STEM is because they feel a low sense of belonging (Seymour & Hunter, 2019; Shapiro & Sax, 2011). Sense of belonging is "one's personal belief that one is an accepted member of an academic community whose presence and contributions are valued" (Good et al., 2012, p. 701). While sense of belonging can be an issue for all students, research suggests that female students typically report a lower sense of belonging than their male classmates in STEM (Rainey et al., 2018; Shapiro & Sax, 2011).

Calculus can be an especially critical leak in the STEM pipeline and thus is a good site to study the role of females' sense of belonging and retention in STEM. Calculus is a required course for STEM majors and is often taken early in students' STEM education. In addition, it is often a prerequisite or corequisite for other STEM coursework, and so students who perform poorly may be prevented from taking other courses in their major. Consequently, Calculus is a key junction at which students, especially females, decide whether to persist in STEM (Ellis et al., 2016; Rasmussen et al., 2019; Seymour & Hunter, 2019).

One way that instructors might be able to support their students' sense of belonging is by incorporating active learning opportunities into the instruction. Active learning is a broad term to describe having students participate in the learning process instead of passively listening to an expert (Bonwell & Eison, 1991). Research suggests students typically prefer active learning to lecture, and active learning has been shown to benefit student performance and persistence more than lecture (Seymour & Hunter, 2019).

The current study is part of a larger study investigating connections between students' sense of belonging and the type of Calculus instruction they experience. In the current study, I address the following research questions: To what extent does female students' sense of belonging change over the course of one semester of participating in an active learning Calculus course? For those female students who report an increase in their sense of belonging, to what factors do they attribute this change? Feminist theorists argue that comparing experiences between genders tends to position the male experience as the "norm," however examining the female experience is important in its own right (Du Bois, 1983; Kitzinger & Wilkinson, 1997). Understanding factors that might boost female students' sense of belonging, and bringing those factors to the attention of Calculus instructors could provide female students with opportunities to further develop a sense of belonging and perhaps persist in STEM.

Related Literature and Theoretical Perspective

Sense of Belonging

One feels a sense of belonging when they feel connected to a particular environment, or feel accepted and appreciated by others in that environment (Rosenberg & McCullough, 1981). Strayhorn (2012) claims that sense of belonging can be so essential that one cannot complete other processes until this basic human need is satisfied. For students, this might mean they have trouble listening to a lecture or studying for a test without feeling that sense of belonging. Therefore, feeling or not feeling a sense of belonging in class can influence how students behave and perform in that class. Good et al. (2012) conceptualize sense of belonging as "one's personal belief that one is an accepted member of an academic community whose presence and contributions are valued" (p. 701). The current study adopts this definition of sense of belonging.

Good et al. (2012) quantified this construct by developing a Mathematical Sense of Belonging (MSoB) survey to measure students' sense of belonging in the mathematics community. After distributing the survey to 997 undergraduate Calculus students, a factor analysis revealed five distinct factors that make up students' sense of belonging. They are Acceptance (e.g., "I feel like I fit in"), Affect (e.g., "I feel tense"), Desire to Fade (e.g., "I wish I could fade into the background and not be noticed"), Trust (e.g., "I trust my instructors to be committed to helping me learn"), and Membership (e.g., "I feel like I am a part of the math community"). They found that the MSoB had good test-retest reliability and that Sense of Belonging to Math proved to be a significant predictor of students' intentions to pursue math in the future.

Research also suggests that one's sense of belonging is manipulable and can be impacted by what happens in the classroom. Hausmann and colleagues (2007, 2009) surveyed students on their sense of belonging to their institution at the beginning of their first semester, and the beginning and end of their second semester. In their 2007 study, Hausmann et al. found that instances of increased sense of belonging were associated with above average *academic integration* (Tinto, 1975), which is comprised of students' intellectual development and faculty concern for students' development. In fact, declining sense of belonging was attenuated by academic integration. Other studies confirm that *social integration* (Tinto, 1975), or interactions with peers and with faculty, has shown to support students' sense of belonging (Hausmann et al., 2007; Hoffman et al., 2003). Further, Hausmann et al.'s (2009) study revealed a link between social integration, sense of belonging, institutional commitment, and intention to persist. Finally, according to Anderman (2003), students' sense of belonging could be protected against decline by instructor practices that promote mutual respect and academic risk-taking. These studies suggest that sense of belonging is not a static construct and can potentially be impacted by how students perceive their classroom experience, specifically their academic and social integration, and the instruction they experience.

Active Learning

Active learning can be broadly thought of as any form of instruction that engages students in the learning process (Prince, 2004). More specifically, active learning is "the process of learning through activities and/or discussion in class, as opposed to passively listening to an expert" (Bonwell & Eison, 1991, p. iii). Some examples of instructional strategies that support active learning include having students work on problems in groups or individually, engaging students in class discussions, and soliciting student questions. Research indicates students who engage in active learning opportunities have higher levels of achievement, sense of mastery, and persistence than students without these opportunities (Freeman et al., 2014; Lahdenperä et al., 2019; Rasmussen et al., 2019). In fact, the President's Council of Advisors on Science and Technology (PCAST) 2012 report recommended implementing instruction that supports active learning to retain more undergraduate STEM students.

Incorporating active learning into Calculus instruction offers promise in providing opportunities for students to feel more socially and academically integrated in their Calculus course. First, instruction that supports active learning offers opportunities for students to interact with their classmates and with their instructor, thus creating potential for social integration. Academically, with more student-to-faculty interaction might come a stronger perception of faculty concern for student development. This, in conjunction with active learning's benefits for student learning and achievement, might leave students feeling more academically integrated as well. As prior research indicates, feeling socially and academically integrated is associated with feeling a stronger sense of belonging, and thus, incorporating active learning into Calculus could potentially support students' feelings of belonging early in their STEM education.

Finally, research suggests that active learning can be especially beneficial for female students (Rainey et al., 2019; Shapiro & Sax, 2011). Rainey et al. (2019) found that females who leave STEM reported experiencing lecture-based instruction while preferring active learning environments. Their results also linked experiences with active learning, perceptions of professor care, and sense of belonging. Those students who experienced more active learning had greater perceptions of professor care, which was in turn related to a greater sense of belonging in STEM. In addition, publicly sharing and critiquing student work has been shown to support self-efficacy and connect mathematical success with effort rather than innate talent, which is counter to patriarchal ideologies (Leyva et al., 2020). Thus, active learning might be especially important for female students' academic integration, and in turn could positively impact their sense of belonging.

Based on this literature review, I hypothesized that students who experienced active learning in their Calculus course would report an increase in their sense of belonging over the course of the semester. Further, I hypothesized that students who experienced an increase in sense of belonging would attribute the increase to both academic and social integration.

Methods and Data

Participants and Setting

This study was conducted at a mid-sized R1 research university in the mid-Atlantic region of the U.S. during the Fall 2020 semester. The university offers a two-semester Integrated Calculus course designed to incorporate frequent opportunities for students to engage in active learning, primarily small-group work and whole-class discussions. The first semester develops differential Calculus and the second semester (which is not the focus of this study) develops integral Calculus. Necessary precalculus topics are woven throughout each semester as needed. The course uses the Stewart, Clegg, and Watson (2021) textbook, as well as a second textbook specifically designed for integrating Calculus and Precalculus. The course is highly coordinated – in addition to common textbooks and exams, the instructors teach from a shared curriculum detailing which math problems to use each class and when group work will be employed. Thus, students' learning opportunities are virtually identical across different sections of this course. The students are typically freshmen considering a STEM major. In the Fall 2020 semester, each section was capped at about 50 students. Typically, this course is taught in a classroom furnished with circular tables to facilitate students working in small groups. However, due to the COVID-19 pandemic, this course was taught in a synchronous virtual format over Zoom. To maintain opportunities for group work, Zoom's breakout room functionality was used in most class meetings.

Participants were students enrolled in both sections (N=91) of this integrated course in the Fall 2020 semester. Because course rosters do not indicate students' gender, all students in the two sections were invited to participate in the study. These students received an email during Week 3 and Week 11 of the fourteen-week semester inviting them to participate by completing a survey about their experience in the course. For this study, only students who completed both surveys, were freshmen, and self-identified as female were considered. The total response rate of the survey was 88% (N=80), and of these respondents, 37 students self-identified as female.

Data Collection

The survey was administered twice – once in Week 3 (Survey 1) and again in Week 11 (Survey 2) of the semester – using Qualtrics, a web-based survey tool. To collect information on students' sense of belonging, Good et al.'s (2012) MSoB instrument was embedded in the survey. The MSoB consists of 30 Likert items asking students to indicate the extent to which they agree with statements about their feelings of belonging in the course on a scale of 1 (Strongly Disagree) to 8 (Strongly Agree). Recall that Good et al.'s (2012) factor analysis on the MSoB identified five factors of Acceptance, Affect, Desire to Fade, Trust, and Membership. This portion of the survey (i.e., sense of belonging and its associated factors) is the focus of the current study.

Female participants who exhibited significant change in their sense of belonging between Survey 1 and Survey 2 were invited to participate in one-on-one interviews which were conducted during Weeks 12 and 13. The purpose of the interviews was to identify factors to which the female students attributed the change in their sense of belonging. Six students were invited, three of whom had experienced a significant increase in sense of belonging. Of these three students, two agreed to be interviewed. Each interview lasted about 45 minutes, was conducted and video-recorded via Zoom, and was transcribed.

Data Analysis

To investigate whether female students' sense of belonging changed over the course of the semester, mean responses were calculated for sense of belonging and tested using IBM SPSS

Statistics 27 software. Dependent samples *t*-tests were used to determine any change in means between Survey 1 and Survey 2 with $p<0.05$. Effect size was calculated using Cohen's (1988) benchmarks for *d*. To inform interview participant selection, individual students' sense of belonging scores were compared between surveys using a dependent samples *t*-test with $p<0.05$ to determine a sample of students who showed a significant change in sense of belonging between the two surveys. In cases when sense of belonging increased significantly, dependent samples *t*-tests will be used to examine changes in factors associated with sense of belonging with $p<0.05$.

Analyzing the interview transcripts involved first chunking the transcripts by protocol question. To code the data, I read each transcript individually and developed an initial set of codes based on what the two female students said about their increased feelings of belonging. I am now at the stage of refining my codes and preparing to test for interrater reliability.

Preliminary Findings

There are two main results from the quantitative analysis. First, there was no significant change in female students' sense of belonging or its associated factors between Survey 1 and Survey 2, as shown in Table 1.

Table 1. Comparison of average scores (standard deviations) on the MSoB portion from Survey 1 and Survey 2.

Construct	Survey 1	Survey 2	$t(36)$
Sense of Belonging	5.965 (1.34)	6.945 (1.07)	0.703
Acceptance	6.171 (1.42)	7.189 (0.96)	0.377
Affect	5.796 (1.58)	6.479 (1.59)	0.484
Desire to Fade	5.375 (1.68)	6.284 (1.71)	0.601
Trust	6.392 (1.55)	7.466 (0.75)	1.297
Membership	5.915 (1.65)	7.291 (1.03)	0.904

Note: Since the results to the paired samples *t*-test were not significant, effect size is not shown.

Second, only 16% of female students (N=6) reported a significant change in sense of belonging (three increased, three decreased) as compared to the total sample average. Those who increased will be referred to as Annie (t=2.13, *df*[36], p=0.02), Carrie (t=1.71, *df*[36], p=0.04), and Maggie (t=1.78, *df*[36], p=0.04). Two of these three students (Annie and Carrie) agreed to participate in individual interviews. To understand their increased sense of belonging more deeply, quantitative and qualitative analyses are being conducted to determine contributors to these changes. Preliminary qualitative analyses indicate that professor concern for students' development might be especially influential in their increased sense of belonging.

Conclusions

This study investigated female students' sense of belonging in an integrated Calculus course that provides opportunities for students to engage in active learning, especially group work. Prior research indicates that sense of belonging is not static, but more work is needed to investigate contributors to increased feelings of belonging. Preliminary results indicate that while the whole sample of female students did not report a significant change in sense of belonging, certain students did experience a significant increase. Providing Calculus instructors with ways to positively influence their students' sense of belonging could potentially lead to more students persisting in STEM majors, and thus more students, especially female students, graduating with STEM degrees.

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Emerging Mathematics Education Researchers' Conception of Theory in Education Research

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While some faculty follow a traditional path into mathematics education through a graduate program or post-doctoral studies, some enter mathematics education research without formal training. These emerging mathematics education research faculty face unique challenges in setting up a research program. In this paper we explore the challenges that theory poses to emerging mathematics education researchers. We articulate three ways that emerging mathematics education researchers struggle with theory, and suggest that learning about theory and overcoming this struggle can be transformative to their work within and perspective on mathematics education.

Keywords: Professional-development, faculty, perspectives, theory

Introduction

Mathematics education research interests many faculty - both with and without formal backgrounds in it. Faculty may have studied mathematics education in graduate school or pursued post-doctoral work in the field before obtaining faculty positions. This prepares them for research through formal training in and practical exposure to ideas, theories, methods, and common practice within the field.

Many faculty, however, become interested in mathematics education after obtaining a faculty position (often, but not always a teaching-focused position) (Bush et al, 2017) sometimes after tenure and promotion. These faculty rarely have formal training in mathematics education, nor do they have much practical experience in the education research field (even as they may have varied experience as educators). As a result, they struggle to catch up with the field's understanding of theory, methods, and research practice. These emerging mathematics education faculty may have unique struggles as they attempt to set up a research program within the field. In this paper, we focus on a particular area of concern: theory. Despite attempts within the field to make theory more accessible (which we discuss more in the next section), theory remains a significant challenge for emerging mathematics education faculty.

Many emerging mathematics education faculty express confusion or worry around the complexity and usage of what researchers term theory, theoretical frameworks, analytical frameworks, and theoretical perspectives. In this paper, we discuss these terms and the literature surrounding them as part of our discussion of confusion among emerging education researchers. We present three ways in which participants in the Professional development for Emerging Education Researchers (PEER) program (Franklin et al, 2018) struggle with theory. For some faculty, learning about theory is transformative for their research within and perspectives on mathematics education.

Conceptual Framing and Literature

Cresswell and Cresswell (2018, p. 95) quoted Kerlinger to provide a definition of theory within social sciences as "a set of interrelated constructs, definitions, and propositions that presents a systematic view of phenomena by specifying relations among variables, with the purpose of explaining natural phenomena." As argued by Stinson (2020), our choices of theory and how we use it are strongly influenced by our philosophical perspectives and worldviews as researchers. Many have attempted to write concise introductions to particular theories and theoretical frameworks, as well as attempting to distinguish between their various uses and how they all fit into the broader literature (e.g. Simon, 2009, Spangler & Williams, 2019, Lester, 2005, Silver & Herbst, 2007). Others have made an effort to make theory more accessible to emerging education researchers (e.g., Doroudi (2021)). For example, Schoenfeld (2000) offers a practical set of standards with which to judge theories and results in mathematics education, grounded in the notion that research and theory in mathematics education is more similar to the physical and life sciences, as opposed to the proof as the standard in mathematics. Even as these authors work to provide detailed introductions to the theory of mathematics education research, they all recognize, as do we, that the roles theory plays in research can vary. The process of making decisions about what role theory plays in your research, and which theory or theories to use can be very confusing and intimidating to emerging mathematics education researchers.

The authors of this paper take a parallel process view of research, seeing the selection of theory, literature review, data analysis, writing, and formation and revision of claims as ongoing, overlapping parallel processes. Different parts of research are not conducted in separate stages, but occur together and inform each other. Theory is considered at all stages of a research project, and choices of theory and its place within a project are constantly deliberated, in line with Simon (2009). We also agree with Stinson (2020) that personal worldviews and philosophical alignment play an important role in the selection and interpretation of theory and data. As we present our findings here, it is important to note that the PEER program teaches theory as something that is inextricably attached to researchers' worldviews, personal and academic philosophies, and which influences and is influenced by their whole research programs.

Study Context: The PEER Program

This study was conducted as part of a grant for improving and expanding the Professional development for Emerging Education Researchers (PEER) program (Franklin et al, 2018). The PEER program brings together emerging education researchers at an extended, experiential workshop for intensive writing and thinking about research questions and research design. The program also covers selected topics in research based on participants' needs (e.g. workshop sessions focused on particular methods or theories, authorship, ethics, etc.). The bulk of the program centers on participants working together and sharing and refining research ideas together with the support of PEER coordinators. PEER facilitators intersperse groupwork activities with instruction and guided discussion around subject matter and research design.

Our data for this paper are drawn from participants in the PEER-Chicago 2021 workshop. This workshop occurred over Zoom through spring 2021 and was attended by 45 emerging mathematics and physics education researchers from a variety of research and teaching institutions across the country. The workshop consisted of a kickoff session, three two-hour sessions spread over 6 weeks, and then a three-day intensive at the end of June. Before the workshop we conducted pre-interviews with fourteen participants. During the workshop sessions all participants were encouraged to record their "burning questions" about workshop topics in a Padlet[1] document, which we collected. Finally, during post-interviews with 8 participants we followed up on their experiences during the workshop.

[1] See their website for more information https://padlet.com/.

Results

Using the data drawn from PEER we were able to identify three different ways in which participants are confused about theory. We also noted that learning about theory appears to be transformative for PEER participants. Our analysis is presented below.

Participants are confused about theory

In the pre-interviews, discussion of theory by participants was rare. Because the semi-structured interviews focused on participants' past experiences, we didn't initiate discussion of theory. Several participants discussed theory during their pre-interviews. It's worth noting that all participants had prior experiences in projects conducted by mathematics education researchers. Ryan and Lily[2] piqued our interest with their comments on theory.

In his pre-interview, Ryan stated: "[There are so many] different theoretical perspectives that one can adopt when you're looking at your data…" and went on to say: "Where you just kind of do what you know and if you don't know it, you can't do it." Ryan's statement was early confirmation that participants find theory overwhelming to confront. In particular, he struggled with finding theory that is a good fit for his project because there are so many kinds of theory**. If you don't know what you're looking for or where to look, good theory is hard to find.**

Ryan continues, asking for: "[A] handful of references that I can go to in the literature and read some more about over the next, you know, three or four months [would be very helpful]." This statement highlighted the value of linking Ryan to the community's conversation on theory. Many emerging mathematics education faculty expressed a similar sentiment to Ryan: having a few curated sources to look at would be highly valuable. Several examples of such sources can be found in the paper's citations and were provided to participants at the workshop.

References may not be sufficient, however. As Lily said about theory: "... [A]t the beginning, the whole idea of a theoretical framework was totally mind boggling to me." and "I was talking a lot with like my grad student friends at the time in sociology and things like this, and like that's their bread and butter". Lily highlighted the importance of having someone to hold a conversation with and bounce ideas off of when trying to understand theory. She also showed a second way in which our participants struggle with theory: **participants aren't sure what theory and all of the terminology surrounding it means within mathematics education.**

When the interviewers (in pre-interviews) explicitly asked participants about different parts of a research project/design in mathematics education research, theory rarely came up. Lily and Ryan provided us with our first glimpse into why interview participants so steadfastly avoided the topic: because it is overwhelming. The meaning of theory and the web of terminology surrounding theory in mathematics education can be very confusing to emerging education researchers. Secondly, even for those who have developed an understanding of what theory is in mathematics education, it can be very difficult to source theory and learn about new theories without guidance.

Echoing Lily and Ryan's challenges in the pre-interviews, participants expressed similar challenges in their Padlet questions during the workshop; however, the Padlet questions also illuminated a third kind of challenge. During the workshop, theory and its role in education research became an emergent topic in response to participants' questions and facilitators' observations of participants' ideas. This focus on theory afforded robust conversations and

[2] All names are pseudonyms chosen either by participants themselves or by researchers to reflect participants' pronouns and ethnicities.

elicited many confusions and concerns from participants, some of which were recorded anonymously in the "burning questions" Padlets.

Within the Padlet data, a common confusion was how and where to source theory. One Padlet question was “How much of your theories should be based on other work (lit review) vs. your own new ideas?” Participants are often uncertain about how their own ideas interact with theory in mathematics education research. They wonder whether theory comes from other published research or if they should develop it themselves. Others similarly wonder where theory should derive from, and how to appropriately build one’s own ideas into theory. One Padlet question said “How is it distinguishable from a lit review? Don't the theories come from published research?”, and another asked, “How does one *develop* a theory from data?” It is clear to our participants that theory must be generated by someone somewhere, however, participants are uncertain about community norms around who is allowed to generate theory and how. Where to source theory and how to do it appropriately is a stumbling block for participants.

Perhaps one of the most illuminating questions asked by a participant was “Does the "theory" for a paper necessarily need to be a complete ~theory~, or can it be a framework or even just a Frankenstein of ideas you were thinking about when looking at your data? Does every paper need a "theory"?” The first part of this comment echoes confusion that we had already heard about from Lily and Ryan. The asker is uncertain about what theory is, what theory means in mathematics education. They are also (as a consequence) uncertain about where it comes from, who makes it, and how. However, this participant is not merely confused about what theory means, or where it should be sourced. The second question here “Does every paper need a theory?” speaks to a deeper concern: what is the purpose of theory?

A participant asks this explicitly: “Why do you even need a theory? This theory stuff is super intimidating, what’s the best way to ease yourself into it?” **Our emerging education researchers struggle to understand the role of theory in our research.** What do we use it for?

The questions asked by participants during the workshop reinforced comments by Ryan and Lily. Participants are confused about where theory comes from, and what theory means. However, the Padlet questions also made clear a related confusion. Participants struggle to understand the role of theory in mathematics education research.

PEER can transform participants' thoughts about theory

After the PEER-Chicago workshop we conducted follow up interviews with 8 participants. These interviews focused on their experiences during the PEER workshop, and the development of their understanding and perceptions of education research as well as their professional identity. Several participants spoke about the how learning about theory during PEER-Chicago had impacted their understanding of education research.

In discussing learning about theory, Peter noted: "The idea of using part of this theoretical framework and combining it with that one, I'd never considered it before". Learning about theory at PEER transformed how Peter thought about implementing theories in his own research. Earlier in his interview he commented: "I know enough about theoretical frameworks to know that there are constructivist theoretical frameworks, and you know there are varieties of those, and then there are other theoretical frameworks which are not constructivist. And I can read about both of those, and I can see what I'm doing in each, and that just doesn't make sense to me." During PEER his thinking on theories moved away from an image of theories as mutually exclusive descriptions of the world to theories as potentially complementary descriptions of the world.

Another participant, Olivia discussed how learning about theory has transformed her view of mathematics education research: "Well I think the most pointed, that first reading specifically addressed the value of a more theoretical approach to math education, how that can inform a more practical approach." Olivia goes on to say: "I tended to come at things as I want to know if this works, I want to know how to tell if it's going to work. Which is implicitly dismissive of people who want to simply ask questions like, you know 'exactly what do students leave with from this particular approach to describing a logical construct?' and I am less dismissive off that [now]." Olivia has become more open to questions which aren't directly about measuring classroom success, and curriculum implementation.

Finally Penelope commented: "The theories, I still don't have a good handle on the theories, but I found that for me, [...] my focus is more like 'let's make this really good' versus some people who are like 'let's make theory really good' if that makes sense. And so I did get some understanding of where I am and why I'm there and being ok with why I'm there." Penelope's view of their own place in mathematics education research and the acceptability of what they do was transformed by taking the time to learn more about theory.

These comments from our post-workshop interviews suggest that for some participants learning about theory is transformative. It has a large impact on their views of mathematics education, and their understanding of how to conduct research. It's very exciting for us to see that learning about theory can have a deeper impact on participants than simply providing a theory paragraph in their paper.

Concluding Remarks

Theory is deeply important to math education research, informing research projects at all stages of development. A researcher's use of theory is impacted by their own worldviews, personal philosophies, and experience. For many emerging math ed researchers, theory is very intimidating, and difficult to approach or understand. In this paper we have explored several ways in which emerging math ed researchers participating in the PEER program struggle with theory. These researchers struggle with 1) understanding what theory means in math ed, 2) what role theory should play in their research, and 3) where and how to source theory.

We also found that after participating in PEER a number of our participants discussed how learning about theory has impacted their engagement with math education. While it was unsurprising that some participants discussed the application and use of theory in their research, their discussion was not limited to this. Some of our participants discuss an important and transformative effect on their personal identity or perception of math education research. Thus, we find that while emerging education researchers may struggle with theory, the process of learning about theory and attempting to overcome that barrier can be transformative.

Questions for the audience.

1. What does theory mean to you? How do you use theory in research?
2. Do our participants' journeys with conceptualizing theory resonate with your personal journey? With what you've noticed in your students / colleagues?
3. Do you have comments or suggestions for us as we extend this work and analysis?

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Investigating the Impact of Training in Metacognition on the Academic Success of a First-Year Student Enrolled in an Entry-Level College Algebra Course

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Research has shown that metacognition is related to mathematics performance and that explicit training in metacognition can be used to help students develop metacognitively. Studies also show that as metacognition develops, mathematical problem-solving improves. A large population of postsecondary students begin in remedial mathematics courses; institutions across the United States are wrestling with the best approach to educating these students. This paper reports on Julia, a student enrolled in an entry-level college algebra course, who participated in a 15-week training in metacognition designed to support her in her mathematics course. Training in metacognition, contextualized in mathematics content, may be a successful approach to supporting these students' academic success in their mathematics courses.

Keywords: metacognition, college algebra, remedial, self-regulated learning

Introduction

The importance of metacognition in the learning of mathematics has been well-established by researchers (Baten & Desoete, 2019; Schneider & Artelt, 2010; Schoenfeld, 1992; Veenman et al., 2006). Broadly, metacognition can be understood to mean "thinking about one's thinking," (p. 393, Dinsmore et al., 2008). Research suggests that metacognitive knowledge and skills do not always develop automatically in learners during cognitive activity and that explicit training can be used to help learners develop these knowledge and skills (de Jager et al., 2005). Desoete, et al. (2003) also found that as metacognitive knowledge and skills develop, mathematical problem solving improves. Veenman (2006) determined that, while both intelligence and metacognitive skills influenced mathematics performance, metacognition was a more significant predictor of mathematics learning performance in secondary school than intelligence was.

It is known that there is a large population of students who begin their undergraduate career in remedial or developmental mathematics courses. Chen (2016) reported that 59.3% of postsecondary students at 2-year institutions and 32.6% of postsecondary students at 4-year institutions who were beginning postsecondary students in 2003-2004 took "remedial math" at some point between 2003 and 2009. 51% and 41% of remedial coursetakers (of any discipline) do not complete their remedial coursework at 2- and 4-year institutions, respectively. (Chen, 2016). From this, it can be concluded that placement in a remedial mathematics course can significantly impact a student's ability to successfully complete the mathematics or quantitative reasoning requirements for their degree program; in fact, students who begin in remedial courses are less likely to earn a college degree (Chen, 2016). According to Chen (2016), "[r]emedial courses are generally associated with such terms as *developmental, remedial, precollegiate,* and *basic skills* in the course name and/or description," (p. iv). Chen includes a detailed definition of remedial math courses in an appendix of his report; included in this definition is "any mathematics course that deals with the topic of intermediate algebra, precollegiate algebra, elementary algebra, basic algebra, preparatory algebra and/or pre-algebra math." (Appendix D, p. 2).

The purpose of this study is to investigate the impact of training in metacognition, contextualized in mathematical content, on the academic success of a first-year student who was enrolled in an entry-level college algebra course that covered precollegiate algebra content. Reported here are preliminary results.

Review of Metacognition and Mathematics

Many researchers conceptualize metacognition as encompassing two domains: metacognitive knowledge and metacognitive skills (Baten et al., 2017). Metacognitive knowledge includes a person's knowledge of their own cognitive strengths and weaknesses, resources and strategies to address cognitive challenges, and *how* and *when* to use those resources and/or strategies. Metacognitive skills including planning, monitoring, control, and self-regulation before, during, or after a cognitive event. Research has shown that metacognitive skills play a larger role in learning than intellectual ability (Veenman et al., 2006). In particular, metacognition has been shown by several researchers to correlate with mathematical performance (Baten & Desoete, 2019; Schneider & Artelt, 2010). Pennequin, et al. (2010) found that third graders who participated in metacognitive training had significantly higher post-test metacognitive knowledge, metacognitive skills, and mathematical problem-solving scores. These researchers also examined the differential impact of this training on children with "low achievement" and "normal achievement" in mathematics; these groups were initially determined by their teacher and confirmed via a mathematical problem-solving pre-test. While all participants in the training improved in all three of their post-test scores, children who participated in the training and were in the "low-achievement" in mathematics group improved substantially more in their metacognitive knowledge, metacognitive skills, and mathematical problem-solving post-test scores when compared with children in the "normal achievement" in mathematics group.

While many students do spontaneously develop metacognitive skills, there is a large number of students who do not, either because they have not been presented with the opportunity or they have not seen a value in developing these skills (Veenman et al., 2006). Studies suggest that through metacognitive instruction, students can improve in both metacognition and in learning (Cornoldi et al., 2015; Donker et al., 2014; Pennequin et al., 2010; Schneider & Artelt, 2010). Veenman et al. (2006) claims that effective metacognitive instruction rests on three principles: "a) embedding metacognitive instruction in the content matter to ensure connectivity, b) informing learners about the usefulness of metacognitive activities …, and c) prolonged training to guarantee the smooth and maintained application of metacognitive activity," (p. 9). Thus, the skills and knowledge covered in a metacognitive training for students in a mathematics course should be integrated with mathematics and that course, rather than taught in isolation.

Theoretical Framing

The terms metacognition, self-regulation (SR), and self-regulated learning (SRL) are sometimes used interchangeably in literature (Lajoie, 2008). According to Lajoie (2008) there are, however, distinctions between these terms. Metacognition has been used more broadly to refer to "thinking about thinking" and is often conceptualized by metacognitive knowledge and metacognitive skills. SR encompasses metacognitive skills (not knowledge) and the interaction between a person, their environment, and their behavior. SRL integrates motivational and contextual factors with the cognitive domain. This study adapts Pintrich's (2004) framework for assessing motivation and SRL in college students for the purpose of investigating metacognition. Pintrich's original framework includes four areas for regulation: (1) cognition, (2) Motivation/Affect, (3) behavior, and (4) context. As this preliminary study is focused solely on metacognition, I have only included regulation of cognition in the adapted framework. Table 1 illustrates the adapted framework.

Table 1. Framework for metacognition; adapted from Pintrich, 2004.

	Area for regulation
Phases and relevant scales	Cognition
Phase 1 Forethought, planning, and activation	Target goal setting Prior content knowledge activation Metacognitive knowledge activation
Phase 2 Monitoring	Metacognitive awareness and monitoring of control
Phase 3 Control	Selection and adaptation of cognitive strategies for learning, thinking
Phase 4 Reaction and reflection	Cognitive judgments Attributions

Phase 1 includes targeted goal setting for cognitive tasks, as well as prior content knowledge activation, and metacognitive knowledge activation. Phase 2 involves monitoring one's progress on, understanding of, and knowledge for cognitive tasks. For example, this phase includes an individual's self-assessment of whether they understand graphing quadratic equations or not. In Phase 3, an individual chooses and implements cognitive strategies for learning or thinking. Phase 4 involves an individual's assessment of their performance of a task and their attributions for their performance. While the four phases that an individual would go through often occur in the order presented here, Pintrich argues that later phases can still occur if earlier phases do not happen. Pintrich (2004) also notes that most models for SRL allow for monitoring, control, and reaction processes to occur simultaneously as an individual's goals and plans often evolve based on feedback from these phases.

Methodology

Participants for this study were enrolled in an entry-level college algebra course at a regional public institution in the western United States in Fall 2019. Students were placed into this course via scores on an online mathematics skills test. Students who were intending to major in a discipline that required some form of calculus course and either 1) did not take this skills test or 2) scored below the first cutoff score were placed in this course. While students received baccalaureate credit for this course, this course did not satisfy their general education quantitative reasoning requirement. According to Chen's (2016) definition, even though this was a baccalaureate credit-bearing course, this course would be considered "remedial" as the content was pre-collegiate algebra.

In Fall 2019 four students currently enrolled in sections of the aforementioned college algebra course (three women, one man; enrolled in different sections) participated in a 15-week long mathematical metacognitive training, which met for 1 hour each week (except for two weeks), for a total of 13 one-hour meetings. These trainings were designed and implemented by the researcher. While some meetings did involve applying metacognitive skills and knowledge to specific algebra tasks, instruction was primarily focused on metacognitive skills and knowledge more broadly for academic success in a college algebra course. Table 2 contains an outline of what was covered during these trainings over the course of the semester.

Table 2. Outline of metacognitive training

Meetings	Activity
1-3	planning for regular studying academic support resources (on-campus tutoring, instructor's office hours, etc.) reflecting on past experiences/imagining future experiences with mathematics
4-13	Planning for exams discussing strategies for preparing for exams reflecting on exam preparation/strategies after students received their scores revising plans/goals/strategies for next exam practicing applying metacognitive strategies to sample problems on exam reviews

The last two weeks were spent preparing for the final and planning for the next mathematics course participants would be taking the following semester. Throughout the semester, metacognition was defined and discussed explicitly and students were asked to reflect on and participate in conversations about what they learned mathematically and metacognitively. Additionally, during Meeting 3, students were given "the locker problem" (Kimani et al., 2016) and asked during that meeting and subsequent meetings through Meeting 13, to apply various metacognitive strategies to this problem and share their experiences applying these strategies with the group.

Each student was given a journal and asked to write reflections to various prompts in their journals during meetings. Students kept the journals in between meetings for reference or for recording information. Data for this study includes video- and audio-recordings of the 13 training meetings, student journals, other student artefacts, such as worksheets and reflections, and student grade data. This preliminary paper reports on one participant, Julia, using artefacts from her journal and her grade data; Julia's journals were transcribed for analysis.

Using the four phases in the above framework as a priori codes, Julia's journal transcriptions were coded for evidence of regulation of cognition. Her final grade was also examined in the context of her journal data.

Preliminary Results

Presented below are excerpts supporting Julia's development of metacognition; for space and ease of reading, pictures of her journal are shown rather than transcriptions.

Figure 1 shows two entries from Julia's journal from Meeting 4. She lists strategies for preparing for exams (Phase 3) and monitors her understanding of course topics (Phase 2).

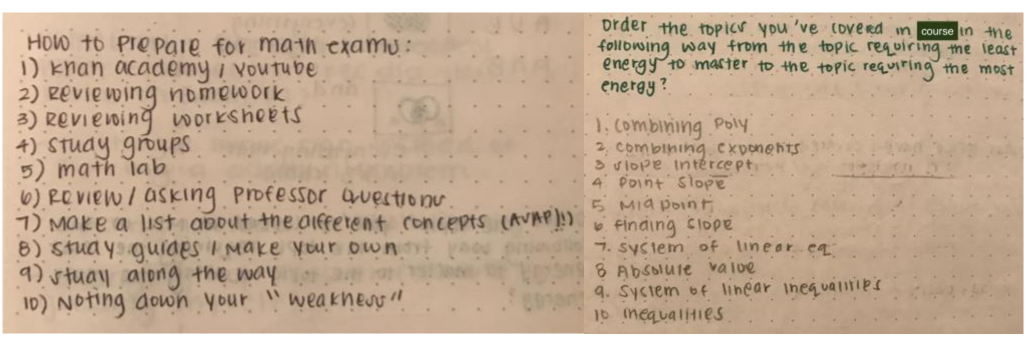

Figure 1. Evidence of Julia engaging in Phase 3 by listing cognitive strategies and Phase 2 by monitoring her understanding course topics

Figure 2 shows two entries from Meeting 6. Julia begins by reflecting (Phase 4) on her performance on and preparation for Exam 1. From this reflection, she makes plans (Phase 1) for preparing for future exams.

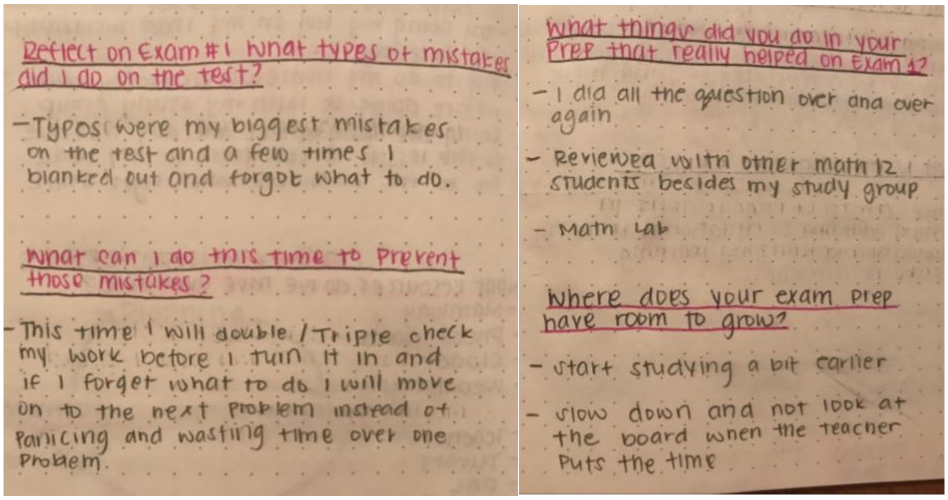

Figure 2. Evidence of Julia engaging in Phase 4 by reflecting on her preparation for Exam 1 and engaging in Phase 1 by making plans for future exams.

Discussion and Future Work

Explicit instruction in metacognition has been shown to improve metacognitive skills and knowledge and mathematical problem-solving skill (Pennequin et al., 2010). Julia's case suggests that metacognitive training may be an mechanism for increasing academic success for students enrolled in entry-level college mathematics courses. Data from Julia's journal indicates that Julia experienced all four phases of Pintrich's (2004) framework for regulation of cognition. Julia also noticed herself developing metacognitively. From her journal:

> I have noticed my ability to understand certain weaknesses and strengths. For instance I know when I am under any time pressure I tend to do worst on test and quizes *(sic)* but when I use different methods to take a test I do better.

Julia also earned an A in her college algebra course in Fall 2019. In response to the question, "What have you learned this semester in this training?" Julia wrote: "Not to focus on the clock (time pressure) by not looking or focusing on the time my scores have actually gone up. And being able to talk to the people in this group has helped." While it is not possible to show a causal relationship between the metacognitive training and her performance in her course with the data presented here, it is possible that the training contributed to her success in the course. In the future, I plan to analyze video- and audio-recordings collected from these trainings in order to more deeply understand how participation in this training impacted participants and their experience in their college algebra course. I also hope to collect new interviews with the participants in order to determine if this training had a long-term impact.

Intended Question for the Audience

A question that I would appreciate feedback on: If I am able to collect new interviews with these participants, how do I frame this training and its impact in relation to the disruptions from the COVID-19 pandemic?

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Conceptions of the Derivative: A Natural Language Processing Approach

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In this preliminary report, we present a novel approach to study student conceptions of the derivative at scale using automated conversation disentanglement and natural language processing (NLP). Using conversation data from an open-access, online mathematics tutoring platform, we use a preliminary process to find conversations pertaining to the derivative. In this report, we introduce two examples of how Balacheff and Gaudin's (2003) conception model will be used to label conversations with conceptions of the derivative. In future work, we aim to extend this work to techniques from machine learning to code the rest of the conversations and to find connections between different conceptions of the derivative.

Keywords: big data, calculus, derivative, machine learning, natural language processing

This study offers two analytic methods from machine learning, *automated conversation disentanglement and natural language processing (NLP),* which reveal the potential to study students' mathematical conceptions using text data. Addressing concerns that prior work on student conceptions has often relied on small samples of students, automated conversation disentanglement and NLP provide a way to analyze mathematical conversations at scale, allow for patterns in conversations to emerge, and can help identify relationships between emerging conceptions. Analysis of conversations from a tutoring platform focusing on calculus is used to exemplify the methods. This study has the potential to: (1) show how descriptive research on mathematical conceptions can be taken to scale; and (2) show how machine learning methods (NLP, more specifically) can play a role in mathematics education research.

Two main research questions emerge from this study:

1. What are the ways in which methods from machine learning can be amenable for studying student conceptions?
2. What are the conceptions of the derivative that students share in this tutoring platform?

Background

How Conceptions of the Derivative Have Previously Been Studied

The conceptions of the derivative remain important yet understudied. In a seminal report, Zandieh (2000) developed a theoretical framework for analyzing students' understanding of the derivative concept. This study used cognitive interviews with nine high school students to describe their understandings of the derivative. Many studies of the derivative concept follow this same methodological approach—studying a small number of students within the same course of study and asking them to perform a task or answer an interview question (see Aspinwall and Miller, 2001; Park, 2015; Vincent et al., 2015). As mathematics education researchers, our ability to collect large amounts of rich data has become easier in the technological age, and in turn, the methods we employ to analyze such data are becoming more advanced. Studies such as Althoff et al.'s (2016) have provided evidence of large-scale analysis of conversation data (in their case, text-message-based counseling conversations). Work of this type that combines the power of natural language processing with rich text-driven data sources provides a new opportunity for mathematics education researchers to grapple onto these techniques to study mathematics knowledge outside normal classroom settings. As described in the data section, we argue that the work of this study offers such an opportunity, in which student conceptions of the derivative can be studied within the context of a large online tutoring platform.

Theoretical Framework

Balacheff and Gaudin (2003; 2009) proposed the notion of conception as a local knowledge of a learner in a specific situation. Important to note here is that conception is not synonymous with *concept* or how one understands a concept. Rather, conceptions are the different thoughts that people can have about something. Further, it provides a way to describe conceptions by taking into consideration the interactions between students and the milieu. More formally, Balacheff and Gaudin (2003) call a *conception* C a quadruplet (P, R, L, Σ) in which:

- P refers to a set of problems that require particular use of a concept, and this family gives a conception its meaning and its usefulness;
- R refers the set of operators, tends to express all the techniques that enable the transformation of the problem;
- L refers to the way in which the problem is represented;
- Σ is a control structure, which validates the use of certain operations and representations to solve a particular problem. It could also be said to describe the components which support the monitoring of the equilibrium of the $[S \leftrightarrow M]$ system.

With this model, Balacheff and Gaudin (2003) create connections between abstract mathematics (concepts), what is taught (knowledge), and what is used and understood by subjects (conceptions).

Operationalizing the Conceptions of the Derivative

We operationalize Balacheff and Gaudin's (2003) theoretical framework in six related conceptions of the derivative. We summarize it in Table 1. The first five conceptions are identified in Zandieh (2000) through her interviews with students. These conceptions are also consistently described across the literature (as shown below in parentheticals next to each conception), as well as appear in conversations in our dataset. We expand Zandieh's framework with a sixth conception—graphical conception of the derivative—as it is discussed in the literature as well as appears in conversations in our dataset. In the next section, we discuss how we conceive of studying student conceptions and operationalizing them in our study.

Table 1. Conceptions of the Derivative

Conception	Main Citations
Taking Derivative Symbolically	Chappell & Killpatrick, 2003; Maciejewski & Star, 2016; Santos & Thomas, 2003
Derivative as a Slope	Christensen & Thompson; 2012; Park, 2013
Derivative as a Velocity	Bingolbali et al. 2007; Chappell & Killpatrick, 2003
Derivative as a Rate or Rate of Change	Bezuidenhout, 1998; Dreyfus, 1991; Orton, 1983; Park, 2012
Graphical Conception of the Derivative	Ubuz, 2007

Formal Definition or (limit of) Difference Quotient	Hähkiöniemi, 2005

Data and Methodology

Data

The data for this project comes from an open-access, online mathematics tutoring platform (MTP, hereafter). To organize the conversations, the platform organizes the text channels by school level—ranging from pre-university courses like precalculus, up through university courses like calculus, linear algebra, and abstract algebra. As we are interested in students' conceptions of the derivative, we have chosen to focus on conversation that happened in the calculus channel. Since 2015, there have been over 700,000 messages shared by thousands of users in the calculus channel.

Automated Conversation Disentanglement

The conversations that take place in the MTP happen within channels as part of one long-flowing chat log, with no indication of separate conversations. There are advantages and disadvantages to this, but one glaring disadvantage is that it is not clear how the conversations are divided up between participants. This is where the work of *automated conversation disentanglement* comes in. Conversation disentanglement enables the identification of separate conversations within a single stream of messages. Kummerfeld et al. (2019) have developed and released an annotated large-scale dataset of Internet Relay Chat (IRC) logs (77k+ messages), as well as code to reproduce their disentanglement experiments. The model considers the author, timestamp, and content of the message when deciding which conversation a message belongs to. We have converted the messages in the MTP to the same format as these IRC messages, and the first author is working with the Kummerfeld et al. (2019) team on finalizing the implementation of the disentanglement process.

Finding Useful Conversations

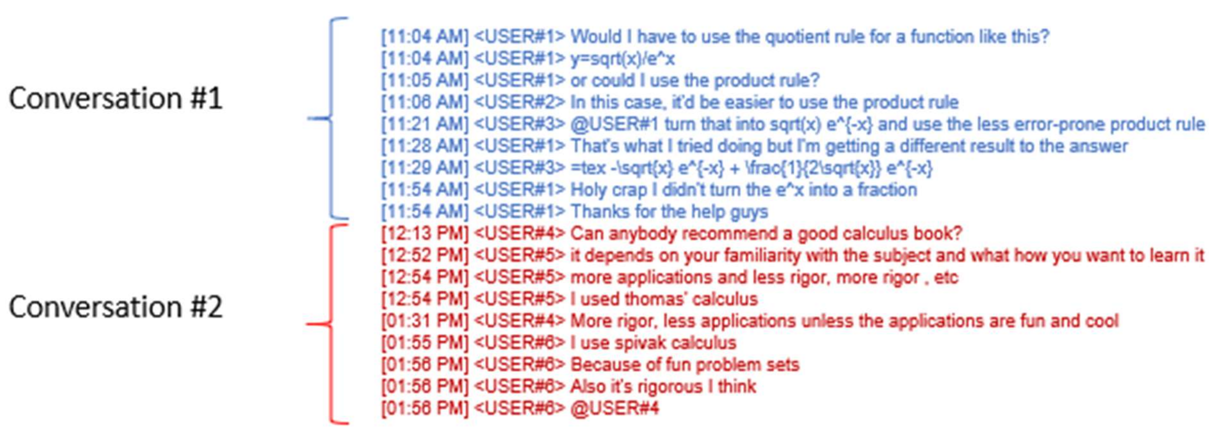

Figure 1. Example conversations from the platform.

Once the conversations are disentangled, it is important to determine which of the conversations pertain to conceptions of the derivative. One simple case could be to run a search for the term, 'derivative'. However, if we are to look Figure 1 above, we would want to filter

Conversation #1 as a relevant conversation and Conversation #2 as an irrelevant one. However, if we were to only search for the term 'derivative', Conversation #1 would be left out. This means there are several search terms associated with each conception of the derivative. For example, including terms such as 'product rule' and 'quotient rule' will likely pick up conversations where students are bringing in problems where they are asked to find the derivative of a product or quotient of two functions.

Once the disentanglement is finished, and assuming we have a large set of conversations associated with conceptions of the derivative, the next step will be to use Balacheff and Gaudin's (2003) conception model to code the conversations with the various conceptions of the derivative, as well as the problems, operators, and controls of each of those conceptions. In the next section, we provide a glimpse into how the coding process will look by discussing two examples that illustrate how the process will be operationalized. This coding process aims to train a machine learning model to classify a conversation based on the features of its text.

Preliminary Coding

Example 1

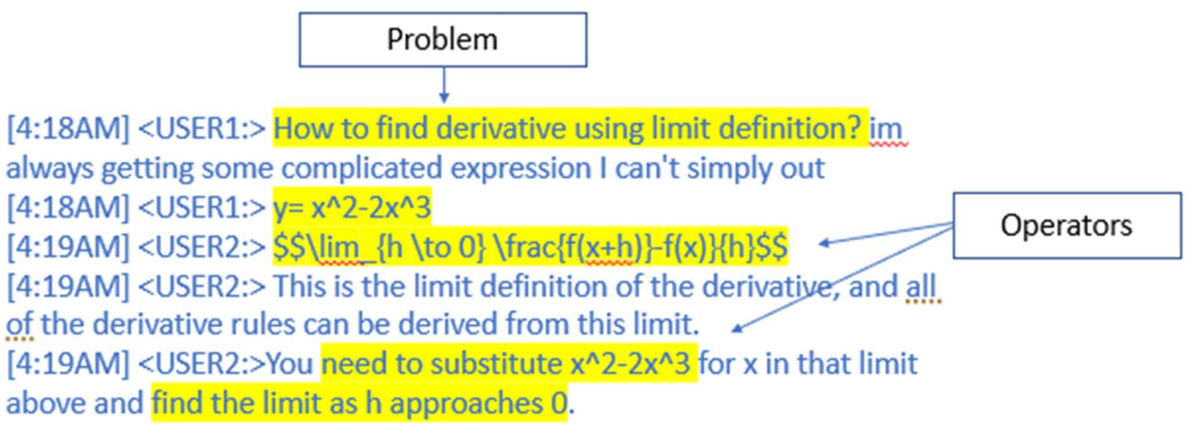

Figure 2. Example conversation between two users involving the problem of taking the finding the derivative of y=x^2-2x^3 using the limit definition of the derivative.

- Conception: Formal Definition or (limit of) Difference Quotient
- P (Problem(s)): Finding Derivative Using Limit Definition (tagged by "how to find derivative using limit definition")
- R (Operators): Plugging function into limit definition of derivative (tagged by "$$\lim_{h \to 0} \frac{f(x+h)}-f(x)}{h}$$, "need to substitute {function}", "find the limit as h approaches 0")
- *Σ* (Control Structure): No controls present in this conversation

Example 2

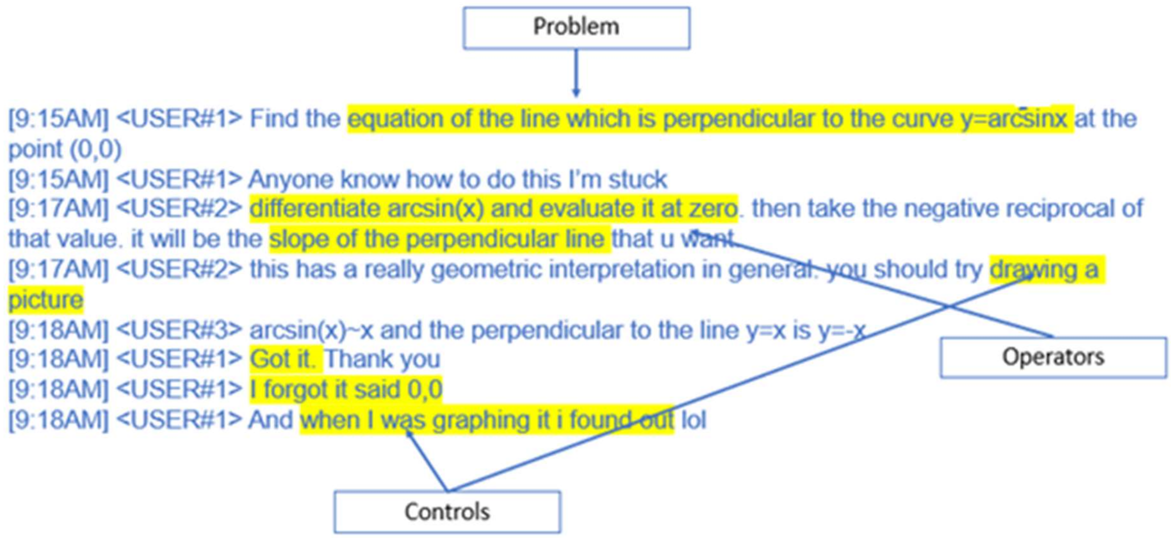

Figure 3. Example conversation between 3 users involving the problem of finding equation of a line perpendicular to a curve at a point.

- Conception: Derivative as a slope
- P (Problem(s)): Finding equation of line perpendicular to a curve at a point (tagged as “equation of the line which is perpendicular to the curve y=arcsinx”).
- R (Operators): Differentiate the function (tagged as “Differentiate arcsin(x)”) and evaluate the function at 0 (tagged as “evaluate it at zero”), take the negative reciprocal to find the slope (tagged as “negative reciprocal of that value”).
- Σ (Control Structure): In the example above, the tutor’s advice of drawing a picture, as well as the student remarking that they checked their work by graphing are indicators of the controls.

By implementing a coding strategy like the ones above, the first author aims to develop a training set (using a sample of a semester’s worth of conversation data pertaining to the derivative) to train a machine learning model to classify the remainder of the conversations from the dataset.

Concluding Thoughts

In this preliminary report, we have provided a glance into two ways methods from machine learning can be amenable for studying student conceptions: (1) given large scale, disentangled conversation data from the MTP, we aim to show how automated disentanglement models can be trained and then deployed to disentangle into separate conversations, and (2) using Balacheff and Gaudin’s (2003) conception model, we aim to show how student conceptions of the derivative arise through conversations in the MTP. As this study is in its preliminary stages as the work of the first author’s dissertation study, we look forward to engaging with the larger undergraduate mathematics education research community on how this study can be improved as the dissertation progresses. Additionally, as the coding is in its early stages, we anticipate that it might be possible for other conceptions of the derivative to show up in the data--- we are eager to have conversations about these additional conceptions with our audience.

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Investigating Individual and Collective Value within a Network of Communities of Practice

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This report examines the value-add of mathematics faculty participating in regional communities of practice (CoPs) embedded within a larger Math CoP network. The CoPs and the network are aimed at fostering the use of teaching with inquiry practices in undergraduate mathematics courses. We examine value found at the individual participant level as well as at the CoP level within the larger network. We present themes identified using Wenger, Traynor and de Laat's (2011) value framework to illustrate how individuals and CoPs find value within the network. In this paper we provide an initial look at how the network can support both regional communities and individual members in finding value and sustaining interaction within the CoP.

Keywords: Community of Practice, Networking, Inquiry, Active Learning

Supporting the achievement of secondary and post-secondary students in mathematics is an ongoing area of concern, particularly due to its impact on recruiting and retaining students in STEM fields (Fayer et al., 2017; Rose & Betts, 2001). One effort to address this is a focus by mathematics faculty on the use of evidence-based teaching practices, such as active learning, to support undergraduate student success. Despite evidence that lecture style instruction is not effective for many students, particularly those from underrepresented populations, it is still frequently used by many undergraduate STEM faculty (Jaworski & Gellert, 2011; Laursen et al., 2019; Stains et al., 2018). Faculty who attempt to employ evidence-based, effective teaching practices often find themselves doing so in isolation, which can make sustained implementation difficult (Banta, 2003). For the purpose of this paper, we use the term "teaching with inquiry" to encapsulate many forms of evidence-based teaching strategies that include: active learning, inquiry-based learning (IBL), project-based learning, problem-based learning, student-centered teaching, ambitious teaching, and team-based learning. Communities of practice (CoPs) provide one avenue of support for like-minded mathematics faculty as they pursue effective teaching and professional development. CoPs focused on teaching with inquiry are positioned to provide support systems by connecting faculty members with common goals and a vision for teaching.

Theoretical Framework

In order for sustainable change to occur, a number of systems need to be in place. Understanding these systems and their influence of the complex nature of teaching and teacher development can be a challenging undertaking. We frame this study by examining CoPs through the lens of value-add, or access to things like funding, resources, and specialized knowledge and expertise (Campbell, 2005), to investigate the ways in which individual members of CoPs and how the broader connecting network facilitates supporting this work.

A community of practice (CoP) is a group of people with common interests who engage in shared learning via ongoing, regular interactions (Lave & Wenger, 1991; Wenger-Trayner & Wenger-Trayner, 2015). CoPs are coordinated around individuals who collaborate toward a common goal, and typically have strong social bonds, active engagement, shared meaning and identities (Henri & Pudelko, 2003). CoPs are often ephemeral- they develop organically out of shared need, but often dissolve, particularly when they do not exist within a formal organization.

Kezar and Gehrke (2017) examined issues with the sustainability of CoPs, particularly those that exist outside the bounds of formal organizations. They found that, "To be sustainable, they [STEM CoPs] had to move from being a loose entity typical of networks and CoPs toward being more like an informal organization...this became a point of tension between becoming more sustainable and losing the value of the loose, informal peer community" (p. 345).

This study examines the Math CoP Network [blinded pseudonym], a National Science Foundation funded project (No. XXXXXXX), aimed at understanding how a network might support, sustain, and promote teaching with inquiry to mathematics faculty participants through engagement in regional CoP activities and interactions. Unique to the research on CoPs, this project examines a group of individual CoPs nested within a larger network. Engel and van Zee (2004) identify elements needed for a successful network, which include having a shared goal, common interests, added value and commitment, capacity to access and contribute to the network, and clarity in planning and management- all focuses of the CoP Network.

Value Framework for Examining CoPs

We utilize the value framework developed by Wenger et al. (2011) to position CoPs within "a dynamic process in which producing and applying knowledge are tightly intertwined and often indistinguishable" (Wenger et al., 2011, p. 21). As Figure 1 illustrates, the framework employs five cycles of value creation - immediate (in the moment resources, information, connections), potential (for the future), applied (tested implementation), realized (actualized implementation), and transformative (broader dissemination to others). In addition to the cycles, value at the CoP level is also supported by strategic value (clarity of the context and vision, ability to engage in strategic conversations) and enabling value (support processes that make network life possible).

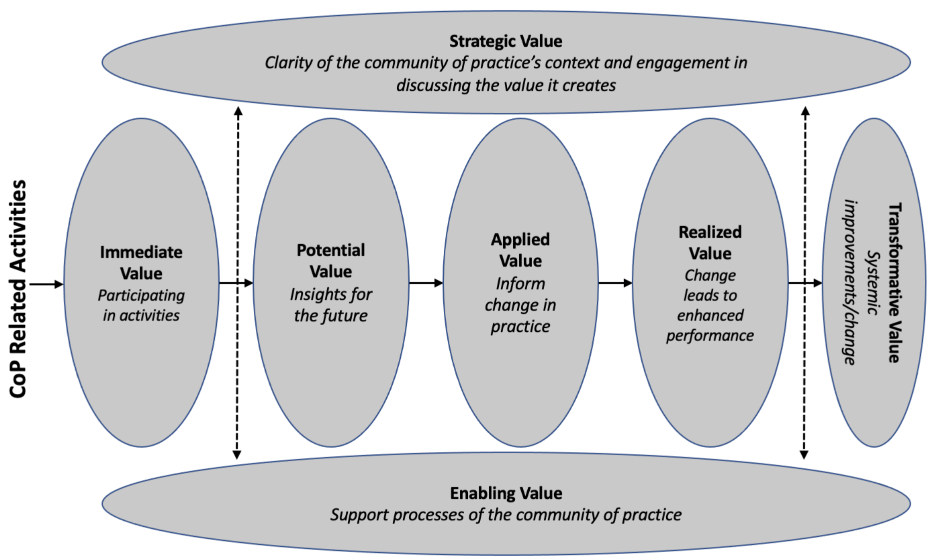

Figure 1. Adaptation of Wenger et al. (2011) Value Framework for CoPs.

Understanding the value-add of individual CoPs and the network that connects them will help identify concrete structures needed to create sustainability. Unpacking the types of value, along with the embedded systems and structures needed to facilitate this value, is therefore vital to supporting active, thriving CoPs. In the first year and a half of this project, our research team has gathered data to identify what individuals find valuable as they engage in activities within their regional CoPs. Moving into the second half of year two of the project, we have expanded our investigation to also examine the types of value regional CoPs find in the network. Therefore, the research questions for this study are: 1) Where do individuals find value participating in Network activities? 2) Where do regional CoPs find value within the larger Network? This preliminary data investigates these layers of value in isolation. In the future, we aim to integrate our findings to provide a more holistic view of how the Network of CoPs impact both individuals and institutions over time.

Methods

Data Collection and Analysis

We utilized two layers of data collection for this study. The first was survey data from faculty who engaged in one or more regional CoP activities between October 2019 to October 2020. Participants submitted 227 voluntary surveys, where 156 individuals provided identifiable information, representing 115 unique faculty responses. The survey consisted of open-ended responses where questions were aligned to gather evidence of specific value types. The second form of data collected was derived from 20, semi-structured CoP leader interviews representing eight regional CoPs (four regions original to the grant, and four added after year 1). The interview questions were framed around value in terms of structures and systems in place for the regional CoPs and the larger Network. Following the interviews, we compiled regional reports that summarized the key activities, strengths, areas of improvement, and future opportunities or threats of the CoP. Through a member checking process (Lincoln & Guba, 1985), each report was verified by interviewees to increase the trustworthiness and validity of the summary.

We adapted the value framework developed by Wenger et al. (2011) to inductively and deductively code open-ended survey responses for evidence of value across three value types: immediate, potential, and transformative. The survey is administered immediately after participation and therefore only these three value types are able to be coded since participants have not had the opportunity to apply their experiences to practice yet. Qualitative data analysis allowed us to interrogate “how people interpret...and attribute meaning to their experiences (Merriam, 2009, p. 5). We initially coded the data concurrently to calibrate and ensure intercoder reliability (Bradley et al., 2007; Krippendorff, 2004). In the second round of coding, we utilized *a priori* codes where we identified responses aligned with the Four Pillars (Laursen & Rasmussen, 2019). We determined that these codes did not completely encapsulate all participant responses, therefore we conducted a third round of descriptive coding (Miles et al., 2014; Saldaña, 2015) to identify additional emergent codes. For the leadership report summaries, we used an adapted value framework to deductively code, this time for the two contextual factors: strategic and enabling value. Using Nvivo software, we simultaneously dual coded (Miles et al., 2014) the summaries to reconcile evidence of value for both enacted and future (anticipated) strategic and enabling value.

Preliminary Results

Individual Participant Value

At the individual level, survey coding showed three types of individual value (immediate, potential, and transformative) in three, key areas: support with resources to improve practice, support through belief shifts in theory and practice, and support of a community of peers. These preliminary results provide baseline data for CoP members’ perceptions of value.

Participants often reported the value of resources to support their implementation of teaching with inquiry (TWI). Learning the "Basics of TWI" was coded 64 times within the 227 survey responses including comments like "Lots of tips and ideas for setting up my first IBL classroom" indicating immediate value. In terms of specific resources and content, technology integration and assessment strategies were frequently mentioned as resources that provided both immediate and potential value for participants. Particularly beginning in Spring 2020 at the onset of the pandemic, immediate and potential value coding became prominent.

Participants also highlighted the value of thinking about teaching using new methods that challenged traditional beliefs of what it means to teach mathematics. For example, participants identified new ideas, such as, "... making the environment conducive to feeling ok to take risks and make mistakes". Participants reflected on how their involvement in CoP activities helped them experience TWI from the student perspective. One participant noted, "The activity itself was not novel, but discussing the activity as a student was the real value. I don't have much opportunity to discuss higher mathematics with others." Participant comments such as this highlight that changing faculty beliefs often includes experiences learning with and from others.

A final and prominent theme from the data emerged around the value in being a part of a community of peers. Participants found immediate and potential value in support from "like-minded peers" where they could be vulnerable sharing their experiences. Although present in the survey results across the full year of data collection, the conditions created by the pandemic intensified the desire for peer support. Participants valued knowing others experienced similar struggles teaching online and commented, "it's a shared experience and a shared concern with other dedicated people." Another participant added, "This was so helpful to learn about strategies for implementing IBL online. I felt like in the winter I was teaching in a vacuum."

Collective Regional CoP Value

The first type of data examined what individuals found valuable participating in Network activities, which aligns to the horizontal value types in Figure 1. From the Network level, we aimed to understand the systems and structures supporting the regional CoPs and so took a broader view. To do this, we identified instances where leaders expressed enabling value (processes that make network life possible) as well as strategic value (promotion of the network's common vision and structures that can make each CoP sustainable long term) as illustrated in Figure 1. Table 1 below shows the coding counts for enabling and strategic value that have been enacted as well as opportunities for future value creation.

Table 1. Coding Counts for Strategic and Enabling Value

Regional COMMIT	Future Enabling Value	Enacted Enabling Value	Future Strategic Value	Enacted Strategic Value
New CoP 1	2	2	0	3
New CoP 2	5	4	2	0
New CoP 3	4	6	3	2
New CoP 4	3	3	1	4
Original CoP 1	4	8	2	1
Original CoP 2	2	13	4	0
Original CoP 3	5	3	3	1
Original CoP 4	2	6	3	1

The regional summaries are, for the most part, focused more on enabling value than strategic value. This perhaps is to be expected as regional CoP leaders spoke more often on what made their regions function rather than the larger network structures. Enabling value focused on the qualities of the leadership team, professional development offerings, and how CoPs are assessing their needs. Original CoPs identified enacted enabling value most often, whereas the majority of New CoPs saw this as a future opportunity. New CoPs are just beginning to create processes to organize leadership teams and consistently implement activities and regional events. They may not yet be positioned to identify features that move toward sustainability. Original CoPs have had more time to develop systems necessary to run a CoP that can sustain as new leaders transition in and out. Preliminary analysis indicates that leaders identified enabling value in having a CoP team composed of other "experienced leaders". Additionally, we found that collaboration was a consistent element of each CoP where participants and leaders are able to seek and share ideas, build relationships with like minded individuals, and through those relationships, develop rapport and trust to engage in vulnerable conversations. In terms of anticipated future opportunities for enabling value, nearly all regional leaders described goals of making connections and collaborating across CoPs, along with finding ways to expand and diversify their membership.

While less prevalent, strategic value was evident in all eight CoPs. All but two CoPs reported finding enacted strategic value in their region, and six of the eight CoPs identified more opportunities for future strategic value than enacted value. Three preliminary themes emerged from the enacted strategic value coding- that of the importance of hosting regular meetings and events within the Network, the role of building common knowledge/vision, and the importance of common structures (e.g., websites, listservs, onboarding tools) in maintaining a functioning Network. Similar to themes around future enabling value opportunities, future strategic value opportunities included devising structures to incentivize and recruit participants, refining communication methods, and identifying systematic ways to enhance cross-CoP collaboration (e.g., hosting a national networking event). This preliminary analysis suggests that CoPs are beginning to share systems and structures in addition to a common vision. However evidence of future strategic value indicates that the Network continues to have areas of improvement for long term sustainability.

Discussion and Questions about the Research

For the MathCoP Network to make intentional and meaningful steps toward sustainability, Network leaders and researchers must understand both individual and regional CoP needs. Thus far, our research has identified a number of individual and regional CoP supports where faculty found value as they work to implement TWI practices into their undergraduate mathematics courses. As we continue to gather data, we are focused on integrating potential value themes from the individual and regional levels to inform Network stability and structures. Our future research will focus on gathering additional, longitudinal data to examine trends over time. Our preliminary results lead to several questions for discussion: 1) Is the value framework an appropriate tool for measuring CoP supports and structures for long-term network sustainability? and 2) In addition to the analysis reported here, we utilize social network analysis to examine social capital and the overall health of the network. Are there additional data worth considering to understand the role of the network for maintaining sustainable CoPs?

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Engaging Problem Contexts in Calculus Textbooks

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Research has shown that a majority of students gain negative attitudes about mathematics as they progress through courses, but we have not explored carefully what might be the main source of these negative attitudes, the problems presented to students, and the ones they are asked to solve. We analyze contextualized problems from two popular textbooks to explore the prevalence of features of engaging problem contexts. Of the 7 research-based features, 2 were not found, and 2 more were found in less than 6% of problems. This is an early attempt at trying to measure features of engaging contexts building on students' perspectives, but not using students to judge individual problems.

Keywords: Interesting Problems, Student Attitudes, Engaging Problem Contexts, Calculus

Students often develop negative feelings towards mathematics, even students majoring in STEM disciplines (Bressoud, Mesa, & Rasmussen, 2015). Researchers have found two possible reasons that students develop these negative feelings. First, students do not see a connection between the mathematics they learn in class and the real world (Boaler, 2015; Boaler & Selling, 2017). Second, few problem contexts are engaging to students (Boaler, 2015; Crespo & Sinclair, 2008. Van den Heuvel-Panhuizen, 2003). Recent research has begun to look at what features of problem contexts help students to engage in the problem and find the problem motivating (Stark & Jones, 2020). However, we do not know to what extent these features are found in textbook problems, in-person instruction, or online instructional videos. Understanding the current state of problem contexts helps us to know how to improve these problem contexts to make them more motivating for students which, we hope, will turn the tide of negative attitudes of so many students. Additionally, there is has been little work on evaluating engaging contexts directly from textbooks so there are methodological issues to be explored as a basis for further research.

Literature Review

Many students see mathematics as hard, boring, and useless and stop taking mathematics classes as soon as possible (Osborne et al. 1997; Nardi & Steward, 2003). Since so much of a student's time associated with school mathematics is spent solving mathematics problems, students' views of mathematics may largely be developed from the kinds of problems they are asked to solve and how they work on those problems. However, relatively little research has explored problems and problem contexts from a student perspective compared to the central role that problems play on school mathematics and the extent that they shape students' attitudes. The work that has been done uses a variety of constructs, so it is hard to understand findings across studies. Some of the constructs used include *fun* (Brown et al., 2008); *challenge* and *control* (Bibby, 2008); *depersonalization* and *tedious* (Nardi & Steward, 2003); *realistic* (Blum, 1993; Boaler, 2015; Gravemeijer & Doorman, 1999); *importance* (NCTM, 2000); and *interest* and *enjoyment* (Schukajlow & Krug, 2014). Moreover, many of these don't focus on problems or problem context characteristics, but general experiences with school mathematics.

One study that did some consolidation and characterization of problem context characteristics is Stark and Jones (2020). Based on common and related constructs in the literature, they created an interview protocol to better understand what made a problem context engaging (their overarching construct). They interviewed 13 calculus students and found that engaging mathematics contexts came in two categories: realistic-and-motivating, and enjoyable-and-motivating. Students discussed features of the problems related to these two categories. We discuss these in detail as part of our theoretical framework.

Research Question

Since research is starting to develop an understanding of students' views of problems and problem contexts, we can begin to use these results as lenses to view the characteristics of problems that students typically encounter in their classes. Our main research question is: *What characteristics of engaging problem contexts do typical calculus problems present?*

One notable aspect of this question is that it has some methodological challenges, namely trying to validly measure or identify constructs/characteristics of problems that originally emerged from student perspectives.

Theoretical Framework

We largely built our ideas from the work of Stark & Jones (2020), using their framework and results. A diagram illustrating their framework and findings is found in Figure 1. The diagram illustrates the two flavors of engaging contexts and the six characteristics mentioned earlier during the discussion of this work in the literature review. We focused on looking at the six characteristics, but also added one more that emerged in our analysis: real-life narrative. Some problem contexts were based not on personal or absurd stories, but events that actually happened, often in a historical context. We included these as another characteristic in enjoyable-and-motivating since these were also stories.

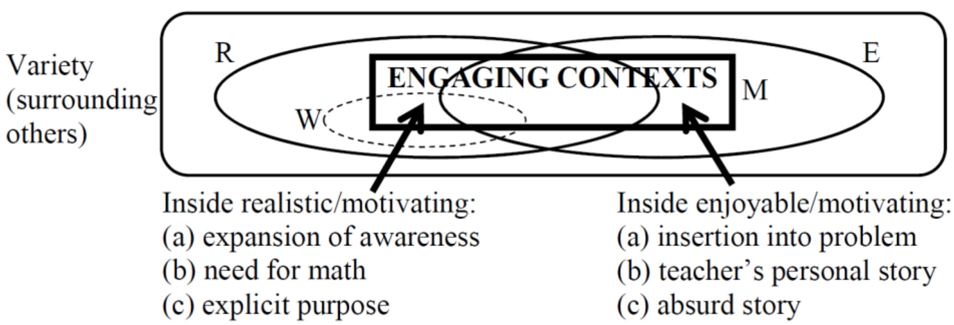

Figure 1. Framework for Engaging Contexts

The definition of each of these seven characteristics are:

- *Expansion of Awareness*: Context that permits students to see mathematics being used in a way they did not previously know it could be used.
- *Need for Math*: Context in which a student can perceive the mathematics as being a necessary tool that a person would use in such a situation.
- *Explicit Purpose*: A specific reason given as to why the answer to such a problem would be useful or beneficial.
- *Insertion into Problem*: A feature of a context that lends itself well to imagining oneself inside that context.
- *Teacher's/Author's Personal Story*: A story whose context comes from the context of the instructor's or writer's life.

- *Absurd Story*: A story whose context comes from a ridiculous situation that is often unbelievable.
- *Real-Life Narrative*: Historical account or description of a sequence of actual, specific events or people that prompts a mathematical problem.

Methods

We selected two popular college textbooks to analyze (Stewart, Clegg, & Watson, 2020; Briggs, 2018). For each book, we looked at all problems that were presented as part of the instructional and explanatory text. We did not analyze the exercises since, from our experience, students are only asked to work a small percentage of the contextualized problems in the exercise sets. Thus, we focused on the problems that students were most likely to see, either from reading the text or from being presented by the instructor from the textbook. Any problem that had a non-mathematical context was coded according to the seven characteristics. Explanations for coding some of the characteristics are below. Not all measures are discussed because we felt identifying the stories and real-life narrative were straightforward.

Expansion of Awareness

To make reasonable guesses about whether a guess would expand the awareness of a student we coded problems from 4 other textbooks. Two of the textbooks were at the college algebra/precalculus level. Two were at the intermediate algebra level. All problems in the expository section (not including the exercises) were coded for a main topic (population, finance, sports, projectile, manufacturing, chemistry, etc) and a subtopic. For example, in the finance category, there were subcategories such as sales, investing, salary, interest rates.

We used this baseline data from previous mathematics courses to make judgments about whether a problem context expanded the awareness of students. If a problem context in calculus was likely to be seen by students in a previous course, we thought it would not be an expansion of awareness. Our rule was that if the context in the calculus book was in 3 of the four lower-level mathematics books, then it was not an expansion of awareness. We developed two possible measures of expansion of awareness, one based on the main category and one based on the subcategory. The main category measure only looked at the main category of problem contexts in the calculus book and lower-level mathematics books, whereas the subcategory measure had to match the main category and the subcategory.

Need for Mathematics

A judgment was made about the problem context to decide if the answer to the question could be found without using the mathematics from that section of the textbook. For example, if a problem had some information that was to be used to find other information, but the information to find would have been easily collected when gathering the initial information, then it was coded as a “no” for needing mathematics. Similarly, if it was a common real-world situation where there were other readily available methods for working through the situation, then it was not considered as having needed the mathematics.

Explicit Purpose

We only considered problems having an explicit purpose if the reason for solving the problem was described in the problem. The purpose had to be a purpose outside of the world of mathematics. We considered reasons such as “minimizing cost” or “maximizing revenue” to be sufficient.

Insertion Into Problem

Our coding of this characteristic used three levels: Yes, No, and Possibly. We looked at 3 features of the problem that were inviting for students to imagine themselves in the context or that would make it easier for students to do so. If a problem had none of the three features, then we coded the context as a “no”, meaning it was not easy for students to insert themselves into a system. One feature of three received a code of “possibly” and two or more received a code of “yes”. The three features were: if a problem context was accompanied by a picture or drawing (not including a graph or basic geometric object), if the context was a common topic that a typical student most likely could have seen in person or through media (i.e. not specialized like a chemical reaction between two solutions and common like going to the store or the speed of a car), and if the context gives an explicit invitation to students to think of themselves performing the action or being in a certain situation.

Results

The first thing of note is that there is a relatively small number of contextualized problems in our data set. There were 174 total contextualized problems across both books, which include chapters often covered in both Calculus I and II. This was only about 20 percent of problems in each book. We sampled two chapters from each book, chapter 3 on Derivatives and chapter 6 on Applications of Integration. The percentages of contextualized problems for each chapter were 20.6% and 17.8% respectively for Stewart et al. (2020) and 16.9% and 22% respectively for Briggs (2018).

The percentage of problems with each characteristic is shown in Table 1. Unless specified otherwise, the percentage represents the percent of problems with that characteristic. We highlight just a few of the results. There are strong similarities between the two books, with most measures being within 5% of each other. The two measures of expansion of awareness are quite different from each other, so budding into lesser-known main categories only happens about 28% of the time but to new sub-categories about 74% of the time. About one-third of the problem contexts don’t require the mathematics that is the focus of the section. There are rarely explicit purposes given for solving the problems or drawn from real-life narratives, and no contexts were from personal or absurd stories.

Table 1. Percentage of problem contexts containing each feature.

Characteristic	Stewart, N=102	Briggs, N=72	Total, N=174
Expansion of Awareness (Main)	29.4%	26.4%	28.2%
Expansion of Aw. (Sub)	75.5%	70.8%	73.6%
Insertion	Y=32.4%, Y or P=84.4%	Y=36.1%, Y or P=100%	Y=33.9%, Y or P=90.8%
Need for Math	61.8%	66.6%	63.8%
Explicit Purpose	7.8%	2.7%	5.7%
Teacher/Author Story	0%	0%	0%
Absurd Story	0%	0%	0%
Real Life Narrative	4.9%	1.4%	3.4%

We searched for the problems with the most characteristics. Four problems, each from Stewart et al. (2020) had four features. One problem explained how Kepler had used data to describe the motion of the planets and led up to a problem to find the polar equation of the elliptical path of the earth around the sun. Another problem found was the pressure of water on the bottom of the swimming pool. The others were an optimization of a cylindrical can and the velocity of blood flowing through a blood vessel.

