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Preface

As part of its on-going activities to foster research in undergraduate mathematics education and the dissemination of such research, the Special Interest Group of the Mathematics Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME) held its twenty-fifth annual Conference on Research in Undergraduate Mathematics Education in Omaha, Nebraska from February 23 - February 25, 2023.

The 25th RUME Conference enabled presenters and attendees the option to participate fully online such that travel was not a requirement, approximately 25% of participants were online.

The program included plenary addresses by Dr. Warren Christensen, Dr. Natasha Speer, and Dr. Hortensia Soto and the presentation of 151 contributed, preliminary, and theoretical research reports and 69 posters. The conference was organized around the following themes: results of current research, contemporary theoretical perspectives and research paradigms, and innovative methodologies and analytic approaches as they pertain to the study of undergraduate mathematics education.

The proceedings include several types of papers that represent current work in undergraduate mathematics education, each underwent a rigorous review by three or more reviewers:

- Contributed Research Reports describe completed research studies
- Preliminary Research Reports describe ongoing research in early stages of analysis
- Theoretical Research Reports describe new theoretical perspectives for research
- Posters may fall into any of Contributed, Preliminary, or Theoretical and were presented in poster format. Authors contributed the poster itself, a summary of the work, or both.

The conference was hosted by the University of Nebraska – Omaha's Department of Mathematical and Statistical Sciences. Many members of the RUME community volunteered for the Program Committee where they reviewed many submissions such that every submission was reviewed by at least one member of the Program Committee. The Program Committee aided us in putting together this program and their hard work is greatly appreciated. The Local Organizing Committee were responsible for the smooth running of the presentations and on-site activities, which would not have been possible without the help of many volunteers, and we thank them for their tireless efforts to host a conference that runs smoothly.

Thank you to all of the researchers who submitted such strong proposals and ultimately made the conference a fun and joyous event.

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Applying Proof Frameworks Is Harder for Students than We Think

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In order to overcome difficulties many students in mathematics and computer science face with rigorous proofs, some higher education institutes present a "Mathematical Reasoning" or "Transition-to-Proof" course. In this paper, we analyzed data from the exams of 310 students in such a course, focusing on the way they approach a proof regarding set inclusion, and whether they use a particular "proof framework" that we taught them when doing so. We found that the students are able to use the proof framework, and that using it is beneficial to them, but that a surprisingly large percentage of them fail to use it consistently.

Keywords: Transition-to-proof courses, Proof frameworks, Operable interpretations of definitions

Introduction

One of the challenges mathematics instructors face is teaching their students to write correct rigorous proofs and to use the definitions properly. Between 1990 and 1999, more than 100 papers have been published on this subject (Hanna, 2000). The question of how students engage with proof and proving is continuously being studied (Reid and Knipping, 2011). These studies have suggested various explanations for the root of the problem, and proposed a variety of methods for addressing it, which have been tested with varying degrees of success. One common conclusion is that the transition from an intuitive answer to the rigorous formulation is one of the students' main sources of difficulty. Vinner (1991), for instance, claims that students tend to ignore the definitions necessary for a rigorous proof, preferring an intuitive "general explanation" answer. Moore (1994) supports this assertion, noting that, while attempting to write formal proofs, students do not necessarily understand the content of relevant definitions or how to use them.

One of the ways institutes deal with this issue is by having students take an "introduction to proof" course. Most such courses emphasize helping students learn the proof construction process (see in particular, Selden, Benkhalti and Selden, 2014 and Moore, 1994). This paper presents data collected in the context of the course "Mathematical Reasoning" - a mandatory course for first year computer science undergraduates. Specifically, our study focuses on the manner and extent to which undergraduate students employ - or fail to employ - a "proof framework" while proving a theorem. The concept of the proof framework was first introduced by Selden and Selden, 1995, and later expanded upon (2014). Similar ideas were presented by Scheinerman, 2013, who refers to it as a "proof template," and Hammack, 2013. In this paper we focus on one specific framework, used for proving set inclusion according to the definition. We analyzed 310 students' answers to questions that require this proof and discovered that a large percentage did not consistently use the proof framework, even when they had presented the ability to apply a very similar proof framework in an earlier part of the question, and despite the fact that this was the only technique they had learned for dealing with a question of that sort.

Proof Framework

Selden, Selden and Bankhalti (2018) define *proof frameworks* as "a way of structuring a proof, in which the student begins by writing the first and last lines of a proof and works towards the middle. It often consists of a first level, in which a student unpacks the logical structure of the theorem statement to write the first and last lines of the proof, and a second level, in which a student unpacks the definitions involved in the conclusion to write the second and second-to-last lines of the proof" (Selden, Selden and Bankhalti , 2018 p.4). To illustrate this point, they provide a sample proof framework for an elementary set theory theorem, which we have copied in its entirety here:

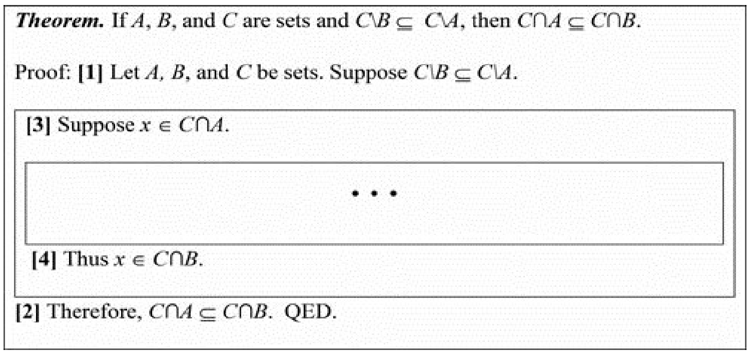

Fig. 1. Proof framework of set inclusion

Clearly, a theorem regarding set inclusion should start with an element that belongs to the included set, and by a sequence of steps ends with the result that the element belongs to the including set. In their study, Selden, McKee, and Selden, 2010 discuss the habits of mind that appear to drive many mental actions in the proving process, and discuss the way feelings can help cause mental actions and also arise from them. They offer the following example of a wrong "proof," in the form of Sofia's answer to the following theorem.

Theorem: *For any sets* A, B, C *if* $A \subseteq B$ *then* $(A \cap C) \subseteq (B \cap C)$.
Sofia's Proof: (Selden et al., 2010, p.13) Let A, B, C be sets
Suppose $x \notin A, x \in B$ *and* $x \in C$
Then $x \notin (A \cap C)$, but $x \in (B \cap C)$
Therefore $(A \cap C) \subseteq (B \cap C)$

We were curious **how common** the approach presented by Sofia might be - namely starting wrong, such that it is impossible to produce a proof from that starting point. These authors' study presented insightful qualitative findings, based on a deep analysis of a small sample. Our study is based on an analysis of hundreds of students' exams, which allowed us to produce, in addition to qualitative findings, some interesting quantitative results. Our primary goal was to understand whether students who fail to present a rigorous proof do so because they do not use the proof framework that we teach them, or because they use it incorrectly.

This information is important in order to improve teaching: Should we be placing more emphasis on the use of the proof framework (i.e. encouraging the students to use it), or putting more effort into the details of the framework itself (making sure that the students use the framework correctly)? It is also interesting to check whether students who do not use the framework (i.e. students who use other proof techniques) are doing well. If we see that using the proof framework does not result in better proofs, compared to other techniques, this might indicate that the effort invested in sticking to a proof framework is redundant, or even harmful.

Our Course

Much like other Transition-to-Proof courses elsewhere (e.g. Selden, Selden and Bankhalti , 2018), our reasoning course is based on students practicing the act of proving. We cover basic classical mathematical concepts and proving methods: The role of definitions, quantifiers, proof by contradiction, dealing with cases, and induction. We rehearse these concepts and methods throughout the course. The course contains only elementary mathematical concepts: arithmetic, basic set operations, and basic linear algebra. It consists of 3 hours of weekly lectures, with an emphasis on independent practice and homework. In every lesson, we solve the homework given in the previous lesson. The students are encouraged to present their proofs in class, and get feedback from the teacher. This usually takes the first half of the lesson, and the second half is dedicated to a method of proving, explained and followed by many examples. The students are first year, first semester, computer science undergraduate students, and the course is designed to help them develop the skill of proving, which is needed in later courses such as Linear Algebra, Calculus, Data Structures and Algorithms. In each genre of proof, we introduce the students to a proof framework (see Section II for details) as a way of getting started writing proofs.

The Study

We teach in one of the largest colleges in the country, and our students have high grades in the acceptance exam and in the matriculation exam in mathematics (significantly higher than the average).

Around 300 students are enrolled in the "Mathematical Reasoning" course every year. When dealing with **basic set theory** we stress that:

1. When proving $A \subseteq B$ one should start with 'Let $x \in A$' and end with 'then $x \in B$'.
2. When proving $A = B$ the technique is to prove that $A \subseteq B$ and that $B \subseteq A$, using the framework of set inclusion.

We chose a non-covid cohort, where all lectures and the exams were frontal (i.e. not via zoom). The class took place between October 2019 and January 2020 and was divided into 5 groups of around 60 students. We analyzed the students' responses to a multi-part question on basic set theory given in the course's final exam. Our goal was to analyze the use of the proof framework in students' answers, specifically: did students use the proof framework that we taught them? If so - did they use it correctly or did they merely write something that looks like the framework but is actually totally wrong? If not - were they able to present a rigorous and correct proof without depending on the proof framework?

The exam question that we analyzed began with a "definitions" section requiring the students to present definitions that were taught during the semester. Specifically, we were interested in the correct definitions of inclusion and equality of sets. The "definitions" section was followed by two "proof" sections, in which the students were asked to prove or disprove certain claims (that were actually both true). We will later refer to the two proof sections as the "first" and "second" proofs. Here is the question (summarized, for brevity):

1. (10 pts) For A, B sets define $A \backslash B,\ A \cap B,\ A \cup B,\ A \subseteq B,\ A = B$
2. (12 pts) Prove or disprove according to the definitions:
   For sets A, B, C, $(A \backslash B) \backslash C = (A \backslash C) \backslash (B \cup C)$
3. (12 pts) Prove or disprove according to the definitions:
   For sets A, B, C, $C \subseteq A$ if and only if $(A \cap B) \cup C = A \cap (B \cup C)$

The "definitions" section is important as it requires the students to recall the correct definitions needed in the proof sections. In the analysis, we set aside the exams of those who did not write the definitions correctly, based on the assumption that we cannot expect students who are not able to present the definition of inclusion correctly to prove inclusion correctly (only a handful of students failed to correctly answer the definitions section, likely because this section is repeated every year).

The first proof is a set equality. This requires, by definition, a bi-directional inclusion. We expected the students to use the proof framework twice, once for each inclusion. The first starts with $x \in (A\backslash B)\backslash C$ and ends with the conclusion that $x \in (A\backslash C)\backslash(B \cup C)$, and the second does the same, but in the opposite direction. Since the question required the students to prove by definition, using set identities was not allowed. However, the students *could* also legitimately prove each of the directions by contradiction.

The second proof is an "if and only if" claim. One direction of the proof required students to prove an inclusion, and the other required them to prove a set equality. We expected the students to use the proof framework three times. Here, as well, proof by contradiction is considered "according to definitions", and was accepted as a correct answer, although it is not based on the proof framework. Note that there are altogether *five* inclusions to be proved: two of them for proving the first proof, and three for proving the second. The students saw at least four examples of very similar questions in class and solved five more similar questions in their homework assignments, which were solved in class after submission. In addition, every exam in the course from past years contained a question of that sort. When analyzing students' answers for the first and second proofs, we first focused on whether the student implemented the proof framework in the answer, to determine the extent of the phenomenon of students not applying the framework. We categorized the students' answers according to the following taxonomy:

1. **Correct use:** The student followed the proof framework correctly.
2. **Misuse:** Presenting incorrect use of the proof framework, (i.e., starting the proof of inclusion by inspecting an element which does not belong to the included set).
3. **No use - right:** Not following the framework and using another technique, however the solution was correct (e.g., proof by contradiction).
4. **No use - wrong:** The student did not follow the framework and used another technique, and the solution was incorrect.

If the student skipped the proof sections of the question or tried to disprove it (for both proof sections, the correct answer was proving), we marked this as well.

Findings

We analyzed 310 exams. We focused on the students who wrote the definitions correctly, meaning they knew that the definition of inclusion $A \subseteq B$ is that every $x \in A$ holds that $x \in B$, and that the definition of $A = B$ is that $A \subseteq B$ and $B \subseteq A$. 10 students failed to write the definitions correctly and were filtered out of the study. A perfect usage of the proof framework consisted of 5 uses, as the first proof required two inclusions to prove an equality and the second proof was an iff proof that required proving three inclusions. Therefore, within a single exam, it was possible for each student to sometimes use the framework right, sometimes use it wrong, and sometimes not use it at all. We started our analysis of the proof sections by inspecting the first proof, which asked the students to prove or disprove an equality (see Table I).

The results seemed promising. A large percentage of the students indeed used the framework we had taught them, and a very large percentage out of those received all the points for that part. It is interesting to note that only two students tried to prove by contradiction, which is allowed (it is considered as proving "according to the definitions"). However, their overall answer was not perfect and they did not receive all the points.

TABLE I
SUCCESS RATE FIRST PROOF

	Num	percentage
Failed on the definition part (out of)	10 (310)	3.2%
Disproved or wrote nothing	2 (310)	0.6%
Correct definition and attempted to prove	298 (310)	96.1%
Two correct uses	274 (298)	91.9%
Received all points (two correct uses)	224 (274)	81.7%

There were two conclusions we could draw right away:

1. Almost all students are **familiar** with the framework for set inclusion.
2. Almost all students know how to prove an equality between sets.

Having established that, and encouraged by the students' success in the first proof, we continued to check the second proof, where we found a drastic drop in framework use. Recall that the second proof is an iff claim with one side being an equality and the other side a single inclusion. Hence, we expected the proof to contain three inclusions. However, fewer than half of the students met this expectation by using the framework three times, as shown in Table II.

TABLE II
SUCCESS RATE SECOND PROOF

	Num	Percentage
Failed on the definition part (out of)	10 (310)	3.2%
Disproved or wrote nothing	26 (310)	8.3%
Correct definition and attempted to prove	274 (310)	88.3%
Three correct uses	125 (274)	45.6%
Received all points (three correct uses)	72 (125)	57.6%
At least one by-contradiction (no-use-right)	34 (274)	12.4%
Received all points	12 (34)	35.3%

The results were surprising. We expected that students who were able to use the proof framework correctly in the first proof could do that again in the second proof as well. It is important to note that the numbers clearly show that using the proof framework is potentially beneficial, with a success rate of 81.7% for part one for those who used the framework (compared to 0% for those who did not use it), and a success rate of 57.6% for those who used the framework in part two (compared to 35.3% for those who did not use the proof framework in the second proof). Having noticed the difference in the percentage of framework use between the two proofs, we proceeded to count the number of students who had shown familiarity with the framework (i.e., used it at least once), but who did not use it all the way through the exam. This finding is presented in the first row of Table III. Then we counted the number of students who used the framework perfectly on the first proof (two "correct use"), but made at least one misuse in the second proof, starting with an element belonging to the set in the hypothesis rather than an element in the included set. This finding appears in the second row of Table III. Both numbers (33.5% and 28.5% respectively) seem to be very high, and point to a difficulty in assimilating and implementing the proof framework, even when you have shown that you know how to use it.

TABLE III
CORRECT AND INCORRECT USES

	Num	percentage
At least one correct use and at least one mistake (out of)	100 (298)	33.5%
Two correct uses in first part and at least one misuse in second	78 (274)	28.5%

TABLE IV
MISTAKES IN BOTH PROOFS

	Num	percentage
First proof At least one misuse (out of)	3 (298)	1%
At least one no-use-wrong (out of)	20 (298)	6.7%
Second proof At least one misuse (out of)	85 (274)	31%
At least one no-use-wrong (out of)	39 (274)	14.2%

The large difference in the number of correct usages of the framework between the first proof and the second is especially striking, as can be seen in the differences between Table I and Table II. This overwhelming difference drove us to look deeper into the answers to the second proof. Two findings arose from that inspection:

1. We first ruled out the effect of the "iff". It turned out that **all** the students who attempted to prove the second part but did not follow the framework wrote that there are two directions to be proved. This means that they correctly addressed the iff part of this proof.
2. The second proof, as opposed to the first one, had a hypothesis. Many students began their answer with an element belonging to the set in the hypothesis rather than to an element in the included set. This was found to be the most common "misuse" of the framework.

In order to conjecture about the reasons for the difference between the two proof sections, we asked ourselves how many students made the mistake of using the framework wrong ("misuse") or not using it at all ("no-use-wrong"). The results are presented in Table IV. The type of "misuse" employed by many students in the second part, which included a hypothesis, reflects a problem described in Selden et al., 2010 as "*Students who focus too soon on the hypothesis.*" There can be several explanations for this common phenomenon. Our conjecture is that the students have not yet acquired the habit of relying on the proof framework, and it does not come naturally to them, so they were distracted by the existence of a hypothesis. Another possibility is that the effort of being strict while solving the first proof was followed by a "loosening" later in the exam. The evident conclusion, however, is that whatever the reasons, applying the framework is harder for many students than we may think.

Examples

In this section, we present snippets from actual exams, focusing on the approach taken.

A. Adam

Adam's answer demonstrates a **correct use** of the framework, 5 times as expected:

1. Prove / disprove according to the definitions: For sets A, B, C, $(A\setminus B)\setminus C = (A\setminus C)\setminus(B \cup C)$.
   We will show a bi-directional inclusion. Let $x \in (A\setminus B)\setminus C$ [...] then $x \in (A\setminus C)\setminus(B \cup C)$.
   Now let $x \in (A\setminus C)\setminus(B \cup C)$ [...] so $x \in (A\setminus B)\setminus C$.
2. Prove / disprove according to the definitions: For sets A, B, C, $C \subseteq A$ if and only if $(A \cap B) \cup C = A \cap (B \cup C)$
   We will prove the two claims.
   - First, assume that $(A \cap B) \cup C = A \cap (B \cup C)$ and we will prove that $C \subseteq A$. Let $x \in C$ [...] and also $x \in A$.

- Second, assume that $C \subseteq A$. To prove $(A \cap B) \cup C = A \cap (B \cup C)$ we will prove bi-directional inclusion [...]
  - First, let $x \in A \cap (B \cup C)$ [...] it holds that $x \in (A \cap B) \cup C$.
  - Second, assume that $x \in (A \cap B) \cup C)$ [...] so $x \in A \cap (B \cup C)$.

We proved bi-directional inclusion so by the definition of set equality it holds that $(A \cap B) \cup C = A \cap (B \cup C)$. We proved both iff directions so this completed the proof.

B. Liam

Liam's proof for the first part was perfect, but in the second part, trying to prove equality between sets, he *focuses too soon on the hypothesis*. This is an example of a **misuse**.

[...] Let us prove that if $C \subseteq A$ then $(A \cap B) \cup C = A \cap (B \cup C)$.
According to the definition of equality we will show a bi-directional inclusions.

1. We prove that $(A \cap B) \cup C \subseteq A \cap (B \cup C)$ -- Let $x \in C$ [...]
2. We prove that $A \cap (B \cup C) \subseteq (A \cap B) \cup C$ -- Let $x \in C$ [...]

C. Sean

Sean, like many others, knew that to prove inclusion he should start with an element that belongs to the included set (he did this perfectly in the first part). However, instead of starting with $x \in C$ he started with an x that belongs to one of the sets in the hypothesis:

[...] Assume $(A \cap B) \cup C = A \cap (B \cup C)$ and prove that $C \subseteq A$. Let $x \in (A \cap B) \cup C$ [...]

D. Rita

Rita's proof of the first part was also perfect. However, in the second part she omitted the framework altogether. Her answer is an example of **no use - wrong**:

Let A, B, C be sets. We want to prove that if $C \subseteq A$ then $(A \cap B) \cup C = A \cap (B \cup C)$. By the hypothesis that $C \subseteq A$ it holds that every element that belongs to C also belongs to A. Let us look at the set $(A \cap B) \cup C$ the set of elements that belong to A and also to B, or belong to C. Now let us look at the set $A \cap (B \cup C)$ This set contains the elements that belong to A and to B or the elements that belong to A and to C. Combining the two, and according to the hypothesis, it holds that $(A \cap B) \cup C = A \cap (B \cup C)$.

Conclusions

Our examination showed that most students are able to use a proof framework correctly and that using a proof framework is potentially beneficial (i.e., significantly raises the chances of presenting a correct answer). However, we also discovered that students who were able to present a correct use of the proof framework in one question suddenly failed to use the same framework in a subsequent question. Analyzing examples of answers, we saw that many students neglected the framework that they had been using in the previous question, while others continued to use it, but incorrectly (starting with an element that belongs to the wrong set). This can be attributed to several reasons. At first glance it may seem that the 'iff' added a measure of difficulty to the question, but our analysis suggests that this was not the actual obstacle. Other possible explanations may be that the heavier "load" of the second proof caused students to lose their way, or perhaps they tired as they progressed through the test, thereby dropping the framework and going back to their bad "natural" habits. Finally, when a question contains a hypothesis, it may distract the students, who focus on it too soon. Making recommendations on how to better make use of proof frameworks may be premature at this preliminary stage. However, we can say that employing proof frameworks can indeed be beneficial for students, but that doing so consistently is harder than it may first appear. It appears that additional practice of the framework's more complicated usage scenarios is required. It may also help to present students with actual mistakes to avoid. It is clear, however, that students are able to apply proof frameworks, and we should aim to help them do so in the more complicated proofs.

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Planting Seeds through Embodiment to Teach Formal Concepts of Abstract Algebra

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In this descriptive case study, we explored how a mathematics educator integrated embodiment into a first semester abstract algebra course. We found that, in addition to gesture, the instructor encouraged students to interact with physical materials and simulate mathematics using their bodies. Our results offer practical implications by illustrating examples of how embodiment can be incorporated in an abstract algebra classroom.

Keywords: Cosets, Embodied Cognition; Equivalence relation, Factor Group, Homomorphism

Introduction and Literature Review

The purpose of this work is to contribute to the literature on how embodiment, beyond gesture, can be integrated in the teaching of collegiate mathematics. Nathan (2022) defines embodiments as the use of "body-based resources to make meaning and connect new ideas and representations to prior experiences" (p. 4). In this case study, we addressed the research question: In what ways does a mathematics educator, who is knowledgeable about embodiment, integrate embodiment to support students' learning of abstract algebra concepts? We found that instructor-led embodied activities planted seeds for formal definitions, theorems, and proofs. Additionally, our results indicate that embodiment served as a tool to disambiguate referents of students' speech, to position students' contributions as legitimate, to strengthen the classroom community, and to include students who were not yet fluent in formal language.

A majority of the research related to embodiment intersects gesture and K-12 mathematics teaching. Studies show that teachers use gestures to convey mathematical ideas (Alibali & Nathan, 2012; Font et al., 2010), scaffold material (Alibali & Nathan, 2007), or establish common ground (Alibali, et al., 2013). In some of their more recent work, Alibali et al. (2019) found that teachers sometimes repeated students' verbal utterances and added gestures or made gestures, without speaking, in correspondence with students' verbal utterances towards referents. Gestures appeared to highlight students' verbal utterances, correct students' verbiage, or develop shared understanding in the classroom. Research at the intersection of embodiment and collegiate mathematics teaching is sparce and primarily attends to gesture (Lee et al., 2009; Stewart et al., 2019; Weinberg et al., 2015; Wheeler & Champion, 2013). For example, Lee et al. described how a differential equations instructor used pointing gestures to connect a mathematical problem with inscriptions. Wheeler and Champion portrayed an instructor's gesture to represent mapping of a function in abstract algebra. Weinberg et al. highlighted how an abstract algebra instructor's speech, writing, and gestures took on meaning in relation to each other. Beyond gesture, Stewart et al. explored how a linear algebra instructor navigated between embodied, symbolic, and formal instruction. Stewart et al. found that the instructor often used embodied explanations before moving to symbolic or formal instruction. For example, after defining span, the instructor asked the students to "imagine beginning with a new empty set and including a new vector in the set, one at a time" (p. 1257) and then imagine the span of the set. This was an attempt to help students experience linear independence before formally defining it.

Theoretical Framework

For this research study, we drew upon Nathan's (2022) four types of embodiments that he claims are useful for educational settings as the framework. The first type, *body form, movement,* and *perception,* make up our primary experiences, as we exist in "a body that has form and movement as well as specific perceptual capacities" (Nathan, p. 87). The second type, *gesture,* entails "movements of our hands and arms, as well as other body parts, that we make spontaneously as we talk with others and think to ourselves, including pointing and tracing actions" (p. 89). Nathan defines *simulations* as mental events that "enable us to extend the cognitive processes we have for things that are present and familiar so that we can think about possible things and unfamiliar ideas" (p. 90). The last category, *materials*, refers to how "we use physical objects and material in the world to perform epistemic operations" (p. 93). Nathan also describes four ways that embodiment supports student learning. First, *grounding* connects abstract ideas to concrete experiences to facilitate making meaning. *Offloading* leverages "external resources to manage highly constrained cognitive and attentional resources" (p. 99). *Cognitive-sensorimotor transduction* occurs when intellectual processes and sensorimotor processes interact and support learning unconsciously. Finally, *participation* entails communities of practice where newcomers become old timers (Lave & Wenger, 1991). Nathan believes communities of practice support learning "because they develop effective and efficient ways to collaborate, use cultural tools, and application-specific discourse practices" (p. 106).

Methods

The second author served a dual role of instructor (participant) and researcher. She has almost 30 years of teaching experience, has taught abstract algebra numerous times, and her research area centers on undergraduate mathematics education where she adopts an embodied cognition lens (e.g. Soto & Oehrtman, 2022; Soto-Johnson & Hancock, 2019; Soto-Johnson & Troup, 2014). She tends to follow Tall's (2013) three worlds to navigate between embodiment, symbolism, and formalism while teaching. She encourages students to use their bodies, tangible materials, and technology to learn mathematics (e.g. Dittman et al., 2016; Soto, 2021, 2022; Soto-Johnson & Troup, 2014). She values community and requires her students to work with their peers throughout the semester. For the course, the instructor adopted the text *A Book of Abstract Algebra* (Pinter, 1990) because as an undergraduate she used this text to learn abstract algebra and, as a mathematics educator, she appreciates its constructivist format.

The first author attended each lecture, had weekly conversations with the instructor on the role of embodied cognition in her teaching, audio- and video- recorded six weeks of class, and kept field notes. To capture recorded data, the first author set up two cameras. One stationary camera framed the front of the room, and the second hand-held camera was used to focus on the instructor or students. After the course, the first author and the instructor met twice a week to review recorded classroom data, which was divided into seven- and ten-minute segments. The two authors time-stamped instances when the instructor integrated embodiment, took descriptive notes on those instances, and summarized major themes from each segment. Themes included: regesturing students' gestures, delaying formal language until students understood relevant concrete examples, and transforming students' contributions into inscriptions.

After reviewing classroom data, the research team selected video segments that exemplified the instructor's use of embodiment. We fully coded and analyzed four episodes, but for the sake of brevity, we only discuss the episode related to acting out conjugation, showing a subgroup is normal, and the Fundamental Homomorphism Theorem (FHT). The two non-instructor research members re-watched the selected video segments and coded the types of embodiments as well as how each embodiment type appeared to support students' learning per Nathan's (2022) framework. They then used the codes to write descriptive results (using pseudo names for students) on how the instructor integrated embodiment. The instructor reviewed written results and the entire team engaged in round table discussions to ensure the accuracy and validity of the results, which served as member-checking (Merriam & Tisdell, 2015).

Results

In this section, we summarize how the instructor introduced conjugation, normal subgroups, and the FHT by leveraging all four types of embodiments, which appeared to support students' learning in each of the four ways described by Nathan (2022). Table 1 is a summary of these findings.

Table 1. Instances of embodiment in exploring conjugation, normal subgroups and the FHT

Forms of learning→ Types of embodiments	Grounding	Offloading	Cognitive-sensorimotor transduction	Participation
Body form, movement and perception	Vocabulary "to the right of" and "to the left of"		Described conjugation in terms of direction	
Gesture	Held imaginary group elements to gesture the action of conjugation			Gesture to establish common ground Orchestration of conjugation
Simulation	Students acted out conjugation on a normal subgroup to develop embodied meaning			Students searching for other students to help them conjugate
Materialist epistemology		Paddy paper to represent of D_4		

The instructor created a simulation in which she assigned students to act as elements of $\langle \mathbb{Z}_8, + \rangle$ and $\langle \mathbb{Z}_4, + \rangle$. She asked students to perform a homomorphism from $\langle \mathbb{Z}_8, + \rangle$ to $\langle \mathbb{Z}_4, + \rangle$; she assigned certain students from $\mathbb{Z}_8$ to map to specific elements of $\mathbb{Z}_4$. Students chose to touch the shoulder of the appropriate student to represent mapping. Together, the class verified that the kernel (K) of the homomorphism formed a subgroup. Then, without using the term conjugation, the instructor explained how to act out conjugation on the subgroup $K = \{Margaret, Cam\}$ of $\mathbb{Z}_8$ by stating, "Margaret [or Cam] is going to be in the middle and is going to be operated on the left-hand side by the inverse and on the right-hand side by the element." While describing conjugation in terms of the simulation, the instructor gestured to "hold" Margaret in the middle, “hold” an element’s inverse on the left, and “hold” the element itself on the right, which encouraged her students to imagine conjugation (see Figure 1). The instructor’s gesture grounded her explanation of conjugation within the students’ performance. Additionally, gesture supported participation because it positioned students as central rather than peripheral participants. Students then created their own sign for conjugation in which Margaret (the identity) stood in the middle, Julie ($5 \in Z_8$) stood on the right, and Jim ($5^{-1} \in Z_8$) stood on the left (see Figure 2).

Figure 1. Gesturing Conjugation of the Kernel

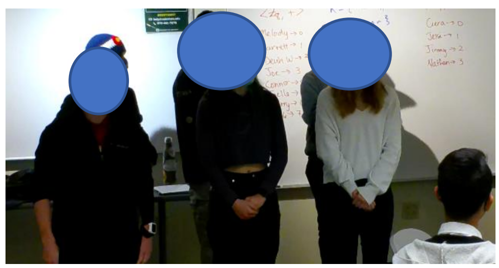

Figure 2. Julie and Jim simulating the action of conjugation on Margaret

The informal language “to the right of” and “to the left of” to describe conjugation is an example of the instructor using body form to support students’ learning because spatial relationships *grounded* conjugation in a primary aspect of the human experience – direction. Body form also supported student learning via *cognitive-sensorimotor transduction* because it reversed the complex notion of conjugation in terms of students’ physical position in space. Given Margaret was the identity element, the class noticed that the action of conjugation by an element (a student) and its inverse (another student) on Margaret always returned Margaret. Then, by acting out conjugation on Cam, students showed that every element-inverse pair in $\mathbb{Z}_8$ returned Cam under conjugation (see Figure 3).

Overall, simulation served as a tool that helped the instructor guide her students in showing that $K = \{Margaret, Cam\}$ was a normal subgroup. The instructor’s use of simulation grounded conjugation and normal subgroup in a meaningful experience before she introduced formal definitions. Moreover, simulation supported student learning via participation. For example, when Patrick conjugated Cam, he looked for his needed classmate (element) to help him perform the operation and students at their desks helped their peers find their inverses, which strengthened the classroom community.

Figure 3. Conjugating Cam by elements of Z_8

After the class (unknowingly) showed that K formed a normal subgroup, the instructor intentionally gestured to recap the simulation. She reminded the class that they began with two groups and "plucked out" a subgroup which generated cosets. Then, via pointing gestures, she emphasized that the two elements in the subgroup they "plucked out" both mapped to the identity in $\mathbb{Z}_4$ (see Figure 4). She also emphasized that the map was a homomorphism and not an isomorphism, which was new to the class. Finally, she emphasized normality of the kernel by stating, "if I take these two elements right **here** [Margaret and Cam] and I operate them by an element from the group and its inverse I somehow end up getting Margaret or Cam." The instructor used gesture to establish common ground because she made sure the class knew that the importance of the simulation was to see an example of a map that was a homomorphism, but not an isomorphism and that the kernel, K, satisfied interesting new properties, i.e., normality.

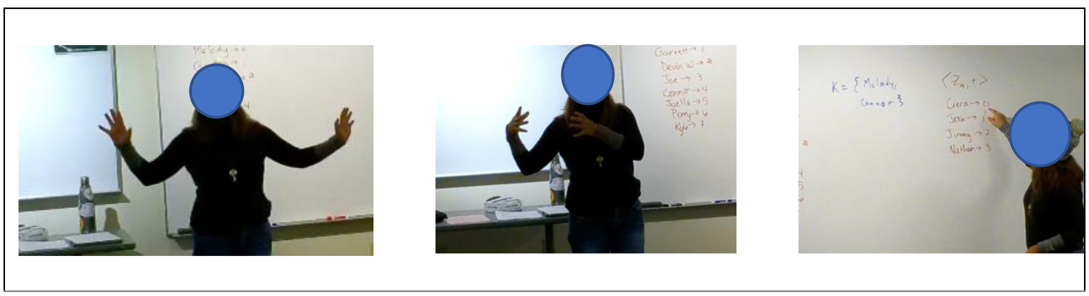

Figure 4. Two groups, pluck out two elements, these elements map to the identity

Following the FHT simulation involving conjugation, the instructor asked the class to complete a problem set related to the group D_4. In conjunction with the problem set, the instructor employed materialist epistemology by holding up a paddy paper and helping the class recall that D_4 represents the symmetries of a square. After distributing paddy paper to each student, she reminded students that all the rotational symmetries keep the square "on the same side." When she noticed Karen flip her square over, the instructor said, "tell me what you just did Karen." Karen replied that she performed a reflection of the square. It seemed that the geometric behavior of the square served to offload information because rather than write down each element of D_4 symbolically, the students could visualize the group D_4 in the material world.

Following this, the instructor encouraged students to look at the Cayley table for D_4 in *Group Explorer* (Carter, 2006), to find patterns in the table, and to reflect on the meaning of the simulation they had just completed. As the students worked in groups, David realized that, looking at this [D_4 Cayley table] … you got your warm and your cool colors here. And earlier in class, … we were talking about, … the cool ones are almost subgroups, but they're not because they don't contain the identity. … now we are realizing that if they have the same number of things like they are kind of subsets, but not subgroups [cosets]. Then that's how you construct a homomorphism to your subgroups or subsets [cosets] … and the homomorphism [image] is isomorphic to S_2 (see Figure 5a).

It appeared that David visualized the FHT. Then, Cam also excitedly changed tabs in his browser to visually show a case of the FHT in a map from $\mathbb{Z}_3 \times \mathbb{Z}_3$ to $\mathbb{Z}_3$ because "you can cut it [$\mathbb{Z}_3 \times \mathbb{Z}_3$] into the nine squares and there is a homomorphism to this one [$\mathbb{Z}_3$]" (see Figure 5b).