Discussion

Our results can help explain previous research findings. Many students find mathematics useless (Osborne et al., 1997). An analysis of these two calculus textbooks shows that only about one-fifth of the non-exercise problems contain a non-mathematical context. With about a third of these contextualized problems not actually requiring mathematics to solve and only about 6 percent done for an explicit purpose, it is little wonder why so many students don't view mathematics as useful.

The four contextualized problems with the most features did not strike us very compelling. For example, they don't seem to prompt the same engagement as the specific problems that students in Stark & Jones (2020) volunteered as engaging problems:

> One student explained, "My calculus teacher gave us this problem about his son who was trying to make the lantern from [the movie] Tangled. He needed to figure out how to maximize the surface area… Even though I don't like calculus, I thought, 'this is a really interesting problem. I kind of enjoy this.'" Another student brought up a situation where his teacher described a "potato gun" his family had and how someone thought it would shoot higher than it really could and ended up making a mess of a gazebo in a park. (p 6).

Both of these examples are personal stories from instructors. Such stories of using mathematics in a real-life context has the potential to capture many of the other characteristics of engaging problem contexts, and tie them together into a coherent experience. It is revealing to see such a lack of personal stories about mathematics used in calculus textbooks, even though calculus is widely used in applications and can be used to make sense of so many everyday situations. If the author, who presumably knows the mathematics well, is not using this mathematics in contextualized situations, how do we expect students to do so? Perhaps personal stories by instructors or authors could combat the depersonalized nature of learning mathematics (Nardi & Steward, 2003).

In this research, we faced a challenge of trying to measure characteristics of contexts that originated from individual student perspectives. We need to do more research to investigate their validity. We have also not privileged any one characteristic, or combination of characteristics, of engaging contexts, but it is reasonable to assume that these characteristics do not have the same effect in making the context engaging. For example, based on the previous paragraph we would hypothesize that a personal story may be a more compelling characteristic (or combination of characteristics) than others.

Questions for Audience

The most questionable measures were for *expansion of awareness* and *insertion into the problem*. How do these measures come across to you? Do you have other suggestions on measuring these without asking students directly?

We plan on doing similar analyses on other instructional materials, such as available class videos (MIT OpenCourseWare, for example, or our own classes) to measure the extent there are engaging problems in classes. Are there other places that would be interesting to know about the problems students experience?

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Justifications of Justifications: Episodes of Meta-theoretic Discussion in Class

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Inspired by Brown's (2018) work assessing the role of a meta-theory in student conviction from indirect proofs, we explore the negotiation of meta-theoretic ideas in classroom discussions of proofs. We use Toulmin analysis to identify episodes in which meta-theoretic topics become overt, and then we present preliminary analysis of themes and patterns in these episodes.

Keywords: meta-theory, proof, Toulmin analysis, geometry

Brown (2018) gives an overview of four hypotheses for students' supposed dislike of or lack of conviction from indirect proofs. The constructive hypothesis (Leron, 1985) asserts that learners construct a mental entity that corresponds to a mathematical object or its symbols on a page and that this construction is harder for indirect proofs because the mental objects cannot be built up directly. The socio-accultural hypothesis (Brown, 2018) asserts that proving is a social practice in a community, a community that has many artifacts that communicate to learners that indirect proofs should be less convincing. And the consistency hypothesis (Sierpinska, 2007) asserts that theoretical thinking requires learners to be sensitive to coherence of the whole system, coherence that is at odds with the semantic interpretation of statements in a proof that uses contradiction. The metatheoretical hypothesis (Antonini & Mariotti, 2008) asserts that mathematical theorems are situated in a reference theory of logic and that indirect proofs are distinctive in requiring learners to use metatheorems in this reference theory, such as the equivalence of a statement and its contrapositive.

During the discussion after Brown's 2016 CRUME presentation about these ideas, it was suggested that only indirect proofs require metatheorems. At that time, the first author hypothesized instead that all proofs require metatheorems but that discussion of this meta layer is suppressed in direct proofs because the structure of these metatheorems is taken as default and normative. Upon further reflection, the first author realized that they taught a course that emphasized epistemological issues in mathematical proof and hence was a promising place to look for examples of students negotiating meanings of metatheorems in both direct and indirect proofs. Returning to Brown's work (2018) more recently while exploring recordings from this course, we realized that the themes from all four hypotheses were overt topics of discussion in this course, a meta-discourse about mathematics. This brings us to our research questions:

1. What is the content of the meta-discourse in this course?
2. How are students negotiating the meaning of the content of this meta-discourse?

Theoretical Perspective

Antonini & Mariotti (2008) model indirect proof as follows. In general, a theorem is a triplet of a statement S, a proof P, and a reference theory T (S,P,T). An Indirect proof of a principal statement S is instead direct proof, C, of a secondary statement S* (such as the contrapositive of S); a meta-theorem of the statement MS=S->S*, a meta-proof MP, and a meta-theory MT; and a principal theorem consisting of S, an indirect proof of S, and both the theory T and metatheory MT. Importantly, the theory T for (S*,C,T) is a theory for a mathematical domain, perhaps an axiomatic framing of Geometry or Algebra, and the meta-theory for (MS,MP,MT) is usually classical logic. This last point is key for how we will generalize Antonini & Mariotti's model for our theoretical perspective: some justifications in this classroom will use a theory of geometry and others will use a theory of mathematics. This analysis will focus on meta-statements and their associated meta-justifications that use a theory of mathematics.

Because our focus is on justifications and the associated theory, we will use Toulmin's (1969) model of argumentation. In this model, an argument consists primarily of a statement that is being asserted, called a *claim,* previously established information, called *data*, and a general principle that explains why the data necessitate the claim, called a *warrant*. For our purposes, an argument may also include support for the warrant, called *backing*, and expressions of concern about the data or the ways that the warrant is applied, called *qualifiers* or *rebuttals*. Building on Antonini & Mariotti's model, within the Toulmin analysis of a justification of a theorem, we can separate individual argumentation triples into the domain and meta-layers based on whether the warrants are in the theory or meta-theory. We also anticipate that qualifiers can require the discussion of the justification to shift from the domain to meta-layer.

Our first goal is to describe the content of the meta-layer discussions. We are sensitized to some potential themes in this layer by our teaching and learning experiences, but we are building a grounded theory (Strauss & Corbin, 1994) from the data and do not have an a priori framework for this content. We will look for themes drawn from Brown's (2018) four hypotheses, but we do not assume that these themes would constitute a framework to describe all of the content of this layer.

Our second goal is to notice patterns in learners' negotiations of the meaning of the meta-layer content in course discussions. The emergent perspective (Yackel & Cobb, 1996) frames this negotiation as happening in the reciprocally related sociological and psychological realm. In particular, we expect to see evidence of the class negotiating and eventually establishing sociomathematical norms whose content refers to the meta-layer of the class discussions. We follow Yackel et al. (2000) by using two kinds of changes in the role of items in the Toulmin analysis across time as evidence of sociomathematical norms. When a claim initially needs an explicit warrant but then later that claim is accepted without warrant, it has become normative; similarly, when a statement that is initially a claim that requires a warrant and later is accepted as a warrant, it has become normative.

Methods

Data from this course are taken from video recordings of a full course in college geometry. Details of the course structures can be found at (Author, YEAR). There were 11 students enrolled in the course. All were junior mathematics majors, with 10 on the track for secondary mathematics education and one who identified primarily as a chemistry major. All students consented to the videos being used for research purposes, and the data collection was approved by the IRB for the institution hosting the course, which was the first author's institution at the time.

These recordings were made for another, related project before the generation of the hypotheses and research questions of this paper. As a result, these data represent a convenient source that was expected to allow the researchers access to the phenomena in question, not a representative sample or otherwise general context that would support generalizations to other contexts.

The entire course was recorded from a single camera, including both small group work and full class congress presentations and discussions. The congress phases in which the class was discussing theorems and proofs was transcribed by an external transcription service, and then the second author corrected these transcripts using her greater mathematical expertise. These transcripts represent more than 10 hours of group discussion of proofs across more than 15 class sessions.

Both authors independently analyzed the transcripts of these discussions using Toulmin's model of argumentation. We then came together to compare our analyses. We do not believe that it is possible determine a "correct" Toulmin analysis of an utterance in many situations; markers of a speaker's intention are often suppressed, different listeners can have different interpretations in the moment, and there may be a sustained difference between individual understandings and the ones that are taken as shared based on the group discourse. The research discussions of these analyses therefore focused on whether the analyses were consistent with the transcript data, and when multiple consistent interpretations were generated, we kept them both for the next phase of analysis.

We used these Toulmin analyses to create a corpus of claim-data-warrant triples that refer to the meta-theory. We added to this corpus any associated qualifiers, rebuttals, or other transcript elements outside the Toulmin analysis, maintaining the labels for their role in the Toulmin analyses. Then we began coding this corpus. Initial passes involved open coding, sensitized by the themes from our theoretical perspective. Subsequent passes involved attempts to define and organize themes around axial codes. This paper is submitted as a preliminary report because this work is ongoing.

Results

This section will give examples of episodes in which the meta-layer became overt in discussion that will illuminate themes in our analysis.

Example 1: Angle-Angle-Side

This first example comes from a student presentation that Angle-Angle-Side is a congruence theorem.

> *We were going to begin ... because we're trying to say that these 2 triangles are congruent, and the only ways that we know thus far that triangles are congruent is by side-side-side or side-angle-side, so we want to seek a contradiction, so we're going to assume that segment AD is not congruent to ED. OK, well, see, when I did [the immediately previous proof of theorem] 44 [Angle-Side-Angle] I then said that WOLOG [with-out loss of generality] one has to be longer than the other.*

The presenter asserts that there are only two results that could conclude that triangles are congruent. Initially, statements like this are prompted by the instructor, but in this presentation, the student makes this statement without any prompting, and it is accepted without issue by her peers, indicating that it is a normative form of reasoning by this point in the course. This statement is a claim about the possible warrants in the proof, so we interpret it as a claim in the meta-theory: if the proof is going to be possible in the current axiomatic system (base theory), it will need to use one of two previous results, which happen to be axioms. Significantly, this move requires the presenter to assess the entire set of accepted theorems in the base theory, which is an example of Sierpinska's (2007) theoretical thinking about the coherence of a system; the presenter then uses this coherence to explain why the proof will "seek a contradiction".

WOLOG?

Example 2: Hypotenuse-Leg

This second example comes from a student presentation that two right triangles with a corresponding congruent leg and hypotenuse are congruent. In the class theory, a right angle is defined as an angle that is congruent to its supplements. Significantly, the students believe that all right angles are congruent, but it has been established that this is not yet a result in the accepted axiomatic system for the class. The proof proceeds by mirroring each triangle across the other leg to produce two larger triangles that are provably congruent.

> *So it's kind of like the first phase. So these two are congruent here. We don't yet know that this side is congruent to that side, but we still have reflexivity, and we also know that for the same reasons that this angle is congruent to that one. So then by SAS again, these two angles, or triangles are also congruent. So because these two triangles are congruent now, we know that by CPCFC, we know that these two sides are also congruent, so they can share a hash. And because this is congruent to that, which is congruent to that, which this side is congruent to this because of CPCFC, and we can add that mark there.*

This presenter makes an interesting move asserting that we "don't yet know" a congruence, and she goes on to repeatedly use the word "know" to describe the results of her sequence of claims and warrants. Viewed through the lens of the constructive hypothesis, this presenter appears to be holding a mental entity related to these congruent triangles. In particular, she appears to hold two versions of that entity, one that exists for her and one that is being constructed toward her personal version in public using only collectively accepted pieces. This move is also normative in this class at this point in the semester. Many of the earlier discussions about diagrams invoked the idea of drawing a generic object that avoids special properties or intentionally makes the target congruences look false. For example, the class proved that any angle that is congruent to a right angle is a right angle using an image of two angles, each with their supplements, only one of which appeared to be a pair of congruent supplements.

Example 3: Squares

In end-of-term reflections, the students identified the day in which they attempted to construct a square as the most memorable. Squares cannot be constructed in neutral geometry. This episode comes after students have identified unwarranted claims in all of their attempts to justify their constructions of a square, and those constructions were implemented in NonEuclid, a dynamic construction environment for the Poincaré disk model of hyperbolic plane geometry.

> *Instructor*: Yes, there are no squares here. Not only is this not working, there are no squares here. What does this mean about what must happen next? That's a more philosophical question.
>
> *Student 1*: You need like more stuff, like more theorems for it.
>
> *Instructor*: Not theorems, axioms. We need a new axiom. We need a new thing that says [Geogebra and NonEuclid] are both examples of what we currently have but it's too general. We need to specify more.
>
> *Student 2*: If we had [theorem] 60 before [this exercise] 58, could we do it?
>
> *Instructor*: Yes, but the thing that happens between 58 and 60 is?
>
> *Student 2*: Axiom Five.
>
> *Instructor*: Axiom Five. Axioms in general should seem like bonfires. They should attract your attention like you're a moth, like woo, an axiom. One of the things I really like about

> this book is that it waits to give these axioms. This would not be clear, this sort of intellectual hygiene the idea that we haven't used, we haven't used it, now we definitely believe that we need a new axiom. That would not have been clear if we stated everything in the beginning. We'll use these words again later but what we've doing up to now is what's called neutral geometry. Basically, both these examples follow it and we're about to move into actual Euclidean geometry.

In this episode, the instructor overtly shifts the discourse from the base theory into a discussion of why and how that base theory will need to be changed in order for squares to be constructible in that system. Two different students engage in this discussion explicitly suggesting that if the class could add a tool to the current theory, they could prove the result. This episode includes the instructor voice, so the evidence of student negotiating is not clear, but it is a very clear moment of discussing ideas outside of the base theory in class. The first author, as instructor, intended episodes about this topic to help learners view axioms as artifacts of human choices and thereby be acculturated into a way of viewing axioms like that held by many professional mathematicians.

Discussion

We see evidence of meta discussions of the elements of Toulmin's model in the transcripts. Example 1 includes a meta discussion of the warrants that need to be present in a proof in the base theory. In many examples, this appeared as foreshadowing about a warrant or claim that a presenter asserted would need to appear later in a proof. Example 2 includes meta discussion of data: being careful to separate what is believed from what is known. Perhaps the role of diagrams is distractive in a straight-edge and compass axiomatic system for constructing geometry. Not shown in these examples, the transcript includes discussions of how a given claim did not contribute to the proof. Example 3 can be interpreted as a discussion of qualifiers or rebuttals that steps outside of the base theory and into a discussion of changing that theory. Also not shown in these examples, the transcript includes multiple discussions in which students make claims about parallels between the structures of different proofs. In particular, they overtly negotiate about the meanings of "without loss of generality" and "similarly" in proofs.

Many of the meta-layer discussions were initiated by an instructor qualifier or rebuttal, who regularly problematized meta-layer ideas that were otherwise implicit in the discussion. Future work will explore instructor moves that supported students in engaging with these concerns and negotiating them as individuals and a group until their resolutions became sociomathematical norms in the classroom.

Finally, a metaphor emerged through our analysis that models the classroom discourse by tying together individual thinking, collective warranting, and classroom facilitation. Each individual has a personal transcript that records their thinking, some of it in pen and some more tentatively in pencil. The class as a whole also has an official transcript as well as an open transcript for what has been said out loud, with entries that can also be written tentatively or permanently. The function of congress discussions is to move ideas from individual transcripts into the collective open transcript and to legislate as the assembled congress about which ideas get moved into the official transcript and why. The content of these transcripts lives in the base theory, and discussions that look at the structure of the transcripts or at the processes for working with the transcripts live in the meta theory.

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Can discussion boards disrupt gendered and racialized discussion patterns in math classes?

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Social Network Analysis is a method to analyze individuals' social accessibility and power. We adapt it to change inequitable issues in STEM postsecondary education. Equity issues in mathematics education, such as underrepresented women and racial disparities, are prevalent. With the social capital perspective, we investigate the demographic characteristics of influential students and their social networks. Seventeen participants are undergraduate students in an inquiry-oriented linear algebra course. The number of nominations on discussion boards as "Shout-out" is data to measure influence and map the social network. By analyzing data with UCINET, we found that (1) the most influential students are non-White males and the principal components of the network are male-dominant, and (2) there is a female-dominant small cluster and female students have reciprocal networks. This study suggests further discussions of (1) how discussion boards position students with the social capital perspective and (2) intersectionality, especially for women of color.

Keywords: Social Network Analysis, Equity, Classroom Discussions, Discussion Boards, Linear Algebra

Introduction

Social Network Analysis (SNA) connotates analysis with the perspective that "individuals are tied to one another by invisible bonds which are knitted together in a criss-cross mesh of connections" (Scott, 1988, p. 109). Also, SNA has been used in various areas such as social mobility, corporate power, and class structure (Scott, 1988). Henderson et al. (2018) suggested using SNA for "change" Science, Technology, Engineering, and Mathematics (STEM) fields in higher education. Specifically, Henderson et al. (2018) argue to improve the dearth of historically underrepresented groups in STEM by using SNA. As a tool and theory, SNA can facilitate uncovering social structure, analyzing engagement of targeted participants, and supporting quality teaching with postsecondary education reform efforts. Thus, we speculate to examine a link from SNA to ways of analyzing student interactions in classrooms and considering issues of equity. On the other hand, Linear Algebra is one of the most important and core courses for STEM undergraduate students. Thus, our assumption is that students' (in)equitable experiences especially regarding race and gender in a linear algebra class are associated with persistence and/or academic achievement. This study aims to answer the following research question: What are the characteristics of influential students, as identified by student nominations, in discussion board posts—with regard to demographic characteristics, especially race and gender in a linear algebra class?

Literature Review

In mathematics education, "equity issues" are a hot topic (Gutiérrez, 2009), especially in terms of race and gender. According to Borum and Walker (2012), "Mathematics is historically a White male-dominated field, so the norms or standards created to center on the ideologies of that specific group" (p. 374). Also, undergraduate mathematics classes are gendered and racialized spaces in general (Leyva et al., 2021).

Women are underrepresented in STEM (Ceci & Williams, 2007; Hill et al., 2010) and it has been an issue especially in higher education (Ong et al., 2011). Around 50% of the U.S. Bachelor's degrees and Master's degrees in mathematics were earned by women in 2014, but only 28.9% of the population of the U.S. doctorate degrees are women (National Science Foundation, 2017). Women keep disappeared through the higher level of education, and this implies that women in mathematics have intensively and gradually underrepresented experiences. This is explained by the phrase 'a leaky pipeline', which refers to "the loss of capable women from more senior roles in STEM disciplines" (Resmini, 2016, p.3533). Also, STEM fields have "chilly" social climates for female students, which means unwelcoming and hostile to women (Ferreira, 2002). Still, sexism in STEM majors exists, and it can be either explicit and flagrant but also implicit and subtle (Ernest et al., 2019). In particular, it has been documented that women sometimes speak less, especially in a public place in inquiry-oriented and/or discussion-based mathematics classes (Leyva et al., 2021).

Furthermore, in terms of race, mathematics is White and Asian dominant. Also, historically STEM is the white-dominant field, so racial disparity is also a crucial challenge (Lee et al., 2020A). As compared to white male students, the women and/or non-white students leave more from STEM majors (Kokkelenberga & Sinha, 2010). Plus, the hierarchical shapes by stereotypes of Black and Latin college students in STEM are perceived to have a lack of innate ability in their major (McGee, 2016). Racial stereotypes are prevalent in mathematics and these stereotypes cause Black undergraduate students in mathematics not persistent by facing lower academic expectations, limited opportunities to engage, and lacked encouragement in STEM fields (McGee & Martin, 2011). Similar to sexism in STEM, racism, particularly racial microaggression is an influential factor in the underrepresentation of college students of color in STEM (Lee et al., 2020B).

On the other hand, we adapted the social capital perspective to examine student ties and influence in the linear algebra class with consideration of possible equity issues in terms of race and gender. The social capital theory refers to inform the value of social connections in "families, youth behavior problems, schooling and education, public health, community life, democracy and governance, economic development, and general problems of collective action" (Adler & Kwon, 2002, p. 17). The social capital theory also views social ties "as avenues through which resources of many different kinds are shared and accessed" (Henderson et al., 2018, p. 4). Between actors, ties provide access to ideas, power, and resources. Thus, actors can look for strategies to be accessible to the new resources and/or power (Henderson et al., 2018). In the linear algebra class, the actors are students, and we assume that the actors will build up ties as their strategical access to other actors as resources. If a student is traditionally privileged or dominant in mathematics, then they may be more accessible, which means they may achieve more strengthened networks.

Study Context, Data Sources, and Methods of Analysis

While the benefits of active approaches to learning are well established (Freeman et al., 2014), the way in which these approaches can be implemented to consistently support different minoritized populations of students remains an open question (Theobald et al., 2020). One particular form of active learning that is popular in undergraduate mathematics is inquiry-based or inquiry-oriented instructional approaches. Such approaches feature student inquiry into mathematics through collaborative problem-solving (see e.g. Laursen & Rasmussen, 2019). However, there is evidence that in some inquiry-based settings students' experiences and learning systematically vary by gender (Laursen et al., 2014; Johnson et al., 2020; Ernest et al., 2019), and we believe that similar variation may exist based on race.

This study was conducted as a part of a broader project aimed at developing curricular materials for inquiry-oriented linear algebra. The linear algebra class was taught in the 2020 Fall semester, as an online-formatted course due to the COVID pandemic, and the application for the synchronous online-formatted classes was Zoom. When students had group activities, they went to the breakout rooms on Zoom so that they discuss. Participants of this research are undergraduate students who took the course "Applied Linear Algebra 1" at a public university in the southern United States. Thirty-six students enrolled in the course, and 17 students consented to use their data. Four of the participants who consented are female, and the other 13 participants are male. More than half of the participants (10 students) are White, including all four female students. Thus, seven non-White participants are all male.

The main data source is discussion boards on Canvas. The discussion boards were assigned biweekly, involved posting an individual write-up to problems they had worked on in groups during previous classes. Particularly, one of the reflection questions includes making a "Shout-out" that aims to celebrate the good ideas and successes of one another. The shout-out does not necessarily have to mention another student in the class, but it can include former teachers or helpful materials in the class. We counted the number of nominations for shout-out of a total of three discussion boards, whose scope was a single unit of instruction. On these "Shout-out" posts, a student can shout out to multiple students. Then, we used UCINET (Borgatti et al., 2002), which is an application to analyze and map networks, to visualize the social network in Linear Algebra class.

Findings

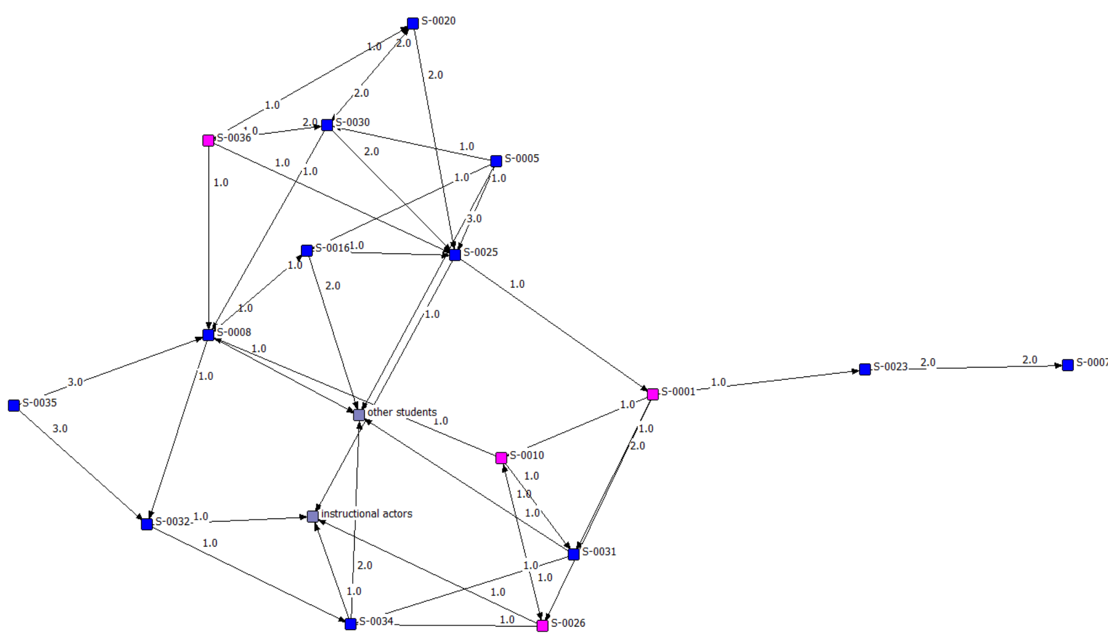

Figure 1. *Social Network in Applied Linear Algebra 1 Class*

Note. Blue dots and pink dots represent male and female students, respectively. Other students are students who did not consent to use their data, and instructional actors mean outer actors such as asynchronous video or a student's former instructor.

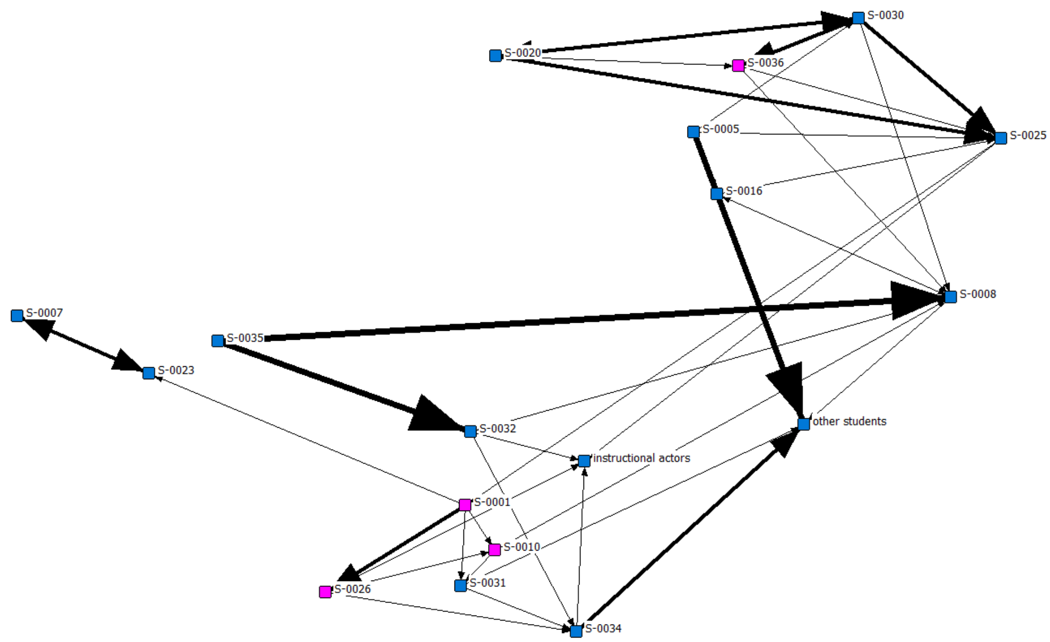

Figure 2. *Principal Components of the Social Network in Applied Linear Algebra 1 Class*

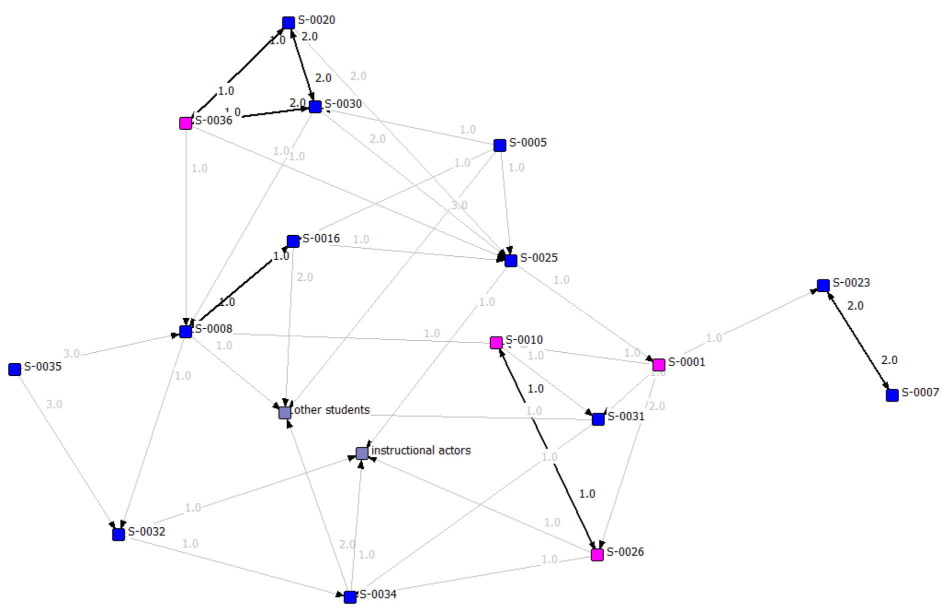

Figure 3. *The Reciprocal Network in the Applied Linear Algebra 1 Class*

Among students, two students were nominated most, S-0008 and S-0025 (See Figure 1). Two students were nominated a total of 8 times out of 37 through three discussion boards, including one time of self-nomination by S-0025. To discuss demographic information of two students, (1) two students are both male and (2) they are non-white students. Specifically, S-0008 is Hispanic who came from a South American country and S-0025 is Black. This racial information is reversed to that of traditionally dominant in mathematics–White and Asian. Also, the principal components (See Figure 2), which show the largest number of nominations, mainly involve nominations of male students including the two most influential students.

Besides, in the social network, we see a small female-dominant cluster among females S-0001, S-0010, S-0026, and one male S-0031 (See Figure 1). As we already discussed, the mainstream was flowing through the male-dominant networks. Though this small female cluster is less principal than male-dominant nominations, they shout out to each other and build up their network. Particularly, S-0001 shout out to S-0010 once and S-0026 twice, and S-0010 and S-0026 shout out to each other once. We could not document how race affects female students because all female students who consented to participate are White.

Overall, the nominations were one-way, rather than reciprocal. Figure 3 shows the reciprocal network and the number of reciprocal ties 6 among the total of 37 ties of the social network. One interesting point from the reciprocal network is that three female students out of four were involved in the reciprocal network. The reciprocal ties connect either male students—A female student S-0036 is connected to two male students S-0020 and S-0030 as reciprocal ties, or another female student—S-0010 and S-0026 nominated each other at separate discussion boards.

Discussion

In this class, it seems racial issues among male students may not be a big deal. This is because the most nominated students have racial diversity—Latino and Black, compared to the traditionally dominant group—White and Asian. However, we think that gendered issues can be discussed more because (1) according to the number of nominations, female students may be less influential in the whole class, and (2) female students have a lack of racial diversity since all female participants are White. However, (1) the small female group implies that "Women Helping Women" and (2) the nominations are more reciprocal than male students so it may be interpreted that female students would attempt to "reciprocate" rather than to "receive".

We speculate that discussion boards may function as a way to interrupt some of the ways in which discussions are gendered and raced, as the mathematical content of posts/contributions is foregrounded since everyone has equal space to contribute, and there may be social positioning that precedes the post when everyone is expected to post and there may be less social positioning that precedes an online post when everyone is expected to post, as compared to speaking during a whole-class discussion. However, still, the finding may indicate less racial diversity among female students than one of the male students, and even female students of color in the course did not consent to use their data for research. This reminds us of the potential discussion of "intersectionality", which means an individual's experience of discrimination or privilege is explained by the intersection of an individual's various identities such as race, gender, class, sexual orientation, and others (Crenshaw, 1989; Coaston, 2019). Our future work can be relevant to the evidence of the following question with the intersectionality perspective: Can the discussion boards reorganize access to social capital in a math class?

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Group Reflection on Mathematical Creativity in Proving

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Proof and creativity are recognized as foundational elements of mathematics, so as university mathematics courses incorporate more collaborative learning strategies, it is essential to support the creative growth of students in group proving settings. We investigate the implementation of a modified version of the Creativity-in-Progress Rubric on Proving (CPR; Savic et al., 2017) as a group reflection tool in an Introduction-to-Proof course. Preliminary results include observations regarding the reflection process of two groups using the CPR.

Keywords: mathematical creativity, proving, reflection

Mathematical creativity is widely accepted as a crucial component of mathematics (Karakok et al., 2015; Mann, 2006; Nadjafikhah et al., 2012; Sriraman, 2004; Zazkis & Holton, 2009). Educators in undergraduate settings have also recognized the importance of creativity to a mathematics education. In particular, the Committee on the Undergraduate Program on Mathematics (Schumacher & Seigel, 2015) asserts that encouraging students to think creatively is essential to successful mathematics major courses.

The domain of mathematics is relatively new to the world of research on creativity (Sriraman & Haavold, 2017), yet the field has grown rapidly in recent years (Heath, 2021; Sriraman, 2017). Despite the rapid growth in research on mathematical creativity, most studies have investigated the creativity of an individual rather than the creativity of a collaborative group or the influence of collaboration on creativity. Mathematicians report that social interaction is an important aspect that influences their creative work (Sriraman, 2004). In the classroom, students' ideas and work are often influenced through collaboration with peers and instructors (Campbell & Hodges, 2020), so it is possible that social interaction also influences the creative work of students. Thus, as mathematics classrooms evolve to incorporate more active learning, it is crucial to understand the relationship between collaboration and mathematical creativity.

Proof is essential to the work of professional mathematicians (e.g., Karakok et al., 2015; Sriraman, 2004) and is often conducted in a social setting (Sriraman, 2004). Proof classrooms may incorporate active learning techniques wherein students work collaboratively on proving tasks (e.g., Bleiler-Baxter & Pair, 2017). It has also been conjectured that student reflection on the proving process can foster mathematical creativity (Savic et al., 2017). Thus, the relationship between collaboration, reflection, and creativity in proving is natural to investigate.

Background

The Creativity Research Group (CRG, 2021) has suggested future research on the influence of socialization and collaboration on creativity in proof (Savic, 2016) using the Creativity-In-Progress Rubric on Proving (CPR; Savic et al., 2017). The CPR was designed as a formative assessment and reflection tool for university students engaged in proving. Although the CPR was designed for individual student use, we have previously investigated the potential implementation of the CPR in a group setting and have posited three suggestions for adapting the CPR: (a) add a category on collaboration, (b) expand the subcategory on Posing Questions to include the act of making conjectures, and (c) ask students to reflect upon their proving process first by viewing the group as a unit and then by evaluating their individual contribution (Heath, 2021). Although these suggestions were grounded in audio and video data analysis of groups working collaboratively on a proving task, we have not yet implemented the suggestions for the modified use of the CPR within a classroom setting. Before we can investigate the impacts of group reflection on creativity in proving, we must determine whether the CPR, with suggested modifications, can be successfully used by students engaged in group proving. Thus, this study seeks to answer the question, how do undergraduate students use the CPR (with modifications) to reflect upon their experiences in collaborative proving?

Theoretical Perspectives

Within the theoretical framing of this study, we address our theoretical perspective on mathematical creativity as well as intersubjectivity and social metacognition.

Mathematical Creativity

Creativity has been notoriously difficult to define. Mann (2005) claimed that there are over 100 existing definitions of creativity in the literature. It has not only been difficult to reach a consensus definition of creativity, but research on creativity also takes many different perspectives. Sriraman (2004) described six categories of approaches used in the study of creativity: mystical, pragmatic, psychodynamic, psychometric, cognitive, and social-personality. Although the focus of our study is on the development of students' mathematical creativity (i.e., the *pragmatic* approach), our rationale for implementing the CPR in collaborative groups is motivated by the *psychodynamic* approach. The psychodynamic approach is based on the idea that the tension between one's conscious reality and unconscious drives spurs creativity. By investigating the use of a reflection tool, we hope that the participants will be able to make connections between their unconscious minds and contributions with the conscious reality of participating in the group and that this will spur growth in creative thinking.

In addition to the joint use of the pragmatic and psychodynamic approaches, we must address a few factors concerning the definition of creativity. In mathematical creativity literature, creativity can be thought of as a process or product, static or dynamic, and relative or absolute (Savic et al., 2017). Aligning with the assumptions of the CPR (Savic et al., 2014; Savic et al., 2017), we assume that mathematical creativity is a process, domain-specific, dynamic, and relative. Note, creativity as a relative construct allows for student work to be considered creative if it is novel to them or their class, but not necessarily novel to the greater mathematics community (Kaufman & Beghetto, 2009).

Intersubjectivity and Social Metacognition

The psychodynamic approach to creativity investigates the tension between unconscious drives and conscious reality (Sriraman, 2004). We contend that in a group setting, this phenomenon can be scaled up to a multi-party scenario to investigate the tension between individual subjective experience and the objective reality of actions within a group. This interplay between multiple parties' subjective experiences is known as intersubjectivity (Matusov, 1996; Sawyer, 2019). For the purpose of this study, we adopt a lens of participatory intersubjectivity, which defines intersubjectivity to be a "process of coordination of individual contributions to joint activity rather than as a state of agreement" (Matusov, 1996, p. 34). This view enables one to account for how something new could be created by group interaction (Sawyer, 2019). Therefore, in a proving setting intersubjectivity can help describe how although students may have different expectations for how their proof will evolve, eventually, students can create a coherent proof despite the differences among their subjective experiences.

Theoretically, reflecting upon the proving process will reveal how individual participation and perception differs from the collective group process in proving, and thus, students can observe how their individual experiences relate to the creation of a proof. These observations will enable students to make their unconscious drives and conscious subjective experiences apparent, to adapt their behavior while engaging in proving, and thus to improve their mathematical creativity. Our theory is that revealing the tensions of intersubjectivity through social metacognition (Chiu & Kuo, 2009) will promote learning and the development of creative skills. Further, the continued use of a reflection tool will promote self-regulation and encourage students to maintain standards and expectations of themselves (Savic, 2016).

Methods

Study Context, Participants, and Data Collection

This study was conducted in an undergraduate Introduction-to-Proof course at a large public southeastern university in the United States. This course facilitated an inquiry-based learning environment centered around small group work on proving tasks and the social construction of establishing norms regarding what constitutes a mathematical proof. Data were collected from 11 students: six white males, three white females, one Asian female, and one white/Asian female. In addition to the demographic survey, data consisted of audio and video recordings of group proving interactions and reflections, completed copies of the group and individual CPR, and a survey to assess students' experiences using the CPR in their groups. Data were collected in the third week of the course. Students were still becoming comfortable with proof and proving; however, classroom norms of collaboration and negotiation were established.

Procedure

Students in the class under study were assigned with reading Savic et al.'s (2017) book chapter, which provides an accessible, detailed description of the categories of the CPR as well as two case examples of implementing the CPR with individual student work. After reading, students provided written reflections on the chapter wherein they described the subcategory of the CPR they found most interesting, a new idea they learned, how they think the CPR might translate to a group setting, and questions they had about using the CPR.

During the next class, the lead author led a discussion with students addressing their written reflections, answering student questions, and redirecting misunderstandings about the CPR. Following this discussion, the instructor of the course facilitated a lesson on logical statements and logical equivalence and then tasked students to work in their assigned groups to prove the statement, "(The Distributive Property). Let A, B, and C be logical statements and prove the following: $(A \wedge B) \vee C$ is logically equivalent to $(A \vee C) \wedge (B \vee C)$." The main mathematical tools they had at their disposal were the definitions of logical statement, conjunction, disjunction, negation, and logical equivalence. The instructor never provided a sample proof for a similar conjecture, so there was room for creativity in students' collaborative proof constructions.

Following this discussion, students participated in a four-part classroom proof and reflection activity (see Table 1). Eleven students consented to be participants in this research, and they were grouped into two groups of four and one group of three. The six remaining students who did not consent to the research were placed into two alternative groups and engaged in the same classroom activity. (Due to space constraints, we have not provided a copy of the modified CPR; however, we refer the reader to Heath (2021) for a detailed description of the original CPR as in Savic et al. (2017) and the suggested modifications for group use.)

Table 1. Description of the four-part classroom proof and reflection activity.

Activity	Description	Time Allowed
Group Proving	Groups worked together on the proving task.	20 minutes
Group Reflection with CPR	Groups discussed and came to a consensus on group placement on the CPR in each subcategory.	10 minutes
Individual Reflection with CPR	Participants privately reflected upon their individual contributions to their group's proving process and reflected upon their strengths according to the CPR.	5 minutes
Group Debrief	Groups reconvened and discussed their individual reflections and strengths.	5 minutes

Following the four-part activity, students completed a post-reflection survey where they described their experiences using the CPR, the challenges they encountered during group reflection, and whether they think reflecting on group proving experiences is helpful.

Data Analysis

Recall, in this study we seek to understand how undergraduate students use the CPR (with modifications) to reflect upon their experiences in collaborative proving. At this stage in our analysis, we have holistically reviewed the video data of two of the three research groups (i.e., group 1 and group 2) as they worked to assess their group's creativity using the CPR. Note: Group 3 did not have sufficient time to complete the group reflection, so we have omitted them from this analysis. Analyzing this video data allows us to gauge student experience using the CPR in groups and to identify any major difficulties. We compare the approaches used by group 1 and group 2 as they worked to reach consensus regarding their placements within each of the subcategories of the CPR. We describe the groups' discussions regarding their placement on the CPR and identify similarities and differences between the discussions of the two groups.

Throughout analysis, we viewed all data collected through the lenses of intersubjectivity and social metacognition. Although intersubjectivity is present throughout all group interactions during this study, including group proving and group reflection, for the purpose of this study, we are interested in analyzing how the tensions of intersubjectivity during the proving process can be revealed through the social metacognition facilitated by group reflection.

Discussion of Preliminary Results

In our initial review of the group reflection process, we noticed two features worthy of attention in this preliminary report. First, we noted a general difference in approach between group 1 and group 2 with respect to how they reached a consensus regarding how to score the group on the CPR. Second, both groups had instances wherein one group member doubted the group's engagement with a subcategory (e.g., tools and tricks, posing questions/making suggestions) of the CPR and another group member reminded the group of a moment in the group's proving process when they did engage with that subcategory.

Two Groups’ Contrasting Reflections

In group 1’s discussion of most of the subcategories of the CPR, after one student read the descriptions of the levels of a subcategory aloud, she would either ask what her group thought or give her own opinion about where the group should be placed within the subcategory. For each subcategory, the members of group 1 engaged thoughtfully by referring back to specific moments during their group proving process to support their argument for where the group should be placed in the CPR category. All four participants thoughtfully engaged in some way.

In contrast, group 2 typically did not support their placements on the CPR with references to their proving process. In most subcategories, group 2 silently read the category descriptions and then one student would suggest a level of “beginning,” “developing,” or “advancing” and the other group members would silently nod their heads or quietly agree with little discussion. Although infrequent, when group 2 made references to their proving process to support their placement, it was only one student doing so.

According to the theory of social metacognition, one might hypothesize that group 1 will benefit more from their reflection than group 2 because group 1 made explicit observations about one another’s ideas during their proving process. One potential implication of this observation is that groups that engage in thoughtful social metacognition and make explicit recollections of one another ideas using the CPR can further their creativity and proving abilities.

Tensions of Intersubjectivity Revealed by Social Metacognition

The second observation made through our preliminary analysis is that both groups had a similar experience: when discussing a subcategory of the CPR, one group member asserted that their group did not engage with that category during their proof, and another group member responded with an example of a moment related to that subcategory that advanced their group’s proof. In group 1, this occurred during their discussion of the Tools and Tricks subcategory, and in group 2 this occurred during their discussion of the Posing Questions/Making Suggestions subcategory. In both cases, reflection using the rubric enabled one student to reveal to another how a contribution of one of their peers advanced their proof and provoked new ideas. In this way, social metacognition revealed the tensions of intersubjectivity present in the group proving: even though during the proving process group members perceived a contribution differently, reflecting upon and monitoring one another’s ideas opened the door for those different perceptions to be revealed and the learning to be advanced.

Conclusion and Future Directions

In this preliminary report, we have described initial observations regarding how students use the CPR as a group reflection tool and used these observations to hypothesize how group reflection on collaborative proving experiences may influence student learning. In our future analysis, we plan to further analyze data of group reflections as well as compare the group reflections using the CPR to students’ self-evaluations using the CPR. We also plan to use survey data to inform the continued refinement of the CPR as a group reflection tool.

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What is a Vector to Students?

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This study presents linear algebra students' vector conception found in the least-squares solution context through an IOLA (Inquiry-Oriented Linear Algebra) CTE (Classroom Teaching Experiment). Students' reflection writings after the CTE are the data source. Using a previously found student conception of vector in another study as a basic framing, the CTE data have been analyzed to investigate how students used the word 'vector' and what they referred to. This study offers a framework, a tool to be useful in a wide range of describing student conception of a vector emphasizing their natural way of thinking of a vector and their use of the vector.

Keywords: Vector, Linear Combination, Span, Linear Algebra, Student Thinking

Vectors are widely used in mathematical sciences. In Calculus, a vector is represented by an arrow or a directed line segment which has both magnitude and direction. Students work with functions of two or more variables incorporating vectors in that course. In linear algebra, vectors are extensively used as students learn the key concepts such as linear (in)dependence, span, basis, and vector spaces. While a significant body of studies have explored student conception of domain-specific contents, less study on student thinking of vectors has been done in linear algebra. This proposal foregrounds vectors and investigates students' notion of vectors to answer the research questions: How do linear algebra students think of a vector? And how do they use the vector and what do they refer to?

Literature

There is little literature exploring students' thinking of a vector in mathematics. Hillel (2000) identified three modes of vector representation: directed line segments (geometric mode), n-tuples (algebraic mode), and elements of vector spaces (abstract mode). In the geometric mode, students interpret a vector as an arrow having the magnitude and the direction with initial and terminal points. In the algebraic mode, students perform vector operations component-wise. In the abstract mode, a vector is described as an element of a vector space, and its vector operation is performed according to the axioms of that vector space. Along the similar strand, Sierpinska (2000) defined the terms, "synthetic-geometric thinking" that mathematical objects are given to students' mind directly being seen as a shape lying in space, and "analytic-arithmetic thinking" and "analytic-structural thinking" that mathematical objects are being component-coordinatized in a given dimension. Watson et al. (2003) found that students have the imagery of "a journey" when adding vectors. For example, students think that the equality in $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ holds because it is moving from A to B and finally to C. But this may become problematic when students interpret the commutative property for addition such as $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{BC}+\overrightarrow{AB}$ since the right side does not give a satisfying meaning to them with the journey reasoning. Kwon (2011, 2013) focused on the progress students make from embodied vector to abstract vector and provided a vector framework attending to the ontological and epistemological aspects. It incorporated Hillel (2000) and Siepinska (2000)'s modes of description and reasoning into the framework. Appova & Berezovski (2013) identified students' misconceptions and error patterns about vector operations. The students in their study did not distinguish between a vector and a scalar in operations, for example, a vector minus a scalar equaling another vector or a scalar. Mikula and Heckler (2017) developed a framework to design essential skills on vector math and implemented an online training for introductory physics students. They categorized the skills required for vector operations such as addition, subtraction, dot products, and cross products using both **i, j, k** and arrow representations.

Theoretical Framework

This study is grounded in the theoretical perspective which makes a distinction between students' mathematics and mathematics of students (Steffe & Thompson, 2000). Students' mathematics refers to the students' mathematical reality which we cannot access directly, whereas our interpretation of the students' mathematical reality is referred to as mathematics of students. Even though models constructed in this study are not representing their mathematics perfectly, it is still worth constructing the models because students' mathematics is indicated by observable behaviors such as the students' gestures, written work, and discourse occurring in their mathematical activities. In this paper, the hypothesis that various ways a student thinks about a vector are closely related to their written description of the vector and to their use of the vector in task setting, is the theoretical foundation in developing the conceptual framework and analyzing data.

The conceptual framework used to analyze data in this study was developed as a tool to be useful in a wide range of describing student conception of a vector. It was initially the result of the author's unpublished class project in which task-based clinical interviews were conducted with students who had no or little experience with linear algebra at that time. The initial framework consists of five different vector conceptions: Vector as a point, Vector as a displacement, Vector as a direction, Vector as an equivalence class, and Vector as a linear combination of other vectors. Its geometric description of the categories is shown in Figure 1. Even though a vector can be in any dimensional space, the descriptions of the categories are illustrated in R^2because the interview tasks mainly included 2D vectors. Vector as an arrow is set as a default because it was the initial description of vector most students made in the study.

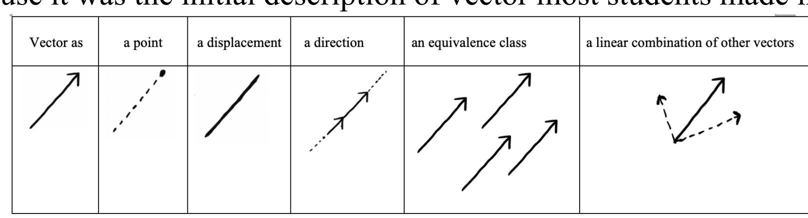

Figure 1. Geometric description of vector conception

Using the initial framework as a basic lens, the author began coding students' data from a classroom teaching experiment (CTE) implementing a recently designed unit about least squares method in linear algebra. The framework was further developed by the author as the vectors in the task of the CTE include various aspects categorized under "student conception of a vector" and "how they use the vector" in the problem context.

Vector conception categories

Vector as an arrow: Students describe the tip of a vector as referring to a line segment.

Vector as a point: Students use the word 'point' to describe a vector or they attend to the tip of a vector considering the tail as if it were at origin.

Vector as a segment: Students describe a vector as a connected segment between two locations.

Vector as a direction: Students use the word vector referring to the span of the vector or the direction of the vector.

Use of vector categories

Over and up alternations: Students use vectors in over and up process alternately.
Scaling: Students use vectors in scaling process.
Constructing a space: Students use vectors in generating a space such as a line or a plane.

Methods

This study reports students' notion of vector and their use of vector in a classroom teaching experiment (CTE) conducted as a part of a broader NSF-funded linear algebra project aimed at developing instructional sequences. A team of mathematics education researchers designed a new unit about least squares methods in linear algebra adopting the instructional design heuristics of Realistic Mathematics Education (RME) informed by Freudenthal (1991). The classroom teaching experiment (CTE) was used as a method to test the instructional sequence of the domain-specific mathematical activities and to see how students in the classroom reason and how the reasoning evolves with the task sequence (Cobb, 2000).

The Task "Meeting Gauss"

The designed task referred to as "Meeting Gauss" begins with an experientially real situation: you want to meet Gauss using three modes of transportation- carpet ($\boldsymbol{v_1}$), hoverboard ($\boldsymbol{v_2}$), and jetski ($\boldsymbol{v_3}$). However, there is no way for you to reach Gauss because Gauss ($\boldsymbol{g}$) is located off the plane spanned by the transportation vectors. Now, Gauss also needs to move to meet somewhere you can reach, but he wants his trip to be the shortest as possible (Figure 2). The big guiding questions students were asked in the task are (a) Where should Gauss meet you? (b) Along what vector would Gauss travel to get to your meeting point? (c) What distance would Gauss's trip be? (d) How would you get to your meeting point using your modes of transportation? To answer these questions, students need to think of vectors and use them in various ways. The instructor in the CTE notated the Gauss travel vector as $\boldsymbol{e}$ and the vector from the origin (Home) to Meetup as $\boldsymbol{p}$.

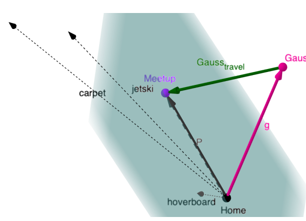

Figure 2. Illustration of Meeting Gauss task in GeoGebra

Data Source and Analytic Method

The CTE was conducted with students in two introductory linear algebra classes (33 students, 38 students) separately at a large public university of the Southeastern United States. The students were STEM majors who had taken Calculus 1 or 2 as a prerequisite. The CTE lasted for four consecutive class days on Zoom at the end of semester. Students' reflection writings on the first two days of CTE is the data source. Students wrote reflections each day after the CTE. The reflections were transcribed into a spreadsheet, and the author highlighted the word vector whenever students mentioned it in their writing. Their reflections were coded line by line using the initial framework, a new set of codes emerged, the author used the updated framework to complete the coding. The author specially focused on students' explanation *before and after* the word vector when coding. When students used the symbolic notations from previous math classes to refer to a vector such as $< a, b, c >$, (a, b, c) or $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$, their meaning is also analyzed. All the names used in Results are pseudonyms.

Results

As shown in Figure 2, the ***e*** vector that Gauss travels and the ***p*** vector from Home to Meetup are important pieces in answering the questions of the task sequence of least squares. While Gauss can travel around freely to meet you, your movement is restricted by the transportation vectors given in the problem context. The analysis on students' reflection writings revealed that their conception of a vector and their use of a vector are multifaceted. In this section, their notion of vector and their use of vectors in the Meeting Gauss task will be briefly presented focusing on the two vectors ***e*** and ***p***.

Vector Conception- Describing the e vector that Gauss travels.

Vector as an arrow: In the initial framework, this category was defined as a default, however, the author noticed that vector as an arrow conception is a way of thinking characterized when students describe the tip of a vector as referring to a line segment. For example, *"Gauss should meet you at the shortest point **from the tip of his vector line** to you"*(Anton). Anton describes the path Gauss travels referring to it as an arrow.

Vector as a point: Students use the word 'point' to describe a vector or they attend to the tip of a vector considering the tail as if it were at origin. Damien wrote, *"Gauss should meet us at the point (1.35,0.06, 2.21) …the vector (Gauss travels) should be [1.35; 0.06;2.21] as that is **the point he needs to reach anyways**"*. Damien found the coordinates of Meetup using the GeoGebra applet and used them as the coordinates for the Gauss travel vector ***e***. The tail of the vector ***e*** is at the Gauss location, not at the origin. Vectors and points have the same coordinates only if the vector has a tail at the origin. Another student, Annalisa noted "*I understand everything up until we had to actually find **the point/vector** to meet at*". Annalisa interprets the Meetup in two ways, a point and a vector. She did not mention anything other than this quote in her writing, but still this is an indication that a vector could be identified as a point or vice versa.

Vector as a segment: Students conceive a vector as a connected segment between two locations. Specifically, they use the word 'displacement' or 'magnitude' to describe a vector attending to the length of the vector. For example, Owen noted, *"Gauss will travel along (-1-a,1-b,4-c) which is **the displacement vector between Gauss's old and new location**."* Owen seemed to write the Meetup as (a,b,c) and subtract it from the ***g*** vector to describe the Gauss travel vector ***e***. His description of a vector includes two locations and a line segment connecting them. Another student, Luka wrote *"We described the entire situation as a right triangle, where **the hypotenuse is the vector between the origin and Gauss**, the vertical component is **the vector between the plane and Gauss**, and the horizontal component is **the net distance travelled via hoverboard and magic carpet***". Luka's description illustrates that each side on the right triangle is a vector which connects the locations among Home, Gauss, and Meetup.

Vector as a direction: Students use the word 'vector' referring to the span of the vector or the direction of the vector. Span of a vector refers to a line scaled either extended or shrunk along the vector. Consider the following examples, *"Gauss would need to move to a point that is **along the line of the vectors** given"(Bianca) "The direction he'd travel can be found with the cross product: [[i j k] [1 1 1] [6 3 8]] = 5i - 2j - 3k =* ***[5 -2 -3] <- so either that direction or***

reversed." (Vaki). Their description for the ***e*** vector indicates that they attend to the scaled line or the direction Gauss travels rather than the length of the vector or the vector itself.

Use of Vector- Traveling from Home (origin) to Meetup

Over and up alternations: Students use vectors to get to Meetup stair-wise. Hanora noted,*"Sometimes it might be shorter if I moved a certain distance so he can reach me without using many **modes of transportation to make zig-zags**."* and Dilon wrote "*Gauss should meet us along one of our vectors. If he wants the shortest trip he should meet. He would use V1 to get there and his distance would be very short, **only 1 unit in each direction**.*." Hanora and Dilon seemed to understand the task context differently than intended, but Hanora's description on the use of transportation vectors indicates that vectors move in over-and-up manner to get to the Meetup. Also, Dilon's explanation reveals that he recognizes $\boldsymbol{v_1} = [1,1,1]$ consists of stair-wise movement 1 unit in each direction.

Scaling: Students use vectors to get to Meetup by scaling. Vaki noted, *"We would get to the meeting place by **using the hoverboard for 1.5 hours and the carpet for 0.5 hours**."* The hours are the amount spent on each transportation, which indicates that Vaki's vector moves in scaling manner. Iliana answered, "*I would get to the meeting point by using a combination of my transportation methods, or by using a **linear combination of the given vectors***." Also, Lurenya mentioned, "*Since Gauss is getting to the plane that is the **span of the three modes of transportation**, **a linear combination of the three vectors** will get us there as well*." In Iliana and Lureyna's description, they would use three vectors to reach the Meetup using the span or linear combination of the transportation vectors, which indicates that they find vectors varying in length.

Constructing a space: Students use vectors in generating a space such as a line or a plane. Ani noted, "*The way I am thinking about this is to locate a spot that is within **the area that the 3 vectors span** and that Gauss could go to. To do that, I would **find the plane that the 3 vectors together span**...*" and Cecil "*What I'm imagining is that if we **created a "basis plane" with these two vectors** then we can map that plane as well as the point Gauss is at"*. There were many students who wrote the plane to be generated by the transportation vectors when they use the word linear combination and/or span.

Discussion

This study, to investigate student conception of vectors and their use of vectors, began as students' data were analyzed with a different focus. The author noticed that the Meeting Gauss task scenario entails various ways of thinking with vectors as exploring ways to answer the guiding questions. By foregrounding vectors, student conception and its use were briefly presented using the conceptual framework developed as analyzing their data. The author finds this study important in developing instructional sequence of least squares and even linear algebra because vector is often backgrounded in studies even though student thinking of vector is multifaceted.

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Using Proof Comprehension Tests in-class to Encourage Student Engagement and Improve Proof Comprehension

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This study considers the use of proof comprehension tests (Mejia-Ramos et al, 2017) as the topic of an in-class student-led discussion. After completing the test, students discussed the items as a group to decide the best answers. Preliminary analysis shows high-levels of student engagement – offering opportunities for questions, discussions, and debates related to the proof and proving in general. As a result of the discussion, students' proof comprehension test scores improved.

Keywords: proof comprehension, classroom discussion, student engagement

Mathematical proof plays an important role in professional mathematical practice and as such in mathematical practice at the advanced undergraduate level. Meanwhile, the undergraduate mathematics education literature has shown time and time again that students struggle with all aspects of engaging with mathematical proof, including but not limited to proof constructing, proof validating, and proof comprehension. While this list of activities is by no means comprehensive nor are these activities mutually exclusive, much of the research regarding the teaching and learning of proof can be classified by one or more of these three activities (Stylianides, Stylianides, & Weber, 2017). Proof construction and validation research is relatively rich with studies considering both experimental and more naturalistic settings, as well as comparing mathematicians' and students' actions while engaging in these activities. Research regarding students' comprehension of proof, however, has focused mostly on the assessment of students' proof comprehension. In this proposal, we explore the use of a proof comprehension test as an in-class activity.

Literature Review and Theoretical Perspective

Mejia-Ramos et al (2012) presented the proof comprehension assessment framework, building on the work of Yang and Lin (2008), suggesting proof comprehension can be measured in both local and global understandings–meaning some understandings can be inferred from a small portion of a proof, say one or two sentences in a proof, whereas some understandings require a more holistic treatment of larger portions of a proof. Mejia-Ramos et al then offered examples of question types to assess the various ways in which one might comprehend a proof. Question types for local comprehension include: meaning of terms and statements, logical status of statements and proof framework, and justification of claims. Question types for global comprehension include: summarizing via high-level ideas, identifying the modular structure, transferring the general ideas or methods to another context, and illustrating with examples.

Mejia-Ramos et al (2017) then developed and validated short-form multiple choice comprehension tests using these question types focused on three particular proofs at the introduction-to-proof level. These proof comprehension tests (and others created using the same framework) have been used by researchers as a measure of success for various interventions (see for example: Alcock et al, 2015; Hodds et al, 2014; Davies et al, 2020). Meanwhile, we believe that proof comprehension is also an activity that should be enacted within the classroom, rather than solely an assessment. As such, this proposal reports preliminary findings from using one of the three tests developed by Mejia-Ramos et al (2017) as the focus of an in-class activity.

Student Engagement in Proof-Based Classes

The study of the in-classroom engagement of undergraduate mathematics students in proof-based courses is a relatively limited, albeit growing, avenue of research. That is not to say that prior studies have ignored the importance of student engagement, in fact, student-centered instruction models, such as inquiry-oriented learning (Rasmussen & Kwon, 2007; Kuster et al, 2018), are built upon perspective that value students' social construction of knowledge.

Meanwhile, the literature also suggests some caution should be taken when lauding the student-centered classroom, as some have found that these approaches may be less equitable than once believed (Johnson et al, 2020). Common concerns in the learning science literature regarding the equity of inquiry-oriented learning and other student-based learning models include that these learning models may favor the already-privileged, already-successful students and that such students may dominate the conversations in-class (Esmonde, 2009). Further, research suggests that student discussions dominated by these stronger students can lead to lower learning gains across the class or group (Theobald et al, 2017).

Research Questions and Hypotheses

In this paper, we explore the following research questions: (1) To what extent can multiple-choice proof comprehension tests be leveraged to encourage student discussion and engagement? (2) To what extent does student discussion around multiple-choice proof comprehension impact students' proof comprehension? We hypothesize that allowing students to discuss the multiple-choice proof comprehension tests will provide an open forum for students to discuss a variety of topics related to not only the specific proof included on the test, but also general proving-related topics. Moreover, by providing these specific items along with concrete answer choices we believe the test offers easier access to the discussion for students who maybe struggling with the material or may be hesitant to engage.

Methods and Analysis

The present study was conducted in a single session of an introduction-to-proof course at a large, public, research university in the United States. The course was taught in a moderate lecture style (Johnson et al, 2019), in which lecture took between 25%-75% of the course time with non-lecture course time spent with students working in small groups on tasks, giving occasional presentations of completed work, and having students explaining their thinking.

On the day of the study, seventeen students were present. Each student was given a copy of a proof comprehension test of the theorem: "Every third Fibonacci number is even. That is, f_{3n} is even for every $n \in \mathbb{N}$." (Where the Fibonacci sequence is defined by $f_1 = 1$, $f_2 = 2$, and $f_n = f_{n-1} + f_{n-2}$ for all $n > 2$ and $n \in \mathbb{N}$.) The proof comprehension test included the theorem and proof, as well as 12 multiple-choice items developed and validated for measuring students' comprehension of this particular proof (Mejia-Ramos et al, 2017). Students were given 20 minutes to read the proof and complete the comprehension test before they were instructed to talk through the test as a class – identifying what answers were chosen, discussing why they chose those answers, and which answer they believed to be correct following their discussion on the previously-selected answers. The student discussion occurred without teacher intervention or participation and lasted 33 minutes.

Prior to beginning their discussion, the instructor-researcher provided pens and asked students to switch their writing instruments to a new color so any notes or changes made in their answer selections as a result of the discussion could be documented. The entire discussion was video and audio recorded and the students' tests were collected for analysis.

For the purpose of analysis, we graded the tests, tracking students' responses and scores prior to and following the class discussion. Analysis of the video data is ongoing, however we have transcribed the discussion surrounding two items of the comprehension test and have begun identifying key moments of the discussion to highlight student engagement in the discussion as well as instances of the students learning from each other in this discussion. In the following section, we present results from our preliminary analyses.

Results

The student discussion followed the test in a linear fashion, discussing each of the items in order. While some students spoke more than others during the discussion, each of the 17 students participated in the discussion to some extent. For each question, a student (not always the same student) would ask what answers were selected. Then students were asked to explain why they chose each of the selected choices. Occasionally, a short debate would occur between factions who selected different choices. After agreeing on a best answer choice (or agreeing that they had reached an impasse), the class would move on to the next item. As a class, they came to a consensus on what they believed to be the best answer choice for 9 of the 12 problems and chose the correct answer for 8 of those 9 items. In this section we first discuss the test scores as impacted by the student discussion, then look closer at the discussion surrounding one test item.

Test Scores

Figure 1 shows the scores of the students pre- and post-whole class discussion. Before the discussion, we can see a spread of student scores. Eleven of the 17 students answered 6 or fewer answers correctly, meaning they initially scored 50% or lower on the 12-item test.

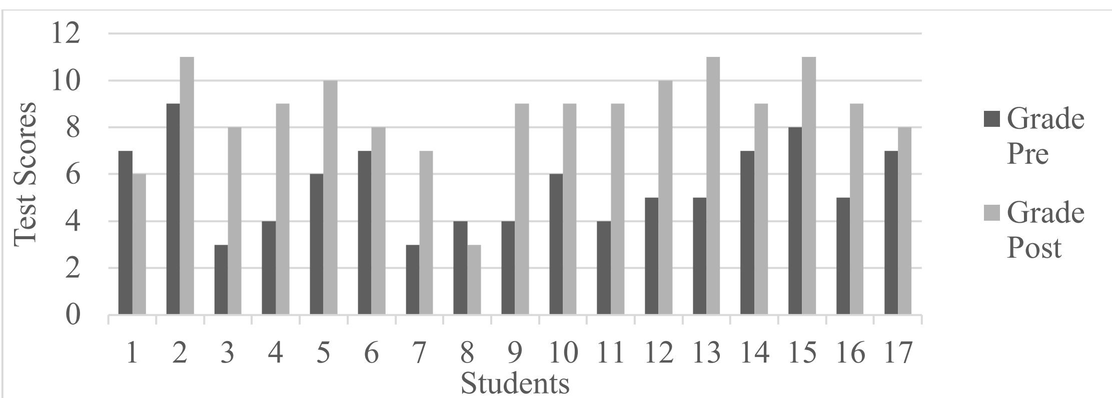

Figure 1. Each of the 17 students' scores before and after their discussion of the test.

Meanwhile, 15 of the 17 students had a higher score following the whole class discussion, with the remaining two students showing a score decreasing by only one point. Across the class, there was a 3.11-point increase in the students' average scores. Thus, almost all students directly benefited from the discussion in terms of their test scores, getting on average three more questions correct in comparison to before the discussion. Moreover, we see an average increase of 4.09 points for the students who initially scored 6 or lower, while the students who initially scored higher than 6 had an average increase of 1.33. Thus, there is a clear benefit of the whole class discussion to students' comprehension of the given proof in terms of their test scores.

Table 1 shows the number of students who indicated an answer choice for each question, both before and after the discussion, as well as the number of students who changed their answer choices due to the discussion, indicating total changes as well as changes to the correct answer. We make several observations from this table. First, not all students answered all questions and the number of answered questions decreased further into the test – suggesting four students did reach the end of the test. Next, nearly every student answered every question post-discussion. Those questions not answered by all 17 students include Q9, Q10, and Q12 – which are the three questions for which the students were not able to come to a consensus on a final answer.

Table 1. The count of students who answered each question (pre/post discussion) and who changed their answers.

Number of students who...	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12
Answered pre-discussion	16	17	17	17	17	16	16	16	14	14	13	13
Answered post-discussion	17	17	17	17	17	17	17	17	15	16	17	16
Changed answers	3	5	6	13	4	5	2	10	3	8	12	5
Changed to correct answer	3	5	6	10	4	4	2	9	0	-2	12	0

Finally, many students chose to change their answer choices due to the discussion. However, not all students changed their answers to the agreed upon answer even when the class's chosen answer was correct (such as Q6 and Q8). Moreover, despite the class's failure to agree on a final answer choice on Q9, Q10, and Q12, some students still changed their answers. We interpret these occurrences as evidence that students were not blindly following the talkative students and accepting their answers – rather we see students thinking about the answer choices and the reasons their classmates gave for all answer choices in order to select a new answer.

Using the ideas of the proof, how could one prove that f_{3k+1} is always odd, for every natural number k (that is, for $k = 1,2,3,4,\ldots$)? Please select **the best** option.

(a) Showing that f_1 is odd to address the base case. Then assuming f_k is an odd number and proving that f_{k+1} is also odd.

(b) Letting k be an arbitrary natural number and finding an integer m such that we can express f_{3k+1} as $f_{3k+1} = 2m+1$.

(c) Proving f_4 is odd and writing f_{3k+4} in terms of f_{3k+1} in such a way that we can conclude that f_{3k+4} is odd whenever f_{3k+1} is odd.

(d) Use the proven statement to conclude that f_{3k} is even. Then, by adding 1 to f_{3k} we can conclude that f_{3k+1} is an odd number. 15

Figure 2. Problem 8 and its answer choices.

Student Engagement in Discussion of Problem 8

Of the 16 students who initially answered problem 8 (shown in Figure 2), two chose answer A, five chose answer B, five chose answer C, and four chose answer D. The students remarked on the wide distribution of the answers then argued the merit of each choice, in order from A to D. For this question, one student took up a leadership role, asking his classmates to discuss why they chose their answers, asking "A's, what's up with A? Why did you like A?", "What about B?", and so on. During this discussion, 10 of the 17 students participated. Through the discussion, the students came to a consensus with answer choice C (the correct choice).

The discussion of choice C lead to an interesting interaction. Student J disliked C because of the word "whenever". Student J stated, "But it says whenever f_{3k+1} is odd. Whenever is like implying that like it may be odd or it may be even." He continued, "But the question is asking for, 'is always odd'." That is, Student J focused on the use of "whenever" and struggled with this alternative presentation of a conditional statement. He believes the "whenever" implies the opportunity for f_{3k+1} to be even which contradicts the desired conclusion.

Several of his classmates responded to Student J's concerns. One student said the choice was written "in a weird way to say that we're assuming the $3k + 1$ is odd", explaining that "it's just putting the inductive step after how you've proved the inductive step". Another student notes that it was simply the word choice confusing Student J, "it's the whenever, you just don't like the whenever". Finally, another student commiserated with Student J offering this description, "Yeah, C feels like the proof that you already wrote and it's like 'that proves that this is, whenever this is'." In this interaction, Student J took the discussion as an opportunity to question not the mathematical content of the item, but the language used in the answer choice. Moreover, we see his classmates working to help him understand the statement, despite its wording.

During the discussion of D, Student R explains, "I just picked D because if $3k$ is even, then if you add 1 to that, regardless of the multiple... (crosstalk)." Another student chimed in supporting Student R's reasoning before another student offered the retort "You see, I don't think that you're adding 1 to the actual Fibonacci number. You're adding it to the k", with another student adding "Because that's what--you're adding 1 to the f_{3k}." The students who chose D seemed to believe the answer choice was implying an addition of 1 to f_{3k}, rather than the index of f_{3k}. However, after some students reminded their classmates the 1 is added to the index k during the induction step, the original student understood their error adding "Oh, that makes sense."

Discussion

Overall, we see an increase in scores following the students' classroom discussion, especially for those students who originally scored below 6 on the proof comprehension test. Similarly, we believe the student dynamics and engagement in the discussion is important to note. Despite the common expectation that students known for their mathematical prowess may overtake the conversation, this data suggests the discussion around this multiple choice test was much more inviting to all the students in the class. Moreover, we saw some of the stronger students helping their fellow classmates better understand the material – as we saw with Student R's and Student J's classmates offering help. We also saw some of the more traditionally quiet students speak up.

When asked what they thought of the test, students attributed the test to be challenging and even exhausting at the end. However, they also remarked that the test caused them to think deeply about the proof in ways they had not done previously. One student remarked that when writing proofs he often did not know where to start, but with the test, "like you can see different ideas and see actually which one makes sense to you." In this way, we believe these multiple choice tests offer many access points for students to engage in discussion and learn from their peers – not only with regards to the mathematical material at hand, but also in regards to mathematical conventions and language choices. In Student R's case, by offering his reasoning for choosing an answer, he had the opportunity to revisit the use of indices in sequences and how that relates to inductive arguments. In Student J's case, the test helped him to question the language used in a mathematical text and potentially learn from his classmates about an alternative presentation of a conditional statement.

This analysis is still in progress and this project is not without limitations. The study in question involves a single activity used in a course taught by the researcher. It is possible that the style of the course primed the students to be particularly conversational. Meanwhile, this project does seem to highlight some advantages of having students discuss full mathematical texts, as opposed to focusing only on producing them from scratch. Some avenues for future research include developing additional activities to help students better comprehend the proofs they read.

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Understanding the Developmental Mathematics Research Landscape: A Critical Look at Intended Audience and Outcomes

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Although a wide variety of reports on developmental math exist, to date there has not been a large-scale examination of existing work from a math education point of view. Towards this goal, we analyzed 426 reports and peer-reviewed journal articles relating to developmental math published between 2000 and 2020. In report, we quantify the publishers and intended audience, examine the types of outcomes reported on and, where possible, examine the type of developmental math model discussed. We find that over the last decade, less than 20% of reports on developmental math have been aimed at math education audiences. While math education publications more frequently examine math knowledge and student experiences, the overall number of reports, compared to those examining pass rates, is relatively small.

Keywords: developmental math; literature analysis; outcomes

Developmental courses, which are taken by college students who have been identified as not yet ready for “college level” courses, enroll a large portion of undergraduate math students (Blair et al., 2018) and pose a conundrum to math education researchers. On one hand, developmental courses generally cover content traditionally labeled as “high school level.” On the other hand, these courses take place in postsecondary environments with different norms, structures, and levels of access than high schools. Low success rates, particularly for students from marginalized populations (Chen, 2016), provide additional complexity to conversations about these classes.

Arguably, as experts in both education and math content, the math education research community is well positioned to contribute meaningfully to conversations about the value and equity of student outcomes and experiences in developmental math, and there is much to understand. The low success rates, combined, perhaps, with increased attention thanks to then President Obama’s “American Graduation Initiative” in 2009, have resulted in a variety of developmental math initiatives. Some of these initiatives provide a mechanism for students to progress through the required content more quickly, including Bridge, Acceleration, Modularization, Corequisite and Emporium models (c.f., Parker, 2012, Twigg, 2003). Other initiatives change or restructure the required curriculum by removing algebra content and focusing more on quantitative reasoning or statistics, with the aim of better aligning the content with students’ ultimate goals, such as the Carnegie Pathways (e.g., Hoang et al., 2017) model. Lastly, some initiatives remove requirements for developmental instruction entirely, such as legislation in California (A. B. 705, 2018) and Florida (S. B. 1720, 2013). However, we have noticed that math education journals rarely publish in this area.

As scholars concerned with developmental education, we were curious as to whether our impressions are reflective of the field. We have thus set out to critically analyze published literature on developmental math. Although many reviews related to developmental math exist, they tend to examine the efficacy of developmental math education (e.g., Davis & Palmer, 2010; Melguizo et al., 2011) or describe developmental initiatives (e.g., Jaggers & Blickenstaff, 2018).

In her comprehensive review, Mesa (2017) provides a discussion of the existing literature on developmental math through 2014, but only reviews documents within math education on community college students. In this review, we aim to understand the developmental math literature landscape across all sectors, with a focus on the extent that existing literature examines questions of interest and importance to math education researchers. While a variety of entities write about developmental education, as researchers we specifically focus on original reports. Towards this end, we examine a combination of peer-reviewed journal articles and reports published between 2000 and 2020 by agencies concerned with developmental math to understand:

1. Who is the intended audience of original research on developmental math?
2. To what extent does existing research define the developmental model examined?
3. Which types of developmental math outcomes are the most widely considered?

For these three questions, we also examine how, if at all, the answers have evolved over time and to what extent the body of existing work seems to contribute to an increased understanding of students' mathematical learning in these courses. Ultimately, our aim is to unpack how researchers define developmental math; analyze the types of outcomes typically discussed, particularly as they relate to learning; and build a foundation for new types of questions about developmental math to be investigated by math education researchers moving forward.

Methods

Our initial sample of records was drawn from the EBSCOhost database, which curates documents from a variety of sources, in the summer of 2021. Given the diversity of stakeholders in developmental education, the wide net provided by the EBSCOhost database was well-suited to our purpose. Our final search criteria included documents with a publication date between 2000 and 2020 (inclusive) and a system-listed document type of journal or report. In addition, the associated abstract needed to include either the word "developmental" or "remedial", a word with the stem "math", and one of the following words: college, university, post-secondary, postsecondary, or undergraduate. These search criteria yielded 1,442 documents.

Initial review of the records suggested some of the documents were unrelated to our interests (e.g., developmental psychology). In addition, some records were duplicates or were published in non-journal periodicals. Thus, we engaged in a review of the abstracts for probable inclusion in our final data set. Our final inclusion criteria for this stage of review included articles and reports with abstracts that indicated the document related to developmental math students, instructors or instruction. Articles concerned with broader examinations or descriptions of course delivery options or curriculum pathways that altered or removed developmental coursework for students were included. We included historical treatments or reviews when the focus of the document otherwise fit our inclusion criteria. To keep the focus on postsecondary settings, we did not include reports or articles that examined initiatives that took place while students were still enrolled in high school (e.g., dual enrollment). We also did not include articles that described students entering developmental math, with no other results provided, as these do not shed light on what happens in developmental math, only on the population enrolling in the courses.