Figure 5a. D_4 is homomorphic to S_2

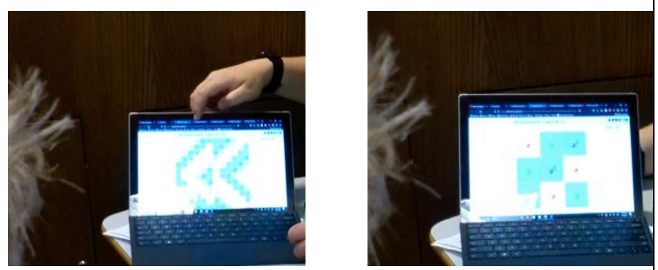

Figure 5b. $Z_3 \times Z_3$ is homomorphic to Z_3

Discussion and Implications

We situate our findings within existing approaches for the teaching of abstract algebra and discuss how gesture, along with other embodiment types, supported shared classroom understandings and helped strengthen the classroom community.

Early instructional design research in abstract algebra was based on actions, processes, objects, and schemas theory (Dubinsky et al., 2001) and the activities, classroom-discussions, and exercises cycle (Asiala et al., 1997). More recently, an inquiry-oriented curriculum was developed by the *Teaching of Abstract Algebra for Understanding* research program, which adheres to the principles of guided reinvention and realistic mathematics education (Gravemeijer & Doorman, 1999; Larsen, et al, 2013; Larsen & Lockwood, 2013). Similar to existing approaches on the teaching of abstract algebra, the instructor in this research study facilitated informal activities before presenting formal definitions, theorems, and proofs. Her approach is novel because she integrated all four types of embodiments described by Nathan's (2022) framework, which also appeared to support student learning in the four ways Nathan described.

Alibali, et al. (2019) drew from a compilation of their research to describe three cases illustrating how teacher gestures can create common ground and support students' contributions. In the first case, students sat at their desks so that they were "physically distant from the referents of their utterances," while the middle school teacher was "proximal to those referents" (p. 350). The teacher regestured and revoiced the student's verbal utterances by pointing at symbolic referents to ensure their interpretation was consistent with the student's intentions before moving on. In the second case, Alibali et al. (2019) identified a variation of gesture by a high school geometry teacher who silently gestured over inscriptions located at the board as a student spoke from their desk to "amplify certain voices" (p. 356). In the third case, both the college teacher and student were within arm’s reach of the referent (an electrical breadboard). Here, the teacher pointed to connect the students’ gestural actions on the electrical breadboard to symbols in a truth table. The instructor from our research study used gesture to establish common ground and support students’ contributions in ways that overlap with and extend Alibali et al.’s (2019) three cases. Specifically, the instructor used various types of gesture (besides pointing), she combined gesture with simulation, and her gestures supported student participation.

The instructor’s gestures that described conjugation by “holding” group elements (students) to the left, right, and center in relation to each other differed from the three cases described in Alibali et al.’s (2019) work on establishing common ground. Compared to the teachers from Alibali et al., this instructor did not sole use pointing gestures; she also used representational gestures, while describing conjugation. Alibali and Nathan (2012) refer to the merging of iconic and metaphorical gestures as representational gestures. McNeill (2005) categorized metaphoric gestures as those which indicate images of an abstract concept such as holding a function in one’s hand and iconic gestures as those which “present images of concrete entities and/or actions” (p. 39). The instructors’ gestures along with simulation closed the distance between students and referents because students were the referents of the instructors’ gestures. Students responded to her gestural description of conjugation by creating a sign: physically standing to the sides of one another. The instructors’ representational gestures, thus, created common ground because students responded with the physical action of standing beside each other and the instructor could ensure that her students understood what she meant by checking that their embodied sign aligned with her knowledge of the conjugation operation.

Nathan (2022) believes communities are helpful for learning because they leverage cultural tools and practices as a means of collaboration. The way that students followed the instructors’ descriptive gesture of conjugation by developing their own tool (the sign of standing beside each other to conjugate) is an example of how gesture can strengthen a classroom community and, thus, establish common ground. Through creating and implementing their own sign, students’ ideas were positioned as significant and actively used by each other to communicate and work as a team. The entanglement of gesture and simulation illustrates how gesture along with other embodiment types can help establish common ground because the entire class adopted a shared sign, which naturally prompted and reinforced shared understandings.

After the simulation ended, the whole class sat at their desks and the instructor stated, “lets recap what we just did.” Unlike the cases described in Alibali et. al.’s (2019) findings on how gesture is implicitly used by teachers to create common ground, this instructor explicitly stated that establishing common ground was her purpose for regesturing. This made the instructors’ intentions clear to her class. The instructor used both pointing and representational gestures to review the significance of the simulation. During her recap, students asked why the order of the kernel divides the order of the group. Student questions suggest that when the instructor explicitly stated that her goal in gesturing was to review, she invited students to engage in reflective practices and students felt that they could ask for help establishing common ground.

Our study provides examples of how embodiment could be integrated in abstract algebra classrooms. In response to Johnson et al. (2020), we believe incorporating embodiment can facilitate understanding and has the potential to lead to equitable outcomes where students’ contributions are supported. Future research may entail creating an embodied curriculum (Wang & Zheng, 2018) and developing embodied simulation activities for abstract algebra using unrefined human experiences, genuine relationships and collaborations, adjustments tailored to the students’ interests and instructors’ expertise, and scenarios that embody real-life experiences.

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Students mathematical working styles and perception of immediate feedback in a digital assessment system

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The use of digital formative assessment in higher education mathematics is increasing throughout the world. Digital tools come with features such as endless ways of creating tasks and assessing in different ways. However, there is a need of research that focuses on what students do when working with digital tasks and how immediate feedback influence their learning process. This study examines students working styles and immediate feedback in STACK (The System for Teaching and Assessment using a Computer Algebra Kernel). The study reports on interviews with five first-year pre-service teachers who had completed a STACK-test for the first time, in order to get information regarding their working styles and perception of immediate feedback. The results indicate that students have issues regarding mathematical representations in a digital environment and a preference for line-by-line solutions in the feedback.

Keywords: Digital assessment, Feedback, Workings Styles, Syntax

Tools for digital teaching as well as digital assessment have become widespread and accessible in mathematics education (Drijvers, 2018). This was accelerated by the Covid-19 pandemic, and now we have reached a point where computer aided assessment is, in some countries, a part of university mathematics. In addition, previous studies (Faber et al., 2017 (Haelermans et al., 2015; Sung et al., 2016) have shown that using digital learning tools for formative assessment may have positive impact on students' achievement. Drijvers (2020) work show that mathematics students' use of digital tools, in addition to pen and paper, lead to new opportunities and constraints. There can also be room for exploration that may lead to explorative and productive activities, whereas the constraints such as syntactical ones, may lead to frustration (Drijvers, 2020). Therefore, research on students' working styles, where they have opportunities to exploit the opportunities and deal with the constraints a digital tool provides, is essential. The aim of this paper is to explore how students work a digital assessment system and how they approach digital mathematical tasks.

In this paper, working styles is used in line with Weigand and Weller (2001) in their report on changes in students' working styles when they worked with mathematical tasks on computers. Working styles were described in three ways, on tool level, representation level and object level. Their research showed that students worked more experimentally and used more search strategies than expected. However, their study also showed that students did not get a better understanding, but a different understanding of the concept of functions. This study will contain observations of how students use the computer and pen and paper, hence tool level. And how students write mathematics using a keyboard, which is related to representation level.

In the study reported from here, the participants worked with tasks in STACK (the System for Teaching and Assessment using a Computer Algebra Kernel). STACK is designed for formative assessment in mathematics, it is internet-based, and the interactions are through a web-browser (Sangwin, 2013). A distinguishing feature of STACK is the opportunity to create detailed and appropriate feedback for each situation (Sangwin, 2015). In STACK, students enter an algebraic expression into a computer using linear syntax. An option within STACK is validating answers, which means invalid answers are rejected and that students confirm their answer is interpreted correctly by the system before submitting.

This is a qualitative study based on interviews with five pre-service teachers who, before the interview, completed STACK-tests within the mathematical topic of algebra. Students were asked about their working styles and perception of feedback in the interviews, and the aim was to answer the following research questions:

1. What working styles do students in higher education use when working with digital tasks in mathematics?
2. How do students perceive immediate feedback in a digital assessment system?

Conceptual framework

To analyse how students in higher education work with mathematical tasks in STACK and get their view on immediate feedback, the concept of "working styles" was operationalised. The concept of mathematical working styles used by Weigand and Weller (2001) is described in three ways:

> First, it has to be seen on the tool level, e.g., whether one works with tools like paper and pencil, computers, or physical models; second, on the representation level, that describes working with representations like symbols, graphs, tablets, diagrams or pictures, and third, on a mathematical object level, where we see a working style as a sequence of actions that operate on objects and transform these objects. (p. 88)

An object can be an equation, an action can be multiplying, and sometimes one step of this sequence involves several sub-steps (Weigand & Weller, 2001). According to Thies and Weigand (2001), the relationship between object and representation is that the mental objects and their real representations are connected inseparably. In this way, operating with objects on the computer is the same as operating with mathematical objects.

The tool in this study is a computer, which entails special mathematical notations for the objects, and mouse-driven or key commands for different actions. This allows a person to work, for some in new ways, with objects or the representations of objects on the screen (Thies & Weigand, 2001). In this study, the concept of working styles will be used to explain how students work with tasks in STACK, and because of the scope of this paper, only the tool and the representation level are emphasised.

On tool level students had to make their answers using a keyboard. However, they had full access to pencil and paper to write notes. Therefore, it is interesting to observe how they use the tools. On representation level, students work with representations of mathematical symbols on a computer and had to use a keyboard and therefore computational notation to give their answer.

Assessment and feedback

Black and Wiliam (2009) define digital formative assessment as a close view of the teacher, technology, and student interaction within a learning process and their functions. Formative assessment may positively impact learning, and teachers, peers, and students are all actors in formative assessment (Dalby & Swan, 2019). Formative feedback is a key element of formative assessment because it provides the learner with information about their current state of knowledge and their learning goals (Narciss et al., 2014). This research conceptualises formative feedback in line with the definition by Black and Wiliam (1998) which "refers to all those activities undertaken by teachers- and by their students in assessing themselves- that provide information to be used as feedback to modify teaching and learning activities" (p. 82).

Digital learning tools might be an essential aspect of formative assessment in higher education. Technology can be used to improve the quality of the feedback received by students during their formative assessment, especially when considering the following three factors according to (Cosi et al., 2020). First, while using a technological tool, the time factor is immediate. Second, students can receive feedback in different formats such as text, video, audio etc., which enrich the message. And third, facilitation of a range of complementary resources is available on the internet.

According to Kulhavy and Stock (1989), the content of the feedback message should comprise two separate components, verification and elaboration. Verification is a judgement of whether the initial response is right or wrong, while the elaborated component consists of more information than just right or wrong. In this study, there were two different STACK-tests. The first one contained a verification component only, which indicated to the students if their response was right or wrong. The second STACK-test contained both a verification and an elaboration component, where the students also got information about the right response.

Methods

Data collection

The participants in this study were first-year student teachers for grades 5–10 at a university in Norway. All participants had deliberately chosen mathematics as their future subject to teach and the data were collected during their first mathematics course. The first lecture, I informed about the project and offered all students to participate in a two-hour working session every week, where they could choose to work with weekly tasks provided by their lecturer or digital tests created in STACK. The students were reminded of this working session each week. The STACK-tests were in addition to the students' regular work and was created as a supplement to their work towards the final exam. The topic of the STACK-test was algebra, which was part of the course content.

In total, 30 students completed the STACK tests, 13 students agreed to participate in the study, and 5 students participated in an in-depth interview. Early in the semester, three students completed the first STACK test and volunteered for interviews. Later in the semester, two more students volunteered after completing the second STACK test. The participants are, in this paper, named Oliver, Roger, Daniel, Harry, and Jack. It is accidental that all participants are male, because in class there was approximately fifty percent female and fifty percent male. Daniel, Harry, and Jack conducted individual interviews and completed the first STACK test, while Oliver and Roger participated in a group interview and completed the second STACK test.

The STACK-tests

STACK uses the computer algebra system Maxima to interpret students typed algebraic expressions and assign outcomes such as feedback and marks (Sangwin, 2019). STACK is equipped with algebraic equivalence, which makes multiple solutions, as long as they are equivalent, correct. STACK has a built-in interpretation of the students' answers; this means that the students' writing is shown interpreted mathematically before the students give their final answers. Regarding students input in STACK, there are six different options for inserting stars (*). In both tests the option of don't insert starts automatically was selected. This means that if there are any pattern identified the result will be an invalid expression.

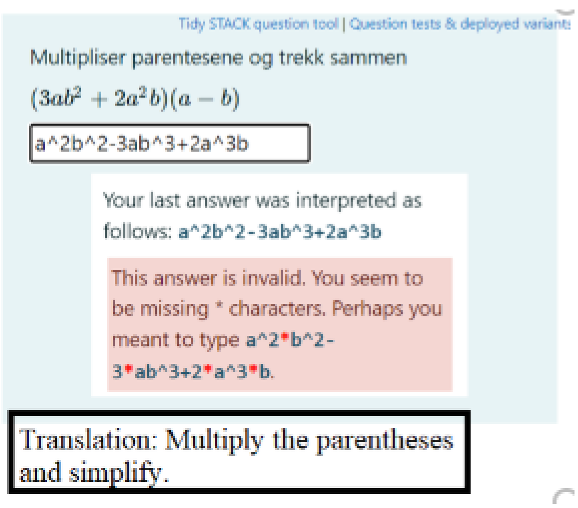

Figure 1. A picture of question 3 which show STACKs interpretations of an invalid expression.

The two tests had 9 similar tasks, and the second test consisted of 6 additional tasks. The first task in both tests concerned acceptance of participation in this study, and students had to actively decide whether to engage or not in this study. The difference between the two tests was primarily the feedback, where the first test had feedback as verification, and the second test had feedback including verification and elaboration. In addition to a difference in verification where the first test a verification was given after each reply, but in the second test, the students didn't get the result until all tasks were answered. All tasks were in the mathematical field of algebra, which was one of the students' mathematical topics this semester.

Because of space limitations and the relevance of this study, not all tasks will be recalled. The aim is to look at what working styles students had in this digital environment and how they wrote the answers into STACK.

Data analysis

To be able to answer the research questions, the interviews were transcribed in their entirety. Afterwards, the transcripts were coded in a process inspired by grounded theory. First, the transcripts were broken into discrete parts, closely examined, and compared for similarities and differences, which Saldaña (2021) refers to as the initial coding process. The transcripts were read statement by statement, and notes were taken on the side. This process created a starting point, which led to further exploration. Some of the notes after this initial process were: improvements, more organized approach to tasks, learning outcome, mastery, multiplication sign, step-by-step response, notation of tasks, and earlier experience.

After preliminary analysis codes that was overarching and related were created (Swanborn, 2018). This led me to two categories which will be elaborated on in this paper: working styles and feedback.

Results

First, I present the findings related to the first research question, addressing working styles students use when solving digital mathematical tasks. Second, findings related to feedback will be presented. Both parts will mainly consist of reproduction of the five interviews with Oliver, Roger, Harry, Daniel and Jack.

Working styles

In this section, ways of working with mathematical tasks digitally are reported. Oliver says that he always writes down each task on his tablet (similar to pen and paper) and calculates line by line. He had a similar approach to the STACK tasks, where he wrote line-by-line what he did in each task. This is a working style he always uses when approaching a mathematical task involving algebra. Oliver says that this way of approaching tasks give him a good overview of what needs to be done in order to solve the task. Roger explains that he tried to calculate in his mind first because he saw some of the tasks to be straightforward. If there were tasks, he could not calculate in his mind, he wrote down some notations to get back on the right track. These notations contained steps he knew how to solve but could not calculate it in his head. He explains that he enjoyed especially a task about quadratic identities, because he immediately saw that this was the case. In that task he had to understand how to calculate with parentheses, which he knew how to do, it became a practice on routine. Roger used STACK to check if his calculations were correct. Harry perceives algebra as a topic he manages well and explains that he did most of the calculations in his mind. He used STACK to guess and check his if the calculations done in his head was correct. If he did not know the answer, he guessed something, in order to look at the right answer after completing the task. He emphasized that since this was an assessment with no impact on his grade, the guessing and checking approach worked fine. If the test had an impact in his grade he would use a different approach, where he would have written down each task and made line-by-line calculations. He had high expectations of his work, especially work that had an impact on his grade. Daniel also explains that since this was a digital test with no impact on his grade, and the level of the tasks was not that high, he did most of the calculations in his mind. He used pen and paper to do small calculations he could not complete in his head. He reports that in this test, and digital tests in general, he does more calculations in his mind and uses the assessment system as a way to check if his answers are correct. Oliver used his tablet, Roger and Daniel used pen and paper in addition to the computer as tools. Harry did use any tool in addition to write notation in this test.

Using STACK for the first time requires writing mathematics with the use of a keyboard. All five students made comments regarding the multiplication (*) sign. The tests were programmed with an option where the program did not insert * characters automatically. Daniel said that remembering to write this sign everywhere because I know it is there. And since we do not usually write it, he forgot to write it in each task. In the interview with Oliver and Roger, a discussion about the multiplication sign occurred because Roger emphasized the importance of knowing that $4a$ actually is 4 times a. However, Oliver said that since he already knows that the sign is there, it is a waste of time to write it every time. The had different views on the importance of this * sign. Oliver was clearly frustrated about this sign, while Roger saw it as a way of writing mathematics digitally. Harry faced a challenge when he was going to write the power of a number. He did not know which symbol belonged to the right mathematical meaning. After spending some time thinking about it, he remembered using GeoGebra in upper secondary school and remembered what key it was. Jack also had some challenges with finding the power of a number. His approach was to write the wrong answer and then look at the right answer in the feedback. Then he could use this to remember which key it was if another similar task appeared. Another challenge Harry reported was that in one task, he wrote the wrong mathematical symbol and got the wrong answer. He said that he did not remember what this symbol was and that this mistake would never happen if he had written these tasks on pen and paper. This shows that students had different approach to finding the right symbol to a mathematical meaning. It is clear that this challenge would not happen if this test was created on paper, and the students were required to write the answers on paper. They knew what the mathematical meaning was, and how to write it on paper, but not the transition to a computer keyboard was difficult. If this test would have had an impact on their grade, Jack would have gotten the wrong answer if he deliberately answered wrong to look at the symbol. It is worth knowing that this can be challenging, because students mark may not reflect their knowledge.

Feedback

Feedback was given differently in the two tests. The first test had a verification component, and feedback was given immediately after each task. In the second test, feedback contained a verification component and details about the correct answer and was given after all tasks were answered. Jack, Roger, and Oliver explained that they would like a step-by-step solution after completing each task. They all agreed that this would help them in the learning process. Oliver and Roger emphasized that it will be easier to know if you are on the right path if you get a response after each task. Oliver said that, in this case, if you have to go through 15 tasks without any feedback, you had no way of knowing you were on the right track. However, if you got feedback earlier, you would know that you were doing right. Jack explains that in some of the tasks he was close to the correct solution, and if he got a line-by-line solution as feedback, he would quickly understand his mistakes. If a similar task would appear, then he could use the feedback in a different situation. He also explains that tasks he did not know how to approach, just by viewing the right answer in the feedback, he would not know how to do it in a similar situation. Daniel elaborated that a line-by-line solution given as feedback would help him figure out exactly where his mistakes were. And that a right answer and a line-by-line solution would help his mathematical understanding and development.

Discussion

This work sheds light on students' own report on working styles and how they perceive immediate feedback in a digital environment. In the case of working styles, the students consider writing mathematics on a computer to be challenging. The students had to use the computer as a tool to communicate mathematics. They also had access to paper and pencil or tablet to write notes during the tests. On tool level, Roger, Daniel, and Harry compared their work done in STACK with mathematical work they usually do on paper. The three of them agreed that if this test were on paper, they would write down more detailed line-by-line calculations on paper. As Usiskin (2004) points out, we have turned to paper and pencil for the last 400 years regarding transformational activities in algebra and arithmetic. Daniel, Harry, and Roger reported that they did most of the calculations in their mind. There are some explanations why this might be the case. First of all, this test had no impact on the students' grades. There might be no incentive for the students to complete all calculations as thorough as they would in a grading situation. Second, there was an opportunity to guess the answers and look at the feedback to see if they were on the right track.

There were some issues on the representation level because the notation used on a keyboard is sometimes different compared to the notation using pencil and paper or a tablet. Some students reported their frustration regarding finding right mathematical meaning, and some experience frustration regarding the multiplication sign, however, using the computer as a tool seemed like something the students are used to in different settings. Harry, Jack, and Daniel report that they have trouble finding the right key on the keyboard that belongs to the right mathematical meaning. The keyboard is the basic interaction with a computer, and according to Sangwin (2013), assigning definite meanings to individual symbols is a problem that needs to be solved.

Since this is the students first meeting with STACK in a test, and with no knowledge about the student's prior experience with writing mathematics digitally, these results were expected.

Students in this study also reported that they forgot to write the multiplication sign in places in line with the regular syntax of mathematics. There is an option in STACK to remove this function; however, as students become teachers, it is fundamentally important to know that the multiplication sign is there. While this issue frustrated Oliver and Roger, Jack saw it as a reminder of the difference between number and variable. This type of syntax error is not interpreted as a wrong result in STACK, but purely a technical error (Sangwin & Ramsden, 2007). A question to be raised for further use of STACK is whether to have this function switched on or off because it might lead to less frustrated students if you remove this function. On the other hand, this function is important, especially for pre-service teachers.

Oliver and Roger completed the second STACK test, where feedback was given after completing all tasks, while Jack, Daniel, and Harry completed the first STACK test where feedback was given after each task. Jack, Daniel, and Roger reported that they would value a step-by-step solution after each task. They all reported they would value more information than just verification or justification and the right answer. Kluger and DeNisi (1996) found that task-level feedback often emphasizes one solution, or strategy, and might therefore be a hinder for the transition to more complex tasks. If a solution to a task is presented to the students, they might lock their future work to only this way of solving a similar task and may therefore not help the students to think about other ways of solving a task. This issue needs more research because what the students want and what they will learn most from is debated.

A report from Shute (2008) says that step-by-step feedback is often more effective than presenting all feedback together. Given a summary after completing the whole test might be perceived as overwhelming, and as a result, students may reject all feedback given. This is in line with what the students in this study report.

Conclusion

Typographical errors can easily be missed by students, resulting in wrong answers. This might lead to frustrated students and should be avoided if possible. There are simple things lecturers can do to avert this problem when introducing STACK to students. When STACK as an assessment system is implemented, students should be taught the usual pitfalls, such as the multiplication sign (if this function is on) and learn which key to push for the most common mathematical operations and meanings. Spending time in class on mathematical representations used in a mathematical digital environment may help the students from spending time on finding the right key for the right mathematical symbol.

Students also report that they would appreciate a line-by-line solution as a part of their feedback. However, previous research is not clear that this is the best for their learning outcome. This study lacks data to conclude about students' learning outcome regarding feedback.

In this study, several students report that they calculated without using written notation. This also raises questions for further investigations. Is this a common case for students when they use computer as a tool to write mathematics? Is this a representations level case? Is this a result only in this study, or are the students working in a digital environment more into guessing and checking? More work needs to be done in order to answer these questions.

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Variation in Students' Thinking about Sameness and Comparisons to Mathematicians

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Sameness is a topic threaded throughout all levels of mathematics and yet it has not received much focus from the mathematics education research community. Our work categorizes the dimensions of variation that students highlighted when asked to describe sameness using a recent framework on sameness in mathematics. We captured students' ideas about sameness through a pair of open response surveys at multiple doctoral-granting institutions across the United States. In this presentation, we report on the variation found in the student surveys and also compare our findings to a prior study of surveyed mathematicians.

Keywords: Sameness, Dimensions of Variation, Example Space, Advanced Mathematics

Introduction and Background

The concept of equivalence is fundamental in mathematics (Asghari, 2009; 2019). Students are introduced early in their schooling to a specific type of equivalence, mathematical equality, and continue to use equality throughout their mathematics courses (e.g., National Governors Association Center for Best Practices, 2010). Despite their familiarity with the concept, students do not always develop a robust conceptual understanding of equality, which can hinder their ability to engage with middle school algebra topics like solving equations (e.g., Alibali et al., 2007; Kieran, 1981). In undergraduate mathematics courses, students encounter more complex ideas of equivalence like the notion of isomorphism in abstract algebra. Studies have shown that students, professors, and researchers all associate isomorphism and sameness (e.g., Dubinsky et al., 1994; Leron et al., 1995; Rupnow, 2019; 2021; Weber & Alcock, 2004).

Early work on students' understanding of mathematical sameness focused on alternative interpretations of operations and objects from those of the researchers (Melhuish & Czocher, 2020). Still, little is known about how undergraduate and graduate students understand sameness. Recent studies focused on mathematicians' understandings of sameness have categorized notions of sameness according to salient features and examined norms while interpreting sameness (Rupnow et al., 2022; Rupnow & Sassman, 2022). While mathematicians are able to recognize and distinguish between types of sameness, it remains unknown whether students do so in similar ways or what aspects they attend to.

Furthermore, math education research has benefited from comparing experts' understandings to those of students, such as in the context of example usage (Lockwood et al., 2016; Lynch & Lockwood, 2019) and proof production (Weber & Alcock, 2004). These types of studies provide opportunities to leverage mathematicians' expertise to find potentially productive connections and ways of thinking to teach students. In this paper we examine the ways in which mathematics undergraduate and graduate students describe the notion of sameness and how these compare to mathematicians.

Theoretical Perspective

In this study, we employed the notion of example space (e.g., Watson & Mason, 2005) to characterize how undergraduate and graduate students view mathematical sameness. Example spaces are the collections of examples that are associated with a particular concept and include the ways of making such examples. The structure of an example space can be described in terms of dimensions of possible variation, which refers to the aspects that are allowed to vary, and the range of variation of a particular topic, which involves the different options in the dimensions (Watson & Mason, 2005). In particular, we drew on the community example space created from a survey of mathematicians, largely algebraists (Rupnow et al., 2022), which characterized dimensions of variation of sameness used and range within those dimensions. Rupnow and colleagues (2022) highlighted five dimensions of variation: universes of discourse, concepts, objects, properties, and qualities. We adopted this framework to analyze our student data. In so doing, we provide a community example space for undergraduate and graduate students to highlight dimensions and range of variation of mathematical sameness. We also explore the differences and similarities between students' descriptions of sameness and those of mathematicians. In particular, we address the following research questions:

1. Which dimensions of variation did students highlight when describing mathematical sameness?
2. In what ways were the examples and characterizations of sameness provided by our students similar and different to those provided by algebraists?

Methods

Data Collection

We conducted a pair of open response surveys at doctoral-granting institutions across the United States. Each survey consisted of fifteen questions aimed to capture students' ideas about sameness. The first survey, Survey-A, invited actively registered abstract algebra students at three different institutions. A member of the research team visited each of the three courses to personally invite students to participate. During data collection, each of the courses offered at the selected institutions were cross-listed; that is, the courses were listed in the course catalogs as available to both undergraduate and graduate students. Therefore, the 15 students who provided responses to Survey-A were either undergraduate or master's students currently enrolled in abstract algebra. The follow up survey, Survey-G, was also conducted at three doctoral-granting institutions across the United States and invited any current master's or doctoral student in mathematics. For Survey-G, students were emailed an invitation through departmental lists of graduate students, there were 12 responses in total. Responses to both Survey-A and Survey-G serve as the dataset, consisting of 27 total student responses.

While both surveys asked how students think about sameness in general and in specific contexts, the wording of the questions was slightly different. The questions we chose as most relevant for analysis are as follows:

- (Q2) What does it mean to be the same in a math context? (both)
- (Q4) How do you know two things are the same in abstract algebra? Is this the same or different from other classes? (Survey A)
- (Q4a) How do you know two things are the same in abstract algebra? (Survey G)
- (Q4b) How is sameness in abstract algebra similar or different from sameness in other branches of mathematics? (Survey G)

Data Analysis

The research team, consisting of abstract algebra education researchers, analyzed the student responses. We used Rupnow and colleagues' (2022) sameness framework describing the various contexts, concepts, objects, properties, and qualities related to sameness. Each member individually coded all student responses to the survey questions provided above. Together, the team compared coding, discussed any discrepancies, and came to group consensus, resulting in total inter-coder agreement. To compare our findings to those of the mathematicians, we considered the differences in prevalence of each dimension.

Results

The results that follow report the universes of discourse, concepts, objects, properties, and qualities used by students when describing sameness. See Rupnow and colleagues (2022) for a detailed definition for each code found in Tables 1-5. Within each table, the frequency ordering of the mathematician paper is maintained to permit comparison of relative frequencies between the mathematicians and students. Following each table, we also provide illustrative examples of common student responses to better characterize the findings.

Table 1. Universe of discourse for sameness

Context[a]	Survey-A	Survey-G	% of total
Linear Algebra	0	4	15%
Topology	2	7	33%
Geometry	0	2	7%
Discrete Math	0	3	11%
Calculus[b]	-	-	-
Arithmetic/pre-university algebra[b]	-	-	-
Analysis	0	5	19%
Category Theory	0	3	11%
Other	0	2	7%
Set Theory	1	2	11%

[a]Some questions prompted consideration of or comparisons to abstract algebra so we do not report frequencies for abstract algebra, but we do discuss ways that discussion of abstract algebra was positioned below.
[b]Not present in graduate student responses

In Table 1, we focus on descriptions of each universe of discourse (context) for discussions about sameness. Among our student participants, topology was the most common context. Some responses broadly noted similarities and differences between abstract algebra and topology: "Another branch of mathematics where I think about sameness is topology. I think sameness in abstract algebra is both similar to and different from sameness in topology." Others linked contexts with particular concepts: "For instance, in topology, we have homeomorphisms that establish the 'sameness' of topological spaces."

Other responses made use of linear algebra and analysis contexts. For instance, a participant focused on co-existence in the same vector space as characterized by the span: "A vector v in a vector space identified with all of its nonzero scalar multiples. In other words, v is the 'same' as w if v = a.w[1] for some nonzero scalar a." Analysis contexts included oblique references to sameness of functions equal almost everywhere (a.e.): "In a more elaborate sense, one can argue that usually in algebra the homomorphisms are bijections whereas in analysis (measure theory) often time ignores sets that are very small."

In general, we note that the prevalence of each universe of discourse roughly mirrors the mathematicians, though our students did not allude to arithmetic or calculus to make points about

[1] We interpret v = a.w as the product $\mathbf{v} = a\mathbf{w}$.

sameness, nor did many use geometry examples. We further note that the preponderance of instances are grounded in areas of math outside abstract algebra and the variety in contexts was driven by the graduate students in the second survey.

Table 2. Concept conveying a type of sameness

Concept	Survey-A	Survey-G	% of total
Isomorphism	9	11	74%
Equality	6	6	44%
Other (e.g., categorical equivalence)	1	4	19%
Equivalence Relation	5	1	22%
Congruence	1	1	7%
Homomorphism	1	4	19%
Homeomorphism	2	6	30%
Diffeomorphism/Homotopy equivalence	0	1	4%
Similar (geometry)	0	1	4%
Linear transformation[a]	-	-	-

[a]Not present in graduate student responses

Table 2 highlights concepts conveying a type of sameness. Among our participants, isomorphism was the most common concept conveying sameness, followed by equality and homeomorphism. Discussions about isomorphism generally were positioned within abstract algebra and emphasized sameness of underlying structures: "We know two things are the same in abstract algebra when we can construct an isomorphism between the two structures. That is, the two structures may not look the same, but they have the same properties." While some references to equality occurred in isolation, many references were used to contrast with isomorphism:

> The meaning of "same" in a math context is a vague one. We can say two groups are the same if there exists an isomorphism between them, but the elements of the respective groups may be different. We may also say things are equal which is a different meaning than isomorphism but also implies a form of sameness.

While this participant's characterization of isomorphism is more specific than that of equality, we believe it is noteworthy that they used multiple concepts to characterize sameness. Similarly, in keeping with the large number of responses (especially in the second survey) positioning discussion in topology, we observed a number of instances of homeomorphism, especially in contrast with isomorphism. For instance:

> They are the same if there is an isomorphism between them. It may be an isomorphism of groups, rings or modules…The sameness in algebra means that the relationships between elements in one structure is the same as the relationships between the elements of the other structure with respect to the underlying operations. In other situations in math, this may not hold true, but two things might be considered the same if they share some property in common or if there is some special bijective function between them. For example, topological equivalence means there is a homeomorphism between the two sets. They can be considered the same in the sense that one can be molded into the other as opposed to algebraic sameness where the underlying mechanics of two sets are the same but they are labeled differently.

Specifically, while we recognize that questions asking about sameness in abstract algebra likely drove the high number of references to isomorphism, we were encouraged by the number of responses that characterized sameness using more than one concept (6 of 15 in the first survey, 11 of 12 in the second survey). We also observe that while there was a higher percentage of references to homeomorphism and fewer to congruence in this survey than the mathematicians, isomorphism and equality were the most commonly discussed in both settings.

Table 3. Objects to which sameness is applied

Object	Survey-A	Survey-G	% of total
Group	4	6	37%
Ring	2	5	26%
Other (e.g., categories, modules)	3	9	44%
Vector Space	0	4	15%
Set	2	8	37%
Number	1	2	11%
Function	0	2	7%
Field	0	1	4%
Quotients	1	0	4%
Elements	3	3	22%
Triangle	0	1	4%

Table 3 highlights objects that could be 'the same'. In addition to references to groups and rings (similar to the mathematicians), our student participants centered sets and elements as well as a variety of Other objects (e.g., categories, graphs, modules, manifolds). Respondents often listed multiple examples of structures and sometimes used these lists to contrast with sets:

> Most areas of mathematics have a notion of isomorphism between the objects of interest. Exact equality is almost always a much too strong condition to be of any significance when studying mathematical objects. In topology, homeomorphic spaces are looked at as "the same"; in graph theory there is a notion of isomorphic graphs; in analysis one usually studies spaces up to an isometric isomorphism; and even in category theory there is a notion of equivalent categories. This does contrast with set theory where in many cases one does care about exact equality of sets and not just a bijection between them.

Others used lists to contrast whole objects with elements, such as:

> In my experience in linear algebra, the "things" that you'd deal with are structures (groups, rings, fields, modules, ...) and their elements. The elements of the same structure are the same if and only if removing both of them is the same as removing one of them (sameness in the sense of set elements). Two structures are equal if they are isomorphic. This guarantees that they share the same algebraic properties. This will mean that sameness is [an] equivalen[ce] relation with equivalence classes.

We again observe, most of the variety in objects, both in the Other category and generally, were driven by responses to the second survey.

Table 4. Properties conveying sameness or a lack of sameness

Properties	Survey-A	Survey-G	% of total
Cardinality / Bijection	3	8	41%
Representation	1	4	19%
Other (e.g., commutativity, generators)	2	5	26%
Dimension	0	1	4%

Table 4 highlights properties of objects that should be shared for objects to be the same or that could vary and still allow the objects to be the same. Similar choices of properties arose in this survey as in the mathematicians' survey. There were numerous references to bijections, especially as they related to understanding concepts listed above, like isomorphism and homeomorphism. Other properties that students noted included orders of elements within a group, commutativity, and topological invariants. One participant highlighting multiple properties wrote:

> Two isomorphic groups that are essentially the same have similar properties (like the same order of the group or the same orders of its elements) and behave in similar ways with respect to how elements are operated together (e.g., there is a homomorphism between the groups, and the generators of one group correspond to the generators of another group).

Note this participant commented on cardinality (orders of groups) as well as Other properties of orders of elements and relationships between generators.

Table 5. Qualities of sameness

Qualities	Survey-A	Survey-G	% of total
Levels of Sameness	5	5	37%
Computability / Construction	2	2	15%
Logical equivalence	1	2	11%

Table 5 highlights qualities of sameness, namely ways to compare the types of sameness conveyed by specific concepts. For instance, the following response received a levels of sameness code:

> Two things are exactly the same if, and only if, all of their properties have the same value. We often discuss sameness without meaning, "exactly the same". Here, we reduce our scope to specific properties and check to see if two things have these specific properties in common. Then we call them the same in some sense. Other times, if two things have exceptionally close property values we might call them (knowing that it is strictly untrue) the same. So, in a math context, when one declares two things to be the same, one may mean they are exactly the same. See above. Or, one may mean only some of their properties are the same. This is almost always intuited, or explicitly given, in context.

This segment, which came at the end of an extended response in which five axioms and six definitions were provided to carefully characterize sameness, highlights a distinction between being "exactly the same" and having many properties be the same, which is often sufficient to be called the same in a particular context.

The computability/construction code and logical equivalence code captured ways of demonstrating sameness of object. Computability/construction focused on creating specific mappings (e.g., "We know two things are the same in abstract algebra when we can construct an isomorphism between the two structures."), whereas logical equivalence focused on the use or existence of theorems that can be used to draw the same conclusions, such as the following:

> Often times in mathematics some results appear to be equivalent to some other result although their proof might be completely different. For example, the axiom of choice is equivalent to Zorn's lemma, Tychonoff's theorem, the Well ordering principle, and a whole lot of results in different fields of mathematics.

Unlike the other dimensions, we saw similar frequencies in the descriptions of qualities from both surveys and these frequencies occurred in a similar ordering to that of the mathematicians.

Discussion

In this paper, we have observed many similarities between the students of our data set and previous work with mathematicians (Rupnow et al., 2022). All dimensions of sameness (contexts, concepts, objects, properties, and qualities) in the mathematician study arose in the student data. Many of the particular examples of each dimension, that together comprise the range of variation for each dimension, were also cited. When aggregating across our students, we see similarities to the mathematicians in the ordering of examples cited, including emphasis on isomorphism and equality as concepts conveying sameness, bijections/same cardinality being an important property for sameness, and nuance in qualities of sameness such as the existence of different levels of sameness. Some differences are explainable by the participants' backgrounds, such as frequent emphasis on abstract algebra and algebraic objects (groups, rings) among the mathematicians, who were largely algebraists. Whereas students, some of whom did research in applied math or analysis, gave more emphasis to analysis and other non-algebraic objects.

Examining the data in closer detail, however, it may not be fair to lump together our two student samples. The respondents to Survey-A were students in an abstract algebra course, which included both undergraduates and first-year graduate students in math education. The respondents to Survey-G were exclusively graduate students and included students farther into their programs. We also see some differences in the range of variation discussed. Much of the range in each dimension of variation, which mirrored the mathematicians, was driven by the Survey-G respondents. Except for the concept *equivalence relation* and object *quotient,* the same or more participants in the second sample provided each example within each dimension of variation, despite fewer participants. Thus in this sample, the Survey-G respondents discussed a greater range of topics per person than students responding to Survey-A.

While we recognize this is a small data set, we believe the differences between our two student samples spark questions warranting further research. In particular, it is possible that some formative experiences happen in early graduate school that lead graduate students to write and act more like mathematicians beyond simply taking more courses. Research might examine the impact of studying for preliminary/comprehensive exams on graduate students' perceptions of mathematics and abilities to make connections across their coursework. It is also possible that students who make stronger connections across courses are more likely to want to pursue graduate work in mathematics; thus, some weeding process may occur before graduate school. Other experiences or mechanisms may also be important to graduate students' development of nuanced views of sameness. While RUME is focused on undergraduates' experiences, we nevertheless encourage such work on the transition from undergraduate to graduate students to be included in the RUME community because of the potential impact on understanding what motivates undergraduates for higher study.

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How do instructors understand unequal representation in STEM?

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While trends over the last few decades show increases in women and non-white scholars in STEM fields, there remains disparate attrition at different transition points within academia. The underrepresentation of certain demographic groups in STEM is broadly understood as rooted in systematic historical oppression and exclusion which continue their presence in various ways – but it is not clear how understood this is by faculty at large. To better understand faculty beliefs related to the (dis)advantagement of race-gender groups in STEM, we conducted a Latent Class Analysis on 945 responses from instructors of introductory chemistry, physics, and calculus courses at a variety of institutions across the United States. We then looked to see how those views relate to faculty support of policies and activities which might advance diversity and equity in postsecondary STEM education. Four clear patterns are discussed.

Keywords: Equity, instructor beliefs, DEI, STEM instructors

One symptom of the systemic inequities plaguing our society is that the race-gender demographics of STEM fields in the United States are markedly dissimilar from those of the country and the undergraduate student population. Though representational diversity is not equivalent to equity or justice, the persistent discrepancies in representation are certainly indicators of a larger problem. Trends over the last few decades show increases in women and scholars of color[1], with variation between different STEM fields (NCSES, 2021). However, there continue to be disparate attrition for women (Shaw & Stanton, 2012) and non-white (Shaw et al., 2021) scholars at different transition points within academia.

The underrepresentation of certain demographic groups is broadly understood as rooted in systematic historical oppression and exclusion which, while now ostensibly outlawed, continue their presence in various ways – but it is not clear how understood this is by most faculty. Current faculty are key actors within the broader educational system: they interact regularly with students, make decisions about who to mentor (and how), and have some say in policy decisions within their academic units. How they engage in these activities has the potential to impact recruitment and retention. While instructors may occupy a role in which they can support diversification of the fields and a shift toward more equitable systems, they may not be aware of this potential (Gandhi-Lee et al., 2017) or even the need for change – and in some cases may push back against efforts to advance diversity, equity, and inclusion. In order to better recruit faculty to the cause of advancing equity in higher education, we must better understand their current views and the organizational culture(s) in which they exist – parallel to the work our field has done to recruit instructors to use research-based instructional practices (e.g., Apkarian et al.,

[1] We note that gender is not a wo/man binary, that race is a fluid and complex social construct which should not be reduced to simplistic categorizations, and the intersectional nature of identity. However, we are limited in some respects by how these terms are used generally in society, federal databases, and prior research.

2021; Chowdhury et al., 2021; Johnson et al., 2018, 2019; Reinholz & Apkarian, 2018; Yik et al., 2022). This study takes a first step toward that end, reporting on a subset of:

1. *Classes (or patterns) of instructor beliefs related to the (dis)advantagement of race-gender groups in STEM which exist among current instructors; and*
2. *Relationships between class membership and support for activities in support of advancing DEI in postsecondary STEM contexts.*

Theoretical Background & Relevant Literature

We take the position that equity is a moral imperative, and that STEM, education, and academia are currently inequitable systems which need to undergo systemic cultural change. Faced with institutional inertia as well as complex logistical, economic, and political considerations, we look in the short-term to the role of faculty to support students and shift disciplinary culture to be more inclusive and equitable. In essence, faculty act as the ultimate gatekeepers of the disciplines, and their actions – influenced by their beliefs – have great power to support or constrain efforts toward diversifying and making equitable the STEM fields.

Among the indicators of inequity in STEM higher education is the race-gender imbalance in representation (Apkarian, in press; NCSES, 2021). Some of the immediate factors impacting students' intention to pursue STEM (e.g., recruitment) include existing interest in STEM, positive math and/or science identity, and academic preparation (Cribbs et al., 2015; Gandhi-Lee et al., 2017; Graven & Heyd-Metzuyanim, 2019; Langer-Osuna & Esmonde, 2017); of course, these are not equally accessible to, nor cultivated in, all students. Beyond recruitment, there are disparate rates of retention at various junctions along the path to a STEM career (Ellis et al., 2016; Hatfield et al., 2022; NCSES, 2021; Shaw et al., 2021; Shaw & Stanton, 2012; Wapman et al., 2022). Additionally, there are cultural systems, specific to mathematics and science. These include a racialized hierarchy of mathematical ability which generally manifests as an expectation that Asian (men) are 'too good' while those who are not White or Asian are 'not good enough' at mathematics (Martin, 2003; McGee & Martin, 2011; Shah, 2017, 2019). There are additional gendered norms and expectations, which often take the form of suggesting that women are not as capable of mathematical or scientific thought as men (Handley et al., 2015; Hottinger, 2016; Leyva et al., 2016). Of course, these are intersecting systems which produce intersectional experiences (Castro & Collins, 2021; Cech, 2022; Covarrubias, 2011; Gholson & Martin, 2019; Williams et al., 2014). These, and additional systems of inequity such as ableism (e.g., Annamma et al., 2013; Bettencourt et al., 2018; Taylor & Shallish, 2019) are often hidden behind a belief that STEM is insulated from racism and sexism, and is instead a true meritocracy (Au, 2013; Cislak et al., 2018; Handley et al., 2015; Liu, 2011; Sobuwa & McKenna, 2019).

While there exist a number of strategies employed by higher education institutions to try to acknowledge, mitigate, and/or address aspects of system inequity (e.g., affirmative action, and similar, policies for student admissions and faculty hiring; diversity statements), these are not uniformly supported by faculty nor is there unanimous consensus among STEM faculty that they have a personal responsibility to address inequities in their disciplines (Johnson et al. 2022). We take support for, and engagement with, policies and efforts to diversify STEM fields as aspects of faculty professional practice and, just as with other aspects of professional practice (e.g., instructional practice) we see faculty beliefs as both a key influence and a necessity for change.

Participants & Data Collection

Data come from a web-based survey of postsecondary instructors distributed in Fall 2020. Participants were instructors of introductory chemistry, physics, and calculus courses at a variety of institutions (including two-year colleges, four-year colleges, and universities). Each had completed a survey in 2019 regarding their instructional practices and factors impacting their pedagogical choices as part of a larger project; they were then invited to participate in this follow-up regarding their experiences with, and attitudes toward, DEI issues and initiatives in higher education. For the follow-up survey, we received 1064 responses from the 2229 invited to participate (47.7% response rate).

We focus this analysis on a series of questions related to beliefs about different race-gender groups and STEM. These items were designed to gauge whether instructors recognize the systemic advantages and disadvantages related to race and gender in the sciences as well as to gauge instructors' beliefs about various explanations for under-representation of demographic groups compared to white cisgender men. Previous analysis (Johnson et al., 2022) showed that, in the aggregate, many STEM instructors recognize relative systemic (dis)advantages based on racial and gender characteristics; few suggested variation in aptitude for STEM; responses were more mixed with regards to the distribution of opportunity and interest in the disciplines. It also revealed that instructors were more likely to support statements in support of DEI than affirmative action style policies. This analysis seeks to understand patterns within these responses in order to better understand clusters of attitudes and positions relevant for faculty support to advance equity and inclusion in STEM.

Latent Class Analysis

We employed *latent class analysis* (LCA) to identify potential groups or classes among our participants, which might drive patterns in their responses. LCA is a type of structural equation modeling used to identify groups (called latent classes) in multivariate categorical data. The data used for the LCA is the 22 items about (dis)advantage, opportunity, aptitude, and interest in STEM for different race-gender groups. In the survey items regarding (dis)advantage, respondents were presented with 8 race-gender pairs (Asian Women, Black Women, Hispanic/Latinx Women, White Women, Asian Men, Black Men, Hispanic/Latinx Men, White Men) and asked "From your perspective, do people from each of the following demographic groups receive a net systematic advantage or disadvantage due to their demographic characteristics in [your field]?" For the survey items regarding opportunity, aptitude, and interest in STEM, respondents were asked "More than half of the PhDs awarded in 2017 in mathematics and the physical sciences were received by white men. As compared to the majority group (white men), how much interest, aptitude, and opportunities do you think other groups have in [your field]?". They were prompted to respond with regard to the seven other race-gender pairs. Each of these 22 items had three response options, more/equal/less. We reduced the 1064 responses to 945 complete cases.

Multiple LCA models were built and tested; model selection involved inspection of model fit parameters and diagnostic criteria (Nylund-Gibson & Choi, 2018; Weller et al., 2020). Analysis was performed in RStudio using the poLCA package (Linzer & Lewis, 2011; R Core Team, 2022; RStudio Team, 2022). For $5 \leq k \leq 10$, 10 models of k-classes were generated using Monte Carlo Markov Chain methods. These models were then compared in terms of their model fit statistics, diagnostic criteria, as well as their interpretability considering existing theory and literature, as incongruent fit indices are common with such analyses (see Table 1 for model fit statistics.) This resulted in a choice of the eight-class model, which had the lowest BIC and highest BF. For this eight-class model, the minimum average latent class posterior probability is 0.940, well above the minimal threshold; entropy was 0.96, well above the .80 standard cut off; the smallest class is 2.8% of the sample population (Class 5); this is 26 individuals and may be admitted for analysis with caution.

Table 1. Evaluating class solutions.

Model	LL	AIC	BIC	G^2	X^2	BF
5	-13247.61	27083.22	28509.47	15075.2	4.4e+18	0.000
6	-12899.78	26505.57	28218.04	14379.54	1.6e+18	0.000
7	-12661.73	26147.46	28146.15	13903.44	2.9e+16	0.000
8	-12395.36	25732.71	**28017.62**	13370.69	**3.6e+15**	**>>15.00**
9	-12281.36	25622.73	28193.85	13142.7	8.3e+17	0.000
10	**-12066.67**	**25311.35**	28168.69	**12713.32**	1.16e+15	--

Note: LL – maximum log-likelihood; AIC – Akaike information criterion; BIC – Bayesian information criterion; G^2 – likelihood ratio/deviance statistic; X^2 – Pearson chi-square goodness of fit statistic for fitted vs. observed multiway tables; BF – Bayesian factor.

Model Description

Figure 1 shows the proportion of participants, in each latent class, who responded "more," "equal," or "less" to four key questions (divided by dashed lines) regarding each of the target race-gender groups; that is, the conditional probability of a participant selecting responses given that they are a member of a particular class. Note that the question at the bottom is the only one which includes White Men, because that's the question about net advantage/disadvantage; the others are asked in reference to White Men (the dominant group). Moving upward, the second question related to *interest* to engage in the field; the third to *aptitude* for the field; the fourth (top) to *opportunity* in the field.

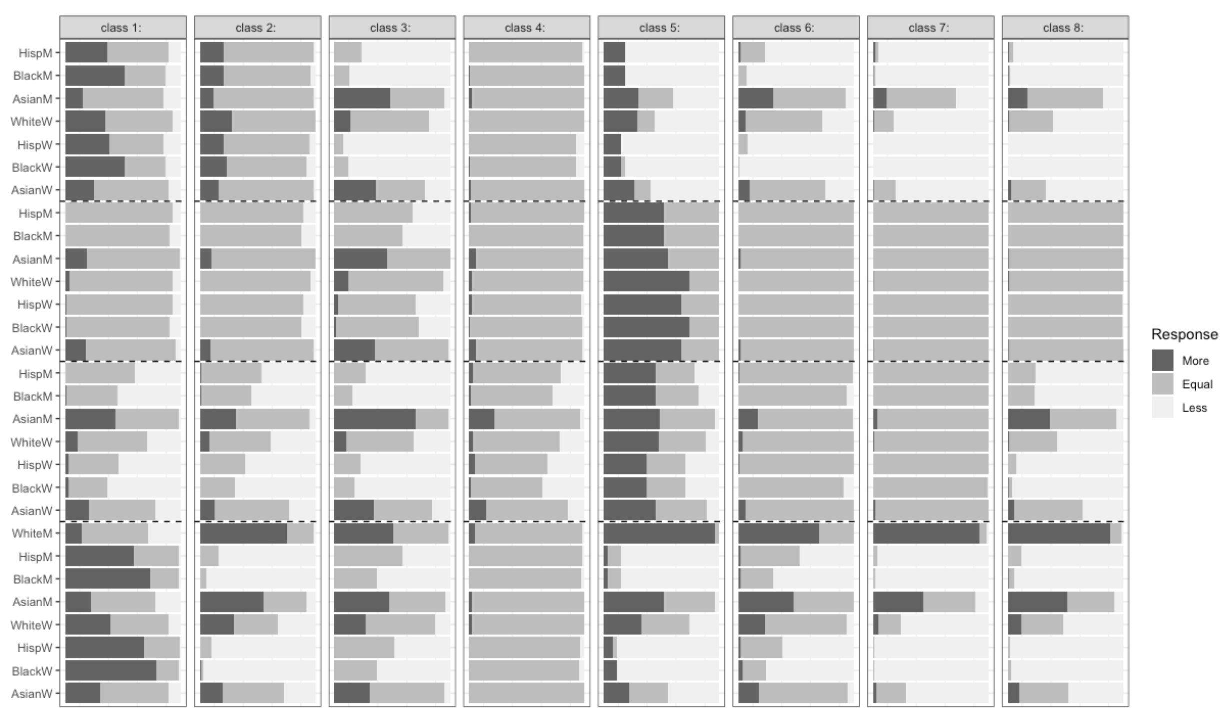

Figure 1. Conditional probabilities of answer choice selection based on class membership.

Model Interpretation

We anticipate that these profiles coincide with beliefs and actions which have direct impact on DEI issues and initiatives in higher education which is of great interest to our work. To that extent, we performed a series of Pearson's Chi-squared tests (with Bonferroni correction). Table 2 reports the results of these tests, as well as the cells in which standardized residuals (*sr*) had magnitude larger than 2, indicating that they are contributing to the statistical significance. This suggests that the model captures some variation in relation to faculty positions on DEI initiatives.

Table 2. Chi-squared tests

	χ^2	df	*p* adj	Contrib.
[A] *Do you, or would you, support AA-style policies in…*				
Student admissions	112.7	14	< 0.001	1, 3, 4, 7
Faculty hiring	132.5	14	< 0.001	1, 4, 7, 8
[A] *Do you, or would you, support DEI statements by…*				
Departments	134.3	14	< 0.001	1, 4, 7, 8
Institutions	135.4	14	< 0.001	1, 4, 7, 8
Professional societies	162.3	14	< 0.001	1, 4, 7
[B] Efforts to increase diversity are […] for my field	187.1	14	< 0.001	1, 3, 4, 7, 8
[C] I have personal responsibility to address DEI	73.01	7	< 0.001	1, 4, 6, 7, 8

N varies across tests by question response rate. All p-values adjusted for repeated testing. [A] Response options were yes; no; no opinion; [B] Response options were beneficial; detrimental; neither. [C] Response options were condensed to Agree (Strongly Agree; Agree); Disagree (Strongly Disagree; Disagree).

Discussion of Results

Classes 1, 4, 7, and 8 contribute the most consistently to associations between latent classes and additional characteristics, account for about 2/3 of the participants, and are easily interpretable within the extant literature. Therefore, we focus on them and suggest that additional, clearer data is needed to understand other patterns of belief. We name these four based on their most distinctive analysis patterns: Class 1 becomes *Advantage to Underrepresented Racial Groups*; Class 4 becomes *Race-Gender Equity in STEM*; Class 7 becomes *Opportunity and Advantage for White and Asian Men*; Class 8 becomes *Opportunity, Interest, and Advantage for White and Asian Men*. We discuss the belief patterns of each group, as well as the relationship between group membership and positions on DEI initiatives.

Class 1: Advantage to Underrepresented Racial Groups

This group of 73 represents 7.8% of the survey respondents. The group presents an inverted pattern of advantage compared to the other groups (see Figure 1); the majority of this group identify Black and Hispanic men and women as having a systemic advantage in STEM and Asian and White men and women as having neither an advantage nor disadvantage. More than a quarter of this group identify White men as *disadvantaged*. As compared to White men, this group suggested that Black men and Women have more opportunity to do science and mathematics, with other groups considered more equal. In terms of aptitude, the majority of this group (as with most groups) viewed everyone as having equal aptitude (though the distribution of the minority is interesting as well). In terms of interest, however, we see that this group believes that some of those least represented are also less *interested* in STEM than White men.

Considering the other questions (Table 2), this group were less likely to support affirmative-action style policies (admissions *sr* = -4.6; hiring *sr* = -5.0) and DEI statements (department *sr* = -7.3, institution *sr* = -7.9, society *sr* = -8.9); more likely to say that efforts to diversify STEM are detrimental for the field (*sr* = 7.8); less likely to say that they have personal responsibility to address inequities (*sr* = -5.7). These responses may reflect some kind of 'reverse discrimination' mentality, in which those with privilege view any reduction in their status (or increase in others' status) as an attack, rather than a leveling.

Class 4: Race-Gender Equality in STEM

This group includes 107 people, or 11.4% of the sample. The vast majority (93-100%) of the members of Class 4 responded that each race-gender group receives *neither a systematic advantage nor disadvantage* due to their demographic characteristics. They report similarly consistent and dominant views that all groups have equal opportunity (91-100%) and aptitude (93-99%). The majority (61-76%) of this group report that all race-gender groups have equal interest in STEM. (See Figure 1).

Like Class 1, members of Class 4 were less likely to support affirmative-action style policies (admissions *sr* = -5.4; hiring *sr* = -6.7) and DEI statements (department *sr* = -5.1, institution *sr* = -4.7, society *sr* = -5.9) and more likely to disagree that they have personal responsibility to address inequities (*sr* = -5.3). They were also less likely to say that efforts to increase diversity are beneficial (*sr* = -7.6), but unlike Class 4 they are more likely to say that these efforts are neither beneficial nor detrimental (*sr* = 8.0) rather than to identify them as detrimental for their field (*sr* < 1).

Class 7: Opportunity and Advantage for White and Asian Men

Class 7 is the largest group of them all, including 283 (30%) of the survey participants. The vast majority (96-100%) of this group indicated that Black and Hispanic/Latinx men and women receive a systemic disadvantage in STEM; a substantial majority indicated that Asian women (71%) and white women (76%) receive a systemic disadvantage. For Asian men, the story was rather split between neutral and advantage. In contrast, 93% of this group indicated that white men receive a systemic advantage. (See Figure 1).

Members of Class 7 were more likely to support affirmative-action style policies (admissions *sr* = 7.6; hiring *sr* = 6.9) and DEI statements by departments (*sr* = 6.9), universities (*sr* = 6.1), and professional societies (*sr* = 5.9); more likely to report that efforts to encourage diversity are beneficial for the field (*sr* = 6.5). In all these responses, members of this group were less likely to hold an opposition *or* neutral stance. This group was also more likely to agree that they have personal responsibility to address inequities (*sr* = 3.9).

Class 8: Opportunity, Interest, and Advantage for White and Asian Men

Class 8 included 158 participants (17% of sample). This group is very similar to Group 7 in their responses to the advantage, opportunity, and aptitude questions, but differs when it comes to the question about interest levels. While members of Group 7 unambiguously noted that all groups have equal interest in STEM, Group 8 is more complicated. Almost all of this group (93-96%) reported that Black and Hispanic women are less interested in STEM than White men; roughly ¾ of the group suggested that Black and Hispanic men are less interested; the majority (57%) suggest that White women are less interested compared with White men. Regarding Asian men and women, we see that the majority (58-59%) report that Asian men and women have the *same* level of interest as White men – but the bulk of the others diverge. A little more than a third (35-37%) of the group thinks that Asian women are less interested, but Asian men are *more* interested than White men. (See Figure 1). This reveals nuance and complexity within instructors' beliefs, evidencing the intersection of racialized and gendered hierarchies in STEM.

Chi-squared testing revealed a bit less about members of Class 8 than Classes 1, 4, and 7. They were less likely to oppose affirmative-action style policies in faculty hiring (*sr* = -2.1), but not more likely to support (*sr* <1); they did not contribute to the significance of responses to affirmative-action in student admissions. Similarly, they were less likely to oppose DEI statements by departments (*sr* = -2.3) and institutions (*sr* = -2.1) but not more likely to support – and they did not contribute to the significance regarding DEI statements by professional societies. They did not contribute to the significance of the personal responsibility question. However, they were strongly overrepresented in saying that efforts to encourage diversity are beneficial (*sr* = 4.1) and were less likely to say these efforts were detrimental (*sr* = -2.1) or not to have an opinion (*sr* = -3.4).

Concluding Remarks

This report is a very brief investigation of STEM instructors' views related to race and gender in STEM, as well how those views relate to activities which might advance diversity and equity in postsecondary STEM education. We detected eight profiles within our sample of 945 faculty. The four detailed here suggest that there are differing views on systemic advantages in relation to race and gender; there are also differing views of potential causes.

The prevailing response pattern found in Class 1 appears to aligns with beliefs that reparative efforts are 'unfair,' which have manifested in court cases opposing affirmative action[2] and may reflect how a loss of privilege can feel like oppression (Klein, 2020; Murrell & Jones, 1996; Yosso et al., 2004). The belief pattern reported by Class 4 is consistent with (inaccurate) beliefs about STEM and academia as race-gender-neutral spaces which function as meritocracies and do not perpetuate societal inequities. The response distributions in Classes 7 and 8 have similar response patterns with regards to systemic advantages and disadvantages; they also report similar patterns in terms of opportunity and lack thereof. However, while nearly all respondents in Class 7 report that all race-gender groups have equal interest in STEM, respondents in Class 8 indicated that they view the race-gender groups that have been systemically disadvantaged also have *less interest* in STEM.

This kind of nuance, even among those who generally recognize the presence of systemic race-gender inequities in STEM, are indicative of the nuances needed to enact systemic change toward more equitable systems. Additional research is needed to ascertain nuances of belief and how to leverage and/or change those beliefs. For example, how do members of Class 8 come to believe that Black and Hispanic men and women have less interest in STEM than Asian and White men and women – and how might that impact their views on who has opportunities to do STEM and net systemic advantages and disadvantages? There are many limitations to this study, but none which suggest that there are *not* such beliefs among instructors – particularly given the extant literature.

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[2] There are some legitimate criticisms of affirmative action policies and their implementation, although such policies have had some positive impacts. We omit a full discussion.

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Variations in Grading and Instructor Feedback

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Although developing proof proficiency is an integral part of undergraduate mathematics, little is known about instructor grading practices and feedback conventions in undergraduate courses in which proof is taught. In recent years, small-scale studies have begun to shed light on how individual instructors at single institutions assign points and provide feedback on proof-related activities. Our study extends this work by describing the results of a large-scale online survey of mathematics educators (n = 86) from multiple institutions. Participants were asked to score and give feedback to hypothetical student submissions of a proof-adjacent assignment; we investigate participants' responses and provide a taxonomy of their feedback.

Keywords: Feedback, Grading, Proof-related activities, Teaching Practices of Mathematicians

Proof is widely agreed to be central to the activity of mathematicians, and therefore argumentation and proof-related activities are an integral part of undergraduate mathematics education. Developing proof proficiency "is often the primary goal of advanced mathematics courses and typically the only means of assessing students' performance," (Weber, 2001, p. 101). In light of this, mathematics educators spend significant time evaluating and responding to students' mathematical arguments—and yet, little is known about instructor grading practices and conventions for feedback (Moore, 2016). In fact, until recently, little has been known about the teaching practices of undergraduate mathematics educators in general (Speer, Smith, & Horvath, 2010). Small scale exploratory investigations have begun to shed light on the criteria professors use to grade proofs (Moore, 2016; $n = 4$ $n = 4$), assign points (Miller et al., 2018; $n = 8$ $n = 8$), and provide feedback (Kontorovich, 2021; $n = 1$ $n = 1$). Findings indicate that proof grading is a complex teaching practice requiring difficult judgements that lead to substantial variation in points assigned and a need for further research.

Our study aims to add to the growing body of literature on undergraduate mathematics teaching practices by investigating the scores and feedback provided to a proof-adjacent activity on a large scale. We conducted an online survey of mathematics educators in which we provided 5 different student submissions analyzing a single mathematical statement. Survey participants were asked to score each submission and provide feedback to the student. Responses to the survey were used to investigate the following research question: *What kinds of feedback accompany different numerical scores on an undergraduate, proof-adjacent mathematics task?*

Background

In general, "feedback is conceptualized as information provided by an agent (e.g., teacher) regarding aspects of one's performance or understanding" (Hattie & Timperley, 2007, p.81). There is an extensive body of research on feedback, including reviews of the literature and meta-studies investigating the effectiveness feedback (Black & William, 1998; Shute 2008; Hattie & Timperley, 2007). Many of these studies span disciplines and provide frameworks for describing particular features and functions of feedback. For example, Shute (2008) compiles several frameworks in her review differentiating *directive* feedback which tells the student what to fix; from, *facilitative* feedback which provides comments or suggestions without explicit direction (Black & Wiliam, 1998, as cited in Shute, 2008). Feedback can also contain elements of both *verification* and *elaboration*. Verification acknowledges whether or not the work is correct, whereas elaboration is a description of the work designed to lead the student to a better answer (Kulhavy & Stock, 1989, as cited in Shute, 2008). While many of these frameworks offer general ideas about feedback, they are often drawn from studies based in secondary or primary school settings and feature participants who are engaged in well-defined tasks or well-structured problems. Literature on feedback in undergraduate settings that features ill-structured or open-ended tasks with multiple solution strategies is needed.

Methods

Data Collection

We conducted an online survey of mathematics educators in which we provided 5 different student submissions to the assignment pictured in Figure 1.

For the following statement, indicate whether it is mathematically true or false. If you claim that the statement is false, refute it with a counterexample. Explain the relevance of your example to the statement and how it disproves the statement. If you claim that the statement is true, prove it.

There exist four different prime numbers the sum of which is prime.

Figure 1. The assignment that engendered the student submissions featured in the online survey.

Survey participants were prompted to "Imagine that you were teaching a math class, and you received the following five homework submissions as an argument or justification for whether the statement ['There exist four different prime numbers the sum of which is prime'] is true or false." For each submission, participants were asked to score it out of 10 possible points and to also provide written feedback for the student. No scoring rubric was provided. The student submissions in question are pictured in Table 1.

Table 1. The five student submissions assessed by participants in the study.

Participant	Submission
Alpha	True. In order to prove this exists we only have to show one valid example of the statement. *a, b, c, d, e* in "primes" $a + b + c + d = e$ Example: 2 + 5 + 7 + 17 = 31 2, 5, 7, 17, and 31 are all primes. Therefore, there does exist a case where four distinct/ different primes sum to equal a prime number.
Beta	True. For example, consider the sum of prime numbers 2 + 3 + 7 + 11 = 23 which proves that there exists four different prime numbers the sum of which is prime. But there also exists four different prime numbers the sum of which is not prime. For example, 2 + 5 + 7 + 11 = 25.
Gamma	True. 2, 3, 5, and 7 are all different positive primes. Their sum, 2 + 3 + 5 + 7 = 17, is also a prime number. This example proves the initial statement that there exists a (at least one) solution. Note that 2 must be one of the primes in the sum: 2 is the only even prime number, so if none of the four primes is 2, then we have odd + odd + odd + odd = (2a + 1) + (2b +1) + (2c + 1) + (2d + 1) = 2(a + b + c + d + 2) = 2k

	where k is an integer, hence the result is even and cannot be prime (unless k = 2, except all odd primes are greater than 2, so the sum will be greater than 2 making this not possible).
Delta	True, 2 + 3 + 5 + 7 = 17, all primes.
Epsilon	This statement is true. Proving by contradiction, let us assume that this statement is false, then it becomes, "There don't exist four different prime (positive) numbers the sum of which is prime." We can disprove this with a single example, such as 2, 5, 7, and 17, which sum to 31, a prime number. As the inverse statement is false, we can see that the original statement, that there exists at least one group of prime numbers that sum to a prime, is true.

The student submissions in Table 1 were inspired by real student work and only slightly re-worded for clarity. These particular submissions were chosen because they all correctly identified the statement as true and provided the necessary example. However, they also represented a variety of different, sometimes non-standard approaches to mathematical proving.

In total, 86 participants provided at least a numerical score to each of the 5 student submissions. Of the 430 individual numerical scores, 69 had no accompanying written feedback. Therefore, in total, we analyzed 361 individual *complete responses* in the sense that they contained both a numerical score and accompanying written feedback. Delta received the fewest complete responses (n = 69), whereas Gamma received the most (n = 74). The other student submissions each received 72 complete responses.

Participant Demographics

Two populations of mathematics educators were invited to participate in the online survey through email invitations distributed via listserv. The first population was comprised of mathematics educators and mathematics education graduate students at an R1 institution in Canada; the second population was an international group of professionals who were subscribed to a listserv dedicated to promoting research in undergraduate mathematics education. Participants were only asked demographic questions related to their education and teaching experience. More specifically, participants described the highest levels at which they had taken courses in mathematics and mathematics education. They also provided the highest levels at which they had taught courses in mathematics and mathematics education (see Table 2).

Table 2. Participants' experience as students and instructors of mathematics and mathematics education.

	Graduate	Undergrad.	Secondary	Primary	None of these
Experience as a student:					
Mathematics	71	15	-	-	-
Math Education	62	6	-	-	18
Experience as an instructor:					
Mathematics	25	46	15	-	-
Math Education	43	14	-	1	30

Data Analysis

To analyze the content of the complete responses, they were subjected to two rounds of coding. In the first round, each researcher used open-coding techniques to apply many descriptive codes (Saldaña, 2016) to the data; these codes sought to comprehensively capture the variety and nuance within the data from multiple perspectives and along several axes. The research team then came together to discuss the variety found in the content. Together, they organized their independently generated codes into a unified set of codes that captured the types of feedback given to the student characters. In a second round of coding, each researcher independently recoded the entirety of the dataset, this time using the newly created codebook. Discrepancies between these codes were discussed on a response-by-response basis until consensus was obtained and complete intercoder reliability was achieved.

In this report, we present a subset of feedbacks that the research team recognized as *actionable feedbacks*. We interpreted feedback as actionable if the student to whom it was directed could interpret that feedback as suggesting a course of action that would bring the submission closer to an implied mathematical standard for proof-writing.

Findings

We identified 62% of written feedback as containing an actionable element. At the broadest level, we categorized actionable feedback as either *prompting an addition, prompting a reduction*, or *prompting a change*. Note that these three types of feedback (and each of their respective subcategories, described in the following sections) are not treated by the research team as mutually exclusive categories. In this study, however, overlap was observed to be minimal.