After reviewing abstracts, 488 records were identified as meeting the inclusion criteria or needing further review. We then engaged in coding of the abstracts. Because coding demanded that we read abstracts deeply, we often picked up on nuances we had not previously noted and thus sometimes marked a record as incorrectly included. After discussion of these records, we removed an additional 62 documents. This left 426 records in our sample for analysis.

Initial Document Coding

We coded the document abstracts and publishing agencies. Here we discuss only those codes used in this report: Publication Agency, Audience, Outcomes and Developmental Model.

A single team member coded the publication agency and peer-review audience using data within the EBSCOhost record. The remaining three authors coded the abstracts for outcomes and the developmental model. Codes were developed iteratively through collective discussion. Fifty-three percent of the abstracts were coded independently by two team members. The remaining 47% of the abstracts were coded by a single team member. Initial agreement between all double-coded records was greater than 80% (and often greater than 90%) with the exception of coding for "other" categories on the outcomes and developmental model codes, where individuals had to describe what belonged there. Prior to the RUME conference, we intend to reconcile all coding disagreements and update our methods and results.

Publication agency. Publication agency was coded based on publisher information provided by EBSCOhost. In cases without information, we included the record if the abstract met our inclusion criteria and left the publication agency code blank. Six publication agencies were identified and coded for: Federal government agency, school or state agency, non-profit organization, professional society, institute or center, or peer-reviewed journal.

Math education audience. For records that were identified as a peer-reviewed journal, we examined the journal and determined the primary audience of the journal, making note of whether the journal was primarily geared towards a math education audience or not.

Outcomes. Abstracts were coded for the nature of the results, outcomes, or products presented. These outcomes fell into seven categories. *Passing* was assigned to reports examining students' course grades or student success rates in a math class, inclusive of either developmental or non-developmental classes. *Finish* was assigned when outcomes related to students' completion of a math class sequence, a degree, a transfer, or the student was retained. We assigned *Performance* when the outcome related to students' scores on assessments such as tests or final exams. *Knowledge* was assigned when the outcome related to assessing student understanding of mathematical ideas or concepts. *Student outcomes* was assigned when the outcome related to the students' attitudes or referred to students' success or outcomes, but did not provide enough detail to be coded as any other category. *Student experiences* was assigned when the outcome related to students' perceptions or to the climate of their program or school. *Faculty/Instruction* was assigned when the outcome had anything to do with the developmental faculty or instruction generally (e.g., students were not the population of interest). For this report, we omit discussion of "Other", but will clarify this variable prior to the conference.

Developmental model. A variety of developmental math content delivery models have emerged over the last 20 years. We coded for named initiatives, including *Pathways, Accelerated, Online (Emporium), Corequisite*, or *Bridge*. In addition to these named initiatives, we had additional model codes: *Traditional, Online (Regular), Policy change, Not Stated*, or *Other.* We only assigned Traditional when the abstract explicitly used "traditional" to describe the developmental classes discussed. This decision reflects the fact that traditional is often used to describe instruction as teacher centered and lecture heavy. However, instructional practices can vary widely (e.g., Mesa et al., 2019), even within named initiatives. When the developmental model was not named or described, we assigned "Not Stated". When the model was described, but did not fit one of the other categories, we assigned "Other".

Results

Research Question 1: Intended Audience

We first considered the proportion of research on developmental math by publication type, graphed in Figure 1. As expected, given our search criteria, a large proportion of research appeared in peer-reviewed journals; however, roughly 40% of the research on developmental math appeared in reports instead (Figure 1a). We note also that the peer-reviewed journals in this category included not just journals intended for a research audience, but also a large number of journals for practitioners (in future analysis we plan to further break down our analysis based on the intended audience of the journal, e.g., researchers vs. practitioners). Of the research on developmental math published in peer-reviewed journals over the last decade, less than 20% has been published in math education journals, even when including practitioner journals in this total (Figure 1b). Accounting for reports, the total proportion of developmental math research published in math education journals (practitioner- or researcher-focused) is then roughly 10%.

Research Question 2: Developmental Models Examined

Next, we considered the specific developmental math model mentioned in research reports over time (Figure 1c). The proportion of research articles on developmental math that did not name the developmental model decreases over time, which is likely explained by the various initiatives sparked by the American Graduation Initiative (The White House, 2009). While we might expect some "types" of developmental math to go out of fashion and others to emerge as more dominant over time, this does not seem to be the case. Rather, there currently appears to be no dominant model of developmental math in the literature, and the landscape appears to be getting more complex over time. We also note that the focus of most of these models is on providing different instructional approaches, modes of instruction, or curricular sequencing at the institutional level (e.g., how many courses do students need to take, or how many credits are attached?) rather than on the content students engage with. The Carnegie Pathways models (Hoang et al., 2017) come the closest to engaging with learning, by modifying the content students were taught (primarily removing algebra content that was viewed as difficult); however, even for this model we noticed that in our analysis of outcomes that the focus of published reports was primarily on course completion and college progress. Measures of students' learning (e.g., of specific mathematical concepts) have been less prominent.

Research Question 3: Outcomes Examined

Our examination of the proportion of outcomes shows that the most commonly measured were passing and finishing, which made up roughly half of the outcomes; the next most common was student "outcomes", with all three combined making up 71.7% of the total (Figure 1d). Only 3.2% of studies overall measured student mathematics knowledge (i.e., specific mathematical concepts or ideas). However, when considering outcomes just for peer-reviewed journals based on discipline (Figure 1d), we noted interesting differences. Articles from math education journals focused attention on students' performance or knowledge, whereas articles for other audiences tended to focus on passing and completion rates (Figure 1d). In math education journals, roughly 14.3% of research measured student learning, compared to only 1.6% in other journals. We note that the focus on learning in math education journals is relatively low compared to other outcomes; however, it is almost 9 times higher than at non-math education journals where the bulk of the research is currently published. Therefore, increasing coverage of developmental math research in math education journals could increase reforms focused on student learning.

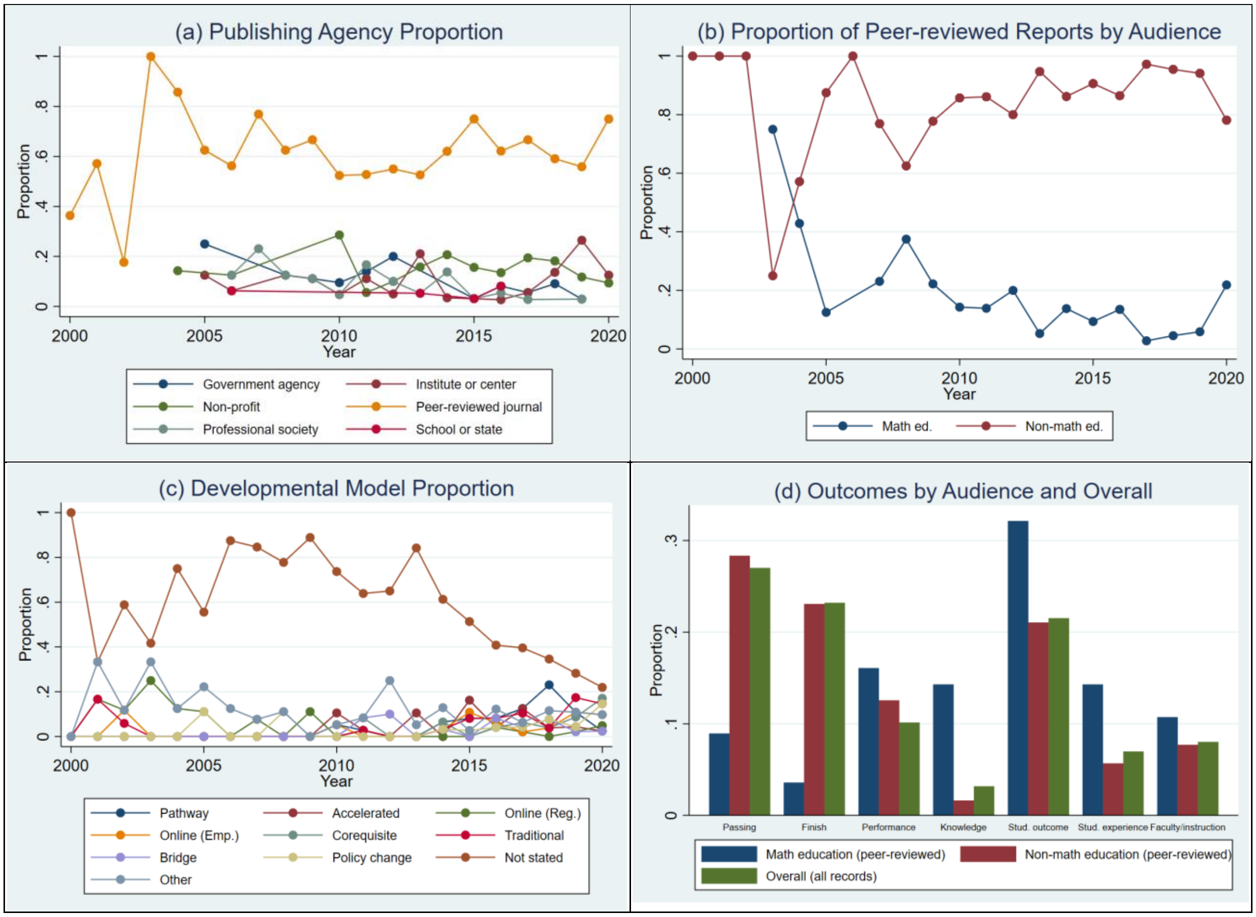

Figure 1. Audience, Developmental Model, and Outcomes Over Time

Discussion, Next Steps, and Questions

All together, these combined results suggest that many models for delivering developmental math content are being implemented, which are perhaps under informed by research on how and what students are learning within development math courses. We are intrigued by these initial results and the questions they raise. As our research continues, we intend to add additional codes where necessary after recognizing the repeated use of "other" codes for the same topics (e.g., credits, enrollment); if anything, this may mean that the incidence of reports which focus on learning are actually currently overcounted. We are currently coding the data for measures of equity, use of deficit language, and how developmental math is defined and operationalized.

Audience Questions

- Are there other analyses that we should consider that we haven't done yet?
- If you currently do research in developmental mathematics, what kind of analysis of existing research would most help further your research?
- If you have not yet done developmental mathematics research, what kind of analysis of existing research would be most helpful to you in starting to work in this area?

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Emergent Modeling: Using Python in an Instructional Task Sequence on Logic and Set Theory

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In this preliminary report, I describe the instructional task sequence and initial findings of a teaching experiment that explores how computing can be leveraged as a way to facilitate and strengthen the connection between set theory and logic. As such, I address the national call for modernizing the undergraduate curriculum to reflect the growing importance of computing and programming. In particular, I examine how programming may support student reasoning and learning of mathematical set theory and logic, with the goal of characterizing students' advancing mathematical activity and growth over the course of a multi-session long teaching experiment. The ultimate results of this study will inform how we may be able to infuse computing into an Introduction to Proofs course in a new and innovative way.

Keywords: Computing and Coding, RME, Set theory, Logic

In this study, I focus on the development and use of ideas commonly taught in an Introduction to Proofs (ITP) course regarding set theory and mathematical logic. The conjecture that I explored in connecting these two topics is that programming/computing can be leveraged as a way to facilitate and strengthen the connection between set theory and logic. In particular, I detail one aspect and preliminary findings of a larger dissertation study which consists of three main research foci: (a) how Python may influence student reasoning and learning; (b) students' advancing mathematical activity and growth over the course of a multi-session teaching experiment (TE); and (c) the impact of programming on students' sense of confidence and enjoyment when doing mathematics. For the purposes of this report, I will highlight one aspect of my second research focus, which is driven by the following research question: *How could an instructional task sequence, built around the use of Python, be used to help develop ideas related to mathematical logic and set theory?* Specifically, the goal of this particular question is to understand the ways in which students develop ways of reasoning about the logical operators \`and\` and \`or,\` and how they are able to flexibly use these operators in Python to understand introductory concepts of set theory such as set intersection. The findings presented here complement prior ITP research (which has often focused on the challenges and difficulties that learners' encounter) by exploring the prospects and possibilities for students' successful mathematical progress.

Theoretical Perspective

Freudenthal's (1971) work on mathematics as a human activity of mathematizing realistic situations eventually led to the development of the instructional theory known as Realistic Mathematics Education (RME) (Freudenthal, 1991; Gravemeijer, 1999; Gravemeijer, 2020a; Treffers, 1987). I draw on this theory in the development of the tasks that were used to guide my TE. From a RME point of view, the design of instruction should be realistic in the sense that the material is imaginable by the students and/or relevant to their experiences (van den Heuvel-Panhuizen, 2003). Foundational to the RME approach for the teaching and learning of mathematics are three main heuristics: (a) guided reinvention, (b) didactical phenomenology, and

(c) emergent modeling (Gravemeijer, 2020b). In broad terms these three heuristics reflect the importance of reinvention and ownership of mathematics by the students through their own mathematical activity. In order to characterize students' increasingly sophisticated ways of reasoning, I highlight the third heuristic, emergent modeling, in more detail below.

Emergent modeling is the gradual process in which learners construct sophisticated mathematical conceptions through the development of models as they shift from "model-of" methods for solving informal mathematics to "model-for" methods used for more sophisticated ways of reasoning. Zandieh and Rasmussen (2010) define a model as:

> Student-generated ways of organizing their activity with observable and mental tools. By observable tools we mean things in their environment, such as graphs, diagrams, explicitly stated definitions, physical objects, etc. By mental tools we mean the ways in which students think and reason as they solve problems—their mental organizing activity. (p. 58)

Gravemeijer (2020b) explains that meaning is not embedded within the mathematical symbols themselves, but is created by the learner through the development of models of mathematics. Additionally, each model is made up of smaller more-comprehensible sub-models. The relationship between sub-models and models can be described as a recursive one in which a smaller model is developed to lead to more mathematics that requires another smaller model that leads to more mathematics and so on. In a sense, one may think of the development of sub-models and models as the transition from a model-of specific, or more informal problem tasks, to a model-for more formal mathematics. An important aspect to highlight here is that each sub-model is built from the previous sub-model. The model-of/model-for transition is described in more detail by Gravemeijer, et al. (2000) as levels of activity, with the four levels being *situational, referential, general and formal.* In the next section I describe the four levels of activity in more detail, as they will be used for the primary method of analysis.

Methods

Participants were recruited from a four-year Hispanic-Serving Institution and were purposefully selected (Patton, 1990) in that they had already taken, or are currently enrolled in, differential calculus and not enrolled in an ITP course. A pilot study was conducted with two students, in which I conducted three one-hour long TE sessions. The main study data consisted of five one-hour sessions with four groups of students. There were 10 students for the main study, two to three students in each group. Due to the ongoing COVID-19 pandemic, the TE sessions were conducted and recorded via Zoom. For each session, I shared my screen which displayed two windows. The first window was a Jamboard slide deck to present the tasks to the students, and to capture their ideas and diagrams. Jamboard is a Google interface that serves as a collaborative and interactive canvas. In a window next to Jamboard, I had an Integrated Development Environment (IDE) that was used to run the Python code.

Data analysis included both (a) ongoing and preliminary, and (b) retrospective analysis of the data corpus (Confrey & Lachance, 2000). Ongoing and preliminary analysis consisted of frequent (after each TE session) and summative reflections of emerging issues throughout the TE related to the conjecture of how computing might be leveraged to connect set theory and logic and be responsive to the students' needs from one TE session to the next.

For retrospective analysis, I am utilizing the four levels of activity as the primary analytic framework to answer the research question that is the focus of this report. While the emergent modeling heuristic and levels of activity were initially developed to provide insight into instructional design, it has also proven to be useful as an analytic tool to frame the model-of/model-for transition (Rasmussen & Blumenfeld, 2007). By documenting the levels of activity, it is possible to gain insight into students' mathematical understanding and growth. The first level, situational activity, describes a situation in which students are engaged in an activity that is grounded in an experientially real context. By experientially real, I mean a problem scenario that the students can imagine in their mind's eye. This problem context is the first step in which students develop sub-models to describe a model-of method for solving a problem. Additionally, the situational activity should be something that the students can come back to as they progress in more sophisticated ways of reasoning. This is where the next level comes in, referential activity. This activity describes a scenario in which students are expected to move away from the context-specific model and take a slightly more abstracted perspective of the mathematical activity. However, any abstraction must not be too far removed from the situational activity so that the students can still see the relationship with, or refer back to, the model-of approach. The next step is to present the students with other problem scenarios that are context-free. This is known as general activity where students use their model composed of sub-models to reason about a mathematical idea without being grounded in any particular context. Formal activity describes a situation in which students are reasoning abstractly about mathematics; they are using their previously established ideas as a model-for higher-level activity. As is usually the case, each group in the TE found their own path through the study, which resulted in not being able to reach the formal activity that I had originally anticipated.

Instructional Task Sequence

Here I provide the instructional task sequence that was developed to support students' reasoning about introductory concepts of set theory and the logical operators `and` and `or.` Given the space constraint of this report, and the preliminary nature of the results, I provide an overview of the most influential tasks for the students in the TE. Along with the tasks, I present some student work representing their responses to the tasks and activities. The primary focus of this report is on the sequence of instructional tasks as they relate to the four levels of activity.

To begin, I introduced the idea of a logical proposition (an expression that can take on the truth value of True or False) through the concepts of set inclusion and set cardinality. In Figure 1 I provide the code that the students were asked to consider.

```
1 setA = {"dog", "bird", "lion", "cat", "fox"}
2 setB = {"dog", "lion", 7, "lion", "red", 4}
3
4 print("dog" in setA)
5 print(len(setB) == 5)
6 print("San Francisco" in setB)
```

Figure 1. Code defining two sets, with three propositions to be printed.

In Figure 1, lines 4 and 5 will produce a True output and line 6 will produce a False output. Lines 4 and 6 are checking to see if the given element is a member of the given set, and line 5 is verifying that the cardinality of setB is equal to five. These propositions were then assigned to variables, `p = ("dog" in setA),` `q = (len(setB) == 5),` and `r = ("San Francisco" in setB).` The students were then asked to consider what `print(p),` `print(q),` and `print(r)` would produce.

The students were also asked to create their own propositions based on the characteristics of setA and setB and were asked to verify the truth value of the propositions that they created. For example, one student verified a false output by suggesting `m = ("LA" in setB)` and then running `print(m).` For the next step, I asked the students to consider the two propositions in the context of the logical operators `and` and `or.` All of the students came to a similar understanding of the function of the operators by running the various combinations of print statements such as `print(p or q)`, `print(m or q),` `print(p or r),` and `print(m or r).` By executing the various print statements in Python for `or` and `and,` all of the students in the study determined that the `or` operator only needed one true premise to produce a True output and the `and` operator needs two True premises to produce a true output (when in the form `print(s `and`/`or` t)` where s and t are propositions. At this stage, the students' mathematical activity would be classified as situational in which the students were able to develop sub-models of how the propositions were taking on either a True or False value through the execution of a print statement as well as developing an understanding of the functionality of the logical operators.

Once the groups felt comfortable with the operators, we moved onto compound propositions where they were asked to create print statements that contained multiple logical operators and multiple propositions. Groups 3 and 4 provided the following two print statements, respectively: (a) print(p and r or q and r), and (b) print(((r and m) or p) and (q and r)). Figure 2 showcases the work that each group was able to arrive at after some initial discussion.

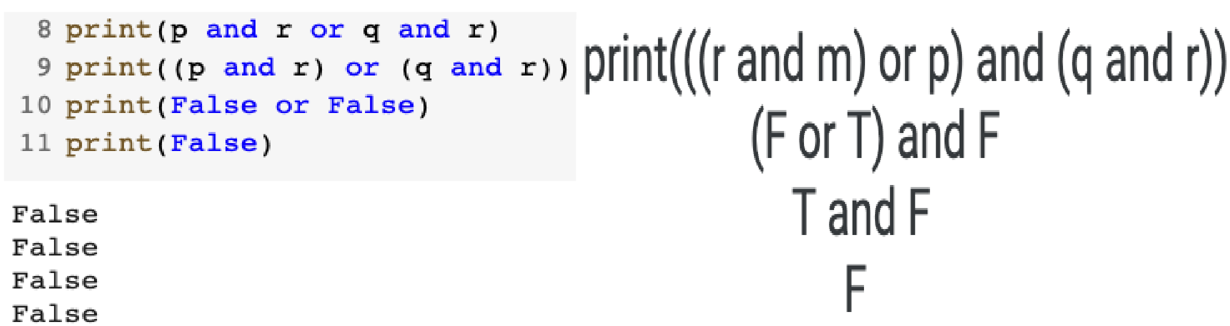


Figure 2. Screenshots of Group 3's work on the left and Group 4's work on the right.

I still consider the students' work in Figure 2 to be situational activity, as they are working with the same propositions. The transition to referential activity occurred when I asked the students to consider two unknown propositions. That is, what would happen if we had *s* `or` *t*? After some discussion of what an unknown proposition is, all of the groups arrived at some variation of a truth table as presented in Figure 3.

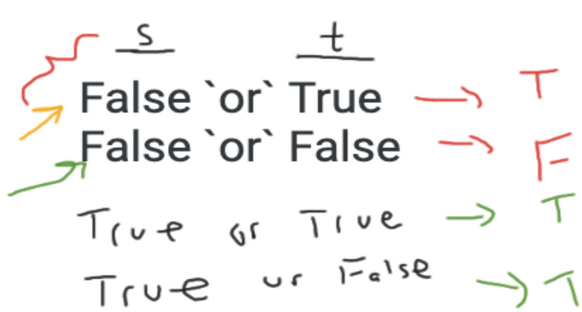
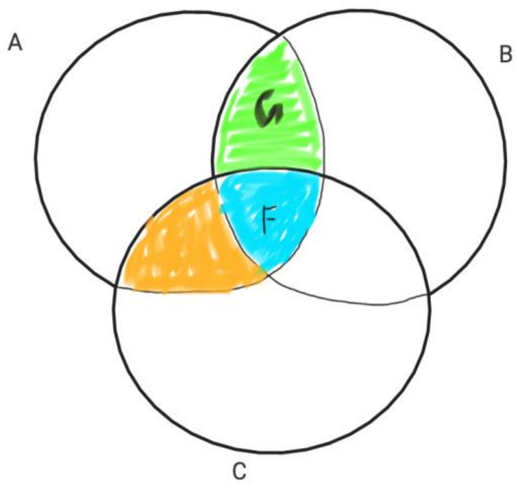


Figure 3. Screenshot of Group 2's work.

The students' work would be classified as referential activity as they were working with a slightly more abstract idea of unknown propositions and logical operators without context, but still close enough to what is seen in Figure 2. Due to space constraints, I will skip the tasks used to introduce the concept of a For Loop and jump straight to the task asking students to find the common elements across three sets. In other words, to find the intersection of three sets. Figure 4 highlights the work that was done by Group 1 as they chose to represent the intersection of three sets using a Venn diagram (shaded in blue).




```
19 F = set()
20 for x in A:
21     if ((x in B) and (x in C)):
22         F.add(x)
23 print(F)
24
25 G = set()
26 for x in A:
27     if ((x in B) and (x not in C)):
28         G.add(x)
29 print(G)
```

Figure 4. Group 1's Venn diagram with the corresponding Python code.

Creating the code corresponding to each shaded region in the Venn diagram was as far as we were able to progress through the TE with Groups 1 and 2. Groups 3 and 4 were able to take this idea a bit further with a number theory problem. Although, what is important to highlight here is that in lines 21 and 27 in Figure 4, the students were able to use propositions with the logical operator `and` to produce a new set, F, that contained the elements of the set intersection between A, B, and C. They also generated set G, which contained elements in A intersect B, not C. This would be an example of general mathematical activity as the students were able to use their sub-models that they created for their understanding of the logical operators in the context of a more abstract problem scenario where the truth values of the propositions were changing. For example, some of the values of *x* may have made the proposition `(x in B)` False, while other values of *x* made the proposition True.

Concluding Remarks

I anticipate that future work will show that Python will be able to support students in their reasoning of mathematical logic as well as other concepts in set theory. In addition, I conjecture that utilizing programming as a means to teach mathematics proved to be a tangible and accessible onramp for all of the students in my study. More broadly, the empirically and theoretically grounded instructional sequence detailed here contributes to the "learning trajectory" research agenda, which is one of four nascent areas of inquiry-based mathematics education research that Laursen and Rasmussen (2019) identify as important for researchers and practitioners to explore in higher education settings where inquiry approaches hold promise. Next steps for this work are to highlight the students' developmental progression through the learning trajectory designed to help support students' understanding of set theory and logic in the context of computer programming.

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A Case Study of an Exemplary Active Learning Mathematics Instructor

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Understanding the student experience in active learning calculus classes requires listening to student voices. This report elevates students' lived experiences in one exemplary active learning mathematics instructors' Calculus I and Calculus II courses. We use a mixed-methods approach to analyze survey data (both free response and numerical items) about students' interpretation of and affective reaction to their current mathematics instruction. The results show that students in this active learning class relate their identity to their ability to do or learn mathematics, and recognize instructional practices that attend to issues of diversity, equity, and inclusion. These results are also compared in reference to other sample populations of students from the same university.

Keywords: Calculus, Student Experience, Active Learning, Instructional Practice, Case Study

Research has found that active pedagogical strategies can be effective for learning and narrowing opportunity gaps (Freeman et al., 2014; Theobald, 2020). However, active learning should not be treated as a monolithic experience for students in every classroom. As research has demonstrated, students in active learning classrooms are most at risk for marginalizing and anxiety-producing experiences (Aguillon et al., 2020; Cooper et al., 2018; Cooper & Brownell, 2018; Shah et al., 2020). This is not entirely surprising given that students in active learning classrooms are generally provided with more opportunity to engage with one another through small group discussions. However, this implies that research still needs to be done to better understand what types of active learning classrooms foster inclusive learning environments for all students. Moreover, there is very little research on how students are reporting their experiences in active learning classrooms.

In this preliminary report, we investigate one active learning classroom taught by a university professor, Professor S, who was recognized within the state's university system as being on the cutting edge of innovative pedagogy and leadership. Professor S was highly committed to the use and implementation of active learning strategies and at the time of the study was leading local professional development efforts to infuse active learning in introductory math courses. Through the use of survey analysis, we describe how students in Professor S's active learning classroom are reporting their time in class, and importantly, compare the survey results from students enrolled in courses with other instructors. The research questions that we will address in this report are the following: (1) How do students characterize the instructional approaches being used in a course taught by an active learning instructor? (2) How are students interpreting and describing the instructional approaches in relation to their sense of math identity?

Methods

The student postsecondary instructional practice survey (SPIPS-M) was administered to several Precalculus, Calculus I, and Calculus II courses at Professor S's university. The SPIPS-M specifically targets students' interpretations of instructional practices, what they found helpful, changes in attitudes towards learning and doing mathematics, and their perceptions of the climate in the classroom (Apkarian et al., 2019). Along with the nearly two dozen 5-point Likert scale items, there are several free response items that include: instructional practices that students found helpful ("Helpful"), practices they found unhelpful ("Unhelpful"), how issues of diversity and equity were being addressed through instructional practices ("Diversity Equity and Inclusion (DEI) Strategies"), and if their identity affected their ability to do or learn mathematics ("Identity Free Response"). Over the Fall 2018 and Spring 2019 semesters, 75 of Professor S's students responded to the survey.

While these 75 students provided an in-depth look into the student experience in Professor S's classroom, a stratified random sample of 80 students outside of Professor S's classroom was created as a frame of reference to contextualize the student experience within the norms of the institution. Both of these samples were analyzed separately with two different approaches: quantitative analysis to determine the results of Likert-scale items and qualitative analysis methods to synthesize and thematically group answers to free response questions. For the qualitative methods, we employed a coding scheme that summarized the student responses into thematic groups based on the given free responses to each of the 6 items.

In addition to the free-response questions on the survey a number of Likert questions asked about instructional practices, activities and attitudes. A prior factor analysis (Creager, et al., under review) identified relevant constructs that are summarized in table 1 along with their construct reliability measures.

Table 1. Survey items from the SPIPS-M that Craeger et al (under review) developed into relevant constructs, with their corresponding Cronbach number.

Relevant Constructs of Practices, Activities, and Attitudes			
Construct Name	Description	Survey Items	Cronbach
Peer-to-Peer	Students collaborate to process mathematical ideas	P6, P7, P8, P10, P15, P16, P20	.87
Math Engagement	Students engage with meaningful mathematics	P1, P2, P5, P11, P12, P17, P18, P19, P22	.87
Instructor Inquiry	Students contribute their mathematical ideas during class and receive immediate feedback	P3, P4, P20	.66
Class Participation	A wide variety of students participate and form community	P13, P16, P21	.82
Math Attitudes Beginning	Students reflection on their affect and attitude towards math at the beginning of the course	A1, A2, A3, A4, A5	.77
Math Attitudes End	Students report their current affect and attitudes.	A6, A7, A8, A9, A10	.80

Preliminary Results

Class Time Spent, Instructional Activities, and a Change in Attitude

Drawing on the quantitative data, we examined how students in Professor S's courses were reporting the percentage of instructional activities that occurred in the class over the whole term (e.g., lecture, whole class discussion, small group, and individual) compared to all other students surveyed as seen in Figure 1. This analysis, which addresses our first research question, provides a broad description of the types of instructional activities that students report experiencing and provides evidence of Professor S's use of active learning approaches. A Welch's two sample t-test demonstrated that students in Professor S's courses reported statically less lecture [$t(154) = 18.1$, $p<.001$, Mean=53% versus 21%], more whole class discussion ($t(96)=-3.42$, $p<.001$, Mean=20% versus 14%) and more small group work ($t(85)=-13.46$, Mean=40% versus 11%). There was no significant difference in students' report of time spent working individually [$t(102)=1.39$, $p=0.16$, Mean = 18% versus 21%].

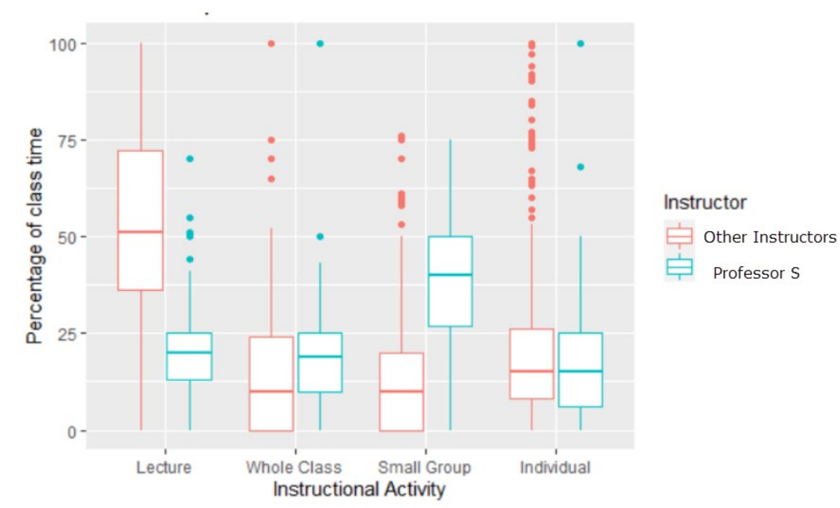

Figure 1. Students reported percent of instructional activities used over the whole term. Turquoise represents Professor S's students and orange represents students in all other classes.

Transcending beyond the rough categories of instructional activities, we examined how students reported various instructional practices to be descriptive of their experiences in the mathematics courses (See Figure 2). These instructional practices are aligned with the pillars of inquiry based mathematics education (Laursen & Rasmussen, 2019) and are detailed in Table 1. Students in Professor S's courses reported statistically greater reports of math engagement [$t(103)=-6.1$, $p<.001$]. Peer-to-peer interaction [$t(117)=-13.7$, $p<.001$], class participation[$t(97)=-7.40$, $P<.001$], and instructor inquiry [$t(105)=-8.6$, $p<.001$].

Furthermore, we wanted to examine the impact that this course had on students' mathematical identity (Confidence, Interest, Enjoyment, and Ability). We asked students on the survey to respond to several items related to affect as they perceived or reflected from the beginning of the course and now (see Figure 2). Students in Professor S's course reported statistically lower levels of math identity at the beginning of the course [$t(85)=3.40$, $p=.001$] and there was no significant difference at the end of the course [$t(96)=-1.26$, $p=0.21$].

Student Interpretations and Descriptions of Active Learning Practices

The following themes emerged from Professor S's student responses. We also compared each theme with what we found in the 80 students outside of Professor S's class.

Group work viewed favorably. Students in Professor S's class who responded to the "Helpful" item, 63.9% (47 students out of 72 total responses) of them thought that group work was particularly helpful in their learning of mathematics. While group work remained the most helpful strategy by frequency, Professor S's extended office hours and explanatory videos, online collaborative platforms such as Voicethread and Titanium, and school resources such as Supplemental Instruction (SI) and tutoring all played a role in helping students succeed in his Calculus courses. Students not in Professor S's class frequently mentioned tutors, SI, lecture notes, and other students in the class as resources but they hardly mentioned group work, specifically in-class group work.

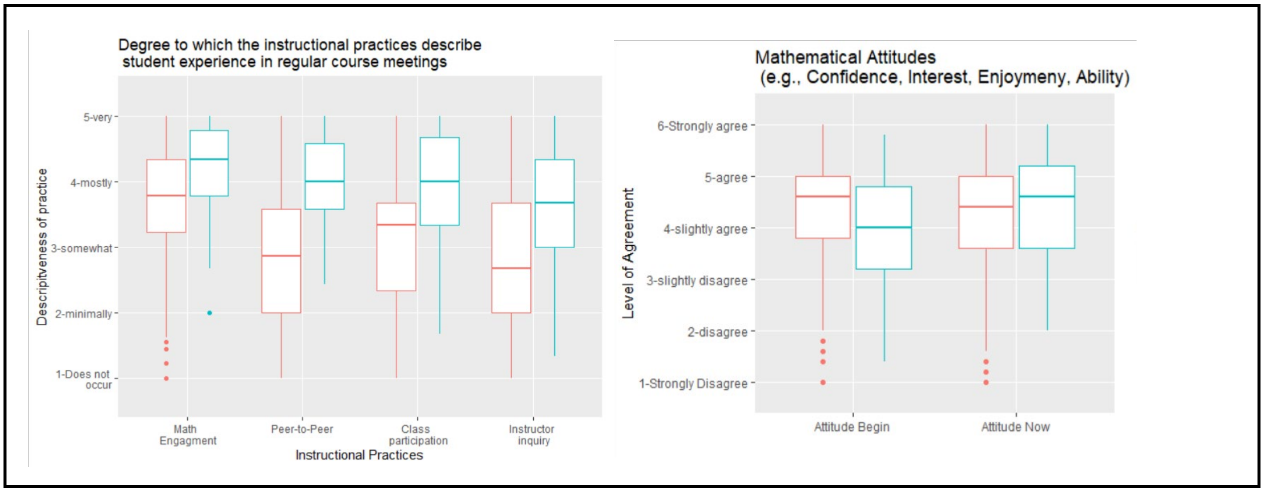

Figure 2. Student experience with instructional practices and students' reported change in mathematical attitudes. Turquoise represents Professor S's students and orange represents students in all other classes.

Some students offered explanations for why they thought group work was a practice that was helpful to them. Out of the 47 respondents for this item, six students mentioned that it was a good way to meet other students and collaborate and another six students reasoned that they had peers to help with questions when they needed help. An additional three students said that it was helpful to see how others think about a problem. One student said that, "if someone didn't feel comfortable asking the professor for help they were more likely to do it in the groups."

Students recognizing DEI strategies. More than half of the respondents to the "DEI Strategies" item also attributed group work as a way to attend to diversity and equity in the classroom. Out of the 22 students who thought that group work helped to address issues of diversity and equity, six of them said that it was because everyone was able to participate. This sense of inclusivity in groups extended to another six students who said that being in a group encourages students to meet and talk with peers. One student specifically mentioned the practice of assigned group roles as well, which is a practice that Professor S employs often. Professor S's attention to diversity, equity, and inclusion manifested in students experiencing positive group work situations and increased opportunities to contribute in class, which were both backed up by Professor S's encouragement and support. Students feel that Professor S knows their name, that Professor S cared about them, and that their contributions were valued. Students outside of Professor S's class did not have much to say about instructional practices that attend to issues of DEI. The most prevalent response for these students was a reference to how their professor "treated everyone the same".

Identity as salient. Out of the 44 students that responded to the "Identity Free Response" item, a dozen students made some connection between a personal trait that they have and their ability to learn or do mathematics. The most prevalent trait was shyness or introversion, which has the potential to be a difficult trait to overcome in a class with added social pressures that emerge from so much group work. However, one of these students explained, "I'm introverted and shy, and this course helped me overcome those obstacles in various ways when I had to discuss the course material with my instructor, or with peers." Another student followed a similar rationale when they said, "I typically keep to myself a lot of the time, but this class forced me to participate more and work with others, something I would normally not do, however, I'm glad I had that experience." While several students inside and outside of Professor S's class commented on their math identity, students not in Professor S's class did not share any instances developing their identity through the instructional practices such as the examples above.

Discussion

Out of the thematic groups of instructional practice constructs in Figure 2, all of these instructional practices were greater in Professor S's course. This suggests that these practices are not part of a zero-sum game where a gain in one implies a loss in the other. Also in Figure 2, we see that students in Professor S's class reported statistically lower levels of math affect at the beginning of the course and there was no significant difference at the end of the course. This result may suggest that upon reflection, students in Professor S's course perceived a level of growth in their math affect and thus reported lower levels at the beginning of the course. Alternatively, students who enroll in Professor S's course may have had lower math affect, but after the course experience, they report similar levels as their peers in other courses.

Our qualitative results show that almost two thirds of Professor S's students mentioned group work as a helpful instructional practice, while students enrolled in other professor's classes often relied on lecture notes and more resources outside of the classroom. This suggests that students outside of Professor S's class require more university resources to be successful in Calculus. Since resources like math and science centers, tutoring, and supplemental instruction are directly tied to university funding, our results suggest that traditional lecture in calculus courses could be costing universities more than just adverse student experiences.

Inclusivity was a significant theme that ran through the student experience in Professor S's class. Whether it was group work supporting a general increase in student contributions or the various modalities these contributions could be performed (online, in person, text, or voice), students in Professor S's class recognize that their voice is valued. In terms of identity development, some students in Professor S's class felt that a personal trait such as shyness, which is potentially damning in an active learning class, could be overcome with instructor support and genuine encouragement. Professor S's students know that he cares about them and thus would not engage in instructional practices that would harm them.

Every aspect of Professor S's active instructional approach has the foundation of empathy and positivity, promoting this instructor from a vehicle for dissemination of content to a vital support structure for success. This instructor put in the time and the effort to establish and maintain instructional practices that align with the pillars of active learning, but we know that Professor S is not a superhero. They are just an instructor who recognizes the assets that all students bring to their class and knows that active learning pedagogy capitalizes on what little precious time instructors have with their students in class.

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Isomorphism and Homomorphism in Abstract Algebra Textbooks

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Despite isomorphism's dual nature as both a property of objects (e.g., groups) and a function, most research has examined "isomorphism" as a singular concept. We analyze how common introductory textbooks structure and relate the ideas of isomorphic objects, isomorphism function, and homomorphism. We incorporate a discussion of the informal descriptions of these constructs and highlight most texts' use of the concept of isomorphic objects to motivate defining the isomorphism function. However, despite the consistency in informal language for isomorphic objects, we observe variety in the informal language used to discuss the isomorphism function and homomorphism.

Keywords: abstract algebra, isomorphism, homomorphism, sameness

Mathematics educators have identified isomorphism and homomorphism as key concepts in introductory abstract algebra worthy of further study (Melhuish, 2015). However, isomorphism is often treated as a single idea despite the word "isomorphism" referring to the property of two structures being isomorphic and the function that witnesses two structures are isomorphic (the isomorphism function). Certainly, there is a connection; two groups (rings, fields) are isomorphic if and only if there exists an isomorphism between them. We might also say that the concept of isomorphism function precedes the concept of isomorphic objects; the isomorphism function must be mentioned (not necessarily by name) in order to rigorously define the concept of isomorphic groups (rings, fields).

This isomorphic-isomorphism duality does not extend to the notion of homomorphism; there is no meaningful interpretation of *homomorphic* when applied to whole groups, rings, and fields. However, the relationship between homomorphism and isomorphism (as functions) has an implicit ordering; an isomorphism is a bijective homomorphism, and any definition of isomorphism must therefore mention homomorphism (although not necessarily by name).

Using the above reasoning, one might argue that a student ought to learn the concepts of homomorphism, isomorphism (function) and isomorphic (objects) in that order. However, this logical ordering does not necessarily entail a conceptual ordering. Some mathematicians prefer to teach isomorphism before homomorphism, while others prefer the opposite ordering (Rupnow et al., 2021). More generally, Tall (1991) suggests that students learn new concepts by generalizing from more specific concepts. Gilbert and Gilbert (2009) explain "Isomorphism is a special case of homomorphism, while homomorphism is a generalization of isomorphism. Isomorphisms were placed first in this book with the thought that 'same structure' is the simpler idea." (p.137). Their reference to "same structure" suggests that they think of the idea of isomorphic objects as conceptually more intuitive or "simpler" than the isomorphism function and as a motivation for defining isomorphism. In this preliminary report, we investigate: How do common introductory abstract algebra textbooks structure and order the introduction of isomorphism functions, isomorphic objects, and homomorphisms?

Background Literature

Student understanding of isomorphism has been studied for over twenty-five years (Dubinsky et al., 1994). Leron et al. (1995) described an intervention in which students were initially taught a “naïve” version of isomorphism, wherein sameness was emphasized by renaming one object to produce a copy that is the “same” (p. 154). Rupnow (2021) and Weber and Alcock (2004) confirmed that many students and professors consider isomorphism as a type of sameness. Various metaphors for isomorphism and homomorphism have been examined, such as “relabeling” for isomorphism and “structure preservation” for homomorphism (Hausberger, 2017; Rupnow, 2021; Weber & Alcock, 2004).

Nevertheless, research on the isomorphism concept has not directly addressed the interrelationship of isomorphic objects and isomorphism functions. For example, Dubinsky et al. (1994), Leron et al (1995), and Melhuish (2018) examine how students determine whether groups are isomorphic, suggesting a focus on isomorphic objects. Melhuish et al. (2020) studies students’ use of their function concept to draw conclusions about isomorphism and homomorphism, indicating a function focus. Rupnow (2021) highlights aspects of both isomorphic objects and isomorphism functions without explicitly distinguishing between these concepts nor analyzing their relationship. Notably, these studies do not attend specifically to how the isomorphism function is used to understand the idea of isomorphic objects. We build on Melhuish (2015), who provides an analysis of four common introductory abstract algebra textbooks in which she discusses isomorphic objects. Our work extends Melhuish’s analysis by also examining homomorphism and the isomorphism function separately and by examining connections between isomorphic objects, the isomorphism function, and homomorphism.

Theory and Methods

Our analysis aligns with Thompson’s (2008) description of textbook analysis as a form of conceptual analysis in the sense that it investigates the conceptual coherence of curriculum. Our work fits in Son and Diletti’s (2017) category of “content analysis”, particularly the subcategory of “introduction and development of concepts and procedures”. Similar to Cook et al.’s (2019) textbook analysis on the structure of logic, sets, and proof techniques, we use our conceptual framing of the relationship between isomorphic objects, isomorphism function, and homomorphism to examine the textbooks’ structuring of these concepts. Like Cook et al. (2019), our methods align with constant comparison methodology (Creswell, 2007).

We examined the textbooks that Melhuish (2015) identified as the most common introductory texts: Gallian (2009), Fraleigh (2003), Gilbert and Gilbert (2009), and Hungerford (2012). We identified where each text first formally defines the notions of “homomorphism” and “isomorphism”, regardless of the type of object (group, ring) it defined first. We note that all of these texts except Hungerford introduce groups before rings. We examined where each text informally introduced these concepts; this includes Melhuish’s construct of Example Motivating a Definition (EMD) to analyze how textbooks frame their introduction of the isomorphism concept. We also visited every instance of “equivalence relation” to see how (or whether) the idea of isomorphism was included.

Results

We discuss the structuring of isomorphic objects, the isomorphism function, and homomorphism within the formal presentations of the definitions and then in informal contexts. We also examine how the books relate the concept of equivalence relations with isomorphism.

Formal Definition

In all four texts, isomorphism is defined before homomorphism. See the definitions of "isomorphism" and "homomorphism" presented by Gallian (Figures 1 and 2) as an archetypical structuring example.

> **Definition Group Isomorphism**
>
> An *isomorphism* ϕ from a group G to a group $\overline{G}$ is a one-to-one mapping (or function) from G onto $\overline{G}$ that preserves the group operation. That is,
>
> $$\phi(ab) = \phi(a)\phi(b) \qquad \text{for all } a, b \text{ in } G.$$
>
> If there is an isomorphism from G onto $\overline{G}$, we say that G and $\overline{G}$ are *isomorphic* and write $G \approx \overline{G}$.

Figure 1. Gallian's (2009) definition of group isomorphism, p. 128.

> **Definition Group Homomorphism**
>
> A *homomorphism* ϕ from a group G to a group $\overline{G}$ is a mapping from G into $\overline{G}$ that preserves the group operation; that is, $\phi(ab) = \phi(a)\phi(b)$ for all a, b in G.

Figure 2. Gallian's (2009) definition of group homomorphism, p. 208.

Hungerford and Gilbert & Gilbert address homomorphism as the next topic in the text after isomorphism. Fraleigh uses the phrase "homomorphism property" in the definition of "isomorphism" but does not expand on "homomorphism" as a separate concept until several chapters later. Gallian does not even use the word "homomorphism" until several chapters after introducing isomorphism. All textbooks include isomorphic (objects) and isomorphism (function) (e.g., Figure 1) under the same definition label. However, all textbooks except Hungerford's define the isomorphism function before the isomorphism property. This is not due to Hungerford introducing rings before groups – the same pattern is followed in the group isomorphism section. As we will see throughout, the Hungerford text tends to be the outlier.

Informal Counterparts

Melhuish (2015) quotes portions of each textbook that she describes as informal characterizations of isomorphism. In all textbooks except Gilbert & Gilbert, they appear in the context of EMDs and precede the formal definition of isomorphism. In Gilbert & Gilbert, such characterization comes after the formal definition and is described as the "fundamental idea behind isomorphisms" (p. 178). For each textbook, these portions focus on isomorphic objects, such as the status of equivalence between the structures (e.g., rings) portrayed through general statements of sameness (e.g., isomorphic rings are "essentially the same", Hungerford, p. 70). In all these texts, this informal description of isomorphic objects as portraying sameness is used as justification for formally defining isomorphism functions.

Before continuing, we explain that there are two broad senses in which the isomorphism function has informal descriptions. The first sense only applies to isomorphism functions and involves an informal description of the function itself: we can exhibit a correspondence (function), wherein we align and rename elements between structures to prove structures are isomorphic (i.e., create a matching or relabeling). We refer to this as the *correspondence sense.* The second involves an informal description of attributes of the function (i.e., "preserve the operation(s)", Gilbert & Gilbert, p. 177). This aligns with the homomorphism property and, thus, we identify *operation preservation* as the informal description of the homomorphism concept which can also be used in an isomorphism function context.

Informal versions of the correspondence sense of the isomorphism function occur in the same block of text as the quotes highlighted by Melhuish (2015). This occurs most prominently in the Hungerford and the Fraleigh texts; for each text, the Melhuish example is in the context of a larger example (EMD) involving Cayley tables. Hungerford and Fraleigh use these examples to highlight the "relabeling" aspect of isomorphism and clarify that one needs to see the relabeling (alignment of elements highlighted by the function) before recognizing that the structures are isomorphic. In other words, there is an informal counterpart (relabeling one table to get another table) to the isomorphism function, and this counterpart is positioned as necessary for understanding isomorphic objects. We contrast this with Gallian and Gilbert & Gilbert, who do not accompany their informal description of isomorphic objects with informal counterparts to isomorphism functions. Gallian provides an example of isomorphic groups and states that the reader should recognize via inspection that these groups are "the same", while Gilbert & Gilbert do not provide an example to accompany their informal description.

Instead, Gallian and Gilbert & Gilbert focus on operation preservation as an informal description of the isomorphism function and homomorphism. Gallian uses this phrase within the formal definitions of isomorphism and homomorphism, whereas Gilbert & Gilbert use the phrase afterwards to explain both definitions. Since both Gallian and Gilbert & Gilbert introduce the definition of isomorphism before homomorphism, they are essentially introducing the informal notion of homomorphism prior to the formal definition. We can contrast this presentation with Fraleigh and Hungerford; neither use an informal description of homomorphism in the isomorphism section. Hungerford uses no informal descriptions for homomorphism in the homomorphism section either, while Fraleigh refers to homomorphisms as "structure-relating maps" (p. 125) throughout the homomorphism chapter.

Equivalence Relations

Equivalence relation sections are another place to see how textbooks discuss isomorphic objects, since "is isomorphic to" is an equivalence relation on any set of groups (rings, fields). Hence, we wish to see how textbooks explain isomorphic as an equivalence relation.

Fraleigh, Gallian, and Gilbert & Gilbert, describe "is isomorphic to" as an equivalence relation. Hungerford instead focuses on the properties of functions (isomorphisms) themselves, such as the fact that a composition of isomorphisms is an isomorphism, without ever mentioning how these properties of isomorphisms relate to isomorphic structures (e.g., without mentioning transitivity of "is isomorphic to"). This function focus is consistent with the fact that "is isomorphic to" is not one of the many equivalence relations mentioned in the equivalence relation section. Both Fraleigh and Gilbert & Gilbert use the equivalence relation notion when describing how "is isomorphic to" is a type of sameness. Fraleigh explicitly links "group identification", "up to isomorphism" and "equivalence relation", whereas Gilbert & Gilbert use equivalence relations to describe the notion of sameness captured by "is isomorphic to". In other words, all texts except Hungerford use the notion of equivalence to describe isomorphic structures, with Fraleigh and Gilbert & Gilbert additionally using it in their informal characterizations surrounding sameness. Instead of characterizing "is isomorphic to" as an equivalence relation, Hungerford focuses on the analogous properties of isomorphism functions.

Summary and Discussion

Textbook analyses have an important role in research on instructional practice; Zhu and Fan (2006) explain that several researchers have cited the need for further work in this area. We have provided such work by focusing on how commonly-used textbooks relate and structure facets of the isomorphism concept. We expanded on Melhuish's (2015) analysis of abstract algebra texts by distinguishing between isomorphic objects and isomorphism functions. Creating this distinction allowed us to carefully consider how textbooks structure and relate the ideas of isomorphic objects, the isomorphism function, and homomorphism. A common pattern was found in which all textbooks introduced isomorphism prior to homomorphism, and all three texts except for Hungerford defined the isomorphism function prior to isomorphic objects.

In addition to the discussion of the formal definitions, we delineated various ways textbooks informally described isomorphic objects, the isomorphism function, and homomorphism. This expands on Melhuish's (2015) work, which treats the informal introduction of isomorphism as a singular concept. We found that, although all texts used sameness language to describe isomorphic objects, there are differences in how the isomorphism function and homomorphism were presented informally in textbooks. Gallian and Gilbert & Gilbert used metaphors illustrating the operation-preserving sense of the isomorphism function and homomorphism but did not address the correspondence sense; Hungerford used metaphors illustrating the correspondence sense of the isomorphism function but not the operation-preserving sense (for isomorphism or homomorphism); and Fraleigh used metaphors illustrating the correspondence sense of the isomorphism function and operation-preserving sense of homomorphism, but not the operation-preserving sense of isomorphism. Here we see that while there is some consensus on informal interpretations of isomorphic objects (sameness), interpretations vary around the isomorphism function and homomorphism. Future research should examine how these differences of presentation impact instructors' teaching, if at all, and students' understandings.

While textbooks provide an intended curriculum for a course and often "have influence on teachers' teaching and students' learning" (Zhu & Fan, 2006, p. 622), they also provide potential alternatives to instructors' conceptualizations of material; for this reason, it makes sense to also consider the relationship between how instructors envision the curriculum and the intended curriculum embodied in the textbook. Accordingly, our analysis occurs in tandem with empirical investigation. Prior work examining instructors' language for isomorphism and homomorphism showed that instructors viewed sameness as the essence of isomorphism (Rupnow, 2021), which suggests an isomorphic focus. However, ongoing research with algebraists has shown a greater variety of perspectives. Some, like the instructors, emphasized sameness of structure, whereas others emphasized exhibiting the isomorphism function. This research highlights the importance of both isomorphic objects and the isomorphism function. Furthermore, this raises questions about how sequencing of instruction and language choices in instruction might differ and potentially impact students' understanding of different facets of the isomorphism concept.

Several professors' view of the isomorphism property as the conceptual center of the isomorphism concept and motivation for defining the isomorphism function aligns with our findings in textbooks. As discussed in "informal counterparts", multiple textbooks use the idea of isomorphism communicating sameness of object (isomorphic) to motivate formalizing isomorphism functions. Future research should examine whether isomorphic objects are commonly used to establish *intellectual need* (Harel, 2013) for the isomorphism function in other textbooks. Furthermore, research could examine justifications professors give for foregrounding the isomorphism function and how that could inform textbook construction.

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Distinguishing between Isomorphism and Equality in Abstract Algebra Texts: This Sameness is not the Same as that Sameness

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Isomorphism and equality are important aspects of mathematics and are both types of sameness. However, these are not identical concepts; objects can be isomorphic without being equal. We discuss the difference between equality and isomorphism as types of sameness and the way that popular introductory abstract algebra textbooks distinguish between and conflate these concepts.

Keywords: sameness, isomorphism, abstract algebra, group theory, textbook analysis

Sameness is an important aspect of mathematics. This is especially evident when we consider the prevalence of the equals sign. Isomorphism, which can be considered as a type of sameness, is crucial to various fields of mathematics such as abstract algebra, graph theory, and logic. In this preliminary paper, we distinguish between these two notions of sameness and examine how mathematics textbooks distinguish between them. This paper has both theoretical and empirical goals, the first of which is to orient the reader to the nuances surrounding equality versus isomorphism, the second of which is to provide a textbook analysis concerning these nuances.

Literature Review

There are different types of sameness. In the strictest sense, we have the notion of true equality. Colloquially, we use the phrase 'the same' to describe equality (Rupnow & Sassman, 2021). In modern mathematics, $x = y$ if and only if x and y are identical, which is the case if and only if 'x' and 'y' denote the same object (Mendelson, 2009). For example, $e^{i\pi}$ and -1 are identical (equal) because '$e^{i\pi}$' and '-1' both denote the same object, the number -1. It bears mentioning that such equations are informative because the *names* or *labels* of these objects are different. For example, '$e^{i\pi}$' and '-1' look different and bring different things to mind (Mirin, 2020). Here, we have the same thing (the number -1) with different names (the names '$e^{i\pi}$' and '-1'); changing a name or label of an object does not change the object itself. Alternatively, we could use the name '$7 - 8$', and it would still denote the same object as '$e^{i\pi}$' and '-1' despite having a different label. Less strict notions of sameness are also captured by various non-equality equivalence relations. For example, the numbers 0 and 4 are equivalent with respect to the equivalence relation of having the same parity, but 0 and 4 are still different numbers.

One important aspect of sameness is that it allows us to make powerful inferences in mathematics. Leibniz' Law of Indiscernibles (Leibniz, 1846/1992) states that two objects x and y are equal ($x = y$) if and only they share the same properties. For example, since $e^{i\pi}$ and -1 are equal and since -1 has the property of being real, it follows that $e^{i\pi}$ has the property of being real. Non-equality equivalence relations allow for some inferences, but not as many. For example, consider the equivalence relation $\equiv_2$ defined on the integers by $x \equiv_2 y$ if and only if x is congruent to y modulo 2. Then we have that $0 \equiv_2 4$, which allows us to make some inferences but not others. For example, we can conclude that both 0 and 4 are one more than an odd integer. Clearly, 0 and 4 do not share all properties. For example, 4 has the property of being positive, but 0 does not. We can make similar statements about the equivalence relation of isomorphism. As

Fraleigh (2003) highlights, some properties (which he calls "structural properties") are preserved by isomorphism, whereas others are not. This role of inferences concerning property preservation highlights the crucial difference between true equality and other forms of equivalence. In algebraic structures such as groups, we have the notion of equality (identical groups), but we also have the weaker notion of isomorphism.

While there is sparse literature on the topic of sameness and equality, there is a little literature that distinguishes between types of sameness. However, as Mirin (2019) notes, the bulk of this literature tends not to differentiate between true equality and other equivalence relations. In fact, the word "equivalence" is often used without specifying an equivalence relation. Wladis et al., (2020) observe that the *Common Core Standards* (National Governors Association, 2010) often use the word "equivalence" to refer to strict equality, and many other equivalence relations are not labeled as such.

We now move to the literature on isomorphism and how it relates to distinguishing between strict equality and other types of sameness. Interestingly, the notion of "relabeling" and "renaming" to describe isomorphism is mentioned in multiple articles (Dubinsky et al., 1994; Leron et al., 1995; Rupnow, 2021; Weber & Alcock, 2004). This notion has the potential to conflate isomorphism with equality - as discussed above, renaming or relabeling an object does not change the underlying object. In fact, Randazzo & Rupnow (2021) confirm this interpretation when they refer to "renaming" using the following language: "renaming of an object or a change of perspective that does not change the object itself" (p. 240). Rupnow and Sassman (2021) discusses the ways that algebraists view the notion of sameness as harmful and helpful when referring to understanding the concept of isomorphism. Some participants noted that it was important for students to distinguish between isomorphism and equality. For example, one participant explained a harmful way of thinking of isomorphism as sameness: "Maybe thinking that sameness = identical in every aspect? ". (p.TBD). Additionally, some of the mathematicians interviewed in Randazzo and Rupnow (2021) are careful to distinguish between true equality and isomorphism. Dubinsky et al's (1994) study gives an existence proof that some students do indeed conflate isomorphism with equality. In this study, students were given questions about isomorphism wherein the concept of isomorphic groups was described as "the same except renaming", which appears to be a conflation of equality with isomorphism (since isomorphisms can change underlying objects, not just rename them). The authors report on students confusing the notion of subgroup with isomorphic subgroup, which we view as a conflation between equality and isomorphism. This confusion highlights the importance of distinguishing between the concepts of equality and isomorphism.

Methodology and Theoretical Background

In this paper, we investigate the following: How do common introductory abstract algebra textbooks use the notion of sameness to introduce the concept of isomorphism? How do textbooks distinguish or conflate the notions of identical groups and isomorphic groups?

The textbooks examined were those identified by Melhuish (2015) as the most commonly-used texts in introductory abstract algebra classes. In order from most popular to least popular, they are: Gallian (2009), Fraleigh (2003), Gilbert and Gilbert, 2008, and Hungerford, (2012). Optimal Character Recognition was used to find all portions of the text that used the string "iso" (in order to capture "isomorphic" and "isomorphism," and "equ" (in order to capture "equal" and "equivalent"). Additionally, we examined every instance of what Melhuish (2015) calls an "Example Motivating a Definition" (EMD) and every instance of an "Example Following a

Definition" (EFD) for the definitions of "isomorphism" and "isomorphic".[1] In the Fraleigh text, there were two introductions of isomorphism - one introduced more informally in an example within the text, and the other later but more formally. The blocks of text quoted by Melhuish (2015) as involving informal characterizations of isomorphism were also examined and included with the EMDs and EFDs depending on the position of the informal characterizations; i.e., whether they appeared before or after the formal definition.

This analysis can be understood as a conceptual analysis of textbooks as described by Thompson (2008). Similar to the textbook analysis of Cook et al. (2019), it describes the way that textbooks relate similar concepts (in this case, isomorphism and equality). Analyzing commonly-used textbooks provides a way of examining the "intended curriculum" (Son & Diletti, 2017) and reflects the way that a topic is taught and hence the way that a student might learn (Zhu & Fan, 2006). Our analysis is consistent with thematic analysis (Rossman & Rallis, 2016).

Before moving forward with our textbook analysis, it is worth clarifying for ourselves the distinctions between equality and isomorphism. For simplicity, we use groups as an example, but rings, fields or other objects to which isomorphism can apply could be used to have the same discussion. Recall the definition of a group; a group $\boldsymbol{G} = \langle G, \cdot \rangle$ is a set G together with a binary operation $\cdot$ on G satisfying certain axioms. As Mirin (2017) notes, the way a class of objects (in this case, groups) is defined is closely related to the equality criteria within that class of objects. Hence, a group $\boldsymbol{H} = \langle H, \times \rangle$ is equal to $\boldsymbol{G}$ if and only if $G = H$ and $\cdot = \times$ (i.e. $\cdot$ and $\times$ agree on all elements in G). With that in mind, consider the following question: How many groups are defined in Figure 1?

G_1

*	0	1
0	0	1
1	1	0

G_2

+	a	b
a	a	b
b	b	a

a=0
b=1

G_3

⋆	[0]	[1]
[0]	[0]	[1]
[1]	[1]	[0]

[0]={0,2,-2,4,-4,...}
[1]={0,1,-1,3,-3,...}

Figure 1. Two different groups of order two.

There are two distinct groups, $\boldsymbol{G_1}$and $\boldsymbol{G_3}$. Observe that $\boldsymbol{G_1}$ and $\boldsymbol{G_2}$ are identical. They have the same elements, and the same group operation. This is despite the fact that the elements in the group table for $\boldsymbol{G_1}$ are written differently than the elements in the group table for $\boldsymbol{G_2}$; note that we know that $\boldsymbol{G_1} = \boldsymbol{G_2}$ because we know (by stipulation) that the element *a* is identical to 0 and *b* is identical to 1. If there were no stipulation, then we would not be able to conclude that $\boldsymbol{G_1} = \boldsymbol{G_2}$, nor could we conclude the negation. It is clear that $\boldsymbol{G_2}$ and $\boldsymbol{G_3}$ are different groups; elements of $\boldsymbol{G_2}$ are integers, and elements of $\boldsymbol{G_3}$ are infinite sets.[2] So, among $\boldsymbol{G_1}$, $\boldsymbol{G_2}$, and $\boldsymbol{G_3}$, there is *one*

[1] For all texts except Hungerford (2012), groups were introduced before rings, so these results pertained to isomorphic groups. For Hungerford (2012), we examined the ring section, since that is where the concept of isomorphism is first introduced.

[2] It is not uncommon to see someone name the set of even integers "0". However, when people do this, they are usually explicit that they are adopting this convention. Furthermore, we are not adopting this convention; we defined the group G_1 to have only the integers 0 and 1 as members.

isomorphism class, *two* groups, and *three* group definitions. How is the "sameness" between $\boldsymbol{G_1}$ and $\boldsymbol{G_2}$ different from the "sameness" between $\boldsymbol{G_1}$and $\boldsymbol{G_3}$? The answer is that $\boldsymbol{G_1}$and $\boldsymbol{G_2}$ are identical, whereas $\boldsymbol{G_1}$ and $\boldsymbol{G_3}$ are merely isomorphic. Our analysis investigates how common introductory abstract algebra textbooks conflate and distinguish these types of sameness.

Results and Discussion

All four texts use the notion of sameness to describe isomorphism. In all the textbooks, the notion of sameness is used to establish intellectual need (in the sense of Harel, 2013) for defining isomorphism (see Table 1). This is done prior to formally defining the notion of isomorphism, and in all four texts is used in the context of Examples Motivating Definitions (EMDs) (Melhuish, 2015).

Table 1. Sameness language textbooks use to motivate the isomorphism concept.

Text	Phrase	Locations
Fraleigh	"Same algebraic properties", "structurally alike"	EMD, p.16; EMD, p.28
Hungerford	"Essentially the same", "have the same structure"	EMD, p.71
Gilbert & Gilbert	"Same structure"	Intro, p.137
Gallian	"Same group is described with different terminology"	EMD, p.127

Now that we have established that the notion of sameness is used to motivate and describe isomorphic structures, we turn to investigating the ways in which isomorphic structures are or are not distinguished from identical structures. While all the texts use the adjective "same", the extent to which they mean truly the same (identical) is a matter of investigation.
The following table (Table 2) provides a breakdown of each EMD and EFD for "isomorphism" in all four texts and how they do or do not distinguish between isomorphism and equality. This table delineates by both percentage and total number of occurences whether these examples conflate isomorphism with equality (*Conflation*), whether they distinguish them (*Distinction*), or whether neither happens (*Neutral*).

Table 2. Conflation vs. Distinction in EMDs and EFDs.

Type	Conflation	Distinction	Neutral
EMD	68.42% (13)	26.32% (5)	5.26% (1)
EFD	16.13% (5)	6.45% (2)	77.42% (24)
Total	36.00% (18)	14.00% (7)	50.00% (25)

The Neutral instances do not have the potential to conflate or clearly distinguish in the sense that no sameness language is used, nor is there any clarification about lack of sameness. These mostly include formal language about isomorphism and mostly take place in EMDs. For example, Gallian's Example 1 (2009, p. 129) provides a formal proof that the positive reals under addition is isomorphic to the reals under multiplication. Since there is no sameness language, and no language comparing these groups, we classify this as "neutral". The Distinction instances clearly differentiate between isomorphism and equality. For example, Gallian provides a distinction when they write "It requires somewhat sophisticated techniques to prove the surprising fact that the group of real numbers under addition is isomorphic to the group of complex numbers under addition". By referring to this result as "surprising", the authors are implying that these groups are truly different and not just the same group named differently.

Conflations occur in all the texts; every text had an EMD that was a conflation. All the texts except for Gilbert & Gilbert use the notion of different names for the same number to characterize isomorphism. For example, Gallian writes "The American says, 'one, two, three, four, five,....,' whereas the German says 'eins, zwei, drei, vier, fünf,..'. Are the two doing different things? No. they are both counting the objects, but they are using different terminology to do so". Recall the earlier discussion about equality versus isomorphism; isomorphic groups do not necessarily have the same objects with different names - generally speaking, they actually have different objects (the same objects is a special case of isomorphism) and hence are different objects. Therefore, by using the idea of *same object different name* as a motivating example for isomorphism has the potential to conflate isomorphism with equality.

Of course, examples (EMDs, EFDs) are not the only occasion that textbooks have to conflate or distinguish isomorphism from equality. There is also plenty of verbiage surrounding informal characterizations of isomorphism. For example, Gallian includes a section where they clearly distinguish between isomorphism and equality and then announce that they will refrain from distinguishing: "(...) algebraists speak of isomorphic groups as 'equal' or 'the same.' Admittedly, calling such groups equivalent, rather than the same, might be more appropriate, but we bow to long-standing tradition." (2009, p.127). Gilbert and Gilbert implicitly differentiate between isomorphism and equality by explaining that they use different names for the operations of isomorphic groups (in the context of defining "isomorphic groups") to clarify that these groups have different operations. Although Gilbert and Gilbert do not explicitly state that these groups are not identical, they are heavily suggesting it by clarifying that the group operations are nonidentical. However, the textbooks tend to use language in informal characterizations of isomorphism that conflate isomorphism with equality. For example, Gilbert and Gilbert write the following in their informal characterization of isomorphic groups: "They are algebraically the same, although details such as the appearance of the elements or the rule defining the operation may vary" (2008, p.178). By suggesting that it's only the appearance of the elements - rather than the underlying elements themselves - that change, Gilbert and Gilbert are conflating the notion of isomorphism with equality. Fraleigh and Hungerford make similar conflations in their informal characterizations of isomorphism, also using the "relabeling" and "renaming" language. It seems possible that, when making such a conflation, such authors might be thinking of relabeling an isomorphism class rather than an individual group. Case in point, Gallian quotes R. Allenby as defining an algebraist as "a person who can't tell the difference between isomorphic systems."(2009, p. 127). The fact that all four textbooks do not seem to agree on the definition of Z_n (Hungerford defines its elements as equivalence classes of integers, whereas the remaining texts have its elements as integers) corroborates R. Allenby's characterization. The reader might be wondering - if algebraists conflate isomorphism class (of groups) with groups, why shouldn't students? We answer this question with two points: 1) there are other fields of math, such as model theory, in which mathematicians do carefully distinguish between isomorphism classes and groups (Hodges, 1997) and 2) why introduce "isomorphism" as a concept if equality suffices? Why, then, define the notion of "isomorphism class"?

Further study of how students interpret these different types of sameness is needed. In particular, would better articulation of the different samenesses produce better results in a real world abstract algebra class, and how could that be integrated into effective pedagogy?

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Disability Accommodations in College: Alarming Discrimination in Mathematics

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Academic disability accommodations are essential for providing equitable access to education. This study examines the barriers that college students face in obtaining their disability accommodations in mathematics classrooms. It shows that even when a student has school-approved disability accommodations, implementing them is often not a straightforward process. In particular, the results of this study suggest that college instructors -- including mathematics instructors -- are a barrier to students receiving the accommodations to which they are entitled.

Keywords: disability, equity, discrimination, accommodations

The SIGMAA on RUME Position Statement on Equity states "An important step towards inclusivity involves identifying and removing barriers for full participation (...). These groups include but are not limited to: people of color, women, people living in poverty, **people with disabilities** (hidden or otherwise) [emphasis added] (...)" (Committee on Equity and Mentoring, 2018). Accordingly, our research examines the following research question: what barriers do disabled college students - in particular, mathematics students - face accessing their student-approved disability accommodations?

Background and Literature

The Americans with Disabilities Act (ADA, 1990) and Section 504 of the Vocational Rehabilitation Act (1973) mandate that tertiary institutions provide reasonable accommodations for students with disabilities. With 10% of college students having a disability (U.S. Department of Education, National Center for Education Statistics, 2009), these accommodations are relevant to a large subset of students. Kim and Lee (2015) confirmed that disability accommodations do indeed improve college students' grade point averages. Mamiseishvili and Koch (2011) found that using disability accommodations during their first year in college increased first-to-second-year retention rate amongst disabled students. Considering this fact, it is paramount that disabled math students have access to their accommodations.

There is literature that, although not specific to mathematics, addresses the various barriers that disabled students face. Sometimes students choose not to implement their accommodations due to having perceived instructors' negative attitudes toward such accommodations or having had past negative experiences with instructors regarding their accommodations (Hartman-Hall & Haaga, 2002; Marshak et al., 2010). Rao (2003) explores the various attitudes that college instructors have towards students with disability accommodations. Although mathematics was not considered specifically, the study found that the faculty of engineering - a field closely linked with mathematics - were less likely than other faculty to report being willing to provide accommodations. While Rao's (2003) study addresses self-reported faculty attitudes, it does not examine student experiences, nor does it specifically address the attitudes of mathematics instructors. Hence, our research aims to narrow this gap by examining what happens when students attempt to implement their school-approved accommodations, with a focus on mathematics classrooms.

Methods

Data were collected via an anonymous online survey during the last week of August, 2021. To be eligible for the survey, participants had to be at least 18 years of age and have or have had school-approved disability accommodations through a US college. Recruitment was done primarily through posts on social media (Facebook, Twitter, Reddit, Google Groups). Special effort was made to recruit within online communities that were focused on disability (e.g., Facebook groups for learning disabilities, neurodivergence, illness, and other disabilities). Additional recruitment was done via announcements from one public university's disability services office, as well as requests to professional and personal connections to distribute the survey via email and social media. In total, 117 eligible participants completed the survey.

The first set of questions concerns demographic information, the second set includes information about academic background (time in school, number of math courses taken, information about in which courses they requested accommodations), and the third set asks specific questions about experiences implementing college-approved disability accommodations. The third set ends with following question meant to capture a range of negative experiences (this question will hereafter be referred to as overall): "True or false: every time that you have requested your college/university-approved accommodations from an instructor for any course in the USA, that instructor has complied without resistance or hesitation, and without making any negative remarks or invasive questions regarding your disability(ies) or accommodation(s)?" (Table 1).

Several questions in the third set allowed the participants to account for experiences specifically in mathematics courses. These included negative instructor experiences when attempting to implement accommodations (refusal, difficulty, skepticism, remarks, Table 1) These questions were placed prior to the culminating overall question in order to encourage participants to reflect carefully on different types of experiences before having to simultaneously reflect on the multitude of experiences captured by the overall question. Additionally, there was a question (disclosure, Table 1) regarding students' refraining from using their accommodations due to fear of their instructor's perception or response.

Table 1. Instructor experience questions and their abbreviations.

refusal	Has an instructor ever refused to comply with your college/university - approved accommodations?
difficulty	Has an instructor ever made it difficult for you to receive your college/university - approved accommodations?
skepticism	Has an instructor ever expressed skepticism about your disability(ies) or disability accommodation(s)?
remarks	Has an instructor ever made a disparaging or negative remark about your disability(ies) or your disability accommodation(s)?
disclosure	Have you ever refrained from mentioning your disability accommodation(s) to an instructor due to concern over the instructor's perception of you, your disability, or use of accommodations?

overall	True or false: every time that you have requested your college/university-approved accommodations from an instructor for any course in the USA, that instructor has complied without resistance or hesitation, and without making any negative remarks or invasive questions regarding your disability(ies) or accommodation(s)?

The survey ended with an open-ended question prompting the participants to "share anything else you would like to about your experience having disability(ies) or disability accommodation(s). What do you want your instructors to know? What do you want your math or statistics instructors, in particular, to know?"

Results

Of the 117 participants who completed the survey, three who reported that they did not ever attempt to enact their accommodations were excluded from the analysis. Our findings suggest that instructors, including math instructors, often impose barriers to students' accessing their accommodations. We consider the results for two populations: the general population (114 participants), and the "math population" (44 participants). Before going to the specific results, it is worth noting that the average number of years the general population participants spent in college is 5.39, their average number of math classes taken is about[1] 4.27, and the average number of math classes in which they attempted to implement school-approved accommodations is about 3.07. Also, 34 members of the general population (29.82%) had not ever attempted to implement their accommodations in a math course and had hence likely never had the opportunity to have a negative experience regarding accommodations with a math instructor. The math population includes only those participants who have completed at least three math courses and have requested to implement their accommodations in at least three courses. Of the math population, their mean completed number of years in college/university is 5.86, the mean number of math courses taken is about 7.30, and the mean number of math courses in which the participants attempted to implement school-approved accommodations is about 6.80. These numbers provide us a sense of how often students have opportunities to have certain experiences in implementing their accommodations, particularly in math classes.

For the overall question (Table 1), 42.98% (49 participants) of the general population answered "true". That means that fewer than half of students answered that they have unproblematically been able to implement their accommodations. Additionally, 15.79% (18 participants) report that an incident leading them to answer "false" occurred in a math course. When we narrow to the math population, we have that 16 out of 44 students (36.36%) answered "true". Twelve students (27.27%) answered "false, and at least one incident occurred in a math or statistics course". For this specific question, we look at a third population: students who have taken at least one math class and have requested accommodations in at least one math class. For this question and population, we have alarming results: 31.25% of this population answered "yes, and this happened at least once with a math or statistics instructor". Due to space constraints, we do not explore this population further. However, the fact that the rate of negative experiences with math instructors is higher for a single math class rather than three is worthy of further exploration.

[1] This is a possible slight underestimate. Two people chose "20+" for "number of math courses taken" and "number of math courses in which accommodations were attempted to be implemented", which were coded as "20".

We now move to perhaps the most severe form of disability discrimination discussed in this study: instructor refusal to implement or allow college-approved accommodations (the *refusal* question in Table 1). Only 64.91% (74 participants) out of the general population claimed that they have never had an instructor refuse their school-approved accommodations. A total of 5.26% (6 participants) have been refused accommodations in a math course. When we narrow it down to the math population, we see that the situation is somewhat worse; 11.36% (5 participants) were refused accommodations in a math class, and 59.06% (26 participants) claimed that they had never encountered a refusal from any instructor. For example, one student reports on a math instructor who refused to allow them to take an exam in the school's testing center (to satisfy an accommodation of a distraction-free environment). There are less severe situations than refusal that still should not occur, such as instructors imposing any difficulty in the accommodation process. Unfortunately, 41.23% (47 members) of the general population answered that an instructor has never made obtaining an accommodation difficult, and 14.91% (17 participants) answered that this happened at least once in a math course. For the math population, 34.09% (15 participants) answered that an instructor had never made it difficult to obtain their accommodations, while 29.55% (13 participants) report that a math instructor did made it difficult. One student explained that a math instructor initially refused to allow her to use her accommodations of recording lectures and taking extra time on exams.