Feedback Prompting an Addition

Feedback prompting an addition recommends, implicitly or explicitly, that something be added to the student's submission to improve its overall quality. Within this type of feedback, four prominent subcategories arose. First, and most broadly, some feedback that prompted an addition simply asked the student to add more without any specific direction about what type of addition would be appropriate; this was a general prompt.

When the feedback prompting an addition made a specific recommendation to the student, it fell into one of the three remaining subcategories. The addition could be an *epistemological justification*; this was a common suggestion when the participant did not feel that the student submission adequately demonstrated why the proposed statement was in fact true. The addition might instead be a *structural justification*, which (in the eyes of the participant providing the feedback) would lead to a submission that had clearer internal logical structure and purpose. Finally, some participants directed the students to provide *additional insight*—that is, the feedback suggested that the submission would be improved if the student added details of their problem solving, examined related corollaries, or demonstrated a deeper understanding of the mathematics than might be required to establish the truth of the statement. Examples of each type of feedback prompting an addition are provided in Table 2.

Table 2. The types of feedback prompting an addition.

Feedback Type	n	Example
General	6	"You should explain a bit more."
Epistemological Justification	24	"Why is it sufficient for you to identify just a single example to establish the statement as true?"
Structural Justification	33	"The argumentation is correct but you should express it through a discourse, and make logical relationships clearer."
Additional Insight	69	"You have answered the question correctly. Can you expand by giving another three examples? What do all of your examples have in common?"

Feedback Prompting a Reduction

Feedback prompting a reduction pinpointed a particular area in which the student's submission could be cut down in some way, rather than expanded. This type of feedback manifested in two ways. First, the reduction might target *excess information*; this type of feedback can be interpreted as the antithesis of feedback prompting the addition of additional insight. That is, whereas some participants posed questions in their feedback that solicited mathematical details that did not directly contribute to a proof of the statement at hand, other participants specifically requested that students remove such details. Second, the reduction might require removing *excess machinery*. In this type of feedback, the respondent typically felt that the information presented in the student submission was relevant but that the proof structure or level of mathematical formalism used to convey that information was unnecessary and could be reduced. Examples of each type of feedback prompting a reduction are provided in Table 3.

Table 3. The types of feedback prompting a reduction.

Feedback Type	n	Example
Excess Information	44	"It suffices to show that one case exists. It's not necessary to show that there are sums of primes for which the property does not hold."
Excess Machinery	51	"Your proof is correct, but it's unnecessarily cumbersome. The original statement is a claim about a single case existing. The most direct proof then just gives an example. The extra layer of disproving the negation is a bit confusing for a reader."

Feedback Prompting a Change

Finally, feedback prompting a change suggested that some already existing aspect of the work was relevant, and thus should not be removed, but instead should in some way be modified. Feedback prompting a change typically targets one of two types of perceived shortcomings. On one hand, the reviewer may have felt that the mathematical content of the student's submission needed to be changed. This type of feedback typically attended to a student's misuse of mathematical vocabulary or symbols; often, it pointed out what the respondent perceived to be a "typo" on the part of the student. On the other hand, the reviewer may have called attention to a way in which the organizational or grammatical formatting of the submission was inadequate. Examples of each type of feedback prompting a change are provided in Table 4.

Table 4. The types of feedback prompting a change.

Feedback Type	n	Example
Mathematical Content	34	"Terse and correct except for the error in mathematical grammar. The equation is a true assertion about primes, but an equation is not a prime."
Formatting	9	"Nice addition. In the future, please separate the proof from additional insight, and mark the proof as such."

Frequency of Actionable Feedback Types

In Table 5, we present the frequency with which of each type of actionable feedback was provided to each student submission. In this table, we also coordinate the type of actionable feedback with whether or not the participant awarded the student submission in question with a full or partial score (FS and PS, respectively). For example, of the 19 feedbacks that prompted Alpha to include additional insight, 9 of these still awarded his submission a full score; another 10 awarded him less than a full score.

Table 5. The frequency of each type of actionable feedback given to each student submission.

Feedback Type	Student									
	Alpha		Beta		Gamma		Delta		Epsilon	
	FS	PS	FS	PS	FS	PS	FS	PS	FS	PS
Prompting an Addition										
General	-	-	-	-	-	-	-	6	-	-
Epist. Justification	1	1	1	3	-	-	2	15	-	1
Str. Justification	2	1	4	3	1	1	1	12	-	8
Additional Insight	9	10	6	9	7	3	6	9	4	6
Prompting a Reduction										
Excess Information	-	-	16	16	8	4	-	-	-	-
Excess Machinery	3	1	-	-	-	-	-	-	25	22
Prompting a Change										
Math Content	3	5	-	1	7	3	-	3	-	12
Formatting	1	1	2	-	2	-	-	3	-	-

Discussion

Miller (2018) observed that, when instructors allotted scores to student-submitted proofs, "the points that they assigned was dependent upon their perceptions of participants' understanding of the proof that they submitted" (p. 30). As seen in Table 5, certain types of actionable feedback are more likely to be attached to partial numerical scores; prompts for additional epistemological justification, for additional structural information, and for a change in mathematical content. These types of actionable feedback appeared 83%, 76%, and 71% of the time (respectively) with a partial numerical score. In light of Miller's finding, the association of these types of feedback with partial scores could indicate which types of errors instructors tend to recognize as indicative of a fundamental misunderstandings on the part of the student author.

On the other hand, actionable feedback prompting a reduction of excess information or machinery did not clearly correlate to either a full or partial score. Respondents in our study did not clearly agree on whether or not "too much" of something demonstrated that the student fundamentally misunderstood the underlying mathematical logic. Beta and Epsilon received prompts for a reduction in information and machinery (respectively) more than any other student submission received any other type of feedback. However, despite how frequently respondents provided these types of feedback, they did not clearly correspond to a partial score.

We note that prompts for additional insight were also more common on partially scored submissions, but only slightly (at 54%)—and this trend was reversed in the case of Gamma's feedback. In fact, because of Gamma's overall high scores, this was not the only trend they reversed. Gamma's submission was the only student's work to receive mostly full scores even in the presence of a prompt for a change in mathematical content which, as discussed in the previous paragraph, was often attached to partial scores. As described in Moore (2016), some instructors, when grading proof-like student submissions, do not deduct points for each error; instead, they attempt to assign a score "by judging the overall quality of the proof" (p. 263). Thus, while Gamma's mathematical typo generated a number of feedbacks prompting a change in mathematical content, they still often received full credit because respondents perceived that Gamma understood the underlying mathematics of the proof.

Prompts for additional insight comprised a large and especially nuanced portion of feedback. This type of actionable feedback occurred more often than any other, and in 30 of its 69 instantiations, it was the only type of actionable feedback offered to a student who received a partial score. Of these 30 feedbacks, 6 are very open-ended prompts (e.g., "Can you dig deeper?"). The rest are guiding questions that ask the hypothetical student to consider or provide material that is beyond the scope of the original prompt (e.g., "What are the reasons of the choice of your primes?"). Occasionally, the respondent instead presents their prompt for additional feedback in the form of an observation; for example, one feedback explains that "this [Delta's submission] is correct but it offers very little in the way of showing where your numbers come from or telling me how you understand this must be true."

Miller (2018) found that, by some instructors' standards, "correct proofs would not necessarily receive full credit" (p. 29). Our study corroborates this finding. Student submissions were chosen for this study on the basis that they each correctly attended to the task at hand. As Miller observed, however, "correctness is not the only criterion that is used by mathematicians for grading proofs" (ibid, p. 30); the student submissions in this study were also often scored based on how well respondents felt they demonstrated the student authors' understanding of why their work was sufficient and correct. Sometimes, this is clear in the accompanying written feedback—see the example of feedback requesting additional structural justification in Table 2.

Conclusion and Limitations

Our main contribution is a taxonomy of the feedback mathematics educators provided to students in proof-adjacent activities. Beyond a more detailed description of the feedback presented above, the presentation will also explore a type of feedback adjacent to the actionable feedback described in this report: that is, when no actionable written feedback is provided and yet the student submission is only awarded a partial score. We wonder, when instructors provide such feedback, what socio-mathematical norms (if any) are they attempting to reinforce? A future study could also explore what actions students take in response to actionable feedback.

We would like to recognize two limitations to our study. First, respondents were not allowed to physically grade the student submissions—that is, they weren't provided with submissions that they could "mark up" with non-textual feedback. Spiro et al. (2019) found that annotations such as checkmarks or arrows account for 50% or more of the recorded feedback when professors grade assignments. Our survey required participants to type feedback in the form of sentences, which might be different from their usual method of providing feedback to their students. On the other hand, our study could contribute to a better understanding of how instructors' feedback practices change in an online environment.

Second, we agree with the sentiment expressed by Sommerhoff & Ufer, who argue that "the generalizability of results in the context of proofs are always questionable, as they rely on local socio-mathematical norms" (2019, p. 729). From that perspective, our study was limited by the decontextualization of the student work from a greater classroom culture. In fact, some respondents indicated that their treatment of student submissions would depend on the socio-mathematical norms that had been established in the classroom. The feedback given by these participants may not be representative of their actual feedback practices.

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Multivariable Calculus Instructors' Reports of Resource Use: Resources Used and Reasons for Their Use

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Given the challenge of visualizing the main constructs of two-variable functions and their differential and integral calculus, it is important to consider the use and perceived potential of resources to contribute to students' understanding of multivariable calculus. This case study considers how four instructors attempt to utilize resources in their multivariable calculus teaching and their motivations to do so. We study how these instructors think about the digital and non-digital resources that they use to foment students' understanding. We also look at their reporting of the ways that instructors, students, and resources interact in multivariable calculus to determine if resource use is meant to facilitate visualization, reasoning, or communication. With this, the study proposes to contribute to the discussion of resource use in multivariable calculus.

Keywords: multivariable calculus, digital resources, visualization

Introduction

Multivariable calculus is an important tool for modeling phenomena in science, technology, engineering, mathematics, and other fields of knowledge. There is a growing research base on the teaching and learning of two-variable functions that includes the study of related definitions, basic ideas, and geometric understanding (e.g., Martínez-Planell & Trigueros, 2021; Dorko & Weber, 2014; Martínez-Planell & Trigueros, 2012, 2019; Trigueros & Martínez-Planell, 2010; Weber & Thompson, 2014; Yerushalmy, 1997). There are also studies about the differential calculus of two-variable functions (e.g., Harel, 2021; McGee & Moore-Russo, 2015; Martínez-Planell et al., 2015, 2017; Tall, 1992; Trigueros et al., 2018; Weber, 2015) and their integral calculus (e.g., Jones & Dorko, 2015; McGee & Martínez-Planell, 2014; Martínez-Planell & Trigueros, 2020). All of this is discussed in a survey article on multivariable calculus (Martínez-Planell & Trigueros, 2021). There are some studies on the use of non-digital resources (e.g., McGee et al., 2012, 2015; Sherer et al., 2013; Wangberg et al., 2013, 2020). However, the literature on the use of digital technologies is somewhat sparse (e.g., Alves, 2014; Habre, 2001; Ingar & Silva, 2019; VanDieren et al., 2020).

To further study the use of digital resources, we seek to study how multivariable instructors report that they use resources in their teaching of multivariable calculus. More specifically, we consider how multivariable instructors report that digital, digitally generated, and other resources are used to nurture students' understanding and their reports of how students, instructors, and resources interact in the construction of multivariable calculus understanding. This study attempts to contribute to answering these questions by focusing on four instructors as they discuss the resources they use and the reasons for using them to determine if they are focusing on visualization, reasoning, or communication.

Theoretical Framing

Didactic Triangle and Didactic Tetrahedron

The didactic triangle is a model that can be used to organize and discuss instructional phenomena by considering three key components, the teacher, the students, and the content, as well as interpreting the links between the three components in the context being studied. Observing that "artifacts are not passive resources that teachers and students draw on but 'actively' shape activities," Rezat and Strasser (2012, p. 644) argue to extend the didactic triangle to the model of the didactic tetrahedron (displayed in Figure 1). The idea of the didactic tetrahedron was previously presented by Tall (1986). This seems particularly appropriate in our study since we focus on artifacts used, including both digital and non-digital resources that instructors leverage in their instruction.

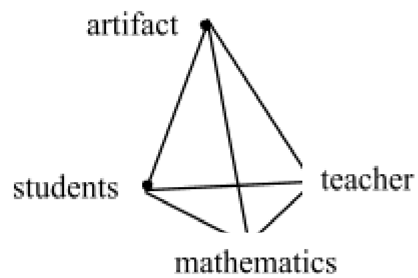

Figure 1: Didactic tetrahedron model (Rezat & Strasser, 2012).

The Teaching Triad

Jaworski's (1994) teaching triad (displayed in Figure 2) may be thought of as an expansion of the "teacher" node in the didactic triangle. It consists of three interrelated elements: sensitivity to students, mathematical challenge, and management of learning (Jaworski et al., 2017). *Sensitivity to students* relates to the teacher-student link. It describes the teacher's knowledge of students and the teacher's attention to affective, cognitive and social student needs. It includes the ways in which the teacher interacts with individuals and how the teacher guides group interactions. *Mathematical challenge* relates to the teacher-mathematics link. It describes the challenges that a teacher offers to students to promote mathematical thinking and activity. It includes the assigned tasks, the posed questions, and any teacher emphasis on metacognitive thinking. The *management of learning* construct extends the didactic triangle to the wider context of the classroom. It describes the teacher's role in the constitution of the classroom learning environment by the teacher and students. It includes, among other things, classroom groupings, instructional planning of tasks and activities, selection and use of instructional resources, and classroom norms that are set. The three components are interrelated.

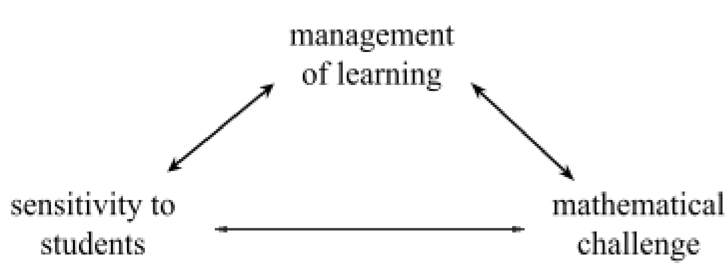

Figure 2: The teaching triad (Jaworski, 1994).

Spatial Literacy

The broader sense of being literate refers to being competent and knowledgeable in a certain area (Moore-Russo & Shanahan, 2014). Spatial literacy includes the abilities to visualize objects in three-dimensional space, to reason about their properties and relations, and to communicate with others about these objects and relations. Moore-Russo and colleagues (2013) introduced a framework that describes spatial literacy in terms of three domains (shown in Figure 3): visualization, reasoning, and communication. *Visualization* is the process of generating cognitive representations of spatial objects through visual images that may or may not be facilitated by external representations or physical actions. For example, a student can internally visualize a paraboloid when looking at a representation of a paraboloid on a computer screen, or when listening to the word “paraboloid”, or when out of habit or by memorization the student imagines the geometrical object given the symbolic representation $z = x^2 + y^2$ without reasoning about or justifying the relationship or communicating about it with others. *Reasoning* is defined as the cognitive processes that individuals use to form conclusions or make judgments from a given set of premises; it is the process of organizing, comparing, or analyzing spatial concepts. For example, a student is reasoning when mentally using transversal sections to justify that the graph of $z = x^2 + y^2$ is a paraboloid. Reasoning may or may not involve geometric imagery, as when using an entirely symbolic mental argument to conclude that $z - x^2 = y^2$ is also a paraboloid. *Communication* is the exchange of information with others using resources such as language, written symbols, gestures, etc. to convey ideas that relate to spatial objects or relationships. For example, a student explaining to another why the graph of $z = x^2 + y^2$ is a paraboloid is engaging in communication. At the same time, the student may or may not be engaged in reasoning and/or visualization depending on the argument in the student’s mind. The different domains overlap as suggested in Figure 3. Although we will not give examples of each possibility, they may be imagined using variations of the above paraboloid examples.

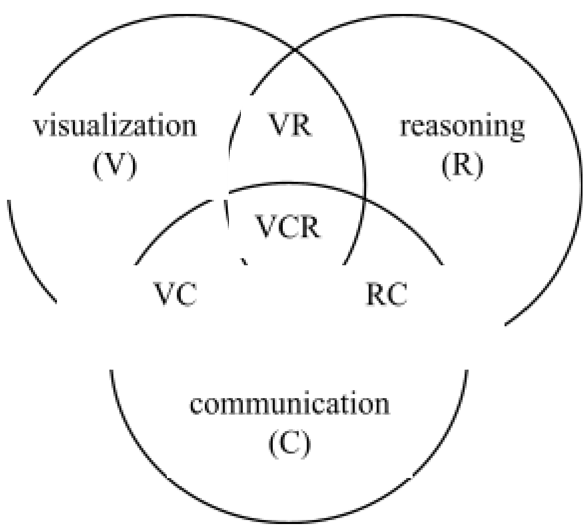

Figure 3: Spatial literacy domains (Moore-Russo et al., 2013)

Research Questions

This case study considers how university instructors reflect on teaching multivariable calculus specifically related to the resources they use. The specific research questions follow.

1. Which **resources** do university mathematics instructors report that they use or have used in the past for teaching multivariable calculus? What reasons do they give for their selections? How do they report that they use the resources?

2. Which, if any, **spatial literacy domains** (i.e., visualization, reasoning, communication) do instructors mention and focus on? Is this the same for all instructors or does it vary by instructor?

Methods

The data reported here are part of a case study that involved four university instructors from four different U.S. universities. We studied written responses to a series of 24 survey questions and present the results of the eight questions that directly address the resources used and the reasons for their use below. The instructors were part of a convenience sample of multivariable calculus professors from four different universities in four different states working on a project whose main purpose was to help students develop a deeper understanding of multivariable calculus concepts through hands-on, physical 3D explorations using guided learning activities and innovative, pedagogically designed 3D-printed surfaces and solids. All four instructors were currently using the digital CalcPlot3D platform (Seeburger, 2018). Since this paper focuses on the instructional decisions that guide the work of teaching, specifically of teaching multivariable calculus, we analyze data from the instructors in regard to their teaching rather than analyzing student thinking, student engagement, or how students make meaning of the content at hand.

An instructor's response to a survey question was considered the unit of analysis. The research team independently reviewed the survey responses multiple times. During their independent review, they marked ideas that appeared frequently. The "bottom-up" analysis yielded possible directions that the analysis could take. The team met to discuss this first pass over the data, comparing their notes. They decided to revisit the data individually while considering possible frameworks that could be used to frame the research. The team met twice to determine the framing of the study and then decided on specific categories to use for the coding. Each team member coded the data set independently before coming together to discuss the coding until a consensus was reached. During this process, the team pulled out representative quotations from the data set. The results were analyzed both in terms of each instructor as well as in terms of general tendencies across the group of four instructors.

Results

We begin by reporting our results for the first series of research questions: Which resources do university mathematics instructors report that they use or have used in the past for teaching multivariable calculus? What reasons do they give for their selections? How do they report that they use the resources? In our discussion, we will refer to the four surveyed instructors as M, P, S, and T, and we consider their use of digital resources then their use of other types of resources.

Instructor M has used CalcPlot3D since initially teaching multivariable calculus. For out-of-class assignment, she tried Survey Monkey and a publisher's online homework system before settling on WeBWorK, since she could embed video lectures and CalcPlot3D labs in that platform. She has also used Jamboard and Flipgrid for more student interaction. She has used Matcha.io as an online, mathematics editor. Instructor P initially used Cabri Geometry and Excel before writing his own Visual Basic, Java, and Java Script applets. CalcPlot3D followed from this. This instructor also used other digital resources like Smartboards, which he used to record lessons and an online custom OER textbook on the LibreTexts platform together with the WeBWorK online homework system and student-produced videos. While some digital textbooks might be what Viglietti and Moore-Russo (2018) call "static resources" (i.e., text-heavy resource that relies on typed text and digital pictures), the custom OER textbook used by P leveraged interactive figures. Instructor S initially used Mathematica and GeoGebra before she adopted

CalcPlot3D. Instructor T initially relied on Mathematica with some incorporation of Desmos and Tikz for two-dimensional plots before he moved to using CalcPlot3D. The reasons given by the four instructors for using digital resources in multivariable calculus and for changing their initial graphing software (which only applied to P, S, and T) are detailed in the next paragraphs.

Even though their initial choices of software varied, their reasons for using digital resources were fairly consistent. The responses from all four instructors showed the importance of *mathematical challenge* to them and the value they placed on students visualizing ideas of multivariable calculus, convinced that doing so would improve student understanding. M mentioned that digital resources "give students a deeper understanding of the material beyond the computational and symbolic which they are used to" stating that digital resources "challenge students to think and reason visually in 3D." P reported that "students need to see the relationships of calculus in motion to develop the kind of intuition for the definitions and constructs involved." S claimed to use digital resources every class "to create illuminating visual demonstrations" and to show students the insights that could be gained through *visualization*. S felt that digital resources help foster "a spatial awareness and understanding that connects to the material they [students] are learning." T felt digital resources helped students "attach geometric meaning to various course concepts."

The reasons that Instructors P, S and T gave for changing from their initial choices of graphing resources to a different digital resource (in this case, CalcPlot3D) also show *sensitivity to students* in their concern for students' affective, cognitive, and social needs. The most cited reasons were that not only is CalcPlot3D free (i.e., students do not incur any cost), but more importantly it requires no specific syntax so that it is easier to use. Instructor P said that he "had to change platforms a number of times to keep my visualizations usable and easier to get students to play with themselves." When explaining why she switched digital resources, S reported that "CalcPlot3D requires no coding syntax, unlike Mathematica, which I have used before for this course. It is more intuitive for users than GeoGebra, which makes it much more accessible to students." She stated, "my students could immediately jump into graphing with the program without any training." Instructor T adopted CalcPlot3D because "the syntax is simple enough that I feel good about having students work with it, and the fact that it's web-based is great."

Issues related to the *management of learning*, that included providing formative feedback, also figure among the reasons given by the instructors for using digital resources. Instructor comments often dealt with software features that allow for the creation and use of in-class instructor demonstrations or independent student work. Instructor M said, "I have embedded video lectures into webwork with 'exit ticket' questions that students answer immediately following watching a video. I also embed CalcPlot3D labs in webwork to give students immediate feedback." Instructor P stated that he would use a scenario presented with CalcPlot3D as a common focus for student discussion board assignments, and he "encouraged students to visually verify their results using CalcPlot3D" during student video presentations. Instructor S reported that she liked "the self-explanatory nature of the [CalcPlot3D] program and the powerful options that are built in to help instructors easily craft in-class demonstrations." Instructor T said, "there are a lot of visualizations I spent hours on creating in Mathematica, only to later realize that CalcPlot3D had already built in…the software is robust enough that I can usually render the objects that I want to render, and I'm calibrating my expectations of what I can ask students to do with the software independently and with how much guidance." T reported using CalcPlot3D as "visuals I include in lectures, assignments, and worksheets."

As for non-digital resources, there was also a wide variety of resources that have been used by three of the instructors including: 1x1x1 blocks (M), Play-Doh (M), paper cylinders to demonstrate parametric curves lying on surfaces and finding tangent planes (M), a hill on campus to explain partial derivatives (S), extendable pointers with arrows (S), a machine-shop-produced tabletop 3D coordinate system (S), coffee stirrers and fishing line (T), giant arrows made from wooden dowels for vectors (T), the K'nex toy pieces for xyz axes (T), various foods that were sliced to represent integration (T), and cardboard sheets for planes (T). The incorporation of such a wide variety of resources shows the instructors' *sensitivity to students*' cognitive needs. These resources shed light on how instructors have changed their *management of learning* for certain topics by bringing what they believed to be the most relevant resources.

The use of 3D-printed surfaces, a non-digital yet digitally generated resource, were reported by all four instructors. This is of particular interest since it is the focus of a project in which the four instructors participate. All four instructors were aware and mentioned that CalcPlot3D has the capability to produce STL files that can be used to print a 3D surface (or curve) model using a 3D printer. The instructors report using 3D-printed surfaces either just a few times a semester or some of the time. The most common reasons given for using 3D-printed surfaces related to *mathematical challenge*; they included their beliefs that: 1) embodied learning is privileged by the potential interactivity and manipulability of the physical object and 2) the 3D model had the potential to bridge the gap between the 2D renderings of a screen and the 3D reality. Instructor M mentioned student engagement when using 3D-printed surfaces, she claimed the engagement came from students being "able to hold and point to them" and believed in the importance of showing "an additional visual representation of the surface beside the 2D version on the screen." Instructor P shared that since a 3D-printed surfaces are "more concrete than the 3D renderings on the 2D computer screen using CalcPlot3D, it can be quite helpful for students, making the surfaces tactile" and "used it to help us confirm our various traces of the surface drawn on the Smartboard." Instructor S mentioned, "I wanted to provide students with a physical object to touch, to hold at various angles, and to point to features as they discuss concepts with their classmates. Having a tactile object removes any ambiguity that could arise from viewing a 3D object on a 2D screen." While Instructor T claimed that both 3D-rendered and 3D-printed objects afford avenues for students to be able to attach geometric meaning, he believed, "3D models have some advantages, like being able to draw on the models; they are easy to handle and view from multiple angles, in a more natural way than computer images." The reasons for using 3D printed surfaces are indicative of the instructors' *sensitivity to students*' cognitive needs. However, there were comments, especially from Instructor T about how they also affect the *management of learning* since there are non-negligible constraints to their use in class (e.g., number of models needed, class time issues in passing out and collecting the models).

We now report our results for the second series of research questions: Which, if any, spatial literacy domains (i.e., *visualization*, *reasoning*, *communication*) do instructors mention and focus on? Is this the same for all instructors or does it vary by instructor? Concerning the spatial literacy domains, the data from all four instructors were similar and provided evidence of a seemingly balanced instructional focus on VR (*visualization-reasoning*) and VC (*visualization-communication*), with some VRC (*visualization-reasoning-communication*) instances.

In the VR overlap, instructor comments focused on issues related to students being able to see the idea and relationships in multivariable calculus as they were trying to make meaning of the topics or trying to solve tasks that had been assigned to them. Instructor M mentioned how "challeng[ing] students to think and reason visually in 3D… gives students a deeper understanding of the material beyond the computational and symbolic which they are used to." Instructor P talked of benefits to student understanding when they are "rotating the surfaces in their hands to gain perspective." S claimed, "Visualization through CalcPlot3D gives students a spatial awareness and understanding that connects to the material they are learning. Instead of going through calculations blindly, students develop a spatial understanding of the concepts."

In the VC overlap, Instructor M related that "Each group got a different model so some students 'connected' with their model referring to it later in the class (e.g., 'Remember in our group's model there was that point that....')." While some of the comments focused on student-to-student *communication*, a number of comments related to instructor-to-student *communication* where instructors used resources in their lectures to help visual clarify the items being presented through demonstrations and examples. Instructor P stated, "I have brought in a couple 3D printed surfaces for students to see and to point out certain features."

The overlap of VRC was also noted, but not as often as comments that involved either VC or VR. As shared earlier, Instructor P encouraged students to use CalcPlot3D as a common anchor that allowed them to visually communicate their arguments in online discussion forum posts as well as in student-produced video presentations. Instructor S observed, "during office hours, students will reach for one of the 3D surfaces in order to ask their question. Likewise, they will reach for a surface to explain the answer to a question that I have posed to them." Instructor T commented that the "models are easy to handle and view from multiple angles (in a more natural way than computer images), and I find they work well for in-class group work" that would naturally involve *communication* among the students and for "office hours" that would involve *communication* between the instructor and a student.

Conclusions

We found that concern for students' learning and the conviction that geometrical understanding is crucial in multivariable calculus, moved the four instructors to consider the use of a variety of resources in their teaching. Three of the instructors started with a graphing resource other than what they were using at the time of the interview (CalcPlot3D). Ease of use for them and for students was the major factor influencing their change in digital technology. It seems that over time (and with more instructor experience) the attention to the use of technology for teaching shifted from being instructor-centered (demonstrations to present in class) to student-centered (activities to engage students and group work). In the spatial literature domains, there was about an equal number of *visualization-reasoning* and *visualization-communication* comments and some which also suggested *visualization-reasoning-communication* activities. The observations resulting from our study suggest that the interviewed instructors were from the start *sensitive to students'* needs and that future faculty adopting digital resources in their teaching could benefit from support in issues dealing with the *management of learning*: the creation of demonstrations and activities using their software of choice, being able to discuss constraints and difficulties using particular software for graphing in multivariable calculus, coordinating the use of graphing software and digital homework systems like WeBWorK, using digital technology in evaluation, sharing materials, and so on. Similarly, the instructors seem to find great potential of digitally generated resources, particularly 3D-printed models, in helping with student *visualization*, *reasoning*, and *communication*. However, the creation and subsequent use of these models in class seems to also require as much support as digital resources.

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Instrumental Genesis: For Loop and ‘len()’ as Artifacts

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In this contributed report I present results from a teaching experiment broadly focused on students utilizing machine-based computing as a mediating tool to learn fundamental concepts related to set theory and logic. Specifically, I highlight the mathematical activity of one student and his unique approach to finding the intersection of three sets. To understand how this student leveraged the computer programming environment, I used the analytical framework known as instrumental genesis, which can be utilized to investigate the confluence of an artifact (often a piece of technology) and the human mind to solve a mathematical problem. Results indicate that the student was able to use computational tools such as For Loops and the len(), or "length," function as artifacts in his solution to finding the set intersection.

Keywords: Computing, Programming, Set Theory, Instrumental Genesis

Introduction

In this report I present the results of an investigation into student thinking in which students were tasked with using Python, the text-based computer programming language, to learn fundamental concepts related to set theory and logic. The call for research to investigate the ways in which technology and, more specifically, computers can be used to facilitate the teaching and learning of mathematics is not new (e.g., Fey, 1989; Papert, 1980; Perlis, 1962). However, previous approaches to address this need have varied in the medium through which the technology is used, such as video games (Levine et al., 2020), graphing calculators (Leng, 2011) and virtual reality (Bogusevschi et al., 2020). Also, these areas of technology integration have been a focus at multiple levels throughout the educational system (Ball et al., 2018; Oates, 2011). Moreover, Lockwood and Mørken (2021) explicitly made a call for research in undergraduate mathematics education with a focus on machine-based computing. They define machine-based computing as "the practice of precisely articulating algorithms that may be run on a machine" (p. 2). An important aspect of machine-based computing is the development of an algorithm rather than its performance or implementation alone. This is an important component of machine-based computing that distinguishes it from other forms of computing such as the use of Desmos where one may only be required to input parameters to create a visual representation of a function. Thus, Lockwood and Mørken consider machine-based computing to be a separate construct that includes the use of programmable calculators, writing packages in Geogebra, and using text-based or block-based programming languages. As such, the focus of my study was to introduce machine-based computing (Python) as a mediating tool to work with mathematical ideas in the context of mathematical logic and set theory.

Additionally, I emphasize a need to focus on machine-based computing in the context of mathematics for three main reasons. The first is that data science, and computer science more broadly, are becoming increasingly valuable skills in an age where fields such as artificial intelligence and machine learning are in position to be the most influential drivers of change for our society (World Economic Forum, 2016). The second purpose draws on Lockwood et al.'s (2019) commentary on the relationship between computing and students' mathematical activity in that computing will be a focus for mathematics educators soon to come. By encouraging the use of computational software as a tool for students to learn and connect ideas in undergraduate mathematics, I am encouraging a movement for where mathematics education will be going in the coming decade. Lastly, many areas of mathematics lend themselves to machine-based computing environments, which suggests that the computational environments may serve as useful tools for learning the mathematical content. For this study, I focus on a fine-grain analysis of how programming may influence student reasoning and learning of mathematical set theory and logic. The research question driving this study is the following: *How does Python mediate students' reasoning of important concepts related to set theory and logic?* In the next section I describe a theoretical analysis framework which can be used to document students' mathematical activity while using a piece of technology.

Theoretical Framework

As alluded to in the introduction, programming is starting to become a fundamental aspect of learning undergraduate mathematics. For example, Buteau et al. (2020) investigates how an undergraduate student may use programming to learn both theoretical and applied mathematics. Rather than investigating student work with a particular mathematical concept, the authors articulated the use of a specific theoretical framework to study the use of programming as a tool for learning mathematics. The framework is known as the *instrumental approach* (Artigue, 2002; Guin & Trouche, 1998; Trouche, 2004). Importantly, Buteau et al. (2020) highlight that the instrumental approach has been used in the past for various artifacts such as graphing calculators, spreadsheets, applets, etc. but has not yet been used in the case where a programming language was considered as the artifact. They do provide illustrative examples of the relationship between mathematical inquiry and programming through the lens of an instrumental approach, but they do not provide an in-depth analysis of one specific mathematical concept. Therefore, to investigate how Python supports students' learning and advancing mathematical activity related to set theory and logic, I utilize the instrumental approach.

The description of the instrumental framework starts with the distinction between *artifacts* and *instruments*. An artifact may be a physical object, such as a graphing calculator, but also may be a formula, graph, or other objects that are central to a certain mathematical task (Roorda et al., 2016). Once the artifact has been determined, the integration of the artifact into a learner's mathematical activity is known as the instrument. This includes the use of the artifact to problem solve and as Trouche (2004) describes it, "an instrument can be considered as an extension of the body, a functional organ made up of an artifact component...and a psychological component" (p. 285). In the context of this study, the artifact is the computer programming environment, Python. As mentioned, programming environments have previously been utilized with this framework in mathematics education, but never with a focus on students' advancing mathematical activity in relation to set theory and logic.

The development of the instrument is known as *instrumental genesis* (Artigue, 2002) and is tied directly to the artifact and to the mental actions of the learner in their use of the artifact to carry out a given task. This means that the learner may be afforded particular lines of reasoning but also may be constrained in their mental activity given the particular features of the artifact itself. This psychological component, the mental processes of the learner to carry out a particular task, is referred to as an *instrumented action scheme* (Trouche, 2004). The idea of a student's scheme draws on the theory of constructivism and is described by Vergnaud (2009) as "the invariant organization of activity for a certain class of situations" (p. 88). To clarify, this definition encompasses the assimilation of familiar situations which learners respond to with their learned rules or already-established ways of understanding as well as addresses the adjustments necessary to address novel situations in which a learner is required to adapt, modify or reorganize their psychological thought processes. In the literature regarding the instrumental approach, schemes are defined as consisting of four main features (Trouche, 2004; Vergnaud, 2009). These features are summarized well by Buteau et al. (2020) as:

1. The goal of the activity, with sub-goals and expectations.
2. Rules of action: stable behaviors of the subject.
3. Operational invariants, which can be theorems-in-action (propositions considered as true) or concepts-in-action (concepts considered as relevant).
4. Possibilities of inferences. These possibilities are essential for the adaptation of the scheme to the specific features of the situation. (p. 1026)

As Buteau et al. (2020) state, the rules of action and operational invariants may present themselves through students' mathematical activity in a situation where a student says "When I want to do this [aim of the activity] … I always act like this [rule of action] … because I think that [operational invariant]" (p. 1027). The rules of action are stable methods of activity that the student will rely on to accomplish a task, based on the belief or conceptual understanding (operational invariant) of how something works. The operational invariants are broken down into two categories, theorems-in-action and concepts-in-action. The theorems-in-action are constructed ways of understanding how something works. The concepts-in-action are the related mathematical concepts that are pertinent to the goal of the activity. For example, if a student was asked to use a graphing calculator to find the local max and min of a function given a certain domain, an example of a rule of action would be 'enter the function in the calculator'. A concept-in-action might be 'zero slope at local min and max,' and a theorem-in-action might be 'the window display of the graphing calculator must capture the specific domain that is asked in the problem in order to be sure of the local max and min.' Lastly, the possibilities of inference would be situations in which the student encounters a problem that might result in new rules of action and operational invariants. In accordance with constructivism, technically these schemes are constructed within the learner's mind and thus are not directly observable, however, they can be inferred by an instructor or researcher based on the "regularities and patterns in students' activities" (Drijvers et al., 2013, p. 27). That is, for the purposes of this study, a "scheme" is a model that I construct based on the actions of the student.

As Roorda et al. (2016) highlight, at the core of the instrumental approach is the relationship between one's scheme and their *technique*, or actions in response to a given situation. In contrast to a student's scheme, the technique consists of the observable actions that are carried out by the student. It is important to highlight that while the observable actions inform the researcher's interpretation of the learner's scheme, their actions may also contradict the scheme that was originally developed based on previous actions. So, the student's techniques are in-the-moment observable actions of the student that may or may not coincide with the student's scheme. The student's techniques are linked with conceptual elements that help frame an understanding for the student's scheme. To capture the idea of instrumental genesis in more detail, it is broken into two subcomponents, instrumentalization and instrumentation, each describing a distinct directional relationship between the artifact and the learner (Artigue, 2002; Trouche, 2004). *Instrumentalization* is the reshaping of the artifact in the learner's mind as to what the artifact can do to accomplish a certain task. Instrumentalization occurs through the use of the artifact and develops over time as the learner understands the functionality of the artifact and its capabilities. This includes the use of an artifact by the learner in unexpected or unintended ways. An example of this could include a learner storing mathematical results and theorems in their graphing calculator as a memory aid for future use. *Instrumentation* works in the opposite direction in which the artifact itself, with its built-in constraints and potential uses, is shaping the ways in which the learner conducts their mathematical activity. In Table 1 I provide a hypothetical example of what a student's subset scheme might look like from an instrumental genesis perspective using the framework provided by Roorda et al. (2016).

Table 1: Hypothetical Example of a Student's Subset Scheme

Instrumentation Scheme	Techniques	Conceptual Elements	Technical Elements
Subset Scheme	Draw a Venn Diagram	To determine if a set is a subset, one must conclude that every element of the set is an element of the other set	Write a For Loop to check that each element is an element of the larger set

Roorda et al. (2016) provide examples of how an artifact may shape a learner's mathematical activity in their description of a student named Andy's schemes regarding the concept of the derivative. One specific example is Andy's use of the dy/dx-option on the graphing calculator to calculate the steepness of a graph at a particular point. Andy used this option in other examples throughout the longitudinal study when they needed to find instantaneous rate of change and did not coordinate between the dy/dx-option on the calculator and the symbolic representations that were covered in class. The authors conclude that while Andy's graphical and numerical representations of the derivative were strong, these representations were being developed independent of Andy's symbolic representations, a negative consequence of Andy's overreliance and strong use of the graphing calculator. Trouche (2004) comments that both instrumentalization and instrumentation work together in tandem as part of the instrumental genesis process, and thus refers to a learner's scheme as the instrumented action scheme as opposed to other terms such as *instrumentation scheme* as used by Roorda et al. (2016). For the purposes of this manuscript, I will adhere to Trouche's approach to the development of a learner's scheme (as perceived by the researcher) and refer to it as an instrumented action scheme or just scheme for short.

Methods

Participants were recruited from a four-year Hispanic-Serving Institution and were purposefully selected (Patton, 1990) in that they had already taken, or are currently enrolled in differential or integral calculus and not enrolled in an Introduction to Proofs course. Thus, the students in this study were not familiar with mathematical logic or set theory. Additionally, the students had little-to-no prior experience programming. The design of the study was a conjecture-driven teaching experiment (Confrey & Lachance, 2000) in which the conjecture was the following: *programming can serve as an experientially real context in which students will be able to connect mathematical logic and set theory.* There were 10 students that participated in the study divided into four groups with two to three students in each group. Due to the ongoing COVID-19 pandemic, the teaching experiment sessions were conducted and recorded via Zoom. For each session, I shared my screen which displayed two windows. The first window was a Jamboard[1] slide deck to present the tasks to the students, and to capture their ideas and diagrams.

[1] Jamboard is a Google interface that serves as a collaborative and interactive canvas.

In a window next to Jamboard, I had an Integrated Development Environment that was used to run Python code.

For analysis, I utilized Roorda et al.'s (2016) framework for identifying the techniques, conceptual elements, and technical elements as the basis for developing the students' instrumented action schemes. Recording the conceptual elements was done through the identification of the mathematical concepts that the students were learning. In analyzing the techniques used by the students (e.g., the output students produced, diagrams they drew, logical statements they considered), I watched the Zoom video recordings to document what exactly the students were producing in relation to conceptual elements. To document the technical elements (e.g., the code the students typed, the mathematics they wrote in Jamboard), I analyzed the Zoom video recordings to document what the students typed and when they typed it. This analysis was done by tagging the audio transcriptions using MAXQDA with Concepts, Techniques, Tech-Elements as the main codes for analysis. If a student typed something without saying out loud what they had typed, I used the 'memo' feature within MAXQDA to document what the student typed. This way I was able to analyze all three aspects of the students' instrumented action schemes within MAXQDA. By coding the techniques, conceptual elements and technical elements, I was then able to construct each students' instrumented action schemes. Importantly, the students' schemes were constructed with the four features as described by Trouche (2004) and Vergnaud (2009) which included the goals, rules, operational invariants and the possibilities for inference. Due to the page length constraints, I highlight the results for only one of the students on one problem within the larger teaching experiment. Additionally, the data presented here came from the fourth session out of five total sessions with this group of students.

Results

In this section I present the work of Alonso (pseudonym), a participant in Group 4 as they worked on a problem in which they were asked to "find the common elements across all three sets." The three sets, A, B, and C were given to the students in the study and contained both strings and integers. The students were also asked to "draw a diagram of what this set relationship might look like first before you write any code." I highlight Alonso's work due to the unique nature of his approach to solving this task. No other student in my study attempted to solve the task in the same way that Alonso did. First, I will briefly present some information that will be pertinent to understanding Alonso's solution method. First, repeated elements in a set, in Python and in mathematics in general, are not counted multiple times for the total number of elements that belong to the set. For example, if we define A = {1, 2, 3, 4, 5, 5}, the number of elements that belong to A, otherwise known as the *cardinality* of A, is five. Second, the cardinality of a set can be determined in Python using the 'len()' function. This function finds the length, or the number of items in the iterable data object. Also, it is important to note that before drawing the diagram, Alonso wanted to clarify what I meant by "diagram." I told him that I would "leave it up for interpretation." Figure 1 is a screenshot of Alonso's diagram, which he used as an opportunity to write pseudocode, a step-by-step process outlining an algorithm that would solve the problem. Of the ten students in the study, Alonso and his partner were the only two to write pseudocode. The other students either drew Venn diagrams, circled certain elements in the sets, or did some type of representation of what the intersection process might look like.

The first process in Alonso's pseudocode was to iterate through all the values, or elements, in A and add those elements to the set B and the set C. This step is fairly straightforward but requires a little interpretation as to what is meant by 'set B and C' in the second line. One could interpret this to mean the logical operator 'and,' because he refers to it as a 'set' instead of 'sets,' and would thus possibly imply the intersection of the two sets B and C. However, in his description of the pseudocode, Alonso clarified this line when he said, “So, if you had set A, you would add it to both set B, and C. So that all the values from set A get cycled through the two sets.” This confirms to me that Alonso was referring to two separate sets, set B and set C.

```
Step through values in set A

add value to set B and C

Check to see if length has changed

if it has not changed add value to set D

Step through values in set B

add value to set A and C

Check to see if length has changed

if it has not changed add value to set D

Step through values in set C

add value to set B and A

Check to see if length has changed

if it has not changed add value to set D
```

Figure 1. Alonso’s Pseudocode for Set Intersection

The next step in Alonso’s pseudocode was to check the cardinality of the two sets B and C. As Alonso described it, “With each iteration you check to see if the length has increased. If the length has not increased, then you know that there has been a repeat.” I am interpreting this statement as a process of adding each element from A to both sets B and C. One then finds the cardinality of the sets B and C. If there is a change in the cardinality from before the element in A is added to the sets B and C, then one can determine that this element is not shared with the set. For example, let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. If we add the element 2 to B, then the cardinality of the set B would change from three to four since 2 is not an element of B. The next step in Alonso’s pseudocode is to add the element to a new set, D, only if the cardinality does not change when added to sets B and C. An overview of the whole computational process is described by Alonso in the following way:

> This would require three For Loops and at the end you would just print the length of D and you would get the number of common elements. Or you could just print D to get which elements are in common.

The three For Loops that Alonso mentioned represent three iterative processes to add all the elements from each of the three sets to the other two remaining sets. Of course, three For Loops are not required. We did not go through the process of writing out Alonso’s code as we were running close to the end of time during the session and we still needed to hear from Alonso's partner about his diagram, but given that Alonso's method was so detailed, my hypothesis is that having the actual code would not change anything about his scheme or his conception of finding the intersection of multiple sets.

Due to Alonso's solution method, I refer to his scheme as the *Monitor Change in Cardinality Scheme.* As for his scheme, it is evident that the goal is to find the intersection of the three sets. I see two rules-of-action for Alonso. First, '*Construct an algorithm that would select each element in all sets'* and second, '*verify whether or not each element existed in each of the other sets*.' One theorem-in-action would be, '*Using a For Loop, one can iterate through every element of a defined set in Python.*' A second theorem-of-action is, '*One can determine the existence of common elements by monitoring any change in the cardinality of a set once an element has been added to the set.*' These theorems in action led Alonso to the solution method presented in his pseudocode. Given that this is just one problem from the larger teaching experiment, there is no identification of a possibility of inference for Alonso. A summary of Alonso's scheme can be found in Table 4.4.

Table 2: Alonso's Monitor Change in Cardinality Scheme

Instrumentation Scheme	Techniques	Conceptual Elements	Technical Elements
Determine Set Intersection by Monitoring Change in the Cardinality	Add an element to a set and calculate the cardinality of the set	The cardinality of a set changes when a new, non-repeated element is added	By writing a For Loop, one can pass through every element in a set. These elements are added to the other sets and the cardinalities of the other sets are calculated. If there is no change in the cardinality from before the element was added to after, for all sets, then the element is a repeated element and can be added to a new set.

Conclusion

Without Python, it is unlikely that Alonso would have come to that solution method to determining the set intersection of three sets, which is evidence on its own that Alonso constructed an instrument. Moreover, one important realization from analyzing these results was that Alonso used the For Loop and the 'len()' function as the artifacts in the construction of an instrument to solve a problem. That is, For Loops and functions are computational tools that were not originally designed to find the intersection of sets, but were co-opted by Alonso to solve the mathematical task of finding the intersection of three sets. This is a nuanced elaboration of how I initially interpreted the role of Python as the primary artifact when I designed this study. Specifically, when I originally conceptualized this study, I considered Python itself as the primary artifact in the students' instrument development. I was thinking about text-based code in general, not specific computational tools like For Loops and functions, which is what emerged through Alonso's work. The brief example presented in this manuscript illustrates that it is possible to understand how machine-based computing can be used to develop and enhance one's understanding of a mathematical idea.

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Mathematics Learning Assistantships and Undergraduate Students’ Identity Development

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Increasing the participation and achievement of students in science, technology, engineering, and mathematics (STEM) in PK-16 education continues to be a focal area of educational transformation and research. Faculty members at Institutions of Higher Education (IHE) plan, implement, and investigate how program structures can support the development, retention, and overall success of undergraduate students in STEM. Active learning classrooms, especially in mathematics, are one way IHE are reforming student learning experiences, and also provide a unique opportunity to engage undergraduate learning assistants with faculty to support near-peer students and deepen their own learning. This report examines the role of learning assistantships in developing undergraduate preservice teachers' identity while participating in a Noyce Scholarship program. Findings from this study show that preservice teacher participants gained competence in a variety of forms and developed their identity as a teacher through their experience as learning assistants in active learning classrooms.

Keywords: Teacher Preparation, Mathematics Education, Learning Assistants

Teacher attrition rates in the United States (U.S.) are approximately 8% annually. However, the highest rate of teacher attrition takes place in high-needs schools and turnover rates for mathematics and science teachers are nearly 70% greater in low socioeconomic schools (Carver-Thomas & Darling-Hammond, 2019). In 2017, 47 states and Washington D.C. identified shortages in qualified mathematics teachers (U.S. Department of Education Office of Postsecondary Education). Additionally, enrollment in teacher preparation programs has declined by more than one third since 2010 (Partelow, 2019). These factors not only contribute to the teacher shortage, but also to a lack of highly qualified, experienced teachers in classrooms. Kini and Podolsky (2016) found that teacher experience is associated with increased student achievement, highlighting the importance of teacher retention. Conversely, recruitment and retention challenges result in approximately 40% of new STEM (science, technology, engineering, mathematics) teachers entering the profession underprepared, yet they are the most likely to teach in high-needs schools, serving a majority of historically marginalized populations of students (Carver-Thomas, 2018; Carver-Thomas & Darling-Hammond, 2019; Sutcher et al., 2016). Teachers who receive less pedagogical training are more likely to leave teaching—especially in mathematics and science (Ingersoll et al., 2012).

This report examines an effort made by a mid-sized, urban university to recruit, train, and retain highly qualified secondary mathematics teachers. The project, supported by a National Science Foundation Noyce Track 1 grant (# 1852908), engages undergraduate dual-major mathematics and secondary education preservice teachers (PSTs). Within the program model, students serve as learning assistants (LAs) in active learning mathematics courses with the support of faculty mentors. In this research report, we utilize Chickering’s Seven Vectors of Identity Development framework (Chickering, 1969; Chickering & Reisser, 1993) to answer the following research question: How do learning assistantships in active-learning mathematics courses develop undergraduate students’ identities as future mathematics teachers?

Literature Review

The development of undergraduate student identity has been recognized as an essential factor impacting students' knowledge/competence, habits, attitudes, dispositions, and emotions (e.g., Allen & Schnell, 2016; Grootenboer & Zevenbergen, 2008). For example, identity development in mathematics begins in early childhood education where dispositions and deeply held beliefs about a student's ability to participate and perform in areas such as mathematics is established (Aguirre et al., 2013). Factors influencing STEM identity development include content competence, the ability to showcase learning to others (performance), and recognition of competence and ability in STEM by others (e.g., Carlone & Johnson, 2007; Herrera et al., 2012). These factors reveal the multifaceted and complex nature of identity development (competence, performance, and recognition) and the interwoven nature of interpersonal interactions involved in developing individuals' social and cultural identities (Herrera et al., 2012).

For undergraduate students interested in STEM education, an entirely new PST identity formation is needed to build upon their background knowledge and experiences, dispositions, roles, and responsibilities as future teachers (Beijaard et al., 2004; Slavit et al., 2016). For example, Ball et al. (2008) posit that effective mathematics teaching requires knowledge within six different domains that encompass both subject matter and pedagogical content knowledge. Not only must teachers deeply understand mathematics content, they must also be familiar with specialized mathematical content knowledge required for teaching and be able to employ effective teaching practices, both content and non-content specific. The K-12 teacher identity literature based across grade bands and contexts are well represented (e.g., Flores & Day, 2006; Friesen & Besley, 2013; Obenchain et al., 2016); however, STEM teacher identity development is still a growing area of educational research due to the increasing workforce demands and changing instructional expectations of STEM teachers to more actively engage students.

Classroom-based practicum experiences have been shown to strengthen some aspects of teacher preparation, yet many key qualities of effective teachers need to be addressed beyond PSTs' technical knowledge of content and for teaching (Banks et al., 2005). PSTs must have "wider-professional experiences" (WPEs) that allow them to consciously grapple and engage with their own attitudes and expectations on teaching and learning and to develop, enact, and reflect on their own professional knowledge in new contexts (Darling-Hammond & Bransford, 2005; Sleeter, 2018). For example, 21st century STEM teachers are called to create active learning environments where students engage deeply with content and collaborate with peers, as instructors uplift student thinking in an equitable learning space (Laursen & Rasmussen, 2019). Despite this aspiration, memorization and lecture-based instruction are still cited as the most commonly used approaches in STEM classrooms (Eurydice, 2015). Preparing PSTs to implement student-centered, active learning environments on their own requires them to experience learning and reflect on these environments through college coursework, field experiences, etc. (McDonnough & Matkins, 2010). Developing awareness through practicum and WPEs helps PSTs see what active learning looks like and how student engagement and learning can be impacted. Research has shown that undergraduates serving as learning assistants (LAs) in active learning classrooms not only shifted their understanding from unawareness to awareness of active teaching practices (Hall, 1974; Tunks & Weller, 2009), but also improved their own development of content understanding and identity, as well as enhanced their communication skills (Close et al., 2016; Goff & Lahme, 2003). Further, LAs recruited to K-12 teaching have demonstrated greater use of reform-based practices than their peers (Gray et al., 2016). While there is growing interest in exploring the processes and outcomes of embedding learning assistantships into undergraduate programs (e.g., Gomez Johnson et al., 2021; Vandergrift et al., 2020), connecting these experiences to PST development is an area of investigation that is needed to develop and retain future teachers.

Theoretical Framework

Undergraduate students undergo substantial changes in the way they think, act, and relate to others and to self during this pivotal time in their lives. Beyond cognitive development through academics, undergraduate education can be enriched by specific learning experiences and interactions in and outside of the classroom (Quinlan, 2011). Seminal work by Chickering & Reisser (1993) integrated undergraduate students' cognitive and psychosocial development theory into a model illustrating the fluid nature of student development along seven core areas, called vectors. The vectors are: developing competence, managing emotion, moving through autonomy toward interdependence, developing mature interpersonal relationships, establishing identity, developing purpose, and developing integrity. These vectors aim to define identity development as a cyclical continuation rather than a linear developmental path, later illustrated by Leggette and Jarvis (2015) as spokes on a wagon wheel to illustrate the interconnectedness of the developmental vectors. Chickering and Reisser's call to higher education institutions is to create environments that foster broad developmental practices and policies to support human talent and potential. By advancing strengths and skills along each vector, students are better equipped to overcome barriers and unexpected challenges through increased cognitive and psychosocial versatility (Chickering 1969; Chickering & Reisser 1993; Evans et al., 2010).

Research Methodology

This study investigates the ways in which engaging secondary mathematics PSTs as LAs in active learning undergraduate mathematics courses influences their development along Chickering's vectors of development. Understanding the lived experiences of the participants is well suited to qualitative research design, as we are able to collect rich and descriptive data (Charmaz, 2008; Yin, 2018) to glean how student participants "interpret their experiences and…[assign] meaning they attribute to their experiences" (Merriam, 2009, p. 5).

Context and Participants

Faculty from the institution's mathematics and teacher education departments developed a program funded by a five-year National Science Foundation Robert Noyce Teacher Scholarship Program aimed at recruiting and retaining high quality secondary mathematics teachers. Undergraduate students participating in the project are paired with a faculty mentor with whom they engage as LAs for active learning mathematics courses. Participants for this study include nine undergraduate students who participated in the program in Fall 2021. Five of the nine participants identify as female and four as male. Five out of the nine participants are upperclassmen (juniors or seniors) who have had at least one practicum experience within their traditional teacher preparation program. Fall 2021 piloted the LA experience in active learning mathematics courses uniformly with all participants.

Data Collection and Analysis

Participants completed a total of ten reflective journal entries across a 16-week semester in Fall 2021. The reflection prompts were open-ended in nature but asked the students to frame their responses around potential implications for them as future teachers. Participants agree at the onset of their acceptance to the program to allow these journals as data collection to inform the research and evaluation requirements of the program. In addition to the written reflections, the participants also take part in a focus group at the end of the fall semester. The following interview prompts were used to facilitate the semi-structured interview with the group:

1. Describe your learning assistant experience.
2. What, if anything, have you learned about mathematics? Did anything surprise you?
3. What, if anything, have you learned about teaching mathematics? Did anything surprise you?
4. What, if anything, have you learned about learners? Did anything surprise you?
5. What, if anything, have you learned about yourself after serving as a learning assistant/teaching assistant?
6. Is there anything you would change to improve your experience as a learning assistant? What advice do you have?

We used Chickering's Theory of Identity Development (Chickering, 1969; Chickering & Reisser, 1993), which is framed around seven vectors, to create a codebook. We used the codebook to engage in deductive, first round coding (Saldaña, 2021), and we added descriptive examples to the codebook to ensure the stabilization of subsequent coding across the data (MacQueen et al., 1998; Roberts, et al., 2019). We read through and coded excerpts from the participants' reflections and the focus group interview that identified moments where students referred to vector-aligned experiences/opportunities, which are: 1) developing competence, 2) managing emotions, 3) moving through autonomy toward interdependence, 4) developing mature interpersonal relationships, 5) establishing identity, 6) developing purpose, and 7) developing integrity. As we coded the data, we identified an additional, emergent code of "self-awareness," which we inductively added to our list of codes (Miles et al., 2020). In a second round of coding, we parsed out sub-codes within the parent codes to help us better understand the aspects of each vector that students self-reported experiencing. To increase the validity and reliability of our results, we coded all data sources together, allowing us to reconcile in the moment any potential differences in coding. The following section presents preliminary findings from our analysis.

Results

Identifying which vectors participants experienced in their role as LAs is an important first step to developing a richer understanding of how wider professional learning experiences can influence PST development. As such, we were interested in the codes that were most highly saturated in the data. Moving through autonomy toward interdependence, developing mature interpersonal relationships, developing purpose, managing emotions, and developing integrity were each coded less than 25 times. Of the seven vectors, the remaining two codes, developing competence (coded 85 times) and establishing identity (coded 41 times) surfaced as the most highly saturated codes in our data, so we sought to further examine what it was about these codes that participants identified surfacing within their LA experiences to better understand how we might leverage such experiences to maximize identity development.

Developing Competence as Future Teachers: Mastering Content, Pedagogy, and Relationships

Chickering sub-divides the vector "developing competence" into several components, two of which surfaced regularly in our data: intellectual competence and interpersonal competence.

Intellectual competence is defined as including the ability to master content, develop sophistication related to intellect and aesthetic, and build a wide range of skills, whereas interpersonal competence involves the acquisition of communication skills such as listening, cooperating, and communicating and responding effectively (Chickering, 1969; Chickering & Reisser, 1993). These sub-codes were the most frequent in our data, with 63 instances of intellectual competence and 22 instances of interpersonal coded, thus we wanted to tease out the aspects of participants' experiences as LAs that resulted in their sense of gained competence.

Intellectual competence. We uncovered several common themes in our analysis of the sub-code of intellectual competence. First, multiple participants described development of their ability to recognize (and sometimes employ) effective strategies for engaging learners in the classroom. One participant reflected on their experiences in an active learning classroom, stating,

> As a LA I have an opportunity to, it's a very interactive class and so I'm having one on one conversations with students or having group conversations with this group and learning a lot about how to guide students' thinking without giving them the answer. I think that's been a big takeaway from the LA experience (Focus Group Interview).

For several of the participants, this was their first-time experiencing mathematics instruction in an active learning environment, as either a learner or a teacher, and it pushed them to rethink their understanding of the structure of a mathematics lesson at the undergraduate level.

This led into our second observation that multiple participants discussed the ways in which observing their mentor teachers' teaching style influenced their thinking about mathematics instruction. They identified specific teaching structures and strategies their mentors utilized, as well as the impact it had on students in their classroom. Participant A explained it thus,

> I have been trying to analyze and pick apart [my mentor's] teaching style… because he is a phenomenal teacher and I want to be the same for my students. One big thing I have noticed is that he… tells the students to do something alone for a couple minutes, then discuss with their partner for a minute, then their group for another minute. He does this everyday… Everything he does is consistent, and it reflects on the students' engagement and learning. (Learning Assistant A).

In addition to thinking about how content is delivered, participants also reflected on the depth of mathematics content knowledge they would be required to have as future mathematics teachers and how to make mathematical connections obvious for their students. A participant described one instance of this he experienced while observing, saying,

> I am going to reach out to my mentor that I am teaching with and ask if they would share the lesson plan ahead of time. This way I can study up on the mathematics I know but forget due to the length of time it has been since I learned or used it. Just because I learned it before, does not mean that it is permanently retained. More practice and re-learning can solidify the mathematics in my brain though (Learning Assistant B).

These recurring themes around intellectual competence illustrate the potential power of using learning assistantships to help students develop mathematical content and pedagogical knowledge for teaching (Ball et al., 2008). As participants continue in their preparation program and future practicum experiences, having the opportunity to recall these WPEs as LAs can help them be more conscious of their teacher actions and leverage their own experiences to help their students learn (e.g., Darling-Hammond & Bransford, 2005; Sleeter, 2018).

Interpersonal competence. In addition to developing intellectual competence, participants also reflected on how their experiences as LAs influenced growth in their interpersonal competence, or their ability to communicate and collaborate effectively, in approximately one-fourth of segments coded for this vector. As participants engaged in conversations with their mentors and with the students in the mathematics courses they assisted as LAs in, they reflected on how these opportunities allowed them to develop their capabilities to engage with colleagues and students.

One aspect of this competence arose through comments that indicated the LAs' learning about the importance of fostering positive relationships with the students in their classes. Receiving advice about how to do this in mathematics courses from veteran instructors can help to develop this vector in preservice teachers. Providing additional and early access through LA experiences can offer these novice teachers more chances to reflect on this. As one LA shared,

> This past week I was able to get to know my mentor teacher…we discussed… her past experiences as a teacher and what I had experienced so far as a future teacher… Her biggest piece of advice was to try and build relationships and trust with the students I will work with… I know that I will have to find my own way here, but hearing examples of what has worked for others helps a bit (Learning Assistant C).

Recognizing the potential impact that the trust and rapport that comes of this relationship building can have on student engagement in mathematics courses was uplifted by another participant, who said, "I want to get to the point where the students become comfortable with explaining their thought process through the work to me. My goal is to increase the student's ability to discuss ideas and their confidence in math" (Learning Assistant D).

Another aspect of interpersonal competence that arose in the data was the idea of vulnerability in front of students as a way to build trust. One participant described this by saying,

> I have gotten more confidence in myself ironically when I'm able to admit when I'm not confident. We were learning about [a math strategy] ... I had never really seen before and… just being real with the students and saying, hey listen, this is not how I learned it when I took this class… and I think the students actually really appreciated that because a lot of them found it really confusing as well. So, I think just, I learned about myself to be more open and straightforward with the students (Learning Assistant E).

This notion of "humanizing" mathematics and the faculty who teach mathematics courses is perhaps a novel revelation for PSTs, who may historically think of the teacher as being the "expert" in all aspects of their field, especially STEM areas (Aguirre et al., 2013; Allen & Schnell, 2016). Taken together, participants recognized that being a "good" mathematics teacher involves more than knowing content, it also requires them to know their students and vice-versa.

Establishing Identity as Future Teachers: Developing a Professional Identity

Establishing Identity was the second most saturated code in our data, with this parent code identified a total of 41 times across the data. Within the larger code of establishing identity, we nested several sub-codes: professional/teaching identity, social or group identity, personal identity, and identity threats. Of these, professional/teaching identity was overwhelmingly the most often coded in the data with a total of 25 counts coded, or just over 60% of the segments (social group and identity threats coded only six times each, personal identity coded only 4 times). A crucial part of helping undergraduate students make the shift from their identity as "student" to their identity as "teacher" is providing them opportunities to gain knowledge of the praxis of teaching and strategies, and also to gain authentic experiences to implement such skills and strategies in real-time. Although most teacher preparation programs offer field experiences where PSTs work in classrooms with students, there is no guarantee that all field placements are with experienced mathematics teachers who incorporate research informed teaching practices, such as active learning and teaching through inquiry. Supplementing these more traditional teaching apprenticeships with learning assistantships can offer PSTs in mathematics rich opportunities to develop not only their praxis, but their self-image as a teacher practitioner.

Participants recognized the importance of this, as well as the impact of engaging as LAs on their professional identities as future teachers. One LA summarized this by saying,

> Additionally, it helps us grow our "teacher voices," which has been my biggest growth. When I presented my first proof to the class, I was nervous to talk in front of others. Now, with my most recent presentation, I was much more confident, was able to face the class and present from the board and engaged with my peers. I noticed my voice was more confident… (my "teacher voice"!). I believe I am more confident now because I have found my 'place' in the class and have gotten to know my peers and professor more, which helps me understand what type of presentation and engagement I should do (Participant F).

Our findings support previous research finding that learning assistantships develop both content understanding and their identity as a future professional (Hall, 1974; Tunks & Weller, 2009). In part, our data show that participants' ability to find their voice, communicate their thinking, and work with others helped them see themselves more as teachers and leaders in the classroom (Close et al., 2016; Goff & Lahme, 2003).

Applications to/implications for teaching practice or further research

Findings from this study support that learning assistantships are advantageous WPE that support the development of PSTs competence and professional identity as second mathematics teachers. In particular, these early program and full semester LA experiences are unique factors of this program design that have the potential to accelerate PSTs development. These LA experiences also provide intentional mentor pairing that are often challenging to control in traditional teacher preparation programs. The sample size and data set presented here are small and lack diverse representation of student participants. However, this study indicates the need for research on the impact of LA opportunities on PST development in mathematics and other STEM areas. This work does add to the body of existing research in this area and presents elements of this apprenticeship model that are important to focus on in future studies:

- leveraging learning assistantships to develop PSTs content and pedagogical knowledge of teaching secondary mathematics using active learning strategies;
- utilizing learning assistantships as models for how to foster positive, trusting relationships with students and allowing LAs to examine its impact on student engagement and achievement; and
- providing additional structured, scaffolded experiences teaching mathematics to facilitate PSTs' sense of "teacher self."

Future work has the potential to extend to other STEM education content areas, which can help to generalize the benefits and challenges of using learning assistantships to develop undergraduate PST identity and support the learning outcomes of mathematics students learning side-by-side with their near-peers as guides.

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Goals, Orientations, and Social Support Offered by a Mathematics Graduate Teaching Assistant

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Research suggests that support offered by an instructor can significantly impact student experiences and that there are different types of support that instructors can offer (e.g., emotional support, informational support, etc.). This case study examines the goals and beliefs of a first-year Mathematics Graduate Teaching Assistant (MGTA) in an attempt to explain their in-the-moment classroom decisions about offering different kinds of support to their students. This study found that the MGTA's beliefs about the nature of mathematics and how mathematics problems are solved were critical in understanding their tendency to offer a particular type of support. Understanding the ways that MGTAs support their students can initiate a valuable discussion surrounding novice mathematics instructors and their professional development.

Keywords: Graduate Teaching Assistants, Social Support, Goals, Beliefs, Decision-Making

In recent years, education researchers have become more concerned with teacher-student relationships and the impact these relationships can have on student experiences. Teachers are no longer considered couriers of knowledge as they have been in the past. Several studies from the past 20 years provide evidence that a positive teacher-student relationship correlates to increased engagement, motivation, effort, achievement, and overall well-being for students (Furrer & Skinner, 2003; Niehaus et al., 2012; Tennant et al., 2014). The present study is concerned with one specific facet of the teacher-student relationship: social support. *Social support* refers to any action intended to alleviate or prevent negative emotions that someone might experience (House, 1981). Numerous studies have explored social support in the classroom, including the kinds of social support that students receive, student perceptions of social support, and the effects of social support on student experiences (Federici & Skaalvik, 2013; Tennant et al., 2014). However, little research has explored social support from the perspective of the teacher, including the decision-making process behind offering different kinds of support. Tardy (1985) claims that the direction of social support (i.e., the distinction between the giver and receiver) is one of the most important characteristics of social support but acknowledges that the receiver is usually the focus of research.

Since research suggests that social support has the potential to positively benefit students and that different kinds of social support affect students in different ways, it is important to understand the elements that contribute to decisions to provide different kinds of support. While much of this decision process occurs subconsciously, research suggests that the decision-making process is determined by things that one can become aware of (Schoenfeld, 2011). An awareness of the decision-making process and the elements that it draws upon can better inform such a decision and lead to more intentional, productive support being offered.

The purpose of this study is to explore how Mathematics Graduate Teaching Assistants (MGTAs) choose the types of support they offer their students in first-year math courses by exploring their goals and orientations about teaching (*orientations* refers to one's beliefs and values) (Schoenfeld, 2011). MGTAs comprise a consequential portion of the teaching force behind undergraduate mathematics education, teaching 17% to 21% of mathematics courses at doctoral institutions (Blair et al., 2013). The long-term intent of this research is to better understand the decision-making process behind offering support, enabling MGTAs to become more aware of the ways they offer support and in turn better serve their students. Such an understanding has the potential to contribute to professional development for all educators, not just graduate students, as making decisions about social support is a ubiquitous part of teaching (Schoenfeld, 2011).

This paper is a part of a larger, semester-long study that explored the ways that six MGTAs conceptualize the types of social support that they offer, as well as the goals and orientations that influence their decisions to offer social support in particular ways. Due to space limitations, this paper focuses on one MGTA and the findings related to the following questions:

RQ1: What goals and orientations are present when MGTAs offer social support to their students?

RQ2: How can MGTAs' goals and orientations be used to explain the types of social support that they offer their students?

Literature Review

This study was designed around two separate frameworks, the first being House's categorizations of social support (1981). House asserts that all examples of social support can be categorized as emotional, informational, instrumental, and/or appraisal support. *Emotional support* is defined as support in which the negative emotion itself is being combatted, such as a teacher providing validation to a student experiencing self-doubt. *Informational support* is that in which resources are provided to combat a negative emotion indirectly. For instance, a teacher referring a student struggling with a homework problem to a helpful example in the textbook would be considered informational support. The other two categories are not discussed in this paper, and therefore will not be addressed due to space limitations. Despite being written more than four decades ago, House's framework is still referenced today and more recent frameworks for categorizing social support are comparable to House's (Wu et al., 2020; Feeney and Collins, 2015; Federici & Skaalvik, 2014).

The second framework around which this study was designed is Schoenfeld's "resources-orientations-goals" framework for in-the-moment decision-making (henceforth referred to as the ROG framework) (2011). Orientations include one's beliefs, values, and other related concepts, while goals are "something that an individual wants to achieve, even if simply in the service of other goals" (Schoenfeld, 2011, p. 21). According to the framework, one's in-the-moment decisions are shaped by these constructs, and if you know enough about one's resources, orientations, and goals, you can understand and ultimately explain the decisions they make (Schoenfeld, 2011). The present study was conducted in a convened course in which lesson plans, student workbooks, and assessments were constant across all participants and were therefore excluded from consideration.