A student is theoretically protected from revealing their specific disability and disability documentation to their instructor. Instead, a school's disability office vets and approves these accommodation requests through documentation from the student's medical provider(s). Despite this rigorous process, some students encounter skepticism from their instructors and might even be asked to reveal personal information. Only 64.91% (74) of the participants of the general population answered that they had never had an instructor express skepticism about their school-approved disability accommodations. A total of 8.77% (10 participants) reported on such an incident occurring with a math instructor. Amongst the math population, 63.64% (28 participants) answered "no", and 18.18% (8 participants) claimed that this occurred with a math instructor. One participant explained that a statistics instructor would question them in front of their entire class about why they needed their accommodations and claimed that such a successful student would not need accommodations. Another participant wrote: "[math instructor] said I was lieing *[sic]* and couldn't be in a PhD program with my disabilities".

In addition to the various immediate obstacles in implementing their accommodations, many students faced disparaging remarks from their instructors regarding their disabilities or accommodations. Only 57.89% (66 members) of the general population had not encountered any disparaging or negative remarks, while 7.89% (9 members) had experienced such remarks from their math instructors. Only a minority (47.73%, 21 members) of the math population reported that an instructor had never made disparaging or negative remarks about their disability(ies) or disability accommodation(s), while 11.36% (5 members) reported that this had happened with a math instructor. One participant wrote "I tried to explain my disability to a [math] teacher, and they said I should not talk about my disability, because they thought it was something to hide". While a minority of students encountered such negative and disparaging remarks from math instructors, the size of this minority is still concerning.

So far, we have documented a multitude of negative experiences in utilizing accommodations. Although a disparaging remark from a faculty member is not necessarily an immediate hindrance to using accommodations (a faculty member could make a disparaging remark but still not resist a student's accommodation), Hartman-Hall and Haaga (2002) suggest that perceived negative attitudes of faculty members concerning disabilities and accommodations make a student less likely to attempt to implement their accommodations in the future. Hence, such disparaging remarks from instructors can indirectly act as a barrier to students having access to their accommodations. The results of this study show that 61.40% (70 members) of the general population answered “no” to *disclosure* (Table 1, “Have you ever refrained from mentioning your disability accommodation(s) to an instructor due to concern over the instructor’s perception of you, your disability, or use of accommodations?”), and 10.53% (12 members) answered that they had refrained from disclosing in a math or statistics course. Only 65.91% (29 members) of the math population answered “no” to disclosure, with 13.64% (6) specifying “yes, and this happened at least once with a math or statistics course”. One participant explained “I was afraid that the professor would think I was making up disabilities or that I was lazy. I ended up barely passing the [math] class”. As we can see from the results of *skepticism*, this student’s fear might not have been unfounded. Expectedly, we see a ripple effect; students who have had negative experiences in implementing their accommodations (as measured by *overall*) are more likely to have refrained from requesting accommodations (as measured by *disclosure*); 12.2% of students without negative experiences have refrained from implementing their accommodations, whereas 49.1% of students with negative experiences have refrained (chi-square = 16.4474, $p<.05$).

Conclusion and Discussion

This study shows that, even when college students have gone through the documentation and vetting process of obtaining college-approved academic disability accommodations, utilizing these accommodations can be a problematic process. Specifically, the results suggest that only a minority of students always had a straightforward experience in implementing their accommodations (as measured by *overall)*, and many of these students had barriers imposed by their math instructors. Students experience widespread discrimination from faculty and math instructors. This discrimination takes many forms and includes direct barriers to receiving accommodations, such as refusal, as well as indirect barriers, such as disparaging remarks and skepticism. When a student has a negative experience with an instructor, that negative experience has the potential to hinder the student from implementing their accommodations in future classes (Marshak et al., 2010). These results are especially concerning in light of the fact that accommodations help disabled students succeed by providing access to education. It seems that, based on the *skepticism* question, some instructors might not truly believe that students are disabled.

We end this paper with a participant’s response to “What do you want your instructors to know? What do you want your math or statistics instructors, in particular, to know?”:

> “I would like instructors and administrators to know that if they refuse to accommodate a student's disability and they don't hear anything else from that student again, it's not because everything was fine after that. It's because that student quietly swallowed the damage to their dignity and their passion and their trust in the institution, and did their best to move on and work around the problem because they simply didn't have the energy and resources to fight about it. And their education likely suffered because of it.”

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Examining the Effects of a Math Intervention Program Designed for Entry Level Undergraduate Mathematics Courses

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This study is an examination of my work as a math intervention coordinator at a small liberal arts college. As part of the Strategic Investment Fund, a math intervention coordinator position was created at our college to provide support for students in entry level undergraduate mathematics courses, often College Algebra. Some students in these courses typically do not have the prerequisite skills to be successful otherwise. As the math intervention coordinator, I worked one-on-one with students to develop a plan for becoming successful and to track students' progress. The purpose of this report is to present results of this intervention program after one semester and to reflect upon its success. Analysis from 15 participants shows that most students reported higher grades, better study and time management skills, and overall more positive attitudes.

Keywords: Math Intervention, College Algebra, Student Success

Oftentimes, students enter college not prepared for what lies ahead of them. Specifically, many students are not prepared to take college mathematics courses. As Tague et al. (2020) note, many colleges and universities require a placement exam to determine which mathematics course students can enter. This often results in students being placed in remedial college algebra courses, due to lacking the necessary prerequisite skills from high school. In addition, these remedial courses sometimes do not count towards college credits and often limits students from completing their degree. Further supporting this concept, Wilkins et al. (2021) found that "students who begin their study of mathematics with a College Algebra course or a Precollege Algebra course have less than a 50% chance of graduating with a degree in engineering even if they receive an A in the course, 40% and 41%, respectively" (p. 628). This means that simply being placed in a College Algebra course lowers students' odds of completing their degree by 50%. Combined with the issue of not being prepared for College Algebra, the chances of these students completing their degree are low.

At our college, similar to others, students are placed in appropriate mathematics based on ACT and/or placement test scores. Although some college and universities have replaced the remedial pre-requisite model with the co-requisite model (Rodriguez et al., 2018; Tague et al., 2020), our college halted both models in Fall 2021 due to state changes and faculty resignations, making College Algebra the entry level course. Because many students come to college not prepared to take College Algebra, as part of the Strategic Investment Fund (SIF; University of Virginia, 2019), a math intervention coordinator (MIC) position was created at our college to provide support for students in those entry level courses. The goal of the MIC was to work one-on-one with students to develop a plan for becoming successful and to track students' progress. The purpose of this study is to examine the effects of implementing this program for one semester and to reflect upon its success, as determined by students' course grade as well as faculty and student feedback.

Relevant Literature

College readiness has been at the forefront of educational research for many years. Much of this research has focused specifically on the mathematics content, since it was a major predictor of student success in the STEM field (Bettinger et al., 2013). In 2018, the National Council of Teachers of Mathematics (NCTM) highlighted the deficiencies of our current high school mathematics programs. In their book, NCTM (2018) argue that many students leave high school underprepared for college. In fact, they claim that 59% of students are not prepared for college math. For example, students in a Virginia high school are only required to complete three mathematics courses to graduate with a standard diploma (VDOE, 2021). This can be a combination of Algebra I, Geometry, Algebra Function and Data Analysis, and/or Algebra II. Students can also substitute an approved computer science course for one of these mathematics courses (VDOE, 2021). This means that some students who enter college may not have taken a mathematics course since their sophomore year, or the highest mathematics course they may have taken is geometry. As a result, college and universities are tasked with supporting these students' needs.

Throughout the literature, there are several supports and interventions that have demonstrated increased success in entry level undergraduate mathematics courses. These supports can be grouped into two categories: *content related supports* and *non-content related supports*. *Content related supports* are those focused on supporting students' content knowledge and understanding of math. These include supports such as tutoring, remediation, and recitation courses (Bettinger et al., 2013; Rodriguez et al., 2018; Tague et al., 2020). For example, remediation and remedial courses have proven success for students who are underprepared for college math. However, as mentioned above, these courses often do not count towards college credit. In addition, co-requisite or recitation courses have proven success, but finding faculty available or willing to teach these courses is often difficult (Rodriguez et al., 2018). Xu et al. (2001) also emphasized the benefits of a tutor center, which helped increase students' College Algebra final exam scores.

Non-content related supports address concerns such as studying, time management, attitudes, and motivation (Credé & Kuncel, 2008; Lane et al., 2017). For example, Downing (2020) noted that some students enter College Algebra with positive attitudes about math, and some enter with negative attitudes. However, after the use of an intervention, focused on culturally relevant pedagogy, many of the students who disliked mathematics changed their attitudes to more positive ones. These students left the class finding more value and enjoyment in math. Furthermore, Lane et al. (2017) noted that oftentimes, students do not know how to study or are afraid to ask for help. Therefore, by creating study habits, developing time management skills, and "normalizing help-seeking behaviors" (p. 172), they were able to effectively support students who were not prepared for college mathematics. From these studies, it is clear that not only do students need support in content, they also often need support in non-content related skills.

Methods

The college at which I work is a small liberal arts college in the Southeastern United States. There are currently 15 faculty members in the mathematics and computer science department. Every student enrolled at the college must take at least six mathematics credits, regardless of the major. The lowest mathematics course offered in Fall 2021 was MTH 1010 College Algebra. In previous years, students could be placed in a lower mathematics course, MTH 0900 Mathematics, a credit/no credit only course, which served as a remedial course to prepare students for College Algebra. Students could also place in MTH 1010 with the one-hour co-requisite course MTH 1011 College Algebra Recitation, also offered as credit/no credit.

However, due to state changes and faculty resignations, these options were removed for the Fall 2021 semester. Therefore, the focus of this study will be on MTH 1010 College Algebra.

In Fall 2021, there were 8 different sections of MTH 1010 College Algebra offered with a total of 125 students. Of these 125 students, 53 had placed into MTH 0900 or MTH 1011 based on their Act scores and/or the department placement test.

Data Collection

The Math Intervention Program. As part of the SIF112 Wise Innovation Ecosystem grant (UVA, 2019), a MIC position was created in 2018 in our mathematics and computer science department to provide support for students in entry level undergraduate mathematics courses. In previous years, the MIC taught and supported students in the MTH 0900 and 1011. However, in Fall 2021, with all students being placed in MTH 1010 and with no opportunities for remediation or recitation, the department had to develop other ways to support those students. The goal for the MIC in Fall 2021 was to revise the old intervention program, increase the number of participants in this program, and to promote student success by helping students improve their course grades.

As the new MIC of 2021, I reflected upon the previous intervention program and found that in 2018-2019, there were three major factors that were identified as negatively impacting student success: (a) life events outside of academics, (b) motivation and not quite understanding how to be a college student, and (c) prerequisite knowledge. To address these areas, students were offered *content related supports* through the MTH 0900 and 1011 courses from 2018-2019. Students were also offer *non-content related supports* such as creating weekly schedules and developing personal and academic goals. In 2021, I felt these factors were still relevant and the interventions were still needed. However, they needed to be revised to accommodate the departments new course requirements.

Student Referrals. Whereas before these students were placed in the MIC's courses, faculty needed a way to refer students to me so I could offer them extra support. As a result, I created three options for student referrals: (1) faculty completed a form and emailed it to me, (2) faculty submitted the same form on our department's Moodle page, or (3) use the math intervention webpage to complete an online submission form for student referrals. Students could also access the online form on the webpage and refer themselves if they felt they needed additional support.

Student Interviews. Once a student was referred to me, I emailed the student to schedule an initial meeting. During this time, I interviewed students to identify their weaknesses, areas for improvements, short- and long-term goals, as well as establish a baseline of their current study habits. This helped me to determine *content related* and *non-content related supports* that would need to be implemented.

Interventions. Reflecting upon the literature, interventions were categorized as *content related* and *non-content related supports*. The *content related supports* that I used were tutoring, faculty office hour visits, online resources such as Khan Academy, as well as online manipulatives. The *non-content related supports* that I used included creating short- and long-term goals, as well as developing weekly schedules consisting of time for class, homework, tutoring, and studying. I also had students keep a log of their hours so they could track their daily progress and ensure they were meeting their goals.

Tracking Student Progress. From the initial referral, I recorded the intervention information in an excel spreadsheet. I then met with each student once a week or every other week to reflect upon their progress and re-evaluate their interventions. These meeting notes were added to their initial interview document, and general information included in the spreadsheet.

Feedback from Faculty and Students. At the end of the semester, I gathered feedback from faculty and students about the intervention process. I asked faculty for feedback on the referral process and areas for improvements to evaluate its success. I also asked students for feedback about the intervention process and areas for improvements. Finally, I revisited each students' goals and determined if they were met or not.

Data Analysis

Preliminary data analysis consisted of analyzing the data in the excel spreadsheet. I first calculated the total the number of students referred, the number of referrals by instructor, the number of referrals made by individual students, the number of meetings per student, as well as the number of tutoring sessions they attended. I also analyzed the different interventions used, evaluating the most commonly used ones, as well as the ones associated with the greatest improvements in grades. In addition, I examined students' goals to determine if they were met. Finally, I reviewed the feedback from faculty and students, identifying the successes and weaknesses of the program. This feedback will then be combined with the aforementioned data to evaluate the overall success of the intervention program.

Results

Initially, there were 25 MTH 1010 students involved in the intervention program. There were 22 students who were referred by three different faculty members as needing intervention support (1, 5, and 16 respectively). There were also three other students who referred themselves. Of the 22 faculty referrals, 4 students were referred for attendance, 2 were referred because they lacked prerequisite skills, 15 were referred for low quiz/test scores, and 1 was referred for "other" reasons: "She seems confident when answering questions in class. This may just be a case of a little math phobia....hoping she just needs some positive reinforcement." Of the three student self-referrals, one student referred themselves for low quiz scores and the other two indicated "other" reasons, including the following comments: "I have extreme difficulty with math to begin with, and this teacher has not been super helpful. I don't think it is possible for me to pass this class without help" and "Feel like I need extra support". However, five students never responded to my emails requesting an initial meeting, five responded but never showed up. And 3 dropped MTH 1010 before meeting with me. This resulted in 15 students actively participating in the intervention program.

Although 12 of the 15 referrals were for content related concerns, when looking at faculty and student comments, 7 students were noted as having low confidence, "math phobia", poor "planning and management" and poor test taking skills. After interviewing 15 students, I felt that 12 needed *non-content related supports* in addition to *content-related supports*. The other three students needed only *content-related supports.* For these students, I referred them to tutoring services. For the other 12 students, we worked on making weekly schedules to better manage their time. I also referred them to tutoring services as an additional resource and scheduled tutoring time in their week. Then, each week I met with students to revise their schedule to fit their needs. For example, just after one week, one student, Erin, wanted to schedule more time in her schedule for studying. She wanted even more structure than the first iteration we developed. Another student Ona wanted a bit more flexibility in her schedule that would allow her to study for specific classes as assignments developed. In this case, we scheduled hour time slots dedicated for studying that she could use for whatever assignments, quizzes, or tests she needed to complete. In addition to developing a weekly schedule, four students needed support with math and test anxiety. For these students, I gave them resources on how to deal with test anxiety, helped them develop study habits, as well as referred them to the counseling.

Overall, the most commonly used intervention was developing a weekly schedule. These students needed structure and specific time designated to studying, doing homework, and attending tutoring/office hours. Having this schedule held students accountable for their actions as they were required to log and track their activities. For example, Erin improved her weekly work and became a better time manager. She noted that she used to stay up until eight o'clock at night reading and studying. When she fell asleep she would have to catch up on her overdue work the next morning. However, after developing her weekly schedule, she found herself becoming more productive and more engaged in her classes. She also noted that logging her hours held her accountable and made her more diligent about work. Erin also reported having a more positive attitude towards math (faculty noted she had "math phobia" in the beginning). Her grade had improved from a C in the beginning to an A as a result of the interventions.

Looking at students' grades, 7 of the 15 students reporting having a F in MTH 1010 when we first met, 2 had a D, 2 had a C, 2 had a B, and 2 did not know their current grade. Of the seven students who reported having an F, at midterm, four still had a F, two had a D and one had a B. Of the two that had a D, one dropped to a F and one improved to a B. Of the two that had C, one dropped to a F and one improved to an A. Of the two that had B, one maintained a B and one improved to an A. Of the two that did not know their grade, one had a D and one had an A. Final grades were not posted at the time of this report but will be analyzed in future analyses.

From student feedback, the interventions helped them become more confident overall. Tutoring helped with the content, but having someone to talk to was beneficial for them. Many students stopped by during the week just to say hello or to discuss serious personal concerns. Students reported that having me as an extra support made their college life less stressful. This then impacted their work and engagement in the classroom. Faculty also reported improvement not only in students' content knowledge but also overall attitudes. One faculty member stating seeing major improvements in some students' test scores as well as class participation.

Discussion

Many students come to college underprepared for the courses they will soon take. At our college, College Algebra is the lowest mathematics course students can currently take. Although some students placed into a lower course, everyone was placed in College Algebra. As the MIC, it was my job to ensure that these students were supported and could be successful. After interviewing students, working one-on-one with them, as well as coordinating tutoring services, results of the math intervention program are positive. Students reported better study skills, time management, and overall more positive attitudes. Although the majority of students were referred for content related issues, many students needed support in other areas outside the classroom.

Questions for the Audience

(1) How are your students placed into mathematics courses? What is the best approach to placement (e.g., ACT scores, placement exam)?
(2) How do you support students who lack the prerequisite skills for College Algebra?
(3) What other supports and resources do you think would be helpful to include in this intervention program?

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Comparing the Mathematical Beliefs of Tutors and Teaching Assistants

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In this preliminary report, we discuss our efforts to compare the mathematical beliefs of tutors and teaching assistants. This proposal grew out of our larger project that has focused primarily on developing a research instrument to measure the mathematical beliefs of tutors. In conducting that research, we realized that a portion of our population not only works as tutors but also teaches classes themselves or provides other types of in-class support for students as teaching assistants. We conjectured that teaching assistants might have a more personal relationship with the students they support (in comparison to tutors) since they work consistently with the same group of students. Due to these differences in job responsibilities and interactions with students, we are curious to know if there are also differences in their mathematical beliefs.

Keywords: tutors, teaching assistants, beliefs, social constructivism

Often, people find mathematics to be difficult. From learning how to solve an algebra problem to learning theorems and proofs, people of every age have struggled with mathematics. Helping students overcome these struggles is the main responsibility of mathematics tutors and teaching assistants. Tutors may work privately with a single student or in a tutoring center helping many students, while teaching assistants are often responsible for working closely with a single class. In either case, tutors and teaching assistants help students who are lost and struggling to understand the material. However, what exactly do tutors and teaching assistants believe about teaching and learning mathematics? And why do their beliefs matter? We propose that examining their mathematical beliefs is important because it allows departments, tutoring centers, and universities to see what their tutors and teaching assistants value when they are interacting with students. With this information, they help tailor training and professional development programs to better prepare tutors and teaching assistants to serve their students.

The purpose of this preliminary report is to describe our study comparing the mathematical beliefs of tutors and teaching assistants (TAs). For the purposes of this study, we use Pajares' (1992) definition of beliefs as "an individual's judgment of the truth or falsity of a proposition, a judgment that can only be inferred from a collective understanding of what human beings say, intend, and do" (p. 316). To help form a collective understanding of our participants' mathematical beliefs, we collected both qualitative and quantitative survey data and are currently analyzing the responses to determine if there is a difference between tutor and TA beliefs. This work stems from a larger project (Pilgrim et al., 2020; Hill-Lindsay et al., 2022) that has focused on modifying Luft and Roehrig's (2007) and Stipek et al.'s (2001) surveys of teacher beliefs. Our hope is that tutors and TAs beliefs align more with a social constructivist view of tutoring as opposed to a more traditional view of tutoring as the transmission of knowledge. Through this study, we aim to answer the following research questions:

1. What are the mathematical beliefs of mathematics tutors and teaching assistants?
2. How well do their beliefs align with social constructivism? And does this depend upon whether they are working as a tutor or a teaching assistant?
3. In what other ways are the beliefs of tutors and teaching assistants similar and different?

Related Literature on Beliefs

Mathematics Teacher Beliefs Instruments

Initially, we went in search of research instruments designed for studying tutor beliefs. When we struggled to find one[1], we expanded our literature search to look at studies of teacher beliefs. As a result of our search, we chose to modify teacher beliefs instruments developed by Luft and Roehrig (2007) and Stipek et al. (2001). We chose Luft and Roehrig's (2007) Teacher Beliefs Interview (TBI) protocol because of the open-ended nature of the questions and the underlying framework. The TBI was designed for science teachers and was based on different views of science (as rules or facts; as consistent, connected, and objective; or as a dynamic structure in a social and cultural context). In addition, each question was designed to align with either belief about learning or beliefs about knowledge. Building on this underlying framework about views of science, Luft and Roehrig developed a coding scheme to categorize interview question responses. In AUTHOR (DATE), we took the questions from the TBI along with the coding scheme and adapted them to fit the context of mathematics and tutors.

After testing our modified questions from the TBI, we realized that coding open-ended survey questions limited the number of questions we could ask. We also wanted to capture some additional beliefs that were not addressed by the TBI. So, we returned to the literature in search of a closed-response instrument to supplement the open-ended questions we had developed. After reviewing several instruments (Stage & Kloosterman, 1992; Dweck, 1999; Mukina, 2017; Sandman, 1980; Sun, 2015; Tapia & Marsh, 2002; Gow & Kember, 1993), we decided to modify the Likert-scale questions from Stipek et al.'s (2001) teacher beliefs survey. While this survey was designed for middle-grade teachers, it focused specifically on their mathematics beliefs. We were drawn to this survey because it addressed some beliefs that were missing from the TBI (e.g., beliefs about the nature of mathematics) but was also built on an underlying framework that was like Luft and Roehrig's (2007) view of science. Stipek et al. (2001) constructed their questions to differentiate between teachers with a more *traditional* "conception of mathematics as a static body of knowledge" versus a more *inquiry-oriented* conception of mathematics as "a discipline that is continually undergoing change and revision" (p. 214).

Mathematics Tutor and Teaching Assistant Beliefs

Before the development of their TBI instrument, Luft and Roehrig's research group conducted a study on the attitudes and conceptions of chemistry graduate TAs (Kurdziel et al., 2003) and found that most of their participants believed that "students learn by having the material digested, organized, and clearly presented to them" (p. 1209). Jelfs et al. (2009) adapted Gow and Kember's (1993) questionnaire and found that many of the mathematics tutors in their sample were less oriented towards knowledge transmission but rather had a task-oriented conception of tutoring. Goertzen et al. (2009; 2010a; 2010b) published several papers on the beliefs of physics graduate TAs. Through their study, they found that beliefs are multidimensional and not easily modified (2009), but they also advocate for respecting the beliefs and experiences of TAs (2010a). Finally, Youde (2020) published a recent study on tutor

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[1] We did find one survey that was designed to measure tutor's self-efficacy beliefs (De Smet et al., 2010). However, this survey was designed for tutors that worked in a remote, asynchronous environment (answering student questions posted on a discussion board).

perceptions, beliefs, and practice within blended learning environments and found that "perceptions and beliefs provided a valuable insight into the actions and motivations of tutors" (p. 1). However, we have yet to identify a study that has conducted a comparison of the mathematical beliefs of tutors and teaching assistants, which is what we aim to do.

Theoretical Framework

As we searched for instruments to measure teacher beliefs, we found that Luft and Roehrig (2007) and Stipek et al. (2001) constructed their instruments based on similar frameworks. We were drawn to these frameworks because they aligned with our own conception of different views or approaches to tutoring. While they used different terms, we have decided to use the ideas of traditional tutoring and social constructivism to describe our theoretical framework.

Traditional Tutoring

On one end of our theoretical spectrum, we define traditional tutoring as focused on the transmission of knowledge, particularly mathematical rules or facts. While the term "traditional tutoring" is not often used in literature, we chose to use it as it mirrors the more popular term "traditional teaching." A tutor or teaching assistant who has more traditional beliefs would value efficiency in performing mathematical procedures and manipulating symbols, regardless of whether the student demonstrates understanding. Because of this, traditional tutors are more likely to focus on answering student questions quickly, instead of helping the student understand the mathematics and connect what they are learning now to what they learned previously or will learn in the future. Traditional tutors would also view mathematics as an innate ability and categorize some individuals as a "math person." Traditional tutors usually control the conversation between themselves and the students, give step-by-step explanations to answer the students' questions, and expect the student to sit and listen quietly. Though traditional tutoring is not all bad, the foundations of these beliefs do not reflect what our discipline has learned about best practices for teaching and tutoring mathematics.

Social Constructivist Tutoring

On the other end of our theoretical spectrum, we describe tutoring beliefs that align more with social constructivism (Vygotsky, 1980). Again, we have not found other references that use the term "social constructivism" to apply to the tutoring setting, but we feel that this concept best fits the underlying foundations of Luft and Roehrig (2007) and Stipek et al. (2001). A tutor whose beliefs align more with social constructivism would encourage students to work together while the tutor provides minimal guidance. Social constructivist tutors would believe that the best way to learn mathematics is to work collaboratively and discuss their ideas. They would act as a guide, helping bring students together and assisting them if they are stuck or struggling to proceed. These tutors also strive to help students understand the connection between the content that they are learning now to the content that they learned previously or in the future. Social constructivist tutors also believe that anyone can be successful at mathematics and deny the notion of being a "math person." While social constructivism is not the only theory of learning, it is supported by the literature and aligns with our own personal beliefs concerning best practices for teaching and tutoring mathematics.

Methods

In the first phase of our project (Pilgrim et al., 2020), the research team modified the open-ended questions from Luft and Roehrig's (2007) and administered a survey to undergraduate and graduate tutors and teaching assistants. Using a modification of Luft and Roehrig's coding scheme, we coded the open-ended survey responses from participants into five categories: instructive, flexible, transitional, responsive, or adaptive. As a result, we were able to plot tutors' and TAs' beliefs as heat maps to see how each participant's answers varied from one another. In the second phase (Hill-Lindsay et al., 2022), we used cognitive interviews to continue refining our open-ended questions and began adapting Likert-scale questions from Stipek et al. (2001). For the third phase of the project (which is the focus of this preliminary report), we administered an online survey in the summer of 2021 that included the modified questions from Luft and Roehrig (2007) and Stipek et al. (2001).

Participants

Initially, we planned only to administer the survey at one institution and compare responses from tutors who worked in the mathematics help center with TA who supported our precalculus classes. However, our response rate was low (N=3), so we decided to send the survey out to tutor center directors across the United States. As a result, we ended up with 66 responses. However, 39 of our respondents either left all open-ended questions blank or responded to them with a lack of serious intent to respond genuinely (e.g., responding to every question with "a"). We also had one incomplete response that was an obvious repeat of the subsequently complete response (the open-ended responses were almost identical). We assume that this individually accidentally ended the survey before finishing and started over. An additional four responses were incomplete and removed from our final data set. This left us with only 22 complete responses to analyze. Demographic information from these 22 participants can be seen in Table 1.

Table 1. Participant demographic information

Demographics	Count
<u>Degree Level (Select One)</u>	
Undergraduate	13
Graduate	9
<u>Current or Recent Job (Select All that Apply)</u>	
Tutor	18
Teaching Assistant	9
Learning Assistant	4
Instructor	4
<u>Job Responsibilities (Select All that Apply)</u>	
I work with students outside of the classroom (in tutoring centers, office hours, etc.)	21
I work with students during class, but don't teach the main lecture or lead breakout discussion sections	9
I don't teach the main lecture, but I lead breakout discussion sections for students	4
I teach the main lecture for the entire class	7

Data Analysis

For the open-ended questions, we are currently conducting a qualitative analysis to categorize responses using Pilgrim et al.'s (2020) coding scheme. Once we have completed our coding, we will group responses in the following ways and look for similarities/differences: by questions, constructs, individual responses, and subgroups (tutor versus TA). For the Likert-scale questions, we plan to conduct a quantitative analysis to calculate basic descriptive statistics (e.g., mean, median, mode, frequency, standard deviation, spread). Like our qualitative analysis, we will group responses and look for similarities/differences. Finally, we hope to have a large enough sample size so that we can use inferential statistics to test our hypothesis that the mathematical beliefs of tutors and teaching assistants are different.

Preliminary Results

Although we have not yet coded all our responses to the survey, we were able to analyze the first three open-ended questions for Participants 1-3. Participants 1 and 3 were both graduate students, while Participant 2 was an undergraduate. Participants 1 and 3 were either currently working or had worked recently as both tutors and the instructor of record for their own course, so we classified them as both tutors and teaching assistants. On the other hand, Participant 2 was just a tutor. Although this sample size is small, it provided us with the opportunity to conduct a preliminary comparison of their responses and begin to gain insight into how tutors' and teaching assistants' beliefs are similar and different. One interesting preliminary result we found was in regard to our question on how they facilitate student learning when they are working with a student. Participants 1 and 3 (the graduate TAs) responded in ways that aligned more with social constructivist beliefs, as they mentioned "reassuring [students] when they make mistakes" and connecting mathematical concepts to everyday life to help the students understand what they are learning. However, they did not talk about encouraging students to collaborate with each other, so we coded their beliefs as transitional. Participant 2, on the other hand, had a more traditional approach to facilitating learning. This participant was very focused on knowledge transmission, stating that they would work out the problems step-by-step and ask the student to listen and repeat once finished. Therefore, we coded Participant 2 as having more traditional beliefs. As we code our remaining responses, we are curious to see if this trend of TAs having more beliefs that align with social constructivism (in comparison to tutors) continues.

We look forward to discussing the following questions with the audience:

1. What types of statistical analyses could we run on our Likert-scale questions, given that we have a small sample size (N=22)?
2. What are the consequences of categorizing participants who have worked recently as the instructor as a "teaching assistant"? Are they "too different" from tutors because they have classroom experience as a teacher of record?
3. Is it reasonable to categorize participants who work as both tutors and teaching assistants as just "teaching assistants"?

Acknowledgments

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Using Network Analysis Techniques to Probe Student Understanding of Expressions Across Notations in Quantum Mechanics

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One important outcome of physics instruction is for students to be capable of relating physical concepts and phenomena to multiple mathematical representations. In quantum mechanics (QM), students are asked to work between multiple symbolic notations, some not previously encountered. To investigate student understanding of the relationships between expressions used in these various notations, many of which describe analogous physical concepts, a survey was distributed to students enrolled in upper-division QM courses at multiple institutions. Network analysis techniques were shown to be useful for gaining information about how students relate these expressions. Preliminary analysis suggests that students view Dirac bras and kets as more similar to generic vectors than to their physically analogous wave function counterparts, and that Dirac bras and kets serve as a bridge between vector and wave function expressions.

Keywords: Network Analysis, Quantum Mechanics, Notation

There has recently been a focus in research at the boundary of physics and mathematics in upper-division quantum mechanics (QM) (e.g., Wawro et al., 2020), including a focus on the three mathematical notations (Dirac, wave function, and vector-matrix) used to describe identical or analogous physical concepts and phenomena (Gire & Price, 2015; Schermerhorn et al., 2019). A comprehensive understanding of how expressions in these notations interrelate and how to translate between them is crucial for a deep understanding of QM (Wawro et al., 2020). One challenge facing upper-division education research is the relatively low sample size when compared to research conducted in the lower division, as this affects the generalizability of claims. This is particularly true in quantum mechanics, where the order in which these different notations are introduced—based primarily on the instructor's choice of textbook—can drastically affect the focus of instruction and thus the eventual conceptual understanding of the students. We have implemented network analysis techniques to probe students' conceptual connections between symbolic expressions, allowing for much larger sample sizes than is typically feasible for this research context. To that end, we address the following research questions: How can network analysis techniques be leveraged to study students' conceptual connections between expressions in quantum mechanics, and what are the connections that these techniques show?

Prior Research on Quantum Notations and Network Analysis

The various notations used in upper-division quantum mechanics have different affordances and limitations for computation, both from an expert point of view (Gire & Price, 2015) as well as in students' work (Schermerhorn et al., 2019). Additionally, incorrect translations between wave function and Dirac expressions causes students to struggle when developing models for determining probabilities (Wan et al., 2019). The ability to reason between and among different mathematical representations has been linked to understanding of QM concepts (Wawro et al., 2020), and work has been done to create instructional materials to aid students in working fluidly among multiple representations (Kohnle & Passante, 2017).

Network analysis techniques such as community detection and cluster analysis have recently been used extensively to study response groupings for various conceptual inventories in physics education research (Brewe et al., 2016; Wells et al., 2019; Wells et al., 2020; Wells et al., 2021; Wheatley et al., 2021; Yang et al., 2020). Members of the research in undergraduate mathematics education community have used social networks among teachers to study community and coaching among educators (Hopkins et al., 2017; Smith et al., 2017), while students' social networks have been studied across multiple fields to study how they impact academic performance, persistence, self-efficacy, and anxiety (Hopkins et al., 2017; Thomas, 2000).

Study Design and Methodology

The survey was designed with two primary goals in mind: to easily collect and analyze responses from many students, and to create a dataset that allows for analysis of students' conceptual connections between mathematical expressions commonly used in QM—particularly those used to express probability concepts. To achieve the first goal, the number of free-response text entry questions were minimized to reduce participant attrition. This meant that the second goal would need to be achieved without much in the way of written responses showing explicit student reasoning. The questions therefore were designed as sorting tasks, where students were given a list of expressions as well as a single concept and asked to select all of the expressions which could represent that concept—see Figure 1 for an example. The survey consisted of 11 different concepts, with the same 16 expressions to choose from. This entirely relation-based dataset—between both expression-concept pairs as well as pairs of expressions used for a given concept—makes network analysis an ideal choice. The survey was distributed to three different institutions, including two public land-grant research universities in the American northeast and one private midwestern liberal arts college, for a total of 27 participants. Of these participants, 20 were in classes that taught Dirac notation prior to wave function notation ("spins first"), and seven were taught wave functions prior to Dirac notation ("functions first").

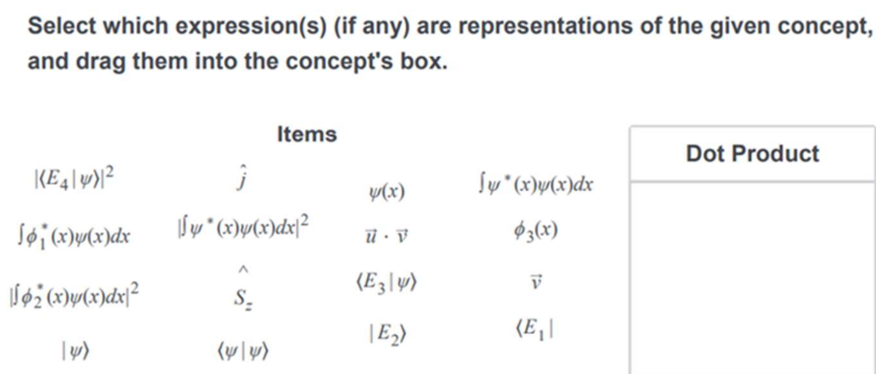

Figure 1. Example of the survey task for the "Dot Product" concept.

Data Analysis and Results

The first step in data analysis was the creation of a weighted network with the 16 expressions as nodes, with the connections between them—known as edges—weighted by the number of students that used the two expressions for the same concept at least once in the survey (Figure 2). The larger network was then broken into communities of more closely connected expression nodes. The method chosen for detecting these communities involved measuring the number of geodesic paths between every pair of nodes in the network that pass through each edge, known as the edge's *betweenness* (Girvan & Newman, 2002). The edges with highest betweennesses are those that connect communities within the larger network. This is because if two communities exist within a network there will, in general, be fewer edges connecting between the two than

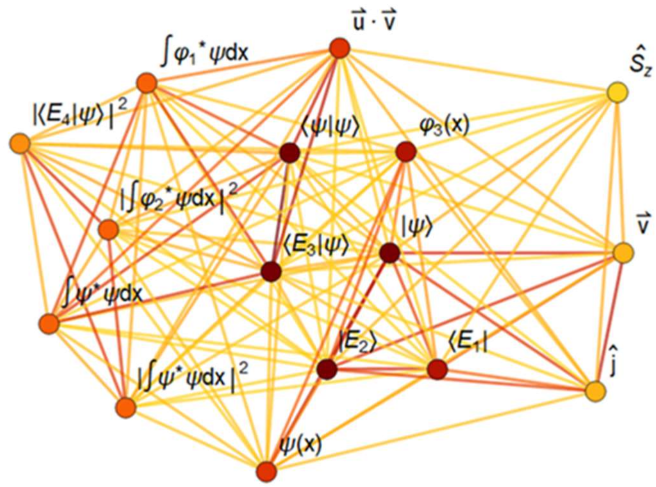

Figure 2. Network with expressions as nodes and occurrences of students using two expressions for a concept as edges. Edge weights are visualized with color, where red corresponds to higher weight and yellow to lower weight.

there are within each community—thus those few connecting edges will bear the load of all the geodesics traveling between the nodes in each community, causing them to have large betweennesses. In this community-detection algorithm, the betweenness of each edge in the network is calculated, and then the edge with the largest betweenness is removed. This calculation and subsequent removal of the edge with maximum betweenness repeats until all of the edges in the network are removed, eventually leaving all nodes fully disconnected. The process of removing one edge at a time gives a cascading hierarchy of communities in the network, with larger communities eventually being divided into constituent sub-communities that are themselves more strongly connected to their own members. The hierarchical structure of the communities found using this method can be visualized by a dendrogram (Figure 3).

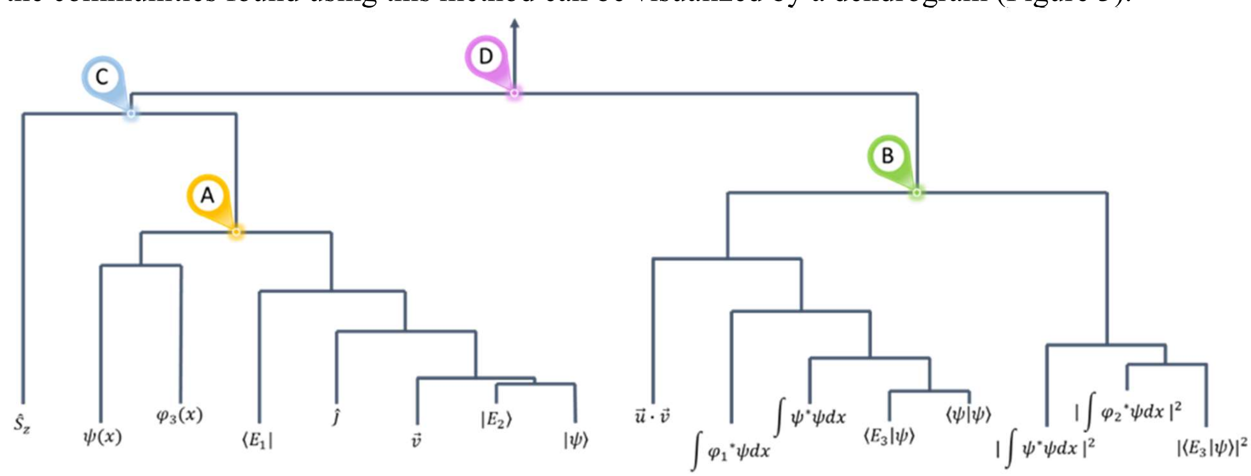

Figure 3 Dendrogram displaying the community structure of the network shown in Figure 2. Vertical height is proportional to number of edges removed between community divisions. Letters correspond to those on Figure 5.

Figure 3 suggests that the first distinction students drew was morphological: the first division of the network was into “single-term” and “double-term” communities. Individual functions, vectors, Dirac bras and kets, and quantum mechanical operators were grouped together, while expressions that contained Dirac bra-ket pairs, two vectors, or two functions—inner products—all shared their own community. However, these community separations clearly have conceptual distinctions to the students as well. Almost immediately upon dividing into single- and double-term communities, the z-component spin operator ($\hat{S}_z$) was excluded from the other single-term expressions. This is of interest for two reasons. First, both physically and mathematically speaking, an operator is very much unlike any other single-term element. Second, $\hat{S}_z$ and $\hat{\jmath}$ share many wholly morphological similarities; the fact that they were clearly not viewed as similar to students is an encouraging sign that the students were attending to actual physical and mathematical meaning and not merely focusing on morphological distinctions.

Figure 3 also gives insight into the relative strengths of conceptual connections between notations. The second division to occur within the single-term community is that of the two wave functions ($\psi(x)$, $\varphi_3(x)$) splitting off from both the Dirac bras and kets and the generic vector expressions. These edges and communities are, due to our survey design, entirely based on expressions that students view as conceptually similar, suggesting that Dirac bras and kets may be more closely associated with vector ideas than with concepts associated with wave functions.

Aside from the information to be gleaned from the hierarchical community structure alone, the structure within the communities as they are being separated into their sub-communities can also be used to investigate how certain expressions are connected, as well as what that means for the expressions' conceptual connections. This connective structure can be teased out by investigating the minimum vertex cut sets (MVCSs) between expression pairs throughout this cascading network decomposition. The MVCS between two nodes in a network is the smallest set of nodes that need to be removed to entirely disconnect the two nodes in question, and can be used to see which nodes tend to connect any pair of nodes. At the stage of the network decomposition shown in Figure 4a, for example, there are two single-node (size-1) MVCSs between $\langle\psi|\psi\rangle$ and $|\int\psi(x)^*\psi(x)dx\,|^2$: $\{\langle E_3|\psi\rangle\}$ and $\{|\langle E_3|\psi\rangle|^2\}$. Likewise, in Figure 4b the $\hat{\jmath}$ and $\varphi_3(x)$ nodes have one MVCS of size 3, made up of the Dirac bras and kets: $\{|E_2\rangle$, $|\psi\rangle$, $\langle E_1|\}$. This is reflective of the apparent symmetry in the community's structure, where the Dirac expressions appear to serve as a sort of bridge between the wave function and Dirac expressions. The size, number, and elements contained within the MVCS(s) between any two nodes is liable to change as the edge-betweenness algorithm plays out and in fact the metric becomes entirely meaningless if the two nodes in question ever become connected directly, as at that point the only vertex that can be removed to separate the two would be one of the two nodes themselves.

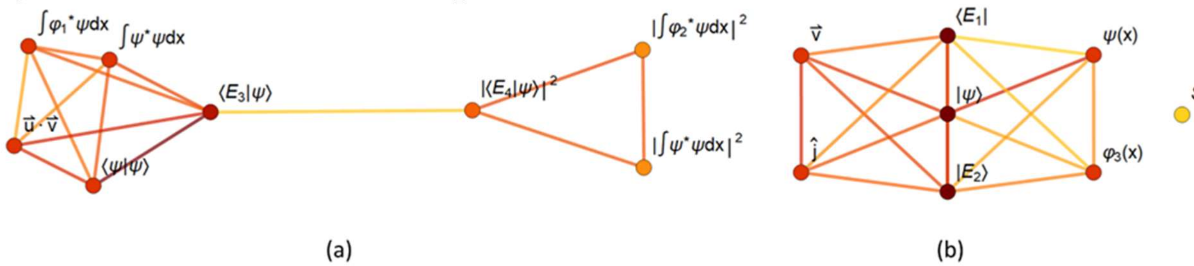

Figure 4. Network partway through betweenness algorithm showing (a) double- and (b) single-term communities.

To investigate whether this apparent Dirac-bridging is meaningful beyond the single cross-section of the betweenness algorithm shown in Figure 4, the MVCS between wave function and Dirac expressions can be examined throughout the community detection process. Moving upwards from the bottom of the dendrogram, where all edges have been deleted and thus all nodes are disconnected, can be thought of as playing the edge-betweenness algorithm backwards; this allows for the MVCSs between the pairs of vector and wave function expressions to be observed as the communities are being "formed." The MVCSs between the $\hat{\jmath}$

and $\varphi_3(x)$ nodes serve as an illustrative example of this, as shown in Figure 5. The 25th edge added into the network provides the first connection between $\hat{J}$ and $\varphi_3(x)$ (point A on Figs. 3 and 5), and the very next then expands the MVCS between the two from $\{|\psi\rangle\}$ to $\{|\psi\rangle, |E_2\rangle\}$. This remains the only MVCS for three more edge-additions before the MVCS expands again to $\{|\psi\rangle, |E_2\rangle, \langle E_1|\}$. This remains the stable MVCS as the next 21 edges are added, during which the two double-term communities merge (B), $\hat{S}_z$ rejoins the single-terms (C), and even the single- and double-term communities reconnect (D). The next edge added directly connects $\hat{J}$ and $\varphi_3(x)$, thus making the MVCS between them meaningless. The $\hat{J}$–$\varphi_3(x)$ pair serves as an illustrative example: the Dirac expressions are always the most prominent connectors between the function and vector expressions, and remain so after all of the communities have connected.

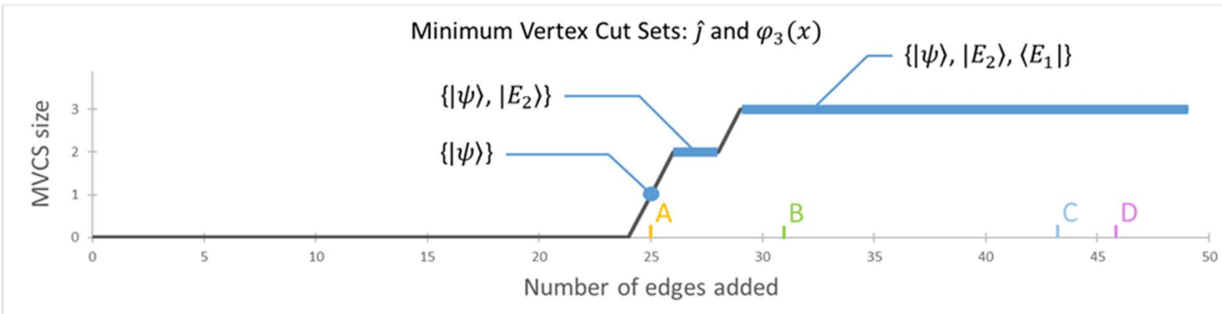

Figure 5. Graph displaying MVCSs for the $\hat{J}$-$\varphi_3(x)$ node pair. Letters correspond with those on Figure 3.

Conclusions, Discussion, Implications for Further Research

Our survey and analysis using network techniques appears to be able to isolate students' conceptual knowledge as it applies to expressions in the various notations used in QM. This combined with the scalability of both the data collection and analysis methods is an encouraging sign of the ability of these techniques to study a large number of students at a large number of institutions. The methods described above will only improve with a larger sample size, and so there is likely more to be learned about students' conceptual connections between expressions across notations in QM if these techniques are applied more broadly. Our data suggests that students think of Dirac bras and kets as a blend of wave function and vector ideas. This is an encouraging finding, as that is effectively exactly why Dirac invented the notation in the first place. What is interesting, however, is that students appear to more closely link the Dirac bras and kets to vector ideas—likely due to their mathematical utility—than to ideas associated with wave functions, the connection to which is almost entirely grounded in a physical understanding.

Within this QM context, network analysis could be used to expose differences in students' thinking about various expressions due to either institutional context or pedagogical focus. We suspect that the networks formed by students in courses where Dirac notation is introduced first would differ greatly from those in courses focused largely on wave functions. Our current data pool is not large enough to make claims in this regard, but future work may show whether any distinctions become apparent. There are a number of areas where this type of data collection and analysis could bear future use, such as with expressions associated with integrals and sums, both in calculus as well as in physics contexts such as electromagnetism or thermodynamics.

Acknowledgments

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Connecting Sameness in Abstract Algebra: The Case of Isomorphism and Homomorphism

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Despite the importance of seeing connections across courses, limited work has examined how a common mathematical theme, sameness, is understood across disciplines. In this paper, we examine how fifteen abstract algebra students explained the nature of mathematical sameness and contextualized isomorphism and homomorphism. Results include students' explanations of sameness, mainly by use of examples, as well as the aspects of homomorphism and isomorphism that the students highlighted when describing the concept to people unfamiliar with it.

Keywords: Abstract Algebra, Sameness, Homomorphism, Isomorphism

Mathematicians recognize and have defined a variety of ways to convey sameness of mathematical objects, including equality, isomorphism, and homomorphism. Furthermore, they specifically identify these relationships as types of sameness and recognize subtleties in the differences among these types of sameness (Rupnow, 2021; Rupnow et al., 2021). However, it is not known whether math majors, including students in abstract algebra, recognize and distinguish between types of sameness in similar ways or to what aspects students attend when considering mathematical sameness. In this paper, we examine the following research question: *How do abstract algebra students describe sameness, isomorphism, and homomorphism?*

Background Literature and Theoretical Perspective

Equivalence is threaded throughout the foundations of mathematics (Asghari, 2009; Asghari, 2019) and notions of equality are central to students' early schooling and beyond (e.g., National Governors Association Center for Best Practices, 2010). However, students do not always have a deeply conceptual understanding of equality, which impacts their ability to engage with algebraic topics (e.g., Alibali et al., 2007; Kieran, 1981). As students reach college mathematics, more complex ideas of equivalence arise such as isomorphism in abstract algebra. Early work on isomorphism understanding showed students and researchers associate isomorphism and sameness, whether through direct instruction on this connection (e.g., Leron et al., 1995) or independently by students (e.g., Dubinsky et al., 1994), and subsequent research has confirmed references to sameness in the context of isomorphism by students and professors (e.g., Rupnow, 2019; 2021; Weber & Alcock, 2004). However, despite early work on students' understanding of mathematical sameness (e.g., Melhuish & Czocher, 2020), how students understand mathematical sameness and draw upon it to understand concepts like isomorphism is not as clear.

In this study, we drew on the notion of example space (e.g., Watson & Mason, 2005) to characterize how abstract algebra students viewed mathematical sameness as well as isomorphism and homomorphism. Example spaces can be viewed as a collection of examples intended to highlight both the dimensions of possible variation and the range of possible variation of a particular topic (e.g., Goldenberg & Mason, 2008; Mason & Watson, 2008). For example, to have a robust example space of operations, one would want to attend to dimensions of variation like the number of inputs, while the range of variation includes the different options in the dimensions (e.g., unary, binary, trinary). Here we provide a composite example space generated by students to highlight dimensions of variation and the range of variation of mathematical sameness as well as ways that isomorphism and homomorphism explanations related to these notions of mathematical sameness.

Methods

We conducted an open response survey at three doctoral-granting institutions across the United States to gain insight on students' understanding of sameness. Students actively registered in abstract algebra were invited to participate. At each institution the algebra course offered during data-collection was listed in the course catalogs as available to both undergraduate and graduate students. Therefore, participants were either undergraduate or graduate students. We are sharing responses to four of fifteen survey questions that best capture students' ideas about sameness. Data consists of fifteen students' responses amongst the three institutions.

Two researchers simultaneously coded the responses based on dimensions of variation highlighted in Rupnow et al. (2021). This coding generated groupings of students' descriptions of isomorphism and homomorphism. In a second round of analysis, all the researchers came together to refine the previous coding, focusing on identifying subcomponents of and/or holistic approaches to isomorphism and homomorphism that were highlighted by the students.

Results

We categorized students' descriptions of sameness from their responses to "What does it mean to be the same in a math context?" and "How do you know two things are the same in abstract algebra? Is this the same or different from other classes?" Highlighting five dimensions of variation used to describe what it means to be the same in math: concept, type of object, properties, discipline, and levels of strength. The subsequent section focuses on students' descriptions of types of sameness, isomorphism and homomorphism, to a ten-year-old.

Sameness

A majority of students (11) referred to defined mathematical concepts to describe mathematical sameness. Numerous students highlighted isomorphism (6) or equality (6):

> The meaning of "same" in a math context is a vague one. We can say two groups are the same if there exists an isomorphism between them, but the elements of the respective groups may be different. We may also say things are equal which is a different meaning than isomorphism but also implies a form of sameness.

Others focused on equivalence relations (3), equivalence classes (2), homeomorphism (2), identicality (2), categorical equivalence (1), homomorphism (1), or bijection (1). For instance:

> We can say two groups are the same if they are isomorphic, meaning there is a bijective homomorphism between the two ... In topology for example we can find a homeomorphism between two topological spaces which is a kind of equivalence relation ... In general we can talk about equivalence of categories.

Notice this participant highlighted a number of concepts, including equivalence relations and categorical equivalence. Most participants who used concepts only focused on one or two; nevertheless, collectively we can see they attended to a variety of concepts.

Other dimensions of variation that arose were related to underlying objects and properties conveying sameness. Seven students noted types of objects to which sameness considerations can be applied: "Same does not just mean identical sets, or spaces, or rings, etc. Same means there exists a bijection between the sets so that one element of a set is mapped onto an element of the second set." This response lists multiple objects (sets, spaces, rings) that could be permitted to vary and still convey a notion of sameness. Other participants highlighted groups, other spaces, and categories. Six students mentioned properties as relevant to sameness--one highlighted cardinality and truth value, and another commutativity and identity, but the other four students did not describe what was meant by "properties." For instance: "Having a common structure and common properties [sameness in math] … They are isomorphic. They have the same structure. I think the concept is different if you look into set theory for instance" [sameness in algebra]. While we might infer relevant properties given the isomorphism response to the second question, which properties exactly should be the "same" were left unspecified.

Given the set-up of the second question, we expected to receive responses contrasting abstract algebra's and other disciplines' versions of sameness. However, while seven participants alluded to disciplines in some way, most provided a general contrast between abstract algebra and other math courses, such as: "In abstract algebra, when two things are the same, it does not necessarily mean that they are equal as it would in other math classes. Rather, it means that their algebraic structures are equivalent." This student seems to be drawing a contrast between isomorphism and equality, which does convey different types of sameness; however, this student did not expand on why we might attend to different types of sameness in different contexts.

Finally, students also attended to different levels that sameness might possess (3). One provided an extended response (multiple paragraphs including five axioms, three definitions, and some additional commentary) in which they attempted to rigorously define their understanding of sameness and in their conclusion noted that the level of sameness we talk about can vary:

> We often discuss sameness without meaning "exactly the same". Here, we reduce our scope to specific properties and check to see if two things have these specific properties in common. Then we call them the same in some sense. Other times, if two things have exceptionally close property values we might call them (knowing that it is strictly untrue) the same. So, in a math context, when one declares two things to be the same, one may mean they are exactly the same. See above. Or, one may mean only some of their properties are the same. This is almost always intuited, or explicitly given, in context.

Notice this student highlights a distinction between being "exactly the same" and "the same" and these notions can be defined on the properties of interest in the given context.

In summary, students highlighted a variety of aspects of sameness, including types of concepts that convey sameness, different levels of sameness, and underlying objects that may be of interest, though some students were more specific than others. We use this characterization of sameness to inform how students interpreted the notions of isomorphism and homomorphism.

Isomorphism and Homomorphism

Turning our attention to the responses to the isomorphism and homomorphism prompts, we expected there to be fewer dimensions present. Due to the questions asked, we anticipated the type of object and discipline dimensions to be absent, and the concept was obvious since each question was specifically focused on a single concept. Therefore descriptions of isomorphism and homomorphism were primarily analyzed for their use of properties and levels of strength.

Isomorphism. When describing an isomorphism to a ten-year-old, students provided a range of responses. Some responses (7) attended to only part of the isomorphism concept: either the bijection or the homomorphism property. For example, two students focused primarily on the bijection without any reference to the homomorphism property, such as the following:

> Consider a collection of dogs and a collection of glasses. If we can find a way to put a pair of glasses on each dog with no dog getting more or less than one pair of glasses, then we have a set isomorphism between the set of glasses and the set of dogs.

The focus here is on matching each dog to exactly one pair of glasses, thus highlighting the bijection. The other responses (5) attended to the homomorphism aspect of isomorphism without explicit attention to the bijection, usually focusing on shared behavior as seen below:

> You are isomorphic to your reflection. Everything you do, the reflection does also, but backwards. … if I first consider the motion, then its reflection, or I consider the reflection first, and then perform the equivalent motion, the result I get is the same.

Observe the notion of operate then map or map then operate of the homomorphism property is modeled in the reflection example as reflect then move or move then reflect.

Some responses (6) attended to both the bijection and homomorphism property. For example:

> Imagine you and your friend open your lunch boxes. You have a peanut butter and jelly sandwich, they have a turkey sandwich. You have carrot sticks, they have celery sticks. You have a cookie, your friend has a brownie. You both start by eating your sandwich, then your vegetable sticks, then … you eat your desserts. You both had the same type of things in your lunch and ate them in the same order, even though the exact items were not identical. Your lunch is isomorphic to your friend's lunch since each of your lunch items behaved the same compared to the other items in your lunch, and you had the same number of items.

The bijection appears through matching parts of the meal (e.g., each have a sandwich) and the homomorphism property through the purpose of each part of the meal (e.g., each have an entree).

The remaining two students focused on relabeling, as can be seen by this response: "An isomorphism allows you to rename everything in a group of things but everything in that group will still interact with each other in the same way." This relabeling could be interpreted as attending to same behavior (related to the homomorphism property) but was not explicit in the component(s) of the isomorphism definition being attended to.

Homomorphism. When describing a homomorphism to a ten-year-old, students struggled to articulate examples that captured the mathematics accurately while remaining relatable to children. Some (5) made no attempt or provided unclear explanations: "I would just give them the basic definition $f(x + y) = f(x) + f(y)$. Then go through some basic examples."

Students who provided context-appropriate examples (10) offered descriptions that varied in their mathematical accuracy while highlighting sometimes multiple approaches to the prompt. Five students incorporated a levels of strength argument as they contrasted their isomorphism and homomorphism understandings: "a homomorphism is when two things might not have the same amount of things or look the same, but still have the same properties. A good example of this would be scrambling an egg or cooking it like an omelette." In their isomorphism response, this student said isomorphic objects "have the same amount of things and same properties but may not look the same" but here weakened the cardinality requirement. Some students (4) focused on same properties or behavior:

> Two things are homomorphisms when they look different, and in fact are different things, but they both do the same thing. For instance, you could have a toy car and a toy truck. They both look different and are indeed different things. But both of them drive!

Notice this student seems focused on behavior invariance, though the specific properties that remain invariant under the homomorphism are no longer evident.

> Others (3) attempted to attend to partition creation under a homomorphism. For instance:
> A homomorphism exists from A to B if A can be folded up and fit inside of B. This is most visually obvious for graph homomorphisms, but showing how Z_8 folds up to fit inside of Z_2 can also be a good way to approach the idea.

While the accessibility of graph homomorphisms to children is a bit questionable, this student made a collapsing analogy to explain what happens under a homomorphism. One student modified their prior isomorphism example with reflections (above) by using shadows instead:

> You are homomorphic to your shadow. It's really similar to your reflection, but unlike your reflection, other things can impersonate your shadow--like cardboard cutouts. In the end, though, moving my hand, and then looking at my shadow will give the same result as looking at my shadow, and then moving my hand.

Similar to their isomorphism response, the homomorphism property arises in their discussion of being able to move and then look at their shadow or look at their shadow and then move. (Notice this also relates to levels of strength, as they contrast isomorphism and homomorphism.)

Discussion and Future Work

Using the theoretical framing of example spaces, we found students' descriptions of sameness were comparable to those of mathematicians (e.g., Rupnow et al., 2021). The same codes that were present in mathematicians' descriptions (concept, type of object, properties, levels of strength, and discipline) were present in the students' responses. Students used a variety of mathematical concepts to describe sameness (e.g., equality, isomorphism, and equivalence relations) across a variety of mathematical objects (e.g., rings, sets, and spaces). Responses highlighted properties such as cardinality along with levels of strength distinguishing exactly the same from just the same. Lastly, several disciplines were mentioned (e.g., algebra and topology).

Based on the set-up, we largely forced students to try to engage with the isomorphism and homomorphism prompts by highlighting levels of strength or properties. Most students were able to describe some properties of isomorphism (thirteen represented the homomorphism property and/or bijection clearly), and all engaged with the prompt. In contrast, students struggled to articulate mathematical properties of homomorphism through examples that were both relatable to children and mathematically correct. While it is possible the students could have identified problems with their analogies in an interview setting, we found it troubling that many could not find a way to engage meaningfully (five unclear/definition-focused responses) or focused on what a homomorphism is not (e.g., lacking the bijection requirement, shown by comparing to isomorphism). This may also be a widespread struggle. Prior work examined two instructors who attended to both the bijection and homomorphism property for isomorphism but struggled to find a relatable example maintaining equal partitions for homomorphism (Rupnow, 2021). Moreover, relatable examples exist. One could modify the isomorphism lunch example given above by having one meal consist of four pieces in each case (a sandwich in lunch one and four chicken nuggets in lunch two, one brownie in lunch one and four cookies in lunch two, etc.), which would maintain the equal partitions and a type of "same behavior" in the meal. Further work should examine how mathematicians and students understand properties of isomorphism and homomorphism, both to help clarify individuals' understandings of these properties and to examine how these properties do or do not permit variation, in keeping with notions of sameness.

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Transitioning from Location-Thinking to Value-Thinking: The Case of Colin

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This report begins documenting how students transition from one way of thinking to another when reasoning with graphs. We describe students' understanding of graphs and graphed quantities using the constructs of location- and value-thinking (David et al., 2019). We draw on constructivist learning theory (von Glasersfeld, 1995) to account for how one student, Colin, transitions from location-thinking to value-thinking. We describe Colin's activity with a task, and highlight key points that supported him in accommodating his graphing schemes.

Keywords: Graph, Location- and Value-Thinking, Representational Activity, Schemes, Accommodation

In precalculus and calculus, students often reason with graphs and link graphs with algebraically defined expressions such as difference quotients. Past research highlighted how students can think about graphs of functions (David et al., 2019; Monk, 1994; Moore & Thompson, 2015) and points on a graph (David et al., 2019; Thompson & Carlson, 2017). However, the research literature has paid less attention to how students transition from one way of thinking to another.

We focus on how a student transitioned between location- and value-thinking (David et al., 2019) using constructivist learning theory to account for changes in the thinking of one student, Colin. In analyzing Colin's thinking, we investigate the question "What aspects of Colin's thinking supported him in transitioning from location-thinking to value-thinking?"

Theoretical Perspective

We understand learning as the process of a person assimilating information to their schemes and accommodating their schemes to make sense of new experiences (Steffe & Thompson, 2000; von Glasersfeld, 1995). A scheme is "an organization of actions, operations, images, or schemes—which can have many entry points that trigger action—and anticipations of outcomes of the organization's activity" (Thompson et al., 2014, p.11). Assimilation involves a learner recognizing information fits into their existing schemes, while accommodation of a scheme involves a learner modifying their scheme to account for new information (Steffe & Thompson, 2000). In a given moment, a learner's scheme for a concept is the result their past accommodations, that is, the reorganization of their previous schemes in ways that the learner has found helpful in their experiences. Hence, a student's thinking that may seem non-standard to an expert has likely proven productive for the student in the past.

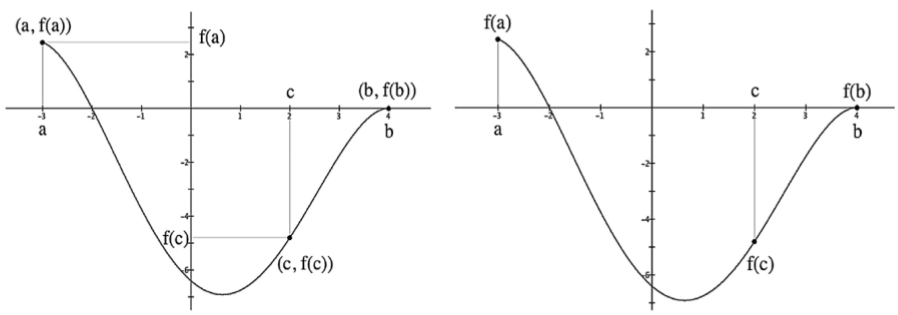
We also use research on students' quantitative reasoning-- a form of reasoning about situations where students conceptualize the quantities involved and relationships between them (Thompson, 2011; Thompson & Carlson, 2017). Graphs represent pairs of quantities, where the graphed quantities can be represented as the measures of directed magnitudes in reference to the coordinate axes (Joshua et al., 2015; Lee et al., 2019).

Location-Thinking and Value-Thinking

We draw on the constructs of location- and value-thinking (David et al., 2019) to describe Colin's understanding of outputs of functions on a graph. According to David and colleagues

(2019), students engaging in value-thinking think of the output of a function for a given input value as a single value. These students further understand a point on a graph as a pair of values [e.g. (input, output)], and a graph of a function as a set of input/output pairs. Students engaging in value-thinking think about points in reference to the axes, conceiving of points as comprised of two quantities measured in orthogonal directions. Students engaging in location-thinking think of the output of a function for a given input value as a point lying in the Cartesian plane along the graph of the function. Further a student engaging in location-thinking treats a point on a graph as indistinguishable from the resulting output for a given input value and a graph of a function as a collection of spatial locations in the Cartesian plane. A student engaging in location-thinking tends to focus on the spatial aspects of a graph, such as the position of points, without focusing on the measured distance between the point and each axes.

One feature David and colleagues (2019) used in their coding scheme was based on where students placed labels for outputs in a graph. For example, placing a label *f(a)* along the vertical axis is indicative of value-thinking, as is labeling a point on the graph as a coordinate-pair *(a, f(a))*. Placing a label *f(a)* on a point along the curve is indicative of location-thinking, as this conveys that the point on the curve is the output. David and colleagues (2019) provided two graphs of a function with labels indicative of value-thinking (
Figure 1a) and location-thinking (
Figure 1b) to illustrate the differences in students' labeling activity in
Figure 1 below.



(a) *(b)*

Figure 1: A visual model of student work indicative of value-thinking (a), and location-thinking (b). (David et al., 2019)

Methods

Data for this study were collected as part of a larger study that aimed to understand how students' understanding of a graph of a function impacted their thinking about the derivative of a function for a fixed input. We focus on Colin's activity on one task he solved during the first week of classes before his calculus course covered material about instantaneous rate of change. Colin was chosen for this analysis because he showed a productive transition from location- to value-thinking that helped him better understand the difference of two outputs. He illustrates the productivity of value-thinking, and the possibility of students to engage in it with relatively minor intervention. The other students in the study did not appear to change their approach to the task or the interviewer intervened in a major way.

The task used Cartesian axes oriented in the conventional manner and asked students to represent both the outputs of two inputs and the difference between those outputs. The task, inspired by Thompson and colleagues (2014), is in the figure below (Figure 2).

Consider the graph of the function *f(x)*. The lengths of colored line segments represent the quantities *a* and *h*. Both the *x*- and *y*-axes have the same scale length.

Represent the following quantities.

I. *f(a)*
II. *f(a + h)*
III. *f(a + h) - f(a)*

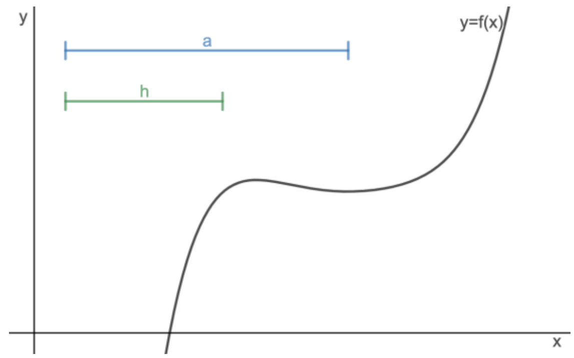


Figure 2: The interview task

The task is intended to provide information about the schemes that a student uses when reasoning with a graph in non-numerical, non-computational ways. This task does not provide algebraic expressions for the function nor numerical or algebraic labels along the axes (besides the independent and dependent variables) in order to help funnel students into using their graphing schemes. These choices were purposefully made so that students' understandings of output in the graphical context were at the forefront of their mathematical reasoning, instead of computation. Analysis from earlier versions of this task revealed that some students computed outputs and differences of outputs of functions numerically and only then matched the computed numerical value to a label presented on the graph rather than reasoning with magnitudes represented in the graph. For this reason, inputs of the function were represented as lengths of line segments to further support students in reasoning with magnitudes depicted in the graph. These decisions also provided practical affordances to our research. Since students needed to reason with the magnitudes represented in the graph and create labels and markings themselves, this allowed the interviewer to inquire about new markings that the student created. Hence, students' resulting activity would support the research team in categorizing students' thinking about the output of functions in terms of location-thinking and value-thinking.

To further investigate the relationship between students' understanding of output and differences of outputs, students were asked to represent *f(a+h)-f(a)*. We anticipated that students using value-thinking would be able to productively represent the difference of the two outputs, particularly where the 'y-coordinate' of a point was a vertical directed distance, either represented from the x-axis to the point on the curve or from the origin along the y-axis. We expected students using location-thinking when representing *f(a+h)-f(a)* to either have trouble conceiving of 'the difference of two points' or to accommodate their graphing schemes (about outputs).

Data was analyzed using theoretical thematic analysis (Braun & Clarke, 2006) where we coded instances of students' reasoning about output as either location-thinking or value-thinking. Colin's activity on the task was divided into two episodes because his understanding of *f(a)* had a marked change during the task. We used thematic analysis to capture keys aspects of Colin's thinking that seemed to support him in transitioning from location- to value-thinking.

Results

Colin's First Episode: Location-Thinking

When beginning his work on the task, Colin's conveyed meaning of output aligned with location-thinking. To represent *f(a)*, Colin first labeled *a* along the horizontal axis by measuring a rightward distance of *a* from the origin and creating a dot labeled 'A' (see Figure 3). Colin then represented *f(a)* by identifying the point on the curve "straight above that (A)". He used a similar process to represent *f(h)* and *f(a+h)*, specifically measuring the input length along the horizontal axis from the origin and identifying the output of the given input as the point on the curve above the corresponding marking. Colin constructed the length *a+h* by concatenating the lengths *a* and *h*. Colin also conveyed a meaning aligned with location-thinking verbally when he was describing his work when representing *f(h)*, saying "so this down here is the value of *h* as an input, and the output using *h* is right there", where he then created the dot on the graph corresponding to *f(h)*. Both Colin's labeling activity and his descriptions suggest that Colin is engaged in location-thinking.

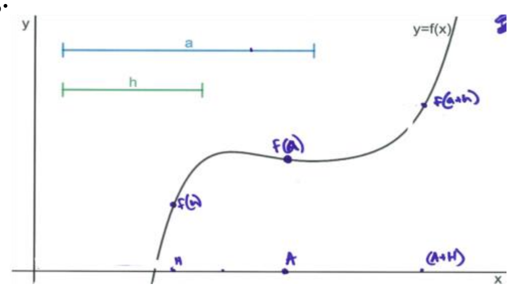


Figure 3: Colin's work after having plotted f(a), f(h), and f(a+h)

When first describing the difference of the outputs *f(a+h)* and *f(a)*, Colin temporarily understood *f(a+h)-f(a)* as the point *f(h)*. However, after reflecting on his work, Colin revised his meaning of difference of outputs as a difference of coordinates pairs.

Colin: But if you're doing this [gestures to the point *f(a+h)* on the curve] minus this [gestures to the point *f(a)*], then you're subtracting two… coordinates. Cause this [gestures to the point *f(a+h)* on the curve] is a coordinate. It's standing in for an '*x*-comma-*y*-value'. So I'm wondering how would you take, um, 'x_1, y_1'[writes '(x_1, y_1)'] and subtract 'x_2, y_2' [writes '$-(x_2, y_2)$'].

Colin's Perturbation – Reorganizing his Schemes

When asked about whether he had subtracted coordinate-pairs before, Colin said that this was not a familiar process to him and indicated that he wanted pairs of values for the points. The interviewer then provided numerical coordinates and asked the student what *f(a+h)-f(a)* would mean to him. Colin then accommodated his graphing schemes. He indicated that he had labeled his graph incorrectly, saying "I marked these [pointing to the point labeled '*f(a)*'] as if they were the outputs, but if they were the outputs, it would just be the *y*." Colin began reflecting and revising his previous responses, and gestured to the prompt *f(a)*. He said,

Colin: This [*f(a)*] should be equal to a *y*-value, because that's the *x* … so like, '*y=f(x)*', and since *a* is what we're putting in the *x*, this [gesturing to *f(a)*] should be the *y*-value, so then I should only be worried about that value [*y*-coordinate of the point currently labeled *f(a+h)*] minus that value [*y*-coordinate of the point currently labeled *f(a)*].

When asked to represent this graphically, Colin at first said that he was not sure how to represent the difference of outputs graphically, but later he drew a vertical line segment from the point $(a+h, f(a+h))$ down to the height of the point $(a, f(a))$. While reflecting on why he represented the difference in this way, he expressed his commitment to sense making, saying "I was just drawing it out and trying to think about some answer that made any sense."

After being perturbed when considering $f(a+h)-f(a)$ as a difference of coordinates, Colin began to consistently think of the output as the y-coordinate of the point he had been thinking about, that is, engaging in value-thinking. At the end of this interview, Colin confirmed this.

Colin: If you'd asked me at the beginning, having not just thought about it for a long time, [I] would have said that it [$f(a)$] was the point. But now that I think about, like, it's like $y= f(x)$, and all we did was substitute the a there, so I would say that this [$f(a)$] is the y-value.

Discussion

During Colin's transition from location-thinking to value-thinking, there were several features of his thinking that seemed critical. The first was Colin's expectation that his activity would align with his prior mathematical experiences. This was clear when Colin became perturbed when considering the difference of two coordinate points, given that he said that subtracting coordinate pairs was not familiar to him, and expressed that he was used to working with numbers. We suspect that Colin had seen and reasoned with quantities written in function notation in prior coursework, particularly with algebraically defined expression and with graphs with number axes. Hence Colin was seeking for his graphical activity to align with his prior mathematical knowledge. In other words, Colin expected that his graphing schemes would align with his other mathematical schemes. This desire to connect his current activity with his knowledge helped Colin become perturbed and reject his current understanding of $f(a)$.

The second key feature was Colin's recognition throughout the task that points in the plane were measured in reference to the axes. When talking about points, Colin would mention coordinates, and these normative frames of reference seemed to support him in leveraging the equation as $y=f(x)$ as a useful tool. We suspect that his measurement activity within these frames of reference also supported him in coming to represent quantities productively. The third feature we'll highlight was Colin's ability to draw on his knowledge of algebraic symbolism to navigate through his state of perturbation and root his activity in his prior mathematical knowledge. Once perturbed, Colin began accommodating his graphing schemes when he revisited how he represented $f(a)$ and connected it to his meaning of the equation $y=f(x)$. Colin justified that $f(a)$ had to be a 'y-value', and then contended that $f(a)$ was the y-coordinate of the point he had been talking about.

Conclusion

While there is much research about students' understanding of graphs and quantities represented in graphs (David et al., 2019; Lee et al., 2019; Moore et al., 2019), less work has been done to understand how students transition between these different ways of thinking. This case study adds to the literature by providing a task that can perturb students thinking in an productive way and beginning to document what features of students' thinking appear to support students in this change. Further work is needed to understand if there are commonalities between these factors or if these factors vary across grade levels. Future research may look to understand how students connect multiple representations, and what sort of teacher moves can help instill dispositions to unite students' mathematical knowledge across different representational systems.

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Ways of Thinking and Ways of Understanding in the Formal World:
Students' Perspectives on Nature of Proofs in Second Courses in Linear Algebra

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Many mathematics departments offer a second course in linear algebra. However, research on teaching and learning the topics in second courses are scarce. To help fill this gap in the literature, in this study, we interviewed 18 students taking a second linear algebra course in both the USA and Ireland. The theoretical framework is based on Tall's (2008) formal world of mathematical thinking and Harel's (2008) ways of thinking and ways of understanding. The goal of the study was to gain an understanding of the teaching and learning of linear algebra proofs. This paper examines the nature of linear algebra proofs from students' perspectives.

Keywords: proof, definitions, formal world, ways of thinking, ways of understanding

Background

Linear algebra is an important topic for many mathematics majors. In a survey paper by Stewart, Andrews-Larson, and Zandieh (2019), the authors summarized some advances in many areas of linear algebra education (e.g., span, linear independence, eigenvectors, and eigenvalues). These studies highlight students' thought processes and difficulties while making sense of these concepts. The authors also identified areas needing more research and revealed some gaps in the literature. For example, research on how students make sense of linear algebra proofs is scarce. Research on topics in second courses of linear algebra, which contain more abstract content, is also desperately needed. The Linear Algebra Curriculum Study Group (LACSG) recommended that "at least one second course in matrix theory/linear algebra should be a high priority for every mathematics curriculum" (Carlson, Johnson, Lay, & Porter, 1993, p. 45). The LACSG 2.0, which was formed in 2018, recommends that mathematics departments offer a variety of second courses (e.g., numerical linear algebra) and include wider topics (Stewart et al., 2022). We have no data on how many mathematics departments in the US offer a second course.