Methods and Procedures

Methodological Approach

This study used a qualitative case study methodology, defined as "an empirical inquiry that investigates a contemporary phenomenon… in depth and within its real-world context" (Yin, 2014, p.16). This methodology is appropriate for this study because the research questions are less focused on the general phenomenon (the decision-making process behind MGTAs offering social support) and more concerned with the individual cases. Each case is an individual MGTA teaching a calculus recitation during their first semester as a PhD student at a large Midwestern public university. A descriptive case study design gives each participant an opportunity to have their story be told in order to “develop a complete, detailed portrayal of [the] phenomenon” (Schwandt & Gates, 2017, p. 346).

Participants

Only MGTAs teaching a recitation for either Calculus I or II were considered for this study. MGTAs were excluded if they were doing doctoral research in mathematics education or if they had significant teaching experience prior to matriculation, as they were not representative of a typical graduate student. This paper focuses on one first-year MGTA in the study, Blake (pseudonymized).

Data Collection

Data was collected in three stages. First, MGTAs were interviewed to explore their goals and orientations related to teaching. Secondly, each MGTA was observed teaching for four 75-minute class meetings; these lessons were audio-recorded while the researcher took note of examples of social support that they noticed the instructor offer. The observations were followed by a second semi-structured interview to discuss a selection of the observed examples of social support. Here, MGTAs were asked to reflect on their motivations and decision-making behind each example and asked to categorize each example within the social support categorization framework (House, 1981).

Data Analysis

A pilot study indicated that multiple coding methods would be needed to analyze the data through the lenses of the different research questions. The purpose of the first phase of coding was to address RQ1 by focusing on the goals and orientations of each participant; this phase used both structural and provisional coding to develop parent and child codes, respectively. *Structural coding* refers to the use of codes relating data to specific research questions; this method allows one to categorize excerpts related to similar topics before moving on to further coding and analysis (MacQueen et al., 2008; Saldana, 2016). In this study, parent codes were used to categorize data related to goals, orientations, and social support, allowing for simultaneous coding. In *provisional coding*, a codebook is developed prior to data collection, but can be “revised, modified, deleted, or expanded to include new codes” (Saldana, 2016, p. 168). Child codes were developed based on the literature review and experiential data (both from the pilot study and the researcher’s familiarity with the data) to capture different types of goals and orientations that participants might express; child codes included things like “Goals – Student Performance” and “Orientations – Nature of Mathematics”. The flexibility of provisional coding allowed the researcher to continuously add codes as new goals and orientations arose. Each interview was coded multiple times to ensure that no codes or excerpts were missed.

A second phase of coding utilized hypothesis coding. In *hypothesis coding*, codes are generated “specifically to assess a researcher-generated hypothesis” (Saldana, 2016, p. 171). The researcher developed specific theories for each participant based on the first cycle of coding, memos that were taken throughout the study thus far, and peer debriefing. An individualized codebook was then developed for each participant to assess the researcher’s hypotheses, and coding was repeated until no new themes emerged.

Findings

What follows is a discussion of Blake. Of the six participants in the larger study, Blake’s case exemplifies how the ROG framework (Schoenfeld, 2011) can be used to relate MGTAs’ goals, orientations, and social support actions. Blake's case ultimately forms the foundation that the rest of the cases expand upon.

A Brief Introduction to Blake

Blake is a first-year MGTA whose enthusiasm for teaching was immediately evident during data collection. When talking about their reasons for coming to a doctoral program, Blake said,

> I want to learn more about math, but then on top of that, I want to become the best math educator that I can be. So teaching for me is extremely important… I find joy in teaching these topics, leading students through problems, helping them understand, and just also in general just helping them enjoy mathematics.

Prior to graduate school, Blake sought out various opportunities to teach in informal settings (such as tutoring) and continued to seek out opportunities to improve their teaching throughout their first semester of graduate school. Blake puts a great deal of energy into their teaching, including developing rapport with their students. Throughout this study, it was evident that Blake knew and cared about their students.

During the observations, Blake gravitated towards offering informational support over the other categories of social support (House, 1981). During data analysis, two compelling themes emerged that could explain Blake's tendency towards informational support: their perspectives on what mathematics is and how math problems should be solved. What follows is a description of Blake's goals and orientations related to these themes and an explanation of how they are reflected in Blake's social support actions.

Goals and Orientations Related to the Nature of Mathematics

When reflecting on what mathematics is and what it means to solve math problems, Blake regularly made comparisons to solving a puzzle. When asked about their goals, Blake said, "I always think of it [math] as a puzzle, so hopefully I can transfer that to them [my students]. Like this is like a fun puzzle, here are the rules that you can use". They went on say that, like a puzzle, much of their enjoyment in math comes from the sense of accomplishment one gets after completing a problem:

> *I* [emphasized by speaker] enjoy doing math, like mathematics is fun to me. It's frustrating, it's annoying, sometimes it's heartbreaking… But I'm here in a PhD program because I enjoy doing and teaching and just like, being a part of mathematics. And so if I can get my students to then feel some of that same enjoyment… If they did a thousand-piece puzzle and like a really cool [math] problem, like if I can get the idea in their head that that is possible? Then I've already won. I've done most of what I want to do.

These ideas—that math is like a puzzle and that it is more enjoyable if it is challenging—are personal beliefs of Blake's that present as goals that they have for their students and are frequently reflected in their day-to-day teaching practices. For example, Blake often encouraged students to try problems that the students may find challenging. Blake suggested different problems for different groups depending on the capabilities of each group (e.g., Blake suggested challenging problems to groups who appeared to have mastered the content for that day while suggesting more standard or introductory problems to groups who seemed to need more practice before attempting the challenging problems). This suggests that Blake was not only aware of the capabilities of individual students and groups, but also that they had a sense of what degree of difficulty would be both challenging and rewarding for different students.

Goals and Orientations Related to Solving Mathematics Problems

Blake frequently referenced a metaphorical "toolbox" full of "tools" that students possess to approach different kinds of mathematical problems. For example, Blake described the process of problem solving in a calculus class as, "You took this set of tools, you applied it to a problem, and then you got to a solution". In the second interview, Blake said,

> We have our bag of tools and then a bunch of math is just figuring out in what ways we can use them, in what ways can we combine them, what are some of the limitations, but also [what] are some of the like further stuff that we can use with these tools?

This belief is reflected in Blake's teaching practices. For instance, during the observed classes, Blake wrote important theorems and definitions on the board before class started; when class began, they encouraged students to refer to these resources as they worked through problems. When asked why they chose to start class this way, Blake said,

> Few things suck and bring up as many negative feelings in math as when you don't even know where the heck to start. So to have at least the definition… You can't work with tools if you don't know what's in your toolbox. So I guess in my case, at the very least everyone has the same set of starting tools and so they can at least get started.

In fact, when helping students one-on-one in class, Blake often explicitly asked students what "tools" they had in their "toolboxes" and asked them which tools they might be able to use.

Blake's Use of Informational Support

Blake's beliefs about mathematics being like a puzzle that can be solved using "tools" from one's "toolbox" are not only present in their goals and orientations but in the social support actions that they offer in the classroom. Blake tends to use informational support more than any of the other categories proposed by House (1981). Table 1 outlines some of the examples of informational support offered by Blake during observations and offers a brief explanation of why informational support is the best categorization for them.

Table 1. Examples of Informational Support Offered by Blake

Example of Informational Support	Justification for Categorization
Before class (and as students started to work individually and in small groups), Blake wrote all the major theorems/rules/definitions that were needed for today's lesson up on the board.	Blake was providing a resource that was not inherently helpful to any particular problem that students were working on; students were responsible for identifying the tool that would be helpful and determine how to use that tool effectively.
While Blake was helping a student, the student kept flipping through their notebook searching for a rule. Blake encouraged them to make a list with all the rules they'd learned recently so that they'd all be in one place.	Advising the student to create a reference page is informational support because Blake is not providing them with something helpful but is suggesting something that would be helpful if the student were to follow through.
Blake checked in with a student who had missed two weeks of class; during that conversation, Blake reminded the student of their office hours and of the tutoring center.	Blake gave this student the tools to seek out help, but it was up to the student to get help if they needed it.

algebraic computations. When the students told Blake they'd decided to skip this problem, Blake made light of it, even going as far as to laugh along with the students when one of them jokingly crossed the entire problem out of their workbook. The researcher categorized this example as emotional support because Blake was empathizing with the students and validating their frustrations with the problem by implicitly giving them permission to skip the frustrating problem. However, Blake interpreted this action as informational support. Blake argued that by encouraging the students to skip the problem after struggling with it for a long time, they were indirectly giving the students information that they could benefit from (in particular, that this problem was not "standard" and that their frustration was probably coming from long computations and not the calculus itself). When categorizing this example, Blake said,

> I want to put it into informational support again because like… I'm not telling them to skip the problem… If you don't think that this is going to be productive use of your time, I'm going to agree with that. And then you should move on to something else. So I'm not fixing the problem… I'm validating their… intuition on like "I could be doing something more productive". Great. Yes… Maybe now think about why this question is maybe not as helpful as maybe some of the other ones are.

Overall, informational support was the most frequent category of social support that was observed, both for the researcher's initial categorizations and Blake's categorizations during the interviews. Moreover, Blake explicitly identified informational support as the category that they believe they provide the most of. The use of informational support is a repeated in-the-moment decision that Blake makes, and therefore according to the ROG framework (Schoenfeld, 2011), any such decision should reflect their goals and orientations.

Explaining Blake's Use of Informational Support

Many of Blake's social support actions are related to their beliefs about the nature of mathematics and mathematical problem solving; in fact, all the examples of informational support discussed in this paper can be attributed to these beliefs. For example, Blake writing theorems and definitions on the board before class reflects their belief about how math problems should be solved. Blake believes that the best way to approach mathematics problems is to have a "toolbox" of "tools" at one's disposal. As a result, Blake perceives it to be their job as an educator to ensure that students have the tools necessary to solve a particular problem. By writing the relevant definitions and theorems on the board at the start of each class, Blake guarantees that every student at least has the required tools to approach any problem that day.

Even the example of informational support that the researcher and Blake disagreed on the categorization of makes sense when juxtaposed with Blake's beliefs about the nature of mathematics. Blake believes that mathematics is like a puzzle and that enjoyment often comes from overcoming the challenge. In this example, the students were frustrated by the problem because it was mired with tedious computations, not with the calculus itself. Blake's argument is that the students weren't benefitting from doing those computations because that problem was not representative of the skill they were supposed to be practicing. In the context of Blake's perception of mathematics being like a puzzle, this problem was like forcing the students to do the puzzle in the dark: it would be an impressive feat to complete the puzzle, but it would not have been representative of what solving a puzzle really is. This particular problem was in contention with Blake's beliefs about what mathematics should be, so it is understandable that Blake would encourage the students to skip that problem with the hopes that the students would recognize what made the exercise futile.

On a larger scale, Blake's general tendency towards informational support over the other categories is aligned with their beliefs about mathematics and mathematical problem solving as well. Many of Blake's goals and orientations reflect their beliefs that mathematics problems are puzzles to be solved, that the puzzles are more satisfying if they are challenging, and that one needs tools to approach the puzzles in the first place. In terms of social support, these beliefs are reflected in the fact that Blake fosters an environment that values productive struggle and has a teaching philosophy in which students cultivate their own "toolbox" of theorems, rules, and definitions; such attributes are closely aligned with informational support. In fact, research has suggested that in mathematics classrooms, informational support is one of the most beneficial categories of social support for developing problem-solving skills (Cutrona & Russell, 1990). When reflecting on social support in general, Blake said, "I should be providing them hints, clues, telling them that they're doing a great job, and giving them the tools to succeed. I should not be sitting down and doing the problems with them", which is indicative of someone who values the use of informational support.

Discussion

Blake serves as an excellent starting point in understanding how Schoenfeld's ROG framework (2011) can be used to understand MGTAs' use of social support in the classroom, particularly in consideration of both the research questions. RQ1 asks, *What goals and orientations are present when MGTAs offer social support to their students?* While the goals and orientations discussed in this paper are certainly not comprehensive, they paint a vivid picture of who Blake is as an instructor. It cannot be said that every first-year MGTA will have the same goals and orientations as Blake, nor can it be said that a MGTA with the same goals and orientations as Blake would offer social support in the same ways. However, discussing Blake's goals and orientations can serve as a starting point for understanding the ways that novice mathematics instructors conceptualize math and mathematics education.

RQ2 asked, *How can MGTAs' goals and orientations be used to explain the types of social support that they offer their students?* Analysis suggested that Blake's beliefs about the nature of mathematics and how mathematics problems should be solved are not only present when they offer social support to their students, but that they are a driving factor in many of the in-the-moment, day-to-day instructional decisions that Blake makes. All the observed social support actions discussed in this paper were considered informational support and could be traced back to these orientations. While Schoenfeld's ROG framework (2011) cannot be used to predict how Blake would respond in certain situations, this does provide insight into how Blake prioritizes and ultimately makes in-the-moment instructional decisions on a day-to-day basis.

The purpose of this study was not to develop a comprehensive understanding of how MGTAs provide social support in the classroom, but to instead begin a conversation about the goals and orientations that are relevant when making in-the-moment decisions. Since the ways that MGTAs offer social support is so heavily dependent on their personal goals and orientations, the results of this paper cannot and should not be generalized. Instead, the findings may inform professional development efforts as mathematics departments aim to understand how MGTAs support undergraduate students in classrooms.

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Student Understanding of Eigenvalue Equations in Quantum Mechanics: Symbolic Forms Analysis

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As part of an effort to examine students' mathematical sensemaking (MSM) in a spins-first quantum mechanics (QM) course, students were asked to construct an eigenvalue equation (EE) for a one-dimensional position operator. Sherin's symbolic forms were used in analysis. The data suggest three symbolic forms for an EE, all sharing a single symbol template but with unique conceptual schemata: a transformation which reproduces the original, an operation taking a measurement of state, and a statement about the potential results of measurement. These findings corroborate prior literature on a construction task rather than a comparison or deconstruction task, and with a continuous variable after instruction on discrete variables.

Keywords: Quantum Mechanics, Student Thinking, Eigentheory

Introduction

Quantum mechanics is one of a handful of topics in which every undergraduate physics major will take at least one course due to its pervasiveness in modern research and applications in physics and beyond. Despite the ubiquity of quantum mechanics courses and the significant amount of work that has gone into improving them, the subject has still proven difficult for students. Learning quantum mechanics has been shown to be a non-trivial task across both traditional (functions-first) (Singh and Marshman, 2014a; Singh and Marshman, 2015a; Emigh et al., 2018) and the more novel (spins-first) (Passante et al., 2015a, 2015b) approaches.

Eigentheory is central to the theory of quantum mechanics; it's baked into the second and third postulates of quantum mechanics. McIntyre (2012) presents the first three as follows:

1. The state of a quantum mechanical system, including all of the information you can know about it, is represented mathematically by a normalized ket $|\psi\rangle$.
2. A physical observable is represented by an operator $\hat{A}$ that acts on kets.
3. The only possible result of a measurement of an observable is one of the eigenvalues a_n of the corresponding operator $\hat{A}$.

The first postulate gives the first exposure to Dirac notation. The ket $|\psi\rangle$ is referred to as the state vector and, when projected into a basis, is often represented as either a column vector or a sum of other basis vectors in Dirac notation with appropriate coefficients. In general, any expression in Dirac notation has a direct translation to more standard linear algebra. Dirac notation however explicitly shows what basis the vector is written in, whereas the basis vectors are only implied in linear algebra. In Dirac notation an operator is denoted with a hat ($\hat{A}$); the linear algebra representation of this operator would be a matrix. The eigenvalues of this matrix are the possible results of measurement referenced in the third postulate. Standard in linear algebra, but not written into the postulates, is that each of the eigenvalues, a_n, of the operator $\hat{A}$, has an associated eigenvector, labeled $|a_n\rangle$. Therefore, one could argue that an eigenvalue equation in quantum mechanics has a fundamentally different interpretation than that of an eigenvalue equation in a mathematics context, and while they could be interpreted the same way, there are more productive interpretations for quantum mechanics.

The majority of research on student understanding of eigenvalue equations has come from the research in undergraduate mathematics education (RUME) community. Henderson and colleagues (2010) found that prior to instruction student reasoning about eigenvalue equations fell into one of three categories: superficial algebraic cancellation, correct solutions but an inability to interpret appropriately, and a correct solution with a correct interpretation. The correct interpretation in this case relates to an operation of a matrix acting on a vector resulting in the scaling of that vector by its eigenvalue. Thomas and Stewart (2011) studied student understanding of eigentheory over a two-year span and found that students tended to continue to think about linear algebra as a set of procedures rather than focus on the concepts. They target specific goals for instruction such as shifting focus toward the ideas of sets of eigenvectors and reinforcing a view that students were lacking: a geometric interpretation.

Physics education research (PER) has studied interpretations of the eigenvalue equation, often focusing on the Schrodinger equation or the eigenvalue equation for the Hamiltonian (total energy) operator, $\widehat{H}|\psi_n\rangle = E_n|\psi_n\rangle$, which is a major focus of QM courses, since time evolution of a system is associated with the Hamiltonian operator. In a study focused on resources students use to understand quantum mechanical operators, Gire and Manogue (2008) identified "quantum measurement as an agent," the idea that taking a measurement changes the system. Students also know that operating on a vector generally transforms the vector, which the researchers labeled "operating as agent." Accessing both of these resources in parallel can lead students to the idea that operating represents measuring. Gire and Manogue (2011) followed this up, noting that navigating unfamiliar language in addition to eigentheory presents additional challenges for students. Singh and Marshman (2013) reported student difficulties determining states for which the eigenvalue equation for the Hamiltonian would be true, some students going so far as to say that it is true for all states, including superposition states.

More recently, researchers have looked at how physics students reason about eigenvalue equations in different formats. Wawro, Thompson, and Watson (2020) investigated how students in a quantum mechanics course were thinking about both traditional mathematical eigenvalue equations ($\hat{A}\,\vec{x} = \lambda\vec{x}$) and a quantum mechanical eigenvalue equation ($\hat{S}_x|+\rangle_{\mathrm{x}} = \frac{\hbar}{2}|+\rangle_x$). They found that the mathematical equation typically elicited a scaling model, which carried over to the QM equation in some cases. They also noted that some students thought of the quantum mechanical equation in terms of representing a physical measurement, similar to prior PER findings. A novel finding was that some students used conditional language describing the equation in terms of potential or possible measurements rather than directly linking operating to measurement; this more subtle interpretation is consistent with the expert interpretation of a QM EE. Notably, some students who seemingly initially equated operating with measuring expanded their interpretation to encompass potential measurement.

The primary goal of this study is to identify cognitive resources students access when constructing and reasoning about eigenvalue equations in quantum mechanics (symbolic forms). This was done by analyzing written data from student responses to an eigenvalue equation construction task. We identified three different symbolic forms for an eigenvalue equation in the data. One is consistent with the mathematical interpretation. Another is reflective of a common misconception in quantum mechanics. The third is connected to the interpretation of eigentheory presented in the postulates of quantum mechanics. The primary delineating factor of the first, and other two are the connections the second and third make to the physical systems being modeled.

Symbolic Forms

Symbolic forms (Sherin, 2001) can be seen as an extension of the knowledge in pieces framework for student understanding (diSessa, 1993), developed as a means of examining how students think about (physics) equations. Sherin intentionally modeled this framework after diSessa's phenomenological primitives (p-prims), which were another set of intuitive chunks of knowledge or ideas (diSessa, 1993). These p-prims were each self-contained and relatively simple; small knowledge structures originating from nearly superficial interpretations of reality. By comparison, symbolic forms are larger structures consisting of two pieces: a pattern of symbols in an expression, the symbol template, and the rough idea expressed therein, labeled the conceptual schema. A conceptual schema is intended to have a fairly simple structure and is not inherently connected to any physical system or reasoning. Given that the schema contains all the meaning, it is possible for a single symbol template to be a part of several different symbolic forms, depending on the interpretation and/or context. To make clear the difference between them, some examples can be quickly explored. A common expression for the total energy of a system in intermediate mechanics is $H = T + U$, where T is the total kinetic energy in the system and U is the total potential energy. The sum on the right side is an example of the *parts-of-a-whole* symbolic form, as the individual pieces being summed compose the total energy of the system. In a senior-level quantum mechanics, class this expression would look quite different, $\widehat{H} = \frac{\hat{p}^2}{2m} + V(\hat{r})$ but could be interpreted in much the same way. In both cases here the symbol template is $\square + \square$ and the schema is "elements that combine to make a whole quantity.'"

Given their intended nature as building blocks, it is possible and likely to find more than one symbolic form in a physics equation. In the examples discussed above the focus was on the meaning of the sum, but other symbolic forms could be used to determine the meaning of the different equal signs. Due to the nature of the framework, it has been used in several contexts to analyze the intersection of math and physics. An example of its more traditional implementation is seen in the work of Schermerhorn and Thompson, who used symbolic forms to investigate student construction of differential length elements (Schermerhorn, 2019a).

Dreyfus and colleagues engaged in a theory-building effort meant to use data to illustrate their conjectures and to explore a method for analyzing student mathematical sensemaking in quantum mechanical problem solving (Dreyfus et al., 2017). They also argue the MSM tools learned in introductory physics are necessary but insufficient for MSM in quantum mechanics; this argument would be consistent with the findings of Kuo and colleagues (2013). The third and final claim of Dreyfus and colleagues' paper is that when students do not succeed in MSM in QM it is not because they are not engaging in the process, but that they are not using particular cognitive machinery (i.e., symbolic forms) needed to engage in expert MSM in QM. They highlight episodes from a study (Bing and Redish, 2012) to illustrate problems students could have interpreting equations with unproductive symbolic.

Dreyfus and colleagues observed two students (electrical engineering (EE) majors) recruited from an upper-level EE course that included a fair bit of quantum mechanics, working on a quantum mechanics tutorial. The researchers identified instances where the mathematics was consequential to the students' reasoning, labeling them as occurrences of potential MSM; they used these instances to conjecture what potential quantum mechanical symbolic forms could be. The primary segment of this interview entails the students reasoning about the time-independent Schrodinger equation or energy eigenvalue equation. The students struggled to determine whether the energy of the ground state of the infinite square well was constant due to the inclusion of the wave function on both sides of equation. The authors suggest that the students were interpreting the energy eigenvalue equation through the *dependence* symbolic form ([...*x*...]; a whole dependent on a quantity associated with an individual symbol) rather than seeing it as an eigenvalue equation, leading to their conceptual difficulties.

These observations were the starting point for the suggestion of new symbolic forms that would be productive for quantum mechanical MSM. The first symbolic form they posit is the *transformation* symbolic form, whose symbol template is $\hat{\square}|\ \rangle$ (an operator acting on a state) and whose conceptual schema is "reshaping" (the idea of stuff getting molded into a different shape). The only other posited symbolic form is the *eigenvector-eigenvalue* symbolic form, whose symbol template is $\hat{\square}|\ \rangle = \mathrm{C}|\ \rangle$, and conceptual schema is "a transformation which reproduces the original". This is a compound symbolic form, containing their *transformation* symbolic form, as well as a new interpretation of the equal sign, and Sherin's *coefficient* symbolic form. In this case the equal sign denotes a relationship between the operator, scalar, and eigenvector, as opposed to other previously proposed meanings of the equal sign (Dreyfus et al., 2017). Expert-level reasoning on quantum mechanical eigenvalue equations adds layers of physical interpretation on top of conceptual understanding, where the operator corresponds to a physical quantity and the solutions are physical states for which that quantity has a definite value, given by the eigenvalue.

Methods

Data were collected in a senior-level, spins-first quantum mechanics course that is required for the completion of physics and engineering physics B.S. degrees at the institution but is also taken by B.A. physics majors, physics minors, and non-physics majors. While not the students' first introduction to quantum mechanics in physics or engineering physics programs, this is the most in-depth study of the topic available to undergraduate students at this institution.

Students were asked to construct an eigenvalue equation for an operator that represents the position of a particle constrained to a one-dimensional system (Fig. 1) as an ungraded quiz. This task falls more in line with the mathematization oriented tasks used to develop the framework, making it better suited to identification of forms than prior, interpretive tasks. An adequate answer to this prompt would use the same symbol template as the *eigenvector-eigenvalue* symbolic form proposed by Dreyfus and colleagues (2017) but specified to the position context.

Consider a quantum mechanical system that is physically constrained to be located along a straight line, as shown below.

Position of object/system constrained to a line

a. Write down an **eigenvalue equation** for an operator that represents the position of this system.
b. Briefly explain what each term in the equation represents.
    i. How do each of these relate to the physical system?
    ii. What, if any, connections exist between the terms in your equation?

Figure 1. Eigenvalue equation construction task.

The authors engaged in collaborative qualitative analysis to refine the codebook until a consensus was reached (Richards, 1981). Student responses to this task were coded with the symbolic forms framework in mind. Due to the nature of the task, most student responses did not fit into existing symbolic forms. As a result, student responses were first coded for a symbol template, and then an associated conceptual schema. The first pass was essentially a binary coding identifying which students provided an expression utilizing the template $\hat{\square}|\cdot\rangle = \mathrm{C}|\cdot\rangle$, where the dots inside the ket symbols indicate identical symbols inside the kets, and thus identical kets. This symbol template is the same as that of the *eigenvector-eigenvalue* symbolic form proposed by Dreyfus and colleagues with the exception that the “empty” kets have been given dots to denote that it must be the same ket on either side of the equal sign.

For those students that used this template, a variety of conceptual schemata were identified, derived from the portions of student responses where they interpreted their expressions. Coding only for expressions that used an eigenvalue equation template proved inadequate however, as a variety of other expressions provided by students did not conform to this symbol template and therefore required additional categorization. Grouping similar student responses by the structure of their equation, and subsequently by ideas presented by the students, resulted in the identification of other potential symbolic forms in addition to those for eigenvalue equations.

Results

Student responses were indicative of three different ways of thinking about eigenvalue equations which are summarized in Table 1. Each of these different symbolic forms shares a single template but has a distinct conceptual schema.

Table 1. Summary of identified symbolic forms for the eigenvalue equation in quantum mechanics.

Symbol Template	Conceptual Schema	Symbolic Form
$\hat{\square}|\cdot\rangle = c|\cdot\rangle$	A transformation which reproduces the original	*Reproductive transformation*
	An operation taking a measurement of state	*Operating as measuring*
	A statement about the potential results of measurement	*Potential measurement outcome*

Reproductive Transformation

The first symbolic form discussed for an eigenvalue equation has the symbol template $\hat{\square}|\cdot\rangle = c|\cdot\rangle$, and the conceptual schema related to a geometric scaling but not rotation of the vector. This is consistent with the traditional mathematical interpretation of the eigenvalue equation (Henderson et al., 2010), and documented in a QM context by Dreyfus and colleagues as the conceptual schema “a transformation which reproduces the original” and labeled *eigenvector-eigenvalue* (Dreyfus et al., 2017). However, because we identify three distinct symbolic forms that all share the eigenvalue equation symbol template, we refer to this form as *reproductive transformation* as opposed to *eigenvector-eigenvalue* for clarity.

These student responses all had the appropriate terms, and some even labeled the elements appropriately (e.g., Fig. 2) but did not provide any physical reasoning or explanation for their expressions. Responses in this group give us insight into the symbol template students are using for eigenvalue equations. These students are showing evidence of the proposed *reproductive transformation* symbol template. Given that this is all some students provided however, it is difficult to determine the exact nature of the schema associated with these students’ understanding of the eigenvalue equation. While they are not demonstrating any additional

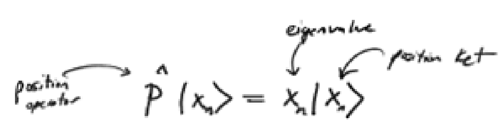


Figure 2. Example student response for reproductive transformation *symbolic form.*

quantum mechanical knowledge in their responses, these students are at least demonstrating that they know that the same ket needs to be on both sides of the equation, consistent with the form.

Operating as Measuring

Some student responses to the eigenvalue equation construction task are indicative of their conflating an operator acting on a state with the taking of a measurement of that state. These students seem to be thinking that the position operator acting on the eigenstate of position represents a measurement of position. While this and *reproductive transformation* share a symbol template, a more appropriate conceptual schema for these students may be “an operator taking a measurement of a state,” which would yield an *operating as measuring* symbolic form for eigenvalue equations.

In one such response and explanation, shown in Figure 3, the student’s expression has all the correct elements, and they are able to appropriately identify the different elements of the expression. However, in addressing how each element of the expression relates to the physical system, the student says that the operator represents “the operation of measuring position,” indicative of *operating as measuring*, a distinct interpretation of an eigenvalue equation.

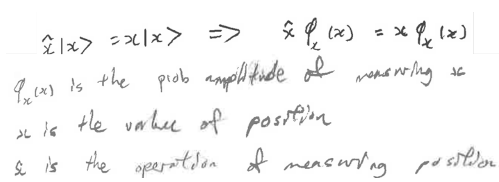
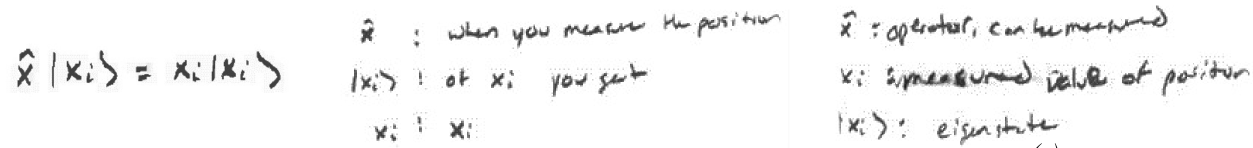


Figure 2. Example of operating as measuring *symbolic form.*

Potential Measurement Outcome

The final interpretation of eigenvalue equations with this symbol template comes from one student. Their eigenvalue equation contained all the correct elements, written as one would expect from convention (Fig. 4a). Figure 4b shows their explanation of what each term represents. When the text alone is read this is a fairly sophisticated statement: “When you measure the position of [the eigenstate] x_i you get x_i.” (The portion in brackets is an addition by the author for coherence, which is supported by the student’s response to the question, “How does each of these relate to the physical system?”, shown in Figure 4c.) This student’s interpretation of an eigenvalue equation goes beyond a geometric interpretation and does not include the notion that operating is the act of taking a measurement. This is indicated by the student’s use of the phrase, “can be measured,” which stands in contrast to the student in Figure 3 who stated that the operator represented “the operation of measuring position.” This is a subtle but important distinction as it separates the idea of taking a measurement from the idea that the operator represents a quantity which can be measured. This student presents a more sophisticated interpretation of the eigenvalue equation than the previous ones shown: that it is a statement about the *possible* outcome of measurement of the position, or more generally, the quantity being represented by the operator. A more concise version for use as a conceptual schema would be “a statement about the potential results of measurement”. This schema is directly connected to the 3rd postulate of quantum mechanics: “The only possible results of measurement of an observable $\hat{A}$ are one of the eigenvalues of the observable a_n” (e.g., McIntyre, 2012). While a single student is not indicative of the greater study population, the sophistication of this student’s response and its alignment with the meaning of a quantum mechanical eigenvalue equation warrant its inclusion as an example of a student-generated expert-level form for an eigenvalue equation.



(a) *(b)* *(c)*

Figure 3. Response related to potential measurement *symbolic form. (a) Student constructed eigenvalue equation. (b) An explanation of individual terms. (c) Relation of terms back to physical system.*

Discussion and Conclusion

Dreyfus and colleagues posited that there may be symbolic forms specific to quantum mechanics that had yet to be seen in student work. Their eigenvalue equation symbolic form, *reproductive transformation,* which is consistent with the desired outcome of mathematics instruction on eigenvalue equations (Henderson et al., 2010) and a productive conceptualization for students (Wawro, Watson, & Zandieh, 2019), is one of three forms identified that all use the same eigenvalue equation template, which follows the canonical structure.

The other two forms, *operating as measuring* and *potential measurement outcome*, both illuminate the physical meaning of the QM eigenvalue equation rather than a mathematical interpretation. *Operating as measuring* instead focuses on the act of taking a measurement "operator as agent" (Gire and Manogue, 2008, 2011) as a lens for interpreting an eigenvalue equation. Wawro and colleagues observed this conceptualization and proposed an explanation for this form: that physics equations typically represent relationships between physical quantities, documenting covariation among the quantities; *operating as measuring* conceptually reflects this relationship and is consistent with physics students' experience.

The third symbolic form, *potential measurement outcome*, is significantly more relevant to and meaningful for the physical interpretation of a QM eigenvalue equation. One could argue it falls into the expert-like additional layered meaning discussed by Dreyfus and colleagues; however, the student response discussed in relation to this form seems completely void of the ideas that form the schema of *reproductive transformation* (Dreyfus et al., 2017). Students could hold these two forms simultaneously (Wawro et al., 2020), and invoke them as needed or convenient, consistent with evoked concept images or resource activation, but *potential measurement outcome* is a more expert-like interpretation of a QM eigenvalue equation.
The way students mathematize a physics problem has been of interest to the PER community since the introduction of symbolic forms. Student construction of equations meant to describe physical systems, both in this and Sherin's (2001) tasks, provides insight into resources accessed by students in mathematizing physics problems. This analysis also opens the door for mathematical sensemaking analysis (Kuo et al., 2013). Some students explicitly attempt to utilize mathematics from other quantum contexts to generalize to this novel system. It is also noteworthy that both interpreting and constructing eigenvalue equations was not a trivial task for students. Some explicitly wrote about the difficulty of the tasks in their responses, while others showed the non-triviality of the task through their failure to provide a classifiable expression in response to the construction task. These data can provide insight into the ways students are reasoning through the discrete-to-continuous transition in a spins-first quantum mechanics course and will be a topic of further exploration.

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Modeling Mathematicians' Playful Engagement in Task-Based Clinical Interviews

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Theoretical accounts of mathematicians' disciplinary practice draw similarities to mathematical play, in that it can entail agency and autonomy, open exploration, creativity, imagination, and enjoyment. We analyzed task-based clinical interviews of 13 mathematicians to determine whether their problem-solving activity was playful and found evidence of three aspects of playful math: agency in exploration or goal accomplishment, self-selection of mathematical goals, and a state of immersion, investment, or enjoyment. We present these findings and introduce a form of problem-solving activity, the Explore-Focus Cycle, as one characteristic of playful math.

Keywords: mathematical play, clinical interviews, mathematicians

One goal of undergraduate mathematics education is to support students' development of productive dispositions towards mathematics (Code et al., 2016; Watson, 2008) helping students engage with mathematics in disciplinarily authentic ways (Lim & Selden, 2009). Work examining mathematicians' practices has highlighted activities such as detecting and formalizing patterns, developing and refining definitions, thinking structurally and generally, selecting and applying proof techniques, and refining earlier reasoning (Burton, 2001; Fernández-León et al., 2021; Melhuish et al., 2021; Martín-Molina et al., 2018). Although it is important to not take an overly narrow view of what it means to meaningfully engage in mathematics as an expert (Stillman et al., 2020), researchers nevertheless argue for instruction that supports the habits of mind believed to be similar to mathematicians' practices, such as generating interesting questions, experimenting, conjecturing, and generalizing (e.g., Bass, 2008; Cuoco et al., 1996).