The current literature on linear algebra proofs is in the context of both first and second courses. For example, a study by Stewart and Thomas (2019) aimed to uncover the perceptions that first-course linear algebra students held of proofs. Data were collected through student interviews, and the study discovered that students viewed proofs differently than mathematicians. The authors concluded that the rigor and structure of certain proofs might hinder students' understanding, although those same traits are preferred by mathematicians. It was suggested that the goal of researchers and linear algebra educators should be to find a better method to bridge this knowledge gap between seasoned mathematicians and mathematicians-in-training. In another study, Cronin and Stewart (under review) analyzed 227 feedback comments from 44 tutors over a period of six years about their interactions in a math help center with 82 students taking second courses in linear algebra. Their findings indicated that the most common areas of difficulty were basis, vector space, subspace, span, and proof. Britton and Henderson

(2009) performed a study focused on students' views on proofs about subspaces (in the context of a second course). They noted that "most researchers agree that it is the abstract and highly theoretical nature of [linear algebra] that is the primary cause of [the course's] difficulty" (p. 964). In particular, students had difficulty moving "from the abstract mode in the definition to the algebraic mode in which the question is framed" (p. 966). Both Hannah (2017) and Britton and Henderson (2009) agreed that the number of new definitions which linear algebra students must learn to begin writing proofs is overwhelming and makes learning proofs more difficult. Malek and Movshovitz- Hadar (2011) used interviews and one-on-one workshops to determine the effect of using their "Transparent Psuedo Proofs," or TPPs, in teaching first-year linear algebra proofs. Their results showed that, for non-algorithmic proofs, students who learned using the TPPs wrote more in-depth and satisfactory answers than students who learned proofs traditionally. For algorithmic proofs, both groups of students performed equally. In their words, the use of TPPs is successful because the "TPP serves as a mediator between the abstract level of the proof and students' abstraction ability, being a concrete model of the flow-of-ideas in the proof, or, in a way, a worked-out example of the problem of proving the theorem" (p. 55). Malek and Movshovitz-Hadar (2011) reflected on "three assertions [which] constitute meta-linear-algebra-knowledge or meta-proof-knowledge that mathematics education researchers and practitioners alike keep searching for..." (p.55). These assertions "have to do with the gain in reconstructing a proof, in explaining the main idea underlying the proof and in constructing a (somewhat similar) new proof" (p. 55). Malek and Movshovitz-Hadar developed a method of teaching proofs that utilized the unique nature of non-algorithmic linear algebra proofs while remaining focused on students' understanding of these proof techniques. Likewise, Uhlig (2002) developed a novel teaching technique compared to the traditional Definition, Lemma, Proof, Theorem, Proof, Corollary (DLPTPC) to teach linear algebra proofs. His technique includes the following questions: "What happens if? Why does it happen? How do different cases occur? What is true here?" (p. 338). In his view, "subject specific 'What, Why, and How?' sequence of exploratory questions generally gives students a deep conceptual understanding because this enforces the first principles of linear algebra and gives them the tools to master the subject matter" (p. 338).

Theoretical Perspectives

As part of the framework of three worlds of mathematical thinking, Tall (2008) asserted that the formal world of mathematical thinking, which is based on formal definitions and proofs, "reverses the sequence of construction of meaning from definitions based on known objects to formal concepts based on set theoretical definitions" (p. 7). Harel (2008) introduced the notion of a mental act as actions such as interpreting, conjecturing, justifying, and problem solving, which are not necessarily unique to mathematics. Harel (2008) also defined the notion of a way of understanding as "a particular cognitive product of a mental act carried out by an individual" (p. 269), and a way of thinking as "a cognitive characteristic of a mental act" (p. 269). For example, "proof schemes are ways of thinking associated with the proving act" (p. 271), and a proof is a way of understanding. Harel asserts that the ability to reason abstractly, generalize, structure, visualize, and reason logically comes under the umbrella of ways of thinking. Considering Tall's views on the formal world of mathematical thinking in conjunction with Harel's (2008) ways of thinking and ways of understanding, as our theoretical framework, the overarching research question for this project is: What are some ways of thinking and ways of understanding necessary for grasping linear algebra proofs in the formal world?

The overall goal of this study was to understand the teaching and learning of linear algebra proofs. In this paper, we examined the nature of linear algebra proofs from students' perspectives and what makes linear algebra proofs arguably different from proofs in other areas of mathematics.

Method

This case study is part of a more extensive study on proofs in linear algebra. The authors were both teaching a second course and collaborated on this study. The research team consisted of a researcher in mathematics education specializing in linear algebra education, a research mathematician working on linear algebra and linear algebra education, and two undergraduate research assistants. The team interviewed 18 undergraduate students after completing the second course in linear algebra, six students from the US, and 12 from Ireland. The goal was not to compare the students from different countries but rather to gain as much understanding as possible by considering both groups of students.

The linear algebra course in the US was proof-based and used the text *Linear Algebra Done Right* (Axler, 2015). Abstract Linear Algebra course is the only second course in linear algebra offered at this mathematics department. As the name of the course indicates, this proof-based course is highly theoretical. The course is also slash-listed, meaning that graduate students can also take it since many do not have an adequate background in linear algebra and often benefit from taking this course. A second course in linear algebra usually attracts mathematics majors primarily. However, because of the increasing importance of linear algebra in business and industry, some computer science, meteorology, and physics majors (to name a few) also take the second course in linear algebra. The US students in this study were all seniors who had taken one other advanced course, such as abstract algebra or analysis. The course covered the following topics: Vector spaces and their properties (including special Vector Spaces such as Isomorphic Vector Spaces and Invertibility), subspaces, span, and linear independence, bases, dimension, linear maps, polynomials, eigenvalues, eigenvectors, and invariant subspaces, inner product spaces/ operators on inner product spaces (The Spectral Theorem, Self-Adjoint, and Normal Operators, etc.), and trace and determinant. The course was taught as a mixture of lectures and group work, as students rearranged the room to sit in groups of 3-4. The instructor handed out several theorems with their proofs and engaged the students in a variety of activities, including evaluating proofs for clarity, elegance, and other criteria. On occasions, students were given paper cuttings of a proof to reassemble. Students also came to the front of the class and presented their own proofs or explained an existing one. Students were also given homework assignments to unpack a proof in their own words and sometimes came up with different proofs and presented them to the class. Both undergraduate research assistants in this study have taken this course in the past with the first author.

The course in Ireland introduced the theory of vector spaces and linear transformations, with an emphasis on finite dimensional spaces. The main topics were: vector spaces over a field, axioms of a vector space, subspaces, spanning sets, linear independence, bases, dimension, linear transformations and matrices, isomorphism, the rank-nullity theorem, eigen theory, diagonalization, inner-product spaces, and orthonormal bases. There was no prescribed text for the course. Lectures were delivered through the straight-lecture route of definition-theorem-proof. For example, the treatment of the concept of basis in lectures followed a traditional approach: (i) introducing the definition of the span of a set of vectors as the smallest subspace containing that set of vectors, (ii) linking in with the previous notion of subspace; and (iii) proceeding to the definition of linear (in)dependence and the definition of basis as a linearly independent spanning set.

The interviews took about 40-45 minutes. They were audio-recorded and later transcribed. A sample of the interview questions was: Which of the following proofs are convincing to you and why (Pythagorean theorem; Gram-Schmidt procedure; Characterization of Isometries, Complex and Real Spectral Theorems)? What is the purpose of the proofs in linear algebra? Describe the nature of the proofs in linear algebra. Is there a difference between linear algebra proofs and abstract algebra or real analysis proofs? How can we best teach linear algebra proof in order to enhance your learning experiences? What were the best techniques that were presented in class that helped you with proofs?

Open coding (Strauss & Corbin, 1998) was performed to analyze the data. The common themes for the question related to examining the nature of linear algebra proofs and some initial codes are shown in Table 1.

Table 1. An initial list of themes and codes for students' perspectives on the nature of linear algebra proofs.

Connections to other concepts; many definitions and theorems;	**Visualization and intuition**	**Similarities or differences of concepts/proof topic regarding other branches of math**	**Style of proof**
Spiderweb of concepts Linear algebra proofs rely on other/prerequisite knowledge Definition heavy	Less algebraic than other branches of math Spatial intuition "Logical leap" vs. trial and error for proof-writing	Unique Self-contained subjects and definitions (not reliant on other branches) Conceptually difficult Different from Applied Math courses and proofs Progressively getting to the destination	Length of proofs The ratio of words to symbols Level of difficulty

Preliminary Analysis

Our preliminary data analysis suggested that students pinpointed some specific aspects of proofs in second linear algebra courses. For example, according to student 1 from the US (S1US), definitions play a significant role in a second course of linear algebra. Also, one needs several definitions together.

> S1US: to me, this was a lot more reliance directly on the definitions and using several definitions together rather than, in prior classes we've just done like this equals this

because of this rule, which equals this view of this rule…working with definitions is, is sort of a new, like we, we saw a little bit in discrete math, and then again in linear algebra a little bit, but, um, with like, this was definitely the most intense definition heavy, math that I've taken.

Student 3 from the US mentioned the structure of proofs and the ability to progressively reach the destination without worrying about small things along the way.

S3US: It's like a lot of structure with a lot of linear algebra stuff.
I enjoyed like being able to, to really not have to worry about the fact that or not having to worry about any type of convergence or doing epsilon delta proofs like you would an analysis. They'd get kind of messy, and you're just trying to almost like the little, the little thing that makes everything fall. It didn't feel the same with Linear Algebra. It felt much more like you're progressively getting to your destination rather than how can I find the one little key that or one little like modification to this Delta or Epsilon to make this work. And I thought that was pretty enjoyable.

Student 5 from Ireland used the analogy of spiderwebs to express his views on linear algebra proofs.

S5IR: Well, in linear algebra, we start with definitions, right? We, we start by defining some things. Now in most cases, it's from, we have some prior knowledge of what we're trying to describe here. So then we have, we, we form our definitions based on these things. Um, then we make other, maybe in the background, again, we're having other observations about what's is how these definitions are interacting or whatever, and we maybe have some conjectures in their heads. Do we want to prove, so in that case, you just take these, you take these definitions, and then you come up with that theorems and proofs and it just kind of spiderwebs outgrows far more complicated. Yeah. Just from where we started.

As we continue on this new terrain of research, we plan to develop the theoretical framework further, complete the coding of the data, and analyze the data by employing the framework. Our analysis will also include some recommendations for teaching proof in seconds courses.

Discussion Questions

1. How to successfully network Tall's (2008) and Harel's (2008) theories?
2. We are in the process of gathering more classroom data. What other additional data should we collect from students?
3. We are also in the process of performing a series of in-depth studies with several instructors of the second course in linear algebra. What are some research questions we should consider as we plan these studies?

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Unpacking a Gateway Mathematics Change Initiative in Response to a State Mandate

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Postsecondary institutions nationwide continue to grapple with issues related to retention and completion within gateway mathematics courses. Many states have put forth legislative mandates to address this pressing issue. The MPIE project investigates the change initiative at one two-year institution, SCHSI, while assisting local stakeholders in developing and facilitating professional development. This article discusses the initial findings from the first year of the project, highlighting silos of knowledge that exist across groups committed to supporting student success in gateway mathematics courses at SCHSI. Recommendations are suggested for how de-siloing might occur within and beyond the MPIE project with the goal of promoting sustainable change in gateway mathematics courses.

Keywords: Gateway Mathematics, Systemic Change, Professional Development

Introduction

Southern California Hispanic Serving Institution (SCHSI) is a two-year Hispanic serving institution geographically located near the southern border of the U.S. Like many two-year colleges, SCHSI has been working on ways to address problems related to student retention and completion. Developmental education has been cited as a pervasive obstacle for students, and a deterrent for future participation in the science, technology, engineering, and mathematics (STEM) fields (Ngo & Park, 2020). Many students begin their SCHSI academic journey by enrolling in developmental mathematics courses which do not provide transfer-level credit. Of these students, less than 10% of them graduate within three years, while 40% of them never complete the developmental sequence (The Campaign for College Opportunity, 2018).

In accordance with state Assembly Bill 705 (AB705), SCHSI has reduced their developmental course offerings which has allowed for students to enroll directly into transfer-level mathematics courses (e.g., for STEM-intending students this includes College Algebra, Trigonometry, and Precalculus). AB705 mandates that two-year colleges “maximize the probability” that students will attempt and complete these courses within their first year of enrollment (AB705, 2012). In addition, this mandate, which was first implemented at SCHSI during the Fall 2019 semester, instructs institutions to eliminate placement exams and instead use multiple measures (e.g., high school GPA, high school grades, previous course enrollments) in student placement.

Mathematics Persistence through Inquiry and Equity (MPIE) is a National Science Foundation-funded project focused on redeveloping gateway mathematics courses at SCHSI. While prior research have documented change initiatives and best practices for implementing sustainable change within organizations (Bolman & Deal, 2008) and departments at four-year institutions (e.g., Reinholz & Apkarian, 2019; Quan et al., 2019), there is little research on how these methods can be adapted or reimagined for the two-year college context. Thus, the primary goals of the MPIE project are (a) to study SCHSI's response to AB705 and provide insight into the classroom environments in gateway mathematics courses, (b) attend to equity in student experiences as they relate to classroom engagement to course outcomes, and (c) study the systemic change effort at the institution and how various institutional stakeholders (e.g., administrators, advisors, instructors, and students) contribute to this effort.

The Six Principles for Department-Level Interventions is an organizational framework that has been used in prior research (Ngai et al., under review; Quan et al., 2019) to guide a group's work towards sustainable educational change initiatives. The six principles include (Quan et al., 2019, p. 2-3):

1. Students are partners in the educational process;
2. The group's work focuses on achieving collective positive outcomes;
3. Data collection, analysis, and interpretation inform decision-making;
4. Collaboration between group members is enjoyable, productive, and rewarding;
5. Continuous improvement is an upheld practice among the group members; and
6. The group's work is grounded in a commitment to equity, inclusion, and social justice.

While there is no apparent "group" or task-force devoted to the institutional change at SCHSI motivated by AB705, there is an alignment of goals among the various institutional stakeholders at SCHSI that have participated in the MPIE project thus far. In this paper, we highlight how instances of the first three principles were apparent across the interviews of various institutional stakeholders.

Methods

The data were collected through the MPIE project at SCHSI, which has high proportions of racially minoritized students (85%) and Latinx students (86%). The MPIE project consists of three phases. Phase 1 consisted of collecting and analyzing data at SCHSI during the 2020-21 academic year to establish a baseline of the institution's response to AB705. Findings from Phase 1 are currently informing Phase 2, the design and implementation of professional development (PD) aimed at transforming instruction in gateway mathematics courses that are more focused on equity, inquiry-oriented instruction, and sustainability. In Phase 3, the project team will investigate the impact and sustainability of the PD. In this paper we discuss results from interviews conducted in Phase 1 with 11 institutional leaders (e.g., administrators and leaders of student support programs), 18 students, and 9 instructors. The purpose of these interviews was to gain a variety of perspectives regarding SCHSI's response to AB705.

Interviews focused on understanding SCHSI's change efforts in response to AB705. Using Quan et. al's (2019) six principles for department-level interventions as a priori codes (Miles & Huberman, 1994), we identified and synthesized instances of each of the principles in the interview data from different stakeholders (students, instructors, advisors, and administrators). So far, the data we collected is rich enough to make claims about Principles 1-3. Interview protocols were not originally designed with the six principles in mind, but future data collection could reflect this in light of our findings. Thus, we focus our discussion on Principles 1, 2, and 3.

Findings and Discussion

Principle 1: Students as Partners

Due to AB705 mandates, SCHSI replaced their mathematics placement exam with a self-assessment questionnaire, shifting the decision of course selection to students. While this afforded students with more agency as stakeholders in their own educational experiences, there also seemed to be confusion in the advising process. In interviews, many students expressed widely varying experiences for how they enrolled into their current mathematics course. While some students partnered with advisors to make choices about their schedules, others chose classes without that guidance. One student's understanding highlights some of the confusion that several students are experiencing:

> So um typically, I know it's like kind of hard and like I mean, I have my student plan. And that's what I'm going off of. So I'm gonna, I'm hoping to meet with the, you know, counselor/advisor. Um, but basically, I'm going to go off of the student plan we had made. But I do know that some of the students have been saying that they haven't been able to get a hold of them. So um I'm not quite sure about that. I haven't tried myself.

With this newfound student agency in choosing classes, instructors are seeing a wider range of ability in their classrooms. This instructor highlighted a commonly expressed concern:

> They're not coming in with the skills they need to pass. There are some who come in poor and do whatever they need to do to catch up, but for the most part those ones who do not come in with the correct skills somehow filter out.

While many faculty members expressed concerns about a greater number of students lacking prerequisite skills to succeed in classes, students generally expressed that they felt that both they, as well as their peers, had been placed in the correct courses. While some students did express feeling less prepared at the beginning of the semester, they cited their hard work and seeking help in order to succeed as reasons for being placed in the right course. These contrasting perspectives between students and instructors may indicate a tension between deficit and asset approaches to teaching and learning. Further, this highlights the disconnects that can occur between teachers and students.

These findings demonstrate that SCHSI's responses to AB705 mandates are moving them closer to alignment with Principle 1, because their changes afford students more power as stakeholders in their educational experiences. This is an example of taking an asset-based approach to student placement. It also indicates a culture-shift opportunity for the college to partner students with other stakeholders so that the student experience is more fully considered in the decision making process.

Principle 2: Achieving Collective Positive Outcomes

While all groups at SCHSI are doing what they can to best support students, silos of knowledge exist that limit the impact of their efforts. In particular, advisors tend to have a more bird's eye view on student success at the college while instructors have a more local view on student success. Through interviews with academic leaders, we learned about some of the main concerns that advisors focus on when counseling students. They are trained to consider *credit creep* that might occur if students are enrolling in prerequisite or support courses. Credit creep occurs when a student's credits approach or go beyond what is transferable to a four-year institution or constrains financial aid awards. One administrator shared some of the concerns they consider from their perspective in relation to support courses - courses taken concurrently with a gateway course to provide students in need with additional academic support:

> So what I mean by that is that the way the discipline faculty have set it up is they have the transfer level course plus the support course. So, and it's a two unit support course which has implications for students' financial aid package. So how do we start really looking at the data to possibly reduce the number of units for the support course. Or if something that could happen at the curricular level to completely eliminate the support course so that what students need, they're getting an regular three to four unit course right.

Advisors are focused on making all of the puzzle-pieces fit in a student's schedule so that they are on track to graduate or transfer in a timely manner. Thus advisors are typically not aware of specific course content and nuanced relationships between courses in the same way that instructors are..

Instructors, on the other hand, expressed a focus on helping students succeed in the courses they are teaching. They are concerned about the prior knowledge that students have coming into a course, and whether their students will leave their course prepared for the next one. Instructors tend to be more focused on the success of students one course or sequence of courses at a time, as highlighted by this instructor's goals for teaching: "So that's what I'm trying to do—understand more of a concept rather than the step-by-step memorizing. So that's kind of my… goal. By the end of the semester, they get out of my class. Next level [math] course, they are ready." Instructors do not necessarily have knowledge of where students are in their respective degree programs, but they are considering their student's trajectories as it relates to mathematics courses.

These findings demonstrate that advisors are making decisions based on long term goals whereas instructors are making decisions based on short term goals. This highlights an important cultural shift that needs to occur at SCHSI that de-siloes the knowledge between advisors and instructors so that both the short and long term needs of students are being considered in tandem.

Principle 3: Data-Informed Decision Making

Various stakeholders at SCHSI use different metrics to measure the success of AB705, creating what we call *the paradox of throughput*. Instructors are concerned about higher failure rates after the implementation of AB705 while administrators simultaneously are expressing positive feelings about higher throughput rates (i.e., more students are attempting and completing gateway courses), which are outlined in Table 1. While more students are taking gateway courses explains both increased failure rates and increased throughput, these different emphases of the data highlight the values of different stakeholders.

School administrators and advisors generally expressed positive feelings with the increased number of students successfully passing gateway mathematics courses. For example, one administrator stated, "Well, some of the positive outcomes [of AB705], is that we're noticing, and forgive me, I don't know the exact details of the throughput rates, but we're noticing that yeah, our students are succeeding." Some administrators also expressed these data trends as hope for the future, citing increased throughput being linked to success in future mathematics courses, while others expressed concerns, asking questions like "how many of them actually are succeeding in terms of the domino effect? How many are actually able to do the follow up coursework …?". Some administrators who work more closely with instructors expressed their awareness of a very different perspective on AB705, and how some have emotionally expressed their feelings of personal failure when students have struggled in their courses.

As instructors have noticed a greater diversity of ability in their classes after the implementation of AB705, some have called AB705 “the worst thing ever.” Many have expressed concerns for students “wash[ing] out because they felt overwhelmed because they felt they were not in the correct course.” One instructor expressed the lack of support they have felt in response to AB705. While instructors continue to teach as they have in the past, they are now witnessing more student failure and withdrawal, which has left a bad impression of AB705 and the measures the college has taken to implement the mandate.

These findings also highlight a culture shift that needs to occur at SCHSI as the college continues to move forward from its early years of AB705 implementation. While it is positive that some administrators are aware of the concerns that instructors have, it will be vital for these perspectives from both groups of stakeholders to be more widely shared so that as a collective, SCHSI can make data-informed decisions that consider the different values of various stakeholders.

Table 1. First attempts and pass rates in College Algebra at SCHSI per academic year

College Algebra	17-18	18-19	19-20
First attempts	121	177	474
Passed	58	85	203

Conclusion

Using principles for institutional change, we identified silos of knowledge across different groups of stakeholders: between students and instructors (Principle 1), between advisors and instructors (Principle 2), and between instructors and administrators (Principle 3). This highlights the need to connect these groups of stakeholders in ways that bring their values and approaches together more effectively. One recommendation for de-siloing these varied perspectives may be for the institution to create a group or task-force specifically devoted to the institutional change at SCHSI. Although the Six Principles were initially conceptualized for a department (or a group within a department), this paper highlights an application to a broader group of individuals that have some shared vision.

The MPIE project team can leverage these findings specifically through PD efforts in Phase 2. One way we can do this is by helping instructors foster asset-based mindsets to de-silo student and instructor perspectives. A second approach the team has discussed is to find a way to incorporate advisors into our PD efforts. Both of these considerations can contribute to the sustainability of student success initiatives at SCHSI. As we move forward in our project, we invite the audience to consider these questions on our minds:

1. What other ways might the MPIE project team facilitate de-siloing efforts to bring different stakeholder groups together?
2. In what ways can Principles 4, 5, and 6 be leveraged to support this group of institutional stakeholders enact systemic change within their gateway mathematics courses?

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Comparing Authenticity in Proof Activity in an In-Person and Online Setting

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The extent to which online course delivery allows students to engage in authentic mathematical activity has yet to be explored. In this preliminary report, we use the Authentic Mathematical Proof Activity (AMPA) framework to analyze data collected in a larger design-based research project. This data consisted of videos of the same lesson implemented online and in-person. Our results show that while it is possible to provide students with opportunities to engage in authentic mathematical activity in an online course, opportunities were limited compared to in-person courses. Researchers may want to use this framework to continue to explore how the dimensions of authenticity are similar or different across online and in-person course settings.

Keywords: online instruction, authentic proof activity

The transition to online courses necessitated by the COVID-19 pandemic has allowed for new exploration into online instruction (e.g., Jung & Brady, 2020). Many courses, such as those in advanced mathematics, are currently being offered online. The efficacy of online mathematics courses is generally poor (Trenholm, et al. 2019); although the majority of this literature focuses on asynchronous courses. Student perceptions of online courses range from quite positive to quite negative (Dobbs, Waid & del Carmen, 2009). For example, 40% of students surveyed by Jacqueline and Smita (2001) indicated higher participation in online courses than traditional courses. However, O'Malley and McCraw (1999) found that students found it difficult to contribute to discussions online whether synchronous or asynchronous. Recent work in the mathematics setting points to ways that "rich dialogic interactions" can be maintained by having students share strategies and engage with them using unique features of online settings such as shared Google Docs and breakout rooms for small group discussion (Jung & Brady, 2020). Further, Öner (2008) suggested the online setting may be particularly conducive to engaging students in authentic proof activity through collaboration and exploration using dynamic geometry software. Similarly, Yopp (2014) illustrated how asynchronous online discussions may serve as a productive space for authentic engagement with quantifiers and tasks via examples and example-generation. The literature further points to ways in which online collaboration may be different and need different support as students engage with features like text-based chat (e.g., Stahl, 2006) or Zoom (e.g., Jung & Brady, 2020).

We aim to contribute to this literature base by situating our study as a direct comparison between a lesson implemented in-person and implemented online. The lesson was developed through an iterative design-based research approach with an explicit focus on engaging students in authentic proof activity defined broadly as engagement in formal mathematics in ways that is consistent with the work of mathematicians. This includes not just creating formal proofs, but the informal activity and alternate goals such as comprehension and validation. The lesson is part of a standard introductory undergraduate abstract algebra course and focuses on comparing between two common proof approaches and analyzing proofs and statements (see Melhuish, et al., 2022 for an outline of the lesson goals.) As our overarching goal was to promote authentic proof activity, we share an analysis of these two lessons to explore how authenticity may have played out differently in the two contexts. We conclude with conjectures as to why the online setting may have led to different instructional choices and different opportunities for students.

Theoretical Framing

Underlying our work is the assumption that advanced mathematical courses provide an opportunity for students to apprentice into the mathematical work of research mathematicians. To this end, we have developed a literature-based framework to describe the activity of mathematicians that has potential for adoption to the undergraduate classroom (Melhuish, et al., 2021). We broadly use activity theory (Engeström, 2000) to frame our approach where activity can be decomposed into goal-directed actions consisting of tools (materials, concepts, procedures) used in service of an objective (including a motive and focal object). Activity occurs in systems that are historically (such as where tools originate) and socially situated (communities with rules and norms divide up labor). Our overarching framework includes three objects: proofs, propositional statements, concept/definitions and three motives: constructing, exploring/comprehending, testing/validating. In service of these goals we include tools: analyzing/refining, formalizing, deformalizing, warranting, analogizing/transferring, examples, diagrams, logic, structure/frameworks, and existent objects (definitions, proofs, statements). We then operationalize authenticity along a number of dimensions to capture multiple, often competing (e.g., Dawkins, et al., 2019; Herbst, 2002; Lampert, 1992) notions of authenticity guided by ideas of content, practice, discipline, and students. See Table 1 for the authenticity dimensions of the *Authentic Mathematical Proof Activity (AMPA)* framework.

Table 1. Dimensions of Authenticity Defined by Tool Use

Dimension	Description	Characteristic
Variety	The degree of variety of tools in use including formal, informal, generating, and translating tools	Disciplinary Tool Use
Complexity	The degree to which tools and outcomes are used in conjunction and succession versus in isolation	Disciplinary Tool Use
Accuracy	The degree to which tools and outcomes are accurate to discipline standards	Discipline Tools and Outcomes
Agency	The degree to which students are the ones generating and using various tools	Student Role in the Division of Labor
Authority	The degree to which students are the ones connecting tools and objectives to determine whether a goal is, or will be, met	Student Role in the Division of Labor
Alignment	The degree to which tools and outcomes reflect student contributions	Student Tools and Outcomes

Methods

This study is part of a larger design-based research project (Design-Based Research Collective, 2003). The focal lesson was developed through an interactive design process and included creating a hypothesized trajectory of student activity linked to particular tasks features and instructional moves. The lesson was developed, evaluated by a panel of experts, and tested and refined over two iterations with a small group of undergraduate students. The goal of this lesson is for students to *comprehend* two different approaches to proving the structural property:

if G is abelian and isomorphic to H, then H is abelian. By comparing approaches (one that begins with elements of the domain and concludes that the image of these elements commute; one that begins with elements of the codomain and concludes these elements commute), students are positioned to attend to structural features of the proof and engage in analysis of which assumptions from the statement are needed and why. This then leads to a discussion of modifying the proofs and the statements through a process of analysis and refinement.

The data for this report comes from two classroom implementations in consecutive semesters. Both implementations were facilitated by the same instructor at the same institution. The class consisted of a relatively even distribution of mathematics and mathematics education majors with less than 20 students in each class. The first implementation, in-person, was videotaped using Swivl. The second implementation was conducted over Zoom and was recorded using the Zoom interface. In both implementations, members of the research team observed small groups and took field notes. We conducted a retrospective video analysis using the AMPA framework to guide analysis. One researcher repeatedly viewed the videos and identified comparable episodes for the two lessons. This researcher and another member of the team independently viewed selected episodes and created analytic memos attending to the various features of the AMPA framework identifying tools at use, objectives (motives/objects), and describing authenticity across the six dimensions. The respective analyses were compared with discrepancies resolved through discussion.

Preliminary Results

The goal of this lesson was for students to analyze and comprehend two different proof approaches for the statement: *if G is abelian and isomorphic to H, then H is abelian*. For the scope of this report, we focus on two episodes. In the first episode, students spent time in their groups coming up with similarities and differences between each proof approach. These were then shared with the whole class to form a list of all the similarities and differences identified by the students. In the second episode, students were asked to discuss with their group whether they thought each assumption (abelian, homomorphism, one-to-one, and onto) was needed. After discussion in groups, the instructor had students share their ideas in a whole class discussion.

The similarity/differences episode consisted of students spending time in their groups attempting to identify similarities and differences between the two proof approaches. In the online section, this was done via breakout rooms on Zoom. After each group was allowed time to discuss, the instructor brought all of the students back to the main room. The instructor then asked the students to state some of the similarities and differences they observed between the two proof approaches. Most of the students' responses were typed in the chat. We observed evidence of students *deformalizing* a proof to explore its components as they used informal language to describe the similarities and differences between two formal proof approaches. We also saw one example of *warranting*: identifying that both proof approaches use isomorphism, and one example of using *structure* by identifying the difference: one proof starts in the domain, while the other starts in the co-domain. The instructor wrote each of the similarities and differences onto a shared document. Therefore, this segment had a high level of *alignment* with student contributions. Additionally, the instructor was not confirming or correcting any of them. Thus, in this segment, students had a high level of *agency* as during the comparison they were spontaneously warranting and identifying proof frameworks (tools), and maintained most of the *authority* as the instructor was not evaluating or linking the students' contributions to the larger motive (in service of exploring/understanding the proofs). At this point, *accuracy* was high as most of the students' noticings were valid, *complexity* was low (different tools/outcomes were not being used consecutively/together to achieve a goal), and *variety* was mid as a number of tools were in play, although they stayed largely formal.

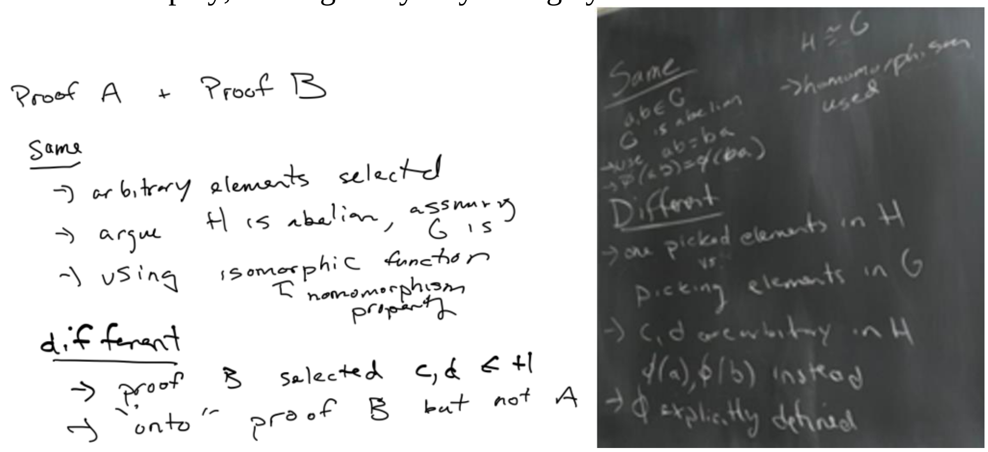

Figure 1. Public Record Documents of similarities and differences in online and in-class setting, respectively

The similarities/differences episode of the in-person section shared many of the same characteristics as the online counterpart. However, we note this occurred earlier in the lesson. Both episodes began with the instructor prompting students to share similarities and differences they observed across the two proofs. The instructor recorded each suggestion on the chalkboard, so this segment also had a high level of *alignment*. However, the instructor also prompted students to explain where each contribution was present in the proof. Thus, students were given slightly more opportunities to warrant in the in-person section than they were in the online section. The frequency of warranting appeared to be the only difference in authenticity across the two lessons. The authenticity dimensions in the in-person section were all comparable to those in the online section.

The assumptions episode consisted of students attempting to determine which assumptions were used in each proof. The online section spent time discussing the assumptions in breakout rooms based on a list of assumptions developed from a poll earlier in the class. When they returned as a whole class, the instructor used the poll function on Zoom to find out which assumptions the students thought were being used. The results of the polls were written down, and the instructor did not endorse any particular answer over another. Thus, students had some *agency* (although highly directed) to *analyze* the proofs, and the segment had a high level of *alignment.* However, due to the use of polls, the *variety* and *complexity* of contributions the students were able to make was limited. They were not given the opportunity to discuss where or why each assumption was being used (*warranting*). In general, students in the online section had limited opportunities to use *authority* -- connect their tools to the larger exploring proof motive -- in this episode *Alignment* remained high as the students' voting was recorded by the instructor in contrast to *accuracy* as many students did not respond in normative ways.

Students' opportunities to engage in authentic mathematical practices differed across the online and in-person sections in the assumptions episode significantly more than they did in the similarities/differences episode. In the in-person section, not only did the students state whether or not they thought each assumption was being used, but they were also asked to point out where each assumption was being used. For example, the students came to a consensus that the assumption that G is abelian is being used in the proof. The instructor then asked them to point out where it was being used in the proof. One student pointed at the proof being displayed on the projector and stated, "*a* operated with *b* equals *b* operated with *a*." Asking students to point out where each assumption was being used resulted in students having more *authority* as they took opportunities to warrant in service of exploring the proof . Additionally, allowing students to point to the proof (*warranting*), describe why each assumption is necessary (*analyzing*), and explain lines of a proof in their own words (*deformalizing*), increased both the *variety* and *complexity* of students' contributions. After some discussion, the class emerged with two conjectures about the assumptions needed for each proof: abelian and homomorphism, and abelian, homomorphism, one-to-one, and onto. The instructor wrote these down, *formalizing* them into conjecture form, and asked the class to discuss them in groups (increasing *complexity*). Therefore, the episode ended with mid *alignment* (as the instructor formalized the students' ideas), but low *accuracy* (as neither conjecture was valid.)

Discussion

Our results suggest that it is still possible for an instructor to provide students with opportunities to engage in authentic mathematical activity in an online setting. This was observed in both episodes we described above. The instructor was successful in giving students agency by providing them with opportunities to warrant and analyze. We also observed that online students had a high level of authority in both episodes, as it was their responsibility to determine what was valid and why. These results are important because they provide evidence that an online setting does not preclude authentic mathematical activity.

Although our results suggest that it is possible to provide students with opportunities to engage in authentic mathematical practices in an online setting, they also provide evidence that the extent to which this can be done may be limited compared to an in-person setting. The first major difference we saw across the sections was that in the online section the instructor tended to invite contributions by using polls and the chat window. This reduced the variety and complexity of students' contributions. Additionally, we saw that the use of the poll and chat in the second episode resulted in students having less opportunities to warrant and less overall authority. However, the use of these features meant that all students contributed, rather than just more vocal students in class. In some sense, this may serve the role of increasing student engagement in activity that parallels the in-class mechanism of a "turn and talk" which is not readily available online. Some of the differences may also be accounted for by pace. In the online version, the group work components of the lesson took more time. This may be partially due to the nature of going between a main room and a breakout room, as well as the time involved for the instructor to move from group to group. The instructor may have opted for polling rather than conversation with warranting due to time constraints. The slower pace accounts for the online version concluding with the assumptions task without further exploration of a conjecture that occurred after this episode in the in-person version. As this work is preliminary and situated in a particular lesson, we hesitate to make global claims about authenticity in activity online. We also acknowledge the difficulty of differentiating between constraints inherent to the course delivery mode and choices made by the instructor as a limitation of this report. However, the common setting, lesson, and instructor provided at least one case that points to similarities and differences across the contexts. Future researchers may want to use a similar approach to analyze authenticity to further understand the affordances and constraints of different class settings.

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Students' Perspectives on Instructor-Modified Pedagogical Proofs

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Mathematical proofs are an integral part of teaching undergraduate mathematics. However, interpreting mathematical proofs is known to be a difficult topic for students to grasp. In the study described in this paper, which is part of a larger study, we asked seven mathematics students to compare several real analysis proofs that had been modified by instructors so that they might be better understood by students. The student participants in our study determined that the proof modifications that they found most useful were when the instructors provided more details within the proofs, gave accompanying examples and diagrams, rearranged the ideas of the proofs, and used colloquial transitions within the proofs. The results of this study have implications for the teaching and learning of proof.

Keywords: Real Analysis, Proof, Advanced Mathematical Thinking

It is evident from the research (Almeida, 2000; CadwalladerOlsker, 2011; Hanna, 1990; Hersh, 1993; Mills, 2011; Weber & Mejia-Ramos, 2014) that mathematical proofs are difficult for many undergraduate students to understand. This may be partially due to the fact that, barring a few exceptions, most mathematics students are generally not exposed to mathematical proofs until enrolling in an introduction to proofs course, most often taken during their second or third year of undergraduate studies (Almeida, 2000). Given the significant role that proofs play in the field of mathematics, we believe that it is critical to investigate ways to improve the teaching and learning of mathematical proof at the undergraduate level. The work described in this paper is part of a larger study and focuses on which instructor-modifications made to proofs that undergraduate students found most useful.

Theoretical Framework

Using the ideas of communities of practice (Lave & Wenger, 1991; Wenger, 1998), we can view mathematicians as forming a community of practice by having their own common activities, thought processes, beliefs, and values. Through participation in community activities, mathematicians solidify their membership to the community by sharing common norms, expectations, understandings, views, beliefs, definitions, symbols, and objects. While conducting these activities, mathematicians reify ideas and experiences into physical artifacts such as textbooks, journal articles, lecture notes, and solutions to problems. Proofs are the most prolific of these objects, being used in nearly every mathematical task that a mathematician performs. These dual actions of participation and reification both help mathematicians situate themselves into the community of mathematicians and helps the community define itself.

Undergraduate mathematics students do not identify as full members of the community of mathematicians, as they do not participate in many of the community activities, and they often do not create reified objects for the community. Rather, they are peripheral members, some of whom are on a trajectory to become full members of the community of mathematicians. As peripheral members, similar to apprentices, students are given small tasks that simulate parts of the practice within the community. From this peripheral perspective, newcomers can see the entire practice and gradually familiarize themselves with all the parts of the practice. As their experience increases, the newcomer is given more tasks to learn that are central to the practice, also providing a more central perspective from which to observe the practice as a whole. To students, instructors act as brokers to the common activities, thought processes, beliefs, and values shared by the community of mathematicians. As brokers, instructors are tasked with demonstrating those shared ideas to boundary members in a way that the boundary members can understand. Similarly, proofs act as boundary objects, providing a link to the practices of full members of the community of mathematicians. When reading a proof in a textbook or written by a professor, students can see the way a mathematician thinks about the task of proving a theorem, and thus learn more about theorems, proofs, and the field of mathematics in general.

Literature Review and Objective of the Study

Proofs serve as boundary objects between students and the community of mathematicians, communicating the ideas of the community with the students. However, scholars have argued that proofs serve many purposes, and some proofs may not be suitable for the classroom (Hanna, 1990; Hemmi, 2010; Hersh, 1993; Weber & Mejia-Ramos, 2014). These proofs vary from simply validating the truth of the theorems, to proofs that explain why the theorems are true. Proofs that help systematize mathematics may be of great use to mathematicians but of little use to a mathematics student. Selecting proofs for their explanatory power, as opposed to only selecting proofs based on conciseness and validity, helps students to better understand the ideas communicated by the proof. For example, while a graphical argument may not be considered a proof by the mathematics community, students can gain understanding from such an argument. Further, students do not necessarily understand the purpose or use of proofs, either in the classroom or in mathematics research (Alemida, 2000; Knuth, 2002; Weber & Mejia-Ramos, 2014). Rather students and mathematicians often have different beliefs and ideas about what proofs are and how they are used.

To better communicate the ideas of a proof, and to mitigate some of these differing beliefs, previous research has examined how proofs are presented in the classroom (Hemmi, 2010; Mills, 2011; Movshovitiz-Hadar, 1988; Weber, 2004). These studies suggest new ways to present theorems and proofs or categorize proofs and their presentations by looking at what they focused on. Some proof presentations found in the literature focused on the logical structures of the proofs; others focused on how the proof works through several examples; still others focused on the theorems that made up the proofs. Another way to categorize proofs uses several axes to measure different aspects of the proofs, by looking at how inductive or deductive the proof is, how formal the proof is, and how visible the proof is (Hemmi, 2010).

One particular study (Lai et al., 2012) asked several instructors to modify two calculus-level proofs to improve student understanding. These instructors were told to prepare the proofs for students who were in their second or third years of their undergraduate degree. The study found that instructors provided various additions, changes, and deletions to the original proofs. Then, this study surveyed 110 mathematicians, providing feedback to the original and modified proofs. The results of the survey suggest that the larger community of mathematicians agreed that the changes made to the proofs by the original instructors should improve student understanding of those proofs.

However, the results of this study may not provide a complete picture of which proof modifications help students understand proofs. Another study (Lew et al., 2016) explored why students struggle in upper-level undergraduate mathematics classes. They studied several factors that might contribute to why students sometimes struggle to understand a mathematics professor, even one that is highly regarded as an outstanding instructor and an excellent lecturer. After having both students and mathematics experts view an advanced mathematics lecture, the students struggled to identify the key points that the lecturer was trying to make. Meanwhile, the mathematics experts were able to identify all of the key points. This implies that members of the mathematics community may not always be able to judge what is accessible to students.

The purpose of the larger study is to fill this gap in the literature. Our goals for the larger study are: 1) to better understand how instructors modify textbook proofs in real analysis to make them clearer to undergraduate students and 2) to capture from undergraduate students which instructor modifications they find most helpful. The first of these goals has been presented previously (Authors, 2021). This paper will continue where the other part of the study concluded, focusing on the second goal.

Method

Seven student participants were recruited through solicitation presentations in junior and senior level mathematics classes. Students then emailed the researcher to volunteer for the study. Participants self-reported a wide variety of understanding of the topics of real analysis: three students had not yet had an analysis course, one student reported to struggle in their analysis course, and the other three reported to have a strong understanding of the topics discussed in the proofs in this study. One student was an applied mathematics major, one was a preservice mathematics teacher, two were double majors in physics and mathematics, one was a pure mathematics major, and the last two were in their first year of mathematics graduate school. Only one of the seven student participants identifies as a woman, two are of Middle Eastern decent, and one identifies as African American.

The researcher conducted a two-hour interview with each of the participants. Due to the Covid-19 pandemic, interviews were conducted through virtual meeting software. After a few questions to establish the students' comfort with the topics in analysis, the students were presented with several original and instructor-modified proofs. In the first stage of the study, we collected 12 instructor-modified proofs, but for this portion of the study we presented each student participant with up to nine of these modified proofs. The instructor-modified proofs that were selected to be given to the student participants included a wide variety of modifications. Some proofs were selected because the instructor introduced new diagrams or examples, others were selected because the instructor restructured the order of the proof. One proof was selected because the instructor used an alternate definition of a set within the proof. The instructor-modified proofs were each placed on a separate sheet of paper, as were the four original proofs. The students were first asked to read the original version of a proof and to comment on elements of the proof that they either found to be confusing or helpful. Then, students were asked to read modified versions of the same proof and again comment on elements of the proof that they either found to be confusing or helpful. Additionally, students were to compare the versions of the proof with each other. These interviews were audio recorded and transcribed. The transcripts were coded inductively using an open coding method (Corbin & Strauss, 2014), with extra attention given to the types of modifications that students mentioned. A second coder was utilized on a subset of the transcripts to ensure coding reliability, as recommended and described by Campbell and colleagues (Campbell, et al., 2013).

Initial Results and Discussion

The researcher used how well the students reported to understand the original proofs to place students on a spectrum from low initial understanding of the proofs to high initial understanding of the proofs. Our analysis found that students with low initial understanding reported to gain understanding from longer modified proofs that contained more explanations, while students with higher initial understanding preferred proofs that were shorter and required them to study the proof in more depth.

Students with low initial understanding struggled to understand some mathematical objects that were defined within a proof. For example, one proof introduced a subset of a sequence that is strictly decreasing. Most students needed to spend time piecing together the definition of that set, and several of the students who were unfamiliar with real analysis did not try to understand the set definition without being prompted to do so. When asked to describe the set, which the proof named P, one student responded, "That's a good point. I don't know what P is. Is it the sequence?... I guess, um, knowing what P is, that'd be pretty helpful." Meanwhile, the students who had the most comfort with analysis proofs recognized that understanding the set definition was a key step to understanding the proof and they immediately tried to understand the definition after an initial reading of the proof. All of the modified versions of this proof contained additional descriptions of this set. One instructor included a diagram and example, one changed the set itself to something they felt was easier to understand and included a colloquial explanation, and the last included a more detailed description of the elements of the set. Different students preferred different modifications. The students who had the lowest initial understanding preferred the diagram and example, students in the middle preferred the simpler set definition, and the students who understood the original proof preferred the proof that included the more detailed description. One of the students that preferred the detailed description said:

> They explain what this set is rather than just trying to interpret what it means…. It's a fun exercise to understand what it means, but also that isn't it just reading the description of what it means is always much easier.

All student participants talked about how important it is for a proof to be "easy to read." For example, they felt that proofs that were too dense were difficult to read because they required the reader to keep a lot of information in their head at one time. One student commented:

> So at one [sentence], it has three parts. You have to think of it at one [time]. It's like when you're, when someone says as like three negatives in a sentence, … it's like a lot of things that I have to keep track of my head at one point, and it makes it difficult to keep it keep tracking.

However, proofs that were very long were overwhelming, and students found themselves easily lost. Similarly, students mentioned that long symbolic expressions can be difficult to understand without a colloquial explanation. One student said, "It's harder for me to understand multiple stage inequalities.… that [long inequality] can be hard, and that took me a minute to generally understand what that means…. So when possible having simpler inequalities are useful." Another student had similar comments about a definition with a lot of symbols. "I like the [colloquial English] one on the left more. I think having that one helps to break it up [instead] of having all these symbols mashing together on the right."

Another finding is that sentences that indicate transitions between parts of proofs in colloquial English were considered very helpful by all students. These phrases often connect ideas from the proof and theorem to each other. This included phrases such as, "this proves part 1 of the theorem," and, "as shown above." Additionally, students found it helpful when proofs included small phrases or symbols that indicate parts of a proof, such as the directional arrows in a biconditional proof. Some of these indicators were subtle. In one proof with four parts, the original proof was written in three paragraphs. Students indicated that this was difficult to follow because they struggled to find which parts of the theorem corresponded to the paragraphs of the proofs. Students preferred a modified version of the proof where each of the four parts of the theorem was proved in a separate paragraph. To quote one of the students, "Trying to equate lines [of the theorem] and paragraphs [of the proof] to [each] other, given I say they split the group into two parts, you can add two paragraphs that very clearly equate to what part."

Possible Implications

With one exception, the student participants felt that the instructor-modified proofs were easier to understand than the original proofs. This implies that the modifications that instructors make to textbook proofs in the classroom are an important part of helping the students succeed in their mathematics courses. Some changes, such as connecting mathematical ideas with colloquial English explanations, seem to help students of all initial understandings to better understand the logical structures of the proofs. Separating a proof into logical paragraphs that match the theorem also seem to universally help the students.

Overall, student participants had a wide variety of opinions on what helped them understand the proofs in this study. Since different students believed that different modifications helped them better understand, it is important for instructors to have a wide variety of tools available to improve or accompany proofs. Where one student might benefit from an example, another might need to see a diagram. However, including all of these tools into every proof will make the proofs too long, which is overwhelming for some students and has the risk of losing the interest of students looking for a challenge in understanding the proofs. Therefore, it is important to find some middle ground. As brokers, instructors can better explain the proofs to peripheral members by adapting the proofs to their students.

It is also important to notice the differences in perspectives between the students who had substantial previous experience with proofs and proof-writing and the students who were more novices to proofs. The student participants who had low initial understanding of the topics in real analysis were also the students who had had the fewest proof-based mathematics courses. These students tended to prefer more examples, diagrams, and longer explanations. This is similar to the introductory mathematics courses that rely on examples and practice instead of proof. Student participants that had more experience with proofs tended to prefer proofs that were closer to a formal proof that a mathematician would use. In practice, the target audience for the proofs in this study would be somewhere in the middle because they would be part way through the course. They would have some exposure to the ideas of analysis but would be seeing these theorems and proofs for the first time.

As peripheral members of the community of mathematicians, students cannot be expected to fully understand proofs as they appear in research. Overall, the results of our study seem to suggest that if one goal of upper-level undergraduate mathematics courses is to recruit and maintain more individuals as members of the community of mathematicians, then it is important for mathematics instructors to include features in their classroom proofs such as colloquial explanations, transitional statements, and diagrams, even if these features would not traditionally show up in their research publications. Introducing such features to novices will enable more students to have access to the cultural practices of mathematicians. As students become more experienced members of the community of mathematicians, their exposure to the common norms and practices increases, these modifications become less important, and instructors can model proofs that are more like the proofs in mathematics research.

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Performing (with) Mathematics through Drawing in Mathematics Education Research

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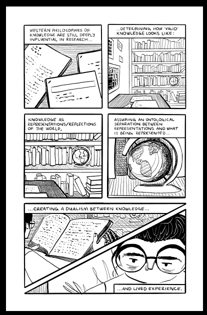
Drawing with/in mathematics education research tends to be understood through representationalist approaches.

Representationalism—the belief that epistemology is separate from ontology (Barad, 2003)—is strongly tied to colonizing worldviews that rely on 'othering,' on hierarchical notions of knowledge and being, and on the erasure of the 'other' (Bell, 2013; Gutiérrez, 2017; Martin, 2017; Million, 2015; Minh-ha, 1988; Santos, 2007; Tuck & Yang, 2014). The lack of philosophical questionings of the representationalist forces guiding our field favors the reification of these worldviews as well as judgmental practices.

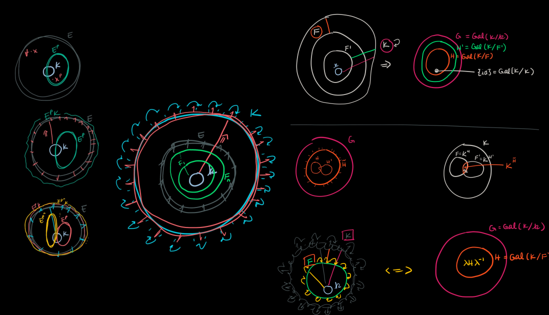


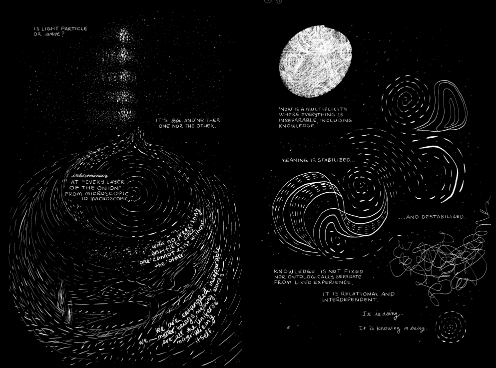
This poster shares an alternative understanding of drawing that can help educators move away from judgmental practices that are reinforced through representationalism and can help embrace the liveliness and uncertainty of the world, attuning to other possibilities, continuities, and complexities.

Motivation

Alternative Understanding of Drawing

1. Rather than a projection of a pre-conceived mental image onto paper or a mediating tool used towards a pre-established goal, the act of drawing is **indeterminate**—with no pre-established end (de Freitas & Sinclair, 2013; Ingold, 2013).
2. **Mathematical knowledge emerges inseparably** from the localized phenomenon of drawing and from the human-more-than-human-bodies-drawing-mathematics-emotions-discourse amalgamations and, thus, does not exist independently from a drawing encounter. Rather, it is **continuously transformed and (re)configured**.
3. Drawing can be perceived as a **dynamic performance** open to not only knowing, but also **expressing reciprocity and experiencing joy and connection through emerging patterns and mathematical concepts** (Gutiérrez, 2017).
4. A mathematical drawing has its own agency and is **never 'finished.'** That is, it is not frozen 'evidence' of a person's thought but is an active force that **continues to transform and to be transformed by the world** (de Freitas & Sinclair, 2012).





Theoretical F(orces)ramework

- **Barad's (2003) agential realism**: Offers important philosophical shifts that resonate with feminist new materialisms:
  a) matter is agentic;
  b) concepts are material;
  c) all entities, including knowledge, are inseparable in the becoming of the universe;
  d) the actualized and the virtual are indeterminate (Barad, 2003; de Freitas, 2017)
- **Gutiérrez's (2017) mathematx**: Offers a dynamic ontology of mathematics:
  - mathematics as performance and attuned to relationality and uncertainty—consistent with agential realism
  - mathematx as "a way of seeking, acknowledging, and creating patterns for the purpose of solving problems (e.g., survival) and experiencing joy" (p. 12), as well as a way of expressing one's sense of beauty and of connecting with others—human and more-than-human—through interconnectedness, kindness, and reciprocity.

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RUTGERS THE STATE UNIVERSITY OF NEW JERSEY

VANDERBILT

COURAGE

challenging, operationalizing, and understanding racialized and gendered events in undergraduate mathematics

A Survey on Students' Perceptions of Racialized and Gendered Classroom Events in Calculus and Precalculus

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Introduction

Undergraduate calculus instruction has an equity problem. Calculus courses have frequently "weeded out" or otherwise discouraged Black students, Latin* students, and white women students from pursuing STEM degrees (e.g. Ellis et al., 2016; Leyva, 2016), and these groups of students have reported having marginalizing experiences (McGee, 2016). Larsen et al. (2017) has called for more research into how calculus instruction can cause inequities, a call our poster will address.

Prior work

In our prior work, we conducted small qualitative studies (student journaling, individual interviews, and focus groups) to investigate Black students, Latin* students, and white women's (heretofore referred to as students from historically marginalized groups) perceptions of marginalizing events in the calculus classroom. The primary output of our research is theoretical accounts of how commonplace events in calculus classrooms can be especially marginalizing to underrepresented student groups (Leyva et al., 2021) and the impact that this has on students' affect and cognition (Battey et al., 2022). However, from this work, we generated two more general hypotheses:

(H1) There are commonplace events in college calculus classrooms, such asking students to drop down a class or ignoring a student contribution, that are discouraging to students from historically marginalized groups.
(H2) Students from historically marginalized groups are more likely to see these events as discouraging *and* gendered/racialized than white men peers.

Methods

Data Collection

- **Student journaling** of instructional events perceived as marginalizing, turned into 9 stimulus events
- Individual interviews around 4-5 of the stimulus events
- Follow-up interviews as member checks for theme analysis
- 9 group interviews with students of same race or same gender focused on 5 stimulus events

- **National Survey - 4-year state universities with at least 10% African American and/or 10% Latin* students**

We tested our hypothesis with a survey. In Fall 2019 and Fall 2021 (the gap was due to COVID), we solicited participation from students at 108 state colleges and universities with similar demographics to the state university where the qualitative study occurred. To date, 800 students have responded. Students were first asked demographic information, including to identify their race and gender. The student then read three instructional events that our prior work identified as marginalizing (Leyva et al., 2021). Students were then asked (i) if they witnessed this type of event in their calculus course, (ii) if the event would make them feel discouraged, (iii) if they felt the professor had acted appropriately, and (iv) if the event would make them consider dropping the course.

Place & Participants

Context

First Study - Qualitative

- Large, public research university
- Diverse student population (46% white, 26% Asian, 14% Latin, 9% Black, 5% multiracial or some other race. 52% female and 48% male)

Second Study - Quantitative

- **National Survey - Sent to the largest state universities with at least 10% Black and/or 10% Latin***

Theoretical Framing

We extend Acker (1990) and Moore's (2008) constructs of mechanisms and emotional and cognitive labor to show how organizational logics lead to inequitable practices. Therefore, the results extend our understanding of how color-evasive and gender-neutral ideologies intersect with instructional practices to discourage students from historically marginalized groups. This in turn contributes to the function of P-16 mathematics courses as gatekeepers to more advanced coursework and STEM careers.

Course Drop Event

The professor asked the class to work on a problem that required multiple steps. After giving some time for the class to solve the problem, the professor said, "If you do not know how to do these steps quickly, you might want to consider dropping down to a lower class or consider not taking Calculus 2".

Ignored Student Event

The professor was going through a problem on the board, and I was sitting towards the front of the classroom. In the middle of the problem, I raised my hand to ask for clarification on a portion of the problem. After two minutes with my hand up, I gave up on asking the question having been completely ignored. This also occurred 6 minutes prior, when I tried to ask a question.

PRELIMINARY FINDINGS

Table 1 supports H1. For each of the six intersections underrepresented in STEM (Black men, Latin* men, and each women group), at least 40% of the participants found this event "extremely discouraging". Table 1 also provides suggestive evidence for H2. Each of the underrepresented intersections found the event more discouraging than White men (although we caution that our analysis is descriptive and exploratory). Table 2 demonstrates this event could have an impact on retention.

"When being discouraged in the classroom, it then causes you to think about whether or not it has to do with the kind of person you are. This specific person must have felt as though they were specifically targeted and then became extremely discouraged by what their professor said."
- Latina student

	White	Black	Latin*	Asian
Man[1]	35%	40%	45%	37%
Woman	51%	64%	63%	51%

Table 1. Percentage of participants who said they would find this event "extremely discouraging"

	White	Black	Latin*	Asian
Man	25%	23%	23%	23%
Woman	31%	31%	43%	35%

Table 2. Percentage of participants who "strongly agreed" that they would consider dropping the class after this event

1: Data in these tables are based on cisgender groups; we are still exploring data for gender expansive groups

Table 3 supports H1. For each of the six intersections underrepresented in STEM) Black men, Latin* men, and each women group), at least 35% of the participants found this event "extremely discouraging". Table 3 also provides suggestive evidence for H2. Each of the underrepresented intersections found the event more discouraging than White men by at least 12% (although again we caution that our analysis is descriptive and exploratory).

"The scenario it's self [sic] is bad to begin with and adding that the student who was being ignored was a person of color makes it worse. The student was already being singled out and if I was in a situation like that, I would immediately assume the reason behind it was either my gender or my skin color. The professor should have stopped and answered the question." - Latina student

	White	Black	Latin*	Asian
Man	23%	43%	35%	31%
Woman	39%	52%	43%	40%

Table 3. Percentage of participants who said they would find this event "extremely discouraging"

	White	Black	Latin*	Asian
Man	23%	43%	35%	31%
Woman	39%	52%	43%	40%

Table 4. Percentage of participants who "strongly agreed" that they would consider dropping the class after this event

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KENT STATE UNIVERSITY | College of EHHS

Student Perception of Automatic Feedback Type in Tertiary Pre-Calculus: A Pilot

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Background

- Web-based homework is now more common than pencil-paper homework with, for example, millions of students utilizing ALEKS (Sun et al., 2021).
- Students' perception of instructor feedback correlates with academic achievement (Núñez et al., 2015).
- Elaborated Feedback consists of hints, tips, written example, etc. (Shute, 2008)
- Elaborated Feedback is the most effective feedback type for the learner compared to Knowledge of Response (correct/incorrect) and Knowledge of Correct Response [correct/incorrect and provides the right answer] (Attali & van der Kleij, 2017; Trenholm et al., 2015).
- Research needs to focus on each format of Elaborated Feedback and attend to specific needs of mathematical content.

Framework

- Feedback Type/Taxonomy (Shute, 2008; Trenholm et al., 2015)
- Mathematical Tasks Framework (MTF) which elaborates on high- and low-level mathematical thinking (Stein et al., 2000)

Research Questions

1. How does tertiary pre-calculus students' perceptions of feedback with connections compare to students' perceptions of feedback without connections?
2. Does the type of feedback (and perception) influence their score on a post-test?

Methods: Pre-Post Design

16 tertiary pre-calc students randomly assigned to one of three feedback groups:

1) Knowledge of Correct Results (control)
2) Elaborated Feedback with connections
3) Elaborated Feedback without connections

Measures: Feedback Perception Inventory (FPI) & 3 question content post-test

Preliminary Results

FPI data is part of an ongoing validation Rasch process where a minimum 66 participants as needed

Non-parametric Kruskal-Wallis test did not detect statistically significant differences between groups on the :

- Piloted FPI (RQ1).
- Post-test (RQ2).

Future Research

- Due to the removal of the study topic from pre-calc, future research will use logarithms as the topic.
- A power analysis for IRT indicates a minimum sample of 66

Intervention Feedback Examples

A circle has a radius of 16 m. Find the length, s, of the arc intercepted by a central angle of 21 degrees.

$$s = r\theta$$

Where, $s = arc\ length, r = radius, and\ \ \theta = central\ angle\ (in\ radians)$

First, convert the angle measure of 21 degrees into radians.

$$\frac{21\ degrees}{1} * \frac{\pi}{180\ degrees} = \frac{21\pi}{180} = \frac{7\pi}{60} = \theta$$

So, $\theta = \frac{7\pi}{60}$

Since the radius is 16 m, we plug both the angle measure and radius into the formula.

$$s = (16)\left(\frac{7\pi}{60}\right)$$

$$s = \left(\frac{16}{1}\right)\left(\frac{7\pi}{60}\right) = \left(\frac{4}{1}\right)\left(\frac{7\pi}{15}\right) = \frac{28\pi}{15} \approx \mathbf{5.9\ m}$$

As defined in geometry, arc length is related to the central angle of a circle and the radius: $s = r\theta$. Where the central angle θ is defined in radians.

Recall, a complete rotation around a circle has a degree of 360 or 2π radians.

Therefore, to convert we can simply begin by setting the two equal:

$$360\ degrees = 2\pi\ radians$$

So, $1\ degree = \frac{2\pi\ radians}{360} = \frac{\pi\ radians}{180}$

Then, $21\ degrees = \frac{21\ degrees}{1} * \frac{\pi}{180\ degrees} = \frac{21\pi}{180} = \frac{7\pi}{60} = \theta$

Since the radius is 16 m, we plug both the angle measure and radius into the formula.

$$s = (16)\left(\frac{7\pi}{60}\right)$$

$$s = \left(\frac{16}{1}\right)\left(\frac{7\pi}{60}\right) = \left(\frac{4}{1}\right)\left(\frac{7\pi}{15}\right) = \frac{28\pi}{15} \approx \mathbf{5.9\ m}$$

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DEPARTMENT OF MATHEMATICS

Understanding Decentering: How Tutors Put Student Thinking at the Front and Center

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Introduction

Background: Existing research has shown undergraduate mathematics tutoring centers' impact on student success in undergraduate mathematics courses (Rickard & Mills, 2017; Cooper, 2010; Byerley et al., 2018). However, minimal research exists in examining the ways undergraduate tutors make instructional decisions in the moment when working with students. This study sought out to better understand how undergraduate mathematics tutors consider and make sense of student thinking.

Theoretical Framework: When an instructor is making sense of student thinking during instruction, they are stepping outside of their own mindset. This action can be classified as *decentering* (Piaget & Inhelder, 1967). Baş-Ader and Carlson (2021) presented a framework introducing five levels of instructor decentering *(Table 1)*. Mills et al. (2019) and Johns et al. (2021) extended Baş-Ader and Carlson's (2021) work to investigate undergraduate tutors' abilities to decenter when working with their students. In their studies, while Mills et al. (2019) and Johns et al. (2021) identified potential opportunities for tutors to decenter, the tutors seldomly probed student thinking to take advantage of these opportunities. This project aimed to further investigate the ways an undergraduate mathematics tutor exhibited decentering behaviors by answering the following research question:

Research Question: In what ways do tutors at a small-scale tutoring center demonstrate decentering in their extended interactions with students?

Methodology

Participants: Three undergraduate mathematics tutors (each with at least two years of experience tutoring) employed in a university mathematics department's virtual drop-in tutoring center

Data Collection:

- 3 recorded real time tutoring sessions over Zoom from each tutor
- The extended tutoring sessions ranged from 8 minutes to 47 minutes
    - The average tutoring session length was approximately 25 minutes, and the standard deviation of the set was 10.55 minutes

Data Analysis: The research team conducted qualitative analysis of the data by creating a codebook used to record instances of the tutors' decentering at high or low levels. The codes were created to depict specific decentering actions the tutors took in their interactions with a student. We grouped the codes into themes and drew connections between our coding and the behaviors outlined by Baş-Ader and Carlson (2021). Following the coding and grouping, the research team analyzed the decentering levels exhibited from each tutor and identified a progression, describing the typical behavior a tutor portrayed in their interactions with the students. The progression of each tutor will be presented in the findings.

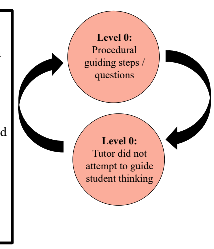
Table 1. Codebook representing specific tutor behaviors compared with the five levels of decentering originally presented by Baş-Ader and Carlson (2021)

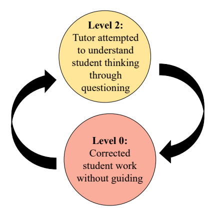
Levels	Description of Behaviors from Baş-Ader and Carlson (2021)	Codebook of Specific Tutor Behaviors During the Tutoring Sessions
Level 0: Shows no interest in student thinking	Does not pose question aimed at revealing student thinking Poses question that focuses on a procedure	Used the tutor's answer as the mathematical authority; corrected student's work without guiding; procedural guiding steps / questions; tutor had an assumption about student thinking; tutor dismissed student work; tutor did not attempt to guide student thinking
Level 1: Shows interest in the student's thinking, but makes no attempt to make sense of student thinking	Guides student toward their way of thinking Poses question to reveal student thinking, but does not attempt to understand student's thinking	Tutor initiated a check for understanding, but the student's thinking was unexplored
Level 2: Makes an effort to make sense of student thinking	Does not use the student's thinking to advance their current ways of thinking Poses question to reveal student thinking and attempts to adopt the student's perspective	Tutor attempted to understand student thinking through questioning; tutor checked for prior knowledge; verbal confirmation tutor understood what student was thinking
Level 3: Makes sense of the student's thinking and uses the thinking when interacting with the student	Leverages student thinking to advance understanding Poses question to gain insights into the student's way of thinking	Tutor was responsive based on a check for understanding (built upon expressed student thinking); tutor asked student guiding questions; verbal confirmation tutor understood what student was thinking
Level 4: Constructs an image of the student's thinking and adjusts their instructional actions	Follows up on student responses to extend student's current thinking Poses question to reveal and understand the student's thinking	Tutor adopted student thinking over their own; the students presented incorrect work, but the tutor did not dismiss their thinking

Results

Each of the three tutors exhibited various progressions in the levels of decentering during their interactions with students. We will present three case studies outlining the behaviors of the tutors and connect the themes to Baş-Ader and Carlson's (2021) decentering framework.

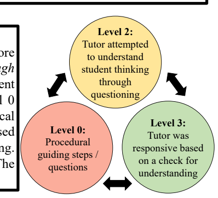
Ken:
In all three tutoring sessions, Ken began by explaining an overview of the concept to the student. After Ken's explanation, he procedurally prompted the student to share their thinking about the problem, asking questions such as "We need to identify how the relationship is going to look like between those variables, right? Can you think of anything that will get those variables to have a relationship from the problem?" This interaction was coded as a *procedural guiding step / question*, aligning with level 0 decentering (Baş-Ader and Carlson, 2021), as the questions Ken asked were directed toward the mathematics rather than inviting the student to share their thinking. Following Ken's questions, the student expressed their inability to identify a relationship between the variables, and Ken answered his own question. Since Ken made no further attempts to access or guide the student's thinking, the remainder of the conversation was also categorized as level 0. We find a similar behavioral progression from Ken throughout the three tutoring sessions – asking the student to share their thinking, but not acting to understand the student's thinking. The figure to the right represents Ken's typical progression working with students during tutoring sessions.





Dylan:
During each tutoring session, Dylan began by asking each student to share their thinking / method of solving the problem. In a particular session, Dylan asked the student "Okay, so what did you try to do before?" The student replied, "Just find the derivative because that's what it's asking". The student did not fully express their thinking, so Dylan asked "Yeah, but how did you approach it? Did you try to simplify it first or did you go straight into the derivative?" We coded Dylan's behavior as *tutor attempts to understand student thinking through questioning*, which aligns with level 2 decentering. The student went on to share their approach of solving the problem, and Dylan identified an incorrect step in the student's work. Dylan quickly pointed out the error and redirected the student's thinking before asking further guiding procedural questions. Although Dylan's behavior was coded at level 2 decentering during his initial questioning, we coded the rest of the interaction as level 0 decentering, as Dylan corrected the student's work without guiding them based on their expressed thinking. We found similar behaviors from Dylan throughout each of his tutoring sessions. The figure to the left represents Dylan's typical progression working with students during his tutoring sessions.

Lisa:
Lisa began her sessions by asking the student guiding questions to understand the student's thinking before explaining the content. This behavior was coded as *tutor attempts to understand student thinking through questioning*, or level 2 decentering. If the student was unable to answer the questions, Lisa provided the student additional information, without revealing the answer. The procedural steps Lisa exhibited was coded at level 0 decentering, as this action made no attempt to access or understand student thinking. Following the mathematical explanation, Lisa invited the student to share their thoughts. This responsive action to the student's expressed thinking was coded as level 3 decentering, as Lisa leverages the student knowledge to advance their understanding. Analogous to Ken and Dylan, we find similarities in Lisa's progression throughout her three tutoring sessions. The figure to the right represents Lisa's progression of decentering.



Conclusions & Future Steps

Our findings outline specific actions a tutor takes when decentering at each of the five levels originally presented by Baş-Ader and Carlson (2021). Each of the case studies outline additional specific actions, describing three tutors' consideration and responses to expressed student thinking. The distinct behaviors shed light on how tutors exhibit high-levels of decentering and gives insight into the ways a tutor is behaving at lower-levels of decentering. Professional development can be developed to support tutors performing decentering behaviors at a high-level, while targeting the low-level behaviors. For example, when we find a tutor dismissing student work, we can encourage them to listen and build upon the student's thinking, thereby prioritizing the student's thinking over the correct answer.

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Measuring Student, Instructor Attitudes and the Learning Environment

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DUE-2013392

Abstract

Attitudes matter in education, and it is crucial for educators to gain a better understanding of their students' attitudes. The Motivational Attitudes toward Statistics and Data Science Education Research (MASDER) team are creating a family of validated instruments to measure students' attitudes toward statistics or data science, the instructors' attitudes toward teaching statistics or data science, as well the learning environment through a National Science Foundation (NSF) grant titled, Developing Validated Instruments to Measure Student / Faculty Attitudes in Undergraduate Statistics and Data Science Education.

MASDER INSRUMENTS

	Student Instrument	Instructor Instrument	Environment Inventory
Statistics	S-SOMAS	I-SOMAS	E-SOMAS
Data Science	S-SOMADS	I-SOMADS	E-SOMADS

GOALS	STUDENT INSTRUMENT	INSTRUCTOR INSTRUMENT	ENVIRONMENT INVENTORY	HOW TO GET INVOLVED
1. Develop instruments that measure undergraduate students' attitudes towards statistics and data science 2. Develop instruments that measure the learning environment, instructor attitudes about teaching introductory statistics and data science, and pedagogical practices that may impact students' attitudes, engagement, and/or achievement 3. Develop and rigorously validate the instruments including expert reviews, pilot surveys, instrument revision, measures of validity and reliability, and transparent reports on this process 4. Create a sustainable infrastructure to facilitate data collection and dissemination	• Measures student attitudes toward statistics or data science • Administered pre and post semester • Can be administered longitudinally	• Measures instructor attitudes toward teaching statistics or data science • Administered perhaps once a year	• Measures institutional and course characteristics, learning environment, and enacted classroom behaviors (Pedagogy) • Instructor completes for each course	• Pilot a survey for us in your Introduction to Statistics courses and/or Introduction to Data Science courses • Serve as a Subject Matter Expert (SME) • Participate in Focus Groups • Pilot the Instructor and/or Environment Data Science Instruments by taking these surveys • Have a colleague in Psychology or Educational Psychology contact us • If you want to get involved, please complete the Google form on our website: http://sdsattitudes.com/

Process

MASDER Instrument → Develop Framework → Write & Revise Items → Pilot Study → Revisions → Operational Study & Use

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SFU

SIMON FRASER UNIVERSITY

Introductory Linear Algebra content coverage as per course descriptions

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MOTIVATION

Linear algebra is widely used throughout science, technology, and engineering fields and therefore introductory linear algebra courses offered in mathematics departments often end up serving students from a variety of client disciplines. In order to meet the many needs of the diverse population in linear algebra courses a number of broad recommendations have been made throughout the years as to what content to cover (Carlson, Johnson, Lay, & Porter, 1993; Diefenderfer, Hill, Axler, Neudauer, & Strong, 2015), all of which specifically highlight the importance of applications to client disciplines. While there is anecdotal evidence based on personal experiences and textbook use as to the content covered in introductory linear algebra, this poster reports on a systematic study documenting content coverage in these courses at the introductory level across the United States. Linear Algebra is not only an undeniably an integral part of undergraduate studies, its usefulness and enrollment are both on the rise. When comparing fall 2015 to fall 2010 courses course offerings, the percentage of colleges offering Linear Algebra increased five points to 25% (Conference Board of the mathematical sciences [CBMS], 2018).

RESEARCH QUESTIONS

1. **What content is covered in Introductory Linear Algebra courses in the United States?**
2. **How prevalent are the topics covered in Introductory Linear Algebra across courses?**
3. How, if at all, does the content covered vary across institutional type?

SETTING & BACKGROUND

Broadly speaking, our research is interested in understanding how Introductory Linear Algebra courses are serving the wide range of students enrolled. We wonder:

What role does Introductory Linear Algebra play in the undergraduate curriculum?

We have begun by investigating:

- What content is currently being covered in Introductory Linear Algebra?
- What client disciplines does Introductory Linear Algebra serve?

OVERVIEW OF METHODS AND DEMOGRAPHICS

Cluster Sample of 11 States across the United States

↓

252 Undergraduate Public Institutions with course descriptions for Introductory Linear Algebra Courses were Identified.

↓

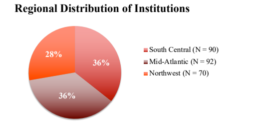
267 Introductory Linear Algebra Courses were Identified and Course Descriptions Collected through Direct Survey Methods (Stefanidis & Fitzgerald, 2014).

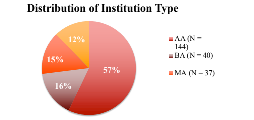
↓

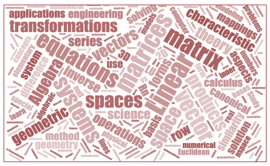
205 Unique Codes were Identified by Analyzing the Course Descriptions Using Open Coding (Glaser, 1978).

↓

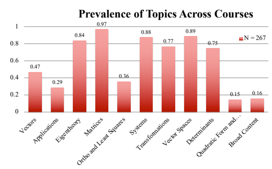
11 Overarching Categories were Identified by Grouping Codes by Topics



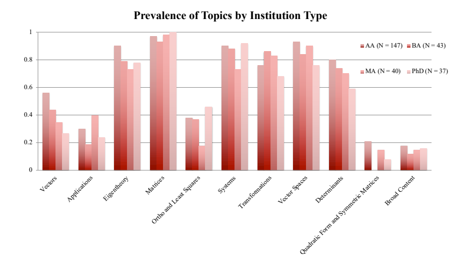




OVERARCHING RESULTS

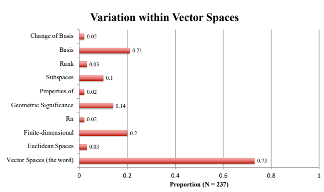


- At least 75% of the course descriptions mentioned each of the following topics: matrices, vector spaces, systems of equations, transformations, eigentheory, and determinants.
- Less than 30% of the course descriptions specifically mentioned applications of linear algebra as a topic covered.

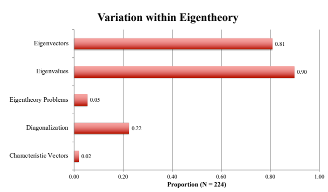


VARIATION WITHIN SELECTED TOPICS

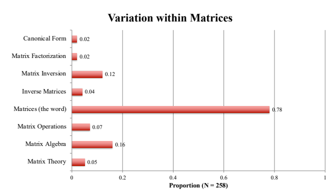
VECTOR SPACES



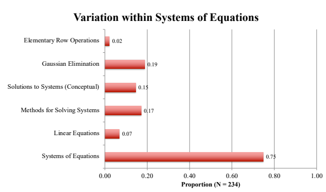
EIGENTHEORY



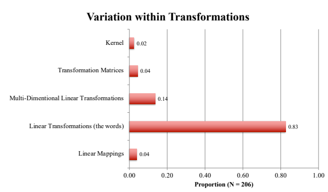
MATRICES



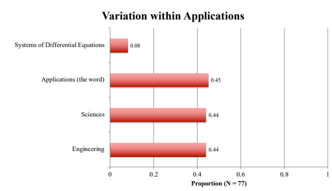
SYSTEMS OF EQUATIONS



TRANSFORMATIONS



APPLICATIONS



FINDINGS

- We recognize that the most frequently mentioned topics in these course descriptions are somewhat unsurprising (especially for those who teach the course).
- Topics that appeared more frequently within these course descriptions were consistent across institution type, while less frequently mentioned topics varied more significantly.
- Few course descriptions mention applications of Linear Algebra concepts. Those that did, frequently mention applications related to the sciences or engineering.
- One surprising finding was the extent to which Introductory Linear Algebra is being offered as a transfer course at Associate-granting Institutions.

NEXT STEPS

- Series of follow up surveys to explore how well the identified topics in these course descriptions align with what is happening in the classroom.
- Study to explore how well the content identified serves the needs of the client disciplines currently leveraging Introductory Linear Algebra.

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Strength- and Weakness-Based Faculty Peer Observation in Undergraduate STEM Instruction

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O. Theresa Ayangbola
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T	Time-Sensitive
R	Reciprocal
I	Inclusive
O	Operative
S	Strengths-Based

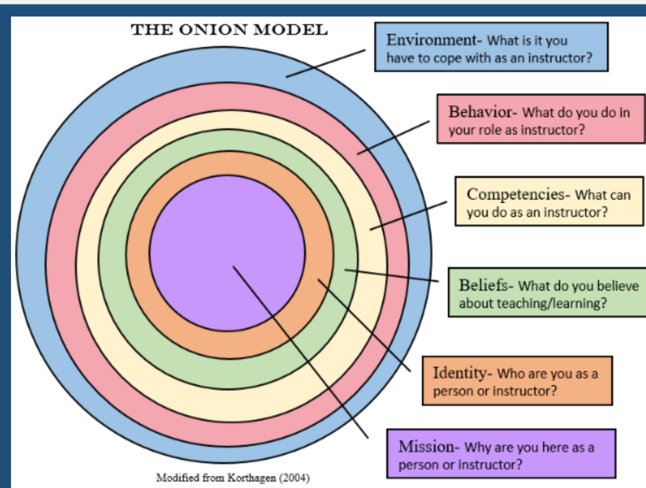
FA2020 — Whole- and TRIOS-Group Discussions Of Inclusive Pedagogy

SP2021 — TRIOS Observation and Debriefing

SU2021 — Dept. Change Social Contagion

What We Asked

In what ways do faculty reflect within TRIOS debrief sessions that utilize a weakness- vs. strength-based approach?



Weakness-Based Reflection

There were some opportunities for the students maybe to share, or maybe for them to have thought in advance about what it meant to them.
- Mathematics faculty member

Strength-Based Reflection

What I really appreciated was how you were encouraging authentic responses in the chat. So you had a very deliberate conversation.
-Biology faculty member

Environment

Well stop [the video] here for just a minute, because it seems to me like Tim is dominating the conversation.
-Mathematics faculty member

Beliefs

It's great because good teaching is generally inclusive teaching.
-Biology faculty member

Behavior

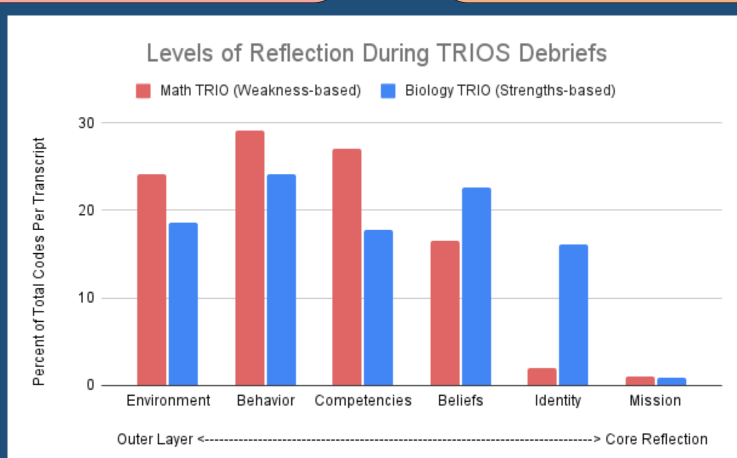
Then what I do is I just assume they're going to read that, and I will kind of skip down to what I think are the big points. - Mathematics faculty member

Identity

If I say "Don't make me use my mom voice," every person in that room knows exactly what I mean.
-Biology faculty member

Competencies

The goal I had for myself in this class is to do a better job of using student thinking to move us forward.... I got to the end of class that day and I went, "Boy, I don't think I did a good job with that."
-Mathematics faculty member



Mission

I care about you if you care about you. The first week is huge for this culture piece, where it's like "I will invest in you. I'm not going away. I am here for you, I'll go one-on-one sessions, we can do face-to-face, we can Zoom. I'm here, but if you're expecting me to drag you to where you want to go, you're in the wrong room." I'm in if you're in.
-Biology faculty member

tbr THE COLLEGE SYSTEM of TENNESSEE

Funded by Tennessee Board of Regents

Mathematics and Science Education Doctor of Philosophy Program

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Tennessee STEM Education Center

Documenting Diversity, Equity, and Inclusion Practices in Community College Algebra Instruction

Claire Boeck[1], Vilma Mesa[1], Mary Beisiegel[2], AI@CC 2.0 VMQI Research Group
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Abstract

We summarize the steps we have taken to revise an instrument that assesses the quality of instruction of algebra lessons taught at community colleges (Evaluating the Quality of Instruction in Postsecondary Mathematics, EQIPM, Mesa et al., 2020) so that it could attend to dimensions of diversity, equity, and inclusion (DEI).

We reviewed: (1) the literature specifically focused on mathematics classrooms, (2) on instruments that attend to DEI practices, and (3) EQIPM. This three-pronged review allowed us to code videos of community college mathematics and other post-secondary mathematics classrooms.

Our review identified three themes, *Participation*, *Social Justice*, and *Empowerment*, but these were difficult to identify in the available videos. We propose some questions for the field.

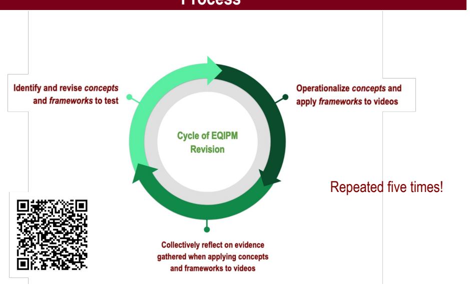
Rationale and Motivation

- Postsecondary mathematics classrooms are sites of inequity (Leyva et al., 2021).
- Current observation instruments assessing quality of instruction in post-secondary contexts:
  - attend to interactions between teachers, students, and content but not at diversity, equity, and inclusion elements (EQIPM, Mesa et al., 2020).
  - quantify participation by inferred student characteristics but do not attend to content (EQIP, Reinholtz & Shah, 2018).
- Diversity, equity, and inclusion postsecondary mathematics classroom require different approaches to describe instruction.
- Observation instruments could contribute to existing research because they can capture contexts and interactions not measured with other methods used to investigate equitable teaching (e.g., surveys, student outcomes).

Research Questions

- How can we capitalize on existing instruments so that they can be used to collect reliable evidence about the ways in which instruction attends to and accounts for diversity, equity, and inclusion in post-secondary mathematics classrooms?
- What type of evidence is needed to make warranted inferences about how diversity, equity, and inclusion in post-secondary mathematics classrooms occur?

Process



Repeated five times!

Identifying Concepts and Frameworks

Reviewed 49 pieces of literature (articles, reports, book chapters, dissertations) on:

- Equitable and/or inclusive teaching in mathematics
- Equitable and/or inclusive teaching in community colleges or commuter institutions
- Classroom observations as a research method, including five observation tools

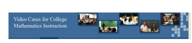
Examples of journals: *Educational Studies in Mathematics, Journal of Teacher Education, Journal for Research in Mathematics Education, Teachers College Record*

Themes from the Literature

Theme	Definition and Subthemes
Participation	+ *Participation*: students have equitable opportunities and invitations to participate in classroom dialogue. (Johnson et al., 2020; Reinholz & Shah, 2018) + *Accessibility*: resources, options, and flexibility are provided so all students can participate. (Moriarty, 2007; Novak & Rodriguez, 2018) + *Opportunity to Learn*: students have equitable access to resources and can engage in the learning process in the manner that helps them understand the content. (Goffney, 2010; Urbina-Lilback, 2018)
Social Justice	+ *Empowerment:* students see themselves as capable of making important contributions to the classroom. (Grisalfi et al., 2009; Hand, 2012) + *Critical Awareness:* teachers explicitly addressing and challenging stereotypes, oppressive structures, and societal inequities related to mathematics education (Bartell et al., 2017)
High Expectations	Affirming that all students, independently of their backgrounds or identities, are capable of succeeding. (Abell et al., 2018; Urbina, 2018)

Applying Frameworks from the Literature to Videos of Lessons

- Videos of lessons taught in algebra courses in community colleges collected as part of the AI@CC project
- Video Cases for Mathematics Instruction



Notes on Videos for Instructor 61122 L1 and L2 — Claire, Vilma, Mary

Equity/Rehumanizing Dimension	Examples of this in the videos (whether positive or negative)?
Gutiérrez: Participation/positioning	-gives students time to respond and addresses all of their comments/questions in an attentive manner (takes all comments seriously) -encourages students to use one another as resources during group work -uses a numbering system for groups so students interact with different peers(?) -affirming statements (e.g., "great way to explain that") -tells students there aren't difficult or easy problems if its they understand it or not (putting onus on students) -tells students they don't need him, they can do it (in an encouraging, praising tone after they give several correct answers) -several times, teacher tells students that they need to know why (positive) -students seek his approval for their work; the teacher, especially in the 2nd video approves their work, tells them what is right or wrong, rather than let them decide that on their own or with other students. seems counter to his intention to have them learn the why -teacher rarely positions students as knowers of the mathematics; teacher does not position students as people who can think about mathematics without his guidance; teacher comes across as ultimate authority of the mathematics, interrupts students' comments to tell students know what the mathematics is (even though they appear to know a lot already);
High expectations	-tells students he didn't teach them everything they need to know to complete worksheet, frames challenges as a positive -gives students the big and small answers in a response to a question about vertical and horizontal lines -inviting students to re-produce sound mathematical explanations. -valuing explanations (why) more than the steps and arithmetic (how)

Operationalizing Revised Concepts from Literature

Classroom Environment Category	Tentative Definition	Examples
Instructor Affect	Demonstrates care for students and enthusiasm for teaching mathematics.	+ Presents mathematics as an interesting topic + Demonstrates genuine concern for teaching and learning + Makes eye contact with students, smiles + Uses students' names + Has conversations about something besides mathematics ("how was your weekend")
Engaging Students in Learning *or* *Equitable Engagement*	Encourages and values students' contributions, holds high expectations and creates supports for students' participation.	+ Encourages and affirms contributions of all kinds (emotions, questions, answers) + Expects all students to participate in learning mathematics + Offers targeted assistance and supports to keep students engaged + Encourages students to develop mathematical ideas
Belonging	Validates and welcomes students who are minoritized within the classroom (women, students of color, students with disabilities).	+ Encourages minoritized students to participate + Positions minoritized students as capable + Affirms minoritized students as being able to learn and produce mathematics + Uses examples that do not assume White middle-class norms

- A short video clip or single lesson is insufficient to contextualize and interpret the evidence of practices supporting equity, diversity, and inclusion
- Having 2-3 video recorded lessons/teacher of various lengths (55mins to 3hrs) is insufficient to rule out that not seeing diversity, equity, and inclusive practices means that they are not enacted.

Questions as We Move Forward

- Limitations of observation or videos: What can we see about DEI with this method of data collection?
- How do we understand DEI and how do our positionalities influence that understanding?
- Loaded terms: How do we define and recognize enthusiasm, positive affect, etc.?

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To Learn More | To Participate

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Experiencing Disability in Undergraduate Mathematics

Kate Cruickshank and Miloš Savić

The University of Oklahoma

Introduction

Dolmage (2017) made an argument that historical studies of disability created a knowledge base devoid of the disabled perspective. The field of disability studies emerged out of "institutions in which disability as a negative concept, as a form of disqualification, was invented and applied and cemented" (Dolmage, 2017, p. 6). The medical model of disability emerged out of this ideology, seeking to pathologize disability, insinuating that the individual is innately flawed. The expectation, then, is that the individual should overcome or compensate for this difference (Lambert, 2019). Another model of disability is the social model, which separates the body (impairments) from disability, defining disability as "the political and social oppression of people with disabilities through lack of access to society" (Lambert, 2019, p. 279). Both models do not consider the lived experience of a disabled person but rather they create a binary between social worlds and impairment, essentially isolating the disabled individual from their own experience.

Disability Theory

We used both the complexly embodied (Siebers, 2008) and political/relational (Kafer, 2013) models that were employed by Lambert (2019). The complexly embodied model understands disability as "a social location complexly embodied," locating it both within the bodymind *and* our social worlds (Siebers, 2008, p.14). Kafer (2013) proposed the political/relational model of disability to assert that disability is socially negotiated, inseparable from politics. This model "views disability as a site of shifting definitions that is felt, particularly in relation to concepts of normalcy" (Lambert, 2019, p. 283). Both theoretical models allow us to investigate undergraduate mathematics education through the lens of disability research. Therefore, we asked the research question: How do disabled undergraduate students understand their identity as disabled in the context of postsecondary mathematics education?

Methods

Five students who self-identified as having a learning, intellectual, or developmental disability from a research university in the Mid-South were interviewed for roughly one hour. Eight interview questions were developed using Siebers' (2008) and Kafer's (2013) models of disability. Hypothesis coding (Saldaña, 2013) was used to analyze the interview data, and the predetermined codes were either complexly embodied, political, or relational.

Quote (Relational, political, and complexly embodied)	Coding and Explanation
"I don't think I ever finished a single test in any of those classes. Um especially at the time I didn't really think anything beyond 'well maybe I just hadn't studied enough.'" "The few exams I was able to review afterward I noticed that the questions that I did answer I largely got right. So that was also a contributing factor in the maybe this isn't such a studying problem but something else. So, I started to become more anxious because I wasn't sure I could deal with it. Going into these exams I would feel more anxious once I started to realize something didn't seem quite right." *(Alex)*	Complexly Embodied: Alex embodied his inability to finish his exams in time as a deficit within him, even though he felt prepared and was answering the questions he did get to, correctly. He embodied the unpredictability of finishing exams, resulting in anxious feelings which may have consequently affected his working memory (Lambert, 2019).
"I feel just as capable [as other students], if not sometimes more. But I feel like the fact that I'm disabled alone means that I still have to fight through a lot more garbage than they do, like.. fighting for accommodations alone is a struggle. I have the benefit of the fact that my special interest happens to be mathematics or physics related stuff … which means I learn it really quickly… I do great in math classes. But I still need accommodations… Having to fight for those accommodations is really annoying. So, like I feel like I'm just as capable as other people, but it doesn't make it any easier you know?" *(Bianca)*	Political: Bianca, restricted by institutional and legal barriers, is unable to receive official accommodations, putting her in a position where she is to negotiate accessibility issues with her professors on a case-to-case basis. Complexly Embodied: This accommodation negotiation process is emotionally and physically taxing, as there is a large power differential between students and professors.
The GPA before and after I started on my anxiety medicines were on the two ends of the spectrum. It was not the same person, and even before in second grade—before I had ADD meds—the grades were bad. Then all of a sudden, I can think without like.. going off into the stars. I don't want to call it like a disability- I mean it is, but like it allows me to think thoughts that other people just don't. *(Connor)*	Relational: Connor is using his Grade Point Average (GPA), a normative measure of "success", to describe his performance in mathematics. Complexly Embodied: This is an embodied description of a trait commonly associated with ADHD. Relational: The use of the word "allows" inferred to me that Connor valued thinking in ways that differ from others, or the "norm."
I have a hard time staying motivated… With math I actually work a bit harder to understand stuff because I want to learn, I want to understand it. That's the main problem—I want to know how to do stuff. Like I can't just be given the answers, I need to know how to do it. *(Tina)*	Complexly Embodied: This lack of motivation, based on context, seems to stem from a lack of expectations within the classroom to learn the material. Political: Tina seems to suggest that she has not been given the opportunity to learn, she has just been given the answers, depriving her the chance to develop her understanding of the material.

Discussion

Using my participants' narratives, I used the theory of complex embodiment and the political/relational model of disability to better understand how disability identities are constructed within undergraduate mathematics courses. My participants, who all claimed a disability identity, presume their own competence, expressing confidence in their ability to understand mathematical material. Simultaneously, they called attention to certain barriers within postsecondary mathematics that seemed to hinder their ability to participate in mathematics. These included time constraints, legal obstacles for obtaining accommodations, and power dynamics with instructors that can surface commonly-held stigmas about disability. There are many ways to think about and know mathematics; these differences in thinking are not deficits, but they are representative of human diversity. Instead of attempting to 'fix' these individual differences, disability should be reframed and relocated "from the individual student to other foci areas as needing to be 'fixed'" (Tan et al., 2019). Academia not only excludes certain bodies through physically inaccessible buildings and campuses, but as evidenced in this paper, academia—specifically mathematics education—excludes certain minds through "mental, intellectual, social, and other forms of inaccessibility" (Dolmage, 2017, p. 9). Until all minds and bodies can access mathematics material—accepting the presence of significant difference—mathematics education will fail to fully include.

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Pre-service Teachers‘ Professional Development

Relating Abstract Algebra and School Algebra

Anna Dellori, Prof. Dr. Lena Wessel – Paderborn University

Background

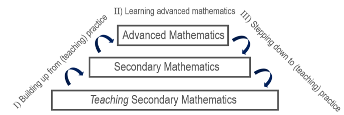
In the German teacher education system many students are dissatisfied with their mathematics content courses (Mischau & Blunck, 2006). Especially in abstract algebra, students experience a discontinuity between the course content and school mathematics (Ticknor, 2012).

Knowledge facets for teaching mathematics

- Content knowledge of the local & nonlocal mathematics
  - "The local mathematical neighborhood has been defined as those mathematical ideas that are relatively close to the content being taught.“ (Wasserman, 2018; p. 3)
- Demand for profession-specific knowledge
  - Mathematics from a higher standpoint: Nonlocal math reshaping the understanding of local content (Hefendehl-Hebeker, 2013; Wasserman, 2018).
  - Teacher pedagogical content knowledge positively predicts student learning (Charalambous et al., 2019).

Designing learning trajectories in intertwined ways

- Instruction model for relating teaching practices to local and nonlocal content. (Wasserman et al., 2019)



Instruction model (Wasserman et al., 2019; p.386)

- Relating registers and representations as a principle for developing understanding of and connecting mathematical content at school and at university level (Prediger & Wessel, 2011; Moreno-Arotzena et al., 2021).
- Comparing and contrasting as cognitive activities having positive effects on the learning process of content knowledge (Lipowsky et al., 2019).

“How can teaching-learning arrangements in an abstract algebra course be designed profession-specifically to promote relations of the different knowledge facets? Which learning pathways, obstacles and potentials can be identified?”

Methodology: Design research project with topic-specific and iterative design research cycles (Prediger, 2019).

Data base: 1st cycle of design experiments with pre-service primary teachers conducted in summer 2021 (4 pairs of two 90 min Zoom sessions).

Preliminary design principles

Sequencing the intertwined learning trajectory by *building-up* from teaching practices, *(re-)learning* academic mathematics and *stepping-down* to teaching practices.

Relating registers and representations from local and nonlocal mathematics in order to establish connections.

Contrasting and comparing local and nonlocal aspects in order to make connections explicit.

Insight into learning pathways: The case of Peter

Mathematical content learning pathway

Definition commutativity:
“If these elements are linked together, exchanging the elements is possible without changing the result.”
First definition associativity:
“If several elements are linked several times, the order of the elements can be changed without changing the result.”
Revised definition associativity:
“If several elements are combined several times and associativity is valid, brackets can be placed arbitrarily without changing the result.” (Group 2 – Part I)

Peter:
“How exactly do we define the associative law? That is actually, I think, only defined with brackets. An in the end, it is actually a certain order of the commutativity. So, you try to exchange something and calculate it at different times. So, it is/”
(Group 2 – Part III, Item 104)

Hypothetical Intertwined Learning Trajectory

I) Building-up

Students define the associative and commutative properties.

Students discover the properties in different local contexts.

II) (Re-)learning

Students explain the non-commutativity of the symmetry group D_3 using a Cayley table. Students give an example of a commutative algebraic structure..

Students use Cayley tables to identify similarities & differences between the symmetry group D_3 and a local algebraic structure.

Students classify various local and nonlocal algebraic structures according to their arithmetic properties.

III) Stepping-down

Students identify local difficulties with multiplication tables by reflecting on their nonlocal use of Cayley tables.

Students evaluate a local textbook excerpt about the arithmetic properties using nonlocal knowledge.

Students create an alternative local textbook excerpt about the arithmetic properties.

Profession-specific learning pathway

Commutative law of addition and multiplication
The addends of a sum or the factors of a product may be exchanged in any way. For two numbers a and b goes $a + b = b + a$ and $a \cdot b = b \cdot a$.
$2 + 3 + 4 = 3 + 4 + 2 \qquad 2 \cdot 3 \cdot 4 = 3 \cdot 4 \cdot 2$

Textbook excerpt (translated: Lambacher Schweizer 5)

Peter:
“Well. Look, these problems can be triggered by the book. (…) That reminds me a bit of the task we did last time, when we found out that it is not associative. I think that is exactly the point. Because in the example it’s not really commutativity that’s being addressed, but rather associativity, when you link three digits together./”
(Group 2 – Part III, Item 41)

Results & Outlook

- Content learning: Pre-service teachers mix up the arithmetic properties. This can be an obstacle for accomplishing profession-specific tasks.
- Profession-specific learning: Pre-service teachers show cognitive activity in relating to nonlocal content when working on authentic professional tasks.
- A redesign of the intertwined learning trajectory with more emphasis on the distinction between the two properties is currently in progress.

References:

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Assessing Mathematical Teaching Knowledge in a Master's Program

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Master of Mathematics Program

With the aim of helping inservice educators–at the secondary and even tertiary levels–continue to deepen their content knowledge, enhance their pedagogical practices, and continue to instill positive, productive dispositions, the faculty at Utah State University developed a master's program focused on mathematics education.

Students enrolled in the Master of Mathematics program focus on the continued development of their knowledge, skills and dispositions for mathematics teaching. They enroll in classes from the Department of Mathematics and Statistics and the College of Education and Human Services to help them continue to develop a "robust knowledge of mathematical and statistical content" and continue to build "foundations of pedagogical knowledge, effective and equitable mathematics teaching practices, and positive and productive dispositions" (AMTE, 2017, pp. 8-9).

Qualifying Exam in Mathematics Teaching

After their first year, students in the program demonstrate their mathematical teaching knowledge by engaging in a set of experiences as part of the program's qualifying exam.

The exam aims to provide synthesizing learning experiences that generate connections across mathematical areas, the classroom, and emerging issues in the field of mathematics education.

For the exam, students are given a set of five prompts and a week to compose responses to them. These prompts, detailed in the poster, range from mathematical analysis to abstract mathematical structures and from mathematical modeling to learning theories.

After the student has submitted their responses, they discuss what they have learned with the exam committee. During this discussion, students may clarify their responses, justify their claims, learn from their mistakes, and demonstrate their mathematical dispositions.

AMTE Standards

The *Standards for Preparing Teachers of Mathematics* (Association of Mathematics Teacher Educators (AMTE), 2017) list a set of standards and related indicators that beginning teachers of mathematics should achieve and possess in order to be effective educators. However, because teaching is a "complex enterprise," many aspects of teaching will need to be nurtured and developed over a period of time (AMTE, 2017, p. 7). Thus, the program and its qualifying exam aim to help inservice teachers to continue their development and assess their achievement of these standards.

In particular, the program helps students achieve the following indicators for the given standard:

Standard C.1. Mathematics Concepts, Practice, and Curriculum

Well-prepared beginning teachers of mathematics possess robust knowledge of mathematical and statistical concepts that underlie what they encounter in teaching. They engage in appropriate mathematical and statistical practices and support their students in doing the same. They can read, analyze, and discuss curriculum, assessment, and standards documents as well as students' mathematical productions.

C.1.1. Know Relevant Mathematical Content

C.1.2. Demonstrate Mathematical Practices and Processes

C.1.3. Exhibit Productive Mathematical Dispositions

Sample Exam Prompts

The 2021 Exam included 5 experiences. Below we share excerpts that illustrate the theme and flavor of each.

1. **Introduction to Formal Systems (Mathematics)**
   You could think of this as a game, but it is intended to be a small *formal system*. This formal system consists of the symbols $M, I,$ and U and strings constructed from those symbols in accord with the five axioms below. Take a moment familiarize yourself with these axioms (Hofstadter, 1979).
   Axiom 1.0: $MI \in \mathcal{S}$
   Axiom 1.1: $xI \in \mathcal{S} \Rightarrow xIU \in \mathcal{S}$
   Axiom 1.2: $Mx \in \mathcal{S} \Rightarrow Mxx \in \mathcal{S}$
   Axiom 1.3: $xIIIy \in \mathcal{S} \Rightarrow xUy \in \mathcal{S}$
   Axiom 1.4: $xUUy \in \mathcal{S} \Rightarrow xy \in \mathcal{S}$
   Prompt: Understand the $\{M, I, U\}$-system well enough to state and prove a metatheorem about it.
   Prompt: (Motive) Prove one of the following claims:
   (a) MU is a theorem; that is $MU \in \mathcal{S}$;
   (b) MU is not a theorem.
2. **Modeling in Secondary Mathematics**
   How does the total mass of all living ants compare to the total mass of all living humans?
   (a) Solve the problem. Write up your solution justifying your result, including illustrations. Work first without consulting any resources. If you decide resources are needed, keep track of what you look up along the way.
   (b) One of the standards for mathematical practices is "Model with mathematics." Explain what this means to you.
3. **Mathematical and Statistical Reasoning**
   Engage in the following:
   (a) Create and give a complete example of a mathematical proof by contradiction.
   (b) Create and give a complete example of a statistical test of significance.
   (c) Referring to specific aspects of your two examples, write a 3 - 5 page essay on the similarities and differences between the conceptual frameworks of "proof by contradiction" and "test of significance."
4. **Analysis**
   Recall the following two definitions:
   Definition 4.1: A sequence $\{a_n\}$ **converges** to L if for every $\epsilon > 0$, there exists a natural number N such that $n > N \Rightarrow |a_n - L| < \epsilon$.
   Definition 4.2: A sequence $\{a_n\}$ is **Cauchy** if for every $\epsilon > 0$, there exists a natural number N such that $m, n > N \Rightarrow |a_m - a_n| < \epsilon$.
   (a) Use these definitions directly to prove that if $\{b_n\}$ converges to B and $\{c_n\}$ is Cauchy, then $\{b_n + c_n\}$ is Cauchy.
   (b) Indicate and elaborate on at least five significant properties that distinguish between the set, $\mathbb{Q}$, of rational numbers and the set, $\mathbb{R}$, of real numbers.
5. **Learning and Learning Theories**
   One of the greatest struggles for a mathematics teacher is the mis-alignment between the preparation of students and the curriculum requirements for a course. What practices are needed to create, support, and sustain a culture of access and equity in the teaching and learning of mathematics? Provide research examples that are specific with examples of various barriers that students face, and possible solutions.

Assessing Responses

After students submit their written responses, a committee of 3-5 will score one or two of the responses and then lead the oral portion of the exam for those experiences.

Experience 1: This experience is designed to measure students ability to reason mathematically. The process which the student engaged in is the highlight of this experience, allowing the student to demonstrate their mathematical disposition and reasoning abilities, Indicators C.1.2 and C.1.3.

Experience 2: Responses are evaluated based on the extent to which the write-up shows evidence of these elements of a typical mathematical modeling process: (1) understand the problem situation; (2) make useful, appropriate assumptions to simplify complexity of the situation; (3) develop a mathematical model or procedure (decontextualize); (4) perform calculations correctly to compute a model solution; (5) interpret the solution and draw conclusions (contextualize); (6) validate the conclusions; (7) communicate the process with full explanations and justifications, and precise mathematical language and notation. This relates to AMTE Indicator C.1.2.

Experience 3: This experience seeks to determine if the student is able to provide appropriate examples of a proof by contradiction and a statistical test of significance and link these two concepts together in a natural way. These prompts help students demonstrate achievement toward Indicators C.1.1 and C.1.2. During the oral portion of the exam, students are questioned about their choices of examples and they are asked to explain their reasoning behind how they chose to relate the two concepts together. Throughout, attention is given to how the student communicates mathematically and how they feel toward their reasoning and mathematical skills, demonstrating their mathematical dispositions which helps demonstrate their achievement toward Indicator C.1.3.

Experience 4: The proof in part (a) is assessed based on its validity and coherence. One would typically show that the convergent sequence $\{b_n\}$ is Cauchy, and then that the sum of Cauchy sequences is also Cauchy. In the oral part we discuss to what degree the proof was determined without referring to outside resources. Responses to part (b) should include correct statements regarding the rational and real number sets, and address the completeness property of the reals. Candidates should show that they appreciate connections from advanced courses (analysis/topology) to secondary school topics (rational and real numbers and their properties), addressing AMTE Indicator C.1.3.

Experience 5: Although not explicitly mentioned in this poster, other AMTE standards aim to ensure educators have a sufficient level of pedagogical knowledge and interest toward improving their own teaching practice. This experience aims to measure the student's understanding of learning theories and best practices. The student is asked to use resources to respond to the given prompt. Their ability to use current resources to support their claims along with their ability to address all aspects of the prompts are assessed.

Conclusions

Students who have completed the program have gone on to teach in secondary schools or colleges, or pursue a doctoral degree leading to further engagement with education.

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Mathematics for Justice & Collegiate Mathematics Education Research

Shandy Hauk, Billy Jackson, & Jenq-Jong Tsay

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Background and Questions

Mathematics *for* and *of* Justice

Poster Purpose: offer and collect ideas for making sense of (and making sense with) justice as an essential component of research in undergraduate mathematics education (RUME).

Background: Consider two paradigms about justice in mathematics education (Larnell et al., 2016). One is based in Freire (1970/1993): mathematics is a tool to critically investigate, critique, and address social/societal issues. The other is based in Moses (1994): mathematical teaching and learning are the foundation for participation in, and transformation of, the status quo of majority society. The first perspective looks at the mathematics ***of*** social justice while the latter is concerned with mathematics ***for*** justice.

Challenging Questions:

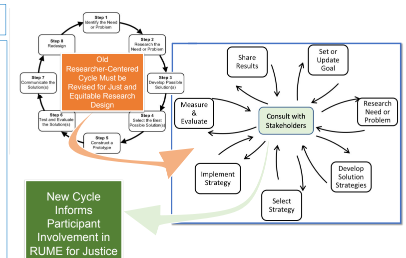
1. In what ways does the research at this conference rely on the paradigm of mathematics for justice? Give an example.
2. How is that example similar to/different from research about equitable instruction?
3. What questions and answers can support building a paradigm of "RUME for justice"?
4. As researchers, how do we attend to our own limitations (e.g., as outsiders looking in) to review and give feedback to those who focus on RUME for justice?

Framework

Necessary Conditions

TODOS/NCSM (2016) have offered three necessary conditions to establish just and equitable mathematical education:

1. **acknowledge** that an unjust social system exists.
2. take **action** to eliminate inequities and to establish effective policies, procedures, and practices that ensure just and equitable learning opportunities for all,
3. be **accountable** by measuring progress so changes are made and sustained.



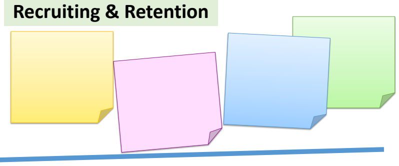
The definition of "RUME for justice" is still emerging. Certainly, it will demand responsiveness to those researched: the transformation of the status quo requires decentering the researcher as authority while acknowledging the various forms of expertise of the researcher and the researched. RUME for justice means civil and human rights are part of the work. Beyond the conduct of research, is the need to build skills for reporting and peer review of such work.

What More? What Next?

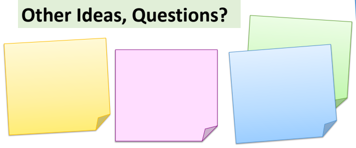
Responses, alternatives, and extensions to the challenging questions.

Study Design

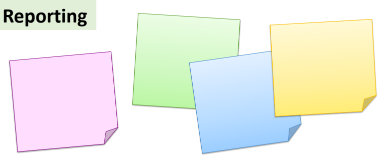
Recruiting & Retention



Other Ideas, Questions?



Reporting



In what ways does work in RUME acknowledge, act, and hold us accountable for increasing justice? What is the evidence of it, for you?

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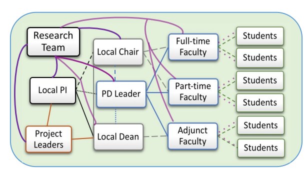
Moses, R.P. (1994). Remarks on the struggle for citizenship and math/science literacy. *Journal of Mathematical Behavior, 13*(1), 107-111.

TODOS: Mathematics for ALL and the National Council of Supervisors of Mathematics (2016). *Mathematics education through the lens of social justice: Acknowledgement, Actions, and Accountability.*

Research Examples

Recruiting

Example: Consequences on Channels of Communication with/among Stakeholders:



Reporting

Reporting on RUME for justice is like and unlike traditional forms. Some differences:

Background: RUME for justice is still emergent so the report must educate the reader about the distinctions of and between justice and equity

Theoretical Framework(s): of the research design, the researcher, **and** of the researched

Methods: includes statement of researcher position

Results: will challenge traditional views of what counts as mathematics, who decides, and what it means to teach, learn, or understand math

Discussion: includes future directions for accountability as well as implications of the work

Reviewing

Some of the skills needed for peer-review of reporting on RUME for justice:

1. ***Engage self-awareness.*** Acknowledge the unfamiliar and be purposeful in noticing nuances (e.g., to provide useful feedback to authors about injustice, inequity, or disrupting the status quo).
2. ***Expect to experience discomfort.*** Embrace the discomfort as an indication of an opportunity to learn even when it challenges life-long assumptions.
3. ***Accept the truths of the researched.*** The goal is to understand the experiences of others (rather than approve or validate them).
4. ***Expect and accept an absence of closure.*** Equity and justice are processes with milestones along the way (not destinations).

Comparing Professors' Intended and Enacted Potential Intellectual Needs for Infinite Series in Calculus II with those Presented in the Textbook

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Research Questions

What potential intellectual needs for infinite series and Taylor polynomials are provided in the textbook?

What potential intellectual needs for infinite series and Taylor polynomials did instructors intend to present to their students in class?

What potential intellectual needs for infinite series and Taylor polynomials are enacted by professors in class?

How do the intended potential intellectual needs for infinite series and Taylor polynomials compare to the enacted potential intellectual needs as conveyed by professors, and how do these compare to the potential intellectual needs presented in the textbook?

Introduction

Infinite series are notoriously challenging for Calculus II students perhaps in part because the content does not resolve a problematic situation for the students (Harel, 2013). Jones, Jeppson, and Corey (2019) analyzed multiple calculus textbooks to identify intellectual needs for infinite series that students could potentially adopt, but they did not investigate how instructors taught from the textbooks and what potential intellectual needs they presented to their students. This study combines a textbook analysis with interviews and observations of lectures of four professors to identify the intellectual needs that professors intend to present and then how they enacted those intentions in their lectures.

Literature Review

Harel (2013) defined intellectual need by saying: "If K is a piece of knowledge possessed by an individual or a community, then… there exists a problematic situation S out of which K arose… Such a problematic situation S, prior to the construction of K, is referred to as an individual's ***intellectual need.***" However, when a student experiences a perturbational state, they do not necessarily construct the intended knowledge. A student may remain in the perturbational state due to a lack of interest or motivation, so Harel (2008) characterized motivation as a manifestation of ***psychological need***, which is the initial interest in a problem that drives a student to pursue a solution, and ***affective need*** which is linked to social and cultural values. While both psychological and affective need "activate and boost" (Harel, 2013) or "stimulate and sustain students' mathematical activity…intellectual need has the potential to enhance the nature and quality of that activity" (Corey *et al*, in preperation). Jones, Jeppson, and Corey (2019) operationalized potential intellectual need to be "motivations contained in a written curriculum that students might potentially adopt as their own intellectual need for the content."

Methods

The data for this study were gathered in three ways: a textbook analysis, instructor interviews, and lecture observations. Four university mathematics instructors participated in semi-structured interviews which lasted about 30 minutes each. Then roughly 35 combined hours of their recorded lectures over infinite series were observed and analyzed.

Results

- The textbook does not provide as many details about the problematic situations as the professors. For example, in the opening of Chapter 10.6: Power Series, the term "represented" is the only description offered to communicate to the student that they are no longer approximating but finding exact values. Professor C on the other hand asked his students, "can we replace that 'approximately equal to' by 'equal to' if I replace 'n' with 'infinity'?" then proceeded to have a discussion about how they are creating exact representations.
- Unlike the textbook, Professor C presented infinite series before sequences so that a local potential intellectual need for sequences to determine the convergence of an infinite sum can be presented at the beginning of the lesson about sequences.

Important Result

The textbook is not a sufficient resource for providing potential intellectual needs for infinite series and Taylor polynomials. Unlike the textbook, professors presented potential intellectual needs through class discussions, through application problems, and by supplementing intellectual need with motivation.

Data

Potential Intellectual Need	Textbook	Professor A		Professor B		Professor C		Professor D
		Intended	Enacted	Intended	Enacted	Intended	Enacted	Intended
Taylor polynomials can be used to approximate functions that are hard to work with.	✓	✓	✓		✓	✓	✓	✓
Linear approximation is not a sufficient approximation in some cases.	✓				✓		✓	
Taylor polynomials can be used to approximate integrals of functions with no elementary antiderivative.	✓	✓	✓					✓
Taylor series provide exact representation of functions.	✓		✓		✓	✓	✓	✓
Taylor series can be used for integrating functions with no elementary antiderivative.	✓	✓	✓	✓	✓	✓	✓	✓
Potential intellectual need presented through an application problem	✓		✓	✓	✓		✓	✓

Results

- Professors pulled from outside resources for examples and applications of infinite series. Examples:
  - Professor B and C presented Zeno's paradox.
  - Professor A presented the problem of computing an infinite sum through an application problem about calculating an investment for a scholarship fund.
  - Professor C presented a position function for at atom particle to motivate an extension of linear approximation.
  - These application problems supplemented potential intellectual need with affective and psychological need.
- The professors emphasized a necessity for infinite series to represent functions and to integrate functions with no elementary antiderivative, but the textbook states that "a main goal of this chapter is to develop techniques for determining whether a series converges or diverges" (pg. 526), which establishes the emphasis on learning the different convergence tests.
- Could presenting infinite series as a method of integration help the students understand how infinite series fit within the rest of the concepts in their course?

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Fostering Student-Centered Online Tutoring Practices with OPTIMUM Interactions

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Introduction

Access to quality tutoring is an issue of equity.

Student use of tutoring resources has been correlated with an increase in
- final course grades (Rickard & Mills, 2018)
- improvements in persistence, retention, and degree completion (Rheinheimer et al., 2010).

Johns and Mills (2021) and Turrentine and MacDonald (2006) identified two primary aspects of online tutoring needed for successful student outcomes:

1. the types of technology tutors have access to
2. communication in the online environment.

Building on this work and on pilot data collected during the spring 2020 semester, we identified practices that are essential to creating student-centered online tutoring sessions and that are distinctly different from in-person tutoring sessions.

The **Online Practices for Tutoring In Mathematics Using Meaningful (OPTIMUM) Interactions** project seeks to understand how Lepper and Woolverton's (2002) Intelligent, Nurturant, Socratic, Progressive, Indirect, Reflective, and Encouraging (INSPIRE) tutoring model can be translated to an online environment.

Methodology

- Data collected during Spring 2020 semester.
- Tutors were trained for in-person tutoring.
- New tutors enrolled in a pedagogy course based on the INSPIRE model (Lepper & Woolverton, 2002).

- Tutoring suddenly moved online in March 2020.
- Video and audio were collected from 29 online tutoring sessions from 7 different tutors.

- Data were analyzed using deductive thematic analysis (Braun & Clarke, 2006, 2012).
- Initial coding based on the INSPIRE model.

- We present some of the themes that were found in the data, and we identify how the skills outlined in the INSPIRE model can translate to the online environment.

The INSPIRE Model for Tutoring

Lepper and Woolverton (2002), in a comparison of the practices of expert tutors, identified characteristics of and techniques used by those tutors during tutoring sessions. The behaviors identified in these expert tutors include cognitive, metacognitive, and affective pedagogical strategies and considerations which focus not only on the academic content under consideration but also emphasize the importance of study skills and student mindsets.

The INSPIRE Model	
Intelligent	**Knowledge of subject matter and pedagogical strategies**
Nurturant	**Developing a personal rapport with the tutee**
Socratic	**Asking questions to foster dialogue rather than telling**
Progressive	**Purposeful selection of problems, systematic feedback, and predictable routines**
Indirect	**Providing appropriate types and amounts of feedback**
Reflective	**Asking the tutee to explain reasoning and generalize**
Encouraging	**Promoting confidence, challenge, curiosity, and control of the learning**

OPTIMUM Interactions

With the rapid shift to providing online tutoring, we observed that many in-person tutoring strategies were not translating to the online environment. We present four key observations and how they might be addressed.

Technology Considerations

The most effective tutors in our study showed comfort and flexibility working with the technology they had available to them and were able to work around the technology limitations of the tutees they were helping. We consider these online pedagogical strategies as instances of the Intelligent aspect of the INSPIRE model.

Resources used by our most effective tutors included

- Shared virtual whiteboard software
- Online graphing tools

Tutors who had access to tablets or touchscreen computers were able to communicate mathematics more easily.

Building Relationships

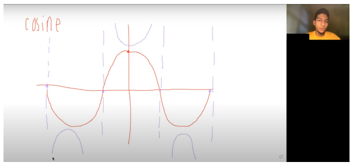
A tutor's ability to build a relationship with a tutee is closely related to the Nurturant and Encouraging aspects of the INSPIRE model. We identified three important points for tutors to consider:

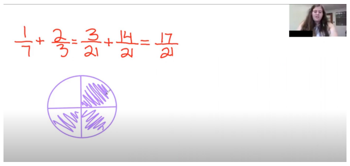
- The student's motivation for attending tutoring
- How to respond to failure/success
- How to work with challenging students

Video Case Studies

Interactions were primarily of the form: ask the tutor for help on a problem, tutor solves the problem, and move on to the next problem. There was limited dialogue between the tutor and student.

As an aid to tutor training, we (with the help of our tutors) created six fictional video case studies demonstrating various desirable and undesirable tutoring behaviors. These case studies were designed to generate discussion among tutors during training.





Scheduling

We identified scheduling considerations that seemed to affect both tutors' and tutees' online tutoring experiences.

- Tutor-to-student ratio – schedule more tutors during peak times
- "Zoom fatigue" – allow for 10-minute breaks every hour
- Tutors' level of experience – have at least one experienced tutor scheduled to assist novice tutors

Conclusion

These results were generated from pilot data collected as part of a larger, ongoing study. These results, among others, informed the production of a set of training materials and resources for both tutors and tutor center leaders. We presented this preliminary set of materials to tutor center leaders in a "Train the Trainer" workshop in August 2021. These materials were also used to train online tutors for the current iteration of this study. Data collection is ongoing and will be used to revise these materials. We look forward to hosting another "Train the Trainer" workshop in Summer 2022.

The full collection of resources generated from our work to date can be found here: https://drive.google.com/drive/folders/1-yucC2Xf5lDrP8FrQcEwg5bHJNKJ4rQK?usp=sharing

Acknowledgements

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University of Nebraska Omaha

Algebra Instruction at Community Colleges: Validating Measures of Quality Instruction

Dexter Lim, Bismark Akoto, Irene Duranczyk, AI@CC 2.0 VMQI Research Group
University of Minnesota

Abstract

This poster presents the research design and data collection strategies for a federally funded project to investigate Mathematical Knowledge for Teaching Community College Algebra (MKT-CCA). This poster focuses on the design of our overall project, a blueprint for MKT-CCA highlighting some problematic issues we are facing while creating the blueprint, and some challenges with writing assessment items in preparation for collecting pilot data.

Project Description

The Algebra Instruction at Community Colleges (AI@CC 2.0: VMQI) project (Mesa et al., 2020-2023) seeks to develop and validate an instrument to assess instructors' Mathematical Knowledge for Teaching Community College Algebra (MKT-CCA) and revising a video coding protocol.

The MKT-CCA instrument sets out to measure the mathematical knowledge for teaching college algebra at community colleges (CC) using multiple-choice or testlet items focusing on the following college algebra **topics**:

+ Linear equations and functions
+ Exponential equations and functions
+ Rational equations and functions

We hypothesize that MKT-CCA will also be organized along the following two ***Tasks of Teaching***:

+ Choosing problems
+ Understanding student work

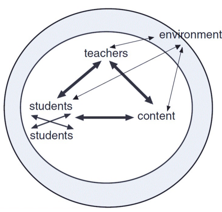


Figure 1. Instructional triangle (Cohen, Raudenbush, & Ball, 2003)

Research Questions for AI@CC 2.0 VMQI

RQ1: What are the dimensions of mathematical knowledge for teaching college algebra at community colleges?

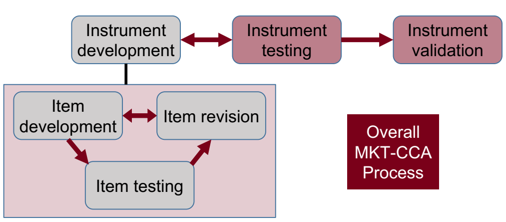
RQ2: What is the relationship between the underlying dimensions of high quality of algebra instruction at community colleges and aspects of diversity, equity, and inclusion?

RQ3: What is the connection between mathematical knowledge for teaching community college algebra and the quality of instruction in the context of CC algebra?

Method

Sources

+ Video recordings of CC algebra instruction from our previous project (AI@CC 1.0)
+ Various college algebra textbooks used in CCs
+ Prior teaching experiences
+ Other MKT instruments
+ Students' written work



Timeline

August 2020 — Collection of ideas and resources
- Looked into textbook, videos, and student work
- Started with Ball, Thames and Phelps's MKT (2008)
- Reviewed other existing instruments: MET, LMT, COACTIVE, TEDS-M, DTAMS, etc.

September 2020 — MKT-CCA development
- Training on item generation by team's psychometrician
- Protocol for drafting items
- Internal review of items in small groups

EQIPM Development
- Reviewed literature on equity

October 2020 — MKT-CCA blueprint development
- Decided on topics/content for items
- Decided on tasks of teaching

November 2020 — Recruitment of MKT-CCA participants
- At conferences
- Via emails from a list of college instructors

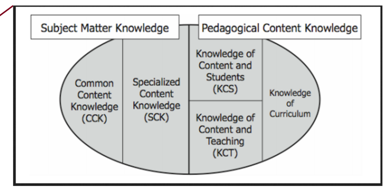
April 2021 — Received feedback of items from advisory board and faculty research associates

June 2021 — Item writing session/camp (Kimani et al., 2022)
- 13 instructors and mathematics educators outside of our research group
- 11 research group members

July 2021 — Cognitive interview (Akoto et al., 2022)
- Interviewed 12 instructors with 36 items

November 2021 — Gathered asynchronous feedback
- 15 instructors with 11 items

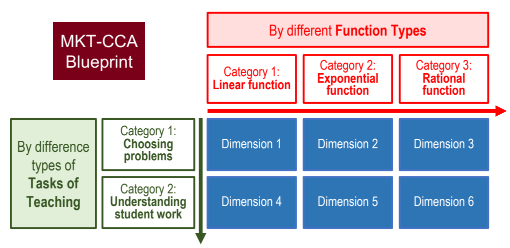
March 2022 — Conduct MKT-CCA Pilot
- ~120 participants with 60 items



Ball et al. (2008)

Choosing Problems | Understanding Student Work

Mesa et al. (2020-2023)



To Learn More

To Participate

Challenges

For MKT-CCA

+ Writing items with a range in difficulty
+ Describing student thinking from prior experience
+ Writing MKT-CCA items about common student errors
+ Creating options that can be mathematically evaluated
+ Ensuring the included topics are part of the college algebra curriculum
+ Discarding some unusable items after revision

For EQIPM

+ Identifying equitable and inclusive practices in our videos
+ Connecting our videos to themes from literature about equitable and inclusive practices

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Investigating a Student's Relative Size Reasoning

Kayla Lock
Under the advisement of Dr. Marilyn Carlson

Motivation and Research Question

- Researchers have documented that students in university mathematics courses have difficulties comparing two *quantities* in terms of their *relative size* (e.g. Byerley, 2019; Tallman, 2015).
- Lobato and colleagues have reported that many students do not view the idea of proportionality, slope, rate of change, and average rate of change as being connected (Lobato, 2006; Lobato & Siebert, 2002; Lobato & Thanheiser, 2002).
- Engaging in relative size reasoning may be beneficial for understanding several precalculus topics including constant rate of change, rational functions, trigonometry, and exponential functions (Lock, 2021)
- Researchers have discussed how productive meanings for ideas in calculus such as the difference quotient (Byerley & Thompson, 2014), derivative (Byerley et al., 2012; Zandieh, 2000), and the fundamental theorem of calculus (Thompson,1994) involve comparing two quantities multiplicatively.
- One's ability to engage in relative size reasoning is dependent on their interrelated meanings for measurement, fractions, multiplication, division, ratio, proportionality, and magnitude which involve *quantitative reasoning* (Thompson, 2011) and *covariational reasoning* (Carlson et al., 2002; Thompson and Carlson, 2017) (Lock, 2021).

Research Question:

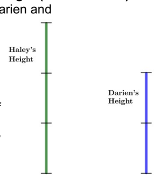
What mental operations do students engage in when approaching novel precalculus tasks that are designed to elicit relative size reasoning?

Relative Size Reasoning

- One who has a productive meaning for the *relative size* of two quantities imagines measuring one quantity's magnitude in units of the other quantity's magnitude, given that the two quantities are of the same unit of measurement (Lock, 2021). If the quantities are not of the same type of unit (e.g., speed which is a relationship between distance and time) it involves thinking of the measure of one quantity's value in multiples of the measure of the other quantity's value (Lock, 2021).
- Relative size reasoning is a *way of thinking* (Thompson, Carlson, Byerley, & Hatfield, 2014) that a person has developed when thinking about comparing two quantities multiplicatively for which they can utilize when conceptualizing mathematical topics (Lock, 2021).
- Relative size reasoning involves imagining the multiplicative comparison of two quantities both in a static context and in dynamic contexts.

Example: Comparing two people's height (static context)

- The students must conceptualize Darien and Haley's height as quantities.
- Measuring Darien's height in units of Haley's height (imagine Haley's height as measurement unit): Darien is $\frac{2}{3}$rd the height of Haley.
- Measuring Haley's height in units of Darien's height (imagine Darien's height as measurement unit): Haley is $\frac{3}{2}$ the height of Darien.



Theoretical Perspective

- Radical constructivism (Glasersfeld, 1995) is the lens used in this study which takes on the perspective that every individual has their own reality, knowledge is constructed based on previous personal experiences, and knowledge lies in the mind of the individual.
- ***Quantitative reasoning*** is the analysis of a situation into a network of quantities and quantitative relationships (Thompson, 1994, 2012).
- A ***quantity*** is a person's conception of a quality of something such that they envision that quality admitting some measurement process. Part of conceiving of a quantity is conceiving of an appropriate unit by which one might measure the conceptualized measurable quality (Thompson, 1994, 2012).
- One constructs a *quantitative relationship* in one's mind when they have conceptualized two quantities and a quantitative operation which together make a third quantity (Thompson, 1990).
- ***Covariational reasoning*** refers to the mental actions involved in coordinating the value of two varying quantities and how the values vary in relation to each other (Carlson Jacobs, Coe, Larson, & Hsu, 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017)
- This work draws on theories of students' understanding in the moment and communicating mathematics:
  - Students explained their meanings in the moment for each response they had to the questions. Each question posed was used as a point to characterize the students' meanings (Thompson, 2013).

Methodology

- I conducted two 1.5hr long clinical interviews (Clements, 2000) where precalculus students from a large southwestern university worked on eight tasks from different topics (measurement, proportional relationships, rational functions, and concavity) designed to elicit students' ways of thinking about relative size.
- First task: a question that presents a static situation and does not require the students to think about varying quantities.
- Second task: presents a dynamic situation and was designed to assess the student's ability to think about a multiplicative comparison of two people's, Joe's and Trevor's, ages as the number of years since Trevor was born varies.
- I attempted to characterize one student's, Morgan (pseudonym), thinking by determining what meanings these students had in the moment while working through the tasks.

Interview Question

Task 1: What does it mean to say that Rihanna is 67 and ½ inches tall?

Task 2: Suppose Joe, who is currently 46 years old, and Trevor want to compare their ages over time in years since Trevor was born in 1996.

What does $\frac{46}{25}$ represent in this problem context? What does $\frac{25}{46}$ represent in this problem context?

a) In the year 2000, describe how Trevor's age compares to Joe's age?
b) In the year 2015, describe how Joe's age compares to Trevor's age?
c) As time increases, how does the comparison of Trevor's age to Joe's age behave?
d) As time increases, how does the comparison of Joe's age to Trevor's age behave?

Results

TASK 1:

- The student read the problem aloud and conceptualized the task as needing to convert Rihanna's height from inches to feet. She wrote the equation $12\text{inches} = 1\text{foot}$, stated "there's 12 inches in 1 foot" and calculated $\frac{67.5}{12} = 5.625$. When responding to why she performed the calculation of $\frac{67.5}{12}$, she responded with, "so how many times 12 fits into 67.5. And since 12 inches is 1 foot, I guess like the 5.625 is how many times 12 goes into 67.5."
- When Morgan was asked how she knew Rihanna was 5.625 feet tall and not another measure (e.g., 7.2 feet tall) she stated, "If I use a ruler to measure Rihanna and a ruler is only 12 inches, I will use it once, twice until its five and then six is too much. So then from there 5 times and a little of the ruler I would know her height."

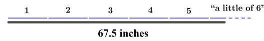
Excerpt 1

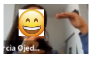
Interviewer: How would you describe the size of one inch in comparison to the size of a foot?

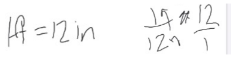
Morgan: Let's say like one inch, like one of them times 12 is one foot. So one inch 12 times is 12 inches or one foot so like one inch 12 times is one foot so then one inch 24 times is two feet, because 24 divided by 12 is 2.

Interviewer: Okay, so can you describe that relationship in the other direction? How would you explain it if you are comparing the size of a foot to the size of an inch?

Morgan: One …one divided by 12 so like $\frac{1}{12}$. Because there's 12 inches in one foot so 12 is how many times … yeah because its either 12 or $\frac{1}{12}$. So it's not how many how many times 12 feet fit into 1 inch. That's not right, its how many times its how many inches fit in one foot and how many …How many .. Oh gosh I am not sure.







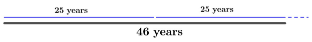
TASK 2:

- Morgan read the question aloud and after reading part (a), she was unaware of what $\frac{46}{25}$ and $\frac{25}{46}$ represented other than writing $\frac{46}{25} = 1.84$.

Excerpt 2

Interviewer: What does each number or symbol in the expression $\frac{46}{25}$ represent? What does 46 represent?

Morgan: 46 divided by 25 is 1.84. What does 1.84 mean? Um I do know it means how many times 25 fits into 46. So how many times ….so 25 fits into 46 1.84 times …so what does 25 mean? I don't know, it's just Joe over Trevor's age, but I don't know what this equation like is trying to tell me. That is Joe's age [46]. Okay so 25 fits into 46 1.84 times and 25 divided by 46, 46 fits into 25 0.5434 times. So what like what does that even mean?



- Parts B and C involved comparing Joe's and Trevor's heights for a specific year after Trevor was born. In part D and E asks about the relative size of the two boys' ages as the number of years since Trevor was born increases.

Excerpt 3

Morgan: You see it here we know 46 minus 25 is 21 so as time increases Trevor's age will always be 21 times …well I mean 21 years younger than Joe's age. So then for (e) as time increases, how does comparison of Joe's age to Trevor's age so and the other way around, so then Joe will always be 21 years older than Trevor.

Interviewer: Is there any other way, you can compare those two ages, as time increases?

Morgan: I guess you could like divide their ages and then. You find out exactly how many times older one is in the other how many times younger the other one is in the other.

Interviewer: Can you describe how that relationship will behave as time increases?

Morgan: Yeah, so Trevor's age to Joe's age. Umm … it'll always be greater than zero because Trevor's age will always be able to fit into Joe's age, but then Joe's age won't be able to fit into Trevor's age, because Joseph older, so it will always be like zero point something.

Conclusion

- Morgan's scheme for fractions entails counting how many of the unit of measure fit into the length in which she is trying to measure rather than thinking about fractions in terms of partitioning.
- The thinking revealed in the interview indicates that a student's ability to conceptualize a unit of measure involves the ability to partition a quantity into equal parts based on the reference length she chooses. This interview also suggests that the mental operations involved in conceptualizing a unit of measure is dependent on one's meaning for each symbol and their relationship in the equation $\frac{a}{b} = c$.
- I argue that in static situations, a student may not have to engage in covariational reasoning in order to think productively about the situation presented, however, in dynamic situations, a student's ability to engage in covariational reasoning influences her ability to engage in relative size reasoning.

Future Research

Future research will aim to investigate

- Whether a student's ability to engage in relative size reasoning is interrelated to one's ability to conceptualize a unit of measure.

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NOTE: More citations available upon request

Calculus 2 Functioning as a Gatekeeper from STEM

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Supported in part by the MSU Graduate School

PURPOSE & MOTIVATION

- Spearheaded by the MAA, the *Characteristics of Successful Programs in College Calculus* provided insight into Calculus I from across the US, and showed the importance of teaching on student achievement, persistence and attitudes (Bressoud et al., 2015).
- Calculus 2 is a natural entry point to consider persistence of STEM-intending students because almost all Calculus 2 students are STEM majors.
- Not all students have historically been equally represented in STEM fields, which has had dramatic economic impacts on individuals and on the country as a whole (PCAST, 2012).
- Perhaps more importantly, this inequity has deleterious impacts on STEM-intending students' aspirations and goals.
- This study looks at achievement and persistence of Calculus 2 students at a predominantly white institution, including comparisons across different races or ethnicities.

RESEARCH QUESTIONS

(1) Is there a difference in the proportion of men and women who discontinue taking math courses beyond Calculus 2?;
(2) Is there is a difference in how students perform in their next math class between students listed as listed as Black, Hispanic, American Indian/Alaskan Native, and Native Hawaiian/Other Pacific Islander and students listed as white/Asian students?; and
(3) Is there a difference in student performance in their next math course based on race and gender for students who earned similar grades in Calculus 2?

METHODS

Data Collection

- Institutional Data that included
  - Gender (provided only as female, male or unknown)
  - Race or Ethnicity
  - Major
  - Course Grade in Calculus 2 and Instructor Name (for Fall 2018)
  - Course Grades for all math classes taken over the next three semesters

Data Analysis

- To analyze this data, we used a two-proportion z-test for each question. All necessary conditions were met, including
  - Sample sizes are much bigger than 30.
  - They are independent samples.
  - We used data for all students who took Calculus 2 in Fall of 2018.
- We find there is a significant difference for all questions.

RESULTS

Calc 2 Letter Grade	Sex	Race	Next Math GPA Grade (median)	Count
A	Female	BIPOC	3.85	2
		W/A	3.70	84
	Male	BIPOC	3.30	10
		W/A	3.70	255
	Unknown	W/A	3.30	2
B	Female	BIPOC	2.00	3
		W/A	3.00	44
	Male	BIPOC	2.30	9
		W/A	3.00	187
	Unknown	W/A	0.00	3
C	Female	BIPOC	0.00	2
		W/A	2.00	17
	Male	BIPOC	0.00	3
		W/A	2.00	90
	Unknown	BIPOC	2.00	1
		W/A	2.85	2
D	Female	BIPOC	2.30	2
		W/A	1.00	5
	Male	BIPOC	0.00	2
		W/A	2.50	16
	Unknown	W/A	3.30	1
F	Female	BIPOC	2.00	3
		W/A	1.50	12
	Male	BIPOC	0.00	8
		W/A	2.00	72

Table 1: Median next course grades (and counts) for students based on Calc 2 letter grades (by sex and race).

- Especially for students who passed Calculus 2, the median letter grade in the next math courses tended to be higher for white/Asian students and the number of students earning those higher letter grades was much greater.
- There is a noticeably large difference in median next math grade by race for students who earned a B in Calculus 2. BIPOC students, as a group, earned one letter grade lower than W/A students.
- A similar distinction for students who earned a C in Calculus 2 emerged as well. However, for this group BIPOC students did not pass their next math course where W/A students earned a median C grade, as a group.

RESULTS

	BIPOC	W/A	Totals
Female	12	162	174
Male	32	620	652
Unknown	1	8	9
Totals	45	790	835

Table 2: Count of students who took a next math course (by sex and race).

- 62% of men took a next math course beyond Calculus 2 compared to 42% of women.
- When controlling for majors that does not require a math course beyond Calculus 2, there remained a significant difference in proportions of men and women enrolled in future courses.

DISCUSSION

- After almost a decade from the PCAST (2012) report, we are still seeing Calculus function as a gatekeeper from STEM fields.
- Continuing to delve into why and how such differences in persistence and achievement exist across students remains an important question in STEM education.
- Even after years with knowledge of good and ambitious teaching to support students, we are still seeing evidence that students from historically underserved groups in STEM continue to be underserved.

QUESTIONS FOR THE READER

- What do you wonder and notice by looking at the tables?
- What are your thoughts on positioning these results, particularly to avoid "gap gazing"?
- What are your thoughts about the language in use as dictated by the restricted way the institution collects and reports students' identification information?

Beyond Race-Gendered computation media and Tech-Tools: Accessing Mathematics and Computations within cultures in a more playful field of re(organizing) activities

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Introduction

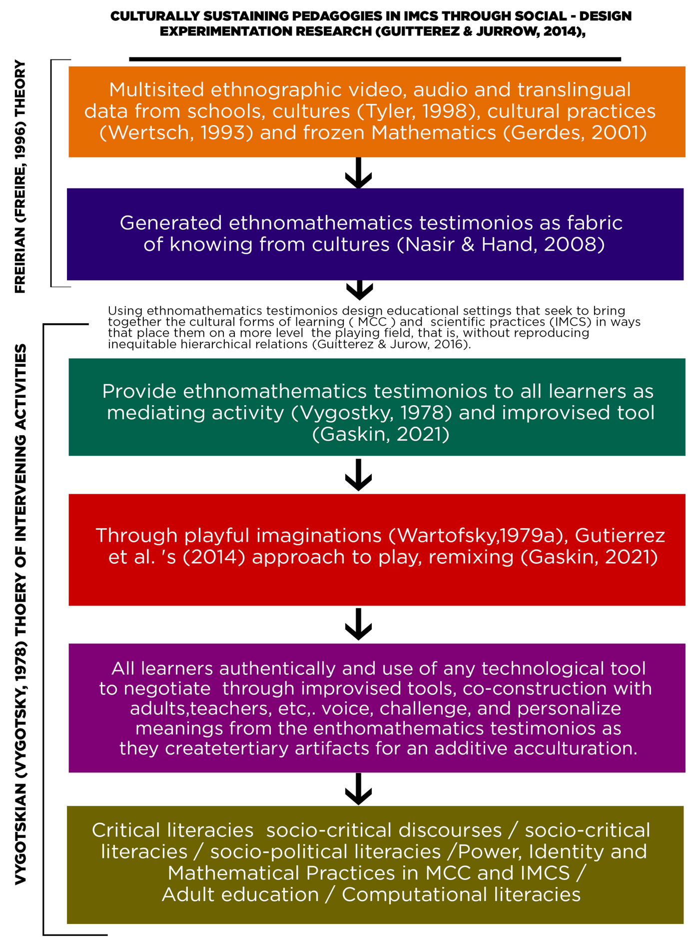
The struggle has been on sustaining cultures since most researchers approach democratization of access to Institutionalized Mathematics and CS education (IMCS) by adding to IMCS to that of the Mathematics and computations within cultures(MCC); thesis designed race gendered computation media and web-based tools.

Some of these web-based tools create a blockade. The users of these tools lack opportunities to personalize, voice, challenge, and negotiate meanings from the natural language and pre-planning thoughts of the MCC to add IMCS.

The majority of these tools hide the thoughts from the MCC. Developers box, restrict, and use only the names and images of MCC thoughts in their professionally designed websites where prescriptive technical literacies like programming and coding are heightened. Hence, most of its users cannot apply these names and images from MCC beyond the race gendered computation media and technological tools to other environments.

In this poster, I propose a new approach that sustains cultures(Tyler, 1998) by giving all learners access to the thoughts of MCC through ethnomathematics testimonios. I employ the Social Design experimentation research (Guitterez & Jurrow, 2014), which is grounded in both the Freirian (1996) and Vygotskian (1978) theories of learning.

Methodology



Conclusion

In the famous Paperts Mindstorm (1980, p viii), he cautioned that;
"If convinced by my story, a modern-day Montessori might propose to create a gear set for children. Thus every child might have the experience I had. But to hope for this would be to miss the sense of the story. I fell in love with the gears. This is something that cannot be reduced to purely "cognitive" terms. Something very personal happened, and one cannot assume that it would be repeated for other children in exactly the same form." As Papert, I challenge the common practice of bringing MCCs from non-dominant groups to the IMCS mainly through race-gendered web-based tools and computation media. I argue that these developers impose their interpretations of the MCC on and for all learners. Above all, such practices marginalize and silence MCC (see Lachney et al., 2016; Scott et al. 2014). I propose the use of social design experimentation and its theoretical underpinnings (Guitterez & Jurrow, 2014: Guitterez, 2008) to provide opportunities for all learners to have access to MCC thoughts. One way of providing access is by using what has been referred to in this poster as ethnomathematics testimonios. An audio, video, and trans lingual tool useful for all learners to personalize, voice, negotiate and challenge meanings from MCCs' thoughts to add the institutionalized forms. This tool helps develop culturally sustaining Mathematics, CS, and Maker education. Learners are able to invent objects to think with. Objects which is at the intersection of cultural presence, embedded knowledge, and the possibility of personal identification (see Papert 1980)

Acknowledgements

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University of Oslo

Four Patterns in Students' Connections Between Mathematics and Computing

CCSE

Motivation

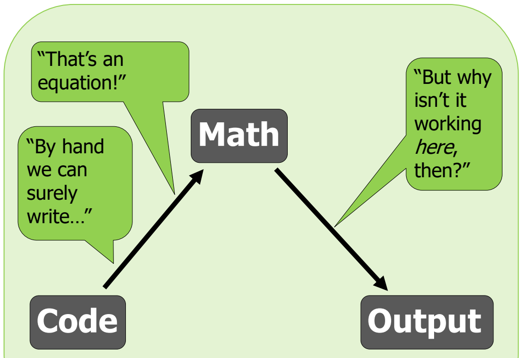
Computers are very common in the practice and teaching of mathematics. But how are students' thinking and learning affected when they use mathematics and *machine-based computing* (Lockwood & Mørken, 2021) together?

Literature

Connections are important for learning and one of several ways in which students *generalise* (Høffding, 1892; Lobato & Siebert, 2002; Ellis et al., 2017). We wanted to know how connections function as *affordances* for students' reasoning (Greeno et al., 1993).

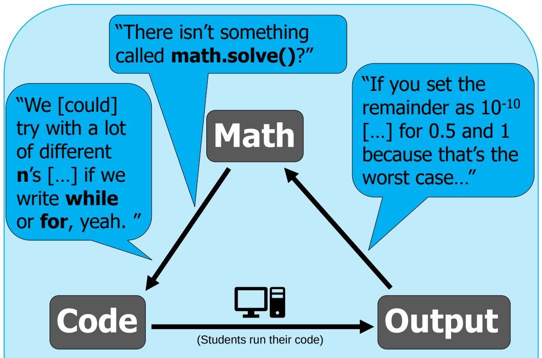
Research Questions

1. What types of cross-domain connections do students form?
2. Which patterns of these connections emerge?
3. What are the affordances of these patterns?



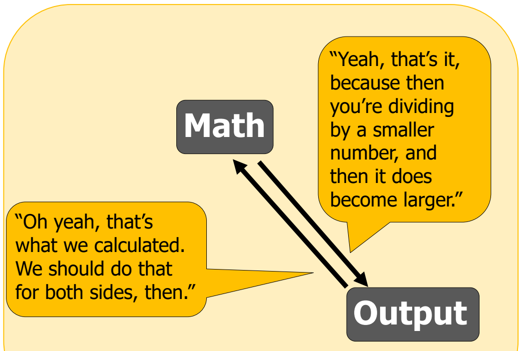
Replicating Program

Code → Mathematical model of code → Mathematical output / proof



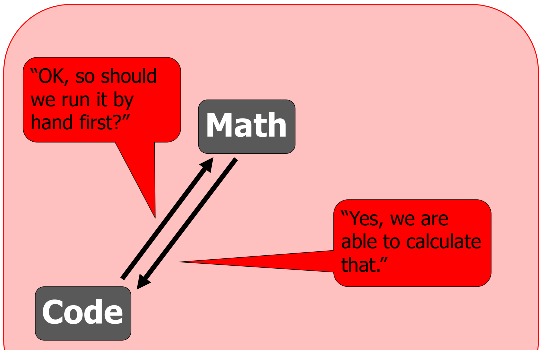
Improvement Cycle

Output → Mathematical model of code → Improved code → Improved output



Justified Improvement

Output → Mathematical model of code → Comparison with program output



Justified Design

Code to be written → Mathematical model of code → Writing code

Methods

We interviewed groups of students in a classroom-like setting, capturing video of them working, audio, and the screen. We then performed a *thematic analysis* (Nowell et al., 2017) to discover emerging themes in the data from the students' points of view. The framework we developed was validated by co-author triangulation.

Discussion

We found that our students were able to shift flexibly between modes of thinking when they were making connections. These shifts afforded the students to prove that a program works, explain how it works, and explain specific results. These affordances seem to be linked to patterns of connections rather than individual connections. It appears that the more similar a mathematical representation and its computational equivalent is, the easier it is for students to connect them.

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[3]Department of Physics and Astronomy & CREATE for STEM Institute, Michigan State University

Digital version: More in paper:

Learning Integrals Based on Adding Up Pieces Across a Unit on Integration

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Background and Guiding Question

Adding up pieces (AUP) is a quantitatively-rich basis for learning the definite integral (Jones, 2015; Ely, 2017; Von Korff & Rebello, 2012). However, most work has examined student thinking and reasoning and little attention has been given to *teaching* integration from an AUP foundation. Further, a full unit on teaching integration cannot stop at definite integrals, $\int_a^b f(x)dx$, and needs to extend to integral functions, $g(X) = \int_a^X f(x)dx$, in order to incorporate the FTC and antiderivatives. In response, we developed a learning trajectory for teaching integration in first semester calculus through an AUP perspective. Our guiding questions were: (a) How do we construct the conceptual pieces of definite integrals through AUP? and (b) How do we build on that same understanding to extend to integral functions?

Comparisons to Previous Learning Trajectories

An existing quantitative approach to learning integration is what we have called *accumulation from rate* (AR), which began with the work of Pat Thompson (Thompson, 1994, Thompson et al, 2013). One might wonder why a learning trajectory based on AUP is needed and whether those from AR can suffice. AUP and AR differ in key ways that have important implications for learning: (a) static partition pieces (AUP) versus dynamic continuous covariation (AR), (b) different roles of infinitesimals, (c) the centrality of definite integrals (AUP) versus integral functions (AR), and (d) the use of rate-only integrands (AR) versus the flexibility of using non-rate integrands (AUP) (see Jones & Ely, in preparation).

Framework

		Numeric	Symbolic	Graphical
Layer 0: Orient (Sealey, 2014; Simm & Oehr, 2019)	Basic Quantities & Basic Model			
Layers 1-3: Definite integrals as AUP (Jones, 2013, 2015; von Korff & Rebello, 2012; Simmons & Oehrtman, 2019)	Partition (infinitesimal)			
	Target Quantity (infinitesimal)			
	Sum (infinitesimal)			
Layers 4-6: Extending to integral functions (Swidan & Yerushalmy, 2014, 2016)	Variable Bound (infinitesimal)			
	Output (infinitesimal)			
	Function (continuous)			

Three Contexts, developed as follows:

1. Constant (basic), 2. Varying (definite integral), 3. Extend (integral function)

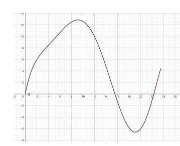
<u>Fuel Flow (interviews 1 & 3)</u>

Consider the flow rate R of fuel through a pipe in L/min as a function of time t in minutes. How much fuel passed through the pipe?

t	0	1.25	2	2.5	3	3.75	4
R(t)	18	12	7	6	4	3	2.5

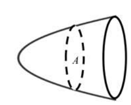
<u>Road Construction (interviews 2 &4)</u>

A road crew is levelling a stretch of terrain. Consider the linear weight of dirt, W, in tons/yd, along the 24 yards. How much dirt is on hand after leveling is done that needs to be hauled away?



<u>Volume of a Solid (interviews 2 & 4)</u>

Consider the round solid shown here is 83 ft long, with different cross sections of area A. Describe how an integral could be used to determine its volume.



Progression of Unit on Integration : Key Activities and Associated Parts of Framework

Interview Lesson #1, Fuel Context

- Constant case: 10 L/min over 4 min. How much fuel? (*Basic quantities*)
- Non-constant case: Data table of different rates. How much fuel? (Numeric/Symbolic: *Partition, target quantity, sum*; scale to *infinitesimal*)
- Non-constant case: If we graph $R(t)$, connect ideas to graph. (Graphical: *all layers 0 - 3*)
- Define definite integral: $F = \int_0^4 R(t)dt$ as partition (dt), amount (Rdt), and sum (∫)

Interview Lesson #2, Road & Volume

- Constant case: If terrain is even, how much dirt? (*Basic quantities*)
- Non-constant case: If terrain varies (see graph), how much dirt? (Num/Symb/Graph: *Partition, target quantity, sum*)
- Reinforce definite integral def'n: $D = \int_0^{24} T(x)dx$ as partition (dx), amount (Tdx), and sum (∫)
- Volume: solidify definite integral $V = \int_0^{83} Adx$ as partition (dx), amount (Adx), and sum (∫)

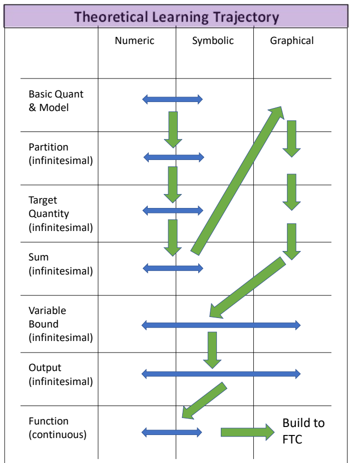
Interview Lesson #3, Revisit Fuel

- Extend: What if time interval extended beyond 4 mins? (Num/Symb/Graph: *Var bound*)
- Extend: For each extra bit of time, what happens with fuel amount? (Num/Symb/Graph: *Output*)
- Extend: Relationship between bounds and outputs? (Num/Symb: *Function*)
- Extend: Construct a graph that shows fuel amount at any time T (Graphical: *Function*)

Interview Lesson #4, Revisit Road/Volume, build to FTC

- Extend: What if road continued? (Num/Symb/Graph: *Bound, output*)
- Extend: Relationship between road end and dirt amount? Graph amount of dirt on hand as road progresses (Num/Symb/Graph: *Function*)
- Build toward FTC: Compare integral function graphs to original context graphs. What do you notice? (*Identify derivative-antiderivative relationship*)

Theoretical Learning Trajectory



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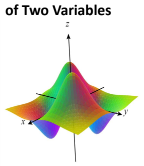
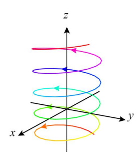
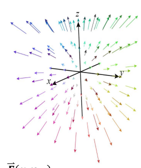
Traditional multivariable calculus (MVC) courses are organized by **3 function types**

Real-Valued Functions of Two Variables	Parametric Functions	Vector Fields
		
$z = f(x,y)$	$\vec{r}(t) = < f(t), g(t), h(t) >$	$\vec{F}(x,y,z) = < M(x,y,z), N(x,y,z), P(x,y,z) >$

But how do *students* organize the course?