Gresalfi and colleagues (2018) note that the above forms of disciplinary engagement involve many of the same features as play including "exercising personal agency to set and reach goals, exploration, imagination, and joy" (p. 1335). Others agree: For instance, Mason (2019) remarked that "playfulness provides an accurate and apt description of what mathematicians do," and Su, in his MAA Presidential Address at the Joint Mathematics Meetings (2017), stated, "We play with patterns, and within the structure of certain axioms, we exercise freedom in exploring their consequences, joyful at any truths we find." He went on to identify play as a key component of mathematics for human flourishing (Su, 2020). Indeed, offering opportunities to develop a mathematically playful disposition can center student voices, invite agency, and increase equitable access to mathematics (Gresalfi et al., 2018; Widman et al., 2019).

These remarks reflect a robust belief that mathematicians are playful in their engagement, and yet, we have found little empirical evidence demonstrating these claims. Consequently, the motivation for this study is to investigate whether and how mathematicians' disciplinary engagement is playful. In particular, we consider the following research questions: Is mathematicians' problem-solving activity in task-based clinical interviews playful? If so, what are the characteristics of mathematicians' playful engagement?

Theoretical Framework and Background Literature: Defining and Characterizing Mathematical Play and Playful Math

Definitions of play vary, but all entail a sense of agency in (a) choosing to engage in an activity, (b) exploring, (c) selecting goals, and (d) accomplishing goals (Jasien & Horn, 2018). Huizinga (1955), one of the early scholars to define play, remarked that play is voluntary, steps outside the ordinary, is bounded in time and space, is orderly, is governed by rules, and involves tension; the player wants to achieve something but wants to succeed by their own expertise. Dewey (1916/1966) similarly noted that people who play “are trying to do or effect something” (p. 203). Thus, play is goal-driven, but as Featherstone (2000) noted, the goals of play, unlike the goals of work, “are always in some sense ‘inside’ the play, they are real and shape the activity of the players” (p. 15).

Mathematical play is a particular type of play. Holton et al. (2001) described mathematical play as the playful exploration that emerges when learners find themselves in mathematical contexts with open goals, and Williams-Pierce (2019) defined mathematical play as voluntary engagement in cycles of mathematical hypotheses with occurrences of failure. Much of the work on mathematical play investigates the mathematics that can arise from playful settings such as preschool children’s block play (e.g., Brown, 2009), or players’ interactions with videogames (Williams-Pierce, 2019). In contrast, our interest is in how to incorporate play into the school mathematics that occurs in classrooms. We use the term “playful math” rather than “mathematical play” to highlight the potential of “playifying” classroom mathematical activity. Following the traditions of researchers who emphasize agency, we define playful math as a form of mathematical engagement that entails (a) agency in exploration or goal accomplishment, (b) self-selection of mathematical goals, and (c) a state of immersion, investment, and/or enjoyment.

Characteristics of Mathematical Play

Characterizations of mathematical play are typically either situated in studies of young children’s activity or in studies of individuals in informal environments. When they concern the mathematical play of undergraduates or experts, characterizations are often, but not always, theoretical rather than empirical. This is likely due to the fact that when students enter high school and college, discussions of learning seldom reference play (Gresalfi et al., 2018). Indeed, play has increasingly been pushed out of school settings (Jerret, 2015), and undergraduate education in particular has seen a separation of play and learning (James & Nerantzi, 2019). Consequently, little is known about the nature of learners’ playful engagement in advanced mathematics tasks (Gresalfi et al., 2018).

Many scholars have called for supporting and legitimizing play in adult lives (e.g., James & Nerantzi, 2019; Sicart, 2014, Swank, 2012). Concerning playful mathematics in particular, researchers argue for the importance of understanding and fostering students’ playful dispositions as a way to encourage engagement that mirrors the activity of mathematicians. For instance, Featherstone (2000) noted a number of similarities between play and the discipline of mathematics itself in that both focus on theoretical entities, are set apart from daily life, create order, and are rule bound. Mason (2019) explained that “mathematicians play with what is given and with what is sought – extending, varying, simplifying, and complexifying” (p. 95), and other mathematicians have described their own practice as playful (e.g., Lockhart, 2009; Su, 2017).

Descriptions of mathematicians’ playful engagement highlight the presence of two phases of activity. Su (2020), for instance, distinguished between the *inquiry phase* and the *deductive reasoning phase*. In the inquiry phase, one engages in pattern exploration and uses inductive reasoning to build conjectures from specific examples. One shifts to the deductive reasoning phase when a conjecture has been selected and must now be proved or disproved. Holton et al.’s (2001) theoretical analysis of mathematicians’ play makes a similar distinction between two phases. In the first, one produces tentative examples without a clear direction. In the second, one becomes more systematic and intentional in generating particular types of cases. These phases are reminiscent of Cai and Cifarelli’s (2005) examination of undergraduates’ mathematical exploration with open-ended problems. Although not specifically referencing play, they distinguished two types of goal episodes, *primitive* and *refined*. Primitive goal episodes entail trying out preliminary ideas without a specific goal other than to generate empirical results to support further exploration. Refined goal episodes are ones in which the problem solver's activity occurs with a well-defined goal in mind.

Although each of these researchers use different terms, all emphasize two characteristics of the first phase: It is empirical, gathering data to inform the development of a pattern, hypothesis, or direction, and it is not goal oriented, in that one is exploring without pursuing a particular, pre-determined direction. The second phase entails goal-directed activity towards a particular outcome in which one's actions are guided by anticipation. There is some disagreement about whether mathematical play is still happening when one shifts into the goal-directed phase (Holton et al., 2001), but there is broad agreement that mathematicians' playful engagement includes open exploration. We therefore attended to these two phases of activity and sought to determine whether mathematicians demonstrated instances of them in their problem-solving activity as a potential characteristic of playful engagement.

Methods

Our data set consisted of task-based clinical interviews from 13 mathematicians (see Lockwood et al., 2016 for additional details). Twelve mathematicians held a PhD in mathematics, and one held a PhD in computer science. During the interviews, the mathematicians worked on the problems shown in Figure 1. Task 1 was taken from Andreescu et al. (2007), and Task 2 was taken from a Putnam exam. All thirteen mathematicians worked on Task 1, and seven also worked on Task 2. We gave the mathematicians time to work on the tasks and asked them to think out loud as they worked. The interviewer generally only interrupted to ask clarifying questions and to answer the mathematician's questions. Each interview lasted approximately 60 minutes and was audio recorded using Livescribe pens and transcribed.

Interesting Numbers Task
Most positive integers can be expressed as a sum of two or more consecutive integers. For instance, 24 = 7+8+9 and 51 = 25+26. A positive integer that cannot be expressed as the sum of two or more consecutive positive integers is therefore *interesting*. What are all of the interesting numbers?

Selfish Sets Task
Define a selfish set to be a set which has its own cardinality as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish.

Figure 1. The Interesting Numbers Task and the Selfish Sets Task.

In analyzing the data, we began with a priori codes based on our three-part definition of playful math: (a) agency in exploration/goal accomplishment, (b) self-selection of mathematical goals, and (c) immersion, investment and/or enjoyment. The first and second authors coded each transcript separately and met weekly to reconcile discrepancies. All three authors then discussed any remaining questions or discrepancies. In addition to coding for playful math, we also analyzed the data for the two phases of activity identified in the literature, which we call *Explore-Focus Cycles* (Ellis et al., 2022). For Mathematician 1, we did this analysis as a whole group. For the remaining mathematicians, the first and second authors conducted their analyses independently, kept notes, and met weekly to discuss and reconcile codes. The project team met weekly to consider discrepancies and finalize codes until the codes had stabilized.

Findings

Playful Math Codes

We found evidence of two out of three playful math codes in all 13 mathematicians' problem-solving activity, and evidence of all three codes in nine of the 13 mathematicians' activity. The first code, *Agency in exploration/goal accomplishment*, occurs whenever a person has the agency within a task to explore their chosen ideas or work towards accomplishing their goals. The mathematicians almost always had agency to explore and accomplish their goals as a result of the open nature of the tasks, the structure of the interviews, and their authority as mathematicians. Even when the interviewer directed mathematicians' actions by asking them, for example, to prove a conjecture, the mathematicians typically retained agency to explore and select their own proof strategies. For example, while working on the Selfish Sets Task, Mathematician 5 (M5) discussed various strategies for answering the problem. She decided between "doing some answers and then prove it inductively" or "try to think about [the task] combinatorially in some way that would help the answer pop out." The interviewer acknowledged these different paths: "Yeah, so there are a lot of different ways that you could go from here" but did not suggest a specific direction. M5 immediately decided to "think about how I'm going to compute this sum". This demonstrated both *Agency in exploration,* as M5 thought through different approaches to solving the problem, as well as *Agency in goal accomplishment,* as she narrowed in on a direction to accomplish the task goal of solving the Selfish Sets Task.

The second code, *Self-selection of goals,* occurs when a person chooses their own goals or sub-goals for a task. These may include exploring the problem, making conjectures, deciding on a strategy, etc. For example, while solving the Interesting Numbers Task, M2 created a generalized algebraic expression for interesting numbers and wondered, "so now the question is what numbers can't be written in that form", which M2 began trying to answer using a parity argument. Similarly, in trying to determine how to prove that the number of selfish sets of $\{1, 2, \ldots, n\}$ is a Fibonacci number, M7 attempted to use a proof by induction on the size of the subsets: "Obviously you've got all the ones before, okay, so the proof would be induction. And, what does this new one add?" She tried to find a recursion but then switched to considering a sum of binomial coefficients. Before stopping the interview, she noted that in order to prove that the sum is a Fibonacci number using induction, she would need to "get a better sense" of the sum. Thus, her self-selected goals included using proof by induction, determining the recursive structure of the problem, and better understanding the sum of binomial coefficients.

The third code, *Immersion, investment, and/or enjoyment,* refers to when a person shows evidence of immersion or investment in the task, or shows that they are enjoying the task. For example, M1 demonstrated investment in the Selfish Sets Task by voluntarily returning to it. He spent 15 minutes on first task, shifted to the Selfish Sets Task for 20 minutes, and then returned to first task. After giving a proof for first task, he noted, "Wait, okay. So now I got a little time to play around with the other one. (Interviewer: Mm-hmm) Might not get it. Sets are not my thing particularly." He expressed an interest in returning to the Selfish Sets Task and attempting to solve it, which indicated investment. Additionally, during the interview, M1 expressed enjoyment with the two tasks, remarking, "These are great problems, by the way." Another indication of immersion or investment is losing track of time, which is one aspect of flow (Csíkszentmihályi, 1990; Liljedahl, 2018). We saw this when M4 was surprised at the time he had spent on the problem: “I've been working on this for 15 minutes, oh my god.” Other mathematicians persisted in solving the problem even when they were given a chance to stop, which indicated enjoyment and investment in the tasks. During M10’s interview, he continued working to solve the problem even when the interviewer provided an opportunity to stop:

Interviewer: Great. And just so you know, so it's been about an hour. So, if you want to stop now, that's fine. But if you want to try...

Mathematician 10: Oh, I'm just getting warmed up.

Interviewer: No please, please do. I just don't, I don't want to take up too much of your time.

Mathematician 10: Okay, no, no problem at all. Okay. Yeah, I should warn you, once I get going, I don't quit.

Thus, M10’s desire to continue working even being provided an opportunity to stop is evidence of investment in the task.

Recall that we found evidence of all three codes for only nine of the 13 mathematicians. For the remaining four mathematicians (M2, M6, M7, and M11), we saw evidence of *Agency in exploration/goal accomplishment* and *Self-selection of goals*, but we did not find evidence of *Immersion, investment, and/or enjoyment.* We found three reasons for the lack of immersion, investment, or enjoyment: (a) problem difficulty, (b) interruptions preventing sustained engagement, and (c) lack of open exploration. M2 was already familiar with the ideas in both problems, and so he was able to achieve solutions to both tasks in about 10 minutes. Thus, the problems were exercises for him rather than problems (Kantowski, 1981). M6 and M7 both experienced multiple interruptions during their interviews, which may have impeded their engagement and the ability to enter a state of flow. Finally, both M6 and M11 had a truncated exploration phase, as they found a viable solution path early on in their exploring. Without the extended exploration, these mathematicians may not have had as many opportunities to become immersed or experience enjoyment. In the next section, we report in more detail on the exploration phase by elaborating the evidence of Explore-Focus cycles in the mathematicians’ problem-solving activity.

Explore/Focus Cycles

We found that Explore-Focus Cycles (Figure 2) occurred in the tasks where mathematicians were playfully engaged. However, the amount of exploring actions versus focusing actions and the amount of time spent on those types of actions varied greatly from mathematician to mathematician. The two mathematicians for whom we found the most evidence of immersion/investment spent a significant amount of time exploring. Additionally, as we noted above, each of the four mathematicians who did not exhibit the immersion/investment code had fewer exploring actions and spent less time in exploration compared to the mathematicians who exhibited immersion, investment, and/or enjoyment. To exemplify the Explore-Focus Cycles, we present one cycle from M1’s engagement with the Interesting Numbers Task (Figure 2).

Within the cycle, mathematicians shift between exploring actions and focusing actions. *Exploring actions* occur when problem-solver(s) either (a) engage in an action with little or no anticipation of an associated outcome, or (b) notice properties of outcomes and wonder about the connection between those outcomes and the actions that led to them. For example, in the two tasks, mathematicians might generate small examples to look for a pattern or notice a general form to help them determine interesting numbers or minimal selfish sets. They may also explore different solution strategies before deciding to try one. A *focusing action* occurs when the action is tied to an anticipated outcome. For example, after generating small examples and noticing a pattern, a mathematician may make a conjecture, and then choose specific examples to test their conjecture. Both conjecturing and choosing examples are focusing actions.

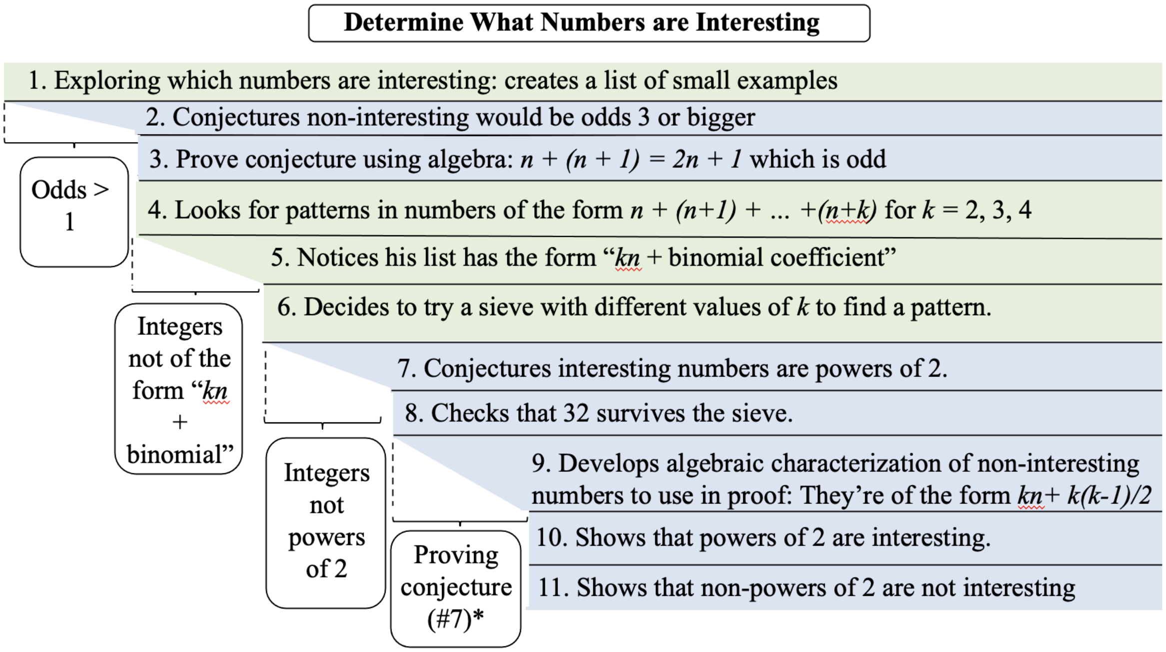

Figure 2. The Explore-Focus Cycle for M1's solution of the Interesting Numbers Task.

In Figure 2, the exploring actions are green and the focusing actions are blue. With our data, the problem space consisted of characterizing the objects specified by the tasks, namely, interesting numbers and minimal selfish sets. Once the mathematicians had a characterization in the form of a conjecture, they shifted to trying to prove their conjecture. The narrowing of the problem space can be seen in Figure 2 as "Odds >1", "integers not of the form kn + binomial coefficient", and "integers not powers of 2". Once M1 had conjectured that interesting numbers are powers of 2 (#7) and checked his conjecture with an example (#8), he shifted to trying to prove his conjecture, which resulted in another narrowing of the problem space. This action is denoted with an asterisk as this narrowing represents a shift from characterizing the interesting numbers to proving his conjecture about interesting numbers.

M1 began the task by exploring the constraints of the problem and generating small numbers (#1). His initial exploration only consisted of sums of two positive integers and led him to conjecture that the non-interesting numbers are odd numbers greater than 1 (#2). During his small example generation, he did not anticipate the form of non-interesting numbers; hence, his engagement was exploratory. He then made a conjecture after generating four examples. At this point, M1 shifted to trying to prove his conjecture, "non-interesting numbers are odd numbers greater than 1." He argued that if we consider two consecutive numbers, x and $x + 1$, then their sum is an odd number of the form $2x+1$ (#4).

At this point, the interviewer redirected M1 to the problem statement, pointing out that non-interesting numbers can be the sum of more than two consecutive integers. This shifted M1 back into a narrower problem space in which he tried to determine which even numbers are interesting (since he had already proved that odd numbers are non-interesting). To do this, he tried small algebraic examples (#4), i.e., he considered numbers of the form $3n + 3$, $4n + 6$, and $5n + 1$, which are the sums of three, four, and five consecutive integers, respectively. He noted that all of these integers are of the form $kn +$ binomial coefficient (#5). While generating these small examples, M1 had not anticipated the general form, nor at this point did he anticipate how it would be useful, thus he was still exploring. Deciding that he wanted to exclude all integers of those forms led him to comment, “this looks a little like the Sieve of Erastothenes … you’re trying to exclude numbers that have certain forms.” M1 then created a list of numbers and systematically eliminated them (#6); in doing so, he realized that the surviving integers are powers of 2 (#7). Although his decision to try a sieve focused his actions, he did not anticipate the outcome of the sieving process, and so we classify it as an exploring action.

At this point, M1 had a conjecture that the interesting numbers were powers of 2. In order to test his conjecture, he continued his sieving process to see if 32 survived the sieve (#8). This was a focusing action because he anticipated that 32 would survive. Having checked that his conjecture held for 32, M1 shifted to trying to prove his conjecture that a number is interesting if and only if it is a power of 2. At first, M1 did not see how to proceed and decided to spend some time with the other task. Once he returned to this task, he reviewed his work and noted that non-interesting numbers can be written in the form $kn + k(k\text{-}1)/2$ (#9). M1 first showed that if a number is a power of 2 then it doesn’t have the form $kn + k(k\text{-}1)/2$ since it would factor as $k/2(2n+k\text{-}1)$ which would be odd (#10). To prove the other direction, he showed that if a number is not a power of 2, then it is not interesting by demonstrating that every odd number can be written in the form $2n+k\text{-}1$ where k is a power of 2 (#11). All of his actions in the proof were focusing actions because he anticipated that they would allow him to reach a logical conclusion and prove his conjecture.

Discussion

Researchers have argued that mathematicians’ disciplinary engagement is playful (e.g., Mason, 2019; Su, 2020). Our study supports these assertions with empirical findings. We found evidence of all three aspects of playful math engagement (agency in exploration or goal accomplishment, self-selection of mathematical goals, and a state of immersion, investment, and/or enjoyment) for nine of our 13 mathematicians and evidence of the first two aspects for all 13 mathematicians. We also found evidence that the mathematicians shifted back and forth between the two types of engagement depicted in theoretical studies (e.g., Holton et al., 2001; Su, 2017), which we call exploring and focusing. Given the constraints present in analyzing task-based interviews, more research is needed to better understand the playful nature of mathematicians’ practice when engaged in their own research activities. Nevertheless, it is encouraging that we were able to verify theoretical claims with our findings.

What is perhaps more interesting is why some of the mathematicians did not demonstrate immersion, investment, or enjoyment. The first two aspects of playful math are ones that are seldom present for students, but we would expect to see in mathematicians. The third aspect, however, appears to be related to not only the ability to maintain engagement without interruption, but also to the level of exploration the mathematicians engaged in. Those whose exploration was truncated may not have found the tasks sufficiently challenging. This finding suggests that an appropriate level of task challenge, one that is within reach but is also not too easy, may be an important factor in fostering flow (e.g., Liljedahl, 2018), and, by extension, playful math engagement.

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Students’ and Mathematicians’ Playful Math Engagement

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Mathematical play is hypothesized to foster student dispositions that mirror authentic disciplinary engagement, such as exploring, conjecturing, and strategizing. However, most research on mathematical play investigates the mathematics that can emerge from children’s natural play or play in informal spaces. We introduce the term “playful math” to highlight the potential of playifying school mathematics tasks and investigate both students’ and mathematicians’ problem-solving activity through this lens. Drawing on teaching experiments and mathematician interviews, we found evidence of playful engagement in both populations and identified similar activity structures across both groups, which we call Explore-Focus Cycles. We present two such cycles and discuss implications of playful math for student activity.

Keywords: Playful math, cognition, mathematician interviews, teaching experiments

Motivation and engagement are critical factors in supporting students’ abilities to understand, perform, and persist in mathematics (e.g., Durksen et al., 2017). Students’ experiences of self-efficacy can foster motivation, which in turn predicts persistence in the STEM fields (Simon et al., 2015). Despite the importance of these factors, however, studies show ongoing challenges with students’ self-efficacy and positive engagement in mathematics (e.g., Martin & Marsh, 2006). Finding ways to help students experience mathematics as enjoyable and worthwhile is an important part of addressing attrition rates (Wilson et al., 2012).

One way to foster enjoyment is to provide opportunities for students to engage in more authentic disciplinary practice, such as exploring, conjecturing, defining, and proving (Jasien & Horn, 2018). In contrast to school mathematics, which can be procedural and routinized, these practices can be simultaneously challenging, creative, and playful. In fact, much of authentic disciplinary engagement involves many of the same features as play (Gresalfi et al., 2018), and professional mathematicians have been described as engaging in mathematical play as part of their practice (Lockhart, 2009; Mason, 2019; Su, 2020). Mathematical play has the potential to center student voices enabling them to create their own challenges, investigate novel ideas, and develop a disposition for innovation (Barab et al., 2010; Gresalfi et al., 2018).

Mathematical play is, therefore, a promising avenue for fostering productive practices (Mason, 2019). However, the bulk of existing research on mathematical play is situated either in early childhood or in informal spaces such as video games. Less is understood about the potential benefits of incorporating play into school mathematics, particularly for older students. Furthermore, the degree to which “playifying” classroom mathematics fosters engagement that is more similar to that of mathematicians is not well understood. Therefore, this study investigates both mathematicians’ problem-solving activity and middle-school students’ activity in playful math tasks to ask, how does students’ playful math engagement compare to mathematicians’ problem-solving engagement? We use the term “playful math”, rather than “mathematical play”, to highlight the opportunities of “playifiying” classroom mathematics tasks.

Background Literature: The Benefits of Playful Math

Playful math offers both conceptual and affective benefits. Playifying mathematics can favor social interaction and communication (Edo & Deulofeu, 2006) and has been shown to foster increased enjoyment (Plass et al., 2013; Sengupta-Irving & Enyedy, 2015), engagement (Barab et al., 2010), and to engender positive attitudes towards mathematics (Holton et al., 2001). Su (2017) noted that it can also foster connection and community: "It's why the MAA supports programs like the American Mathematics Competitions and the Putnam Competition. We help kids flourish through building hopefulness, perseverance, and community" (p. 486).

Researchers have also identified a number of learning benefits of playful math for children, including increased mathematical fluency (Plass et al., 2013), improved knowledge in geometry (Levine et al., 2012; Sedig, 2008), improved spatial skills (Casey et al., 2008; Levine et al., 2012), and improved number development (Siegler & Ramani, 2008; Wang & Hung, 2010). The degree to which these learning benefits extends to older students is unclear, but researchers have identified a number of ways in which playful math can foster forms of engagement that is thought to more closely mirror mathematicians' disciplinary practice. This includes activities such as abstracting and generalizing (Featherstone, 2000; Melhuish et al., 2021), exploring patterns within axiomatic structures (Fernández-León et al., 2021; Su, 2017), conjecturing (Jasien & Horn, 2018), and "extending, varying, simplifying, and complexifying" (Mason, 2019, p. 95), among others. However, the manner in which this actually occurs, particularly with classroom tasks, is not well understood. Therefore, we sought to better understand both the playful nature of mathematicians' problem-solving activity, as well as the activity of students when engaging in playified tasks.

Theoretical Framework: Defining Playful Math and Principles for Task Design

What makes mathematical engagement playful? Definitions of play vary, but all emphasize the learner's freedom and autonomy (Holton et al., 2001; Su, 2017). Jasien and Horn (2018) described mathematical play as entailing agency in exploration, self-selection of goals, and self-direction in how to accomplish them. Davis (1996) described play as an acceptance of uncertainty and a willingness to move, which is similar to Mason's (2019) depiction of play as trying out actions and exploring consequences. Play is not aimless, but rather, it is goal driven and occurs within a structure (Featherstone, 2000; Huizinga, 1955; Su, 2017). Following these researchers, we define playful math as a particular form of engagement in mathematics, one that entails (a) agency in exploration and goal accomplishment, (b) self-selection of mathematical goals, and (c) a state of immersion, investment, and/or enjoyment.

One common feature of playified tasks is that they are typically goal-free (Kapur, 2015), open-ended problems (Cai & Cifarelli, 2005). In goal-free problems, learners only have to attend to the present state and consider how to make any possible move from that state. This is in contrast to goal-specific problems, in which learners must consider a problem's givens, the pre-determined goal, and how to connect the givens to the goal. Open-ended problems are similar, in that one could construct many different problems and solutions within the given situation. In constructing goal-free and open-ended problems, we followed four design principles for playifying our tasks (Plaxco et al., 2021): (1) allow for free exploration within constraints; (2) allow the student to act as both designer and player; (3) engender anticipating within the task; and (4) provide a method for authentic feedback.

Methods

Study 1: Middle-School Teaching Experiments

We conducted two teaching experiments (TEs) over the course of 5 weeks during the summer of 2020 (Steffe & Thompson, 2000). Sessions for the TEs occurred weekly for approximately one hour; the third author was the teacher-researcher. We video and audio recorded both TEs, with the exception of the second session, when a technical error prevented video collection. We also collected student work at the end of each session. The first TE was with two rising 7th grade students, Stewie and GJ, and the second TE was with three participants, Artemis, a rising 7th grade student, and Apollo and Frances, rising 6th grade students (all students chose their own pseudonyms). The students had limited graphing experience and had received no prior instruction on linear functions.

To investigate students' activity when exploring rates of change within playified tasks, we drew on a set of established research-based linear and quadratic growth tasks (Ellis et al., 2020; Matthews & Ellis, 2018). In these activities, students investigate shapes via dynamic geometry software, determining the rate of change of a shape's area compared to its changing length as it sweeps out from left to right, and then graphing that relationship. In following the above-described four design principles to playify the tasks, we developed a series of activities called *Guess My Shape*, in which the students created secret shapes of their choice (design principles #1, #2), constructed graphs comparing length and area (design principles #2, #3), and then challenged one another or the teacher-researcher to determine the secret shape based on the graph alone (design principles #2, #3, #4). Because playful math describes a form of engagement rather than the task itself, we hypothesized that the playified tasks would encourage playful engagement, but we did not assume that they would. For Session 1, we only used the extant covariation tasks, which we call standard tasks. For Sessions 2 - 5, we used both the standard tasks and the playified math tasks. For each of those sessions, we spent the first two-thirds of the session working with the standard tasks and the last one-third implementing the playful math tasks

Study 2: Mathematician Task-based Interviews

Additionally, we analyzed task-based interviews with 13 mathematicians (see Lockwood et al., 2016 for details). We gave each mathematician one or two tasks and asked them to work on the task(s) while sharing their thinking out loud. The interviewer only interrupted to ask clarifying questions or to address participant questions. We audio recorded the interviews with Livescribe pens, which also captured the mathematicians' written work as they spoke. All mathematicians worked on the *Interesting Numbers* task, and 7 of the mathematicians also worked on the *Selfish Sets* task. The Interesting Numbers tasks asked, "Most positive integers can be expressed as a sum of two or more consecutive integers. For example, 24 = 7 + 8 + 9 and 51 = 25 + 26. A positive integer that cannot be expressed as a sum of two or more consecutive positive integers is therefore *interesting*. What are all the interesting numbers?" The Selfish Sets task asked, "Define a *selfish* set to be a set which has its own cardinality as an element. Find, with proof, the number of subsets of {1, 2, . . . , n} which are minimal selfish sets, that is, selfish sets none of whose proper subsets is selfish."

Analysis

We analyzed the TE data for aspects of playful mathematics using a combination of a priori and emergent codes (Strauss & Corbin, 1990). We drew the a priori codes from our three-part definition of playful math, and we developed emergent codes to address aspects of students' playful engagement that were not captured by the initial set of codes. All three authors coded the transcripts independently and met weekly to refine codes and to adjust and resolve discrepancies. This process continued through ten rounds of adjustments until all codes had stabilized. This analysis of the TEs also produced a cyclic model of exploring and focusing actions (Ellis et al., 2022), which we then used as an analytic tool for the mathematician data. We discuss this Explore-Focus Cycle in more detail in following sections.

We then turned to the mathematician data and analyzed the interviews for playful engagement using the codes we developed from the TE analysis. We also developed descriptive notes and diagrams that described each mathematician's actions in terms of the Explore-Focus Cycles. The first and second author coded the interviews separately and met weekly to resolve discrepancies until the coding scheme had stabilized.

Results

Explore-Focus Cycles

In the Explore-Focus Cycle, one shifts back and forth between exploring actions and focusing actions in order to make sense of a problem and determine a solution path. *Exploring actions* occur when the problem solver either (a) engages in an action with little or no anticipation of an associated outcome or (b) notices properties of outcomes and wonders about the connection between those outcomes and the actions that led to them. As such, exploring actions are empirical; they entail gathering information without a pre-determined aim, in order to develop an idea, direction, or conjecture. In contrast, *focusing actions* are ones that a problem solver makes with a specific anticipated outcome in mind.

For example, in the Interesting Numbers task, many of the mathematicians engaged in exploration by generating examples of interesting and non-interesting numbers in order to see if any useful patterns would emerge. Others explored by generating the algebraic formula for the non-interesting numbers and then seeing if they could identify a useful structure in the algebra. These actions typically resulted in the development of a conjecture that the interesting numbers are the powers of 2. At this point, the mathematicians shifted to focusing actions, pursuing strategies with anticipated outcomes such as an algebraic proof that powers of 2 are interesting.

In examining the students' activity, we also found evidence of Explore-Focus Cycles but only in the playified tasks. In the standard tasks, the students engaged in focusing actions but not exploring actions. In order to exemplify the Explore-Focus Cycles for both the students and the mathematicians, we present a truncated version of one cycle for a trio of students, Apollo, Frances, and Artemis, and for one mathematician, M10. The students' cycle occurred during a Create a Shape playified task, and M10's cycle occurred during the Interesting Numbers task (Figure 1). In the figure, the exploring actions are in the green cells and the focusing actions are in the blue cells.

Let us first consider the mathematician's engagement. M10 began with an exploration of possible approaches in order to determine which approach might be most productive (A1). She did not attempt any of the approaches but just listed and considered which ones might be feasible, and so there was not yet an anticipated outcome. She then focused on one strategy that allowed her to generate an algebraic formula for non-interesting numbers (A2). M10 then switched away from the algebraic form and instead generated and explored examples of non-interesting numbers to "see if it makes sense" (A3). After generating examples, she did not notice any patterns, and so she returned to the algebraic formula (A4). Here, she explored the formula to determine helpful structures that might indicate restriction on non-interesting numbers. From here, she switched again to the generated examples to look for patterns and structures and determines empirically that all odd numbers will not be interesting (A5).

For the *Create a Shape* challenge, the students began by exploring possible shapes to use in making a challenge for the teacher-researcher (TR) (B1). Here, they generated options without considering the associated length/area graph. However, once the options were suggested, they considered the reasonableness of the associated graph. They anticipated that some shapes were beyond their ability to graph and returned to exploring other possible shapes (B2). Apollo suggested a spiral and then a C-shape (B3, B4). Then, they applied a grid structure and a "connect the dots" structure to the C (B5). They were not anticipating the outcome of the graph or measurements but were looking for structures that might help them to do so. Artemis then considered the connect the dots method and suggested the need to number them (B6). Artemis was now anticipating how to use this structure for measurement and thus was engaged in a focusing action. Not included in the diagram below, the girls continued exploring and narrowing in on their chosen shape and ended with creating a "wave" shape and corresponding graph that introduced shapes shifted off the horizontal axis (Figure 2).