Mathematics is the art of giving the same name to different things

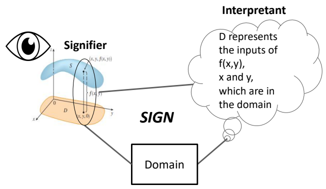
H. Poincaré

To what extent do students recognize the above objects as being of the same form, f : D → C, with a domain and a codomain?

QUESTION:
What are some common student conceptions for the domain and range across different types of functions that appear in MVC?

Theory

- Lens of signs & cultural **semiotics**
- **Asset-based** approach to students' representations
- Set-based focus: **Element ∈ Set ⊆ Superset framework**
  - How do students represent elements of sets? The sets themselves? What ambient dimension(s) do they consider the elements/set to be in?



"Judging what is good and less good in a representation is an error-prone pursuit."
–diSessa (2004)

Methods

- Participants: 4 students in a traditional MVC course
- 3 semi-structured, task-based interviews
  - 1st: Single-variable, real-valued functions
  - 2nd: Multivariable, real-valued functions
  - 3rd: Vector-valued functions
- Questions about representing & interpreting the D&R of different function types in different modalities (graphically, symbolically, or verbally)

Students' conceptions of the domain and range of different types of functions

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Michigan State University

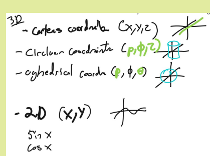
MAIN TAKEAWAY

Participants' **conceptions of domain & range** were often **intimately intertwined** with their **conceptions of specific types of functions and** their **preferred representations** of these functions.

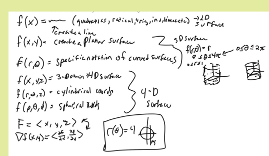
Some Results & Implications

Students mostly attended to factors other than domain and range when listing types of functions in MVC, including the coordinate system and the dimension of the function's graph

Q: Tell me about the different kinds of functions you've encountered in MVC.

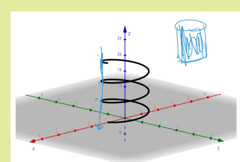


Amelia response



Robin's response

All but one student treated the range as being 1-dimensional, even when considering vector-valued functions

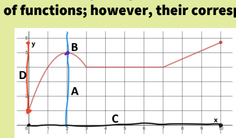


Two students' representation of the range of a parametric function, **r**(t).

Two students took the magnitude of vector outputs to determine the range of a vector field on $\mathbb{R}^2$:

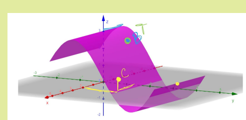
Trinidad: The range is (-∞,∞). I'm thinking of all the numbers you could get out. When you get the vector as an output, you get some number, some number [she writes <#,#>] and then you can account for the magnitude of that vector as well. . . . This means that the magnitude can be infinitely small or infinitely large as you can see [from the diagram of **F**(x,y)].

Students frequently used non-normative *signifiers* for the domain and range of functions; however, their corresponding *interpretant* was often normative



Robin's signifiers for
A: "x=2 is in the domain of f(x)"
B: f(2)
C: The domain of f(x)
D: The range of f(x)

Trinidad's signifiers for
B: Something in the domain of f(x,y)
T: f(B)
C: Something in the range of f(x,y)



Implications

- If we hope for students to organize MVC by function type, we must make this organization more explicit
- Students may benefit from explicit guidance when generalizing from real-valued to vector-valued functions
- Effort spent helping students organize their experiences by function type so they develop a unified notion for function (Zandieh et al., 2017) could pay off in future courses
- It is important to look beyond students' potentially non-normative representations when interpreting their thinking about multivariable functions

Works Consulted

Don't hesitate to reach out with comments or questions!

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Impact of Calculus Coordination on Instructional Practices: A Preliminary Investigation

Background

Calculus course coordination systems attempt to ensure uniformity in course elements such as textbook, and exams; create consistency in student learning opportunities; and transform learning experiences through active learning (Apkarian & Kirin, 2017; Rasmussen et al., 2021; Williams et al., 2021).

Course coordination has the potential to initiate a community of instructors working towards high quality instruction and positively impact student learning (Rasmussen & Ellis, 2015; Rasmussen et al., 2014).

Little is known about how similar or different coordination styles are across contexts or in what ways does the coordination impact the instructional strategies of instructors.

Goal

We aimed at understanding the use of research based instructional practices in introductory mathematics, physics, and chemistry courses at the post-secondary level in the U.S.

Specifically, we sought to examine patterns (if any) of course coordination among calculus instructors at four-year institutions and how might different aspects of coordination impact the instructional practices of instructors.

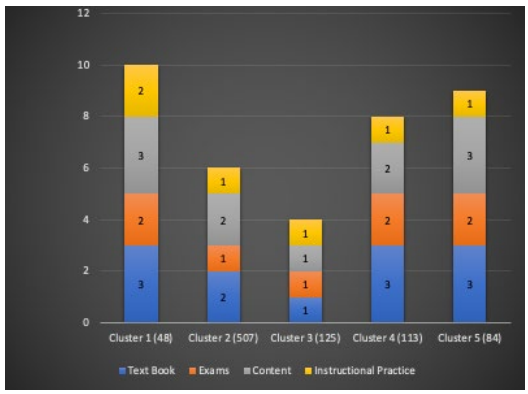
Research Questions

1. What patterns of course coordination, if any, exist among calculus instructors at four-year institutions?
2. How do different aspects of coordination impact the instructional practices of calculus instructors?

Survey Prompts

1. Who are the primary decision makers for (i) textbook; (ii) exams; (iii) content and topics covered in class; and (iv) instructional methods used? Instructors reported on on a four-point scale: (1) myself; (2) myself and others; (3) one or more other people; and (4) does not apply.

2. What proportion of time during regular class meetings (i.e., lecture sections) do students spend (i) working individually, (ii) working in small groups, (iii) participating in whole-class discussions, and (iv) listening to the instructor lecture or solve problems?



5 Clusters based on Prompt 1 with the mean score for each coordination component. The average scores are rounded up to whole number values.

Description of the Cluster	Number
The instructor collaborates with others in deciding the exams and instructional practices whereas a group of others decide the textbook and content to be taught.	48
The instructor collaborates with others in deciding the textbook and content to be taught but is the primary decision maker for exams and instructional practices.	507
The instructor primarily decides the textbook, content, exams, and instructional practices.	125
The instructor collaborates with others to decide the content and exams, is the primary decision maker for instructional practice, and is sometimes on a group of others to decide the textbook.	113
A group of others decide textbook and content and while the instructor is sometimes on a group that decides exams they decide the instructional practices.	84

Participants

900 instructors responded to the two prompts. We reduced the data-set to include only 877 respondents; we excluded 23 instructors who reported "does not apply" to Prompt 1.

We conducted a *k*-means cluster analysis to identify categories of instructors.

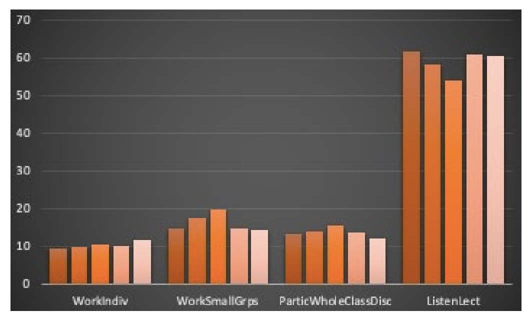
Results

Five clusters of instructors based on who the primary decision makers were for the four coordination components.

We found no statistically significant differences between the group means of the 5 clusters.

We found no statistically significant differences between group means based on the proportion of in-class time students spent on four different activities.

Proportion of in-class activities across the 5 clusters Based on Prompt 2.



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Madhavi Vishnubhotla, Ahsan Chowdhury, Naneh Apkarian, Estrella Johnson

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MOTIVATION FOR COLLABORATIVE LEARNING IN UNDERGRADUATE STATISTICS

Temple University

Aaron F. Wade

INTRODUCTION

The study's goal was to explore undergraduate statistics students' motivation for collaborative learning and explore the effects a particular intervention (Collaborative Learning Values Workshop, CLVW) had on their motivation for collaborative learning. The author/investigator conducted the study with the purpose of enhancing students' motivation (values-utility, importance, cost) for collaborative learning. With students' values for collaborative learning enhanced, they would be more receptive and appreciative of future collaborative learning experiences. In turn, increasing their receptiveness for collaborative learning would increase the effectiveness of collaborative learning opportunities. Study results suggest that statistics students' motivation for collaborative learning can be enhanced via the CLVW intervention.

Keywords: Collaborative Learning, Statistics Undergraduate Education, Utility-value Intervention

Pedagogies that employ collaborative learning opportunities have been shown to positively increase students' academic achievement (Whicker, Bol, & Nunnery, 1997). In fact, a plethora of studies have replicated results of the same nature, as confirmed via a meta-analysis (Springer, Stanne, & Donovan, 1999), but there exists an atypical group of students who are not motivated to learn collaboratively (Plass et al., 2013).

A student can lack motivation towards collaborative learning when they attribute greater value to individualised learning experiences as opposed to collaborative learning experiences. Students who lack motivation (for example, because they believe they learn better from lecture) for collaborative learning are less receptive to the collaborative learning opportunities in which they partake in. Educators who seek to harness the benefits of collaborative learning are, therefore, presented with a challenge—how can students' motivation for collaborative learning be increased?

In an effort to increase undergraduate statistics students' motivation for collaborative learning the study investigated whether statistics students' values (utility, importance, costs) for collaborative learning could be increased via the Collaborative Learning Values Workshop (CLVW). The CLVW intended to assist students better understand both their subjective experiences of learning statistics as well as their learning outcomes when engaged in classroom environments which employed individualised learning in comparison to learning which embraced collaboration as well.

Through an increase in undergraduate statistics students' values for collaborative learning, via the CLVW, (a) their motivation for collaborative learning increased, (b) enabling said students to be more receptive of future collaborative learning experiences, (c) leading to their increases in learning effectiveness for collaborative learning experiences, before (d) promoting achievement in undergraduate statistics.

DATA

The study's theoretical framework was that of Situated Expectancy-Value Theory (SEVT), Eccles & Wigfield (2020)--a prominent motivational theory in the field of education. Statistics students completed pre/post self-report and assessment measures. Self-report measures collected data on their motivation for collaborative learning. SEVT motivational constructs operationalised for measure were that of student success expectancy, interest, values (utility, attainment, cost), while the assessments consisted of quizzes, achievement data.

METHODS

Participants comprised of 54 business statistics undergraduate students from a north eastern United States 4-year university from two differing sections of the same course. During the course of chapter 1, all pre-measures were collected. Mid-semester the CLVW was carried out via random selection. Towards semesters end all post-measures were collected.

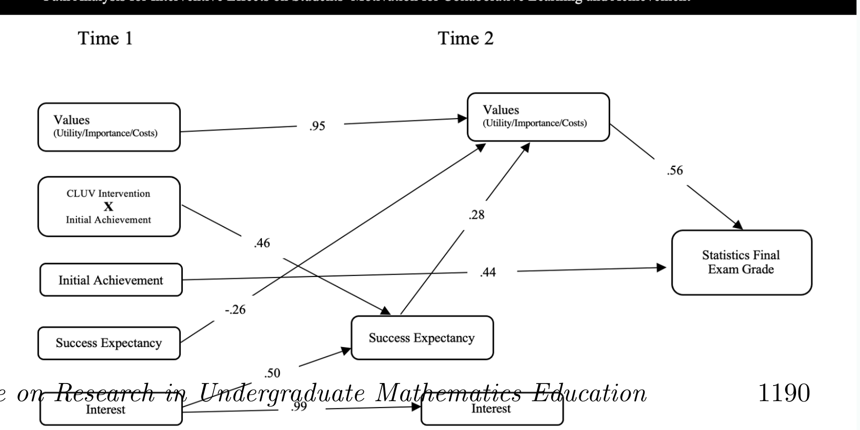
ANALYSIS

Cronbach alphas assessed the motivation constructs for reliability. Strength of correlations between constructs concurred with current research in the field. Regression was conducted. Final path analysis is shown below.

CONCLUSION

Although the CLVW intervention did not directly effect students' achievement, the CLVW, when moderated by initial achievement, increased students motivation (expectancy). Expectancy increases, in turn, increased values (utility- importance-costs) which was the path of association with promoting increased student achievement.

Table 5.
Path Analysis for Interventive Effects on Students' Motivation for Collaborative Learning and Achievement



A Sociocultural Perspective on Beginning Teachers Enacting Reasoning and Proving Practices

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University of New Hampshire

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The opinions expressed herein are those of the authors and do not necessarily reflect the views of the NSF

Background

Teaching Mathematics via Reasoning and Proving (Buchbinder & McCrone, in press)

- Integration of reasoning and proving within the mathematics curriculum
- Emphasis on deductive reasoning for producing and validating mathematical results
- Use of precise mathematical language but within the conceptual reach of the students

- Capstone course ***Mathematical Reasoning and Proving for Secondary Teachers***.
- Prospective Secondary Teachers (PSTs) can develop knowledge, dispositions and skills for teaching mathematics via reasoning and proving (Buchbinder & McCrone, 2020)
- Little is known about long-term development of beginning teachers' learning to teach mathematics via reasoning and proving, and what factors affect this development (Stylianides et al., 2017).
- Beginning teachers experience tensions between their commitments to the university, their cooperating teacher, and developing their own teaching styles (e.g., Smagorinsky et al., 2004)

Goal: To examine how sociocultural contexts of the teacher preparation program and of the internship school, supported or inhibited teaching mathematics via reasoning and proving of beginning secondary mathematics teachers.

The Study

Participant:
Olive – a beginning teacher, interning in a local high school, supported by her cooperating teacher (CT). Olive participated in the capstone course, a year prior to the internship.

Data sources:

- PST: Four lessons: lesson plans, reflections
- Intern: Two lessons: lessons plans, observations and follow-up interviews.

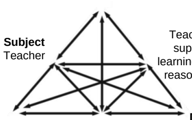
Theoretical frameworks

Activity theory (Engeström, 1987)

Tools
Mathematical tasks

Subject
Teacher

Object
Teaching actions for supporting student learning mathematics via reasoning and proving



Rules
Classroom norms, Teaching style, Adopted curriculum, School-standards

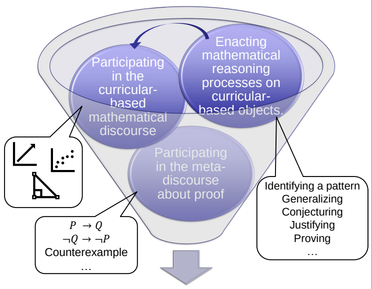
Community
support personnel (the cooperating teacher)

Division of Labor
Support personnel involvement during the lesson's planning and teaching

Outcome
Teaching mathematics via reasoning and proving

Commognitive theory (Sfard, 2008)

Teaching mathematics via reasoning and proving includes providing students opportunities for:



Learning mathematics via reasoning and proving
(Weingarden & Buchbinder, submitted)

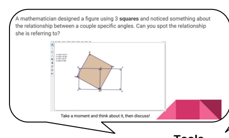
Data analysis:

- Olive's teaching was examined by the activity system in two settings: as a PST and as an intern.
- Olive's tasks and teaching actions were analyzed regarding the opportunities provided to students to learn mathematics via reasoning and proving.
- Tensions between the activity system components were identified

Results

Olive's teaching as a PST

The sociocultural contexts **supported** teaching mathematics via reasoning and proving



Let's discuss.

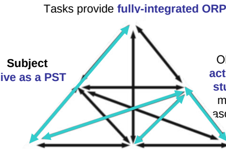
Can we make a conjecture together?

"So, let's look at proof one. I started with the statement that angle ADC is 90 degrees. Can anyone identify **why that might be true**?"

Tools
Tasks provide **fully-integrated ORP**

Subject
Olive as a PST

Object
Olive's **teaching actions supported students** learning mathematics via reasoning and proving



Rules
Olive **was not expected to adhere her CT's** teaching style and to strictly follow the schools' standards and curriculum.

Community
The **CT did not restrain or limit** Olive's teaching and planning. They hardly communicated (only technically)

Division of Labor
The **CT was not involved** in Olive's teaching and planning the lesson.

Olive's imaginary teaching

Teaching mathematics via reasoning and proving

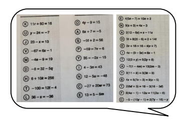
Subject
Olive as an intern

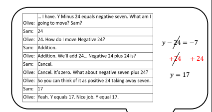
Object
Olive's **teaching actions supported students** learning mathematics via reasoning and proving

"**I personally would love to … put something on the board and they tell me all the things that they wanna do [operations on equations] that don't necessarily make sense** for them to grasp... I think they're stuck to doing only moving this, but if you can conceptually understand that we have a scale, and as long as you're doing both things to both sides, the answer is going to be the same at the end of the day. **I think that would be cool for them to see, like, let two people do entirely different things and show them that they're gonna get to the same answer at the end of the day, as long as it's balanced on both sides**"

Olive's teaching as an intern

The sociocultural contexts **inhibited** teaching mathematics via reasoning and proving





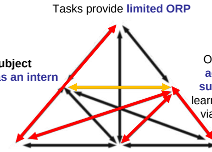
Olive:	… I have. Y Minus 24 equals negative seven. What am I going to move? Sam?
Sam:	24
Olive:	24. How do I move Negative 24?
Sam:	Addition.
Olive:	Addition. We'll add 24… Negative 24 plus 24 is?
Sam:	Cancel.
Olive:	Cancel. It's zero. What about negative seven plus 24?
Olive:	So you can think of it as positive 24 taking away seven.
Sam:	17
Olive:	Yeah. Y equals 17. Nice job. Y equal 17.

$y - \cancel{24} = -7$
$+\cancel{24} \quad +24$
$y = 17$

Tools
Tasks provide **limited ORP**

Subject
Olive as an intern

Object
Olive's **teaching actions did not support students** learning mathematics via reasoning and proving



Rules
Olive **was tacitly expected to adhere her CT's** teaching style and to **strictly follow** the schools' standards and curriculum.

"They have the curriculum planned out for the year, so I'm pretty much facilitating it… I'm welcome to make things, you know, she [CT] included my input and stuff, but she's been teaching this class for seven years. So, she has like a system that tends to work for her. So that's what I've been doing…"

Community
The **CT implicitly restrained and limited** Olive's teaching and planning.

"I disagreed with [the CT's principle of] moving the things [variables and constants] to the left and right. I don't personally care which side the variable is on because I think it's the right answer at the end of the day. So, if they want to write 12 equals N, I'm okay with that… I didn't immediately change anything… There was one example that I was like, I would move this five over here and put a zero over here, but I know she [CT] didn't do that. So, I was like, uhoh I don't want one group of kids to get a different system than another group".

Division of Labor
The **CT played a significant role** in Olive's teaching and planning the lesson.

"…This isn't my lesson plan. This is Anna's [CT] lesson plan. So, it's me applying her lesson […]"

Conclusions

- Beginning teachers implementing teaching mathematics via reasoning and proving needs to be examined while considering how teachers navigate the tensions between the proof-related teaching practices adopted during their teacher education program, their developing personal teaching styles, and the sociocultural components of learning/ teaching environments.
- Contribution: developing theoretical and analytical tools for analyzing beginning teachers' learning how to teach mathematics via reasoning and proving.

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Utilizing Cognitive Interviews to Evaluate and Improve Items for an Instrument to Measure Mathematical Knowledge for Teaching Community College Algebra

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AI@CC 2.0 VMQI Research Group[1]
EHR #2000602, 2000644, 2000527, 2000566

As part of the Algebra Instruction at Community Colleges: Validating Measures of Quality Instruction project to develop an instrument to measure mathematical knowledge for teaching community college algebra (MKT-CCA), cognitive interviews were conducted with community college instructors of College Algebra. The findings and lessons learned from these interviews will be presented in this poster.

Keywords: Cognitive interviews, Mathematical Knowledge for Teaching, College Algebra, Community College

Research has shown that there is a connection between the quality of instruction and the instructor's mathematical knowledge for teaching (MKT), and that students' achievement gains are significantly tied to their teachers' mathematical knowledge (Hill et al., 2005). The Algebra Instruction at Community Colleges: Validating Measures of Quality Instruction (AI@CC 2.0) project is developing an instrument to measure mathematical knowledge for teaching community college algebra (MKT-CCA). The development of an instrument involves a number of processes essential to the construction and selection of good items. This process involves five main stages: conceptualization, construction, tryout, analysis, and revision (Cohen & Swerdlik, 2009). In the construction stage, cognitive interviews play a significant role and have become an integral part in the development of assessment instruments since its introduction in the 1980's (Meadows, 2021; Willis, 2005). Cognitive interviews are usually used by researchers to gain insight into respondents' understanding of survey items (Ryan et al., 2012; Willis, 2015).

We conducted cognitive interviews with community college (CC) college algebra instructors to understand whether the items are interpreted by participants as intended and used the knowledge we thought was needed (Mesa et al., 2020-2023). Eighteen drafted MKT-CCA test items were reviewed by 12 CC instructors with each item being reviewed by two instructors. The data from these interview sessions were analyzed and the strengths and weaknesses of the items identified were used to improve on the items and also guided in the drafting of subsequent items.

This poster will present the findings and lessons learned from this first stage of cognitive interviews with the instructors as part of the instrument and item construction stage. We will also highlight how the feedback received during the cognitive interviews were integrated in the items along with challenges that we faced during the process.

[1]The AI@CC 2.0 VMQI Research group includes: Megan Breit-Goodwin, Anoka-Ramsey CC; April Ström, Chandler-Gilbert CC; Patrick Kimani and Laura Watkins, Glendale CC; Nicole Lang, North Hennepin CC; Mary Beisiegel, Oregon State University; Judy Sutor, Scottsdale CC; Claire Boeck, Inah Ko and Vilma Mesa, University of Michigan; Bismark Akoto, Irene Duranczyk, Siyad Gedi and Dexter Lim, University of Minnesota. Colleges and authors are listed alphabetically.

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Students Generated Observations of a Function Represented as a Graph and Symbolic Equation

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University of New Hampshire

Historically, mathematics was perceived as sets of calculations, numerals, or numerical operations mainly with a single correct answer. This study breaks this cycle by having 600 STEM undergraduate students make accurate, distinct, and relevant observations or inferences about a function represented as a symbolic equation and graphical representation in parallel. The initial analyses indicated that; (a) the students rely on graphical representations more than the symbolic equation of the function, (b) the students indicated that the graph of the function represents more information about the function compared to the symbolic equation. We concluded that getting students to write their observations is far richer than having them answer a single correct answer.

Keywords: Representations, Functions, Graph, Symbolic Equation, Sense-Making

Historically, learning and teaching functions focus on a single correct answer mainly on a single representation: graph, symbolic equation, or table (Altindis, 2021; Altindis & Fonger, 2019; Fonger & Altindis, 2019). However, as noted in the literature, meaning-making is a creative sense-making process. According to Voigt (1994), the mathematical meaning is "individual sense-making process" and "development of mathematical knowledge" (p. 276). Sfard and Linchevski (1994) further posit that students' construction of meaning evolves with a skill of recognizing "abstract ideas hidden behind symbols." (p. 224). Making accurate, relevant observations are processes that use our senses by interacting, touching, seeing, and giving meaning to what we see, feel, and touch, then creating new images. In other words, making accurate observations about function is a process that results in the act of creation. With that in mind, we explored undergraduate students' accurate, precise, relevant observation of a function presented as a graph and symbolic equation. To understand students' meaning-making about a function, we distributed pre-and post-survey asking 600 STEM undergraduate students to make observations or inferences about the graph and symbolic equation in parallel. This poster reports students' written responses related to Figure 1.

A function f is defined by $f(x) = \frac{1}{(x-2)^2}$, and its graph is given below. From the equation and graph write down up to 5 accurate, distinct, and relevant observations or inferences.

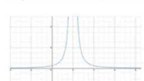

Figure 1: Survey Item 1

For the analyses, we analyzed students' responses to the survey item by employing Corbin and Strauss' (2008) constant comparison analyses. The findings indicated; first, the students were inclined to make a coordinated change on a graphical representation compared to symbolic representation. Second, the function graph represents more information about the function than the symbolic equation. And students coordinated change in the independent with the change in the dependent variable of function on graphical representations by stating, "While the x-axes approach 2 both from left and right, the function approaches infinity". We concluded that getting students to write their observations is far richer than having them answer a single correct answer. However, further research is needed to explore how students' accurate, distinct, and relevant observations about functions can inform learning and teaching functions.

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Design Possibilities: Opening a Door to More Equitable Mathematical Discussions

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Keywords: Professional Development, Teacher Education, Digital Clinical Simulations

Mathematics educators' ideological perspectives that position mathematics and students within a binary of right or wrong (Christensen et al., 2008) can be disrupted by examining the laden social and cultural practices and expectations embedded within mathematics and mathematics learning (Gutiérrez, 2013). In light of this need, the popularity of digital experiences for practice-based teacher education (Cohen et al., 2020) presents an opportunity to design mathematics teachers' instructional simulation towards disrupting settled notions of mathematics learning. Our work explores the design choices of a digital clinical simulation for in-service middle years mathematics teachers that intentionally surfaces opportunities and tensions of teacher moves within a students' small group discourse. In this report, we detail the theoretical underpinnings when designing this three-fold digital simulation within the Teacher Moments (TM) simulation tool, and discuss how the designed simulation within TM is oriented towards disrupting teachers' inequitable decision making in students' small group discourse.

In order to design simulations in which teachers can develop expansive pedagogical attunements, we created three discursive scenarios meant to model a small group challenge in terms of either student status, cognitive demand, or student (mis)conceptions. When imagining an unbalanced student status, we designed for a teacher to respond in ways in order for all students to feel and be perceived as competent by their teacher and classmates at moments when a student was dominating the group discussion (Horn, 2008). In order to practice maintaining cognitive demand, we designed for a teacher to identify where that student was struggling conceptually and then support a student's access to a task as opposed to providing a step-by-step procedure (Stein & Lane, 1996). We also designed for a teacher to support students in their tentative, exploratory ideas while revisiting those ideas through engaging with their group and resist the need to "correct" student thinking (Horn, 2012; Jansen et al., 2016; Kazemi & Stipek, 2001). These multiple moments of challenge and choice serve as powerful sites of learning for teachers experiencing the simulation, as they are opportunities to experience and reflect on a potential perpetuation or disruption of normative ways of facilitating mathematics.

The TM digital simulation design shared here makes space for teachers to experience critical moments within small-group discussions and explicitly practice ways to respond, thus serving as a way to shape teacher discretion. While experiencing the entire simulation, teachers either choose or auditorily record responses at multiple moments; later, they can examine how their responses disrupts or perpetuates the given discursive challenge in a particular group experience by listening to their own responses or visually seeing how particular key decision moments led to particular student responses in each group design's trajectory. The potential for small-group discussions to be a place for generative ideas to take root or not reveals how critical it is to have professional opportunities for teachers to practice how to orient themselves for real-life classroom discourse. TM is not a tool to illustrate instructional "rights or wrongs," but as a lens to surface complexities that arise within micro-moments of mathematical discourse. Therefore, we believe designing digital simulations with TM can help foster critical conversations that expand teachers' conceptions of students' mathematical doing and learning when designed to particularly surface the potential expansive moments of teacher decision making.

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Undergraduates Transition to Formal Proof-oriented Mathematics

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Keywords: proof, proof-writing, undergraduate, transitional proof courses

Most universities introduce students to formal proof-writing norms through transitional proof courses (Selden & Selden, 2008; Stylianou et al., 2015). However, the intent of these courses is commonly not aligned with the experiences that undergraduates have when acclimating to this genre (Bleiler-Baxter & Pair, 2017; Stavrou, 2014; Stylianou et al., 2015). I have reviewed 31 sources from over ten journal venues related to undergraduate proof and proof-writing. I then used inclusion criteria such as impact factor, SCOPUS index, and citation counts to identify significant pieces in this line of inquiry. In this poster, I use Tall's (2008) framework to anchor the findings from my review and to propose some implications for proof instruction.

Challenges of Learning Proofs

The transition to formal mathematics relies on developing three cognitive structures: *conceptual embodiment, proceptual symbolism, and axiomatic formalism* (Tall, 2008). Students' axiomatic formalism involves their engagement in formal proofs in advanced mathematics courses, which thus relies heavily on their development of representations of concepts and symbols for describing concepts dually as a process and a structure (Tall, 2008). However, if students' development of the first two structures is limited, they will encounter many difficulties in proofs (Stavrou, 2014; Tall, 2008).

Students' transition to formal mathematics is often abrupt (Stylianou et al., 2015), which limits the development of conceptual embodiment. For example, instructors often provide students with a repertoire of abstract definitions and theorems and expect students to apply them in proofs in their advanced mathematics courses (Stavrou, 2014) but without sufficient attention to supporting students to develop prototypical representations of the objects and relations described by definitions and theorems. One result is that students struggle to correctly apply definitions and theorems in proofs (Selden & Selden, 2008; Stavrou, 2014).

In many proof-oriented mathematics courses, a large instructional emphasis is given to a proof's logical structure (Baker & Campbell, 2004). The emphasis on logical structure focuses on a student's development of axiomatic formalism, which leads to a deviation from conceptual embodiment and perceptual symbolism. This has been shown by Baker and Campbell (2004) to limit students understanding of the importance and uses of proof, for example, by viewing the role of proof as an end-product in comparison to a sense-making tool for mathematical thinking.

Implications

This review suggests that for students to fully develop axiomatic formalism, educators should provide students with opportunities to *embody* and *symbolize* essential concepts in proofs used in proofs. The development of a student's conceptual embodiment and proceptual symbolism could then benefit from problem-solving pedagogies, such as small-group explorations and in-class discussions, which aim to develop rich understandings of mathematical concepts (Hiebert & Wearne, 2003). This could then promote students' abilities to apply definitions and theorems in proofs and shift their perspective towards proofs as a tool for sense-making and mathematical thinking.

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A Survey of Programs for Preparing Graduate Students to Teach Undergraduate Mathematics

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We present preliminary results of a census survey of mathematics Ph.D programs conducted in Fall 2021. The survey gathered information about how (or if) these programs prepare their graduate students to teach undergraduate mathematics. This survey follows up and extends a similar survey conducted in 2015. Analysis of the survey data was used to examine the extent to which the cultural landscape about preparing graduate students for teaching has changed since 2015.

Keywords: Graduate Student Teaching Assistants, Professional Development

This work is part of a larger effort by MAA CoMInDS (NSF DUE-1432381) to support providers of professional development for teaching (PDT). Since 2014 CoMInDS (College Mathematics Instructor Development Source) has provided that support by (1) conducting workshops for faculty providers of PDT for graduate students (GTAs); (2) building an online resource suite of instructional materials and research products to support the work of preparing GTAs to teach undergraduate mathematics; and (3) promoting a professional community of practice comprising these providers. Each summer since 2016, CoMInDS has offered workshops for providers. A total of about 200 participants from 115 institutions have attended the workshops (Bookman & Speer, 2021).

In 2015, CoMInDS, working with the Progress through Calculus project (Rasmussen et al, 2019), conducted a census survey of graduate mathematics programs. One focus of that survey was how departments prepared their GTAs for teaching. There was a 75% response rate from Ph.D granting institutions. A majority of mathematics departments reported that they conduct a PDT program for their GTAs, however the depth of that PDT varied widely. Another finding was that evaluation of these efforts is limited and the primary means of evaluation were student evaluations (Speer et al. 2017).

In order to gauge the extent to which things have changed since 2015 and to get a more in-depth view of the PDT graduate programs offer, we adapted the 2015 survey and sought data from all US institutions with mathematics Ph.D programs. Some survey questions are identical or similar to questions asked in 2015. Other questions probe more deeply so we can better gauge the nature and extent of the PDT. Respondents were asked to choose from a list of 19 topics that GTAs typically learn about in their programs (e.g., facilitating group work, university resources for students) and are asked to choose from a list of 11 activities in which GTAs participate (e.g., develop a lesson plan, deliver a non-lecture based practice lesson). We also asked what were the sources of the instructional materials and activities used in the PDT.

Part of the 2021 survey asked about the providers of PDT for the graduate students. This information was not gathered in the previous survey. Among other questions, we asked, "What kind of faculty positions do each of the providers of GTA preparation hold?" "What factors were taken into consideration when choosing who will provide GTA preparation?"

Findings from the 2021 survey will be used to inform the mathematics community about the current state and needs of providers delivering the PDT.

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Instructors' Dispositions to Incorporate Data Science in Mathematics Courses

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Keywords: data science, technology, computing, calculus, statistics

Data science involves the application mathematical and statistical concepts, distinguished by the use of technology, particularly the use of software to visualize and structure data and to enact machine learning. Our research aim is to understand how to prepare mathematics instructors to incorporate data science learning goals in grades 11-14 courses (e.g., pre-calculus, calculus, linear algebra, and statistics). We engaged a group of four instructors of high school and community college mathematics in learning some technologies used in the discipline of data science and identifying opportunities to incorporate those technologies to concurrently serve mathematics learning goals. The participants each had prior experiences with *Python* or *R* and at least ten years of teaching experience at the high school or community college level. We first engaged the four instructors in collaborative learning about data science, via their working through an introductory open-source text, *How to Think Like a Data Scientist* (HTTLADS) (Miller, Boggs, & Pearce, 2020) over the course of five weeks during Summer 2021.

We next embarked upon a modified lesson study (Rock & Wilson, 2005) in which each instructor began by preparing a lesson incorporating data science knowledge they learned to fit within learning objectives of mathematics or statistics course they would be teaching in the academic year. Participants met weekly for an hour to discuss their learning of the data science modules, interacted via shared annotations between meetings, and shared an approximately one hour video of themself engaging with HTTLADS. They were also individually interviewed for approximately one hour prior to the start of the school year by one of the authors. Interview questions included opportunities for reflection on their video submission as well as prompts for their dispositions for introducing data science in their mathematics teaching.

To analyze the participants' dispositions to incorporate data science in their mathematics teaching, we adapted Niess, Sadri, and Lee's (2007) five-stage developmental model for incorporating technology in teaching and learning mathematics. At stage one, instructors learn to use the technology for themselves and recognize the alignment of the technology with mathematics content but are yet to integrate the technology in teaching and learning of mathematics. For most data science tools instructors were in stage one, but for some data science tools they were in stage two (during which instructors engage in activities that lead to a choice to adapt or reject introducing the data science tool). We used the constant comparative method (Strauss & Corbin, 1994) to further analyze commonalities and distinctions in their dispositions from the interviews. The poster will display our analyses of the interviews.

One interesting result was that each participant described the use of *Google*'s search engine as an invaluable tool in their learning. They expressed diverging perspectives on expectations for mastery of a data science tool before introducing it to their classes. Two participants expressed a belief that their students would assist them with any issues that arose (such as syntax), while the others wanted to avoid software tools for which they felt less expertise. We look forward to further discussions with conference attendees about how the participants are identifying ways to align data science learning goals and mathematics and statistics learning goals.

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Prospective Secondary Mathematics Teachers' Understanding of the Role of Examples in Proving: Dealing With Conflicting Evidence

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Keywords: Prospective Secondary Teachers, Reasoning and Proof, Logic, Geometry

Secondary teachers' knowledge of reasoning and proving is fundamental to supporting students' opportunities to meaningfully participate in proof-related practices. Prior research identified several gaps in teacher knowledge of proof, such as misunderstanding the role of examples and counterexamples in proving. Still, intervention studies addressing these gaps are scarce (Stylianides, Stylianides & Weber, 2017). We designed a capstone course *Mathematical Reasoning and Proving for Secondary Teachers* and studied how it affects prospective secondary teachers' (PSTs) knowledge of proof (Buchbinder & McCrone, 2020), specifically, their understanding of the roles of examples in proving. We report on data collected over several years from about 50 PSTs completing an online, scenario-based task *What Can You Infer from This Example*. The PSTs had to decide whether a statement "A quadrilateral whose diagonals are congruent and perpendicular to each other is a kite" is true or false and justify their reasoning. Then, the PSTs examined six quadrilaterals proposed by hypothetical students, and determined whether each one proves, only supports, disproves, or neither proves nor disproves the statement.

We analyzed the PSTs' responses with Buchbinder and Zaslavsky's (2019) Role of Examples in Proving framework, and used open coding (Patton, 2002) to categorize PSTs' justifications. The results suggest that the PSTs have strong declarative knowledge, e.g., "supportive examples do not prove", "one counterexample disproves". Yet, many PSTs struggled to discern between supportive, irrelevant and counter-examples. Since definitions of quadrilateral may vary, (Usiskin, 2008) it is important for teachers to apply a particular definition consistently when evaluating student contributions; but this was not always the case for our PSTs. About 33% of our PSTs struggled to maintain logical consistency throughout the task. For example, they correctly identified an isosceles trapezoid with perpendicular diagonals as a counterexample, but also thought that a general kite disproves the statement, when in fact it disproves the converse. After analyzing student work, PSTs were invites once again to decide whether the statement about quadrilaterals is true or false. Almost all PSTs correctly identified the statement as false, and the number of correct justifications increased by 65%. The PSTs and the course instructor (the second author), discussed the mathematical and pedagogical matters invoked by the task and collectively resolved any remaining confusion.

The scientific significance of this work is twofold. First, we illustrate an example of a rich task, embedded in pedagogical practice that elicits and enhances PSTs' knowledge of proof. Second, from the research perspective, we provide a nuanced analysis of how PSTs deal with examples that constitute presumably conflicting evidence for truth-falsehood of a given statement, and report on aspects of PSTs' understanding of roles of examples in proving which, to our knowledge, have not been previously reported in the literature.

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Identifying the Language Demands of Inquiry-Oriented Undergraduate Mathematics Courses

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Keywords: Classroom Discourse; Undergraduate Education; Equity, Inclusion, and Diversity

Undergraduate STEM courses are increasingly adopting active learning instructional approaches (Cooper, Downing, & Brownell, 2018) based on prior research findings that active learning is more effective overall than traditional lecturing (Freeman et al., 2014). In particular, one active learning approach that has gained traction in undergraduate mathematics is inquiry-oriented (IO) instruction (Rasmussen & Laursen, 2019). According to Laursen and Rasmussen (2019), activities in an inquiry-oriented approach engage students in doing and thinking about mathematics (through reading, writing, discussing, or solving problems) as well as in talking with their peers and the teacher about what they are doing and thinking. In addition, the IO curricula consist of instructional sequences of daily tasks that lead students toward creating or reinventing big ideas, such as a major theorem, a definition or a procedure. IO classes engage students in authentic mathematical practices, such as explaining, justifying, conjecturing, proving, and defining (Moschkovich, 2007; Rasmussen et al., 2005).

However, while IO classes may be beneficial on average, their impact may not be equitable on certain groups of students. For example, Johnson et al. (2020) found that IO abstract algebra benefited men more than women, while women performed equally well in inquiry-oriented classes and lecture-based classes. Findings such as this one raise questions about which other student groups might also be differentially impacted by IO classes. One important group to consider is multilingual students who are learning the language of instruction. Attending to this group is critical because, while the undergraduate population is becoming more linguistically diverse, IO courses are usually taught only in English and they may induce different language demands than lectures do. To this end, I present and illustrate an emergent conceptual framework to identify the language demands in IO undergraduate math classes.

Drawing primarily on a situated sociocultural theory of learning (Moschkovich, 2015) and K-12 mathematics education research on language and mathematics, the framework has three dimensions: systemic, discursive, and lexical. The *systemic* dimension attends to which linguistic systems are used, including mathematical language systems (Moschkovich, 2015), communicative modes and text genres (Lyon, Bunch, & Shaw, 2012), participant structures (Chapin, et al. 2014), and named languages. The *discursive* dimension attends to the patterned ways of being and acting – e.g., sociomathematical norms (Yackel & Cobb, 1996) and discourse practices (Moschkovich, 2007). The *lexical* dimension attends to the uses of key mathematical terms or any unfamiliar or ambiguous terms (Kaplan, Fisher, & Rogness, 2009). I will illustrate the use(fulness) of this framework using classroom interaction data from an IO undergraduate math class. Revealing the language demands of IO instruction will suggest the need (and possibly ways) to integrate linguistic supports to address those demands, making undergraduate mathematics learning more equitable for all students, including multilingual students.

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Understanding Perceptions of an Innovative Active Learning Approach in Calculus Through a Learning Assistant's Perspective

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Learning Assistants (LAs) are becoming more popular in undergraduate STEM courses. LAs have become vital resources for instructors who have implemented active learning strategies in the classroom. The purpose of this study was to further explore LA perceptions of active learning in an introductory calculus course through focus groups.

Keywords: learning assistants, active learning, calculus

The use of Learning Assistants (LAs), trained 'near-peer' undergraduate classroom facilitators integrated into classrooms to support learning with groups (Otero et al., 2010), is becoming a more common practice across STEM departments in the United States. Research on LA impact shows the LA model improves student outcomes, including increasing conceptual understanding (Sellami et al., 2017; Van Dusen & Nissen, 2019), and lowering drop, fail, and withdrawal rates (Alzen et al., 2018, Barrasso & Spilios, 2021). The purpose of this qualitative study is to further explore LA perceptions of their participation in an innovative active learning approach in calculus.

The Modeling Practices in Calculus (MPC) model is the innovative active learning in mathematics approach that accompanied this study (Castillo et al., 2020). The MPC model integrates three core elements: cooperative learning, social metacognition, and a culturally appropriate learning environment. This environment is also enhanced as LAs are incorporated into the classroom to support learning with groups. LAs are natural agents of this culturally appropriate model, as their demographics are that of the students, who provide insights and connections from the perspective of a recent participant in the course.

Two pilot focus group interviews, lasting 45 to 60 minutes each, were conducted with 16 new and returning LAs in Spring 2020. In Fall 2020, six focus group interviews were conducted with a total of 14 new LAs and 16 returning LAs. Focus groups were guided by a semi-structured interview protocol based on the aforementioned core elements of the MPC model. Interviews were analyzed using a grounded theory approach (Strauss & Corbin, 1994). The two pilot interviews were used to develop a codebook of perceptions related to LA experiences and perceptions of active learning. The codebook was modified and updated after two rounds of coding (interrater agreement of 71.1%) and was used to code the six Fall 2020 focus group interviews. After codes were subgrouped to exclude unrelated codes a total of 542 level-1 codes were included in the analysis.

The top three main level-1 codes refer to the perceptions that LAs have about the development of their own instructional practice (22%), their experiences with students in the classroom (20%), and the significance of students' conceptual understanding in the learning process (18%). Further discussion and breakdown of these overarching themes, as well as implications and areas for future research, will be presented in the poster.

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An Investigation of Active Learning Impacts on Student Understanding of Infinite Series Convergence

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Keywords: Calculus Education, Problem-Based Learning, Infinite Series, Concept Image

The purpose of this study is to investigate the progression of student understanding throughout a three-week unit about infinite series convergence when active learning techniques are applied in a Calculus II classroom. The research question is: How does a problem-based curriculum implemented in a classroom affect concept images students create when learning about infinite series convergence? Research has been completed before regarding active learning techniques using problem-based learning and student understanding of infinite series convergence; however, the combination of these two areas has not been studied extensively. The poster will include samples from the curriculum designed with a problem-based learning approach, and a description of the research methods that will be applied to better understand student concept images of convergent and divergent sequences and series.

Infinite Series and Curriculum Design

When encountering infinite series and sequences for the first time, students have fundamental understandings about what it means for an infinite series to converge (Ergene & Özdemir, 2020). In contrast to the lecture with guided notes method of course delivery typical at the research university where this study took place, students enrolled in a particular section of Calculus II participated in a three-week study where they engaged in interactive problem sets in the style of the Park City Mathematics Institute program for teachers (see Kerins et al. (2017), for example). During each of the twelve class periods of the study, students worked at their own pace solving problems in small groups. Following some time in small groups, a whole-group discussion was employed to emphasize and formalize key ideas students developed. Focus was given in the problem sets to analyzing geometric series, p-series, and using integrals and comparisons to determine series convergence. Students were also introduced to alternating series and relevant theorems (“tests”) to determine convergence. Every four sessions, students were given a quiz to complete. Additionally, in the final in-class session, students were given a performance assessment regarding fractals to apply their learning.

Methodology

This study will address the research question with a qualitative analysis of student data generated in class and outside class using thematic analysis (Braun & Clarke, 2006) to make sense of students’ concept images of infinite series. After anonymizing and chronologically organizing student work, the researchers will identify changes in student thinking over time. Student work will be triangulated across both formative (problem set data) and summative (quizzes, performance assessment data) assessment, and literature on infinite series conceptualization. A brief summary of preliminary results will be provided on the poster and the researchers will seek feedback on improvements to analysis techniques.

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Concept Maps of Sequences and Series

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Keywords: Sequences, Series, Calculus, Concept Maps

In this poster, we examine student concept maps of sequences and series. By analyzing student concept maps, we hope to gain insight into student thinking about sequences and series. We then compare these maps to an optimal map designed by experts in the hopes of understanding what students know versus what experts expect students to know at the conclusion of a second semester calculus course. In particular, we look to address the research question, "How do students' concept maps of sequences and series differ from an optimal map based on experts' expectations of second semester calculus students?"

Literature Review and Conceptual Framework

Students tend to struggle with the topics of sequences and series (Earls, 2017; Martinez-Planell et al., 2012; Nardi and Iannone 2001). One way of gaining insight into student thinking about a topic is to use concept maps (Williams, 1998; Coutinho Da Silva 2014). Hence, concept maps serve as our framework for analyzing student thinking about and struggles with sequences and series.

Methodology

This study was undertaken during the Spring and Summer of 2021. Participants included six students from a small, private college in a Northeastern urban center. After being provided with examples of concept maps, students were asked to draw their own concept maps of sequences and series. These maps were then compared to an optimal concept map developed by experts through an iterative process. In order to do the comparison between the optimal and student maps, we developed a scoring scheme using Cronin, Dekker, and Dunn (1982) and Bartels (1995). In this study, points were calculated across four categories: concepts, linking words, mathematical definitions, and holistic assessment. Each map was coded by every member of the team and all scores reached a consensus. Scores were then converted to a Likert rating.

Results, Discussion, and Implications

Of the six concept maps, one was missing several key concepts, including tests for convergence, and was considered "poor." Three of the maps showed more concepts than this first map, but still struggled to join concepts with linking words. These maps were considered "average." The last two maps were consistent in meeting experts' expectations for concepts as well as linking words between the concepts and were considered "good." These results suggest that students may not be learning what experts expect them to learn about sequences and series. This has implications to other areas of mathematics and other disciplines. Are students really learning what we expect them to at the end of a course? Perhaps concept maps can be a tool to help determine if students are meeting expectations at the end of a course.

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Undergraduates' Conceptualizations of the Functions and Forms of Mathematical Definitions

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Keywords: mathematical definitions, mathematical practices, sociocultural theory

This research consists of a comparative analysis of undergraduate mathematics and undergraduate humanities majors' conceptualizations of the function and form of mathematical definitions. Analyses provide evidence of two distinct approaches.

This study takes a sociocultural approach to investigating mathematical thinking and learning, conceptualizing mathematical definitions as cultural forms that serve varying cognitive and communicative functions in activity (Saxe, 2012), and mathematical defining – the process of formulating, reasoning about, and refining definitions over time – as a collective cultural practice (Lave, 1993). Data consists of one-on-one semi-structured interviews with 12 mathematics and 12 humanities undergraduate participants.

Functions of Mathematical Definitions

I found differences in participants' responses to the question "What are mathematical definitions usually used for?" as displayed in the left panel of Figure 1. While mathematics majors tended to see definitions as serving to facilitate communication and proof across a disciplinary community (6/12 participants), humanities majors tended not to (1/12 participants). By contrast, the majority of humanities participants (7/12) saw definitions as assisting them to find answers to routine exercises or problems with numerical or algorithmic solutions, while mathematics majors did not (1/12 participants). Both cohorts mentioned the role of mathematical definitions in aiding understanding of novel content (4/12 in each cohort).

Forms of Mathematical Definitions

I found differences in participants' responses to the question "What makes for a good mathematical definition?" as displayed in the right panel of Figure 1. Mathematics majors voiced a preference for mathematical definitions that were comprehensive, precise, or unambiguous (5/12 participants), while humanities majors tended not to (1/12). Humanities majors voiced a preference for definitions that provided procedural instructions for problem solving (5/12), while mathematics majors did not (0/12). Both cohorts voiced a preference for definitions that were readable or understandable (5/12 in each cohort).

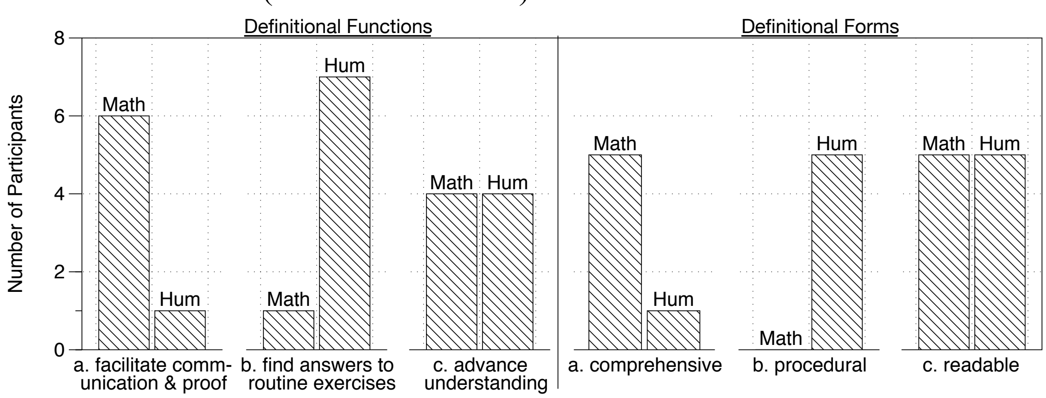

Figure 1. The number of mathematics participants' (n=12) and humanities participants' (n=12) who referenced each function served by mathematical definitions (left panel) and each preferred feature of definitional forms (right panel).

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Secondary Mathematics Teacher Interns' Learning Through Teaching: A Case Study

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Keywords: Secondary Mathematics Teacher Learning; Teaching Practices; Noticing; Reflection

The student teaching internship serves as the capstone experience for many undergraduate mathematics students. Further, student teaching marks an important transitional period in their developing mathematical identity from that of a student to that of teacher. With the secondary mathematics internship serving as the context for this study, this proposal details my ongoing work and initial findings for my dissertation. More specifically, this proposal outlines my efforts to understand the perceived influence that teaching practice has on novice math teachers' teachers' sense of understanding math concepts.

A teacher's mathematical identity is comprised of two interacting and overlapping categories, namely *Aspect-of-Self in Mind* (Shulman, 1986, 1987) and *Aspect-of-Self in Community* (Wenger, 1999)*,* according to Van Zoest & Bohl's (2005) Mathematics Teacher Identity Framework. In this framework, the authors organized a teacher's identity into their "in the mind" perception of self, based on their knowledge, beliefs, commitments, and understandings (Shulman, 1986, 1987), and into their perception of self within communities of practice (Wenger, 1999), based on the teacher's participation and varying dimensions of competence. My project sets out to provide narrative accounts as evidence of the interacting relationship between one's social identity and one's conceptual identity.

This study takes place in the student teaching experience of three secondary mathematics teacher interns throughout the 2021-2022 school year. Their student teaching experience takes place over the course of a full-school year, and they are also concurrently enrolled in university courses. Throughout this project, I took on two roles in relation to these interns; I served as their university field instructor as well as a researcher. This dual role has allowed me the unique opportunity to view their internship with detail, as needed for case studies.

To understand how the interns viewed their own content understanding in relation to their teaching practice, I rely upon case study methods (Yin, 2017), specifically I draw upon the interns' guided reflection journals (Akinbode, 2013; Johns, 2010) as my main data source, as well as observations, field notes, and interviews. Data collection occurred in two phases: a preparatory phase, and two two-week collection periods in the Fall and Spring semesters. During the first phase, I met with the interns as a professional learning community to develop skills for noticing key classroom interactions and reflecting on their practice (van Es et al., 2017). This preparatory work laid the foundation for more in-depth collection periods. Towards the end of their first semester of the internship, they embarked on building a reflective narrative of an instructional unit. Using dialogical movements (Akinbode, 2013; Johns, 2010), the interns recorded daily synopses of their teaching interactions and responded to my follow up questions to begin distilling out themes that they noticed emerging from the texts in order to create their reflective narrative. These narratives serve as my primary data source in getting a sense of their lived teaching experiences and how they viewed the work of teaching as influencing their perception of their own content knowledge.

To conclude this poster presentation, I will present my early dive into analysis of the data. By presenting excerpts from the reflective narratives, I will demonstrate my developing coding process and put forth early claims that will help to answer my research question.

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Math Teacher Technology Self-Efficacy

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Keywords: math teacher, self-efficacy, principal, classroom technology

Leadership goals and a desire to drive change are critical for successful technology integration (Tyler-Wood et al., 2018). In addition, principals play an essential role in supporting teachers' work by supporting professional development efforts (Sterrett & Richardson, 2020). Due to the overwhelming amount of technology resources available, teachers need professional development (PD) that emphasizes the pedagogical use of technology in teaching. Such PD would help them choose the software that best supports instruction in their classroom; this could increase their understanding of the relationship between pedagogy, technology, and the mathematical concept (Getenet, 2017; Koehler & Mishra, 2009).

The purpose of this study was to explore how a teacher's technology self-efficacy influences decisions they make about the use of technology in their classrooms. This study's theoretical framework was based on the combination of social cognitive theory and self-efficacy theory, providing insight into how external and internal factors influence a person's perception of their abilities. A qualitative design examined the factors influencing teachers' technology self-efficacy. A convenience sample of 10 high school math teachers from the five rural high schools in Virginia was selected to participate in the study.

Participants shared that practicing before using new technology with students was a critical experience used to increase their self-efficacy. The participants learned through mastery experiences 47% of the time. The participants did not share receiving any feedback from the administration regarding their use of technology. These findings are conclusive with the findings of Gross and Opalka (2020), who reported a lack of communication of expectations and support from school districts.

The continued reference to practice lends evidence to a need for a change in professional development from presenter-led with the teachers as passive learners to allowing the teachers a more active role in the professional development. Several participants in this study were well-versed in finding quality technology software, which technology best blended the content with pedagogy, and the experiential learning process involved. Therefore, one recommendation of this study is for districts to provide math teachers with specific pedagogical-based technology professional development that involves hands-on learning opportunities for teachers and continued support throughout the school year. In addition, districts would benefit from more intensive and practical training to build on models such as the technological pedagogical content knowledge (TPACK), created by Mishra and Koehler (2006), which explains the skills teachers need to teach a subject and use technology effectively.

The participants did not receive any feedback from the administration about their technology use during the 2020-2021 school year. However, Sterrett and Richardson (2020) claim that principals play an essential role in supporting teachers' work; therefore, it is recommended that principals take a more active part in modeling technology use and providing critical feedback to teachers on their technology use. In addition, principals can strengthen teachers' best practices through relevant, timely, and individualized professional learning opportunities (Sterrett & Richardson, 2020).

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Co-Requisite College Mathematics with Undergraduate Learning Assistant Support: A Pilot

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Keywords: calculus, developmental mathematics, co-requisite

Undergraduate programs in the United States aimed at “developmental,” “foundational,” or “remedial” mathematics are vital to the quantitative reasoning of a well-educated citizenry. Traditionally, these programs have involved a sequence of courses students take, each course serving to prepare them for the next. Often, developmental programs begin with non-credit bearing courses that include mathematical content students were exposed to in their elementary and/or secondary education (or even elementary education in some cases) but for which they have not yet illustrated mastery of to various stakeholders at various moments. Such models have undergone scrutiny as of late both in terms of traditional markers of programmatic success (e.g., success in subsequent courses) as well as in terms of the lived experiences of people who are required to participate in such models (e.g., Larnell (2016)).

Additionally, undergraduate calculus serves as a gatekeeper to many university programs (e.g., engineering, computer science programs) that include it as a pre-requisite for coursework in the major (e.g., Bressoud, Carlson, Mesa, & Rasmussen, 2013). Students who do not place directly into calculus in their first year can be, and often are, unable to complete their required coursework in four years since so much of their program begins after calculus.

The coronavirus pandemic challenged many paradigms in education and new support structures are needed. Administrators and educators are concerned with the academic progress made by students during the coronavirus pandemic, specifically those who were studying secondary mathematics in Spring 2020. At [blinded], administrators and faculty designed a university-wide undergraduate learning assistant program aimed at helping provide student support in large-lecture classes.

In this poster, we share details of a pilot co-requisite program in the Department of Mathematics and Statistics across multiple campuses of Washington State University in Fall 2021. There are two elements of the program (1) students who would have been placed in MATH 100 (Basic Mathematics) were offered the opportunity to also enroll in MATH 103 (College Algebra) simultaneously and (2) students who would have been placed in a pre-requisite course for MATH 171 (Calculus I) were offered the opportunity to enroll in calculus along with a one-credit hour support course (MATH 110). Each program is being supported by several undergraduate learning assistants.

Initial results from program evaluation indicate the students in the co-requisite pilots were performing at approximately the same levels in traditional course success markers (e.g., exams, quizzes, homework). Data will be collected at the conclusion of the semester and include weekly projection and reflection discussion posts, final course grades in co-requisite courses, and survey results regarding their experiences. Specifically, we investigate the self-efficacy, belongingness, mindset, and metacognitive actions of students in the co-requisite pilot compared with students from the general population.

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Undergraduate Students' Use of Everyday Language to Make Sense of Indirect Proof

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Keywords: Pre-service Teachers, Indirect Proof, Everyday Language & Mathematical Language

The formal language of mathematical logic, which is precise and eschewing ambiguity, is often at odds with informal, everyday language. Despite this fundamental difference, the two are intimately connected. Indeed, one can think of mathematical logic as a subset of everyday language: the terms and rules used in mathematics are part of everyday language, but everyday language contains additional rules and conventions (Epp, 1999). Furthermore, both instructors and students often use a combination of everyday language and mathematical language when discussing mathematical concepts. But the switching between mathematical language and everyday language, if not done with care, can muddy students' understanding of these concepts.

Our study explores the relationship between mathematical language and everyday language, as used by prospective secondary teachers (PSTs) in the context of indirect proof by examining how they describe contraposition equivalence, negation in proof by contradiction, and the converse. Despite these concepts being fundamental to proving, prior research on how students come to learn and make sense of these concepts is sparse (Antonini & Mariotti, 2008; Thompson, 1996; Stylianides et al., 2004). Some recent studies examined understanding of contrapositive by 8th graders (Yopp, 2017) and of proof by contradiction by undergraduates (Brown, 2018) but only few studies focused on PSTs (Buchbinder & McCrone, 2019).

We analyzed responses to a Mathematical Knowledge for Teaching Proof questionnaire of 35 PSTs enrolled in a capstone course Mathematical Reasoning and Proof for Secondary Teachers (Buchbinder & McCrone, 2020); they took the questionnaire twice: once at the beginning of the semester and once at the end of the semester. We used grounded theory and constant comparison method (Strauss & Corbin, 1994) to analyze participants' written justifications to analyze a set of items dealing with proof by contradiction, converse and contrapositive equivalence. Specifically, we focused on how participants used certain words in everyday language, such as *opposite* or *similar*, to describe logical operations and logical forms. For example, despite correctly identifying the contrapositive to a given statement, a participant justified their equivalence by saying "because it's the opposite". When used solely as a stand-in for negation, the use of the word "opposite" was relatively harmless; but when used in a context such as the reordering of the antecedent and consequent to create the converse, more incorrect answers emerged. The use of everyday language declined overall between pre- and post-questionnaire, as the PSTs adopted more precise mathematical language to describe different concepts relating to indirect proof.

Our study contributes to the literature on the relationship between mathematical logic and everyday language by identifying particular patterns, both correct and problematic, in the PSTs' use of language related to indirect proof. Becoming aware of these patterns can help instructors of proof courses to reflect on their own use of everyday language and to help undergraduates in developing more sophisticated mathematical language.

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The Efficacy of the Flipped Classroom Technique in Undergraduate Mathematics Education: A Review of the Research

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Keywords: Flipped Classroom Technique, Research Review

The flipped classroom technique has recently been a focus of attention for many math instructors and pedagogical researchers. Although research on the subject has greatly increased in recent years, it is still debated whether the flipped classroom technique can significantly increase the overall success of students in undergraduate math courses. While there have been meta-analyses that consider the efficacy of the technique across university disciplines and within other STEM fields, there has not yet been a systematic review within undergraduate math education. By analyzing the existing research and compiling the quantitative and qualitative data, this project examines the efficacy of the flipped classroom technique in undergraduate math courses, ranging from introductory calculus to transition-to-proof courses, in regards to students' performance in the classes, perceptions of the technique, and associated self-efficacy. This project also introduces the current use of the technique and covers successful implementation methods. Additionally, it highlights the flipped classroom technique's potential for improving retention of members of underrepresented groups in math by increasing their sense of belonging as well as discusses the efficacy of the method in early proof-based courses in regards to students' acquisition of sociomathematical norms.

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Focusing on Multiplicative Foundations Essential for Calculus

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Keywords: Multiplicative Structure, Calculus, Differentiation, Integration

Reviews of teaching and learning calculus have concentrated on co-variation, functions, limits, differentiation, and integration (e.g., Frank & Thompson, 2021; Larsen, Marrongelle, Bressoud, & Graham, 2017). These foci are central to the subject but overlook a further potential source of difficulty—students' understanding of multiplication and division with quantities. Such oversight is consequential because (a) multiplication and division are foundational for differentiation, integration, and various families of functions central to most calculus courses and (b) reports exist (e.g., Izsák, Beckmann, & Stark, 2021) in which college students found it effortful to explain how multiplication and division fit diverse problem situations.

I report on an innovative 1-semester calculus course offered at a selective university in the United States. None of the 18 students was majoring in a STEM field. Yet, all but 3 had completed at least one calculus course (e.g., AB calculus in high school, Calculus 1 at university, or similar). The course had three main features. First, the course began with an examination of multiplication understood in terms of measurement. Students worked to explain how measurement interpretations of multiplication equations fit situations involving both isomorphisms of measure spaces and products of measure spaces (Vergnaud, 1983). Second, the course made no mention of limits and, instead, relied on situations that could be modeled by linear, piecewise linear, and step functions. This allowed students to focus on multiplicative structure in the context of functions used to approximate derivatives and integrals more generally. Third, other than introducing a measurement perspective on multiplication equations, the course introduced no formulae: Students derived versions of central calculus results by coordinating natural language with measurement expressed in both equations and drawings.

Results: (a) students demonstrated a range of facility explaining how multiplication and division fit diverse situations, suggesting the course provided appropriate challenge, (b) students were able to reason about multiplicative structure and drawings to understand slope in terms of measurement (how many of the horizontal length make the vertical length) instead of as a unit rate (so many vertical units for every one horizontal unit), (c) students built upon this measurement understanding of slope to derive the point-slope formula, a version of the chain rule, and a version of integration by substitution. In particular, students were able to reason about multiplicative structures in ways that support the chain rule (e.g., understanding why $g'(x)$ appears in $f'(g(x)) \cdot g'(x)$) and integration by substitution (e.g., understanding why inversely proportional relationships applied to areas of rectangles can justify this result). At the same time, their memory of calculus was often vague, and most did not see connections to prior courses immediately. They were, however, able to see connections with assistance from me. These results suggest that college students can learn to reason about multiplicative structures to explain core calculus concepts, which is in contrast to simply remembering formulae and rules. Important next steps for future offerings including making further connections to calculus—for instance, by using piece-wise linear and step functions to approximate derivatives and integrals of more complex functions—and examining whether a similar course would provide appropriate challenges for STEM majors.

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Different Types of Mathematizing as Captured by a Novel Script Writing Activity

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In this report, I describe how student-reconstructed dialogues that capture problem solving processes extend the applications of scripting tasks. Analysis of these dialogues reveals how mathematizing can play a role in the symbolizing, algorithmatizing, and defining activities of students as they confront unfamiliar mathematical constructions—in this case, star polygons. For participants in this study, active teachers enrolled in graduate coursework, mathematization was sometimes supplanted by the use of technology.

Keywords: Scripting Task, Mathematizing, Teacher Education, Technology, Guided Reinvention

Script writing tasks have been used in a variety of research and didactical contexts: in constructing and clarifying proofs (Zazkis, 2014), in anticipating and planning for student-teacher interactions in mathematics classrooms (Zazkis, Sinclair, & Liljedahl, 2013), and in probing mathematical understandings (Zazkis & Zazkis, 2014). Adding to these uses, the script writing task that forms the foundation of this research project asked participants to capture their mathematical exploration, refutation, and discovery in a problem-solving script. Analysis of these scripts allowed us to answer the following research question: *What types of mathematization facilitate the problem solving work of students, and in what way do they do so?*

As described in Gravemeijer & Doorman (1999), mathematization is often both a necessary prerequisite for and a byproduct of the type of guided reinvention of concepts that plays a central role in realistic mathematics education (RME). Rasmussen et al. (2005) use excerpts from teaching experiments grounded in RME to illustrate different types of mathematizing; namely, how there exists both *horizontal* and *vertical* mathematizing, and that both types of mathematizing appear across different manifestations of students' mathematical activity. Rasmussen et al. attend to mathematizing in the context of symbolizing, algorithmatizing, and defining; I consider these activities as well as conjecturing and justifying.

The scripting task in this report was completed by a population of practicing mathematics teachers enrolled in a graduate-level mathematics education course. Participants first generated an empirical definition of a *star polygon* by characterizing differences between different types of self-intersecting polygons. Then, they answered the following questions:

- Does there exist a star polygon on N vertices?
- How many different star polygons on N vertices are there? What do you consider "different"?

Participants' scripts were analyzed for instances of both horizontal and vertical mathematization as well as how students leveraged symbolizing, algorithmatizing, defining, conjecturing, and justifying while exploring the properties of star polygons.

In this poster, I will share instances of each type of mathematization and how they facilitated participants' problem-solving efforts. I will also explore how mathematization may have been inhibited when participants relied on technological applets to produce diagrams of star polygons. Finally, I consider implications of the star polygon activity as an experientially real starting point for an abstract algebra lesson that leverages RME principals to help students reinvent ideas of finite cyclic groups and their subgroups (cf. Larsen, 2013; Larsen & Lockwood, 2013).

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Exploring Student Generalizations About 2x2 Determinants from using a GeoGebra Applet

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Keywords: Linear Algebra, Determinants, Generalization, Inquiry-Oriented

Determinants are often presented in a formulaic way that obscures their rich connections to graphical interpretations of linear transformations. The Inquiry-Oriented Linear Algebra (IOLA) curricular materials (Wawro et al., 2013) build from a set of experientially real tasks that allow for active student engagement in the guided reinvention of key mathematical ideas through student and instructor inquiry (Gravemeijer, 1999). The IOLA determinants task sequence (NSF DUE #1914793, 1914841, and 1915156) uses distortion of space as an experientially real starting point. Students build a conceptualization of matrix determinant as a measure of (signed) multiplicative change in the area and discover its formula for a general 2 x 2 matrix. The sequence utilizes GeoGebra applets that allow students to actively explore the geometric effects of changing 2x2 and 3x3 matrix transformations and note their relationship with the determinant. Students make conjectures that link concepts such as linear independence, inverses, and column operations to changes in the determinant.

In this poster, we explore two research questions: (1) What observations and generalizations about 2x2 determinants do students make from exploring a GeoGebra applet? (2) What is the nature of the students' generalizations? To answer the second research question, we analyze the forms and types of students' generalizations according to Ellis et al.'s (2021) Relating-Forming-Extending (RFE) Framework, which defines generalizing as "identifying commonality, deriving broader results from particular cases to form general relationships, rules, concepts, or connections, or extending one's reasoning beyond the range in which it originated." (p. 9).

Data were collected from two sections of an online, synchronous IOLA course taught by the same instructor; class sessions included whole-class discussion and small-group breakout rooms. The new unit leverages a Geogebra applet in which students change the four entries of a 2x2 matrix and two vectors that define a parallelogram in the domain. The applet displays a real-time mapping of the vectors and parallelogram under the matrix transformation as well as the matrix's determinant. In the post-class discussion board, students individually explained at least one observation or conjecture that their group made while working on the applet and one thing they wondered. Thirty discussion board postings were coded individually by the research team using the RFE framework (2021) before discussing codes together.

Thus far in our analysis, we have found that students generalized relationships between the matrix entries and geometric properties of the associated linear transformation, ratio between the pre-image and image areas, or determinant. These observations and generalizations largely fell into the larger *Forming* and *Extending* categories of the framework. For example, a student posted "One conjecture my group made from the applet was that if the determinant is 0, the image was squished onto a line." This was coded as *Forming* under *Identifying a Regularity: Extracted*, because this student noticed a pattern connecting the value of the determinant with the geometric transformation output. The poster will include analysis of the complete data set and will extract themes in students' generalizations about determinants.

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Authority Manifestations in an Introduction to Proof Course

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Keywords: Authority, Proof, Inquiry-Based

Authority in a classroom becomes shared when participants of a community beyond a textbook or instructor present valid mathematical ideas that hold authority in their context (Amit & Fried, 2005; Gerson & Bateman, 2010; Langer-Osuna, 2016). Mathematical proof presents an opportunity for shared authority when participants in a community communicate their ideas and develop an argument for the validity of a statement (Burton, 1998; Inglis & Mejia-Ramos, 2009; Weber et al., 2014). When mathematicians participate in mathematical proving activities, they often focus on participating in communities and build upon definitions to validate a statement (Burton, 1998). We are interested in what authority sources students rely on to validate a mathematical claim when authority becomes shared in an introduction to proof classroom. Further, we are interested in comparing the practices of students in this shared setting to how mathematicians practice mathematical proving. To answer this question, we have the following research question. (Bleiler-Baxter et al., under review)

1. What are the sources of authority manifested within small-group conversations related to proof construction? (i.e., *Upon what do students base their decisions?*)

To answer this question, we recorded video data of students in small-group settings as they worked together to prove tasks in an introduction to proof course. Students were placed in groups of 3-4 to prove conjectures in 9 different proving episodes. After working collaboratively on their proofs, student groups presented their proofs to their peers for feedback. We then as a research team looked over the 9 episodes holistically to examine themes that emerged for authority sources of students in this shared environment. This was accomplished by looking for times in the episodes when students made decisions as a group to move their proof forward before presenting their group's proof to their peers.

Our results produced four themes of authority attended to by students: (1) The course-developed rubric for proof writing as authority, (2) Peers' confidence (and the need to produce a product) as authority, (3) Form and symbols as authority and (4) Logical structure and mathematical definition as authority.

This poster will provide excerpts from group-proving discussions that highlight the themes developed from our inductive analysis. Then we will make connections between students' sources of authority and mathematicians' sources of authority, informed by previous empirical work on mathematicians' practice (e.g., Burton, 1998).

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Comparing the Mathematical Knowledge for Teaching Geometry of Preservice and Inservice Secondary Teachers

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Keywords: Mathematical knowledge for teaching, Measurement invariance, Pre-service teachers, Teaching geometry

In this poster presentation, we share what our research team has learned by collecting responses from Geometry for Teachers (GeT) students who have taken a mathematical knowledge for teaching geometry (MKT-G) assessment before and after taking the GeT course. By following the definitions of mathematical knowledge for teaching from Ball, Thames, and Phelps (2008), Herbst and colleagues developed an instrument to measure MKT-G called for in the mathematical work involved in tasks of teaching nested in different instructional situations. We used a unidimensional item factor model with 17 items selected from the instrument to understand the participating GeT students' (preservice teachers) MKT-G growth over the duration of the course. GeT students' MKT-G scores were estimated using a distribution of in-service teachers' MKT-G scores. Specifically, to estimate the growth in GeT students' MKT-G scores using their pre-test and post-test MKT-G scores and interpret the growth in the scale of inservice teachers' MKT-G, we first tested multiple-group measurement invariance between the group of GeT students and inservice teachers. After confirming that the same knowledge construct is being assessed across the groups, the averages of 435 GeT students' pre-test and post-test scores were estimated relative to the inservice teachers by setting the average of 405 inservice teachers' scores to be referenced as zero. Next, the GeT students' growth in MKT-G was estimated by the difference between the estimated post-test and pre-test scores.

Four main results emerge: (1) On average, GeT students score about 0.23 standard deviation units higher on the MKT-G test after completing the Geometry for Teachers course. (2) GeT students taking the MKT-G test score about 0.98 standard deviation below inservice teachers (with an average of 14.2 years of mathematics teaching) that took the same test, on average. (3) The analyses showed that partial measurement invariance was attainable between the groups, meaning that the relationships of the items to the measured knowledge were equivalent between preservice (GeT students) and inservice teachers. (4) the growth in GeT students' MKT-G is equivalent to the difference in MKT-G scores between teachers who differ 3 years in experience teaching geometry.

This study shows the positive association between the college geometry courses designed for future teachers and the mathematical knowledge for teaching geometry in terms of the growth in the knowledge of the students who took the courses. Also, this study contributes to the methodological approach measuring knowledge gains of one teacher population (e.g., preservice teachers) in terms of a scale from a different teacher population (e.g., inservice teachers).

Acknowledgments

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Prospective Teachers' Knowledge of Students' Understanding Concept of Area

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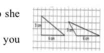
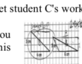
Student struggles are not dead ends, they are foundations for the inquiry into mathematical concepts (National Council of Teachers of Mathematics [NCTM], 2000). In the literature, most studies have focused on numbers and operations, and have generally overlooked geometry and measurement; specifically, area (Ball et al., 2001). In this qualitative study, seven prospective teachers were given three tasks (see about the area of a triangle. The purpose was to understand how prospective teachers interpret the given elementary and middle grades students' answers.

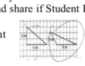
Task 1: Please read and answer the problem. Problem: Anna has 2 corners of her yard that she could use for a garden, so she needs to decide which one has the larger area. Which one is bigger? How do you know? Please share your answer.



Task 2: Could you please interpret students' (A-E) work and share if they have any struggle/ misconception? How could you help with their struggles/misconceptions?
Student A: "The first one is bigger because base times height divided by two gets the answer". Could you please interpret student A's work and share if Student A has any struggle/ misconception? How could you help with this struggle/misconception?
Student B: "They are same because you do (3x6)/2= A and since you have a triangle that is half a rectangle or square and that is why you divide it in half". Could you please interpret student B's work and share if Student B has any struggle/misconception? How could you help with this struggle/misconception?
Student C: "They are the same because you have to do b x h x 1/2 and 6x3= 18 and 18x 1/2= 9". Could you please interpret student C's work and share if Student C has any struggle/misconception? How could you help with this struggle/misconception?
Student D: "You make the triangles into rectangles then divide it by 2". Student D also drew the picture below. Could you please interpret student D's work and share if Student D has any struggle/misconception? How could you help with this struggle/misconception?



Student E: "I think triangles are the same area but of them is stretched out and one is its real size. I know this because they both have the area of 18. One side of each is 6 cm and the other is 3 cm". Could you please interpret student E's work and share if Student E has any struggle/misconception? How could you help with this struggle/misconception?
Student F: "It has more room and much wide". Student F also drew the picture below. Could you please interpret student F's work and share if Student F has any struggle/misconception? How could you help with this struggle/misconception?



Task 3: Could you please share if this activity could be helpful for a prospective teacher and why?

*Figure 1. Area of Triangle Task for Prospective Teachers (Adapted from Kim. 2002)**

Findings revealed that only 14% of prospective teachers responded to task 1 correctly. For task 2, although only 14% of prospective teachers correctly interpreted the solutions of students A, B, C, and E, 29% of prospective teachers correctly interpreted Student D's solution and 43% of prospective teachers correctly interpreted Student F's solution. For task 3, prospective teachers mainly stated that seeing multiple solutions for the same problem from multiple students and interpreting them is very helpful. Results showed that there is a need to support prospective teachers' understanding of deep fundamental content and concepts of the disciplines that they will teach. One possible way to do this could be via making content accessible through the courses in mathematics with methods courses and field experience. If prospective teachers can learn the content of geometry with problem-solving, connect the content courses with their interests, practice teaching in various diverse school settings, they will be more ready to tackle the challenges that their students may face in the future. Having a solid knowledge of students is an essential component of good teaching (Wilson et al., 2005). Students' experiences can be developed through their teachers' teaching in a classroom (Martin, 2006). However, prospective teachers do not have much opportunity to accumulate knowledge of students from their teaching. Although teachers develop their understanding of their students' geometrical thinking through the years, novice teachers may not have such a background. That is why more opportunities to build knowledge of students' geometrical learning and thinking should be provided to prospective teachers. For example, providing more opportunities to learn about learning theories (e.g., Van Hiele, 1986) that can help them understand students' thinking and struggles based on their developmental stages of knowledge). Prospective teachers can also develop their geometrical knowledge of and knowledge of students' geometrical thinking through their coursework with tasks that engage them in analyzing students' thinking. Due to this kind of learning opportunity, prospective teachers can relate their knowledge to their future teaching. More research studies investigating teacher education programs exploring prospective teachers' responses to students' struggles with the use of interviews, observations, and other data collection tools are needed. Research examining how prospective teachers respond to students' errors would also allow researchers to understand prospective teachers' ways of thinking. Research studies exploring prospective teachers' interpretations and responses to students' struggles will enhance communication among various stakeholders such as teacher educators, professional development experts, teachers, and policymakers.

Two readable tables including prospective teachers' responses and tasks items will be provided clearly on the poster presentation for audience.

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An Analysis of Eleven Department Change Initiatives

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Keywords: Active learning, Departmental change, Calculus

Mathematics departments across the country are working to innovate and improve their introductory precalculus and calculus courses. There is an emerging body of research to which we aim to contribute that is focusing on department change efforts to improve student success in these gatekeeper courses (Reinholz et al, 2020). The analysis reported here is part of a larger study, Student Engagement in Mathematics through an Institutional Network for Active Learning (SEMINAL) (Smith et al., 2021). This larger project included longitudinal investigation of nine departments that received financial and network support to implement active learning per their self-designed change initiatives. These nine departments were selected from 37 proposals to join the SEMINAL project. Of the 29 unsupported proposals, we identified 11 departments that were able to follow up in varying degrees on their proposed change initiatives. This report focuses on these 11 departments with the following research question: What characterizes these 11 change initiatives and what levers and constraints did these change initiatives use or encounter? In addition to providing insight into the change process, this report also lays the groundwork for subsequent analyses that will compare and contrast the 11 stories of change to the strategies, successes, and challenges of the nine departments that were involved with the SEMINAL network.

To learn about their change implementation efforts, we conducted interviews with key personnel at each of the three sites. Interviews were transcribed and summary reports were sent to each site for member checking purposes. We used the four frames model (structures, symbols, people, and power; Reinholz & Apkarian, 2018) as an a priori coding scheme to analyze the culture and subsequent change strategies in each department. The features of each strategy were further organized according to the following Four Categories of Change Strategies: Individual/Prescribed, Individual/Emergent, Environments and Structures/Prescribed, Environments and Structures/Emergent (Henderson et al., 2012). Results revealed that individual mathematics departments may rely on multiple approaches to change and that their desired strategy may not be aligned with their current strategy due to various constraints. For example, the change strategy at Blue State is partitioned into two approaches, one executed through the Math Learning Center (MLC) and another through the mathematics department. In the MLC change efforts can be classified as an Individual/Prescriptive approach, with the coordinator overseeing exam-writing and providing required activities for those teaching courses under his purview. The approach in the math department can be described as an Individual/Emergent approach with the coordinator leading a group of instructors in regular meetings to develop shared course materials. Through the lens of the four frames both approaches are supported by the structure of coordination, but the enactment of this structure varies due to differences in the people, power, and symbols. The poster presentation will provide descriptive accounts and implications of the varying change strategies in each department. This is particularly significant for mathematics departments looking to support the implementation of active learning.

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(SEMINAL, DUE-1624639)

Exploring the Interactional Dynamics of Undergraduate Students' First Mathematics Advising Experiences

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We examined the advising practices for incoming, first year undergraduate students by a mathematics department. Our analysis indicates that gender influences the interactional dynamics in ways that can have a longstanding impact on students' mathematical futures.

Keywords: Gender, Advising, Women, Interactional Analysis

This poster reports on preliminary findings from a study of the advising practices for incoming first year undergraduate students by a mathematics department at a large public university. The primary objective of this analysis is to explore the mediating role of gender in mathematics advising sessions, and implications for equity-oriented advising practices are considered as a result of the findings.

A variety of factors have been theorized to result in attrition of STEM-interested and STEM-capable women at various points along the pipeline, some of which are during the college application process, during or after the first year of undergraduate courses, or after graduating with a STEM degree (Blickenstaff, 2005; Chen, 2013). While many of these efforts have focused on factors that impact the decision-making of individual women over the course of weeks, months, and even years, this study focuses on a much smaller scale—an event that occurs in minutes between two people—as a uniquely critical site of attrition for STEM-interested women.

The mathematics advising sessions we examine take place over a very short timescale (five to ten minutes) but have significant implications for the mathematical trajectories of the students involved. In this brief interaction, the advisor needs to gather relevant information from the student, such as their mathematical background and academic or career goals, provide information on the available courses, and arrive at a consensus with the student about what course is most appropriate. The student's main responsibility is to share the relevant aspects of their background and gather information about the available courses from the advisor. Although this seems like a relatively straightforward interaction, all the advisor has to go on is the student's *performance* of their mathematical background and goals. Both the advisor and the student use their perceptions of the other to inform how they engage in the encounter through verbal and nonverbal communication.

We seek to understand how particular advising structures and practices result in advising sessions where gendered discourses influence (or do not influence) the placement recommendations for students. In this study, we analyze cases in which a pair of students enters the advising context with fairly similar qualifications (i.e., AP scores, placement test scores, academic and career goals) but end up with significantly different course recommendations. Preliminary analysis suggests that course recommendations are influenced by a set of gendered interactional dynamics (i.e., talk, gesture, and body comportment) that strongly influence narratives of mathematical competence and, consequentially, mathematics course recommendations.

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Student Grade Trajectories through the Precalculus, Calculus 1, and Calculus 2 Sequence

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Keywords: Calculus, Pathways, Student success

The Precalculus, Calculus 1, and Calculus 2 sequence (P2C2) remains a gatekeeper into STEM, effectively discouraging students from pursuing STEM majors (Seymour & Hunter, 2019). Working toward a more inclusive STEM educational system necessitates understanding STEM systems as they currently exist. This work investigates the transcript grade trajectories of undergraduates as they navigate the P2C2 sequence using data drawn from the Progress through Calculus project. In the analysis presented on this poster, we investigate the research question: What are the grade trajectories of students who start with Precalculus in the P2C2 pathway? This question then sets the stage for analyses on grade trajectory as stratified by demographic markers.

We conducted this analysis by creating a Sankey graph (Figure 1) using R software to visualize student grades across the P2C2 trajectory for 1171 unique students from eight universities across the U.S. Grades are stratified into three categories: “A/B”, “C”, and “DFW.” The Sankey graphs allow us a powerful visual with which to analyze patterns across courses (e.g., how many students passed Calculus 1 with an A or B out of those that passed Precalculus with an A or B, what proportion of students who started in Precalculus passed P2C2 after three terms).

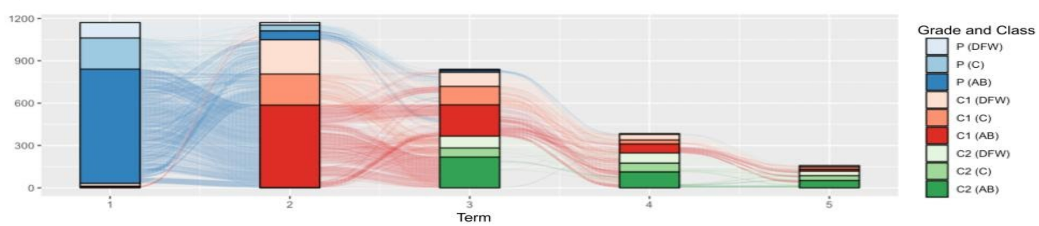

Figure 1. Flow of grade trajectories beginning from Precalculus (in shades of blue labelled P), to Calculus I (in shades of red labelled C1), to Calculus II (in shades of green labelled C2). Each stack represents one term, and each flow line is a student.

This analysis resulted in a detailed visual depiction of how students are moving through P2C2 in terms of grade earned. For example, of the 809 students that received an A or B in Precalculus in their first term, 522 (65%) also received an A or B in Calculus 1 after one term. Of the 449 students who completed the P2C2 sequence within three terms, 202 (45%) of those who received an A or B in Precalculus received an A or B in Calculus 2. However, of the 221 students that received a C in Precalculus during their first term in the sequence, only 26% received an A or B in Calculus 1 in their second term. We will also present pathways as disaggregated by gender and ethnicity. This data could be used to inform development and specificity for various courses in the P2C2 sequence, as well as identify places within the sequence that warrant mathematical intervention.

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Exploring Students' Problem Posing Abilities and Difficulties in Differential Equations

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This exploratory study investigated an undergraduate student's problem posing abilities and difficulties when tasked with posing a real-world problem that can be modeled using a first-order linear initial value ordinary differential equation. A task-based interview was conducted with the student. Analysis of verbal responses and work written by the student revealed that while he was familiar with solving real-world problems that involve initial value first-order linear ordinary differential equations, posing one was problematic. Recommendations for instruction are discussed.

Keywords: Problem posing, problem-posing tasks, differential equations, initial value problems, student thinking

Problem-posing tasks (hereafter, PPTs) have many benefits in mathematics education, including developing and strengthening students' critical thinking skills (Nixon-Ponder, 1995), developing students' understanding of mathematical ideas (cf. Cai & Hwang, 2002; English et al., 2005), and gauging students' understanding of mathematical ideas when used as assessment tools (cf. Arikan & Ünal, 2015; Stoyanova, 2003). Evidence from research shows that there is a paucity of research that has examined student thinking about PPTs at the undergraduate level (cf. Ghasempour et al., 2012; Nedaei et al., 2019, 2021). Contributing towards narrowing this knowledge gap, we conducted a task-based interview (Goldin, 2000) with a student (pseudonym Jude) who had previously completed an ordinary differential equations course. The interview was based on the following PPT:

Give an example of a real-world problem that can be modeled using the initial-value problem:

$$\frac{dy}{dt} = 80 - \frac{45y}{2000-5t},\ y(0) = 100.$$

The following research question guided this study: What mathematical abilities and difficulties do students exhibit when tasked with posing problems that can be modeled using first-order linear initial value ordinary differential equations?

Among other things, findings of this study provide an insight on the importance of using PPTs in the teaching and learning of mathematics at the undergraduate level. While reasoning about the PPT, Jude claimed that working with the PPT forced him to think differently. In fact, Jude even posited that "if you can go backwards like this to make the situation [i.e., generate a PPT], then you truly understand the material," suggesting that PPTs could be used as assessment tools of students' understanding of mathematical ideas, an observation that has been made by some researchers (cf. Arikan & Ünal, 2015; Stoyanova, 2003). At another time, Jude expressed a positive attitude towards PPTs when he said creating a PPT was "very interesting" for him. Overall, Jude struggled to generate a PPT-this is not surprising as the student had not been previously exposed to working with PPTs during course lectures or on homework assignments. Given the numerous benefits of using PPTs during classroom instruction such as developing students' understanding of mathematical ideas (cf. Cai & Hwang, 2002; English et al., 2005), we recommend the inclusion of PPTs in the teaching of ordinary differential equations, and more broadly, in the teaching and learning of undergraduate mathematics.

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Examining Success in Mathematics Course-Sequences: An Exploratory Statistical Analysis

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Keywords: Mathematics, Course-Taking, Course-Sequences, Prerequisites, Curriculum

Success in first-year mathematics courses is commonly measured by higher passing rates and lower course DFW rates. Yet, mathematics courses are designed to be taken in a particular sequence to ensure student preparation and to enhance student success in requisite courses. In most mathematics programs, students select a mathematics course-sequences dependent upon their educational and career goals, such as Liberal Arts, Business, and STEM majors. How can we analyze and quantify student success in a sequence of mathematics courses? How many paths do students take through sequences of mathematics courses? How might the analysis of student success in a sequence of mathematics courses aid in the identification of courses that serve as a barrier to student success? How might it also aid in targeting interventions to support student success?

This research examined 10 years of de-identified registrar data (Fall 2010 through Spring 2020) regarding 100-level mathematics course-taking at a public university in a Western state. This data set included 86,840 data points for 26,129 individual students.

The open-source programming package, R, was used to distill the data down to unique paths through 100-level mathematics course-sequences. A path is different from a course-sequence in that it includes repeated courses (Figure 1) or divergences from the typical course-sequence (Figure 2). The most common paths were identified and the percent of students successfully completing each path with a grade of C- or higher was calculated.

College Algebra → Precalculus 1 → Precalculus 1 → Precalculus 2 → Calculus 1

Figure 1. Path illustrating repeated course taking in a course sequence.

College Algebra → Quantitative Reasoning → College Algebra → Precalculus 1

Figure 2. Path illustrating divergent course taking in a course sequence.

Preliminary Findings and Future Analysis

The initial data analysis produced 2193 unique mathematics course-taking paths. That result posed a challenge because 1330 of the unique paths had a sample size of one student. Thus, we have begun analyzing data based on path clusters, a set of paths defined by a common starting-course and ending-course. The most common clusters of paths will be identified. The percent of students successfully completing the ending-course of a cluster with a grade of C- or higher will be calculated. In addition, the variance of the length of the paths within the cluster will be calculated to identify course-sequences with higher numbers of repeated courses. Analysis is ongoing and further results may be shared.

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The Professional Identity Development of Mathematics Teaching Assistants

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Keywords: Teaching Assistants (TAs), Professional Identity, Roles, Perceptions

Mathematics Teaching Assistants (TAs) play a crucial role in undergraduate instruction but there is still limited work examining their professional identity (Clark et al., 2013) development. Professional identities – including self-beliefs and perceptions of one's roles – directly shape one's instructional practices (Sachs, 2005). The development of professional identity is a complex, dynamic process that is heavily influenced by contextual, social interactions. The present study examines the professional identity development of mathematics TAs as it relates to students, faculty, and other TAs. The research question is: How do mathematics TAs describe their professional identities relative to students, faculty, and other TAs?

Framing

We draw on the perspective that professional identity development in higher education is a social and contextual process (Clarke et al., 2013). Professional identities of mathematics TAs are the product of their prior knowledge, experiences, university contexts, and relationships with students, faculty, and other TAs. More specifically, the conceptions and expectations that these individuals have for TAs shape their professional identities (Schepens et al., 2009).

Method

The study was enacted within a larger research project examining the experiences of transfer mathematics students at a Minority-Serving Institution. Purposive sampling (Miles et al., 2020) was used to recruit five PhD students in the mathematics department who served as TAs for an introductory proof course. We open-coded (Miles et al., 2020) their responses to semi-structured interviews (Rubin & Rubin, 2011) about their experiences and perceptions of their roles and identities as TAs. Then, we wrote memos to identify the common themes in their identities.

Findings

TAs often described their professional identities relative to the students, faculty, and other TAs. The most common identity described was that of *Content Deliverers* – directly related to the perception of their roles as bridging students and faculty. They described needing to present material as "accessible and interesting." The TAs considered themselves as the students' first point of contact for mathematics content and general support. Additionally, TAs described being prompted to be *Mentors* and *Encouragers* by students as they were asked to provide academic and career advice, as well as emotional support. TAs embodied the professional identity of *Assistants,* which was directly related to their job title and perception of their position in relation to the faculty. TAs mentioned attending to the expectations of the faculty, "doing some of the grunt work like grading", and overall "being a solid helper."

Discussion and Conclusion

The TAs' professional identities were closely tied to their perceptions of their position within the network of students, faculty, and TAs. A deeper understanding of the development of mathematics TAs' professional identities can better inform the training and support they receive.

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TA Coaches in the Mathematical Sciences: Exploring Their Roles and Their Perceptions

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Keywords: TA Coach, Peer Instructional Coach, Graduate Teaching Assistants

In recent years, the focus on practice-based, inclusive pedagogy in mathematical sciences has increased as research demonstrated that active learning pedagogy in STEM classrooms narrows achievement gaps (Freeman et al., 2014; Theobald et al., 2020). In response, instruction-related training programs for graduate teaching assistants (GTAs) now emphasize classroom pedagogical techniques that graduate students themselves may not have experienced during their own undergraduate mathematical training. Peer mentoring programs support new GTAs as they move from learning about pedagogy in their teaching training seminar to their first undergraduate mathematics classroom experiences. These peer-relationships, designed as weekly or biweekly interactions between mentors and mentees, develop across the span of an academic semester or year (Lorenzetti et al., 2019). While the mentor-mentee conversations facilitate support and resource sharing (Browne-Ferrigno & Muth, 2012) and provide mentees with constructive criticism after classroom observations (Yee & Rogers, 2017), mentors are not necessarily tasked with being instructional coaches to assist with entire lesson plan development and implementation, like K-12 instructional coaches. Similar to the paired teaching approach to help new faculty adopt active learning strategies (Strubbe et al., 2019), a peer instructional TA coach (hereafter "TA Coach") is a unique element of a graduate peer mentoring program.

This poster focuses on the experiences of TA Coaches in a comprehensive graduate teaching assistant training program in mathematical sciences that was designed and refined at one institution and is being replicated at two peer institutions. During program development, TA coaches were tasked with working with GTAs teaching recitation sections of college algebra and calculus I to facilitate active learning pedagogy and were asked to free-form journal about their experience. At the two institutions replicating the program, the duties changed to support the structure and needs of each department. Recent TA coaches at the three institutions participated in interviews about their experiences. This poster summarizes the roles of the TA Coaches across the three universities and explores their perceptions of the unique benefits that the TA coach role provides to the GTAs they assist and their own instructional experiences.

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A Local Instruction Theory for Emergent Graphical Shape Thinking

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Keywords: Covariational Reasoning, Emergent Thinking, Graphs, Local Instruction Theory

Moore and colleagues (Moore, 2021; Moore & Thompson, 2015) have described *emergent graphical shape thinking* (or *emergent reasoning*) as conceiving of a graph as a trace representing a covariational relationship between two quantities' magnitudes or values. Although such thinking is critical to graph construction and interpretation in mathematics and other subject areas (e.g., Glazer, 2011; Paoletti et al., 2020; Potgieter et al., 2008), researchers have indicated emergent graphical shape thinking is non-trivial, even for U.S. teachers (Thompson et al., 2017). However, there is some evidence that students from middle school (e.g., Ellis et al., 2015) through undergraduate mathematics (e.g., Paoletti & Moore, 2017) can engage in elements of emergent reasoning. These studies suggest that emergent reasoning is within reach for students if there is deliberate efforts to support their development of such reasoning. In this theoretically-oriented poster, we present a local instruction theory to describe ways to support students' developing emergent graphical shape thinking.

A *local instruction theory* (hereafter, LIT) aims both to incorporate and generate generalizable theory to anchor the design of specific learning experiences (Gravemeijer & Cobb, 2006), specifically addressing *how* learning might occur for students as opposed to *what* precise tasks should be used (Prediger et al., 2015). Hence, a LIT strives to reach a broad audience (e.g., teachers, researchers) who can adapt the theory to meet specific classroom needs (Nickerson & Whitacre, 2010). Particular to this poster, we initially developed a LIT by leveraging the extant research on students' quantitative and covariational reasoning (Thompson & Carlson, 2017), understandings of coordinate systems (Lee, 2016; Lee et al., 2020), and emergent thinking (Moore, 2021). We iteratively refined the LIT through six small group teaching experiments (Steffe & Thompson, 2000) and a whole class teaching experiment (Cobb et al., 1995) over the course of three years.

The LIT is composed of three main layered components: (M1) quantitative and covariational reasoning, (M2) (a) constructing a Cartesian coordinate system and (b) representing quantities in the coordinate system, and (M3) emergent graphical shape thinking. These components each have several sub-components. For example, within M2, our LIT includes the following sub-components: M2.1. (a) conceive of varying segment lengths and (b) consider how they can represent a quantity's magnitude (M1.1); M2.2. (a) overlay segment lengths on axes in a Cartesian coordinate system and (b) consider their variations in relation to two covarying quantities; M2.3. (a) construct a point as a multiplicative object in the coordinate system simultaneously representing the two segments' magnitudes (M2.2) and (b) conceive this point as representing the multiplicative object constructed situationally (M1.3).

In this poster, we present the full LIT. We also provide examples from students as they engaged in tasks designed in alignment with our LIT to highlight the interrelationship of the ways of thinking we describe. We present implications that span both research and practice, with emphasis on designing instructional supports for supporting students' graphical fluency.

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Investigating students' interpretations of points and trends on a reaction coordinate diagram by combining analytical graphical frameworks

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Reaction Coordinate Diagrams (RCDs) are chemical representations that encode information about the energy of an evolving molecular trajectory via an abstracted Cartesian coordinate space. Previous research has revealed that students often conflate the x-axis of an RCD with "time." In this study, the location-thinking and value-thinking and graphical forms frameworks were applied to investigate students' interpretations of points and trends along a reaction coordinate diagram to characterize the nature and character of this alternative conception. Results derived from this study demonstrate that students' alternative conceptions about the x-axis arise from the application of unproductive value-thinking and graphical forms reasoning approaches, while productive reasoning approaches were supported by location-thinking.

Graphical Representations, Visualization, Physical Interpretation, Chemistry

Study Overview and Research Question

Reaction coordinate diagrams (RCDs) are abstracted graphical representations used within the general chemistry classroom to compare kinetic and thermodynamic parameters of chemical reactions. Previous research has demonstrated that students often conflate the x-axis of a RCD with "time" (Atkinson, Croisant, & Bretz, 2021; Lamichhane, Reck, & Maltese, 2018) and associate graphical ideas with perceived trends in a RCD (Rodriguez, Stricker, & Becker, 2020).

To better understand the mechanism behind these alternative conceptions, a qualitative interview study was undertaken to assess how chemistry students (N = 16) interpret the points and trends along a RCD. The location-thinking and value-thinking (David, Roh, & Sellers, 2018) and graphical forms (Rodriguez, Bain, & Towns, 2020) frameworks were simultaneously applied to investigate and understanding students' interpretations of points and trends along a RCD given the ability of both frameworks to bridge fine-grain mathematical resources (Hammer, Elby, Scherr, & Redish, 2005) to the conceptual ideas students associate with graphical features. Guided by the actor-oriented model of transfer (Lobato, 2012), the following research question was used to frame the applied methods and analysis: What mathematical resources do students transfer to make inferences about the points and trends on a reaction coordinate diagram?

Results and Implications

Analysis of findings revealed six distinguishable graphical reasoning approaches students adopted to interpret RCDs that incorporated location-thinking, value-thinking, or graphical forms. Additionally, students were also shown to re-frame the RCD at times into a kinematic system to associate intuitive meaning with the diagram. Similar to previous research, students were found to associate the x-axis of the RCD with time. Further examination of these x-axis interpretations revealed that these alternative conceptions were supported by value-thinking and graphical forms approaches. Productive instances of graphical reasoning instead involved the application of location-thinking by mapping unique physical states to points or regions along the diagram. This emergent form of location-thinking has been designated as "location-thinking, states." Suggestions are made to practitioners on how to depict RCDs in the chemistry classroom to drive students to apply location-thinking, states reasoning when interpreting RCDs.

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Interplay of Mindset and Metacognition: A Pilot Study

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Individually, growth mindset and metacognition are powerful learning influences. However, there is little to be found in literature regarding their interplay. This poster reports on preliminary findings of a pilot study which aims to explore the interplay of these two constructs as they affect mathematics learners. Quantitative findings addressing the statistical relationship between constructs are presented in this poster. At RUME 2022, we seek feedback on future directions for this study and its research design.

Keywords: Mindset, Metacognition, Self-Regulation, Learning

Metacognition (Flavell, 1979) and growth mindset (Dweck, 2006) have been found to positively influence learning individually (Dweck, 1986; NAS, 2018), yet there is little to be found in the literature about their intersection. As many are aware, failure and withdrawal rates in undergraduate mathematics course are alarmingly high (Barr & Wessel, 2017) with one contributing factor being that many students arrive underprepared and struggle to adapt to the post-secondary learning environment (Burrill, 2016). Moreover, to combat this difficult post-secondary transition, the ability to persist through challenges and regulate learning—promoted by the attainment of a growth mindset and metacognitive practices—becomes critical for undergraduate mathematics students. Thus, investigating the interplay of mindset and metacognition may help determine what interventions and support is needed for students varying across these two factors in the face of academic challenge and failure.

In this poster, we will present preliminary findings of a pilot study aiming to answer the following research question: Is there a relationship between mathematics students' level of metacognitive action and mindset type? In this study, a nonidentifying survey consisting of demographics, the Metacognitive Awareness Inventory (MAI) (Schraw & Dennison, 1994), and an adapted version (P'Pool, 2012) of the Dweck Mindset Instrument (DMI) (Dweck, 2006) was assigned to calculus students enrolled in a corequisite support course ($N = 49$) as part of their regular coursework. Students were assigned to a mindset type—growth, fixed, or undecided—and a level of metacognition—high, medium, low—based upon the scoring of each instrument. A chi-square test of homogeneity showed no significant relationship between mindset type and level of metacognitive action ($p = 0.82$). This result indicates that students with a particular mindset type (e.g., fixed mindset) are equally distributed among levels of metacognitive action. We note that excellent internal consistency was found in both the MAI and DMI. Due to the small sample size of our pilot, an a priori power analysis was conducted to compute the projected sample size needed to reject the null hypothesis at this effect size in future studies.

As we move forward with this study, we seek to unpack the nuances of this interplay through a qualitative study. For example, as we consider these constructs, we may ask ourselves how supporting a highly metacognitive student with a growth mindset differs from support of a highly metacognitive student with a fixed mindset. More broadly, our research questions are as follows: (a) For various intersections of mindset type and metacognition level, how does affect, attitude, and behavior differ in the mathematics classroom? (b) How does supporting these various types of mathematics students differ? At RUME 2022, we will discuss future directions for this study and seek feedback on the details of the research design.

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Research-Based, Inquiry-Oriented Linear Algebra Videogame - Vector Unknown: Echelon Seas

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Keywords: Design-Based Research, Inquiry-Oriented Linear Algebra, Game-Based Learning

We report on the newest iteration of a videogame to support Linear Algebra students' understanding of linear algebra. We drew on design principles from Game-Based Learning (*GBL*; Gee, 2003; Williams-Pierce & Thevenow-Harrison, 2021), Inquiry-Oriented Instruction (*IOI*; Rasmussen & Kwon, 2007; Zandieh, Wawro, & Rasmussen, 2017), and Realistic Mathematics Education (*RME*; Gravemeijer, 1994; Freudenthal, 1991) to develop the videogame (Zandieh, et al., 2018; Mauntel et al., 2019; Mauntel et al., 2020; Mauntel et al., 2021). To build this iteration of the game, we leveraged our experience with students playing prior versions to extend the gameplay experience in multiple ways. Most importantly, the current game [*Echelon Seas*] takes place in a 3-dimensional environment, increasing the ways in which vectors might be used during gameplay. Beyond this, the current game integrates mathematics and puzzles towards an overarching goal. For instance, *Echelon Seas* contains levels in which the player uses matrix multiplication to fire cannonballs and spans of vectors to solve a tilting labyrinth.

During the session, to compare and contrast the two versions, we will allow audience members to play a prior version of the videogame [*Vector Unknown*] as well as *Echelon Seas* [Puzzle 4]. In both versions of the game, the goal is to use vectors to help an avatar reach a goal position. In *Vector Unknown* (Figure 1a), players could only use vectors in the *xy*-plane and all puzzle levels were restricted to 2-dimensional vectors. In *Echelon Seas*, there are stages for 1-, 2-, and 3-dimensions (Figure 1b). Players can also toggle visual representations of the *x*-, *y*-, and *z*-axes to provide geometric references for real-time, dynamic linear combinations of vectors. As in the Inquiry-Oriented Linear Algebra instructional materials (Wawro, et al., 2012), both versions support understanding linear combinations as modes of transportation to reach a destination.

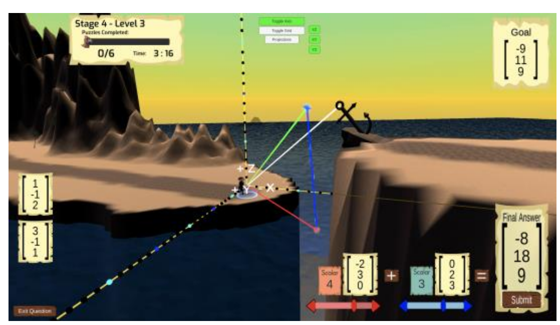

Figure 1: Gameplay images of *Vector Unknown* (left) and *Echelon Seas*, Puzzle 4, level 3 (right)

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Efficacy of a Three Factor Assessment Method in Teaching Undergraduate Linear Algebra

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Knowledge requires assessment of information correctness and justification. Post-item certainty has been used to measure response justification, but pre-assessment confidence has yet to be simultaneously investigated. The objective of this study is to implement a novel three-factor assessment method which simultaneously collects student confidence, certainty, and response correctness in an undergraduate linear algebra course to more comprehensively assess student performance and competency in multiple choice, end-product, and process-based questions.

Keywords: Metacognition, Assessment, Linear algebra, Multidimensional knowledge

Knowledge is information which is both true and justified (Hunt, 2003). Self-reported certainty (*i.e.,* how sure a student is in their provided answer) has been used to justify correctness of information (Hunt, 2003; Snow, 2019; Gardner-Medwin et al., 2003). Evaluating certainty in conjunction with response correctness (2C) accurately assesses absent, partial, complete, and flawed knowledge (Snow, 2019). Confidence, another element of metacognition which indicates how sure students are in their abilities to perform correctly given well-defined criteria, provides critical information about student beliefs prior to viewing assessment items. Confidence, certainty, and correctness (3C) have not been effectively assessed at the same time for strategically identifying outlying student performance behaviors and assessment item efficacies. In this poster we discuss the outcomes of implementing a novel 3C assessment method in an undergraduate linear algebra course and the efficacy of an index for measuring students' metacognitive accuracy which we developed for this method.

Immediately preceding each of six examinations, students in an undergraduate linear algebra course were asked to self-report their levels of confidence for correctly answering forthcoming exam questions pertaining to given learning objectives. Students were then administered the respective examination consisting of multiple choice, end-product, and process-based questions. Immediately after each assessment item, students reported how certain they were that their provided response was correct. Accuracy indices developed to measure the alignment of reported high and low confidence and certainty with answer correctness (*i.e.,* earned credit) were used to determine student and item performance efficacies. Pre-established critical limits were used to determine consistent over- or under- confidence/certainty.

The 3C assessment method employed in this study produces a more meaningful and more accurate assessment of student performance than 2C or correctness-only methods have generated. While high and low confidence and certainty were separately accurate for 33% and 47% of responses respectively, they were simultaneously accurate for only 25% of responses. Students more frequently exhibited consistent under- and over- confidence than consistent under- and over- certainty, respectfully. Early intervention for students exhibiting concerning behaviors highlighted by this assessment method could be critical for advising and guiding students toward more successful learning. The 3C method also provided new and valuable data for analyzing assessment item efficacy. Most notably, 19% of items with high discrimination indices (DI) exhibited inaccurate confidence or certainty. Furthermore, 42% of assessment items with low DI demonstrated accurate confidence or certainty. Future directions for this project include further refinement of the 3C method and analysis of the 3C method in other course frameworks.

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Use of a Three Factor Assessment Method to Investigate Proof Comprehension in Undergraduate Mathematics

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In this poster we discuss the use of a three-factor assessment method to empirically investigate student metacognition (and accuracy thereof) at each level within an established proof comprehension framework. We also observe the progress of student metacognitive accuracy from beginning to end of an introductory proof course, identifying specific areas of this proof comprehension framework in which students exhibit significant frequencies of particular knowledge deficiencies.

Keywords: Metacognition, Assessment, Proof comprehension, Multidimensional knowledge

The proof comprehension framework (PCF) developed by Mejía-Ramos et al. (2012) identifies seven aspects of proof comprehension that occur at the local and holistic levels for proving mathematical statements. This PCF serves as a guide by which to construct assessment items for proof comprehension assessments (PCAs) in advanced mathematics. Student metacognitive behaviors at each level of the PCF have not previously been empirically investigated but undoubtedly contribute to student competency within the PCF levels. A multifactor assessment method which simultaneously collects pre-assessment confidence, post-item certainty, and response correctness (3C) has shown proof of concept by generating comprehensive student performance information in undergraduate mathematics education. Applying 3C assessment methodology to PCAs which adhere to the PCF identifies which areas within the PCF students experience misalignment of confidence, certainty, and correctness.

Multiple choice PCAs developed by Mejía-Ramos et al. (2017) which adhere to the PCF were adapted to include the 3C assessment methods. Pre-assessment confidence was collected on established PCF subitems (Mejía-Ramos et al., 2012). Students in an undergraduate mathematics introductory proof course were trained on the PCF levels as well as metacognitive self-assessment to ensure foundational understanding. These students were then administered two of the above-described 3C PCAs, one near the beginning of the course and the other near the end. Frequencies of each response type were summed and compared. Confidence and certainty accuracy indices were considered according to the PCF levels to which they pertained.

In this poster we report and compare 3C assessment data at each level of the PCF. General confidence and certainty data at each level is compared to identify at which levels students exhibit high or low perceived sureness in their abilities. Accuracy indices among questions at each level of the PCF are analyzed to detect frequencies of over- and under- confidence and certainty among questions at each level of the PCF. Similarly, viewing the frequencies of more specific response types among questions at each level are considered. We also observe progress of student metacognition from the start to the end of the course. Identifying misalignment of student confidence, certainty, and correctness within the levels of PCF better enables mathematicians and mathematics educators to effectively adapt proof comprehension teaching strategies and employ early intervention efforts for students demonstrating consistently low confidence or certainty. This outcome may prove especially effective for underrepresented and minority populations who have not had the same thinking and learning privileges as other students.

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Argumentative knowledge construction in asynchronous calculus discussion boards

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Keywords: Calculus, Knowledge Co-Construction, Asynchronous Instruction, Design Research

A Focus on Mathematical Discourse in Asynchronous Discussion Boards

Classroom discourse constitutes a fundamental activity in which learners can acquire knowledge. A multifaceted phenomenon, any enacted classroom discussion entails the enmeshment of social, cultural, curricular, and modality factors. Focusing specifically on discourse in the context of mathematical discussion activities in the asynchronous online modality, we propose use of Weinberger and Fischer's (2006) *Argumentative Knowledge Construction* framework for design research. We contend that this framework, suitably amended to meet the particular needs of mathematics courses, may enable in-depth analysis of major dimensions of students' knowledge construction as they engage in activities in an asynchronous modality. Research using this framework in the context of face-to-face mathematical learning (Author, Date) and in online settings in other disciplines (Schrire, 2006; Clark & Sampson, 2008; Dubovi & Tabak. 2020) has been reported.

Dimensions to Argumentative Knowledge Construction

Weinberger and Fischer proposed that computer-supported collaborative learning could be analyzed according to four dimensions: *participation, epistemic, argument,* and *social modes of co-construction* (Weinberger and Fischer, 2006).

The *participation dimension* examines the quantity and heterogeneity of students' contributions to the discussion board for each discussion activity. The *epistemic dimension* focuses on the content of students' contributions, attending particularly to the degree to which students' contributions adequately relate the particulars of a problem with the intended concepts that the problem engages. The *argument dimension* derives from Toulmin's (1958) model of arguments to qualify the types of micro and macro argument moves put forth by students in pursuit of a solution. Finally, the *dimensions of social modes of co-construction* "describe to what extent learners refer to contributions of their learning partners" (Weinberger & Fischer, 2006, p. 77). In an asynchronous modality, participants' textual, imagistic, and video submissions can be retroactively analyzed to build group-by-group comprehensive accounts of the knowledge construction associated with a particularly designed prompt.

Use of the AKC Framework for Design Research

We intend to adapt Weinberger and Fischer's (2006) AKC framework in order to conduct design research in asynchronous online mathematics courses. In the theoretical poster herein proposed, we will diagrammaticaly present the dimensions of the AKC framework as conceived by Weinberger and Fischer. We will also identify aspects of the framework that we believe to require adaptation to suit the needs of mathematics-specific courses, will present various pilot discussion prompts, and will invite critique and commentary regarding our proposed use of this framework for our purposes.

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Students in an Introduction to Proofs Course Recognizing When They Are 'Stuck'

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Key words: Introduction-to-Proof, Proof, Proving

With students' demonstrated difficulty in transitioning to proof-based mathematics courses (Stylianides et al., 2017), many universities now offer Introduction to Proof (ITP) courses to help students learn the mechanics of proof writing and underlying logical principles (David & Zazkis, 2020). Many studies have shown the various difficulties which students face as they learn to write proofs, but less research has been done to demonstrate what students learn as they make this transition (Stylianides et al., 2017). In this study, we describe one way in which students learn to write proofs in mathematics, namely by recognizing when they are stuck (i.e., they have made some progress on their proof and have explicit recognition that they are limited in their ability to make further progress). In Reed's (2021) dissertation study, it was found that meaningful learning occurs when students recognize when they are stuck or limited in some way when writing a proof. Indeed, Andrew Wiles describes, "accepting this state of being stuck," as an essential component to doing mathematics (plusmathsorg, 2016, 1:00).

The purpose of this study was to further refine and categorize what it means for a student to be stuck when writing a proof. Thus, our guiding research question was: What are the ways students express that they are stuck when collaboratively writing proofs in an ITP course? To answer this question, we identified and examined six cases where students have recognized that they are not making progress in their proof, oftentimes in these cases they explicitly mention that they are stuck, but that is not the distinguishing criteria to be 'stuck.' To guide our analysis, each researcher wrote their own qualitative description of each case in order to collaboratively describe and categorize the ways students expressed they were stuck. In the synthesis of our descriptions, we found 5 distinct ways throughout the six episodes that students get stuck when writing proofs: (1) Students recognize the lack of operability of mathematical objects; (2) Students are stuck in understanding a peer's proof; (3) Students are stuck in understanding what statements must be justified within a proof; (4) Students are stuck in how to 'write up' the proof (i.e., translating conceptual insight into technical handle); (5) Students are stuck in the algebraic manipulations of a proof.

In this poster session we hope to prompt the following related discussions: *What other ways may students get stuck when learning to write proofs? And how might we help students to productively move forward from these instances?*

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Towards An Operationalization of Mathematization

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Keywords: mathematical modeling, quantitative reasoning, covariational reasoning

Mathematization, the process of transforming a real-world situation into a mathematical model, is historically difficult for students, called horizontal mathematising by Freudenthal (2002). Literature on mathematizing typically frames the research problem as identifying some sort of "blockage" or that an action associated with mathematizing was "difficult" to perform (Brahmia, 2014; Galbraith & Stillman, 2006; Jankvist & Niss, 2020; Stillman & Brown, 2014). While these studies further our understanding of students' difficulties with the mathematizing step in modeling, there is more to be learned about how students' mathematical reasoning influences mathematizing a dynamic situation. Previous studies have indicated that quantitative reasoning promotes mathematization (Ellis, 2007; Ellis, Ozgur, Kulow, Williams, & Amidon, 2012; Mkhatshwa, 2020), and is a lens with which to understand students' reasoning while mathematizing during a modeling task (Carlson, Larsen, & Lesh, 2003; Czocher & Hardison, 2021; Larson, 2013). Given these arguments, it is appropriate to use a quantitative, covariational, and multivariational reasoning theories to describe students reasoning when mathematizing. The goal of this poster is to integrate and synthesize a definition of modeling (Lesh & Doerr, 2003), Thompson (2011)'s conception of quantities and quantitative reasoning, Carlson, Jacobs, Coe, Larsen, and Hsu (2002)'s conception of covariational reasoning, and Jones (2018)'s conception of multivariational reasoning to operationalize mathematization for future study of the mental processes students exhibit while performing a modeling task.

According to Lesh and Doerr (2003), a mathematical model is a conceptual system consisting of elements, the relationships between elements, operations, and rules of governing interactions. Because a mathematical model is a conceptual system, it is held, at least partially, internally and is expressed into the world through different representations. These different representations are dictated by a student's use of any external notation systems (Lesh & Doerr, 2003). The individual's mental representation of the real-world situation dictates the objects with attributes that the individual has the intention of measuring, and the rules governing interactions between objects. Through quantification, the act of conceptualizing the object with an attribute with a measure so that the measure has a proportional relationship with its unit, the objects with attributes are conceptualized as quantities. The quantities then define the elements of the model. The relationships between quantities can then be described by an individual's covariational reasoning, defined as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson et al. 2002 p 354), and multivariational reasoning, is the extension of covariational reasoning to two or more varying quantities (Jones, 2018). The operations on the quantities are determined by the schema of action employed by the students, where schema of action is defined as organized pattern of thoughts or behaviors (actions) that can be applied to different cognitive objects in different situations (Nunes & Bryant, 2021).

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Combing Eye-Tracking with Coordination Class Theory to Analyze Students' Conceptions Related to Graphical Models in Chemistry

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In this work we describe preliminary results from a study that analyzes students' ideas related to graphical models in chemistry. Our project was designed using the theoretical assumptions outlined by knowledge-in-pieces and coordination class theory. Knowledge-in-pieces emphasizes the manifold ontology of cognitive structures and the context-specific nature of knowledge, with coordination class theory building on these ideas to define a concept in terms of the combination of features attended to and inferences. These theoretical commitments informed our methodological approach which involves using eye-tracking to draw conclusions about the features students attend to as they interpret graphical models. Using a first-year chemistry course as the context for our work, we conducted semi-structured eye-tracking interviews that prompted students to interpret graphs centered around particulate-level variation. The current work presents on pilot data that illustrates how eye-tracking compliments coordination class theory.

Keywords: Coordination Class Theory; Graphical Models; Chemistry; Eye-Tracking

Across science and mathematics contexts there is a large body of literature that emphasizes students' reasoning related to graphs and graphical representations. The general consensus of this research is that students have difficulty integrating mathematical reasoning with discipline-specific principles (Bollen et al., 2016; Glazer, 2011; Phage et al., 2017; Planinic et al., 2013; Potgieter et al., 2008). While some work has emphasized the challenges students have (Beichner, 1994; McDermott et al., 1987), other work focuses on the productive ideas students bring to interpreting graphs and representations in a problem-solving task (Elby, 2000; Lee & Sherin, 2006; Nemirovsky, 1996). Research that focuses on leveraging students' ideas is rooted in theoretical commitments that emphasize a manifold view of cognitive structure in which ideas are emergent and activated in response to contextual cuing (diSessa, 1993; Hammer et al., 2005).

Central to the design and analysis of the proposed work is the knowledge-in-pieces view of cognitive structure, which models knowledge as an ensemble of local knowledge clusters that are activated in an emergent process in response to contextual cues (diSessa, 1993). Within the knowledge-in-pieces perspective, coordination class theory models a concept as involving extractions (features attended to) and the inferential net (a knowledge structure that contains relevant ideas for drawing conclusions) (diSessa et al., 2016; diSessa & Sherin, 1998). In the context of typical qualitative research, there are limitations related to drawing conclusions about the features students are attending to in an interview context (i.e., identifying extractions). To this end, we describe preliminary results from a mixed-methods study involving student interviews that combines the quantitative metrics of gaze patterns from eye-tracking with qualitative analysis of students' verbal discussions. Students were sampled from a first-year general chemistry course, with the prompts focusing on graphs and graphical representations. The prompts centered on the threshold concept in chemistry that variation exists at the level of atoms and molecules, the *varied population schema* (Talanquer, 2015). Presentation will emphasize the role of integrating theory with methods, illustrating the affordances of combining eye-tracking with interviews to draw claims about students' coordination class for the varied population schema concept.

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Proof Without Claim

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Keywords: Reading of mathematical proof (ROMP); Proof frameworks; Transition-to-proof

Our work addresses the question: How do students determine what is being proven by a given proof? If students are given a valid proof that does not explicitly state what is being proven, can they ascertain the proof's claim? Investigating these questions sheds light on how students conceptualize proof and proving. We report findings on a student who recently completed a transition-to-proof course. They were presented with *proofs without claim*: valid proofs in which the claim being proven is replaced with blank spaces for them to fill in the claim (see Figure 1).

We will prove that

Suppose there exists a non-prime natural number that is not the product of prime. Consider all such numbers, and let n be the smallest of these. Then, for some natural numbers a and b with $a \neq b \neq n$, we have $n = ab$, since n is not prime. However, a and b are smaller than n. Since a and b are smaller than n, a and b must be products of primes. Thus, n is the produce of primes, a contradiction.

We have proven that

Figure 1. An example proof without claim task.

These tasks give access to how students read and interpret the proofs. This includes (1) the attention, such as verification, that they give to intermediary steps, (2) how they determine what text goes in the blank spaces, and (3) whether what they write in one blank space differs from what they write in the other. Our tasks are informed by the *proof frameworks* in Selden and Selden (2016). We have included typical logical structures of claims and have employed all the basic proof strategies, such as proofs of universal claims via arbitrary instantiations. We also include a proof by counterexample: an opportunity to examine whether students consider the proof as a proof of the negation of a claim or as disproof of the veracity of a claim. We examine how fluently a student can generalize about a claim being (dis)proven and with what logic they state the claim. This study integrates two bodies of literature: that on proof construction (e.g., Selden & Selden, 1995) and that on reading of mathematical proofs (Dawkins & Zazkis, 2021, Inglis & Alcock, 2012, Weber, 2015). We see our work through two lenses. The cognitive lens looks at the relationship between proof frameworks (Selden & Selden), the surrounding logic, and the claim. The second lens is epistemological: what *is* it that a proof is and what is it doing? Both are central questions to mathematics education with a rich history that our work fits into.

Our data suggest that some students view proofs as containers for their constituent calculations. That is, they view these *proofs without claim* as instructions to perform calculations, and ascertaining the claim amounts to determining to which 'problem' these calculations correspond.

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Examining Student Experience in an Inquiry Mathematics Classroom

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Keywords: Calculus, Inquiry-based Mathematics Education, Student Experience

With mathematics playing such an important role as a gateway to other STEM fields, finding ways to improve mathematics instruction has long been a focus of education researchers. This work examines students experiences in an active, inquiry-mathematics classroom which implemented pedagogical techniques shown in prior research to be associated with better outcomes for students.

Research Questions

The research questions for this work are focused on student experiences with particular focus on students who had a negative experience in the course. The questions are: Were any characteristics of students' prior experiences, identities or beliefs entering the course or early experiences in the course likely to associate with students having a negative experience in the course? To drive divergent experiences?

Theoretical Framework

Inquiry-based mathematics education (IBME) has been defined by four pillars outlining the main tenants of this instruction (Laursen & Rasmussen, 2019). For this work I focused on the two student centered pillars shown below in Figure (1) and include additional thinking on how students are entering and leaving the classroom.

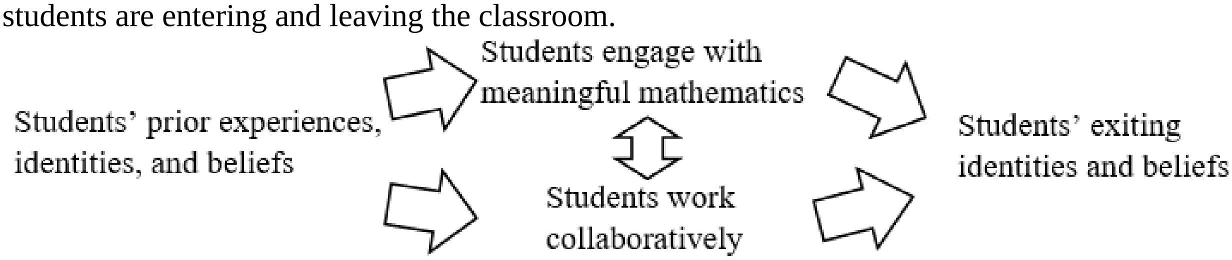

Figure 1. An image of the framework focusing on students identities and beliefs and the two student focused pillars.

Methods and Data

This work took place in the context of a summer bridge program at a university. The data for this work includes math history essays completed prior to the start of class, multiple in class reflections and a final reflection written at the conclusion of the class. This author was involved with the designing and instruction of the course.

Findings and Discussion

Unfortunately, the students' writings reveal how common non-availing beliefs (Muis, 2004) were among students entering the course. In addition to a fixed mindset, a few students described feeling anxiety around mathematics in their math histories. Students who entered the course with math anxiety were more likely to have a negative experience. Early experiences with group work seemed to be a driver of divergent experiences. These findings have important implications for inquiry-mathematics instructors.

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How Undergraduate Mathematics Instructors Assess their Impact at Hispanic-Serving Institutions

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Keywords: Concept inventories, LASSO, decision-making, beliefs

Undergraduate mathematics instructors largely rely on lecture rather than student-centered teaching methods (Larsen et al., 2015; Woods & Weber, 2020). Studies have shown that instructors' behavior is linked to the perception of the necessity of change (Woodbury & Gess-Newsome, 2002) and how instructors make instructional decisions depends on their perceived impact on individual students' learning (Herbst & Chazan, 2012; Schoenfeld, 2011). Thus, we aim to investigate what instructors are using to assess their impact on students.

Many instructors and researchers use research-based assessments (RBAs) to assess instructional impacts. RBAs have been identified as resources for teaching (Furrow & Hsu, 2019) that allow instructors to reflect on their teaching over time and in comparison to other instructors and institutions (Madsen et al., 2017). For example, the Force Concept Inventory has shifted physics teaching from lecture to more student-centered methods (Mazur, 1997).

Our research questions are (1) How are undergraduate mathematics instructors at Hispanic-serving institutions (HSIs) assessing their impact on students, and (2) How can we support the use of research-based assessments for faculty to self-assess their instruction?

Methods

The data for this study come from semi-structured interviews with 10 mathematics instructors from HSIs, from a larger study funded by NSF DUE 1928596. The question that elicited responses relevant to this study was: "How do you assess the impact of your course on students?" Interviews were transcribed with an automatic service and coded by the first author.

Results and Discussion

We found that undergraduate mathematics instructors at HSIs assess student success by the following criteria: student performance on exams, homework, and their subsequent final grade; listening to student thinking during classroom interactions; comments from current or previous students; and department-administered student evaluations. The assessments that instructors used appear to be highly influenced by their contexts, and dependent on locally created sources. We posit that the use of external, less locally dependent, assessment sources could provide an inspiration for change.

We use Learning About STEM Student Outcomes (LASSO) as an example of one tool that can lower barriers for instructors using RBAs. LASSO currently hosts 65 RBAs, with 6 focused on undergraduate mathematics content and 5 cross-disciplinary assessments. Instructors can use LASSO's data analysis tools to explore their student outcomes, compare student outcomes across institutions, and gain another perspective on their courses' impacts on students.

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Shifts in Student Pre-Assessment Confidence and Post-Response Certainty when Evaluated at 3 vs. 4 Levels in an Undergraduate Linear Algebra Course

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Students' confidence in their abilities and certainty in exam responses are two elements of metacognition that influence teaching and learning efficacy. Some epistemologists have studied how confidence and certainty can impact student learning in mathematics education, but none have empirically compared these metacognitive states with respect to specific mathematics content (i.e., learning objectives) and problem solving methods (e.g., multiple choice (MCQ) vs. process based (PBQ) questions). In the present study, we collect students' pre-assessment confidence and post-response certainty during routine exams in an undergraduate linear algebra course to investigate shifts in these perceptions at critical time points and from evaluation at 3 vs. 4 levels. Results of this study suggest an effective method for identifying students with "red flag" metacognitive behaviors and provide meaningful information about student perceptions of specific content in linear algebra that can be used to guide corrective interventions.

Keywords: confidence, certainty, metacognition, epistemology, multidimensional assessment

Motivation and Study Objectives

Most mathematics educators are familiar with "red flag" student behaviors – students who are confident before an exam but then perform poorly, students who are not confident in their abilities yet perform outstandingly, and others between (Cheema & Skultety, 2016; Özsoy, 2011; Parsons, Croft, & Harrison, 2009). The purpose of this study is to examine student confidence and certainty in undergraduate linear algebra relative to specific content and assessment variables to better understand these behaviors. Our objectives are to identify significant metacognitive shifts between student confidence and certainty for individual learning objectives, determine how MCQ vs. PBQ responses influence post-response certainty, and conclude whether a 3- or 4- level evaluation of confidence and certainty produces more reliable data.

Methods and Preliminary Results

Students' pre-assessment confidence and post-response certainty levels were collected throughout six exams in an undergraduate linear algebra course. The first semester evaluated these variables at 3 levels (low, moderate, and high), and the second semester evaluated at 4 levels (low, moderately low, moderately high, and high) to examine the distribution of shifts in moderate and extreme levels of confidence and certainty. Preliminary analyses show significant metacognitive shifts in certain learning objectives, MCQ vs. PBQ assessment methods, and among top vs. bottom performers. The poster will include overall and exemplary analyses of these outcomes while demonstrating how using a 4-level evaluation yields more reliable data.

Discussion and Conclusions

The impact of metacognition on student learning has been extensively studied, especially in the field of mathematics education (Desoete & Craene, 2019; Özcan, 2014; Schoenfeld, 1992; Snow, 2019). Our study utilizes the foundational principles described in the literature to investigate the comparison of pre-assessment confidence and post-response certainty among undergraduate linear algebra students to guide corrective interventions in mathematics education.

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Professional Development and Systemic Change

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Keywords: Professional development, student-centered teaching, policy, institutional change

Professional development (PD) designed for mathematics instructors, and research studying the effectiveness of these programs, continues to grow (Scher & O'Reilly, 2009; Foster et al., 2013; Borrego & Henderson, 2014; Deshler et al., 2015). Our research explores how PD for mathematics instructors can influence departmental change regarding student-centered teaching and learning. The setting is the *PROMESAS SSC*[1] PD, a regional STEM initiative aimed to inspire systemic change in teaching collegiate mathematics through building classroom community, emphasizing student-centered teaching, and promoting rich mathematical tasks. In this research, we examined (1) How the teaching culture of the participating institutions transformed, if at all, as a result of faculty participation in *PROMESAS SSC* and (2) How the culture of the departments transformed, if at all, as a result of faculty participation in *PROMESAS SSC*? To frame this work, we adopted the Four Categories of Change Strategies model (Henderson et al., 2011) along with eight change strategies relevant to STEM fields as described by Borrego and Henderson (2014). They argue that change occurs by (1) disseminating information about curriculum and pedagogical practices, (2) developing reflective teachers, (3) enacting policy, and (4) developing a shared vision. This model greatly informed data collection and analysis of the results. Data collection consisted of five 90-minute focus groups including administrators from participating institutions, PD facilitators, and PD participants. We constructed and organized the focus group questions around Henderson et al.'s four change categories. We began the analysis with open-coding, where code-words emerged from both the interview questions and participants' responses. After this, we categorized the open codes into umbrella codes and then categorized the umbrella codes into themes within each change strategy. Within each of the four categories, we found results similar to those documented in the recent publication, *Transformational Change Efforts* (Smith et al., 2021). Here we document a few highlights. First, as part of implementing new pedagogical practices the participants learned as part of the PD, we found some spillover onto other instructors. We learned that non-participant instructors and administrators were intrigued when they walked by a classroom and observed students talking to one another. This curiosity prompted administrators to ask the participants to share what they learned at the PD at a faculty meeting. The participants also conveyed that they felt as though they were more reflective teachers and began to perceive teaching as a scholarly endeavor. They mentioned how they think about how to modify a lesson immediately after teaching and this was a novel practice for them. In terms of enacting policy, the administrators indicated that the PD "changed their hiring practices" because they wanted instructors who could teach using methods that emulated the PD themes. Within the developing a shared vision category, all research participants described the value of a community and how they appreciated learning from and sharing resources with colleagues from the various institutions. They also conveyed that this made transforming their practices more feasible because they had internal and external support. It appears that the PD acted as a catalyst that brought departments and institutions together to introduce student- and teacher-centered learning.

[1] For more information, visit https://www.csuci.edu/promesas/pathways-outreach-math-excellence-stem.htm#SSC

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Active Learning in a Dynamic Textbook Needs Student Feedback.

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Abstract: *This poster presents a process for development of a dynamic textbook allowing its authors to provide students with the authority of a textbook along with integration of learning as directed by the author/instructor in a manner receptive to the students. The presenters have followed these principles to develop a business mathematics and business calculus texts using the PreTeXt formatting system. The content is aligned with statewide content standards and reviewed by professional mathematicians but is revised with student feedback making the dynamic aspects vary over term to remain viable.*

Keywords: dynamic textbook, didactic contract, instructor as author

Economic pressures have led to the development of open educational resources which are generally available as dynamic textbooks. The presentation, in the format of a textbook except as a webpage or pdf file-- making it mostly static, is made dynamic through the use mathematics analysis software, or MAS (Pierce et. Al, 2010) embedded in the resource. As noted in (Mali et al., 2018) the instructor and the students must be essentially engaged with the dynamic aspects of the materials to capture the full value intended by the author. Incorporating the instructor as an author ensures complete instruction using the materials. Students recognizing the talent of the instructor strengthens the didactic contract.

To balance the didactic contract expanded by valuing the instructor as author, students are afforded the responsibility of providing feedback on the content as presented in the dynamic textbook. This student commentary on the dynamic materials will contribute to the craft knowledge of the instructor (Chorney, 2021).

In the poster itself we describe the formatting and content of the materials, the dynamic aspects of the content and the process for including student feedback into revisions of the materials.

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"Here I Am": Using Poetic Transcription to Explore Students' Narratives of Mathematical Success

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Keywords: Research Methodology, Poetic Transcription, Student Success, Qualitative Methods

As arts-based analysis gains popularity and credibility in research spheres, it is prudent to investigate its efficacy in the context of mathematics education research. Poetic analysis, one such arts-based research methodology, can take several forms that draw from literary poetic traditions (Lahman et al., 2010). In the context of mathematics specifically, researchers have used poetic analysis to analyze the ways in which mathematical discourse and reasoning mirrors poetic structure (Holden, 1985; Staats, 2008; Staats, 2021), or have incorporated poetry into the teaching of mathematics through the creation of poetry-based mathematical tasks (LaBonty & Danielson, 2004; Triandafillidis, 2006). However, mathematics education literature has seldom utilized poetic transcription, a specific type of poetic analysis utilized in social science inquiry, as a form of research methodology.

Poetic transcription offers mathematics education an imaginative and simultaneously grounded "method of knowing" (Richardson, 2003, p. 379). It draws from the literary tradition of a *found poem*, in which excerpts from existing written media are juxtaposed to create poetry with distilled meaning. In poetic transcription, participants' words from interview or focus group transcripts are pulled in such a way as to honor and reflect the participant's meaning and narrative style (Glesne, 2007). Prendergrast (2009) emphasizes that these poetic transcriptions are built from the participants' words but filtered and "re-presented" through the lens of the researcher by referring to these poems as "participant-voiced poems." Via use of member-checking practices and researcher-participant co-construction of such poetry, poetic transcription can provide direction for research that is "genuinely inclusive and democratic" (Levinson, 2020, p. 195). Further, via the artistic embodiment of experiential knowledge, poetic transcription can present nuances of qualitative data in a way that the quoting and analysis of traditional prose may not, particularly when viewed through a lens of feminist theory (Faulkner, 2018) and critical qualitative research (Keith & Endsley, 2020).

This poster will present a use case for poetic transcription by analyzing four students' narratives of their experiences of and definitions for mathematical success. Data for this study involved interviews from four first-year undergraduate women students, all of whom were Pell-grant eligible, First-Generation, or students of color, or held some subset of these three identities. During a semi-structured interview, each student was asked the questions "Describe a time in which you felt successful in mathematics" and "How would you define success in undergraduate mathematics?" This poster describes the process of poetic transcription using Feminist theory (Burton, 1995) to present counterstories of what can be conceptualized as mathematical success.This poetic transcription process included parsing of the data into meaningful phrases, extraction and distillation of such phrases, and rearranging the phrases in poetic juxtaposition. The final step of the process incorporated member-checking through participants' affirming and editing of the poems to construct finished free-verse poetry as re-presentations of the data. In the poster, I share these poems, discuss the opportunities and limitations afforded by such re-presentation, and set the stage for future work surrounding poetic analysis as a viable methodology in the field of mathematics education.

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“But Again, We’re Human”: Engaging Undergraduate Students with the Social Context of K-12 Mathematics

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Keywords: Curricula development; Human context of mathematics; Undergraduate mathematics

Mathematics is a human activity, existing in a socio-cultural space, and today, instructors are increasingly aiming to implement humanizing approaches to learning mathematics in their courses. One approach is to provide context for mathematics curricula by developing and incorporating mathematical tasks which prompt students to connect mathematics to familiar social and cultural contexts (Chapman, 2006; Díez-Palomar et al., 2006; Leonard & Guha, 2002; Niss, 1994). Currently, much of standard undergraduate mathematics incorporates contexts based in physics, engineering, or other science-related applications (e.g., calculating the maximum height of a ball thrown in the air with some velocity). While these types of applications resonate to those students for whom the context is a primary course of study or personal interest, they often do not provide opportunities for students to engage with the human context of mathematics. Curriculum development can include applications of mathematics that hold the human context on par with the mathematics content. Further, these applications can focus on contexts that resonate with most undergraduate students - the shared social context of secondary mathematics education (see Álvarez et al., 2020).

In the META Math project, we explored connections between undergraduate mathematics and secondary mathematics and developed curricula materials for use in standard undergraduate mathematics courses that embed *applications to teaching* (i.e., mathematical tasks whose context is related to teaching) in an explicitly human context (see Álvarez et al., 2019; Álvarez et al., 2020; Arnold & Fulton, 2021). Many of these mathematical tasks asked undergraduate students to examine, complement, and critique the work and mathematical understanding of a hypothetical school student. Additionally, the tasks prompted undergraduate students to pose guiding questions to help guide the hypothetical students’ mathematical understanding.

In this poster, we examine undergraduate students’ written responses to such tasks, as well as their interview responses about such tasks, to investigate the following research question: *What is the nature of undergraduate students’ perceptions of mathematics tasks designed to include the social K-12 human context of mathematics?* Through the lens of three principles—*Habit of Respect, Active Engagement*, and *Recognition of Mathematics as a Human Activity* (see Álvarez et al., 2020)—we report on how undergraduate students qualitatively embodied these principles through their written responses and verbal perceptions of the hypothetical students in the tasks.

The undergraduate students’ responses revealed different emphasis for each of the four different content courses the tasks were implemented in. For example, responses from undergraduate students in discrete mathematics indicated respect for the hypothetical students’ mathematical work and capacity for knowledge building, while undergraduate students in abstract algebra positioned the hypothetical students’ work as a way to identify common misunderstandings to avoid in their own mathematical work. Further work is presently being done to analyze *how* undergraduate students attributed humanness to such hypothetical students in these tasks; this is being done through a lens of mind perception, with specific attention paid to the agency given to the hypothetical students (Gray et al., 2007). Our hope is that such work can provide further grounding on the importance of work done engaging students with meaningful contexts through mathematical tasks (Bright, 2016; Yeh, 2017).

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Adapting to Challenges in Undergraduate Pre-Calculus: The Cases of Bailey, Rose, and Toby

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Keywords: Self-Regulation, Preparation to Calculus, Undergraduate Mathematics

Transitioning to undergraduate mathematics from secondary mathematics may trigger obstacles that make this transition more difficult for students (Gueudet, 2008; Sonnert et al., 2020). For example, Gueudet (2008) and Sonnert et al. (2020) report students' feeling unexpectedly ill-prepared for the material presented in a college classroom. To navigate these obstacles and synthesize and process new information, students need to develop self-regulation strategies. That is, the appropriate skills and actions to address these challenges (Zimmerman & Pons, 1986). This poster analyzes interviews conducted with three first-time freshmen in precalculus aimed at addressing the following research question: In what ways do students adapt their self-regulation strategies from high school mathematics to college mathematics?

To investigate students' adaptation of their self-regulation strategies, a sequence of five interviews were conducted with three students, Bailey, Rose, and Toby. These students were among a total population of twenty-five precalculus students who completed a survey within the first week of the semester at a large, urban research university in the Southwestern United States during Fall 2021. Students were invited to interview based on their responses to the survey. Bailey, Rose, and Toby participated in two individual interviews, one within the first few weeks of the semester which focused on expectations of the course and background information about their prior school experience, the second, occurring shortly before the end of the semester, reflected on the course and how the students may have adapted to any challenges that arose. The remaining three group interviews were task-based, occurring throughout the semester. Group interviews focused on the use of self-regulation strategies during weekly in-class group departmentalized lab activities. Audiovisual recordings of the interviews were made and transcribed verbatim.

Preliminary data analysis of the interviews reveals the following self-regulation strategies used by the three students: seeking information from external sources, keeping records of progress, seeking assistance from peers and instructors, self-evaluating own understanding, providing extrinsic rewards, reminding oneself of the value of the task (or course), and changing their environment. While each of the three students reported using most of these strategies, each mentioning that they would often ask a peer or the instructor for help when stuck on a problem, observations from the task-based interviews revealed that these students responded by seeking external sources of information primarily from online sources, rather than asking a peer for help, even when the peer was nearby in-person. Adaptations identified in the interviews included adopting new study habits to account for increased difficulty of the college course, working practice tests, reviewing notes after lectures. Rose, in particular, reported the most substantive change of study habits, with ongoing adjustments throughout the semester. The fundamental knowledge she believed she needed from the precalculus course motivated her to adopt new strategies and refine old ones.

Findings regarding how Bailey, Rose, and Toby develop and acquire self-regulation strategies as they navigate their first undergraduate mathematics course provides further insight into Johns' (2020) findings of a correlation of high strategy usage in an undergraduate mathematics class with higher performance.

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An Exploration of Caring Relations Through the Energy Exchange Cycle

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Keywords: Caring Relations, Energy, Mathematical Caring Relations

Caring relationships are a critical element of a supportive mathematics learning environment, including undergraduate classrooms. However, undergraduate education is often left out of care ethics research. A novel aspect of this poster brings the work of Noddings (2012a) and Hackenberg (2005, 2010a, 2010b) from a K-12 focus to encompass undergraduate mathematics education.

Hackenberg (2005) presents a mathematical framework based on Noddings's (2012a) caring relations and scheme theory to create student learning models she titled Mathematical Caring Relations (MCR). The Energy Exchange Cycle (Figure 1) is an extension of Hackenberg's caring cycle, a visual adaptation I created to match Hackenberg's (2005) description of MCRs between a teacher and a student during interviews that comprised of a student working through a mathematics problem. Within the description of her caring cycle, Hackenberg describes social interactions as encompassing stimulation and depletion. *Stimulation* is defined as a "feeling of being excited or awake, usually accompanied by a boost in energy or a stronger sense of aliveness" (p. 45). *Depletion* is defined as a "feeling of being taxed in some way, usually accompanied by a decrease in energy or diminishment of overall well-being" (p. 45). Later, Hackenberg (2010a, 2010b) added the notion of subjective vitality to the dualism of stimulation and depletion. *Subjective vitality* is a "physical experience of energy and aliveness, characterized by enthusiasm and vigor" (Hackenberg, 2010a, p. 241). Each of these definitions describe an increase or decrease in energy during a social interaction. The Energy Exchange Cycle shows a full cycle of caring relations between teacher and student in a mathematical interaction.

This poster begins a discussion of how a caring relations model can be used as a supportive element in undergraduate mathematics.

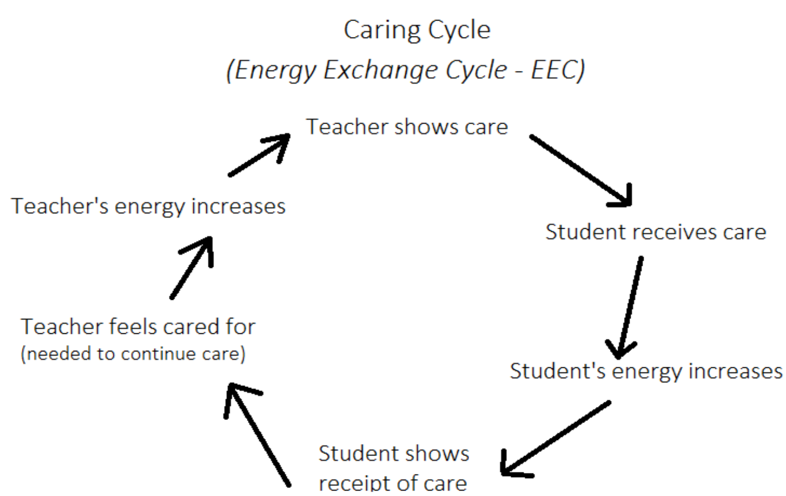

Figure 1. Energy Exchange Cycle

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Alternative Measures of Effectiveness in an Innovative Active Learning Calculus Course

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Results of an implementation of the Modeling Practices in Calculus (MPC) curriculum, an innovative active learning approach, in Calculus I at a large, urban, research-intensive (R1) institution are presented. Using a randomized-control trial research design, students were randomly assigned to either traditional, lecture-based classrooms, or MPC classrooms. Two alternative measures of effectiveness of the curriculum were examined: students' attitudes towards mathematics and precalculus proficiency. Our findings show that active learning classrooms lead to significant effect sizes in attitudes towards math and precalculus proficiency gains when compared to traditional lecture-based classrooms. Furthermore, active learning was found to diminish gender gaps in both measures, acting as a gender equalizer.

Keywords: Active Learning, Attitudes Towards Mathematics, Calculus, Precalculus Proficiency

Recommendations to incorporate student-centered instructional approaches have recently been made by a number of scientific associations (Bressoud et al., 2013; Rasmussen et al., 2019). However, efforts to assess the effectiveness of these approaches have primarily focused on students' achievement (Freeman et al., 2014), somehow neglecting important measures of effectiveness including students' affect and their development of certain foundational skills. These alternative measures are extremely important since they have been linked to persistence in STEM careers (Bressoud et al. 2013) and issues of equity (Ellis et al., 2016). We will present preliminary results from a three-semester RCT study on the adoption of the Modeling Practices in Calculus (MPC) curriculum, an innovative active learning approach, in an introductory calculus course. The main measures of student outcomes considered in this study were student attitudes towards mathematics and precalculus proficiency. Students' attitudes towards math was measured using the Attitudes Towards Mathematics Inventory (Tapia & Marsh, 2004), and precalculus proficiency, using the Precalculus Concept Assessment (PCA) inventory (Carlson et al., 2010).

Data from both inventories were compared across the two groups, finding significant effect sizes in both attitudes towards math and precalculus proficiency development. In terms of attitudes towards math, the MPC curriculum was found to mitigate the detrimental effect of lecture-based calculus on students' self-confidence (Sonnert & Sadler, 2015). Furthermore, students with low values of self-confidence were found to benefit the most from the model. In terms of foundational quantitative skills, MPC students were more likely to succeed in their calculus course even if they started with lower precalculus proficiency, and they also develop these skills significantly more than those from traditional lecture-based classrooms. The MPC curriculum also led to eradicating gender gaps in self-confidence that were found at the beginning of the semester. Our results provide further evidence of the positive effect that well-designed active learning classrooms can have on university calculus students.

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Mathematical Knowledge for Teaching Community College Algebra: The Life Cycle of an Item

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Keywords: college algebra, community colleges, Mathematical Knowledge for Teaching

Research has shown a positive relationship between quality of instruction, instructors' mathematical knowledge, and student achievement in mathematics (Hill et al., 2005; Hill et al., 2012). The *Algebra Instruction at Community Colleges* project (AI@CC; Watkins et al., 2016-2020) sought to establish a similar connection in the community college algebra classrooms but faced the challenge of lacking an instrument to measure the mathematical knowledge for teaching for community college faculty. The *Algebra Instruction at Community Colleges: Validating Measures of Quality Instruction* (AI@CC 2.0:VMQI) project (Mesa et al., 2020-2023) sets out to develop and validate such an instrument, Mathematical Knowledge for Teaching Community College Algebra (MKT-CCA). The MKT-CCA instrument seeks to measure the mathematical knowledge for teaching college algebra at community colleges using multiple-choice or testlet items focused on linear, exponential, and rational equations and functions as taught in the college algebra setting.

As we drafted items, the AI@CC 2.0:VMQI project team needed to create processes for reviewing and revising items. These processes include using feedback provided by our advisory board, creating a system for internal review and revision of items by pairs of team members, drawing upon findings from cognitive interviews, and shifting to internal review and revision through a tiered team system. Throughout these various processes of collecting and applying feedback, we revised items to maintain the original goal of the item and the mathematical content, and when items could not be revised, we took components that allowed us to develop new items.

In this poster we will present the life cycle of an MKT-CCA item. We will explain how our process influences how items evolve and change based on the nature of the feedback and our interpretation of the suggestions, how that influenced an item's longevity and content, and the challenges we faced in the process.

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Characterizing Community College Instruction in Response to State-Mandated Policy and a Global Pandemic

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Keywords: Inquiry, Equity, Teacher Practice, Gateway Mathematics

Introduction

This poster describes the instructional styles of four mathematics instructors at a community college, who were responding to a state-legislated mandate to overhaul developmental mathematics to increase learners' success rate taking remedial math courses (The Campaign for College Opportunity, 2018), and a global pandemic. These external agitations have the potential to catalyze shifts in instruction to be more inquiry-focused and equitable (e.g., through the use of technology), but they also pose considerable challenges. Here, we detail how these four instructors navigated this situation by describing instructional practices in their classrooms.

Methods

We coded four classroom observations of each instructor using an equity-based observation protocol (Reinholz & Shah, 2018). Upon coding the classroom observations, we ran several different crosstabs using *R Statistics* to characterize the quality and distributions of participation. Our statistical analysis was triangulated with instructor interviews to better understand patterns in the data (Miles, Huberman, & Saldaña, 2014).

Findings

Findings suggest that instructors' practice was typical (Hiebert & Stigler, 1997). We noticed that students mostly provided "what" answers during instruction across instructors. "What" statements are more procedural in nature, focusing on providing "the answer." Yet, students did not provide answers proportionally across venue types (e.g., 95% of contributions for Arenas were in chat, whereas 61% were in whole-class discussions, for Martinez).

Table 1. Student Talk Type Across Venue.

	Venue			Student Talk Type			
Instructor	Chat	WC	Breakouts	Why	How	What	Other
Arenas	403	19	0	0	20	320	82
Martinez	53	342	163	14	48	278	218
Peterson	98	2	81	7	16	70	88
Lee	39	21	22	2	4	44	32

Discussion

Our findings showed both commonalities in the procedural nature of instruction, and also differences, insofar that instructors leveraged technology to provide different venues of instruction for different students. In the full poster we also describe patterns disaggregated by race and gender for student participation. On the whole, we found that the external agitations did little to change the status quo of teaching. We will comment on implications for practice.

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The Roles of Formal and Informal Resources in Mathematical Activity: A Case Study of a Mathematician Scaffolding Students' Proof-Writing in Real Analysis

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Keywords: Formal-informal, Real Analysis, Proving, Teaching practice, Sociocultural theory

Effective use of formal and informal resources in mathematical activity is of central concern in advanced math courses such as Real Analysis. Lecturing in advanced mathematics has been criticized for focusing exclusively on "polished formalism" (Davis & Hersh, 1981; Dreyfus, 1991). However, observational research of teaching demonstrated that instructors routinely go beyond the formal text and do engage with informal aspects of the discipline too (Weber, 2004). Yet, formal and informal resources are not situated in lectures in the same way. Formal aspects of activity get written down on the blackboard, while informal aspects come through in oral commentary, gestures, and other ephemeral dimensions of discourse (Greiffenhagen, 2008; Lew, Fukawa-Connelly, Mejía-Ramos, & Weber, 2016). This pattern is both systematic and consequential. Students tend to copy down only what instructors write on the board and "don't get" most of what instructors try to convey through non-written discourse (Lew et al., 2016).

The present case study aims to extend this line of work by exploring the roles formal and informal resources play in a context not yet extensively studied, that of an instructor interacting with student groups. The primary data source for the analysis reported here is a video-recording of a classroom episode from a Uniform Convergence curricular unit of a Real Analysis course in which a mathematician-instructor interacts with five student-groups working on a proof construction exercise. This episode was selected from a larger corpus of ethnographic data because it featured use of both formal and informal resources and thus allowed me to interrogate: What mediates students' access to using formal and informal resources in advanced math?

In line with the methodological tradition of video-based classroom ethnography (Derry et al., 2010; Erickson, 1992), I created detailed content logs of classroom videos and segmented them into a nested structure of episodes and sub-episodes. Drawing on sociocultural theory, I took *social practice* – mediated, goal-directed, joint action – as the basic unit of analysis in this study (Cole, 1998; Wertsch, 2012). Thus, for each sub-episode, I noted shifts in resources (formal or informal), disciplinary functions (e.g. proposition-reading, proof-writing) and participation structures. I created multimodal transcripts that helped determine the discursive resources (e.g. types of words, gestures) participants used. Finally, I abstracted patterns of *what type* of resource gets used *for what function* and *by whom*, checking them against all transcripts of the episode.

I found that while both formal and informal resources served important functions in activity, the two form-types were deployed toward different ends. Formal resources were consistently used to support proof-writing functions, whereas informal resources were consistently used to support other goals such as: proposition-interpretation, proof-validation, and phenomenon-observation. I further found that the enactment of these disciplinary functions involved different division-of-labor arrangements between the instructor and students: students were invited to *actively contribute* to proof-writing, but only *observe* the instructor enact other functions. These findings suggest that positioning proof-writing as the main disciplinary task students are accountable to in Real Analysis courses, as opposed to practices such as text interpretation, validation or phenomenon observation, can explain why advanced math courses are so often experienced as focusing solely on "polished formalism," despite instructors' best intentions.

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