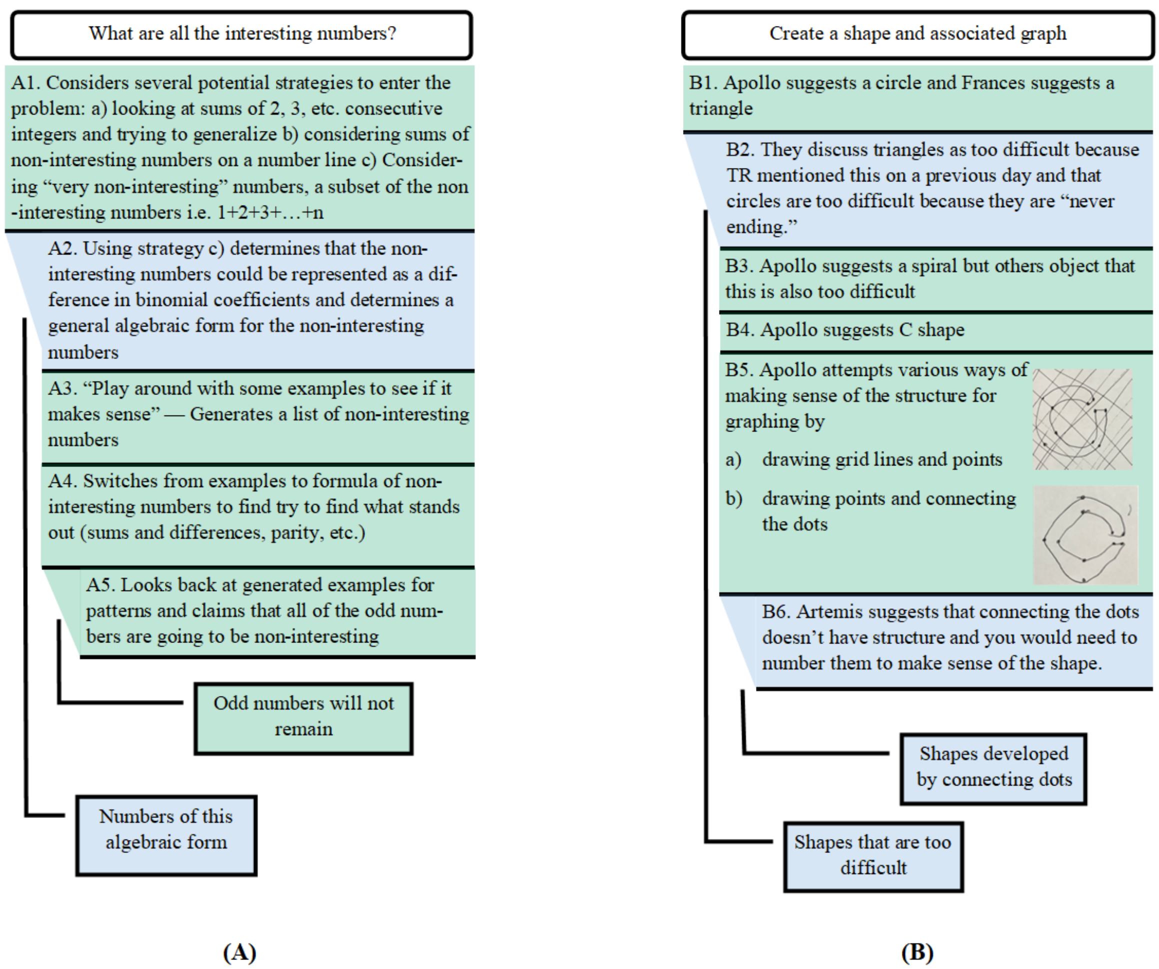

Figure 1. (A)Mathematician Explore-Focus Cycle and (B) Middle-School Explore-Focus Cycle

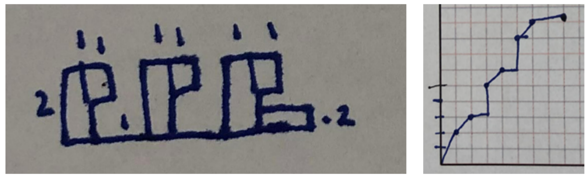

Figure 2. Artemis, Apollo, and Frances's final shape and graph for the Create a Shape Challenge on Day 3

For both the mathematicians and the students, we saw similarity across the presence and the function of exploring actions. Exploring actions can serve to help determine intial goals to begin narrowing in on the problem. M10 began with an exploration of possible approaches in order to determine which approach might be most productive to begin the problem, and then, she focused on one strategy that allowed her to generate information about non-interesting numbers. The exploration allowed her to select an initial strategy. Similarly, the students began by exploring possible shapes to use in their challenge. The exploration of shapes allowed the students to choose a goal for graphing and also helped them focus on the constraints of particular shapes. We saw this when Frances mentioned that triangles were too difficult, and when Artemis suggested, "Circles are never ending and that's impossible to graph a circle." Additionally, we saw that sometimes exploration serves as a way to search for structures. M10 explored structures when looking back at the algebraic formula of non-interesting numbers and when looking across her generated list of non-interesting numbers. The students did this when trying to work with a shape but were unsure of how to graph it. Wanting to graph the C-shape, Apollo tried overlaying a grid and "connecting the dots" between predetermined points. Here, she was trying to make sense of the structure of the C in a way that may be productive for graphing, as using points or grids may be natural ways of creating measureable areas and lengths.

Although similarities exist across the cycles, the cycles only occurred for the students during the playified tasks; whereas, they modeled the mathematicians' engagement across all of the tasks and interviews. One difference in the cycles is the novel, creative, and difficult mathematics that the students created for themselves. In this example, Apollo, Artemis, and Frances had only experienced linear and piecewise linear changes with shapes that were rectangles. However, to make the problem sufficiently challenging for the TR, they worked hard to create something different than what they had seen before, suggesting triangles and curved shapes such as circles, spirals, and the C-shape. Even though they moved on from using curved shapes, the desire to construct something novel and challenging persisted throughout the sessions. The mathematicians did not demonstrate creativity or make the mathematics more challenging, likely because the nature of the tasks did not call for designing or exploring their own questions.

Playful Engagement Codes

Recall that our three-part definition of playful math is (a) agency in exploration and goal accomplishment, (b) self-selection of mathematical goals, and (c) a state of immersion, investment, and/or enjoyment. Three additional codes emerged during our analysis: (d) taking on authority, (e) creative/unusual, and (f) harder math. *Taking on authority* refers to either a shift in authority from the TR to the students, or to an instance of students taking on mathematical authority with one another or with the TR. *Creative/unusual* applies when students have developed an idea, representation, connection, or task that we perceived as creative, novel, or atypical, and *harder math* occurred when students created a goal or task that introduced new challenges for them and represented mathematical ideas that surpassed the TR's intended set of topics or concepts.

Across the 10 sessions of the two TEs, we implemented 40 standard tasks and 9 playified tasks. Students completed one to two playified tasks at the end of each session, beginning with the second session. We coded any evidence of the six codes across both the playified and standard tasks. We then examined the rate of occurrence of each of the codes with respect to task type (Figure 3). For instance, self-selection of goals occurred on average 0.075 times per standard task and 3.11 times per playified task. We found that all six codes occurred with greater frequency on average within the playified tasks compared to the standard tasks. Furthermore, two of our codes, *creative/unusual* and *harder math*, occurred only in the playified tasks.

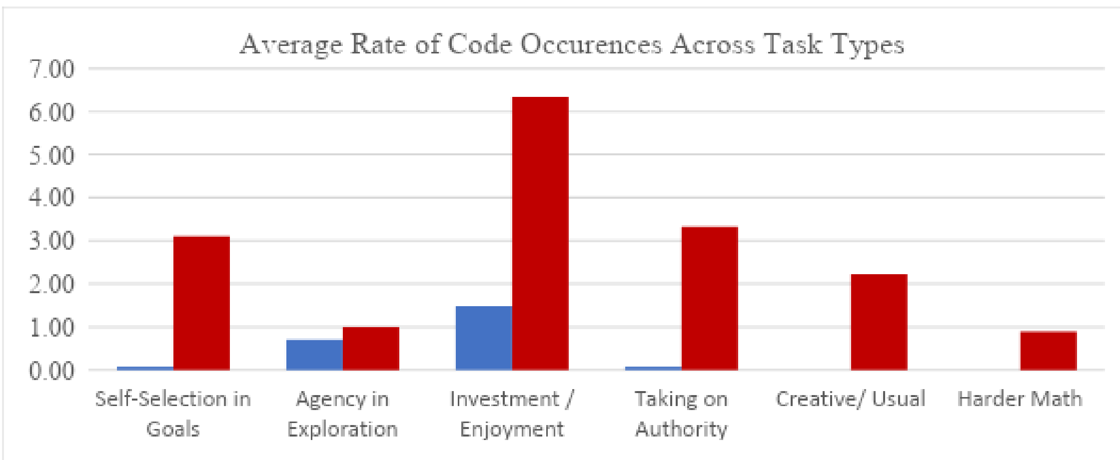

Figure 3. The Average Number of Occurrences of each code per Standard and Playified Task

We also investigated the presence of the playful math codes (a – c) in the mathematician interview data. We did not include the three emergent codes (d – f), as the nature of the interview tasks, unlike more authentic research activity, did not necessitate creative, unusual, or harder math (and the mathematicians already held mathematical authority). We found evidence of (a) *self-selection of goals* and (b) *agency in exploration/goal accomplishment* for all 13 mathematicians, and evidence of (c) *immersion, investment, and/or enjoyment* for nine of the 13 mathematicians. The presence of these codes and the Explore-Focus Cycles in both data sets suggests that playful math can be one way to support the development of productive practices.

Discussion

Playful math is a dispositional stance towards an activity, rather than a description of the activity itself. Thus, we did not know whether our design principles for playifying tasks would necessarily support more playful engagement. We found that they did, and they also supported students' development of novel, creative, and challenging ideas and strategies. It is possible that other open tasks may do the same thing; more research is needed to understand the unique affordances of a play-based approach. Nevertheless, this study offers a proof of concept that playful math can be one fruitful avenue for necessitating (Harel, 2008) more spohisticated concepts and connections.

We also found that mathematicians' problem-solving activity is not only playful, it also entails open exploration, with mathematicians shifting between phases of exploring and focusing. Notably, the students demonstrated similar cycles, but only in the playified tasks.

These findings provide evidence to support theoretical arguments that incorporating playful math into classroom activities can foster forms of engagement similar to mathematicians' activity (e.g., Mason, 2019).

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Students Leverage Struggles with Proof to Script Fictional Dialogues about the Rules of Proving

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Research has noted that some aspects of proof remain implicit in undergraduate instruction. Accordingly, we propose that direct interaction with the rules of proving (e.g., deductive reasoning, proof-writing conventions) is of didactical value to students transitioning to proof-based mathematics. Using the commognitive framework, we aim to delineate the rules that newcomers to proof find challenging. These were solicited with a scriptwriting task that asked students to create a fictional dialogue about a proof-related mistake that they had experienced. The task was embedded in a homework assignment in an undergraduate course for academically motivated high-schoolers. Thematic analysis of 61 scripts resulted in a range of rules that concerned the use of examples, logical circularity, and proof layout. We discuss these findings in relation to students' instruction and the existing literature on proof.

Keywords: Commognitive framework, proof, rules of proving, scriptwriting, transition to proof

Introduction

Dreyfus (1999), Selden and Selden (2015), and other researchers have argued that many aspects of proof remain implicit in undergraduate teaching and learning. Consistently, the lion's share of proof research can be reframed as students grappling with new mathematical rules that are not as explicit as they could be (e.g., Balacheff, 2008; Harel & Sowder, 2007; Moore, 1994; Selden & Selden, 1987; Weber, 2002). Overall, the shift to proof-based mathematics requires a fundamental change in the rules of argumentation and justification. Kjeldsen and Blomhøj (2011) maintain that developing appropriate rules for activities of this sort is indispensable for mathematics learning. In the context of proof, such rules can be associated with deductive reasoning, propositional calculus, logic, and generally accepted proof-writing conventions. Accordingly, we propose that a direct engagement with the *rules of proving* (*RoP* hereafter) is of didactical value to newcomers to proof-based mathematics. This study comes from a larger project that explores how newcomers articulate, apply, and make sense of RoP while making their first proving steps.

Background

A "solid finding" of mathematics education research is that "many students rely on validation by means of one or several examples to support general statements, [and] this phenomenon is persistent in the sense that many students continue to do so even after explicit instruction about the nature of mathematical proof" (ECEMS, 2011, p. 50). Another occasionally reported issue pertains to logical circularity. In abstract algebra, Selden and Selden (1987) found that some students use one version of the conclusion of the assigned statement to prove an equivalent version of the same conclusion. Additional findings of a similar type have been reported in the literature (e.g., Weber, 2002).

Emerging from cognitive studies, these findings are often framed in terms of students' errors and misconceptions since they have to do with proof validity. We adhere to the social perspective on proof (Stylianides et al., 2017). Within this perspective, the findings can be viewed as students' deviations from the rules that are broadly accepted in professional mathematics communities. And while it is not rare for mathematicians to disagree on proof-related issues (e.g., Lew & Mejía-Ramos, 2020), abiding by such rules as modus ponens and propositional calculus is likely to be expected in all communities that practise proof.

Generating proofs that abide by these rules requires additional rules that are less universal. For instance, Stylianides (2007) argues that in a classroom, a proof uses accepted statements and employs forms of reasoning and expression that are known to, or are within, the conceptual reach of the students. Accordingly, proofs can vary depending on what each classroom renders "accepted," "known," and "within students' reach." Furthermore, guidelines are needed to compile mathematical statements, forms of reasoning and expressions into a proof that the classroom community will endorse. In terms of Yackel and Cobb (1996), RoP of this kind are not very different from sociomathematical norms that determine "what counts as an acceptable mathematical explanation and justification" (p. 461). While using different terms, Dreyfus (1999) discusses instances where students grapple with such community-specific rules in the context of Calculus and Linear Algebra. The instances contain explanations and proofs that Dreyfus describes as insufficient: they present computations without accompanying explanations, draw on theorems without spelling them out, and, overall, do not "go back" enough and do not "go deep" enough.

The findings overviewed in this section come from research where students did not follow RoP that the researchers viewed as conventional. This study endorses an anti-deficit approach that foregrounds productive resources that all students bring to mathematics learning. The approach acknowledges that "learning takes time and imperfect articulations of mathematical ideas and some inconsistencies in the student's current conception are a natural part of the process of learning" (Adiredja, 2019, pp. 416–417). Specifically, we interpret students' deviations from conventional rules as a marker of their ongoing transition to proof-based mathematics. We also see value in encouraging students to notice and reflect on such deviations as a stepping-stone towards the conventional RoP (e.g., CUPS, 2004). This study aims *to delineate RoP that newcomers to proof-based mathematics find challenging*.

Commognitive Framework

We follow in the footsteps of previous studies that mobilized the conceptual and analytical potential of commognition to study university students' engagement with proof (e.g., Brown, 2018 Kontorovich, 2021, Kontorovich & Greenwood, 2022; Kontorovich et al., 2023). Commognition construes mathematics as a *discourse*. Discourses are defined as, "different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard 2008, p. 93). Then, *learning* is a process through which one's communication starts abiding by the rules of the target discourse.

Proof and proving concern *endorsable narratives*—narratives that can be rendered as true "according to well-defined rules of the given mathematical discourse" (Sfard, 2008, p. 224). Sfard (2008) acknowledges that "being dependent on what participants find convincing, routines of substantiation are probably the least uniform aspect of mathematical discourses. The very term endorsement may be interpreted differently by different people" (pp. 231–232). Indeed, the mathematics community has been revising these interpretations of what constitutes endorsement throughout history (e.g., Kleiner, 1991). Notwithstanding, Sfard (2008) proposes that "for today's mathematicians, the only admissible type of substantiation consists in manipulation on narratives, and it is thus purely intradiscursive" (p. 232). This manipulation unfolds as a sequence of utterances (or proof components), each either an "accepted fact," or derived according to some generally accepted rules (e.g., modus ponens). In this way, we associate RoP with intradiscursive principles for endorsing true mathematical statements and rejecting false ones. Colloquially speaking, these rules pertain to “when to do what and how to do it” (Bauersfeld, 1993, p. 4).

Commognition distinguishes between *object-level rules* and *meta-discursive rules* (or *metarules* for short). The former rules attest to the properties of mathematical objects and take the form of narratives on these objects (e.g., “the product of rational numbers is rational”). Metarules focus on how discourse participants formulate and substantiate object-level rules. Thus, RoP are meta-discursive by definition. Metarules are often enacted tacitly. Once a rule has been captured in words (i.e. when a *rule-narrative* is generated), commognition refers to it as *endorsed* by the interlocutor (Sfard, 2008). In this study, we scrutinize RoP that newcomers to the discourse of proof-based mathematics endorse.

The Study

Our participants were students in a special program entitled “Mathematics extension and acceleration” in a large New Zealand university. The program is intended for mathematically motivated and academically inclined students (usually 17-year-olds) in their final year of high school. As part of this program, the students take a special semester-long mathematics course that gives them academic credit for a bachelor’s degree in mathematics or engineering.

Proofs play a marginal role in New Zealand school mathematics (Knox & Kontorovich, 2022). The first part of the course ushers students into proving, while leveraging their familiarity with different kinds of numbers. In the first two lectures of the course that we studied, the instructor introduced proof as “an argument that mathematicians use to show that something is true.” Then he presented three true mathematical statements and led a whole-class discussion about what he positioned as “broken proofs.” In the following three lectures, he introduced selected properties of real numbers (e.g., distributivity) and illustrated how these could be used to derive additional properties (e.g., $-a = -1 \cdot a$); the illustrations were discussed as instances of “good proofs.” In these lessons, the instructor shared many proof-related insights and guidelines, including: examples do not prove universal statements, avoid circularity, start with true things, delineate the statement to be proved before proving it, the scope of the introduced variables should be defined, use words to explain the proof (not just formulas), avoid proofs that prove without explaining. In the last section, we draw on these insights and guidelines to reflect on RoP that students endorsed.

A *scriptwriting task* was designed to pursue the aim of this study. In tasks of this sort, students are provided with a thought-provoking prompt and asked to offer a resolution by scripting a fictional dialogue between made-up characters (Zazkis et al., 2013). Research has used scriptwriting to investigate students’ proving (e.g., Gholamazad, 2007). Brown (2018) argued that by making students’ envisioned interactions about a proof public, scriptwriting affords students a way to make their thinking visible and fosters reflection. In turn, researchers get access to proof-related aspects that students render important.

The course instructor and the first author collaboratively developed a scriptwriting task to provide students with opportunities to engage with RoP and collect data (Kontorovich & Bartlett, 2021). Figure 1 presents the task eventually developed. It was embedded in an individual homework assignment together with proof-requiring problems that are typical to transition-to-proof courses.

The second author conducted an initial analysis of students’ submissions (Liu, 2018). The analysis started with an overview of the 74 scripts that were collected to develop a general impression of whether they pertain to RoP. This process converged into 61 submissions, which underwent a thematic analysis (e.g., Braun & Clarke, 2006). The initial themes were iteratively separated and merged to develop general categories and descriptions of rule-narratives that the students generated through the voices of characters that represented them in their scripts (*Student-characters*, hereafter). This process combined deductive and inductive techniques as the initial themes were informed by the literature as well as by the content of students' scripts. When characterizing students' rule-narratives, we distinguished between the *restricting* and the *guiding* version of a rule: the former posited that something should not be done, while the latter offered guidance for how proving should be pursued.

> Writing proofs can be quite difficult, especially when you're new to them! One of the best ways to get better at writing proofs is to **look back** at past mistakes and think about how to fix them. This exercise is meant to help you do this, as follows:
> (a) Come up with a mistake you've made (possibly while working on this assignment!) or seen others make when trying to prove something.
> (b) Write a dialogue between you and an (imaginary) friend, where they make this mistake and you explain what's wrong with that idea and how to fix it: i.e. something like
>
> *Paddy's imaginary friend*: So, I was trying that checkerboard problem above[4], and I figured that if I wrote that I tried three different ways to do it and none of them worked that would totally count as a proof.
>
> *Paddy*: Well, actually it probably doesn't. You see . . .

Figure 1. A scriptwriting task[1]

Findings

This section presents six categories of the most common rules that emerged from analyzing students' scripts. We use rule-like formulations for the first five kinds due to the high consistency of the rule-narratives that the students' generated. Next, we elaborate on each kind and illustrate it with representative examples from students' scripts.

Examples are insufficient to prove a universal statement

This type of rule featured in 16 scripts. There, the students' rule-narratives stressed the insufficiency of drawing on specific examples to prove a universal statement that pertains to infinitely many cases. For example, one script revolved around the statement that the product of two rational numbers is rational. To prove it, "an imaginary friend" multiplied the rational $\frac{3}{4}$ and $\frac{4}{2}$ to obtain $\frac{12}{4}$, which he identified as a rational number. The Student-character did not disagree with the rationality of the product, but he did not accept that this substantiation can be endorsed as a proof. Specifically, the Student-character generated a rule-narrative, stating that "Just because you gave an example where the claim is true, doesn't mean it's always true". In other scripts, the Student-characters explicated the insufficiency of supporting examples by stressing that "a proof cannot be made of citing examples only". Some rule-narratives highlighted that even a large number of supporting examples would not make a proof (e.g., "It doesn't matter how many examples you give […] that would still not be satisfactory enough to say that the claim is true"). Guiding rule-narratives featured in 5 scripts and suggested considering "the full range" and "all possible values" that are relevant for the focal statement.

[1] The task refers to a famous tiling chessboard problem that was part of students' homework assignment.

Failures to produce an example are insufficient to prove a non-existence statement

Six scripts highlighted the insufficiency of failing attempts to exemplify a non-existence statement to be considered as a proof. Consider an excerpt:

Friend- character: When I was attempting the chessboard problem, I tried to manually place rectangles onto the box and fit them in. Since I couldn't solve it, it proves the problem is unsolvable.

Student-character: When you tried to place rectangles on the board, it only proves the specific arrangement doesn't work. But there are many different arrangements which can be attempted, and the result of one cannot be assumed to be the same as all the others.

In this exchange, we see the Student-character generating a rule-narrative, positing that the friend's unsuccess in covering the board does not account for all possible arrangements. Then, her attempts cannot be viewed as a valid proof that the board cannot be covered. In the first part of her submission, the same student wrote this explicitly, "[the specific arrangements] showed that they could not fit, but it did not prove the problem is unsolvable". Another Student-character argued that "no matter how many examples you provide, it's possible for a solution to exist". In this way, other Student-characters also stressed that specific attempts to generate examples for a non-existence statement does not extrapolate to other possibilities. Contradiction-based guiding rules-narratives appeared in three scripts in this category.

Deriving a true claim from a statement does not prove it

This type of rule emerged from seven scripts where "imaginary friends" manipulated the assigned statement until the obtainment of a true claim. Then, it was interpreted as a warrant that the initial statement has been proved. For instance, one "imaginary friend" drew on her deriving that "$0 = 0$" to argue that some initial (and never explicated statement) was true. But the Student-character insisted that "Proofs don't work like that […and] you definitely cannot say $0 = 0$ so it must be true". In what appears as an attempt to replicate this approach, the Student-character wrote "1=2" and multiplied both sides by zero to end up with "$0 = 0$". This move illustrated that a deviation from the focal rule opened the door to the endorsement of false statements. Similar rule-narratives featured in other scripts, arguing that "Assuming the truth of the claim and manipulating it to reach a true conclusion does not prove that it is true." Instead, four scripts contained guiding rules, suggesting one should "start with what you know is truth."

A proof must contain no invalid moves nor false claims

This type of rule grew from 12 scripts where "imaginary friends" conducted a mathematically invalid move to prove a statement. These moves included multiplying both parts of the equation by zero, manipulating with infinity and 0^{-1} like these were real numbers, and building on false assumptions that steered proving towards a problematic course.

To diversify the scope of illustrations, we refer to a script where this rule was enacted silently without being endorsed. There, the 'Imaginary friend' proposed to prove the irrationality of $2^{\frac{1}{\sqrt{2}}}$ by first showing that the exponent is irrational, and then proving that a rational number raised to an irrational power is irrational. The Student-character rejected "that method" by counter-exemplifying the second step. In this way, the scriptwriter implicitly suggested that building on a false assumption disqualifies the proof. Other students were more explicit regarding this rule. In one script, the Student-character posited that "Unfortunately, we can't divide by 0, which kills the whole argument". In another submission, a student shared "I made a false assumption that made the proof false”, which tacitly foregrounds the focal rule. A guiding rule-narrative featured in a single script, and it suggested “always start with something that you know is true!”

A proof should not include circularity

This type of rules emerged from six scripts where an “imaginary friend” drew on the assigned mathematical statement or some variation of it within the proof itself. Then, the Student-character called out this circularity, arguing that it disqualified the proof. For instance, in one script, a “friend” attempted to prove that the sum of irrational numbers is irrational. As part of it, they defined the irrational x and y, and sequentially relied on the irrationality of $x + y + (-x)$ “as claimed” in the original statement. In turn, the Student-character responded with a rule-narrative stating that “you cannot make assumptions based on the claim [since you] are proving the claim to be true”. Additional examples of rule-narratives that the students generated included “using what is required to be shown/proven in the proof (!), which is clearly invalid” and “Well you are trying to prove that the product of two rational numbers is also rational. So how can you say that $a \cdot m$ is rational, since this is the fact you are trying to prove [a and m were previously assumed to be rational]?” Only in one script, a Student-character proposed a guiding rule, calling their fictional peer to “stick to things that you 100% know are true: like our axioms”.

The rules of proof layout

This category emerged from 18 scripts. The rule-narratives that the students generated in these scripts were diverse, and we sub-divided them into four types: (i) the statement to be proved should be explicated, (ii) the numeric sets of the introduced variables should be specified, (iii) the proof moves need to be explained, and (iv) sub-narratives constituting a proof should be proved. Table 1 illustrates each rule with excepts from students’ scripts. Overall, these rules regard the layout of different proof components. Note that all these rules are guiding.

Table 1. Rules of proof layout

Sub-categories of the rules	Excerpts from students’ scripts
(i) The statement to be proved should be explicated	“You must have a claim at the top of your proof so that other readers and markers can understand what you are trying to do. Plus your conclusion may not make much sense without any claim to measure it against”
(ii) The numeric sets of the introduced variables should be specified	“I forgot that when saying something is rational, you say it $= \frac{m}{n}$ where $m, n \in \mathbb{Z}$ you say $n \neq 0$”
(iii) The proof steps need to be explained	“You need to write down your logic and reason step-by-step to backup your proof.”
(iv) Sub-components of a proof should be proved	“We cannot just assume that $\sqrt{8}$ is irrational. You must prove it first, otherwise the rest of the proof is invalid.”

We notice the categorical tone of the rule-narratives in this category, implying that the rules are unambiguous and can be enacted straightforwardly. This appears to be the case for (i) and (ii). Regarding (iii), in some scripts, Student-characters insisted on introducing “math terms” and “explanations” to symbolic substantiations. This afforded students to maintain that a proof that “doesn’t use words” is problematic, without discussing how “wordy” a proof should be.

The issue of ambiguity is also relevant to rule (iv): what mathematical claims can be used without any substantiation, and what claims need to be proved separately? Indeed, one script revolved around the rejection of “the sum of rational numbers is rational.” There, the Student-character criticized a counter-example where $1-\sqrt{2}$ was treated as an irrational number without explicit proof. Then, the same character suggested using $-\sqrt{2}$ instead, arguing that "it's still an irrational number and $\sqrt{2}+-\sqrt{2}=0$". What makes this script interesting is that in the course, neither the irrationality of $-\sqrt{2}$ nor the rationality of 0 were proven, but the student still endorsed these numbers as such.

Summary and Discussion

Building on the literature and the commognitive framework, we argued that direct engagement with RoP is of didactical value to students in transition to proof-based mathematics. To trigger this engagement, we asked students to script a fictional dialogue about a proof-related mistake that they experienced. We used the students' scripts to explore the metarules that newcomers to proof can find challenging.

The analysis resulted in six kinds of rules, none of which are foreign to the mathematics education literature. The innovation is in their epistemological status: while in most studies, the rules emerged from students' infelicitous proof attempts (e.g., Selden & Selden, 1987; Weber, 2002), in our case, the students were the ones to position these rules as something that a mathematical proof must avoid or abide by. Many students went beyond the rule formulation to illustrate, elaborate, and substantiate these rules. Accordingly, a key contribution of our study is in showing that newcomers can not only struggle with proof, but also communicate about it on a meta-level after a relatively short acquaintance period. That said, we acknowledge that our findings come from a special cohort that is hardly representative of a broader student population. Thus, we call for further anti-deficit research into how students learn RoP.

Sceptics may suggest that we should not "read too much" into students' endorsed rules as they could "simply" be drawing on what the course instructor demonstrated. Indeed, the overlap between the instructor's and students' rules cannot be ignored. However, even if the sceptics are correct and this is how students' rules came about, we argue that it is legitimate and even expected. Indeed, Sfard (2008) maintains that transitioning to a new discourse is impossible without a ritual phase, where learners imitate the discourse oldtimers. To do so, the students still needed to craft relevant proof attempts, generate their versions of rule-narratives, and apply them in a different context. Currently, we are developing an analytical framework to distinguish between cases of rigid imitation and agentive application of RoP. Overall, the identified overlap draws attention to the impact of proof instruction on students' communication about proof.

Each RoP can be captured in a guiding or in a restricting form. However, most of our participants adhered to the latter rather than the former. This result may be a side-effect of the assigned task: asking about a mistake may have nudged students to emphasize what is not supposed to happen in a proof. Alternatively, many guiding rules are based on deductive reasoning, communication about which requires a special vocabulary that the students may not yet possess. Accordingly, it could be interesting to explore whether students' capabilities to generate guiding and restricting rules develop in parallel.

We must be cautious when offering implications based on this exploratory study. Yet, we hope that its findings will encourage instructors to consider locating RoP at the center of their proof teaching. Furthermore, scriptwriting tasks may offer valuable information about students' emerging RoP: a carefully designed task can mirror the rules that students find challenging or ambigous, and can also point at the rules that currently escape students' attention.

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Teaching Assistants' Use of Students' Work When Planning How to Address Misconceptions Inferred from It

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Given the crucial role that the graduate teaching assistants (TAs) play in undergraduate mathematics education, this study investigated how TAs interpret students' work and plan to address students' work in teaching. We analyzed TAs' interpretations and plans according to approaches towards addressing students' misconceptions discussed in the literature: viewing them as simply incorrect regardless of students' thinking behind them, viewing them as a reflection of students' flawed ideas to be confronted and replaced, and viewing them as resources for future learning. Our analysis shows that while TAs' interpretations were often sufficiently rich to support using student misconceptions as resources for future learning, none of their plans used them in this way. Rather, most TAs' plans were aligned with the confront-and-replace approach. Our results suggest that professional development could be designed to help TAs convert their rich interpretations into plans for using student misconceptions as resources for future learning.

Keywords: Graduate Teaching Assistants, Use of Student Work in Teaching, Interpretation of Student Work

Introduction

There has been growing interest in the role that graduate teaching assistants (TAs) play in undergraduate mathematics education (Kung & Speer, 2009; Speer et al., 2005). TAs have direct impact on their current undergraduate students' learning of mathematics by teaching classes, holding office hours, and working in tutoring centers (Kim, 2014; Ellis, 2014) and also will have long-term impact on mathematics education at the university level as the next generation of instructors (Ellis, 2014; Kim, 2014; Musgrave & Carlson, 2017). Recently there have been calls for more research especially about TAs' teaching (American Association for the Advancement of Science (AAAS), 2019). The current study addresses an important aspect of TAs' teaching: how TAs interpret and address their students' work. How teachers interpret students' work and plan to address errors has been studied intensively in K-12 teacher education literature (see the review by Stahnke et al., 2016). However, in comparison, we as a field still have limited understanding of whether and how TAs' interpretations of students' work relate to their instructional decisions. To understand how TAs make instructional decisions based on students' work we conducted a study where TAs analyzed students' written work on non-routine problems and planned to address the issues that they identified. The following research question guided our study:

> How do TAs use students' written work in planning how to address with the students any misconceptions reflected in that work?

Our question focuses on misconceptions because we will consider TAs inference of student understanding from written work and whether/how such inferences are related to their plans to address errors. To address this question, we adopted three ways to view students' misconceptions that we identified in the existing literature on how to use student errors or misconceptions in teaching, which we will detail in the following section.

Theoretical Background

Views on Misconceptions

Three different views of students' misconceptions that we identified in the literature (Smith et al., 1993; Li & Schoenfeld, 2019; Rake & Ronau, 2018) guided our analysis of TAs' interpretation of students' work: viewing misconceptions as simply incorrect regardless of students' thinking behind them, viewing them as a reflection of students' flawed ideas or thinking that need to be replaced, and viewing them as resources for future learning. The differences between these views of misconceptions are significant because they suggest different instructional approaches for dealing with misconceptions, such as simply reteaching the content that students showed the error for, confronting the error and replacing it with the expert counterpart, and providing context to reorganize and refine students' existing knowledge. We analyzed TAs' plans to address students' misconceptions in terms of these three instructional approaches. As summarized in Smith et al. (1993), studies that viewed students' misconceptions as "flawed ideas that are strongly held, that interfere with learning, and that instruction must confront and replace" (p. 115) represented a "fundamental advance from previous approaches that essentially divided student responses into two categories, correct and incorrect" by focusing "attention on what students actually say and do in a wide variety of mathematical and scientific domains" (p. 117). However, they also pointed out that "confront and replace" is problematic in practice because "misconceptions continue to appear even after the correct approach has been taught" (pp. 120-121). Smith et al. (1993) provided a constructivist view of "learning that interprets students' prior conceptions as resources for cognitive growth within a complex system of knowledge" (pp. 115-116) and suggested that refinement of students' existing knowledge for learning was more effective than replacement. Specifically, they argued that "instruction should help students reflect on their present commitments, find new productive contexts for existing knowledge, and refine parts of their knowledge for specific scientific and mathematical purposes" (p. 150). This reconceived view of misconceptions has been advocated for in recent approaches emphasizing sense-making in mathematics and understanding (Li & Schoenfeld, 2019) and in empirical studies investigating student reasoning behind incorrect responses (e.g., Makonye, 2012; Rake & Ronau, 2019). It should be noted that, from this view, errors or error patterns are different from misconceptions. Smith et al. (1993) used misconception as "a student conception that produces a systematic pattern of errors" and differentiated it from "simply designat[ing] a pattern of errors" (p. 119).

Given that our TAs sometimes just noticed errors or error patterns from student work samples (and/or connected it to error patterns that they noticed from their previous teaching experience) and proceeded with their plan to address the errors without providing interpretations, in our current study, we analyzed TAs' interpretation of students' errors or misconceptions in terms of the three views described above, focusing on whether it was viewed as a flawed idea to be replaced or a reflection of their knowledge to be used as a resource of learning, and whether and how it was reflected in TAs' planning of how to address the students' errors or misconceptions, i.e., reteaching, confronting-and-replacing, or reorganizing and refining.

Perceiving, interpreting, and responding to students' thinking

Addressing students' misconceptions starts with perceiving and understanding those misconceptions. Given that TAs spend a lot of time trying to make sense of students' written work while they are grading or during small group work in class, our study design adopted a setting where TAs would have opportunities to read written students' work samples, interpret what is written, and devise plans for their next instructional moves. Several existing studies have similarly adopted this setting; they involved student work samples, most of which included errors, that were developed, based on, or adopted from, existing research (e.g., Son, 2013; Cooper, 2010). As noted in the recent review by Stahnke et al. (2016), most such studies focus on content at the elementary level, such as proportional reasoning (Hines & McMahon, 2005; Son, 2013), geometry (Son & Sinclair, 2010), fractions (Jakobsen et al., 2013), algebraic or numeric computations (Cooper, 2010; Magiera et al., 2013) and have involved preservice teachers (PSTs). Stahnke et al. (2016) made common observations based on the results of these studies: (a) "pre-service teachers had difficulties in perceiving and interpreting" among other things "common misconceptions and student errors", and (b) teachers' own "difficulties with mathematics tasks influenced their perception and interpretation" (p. 14). They also showed that (c) teachers' proposed approaches to addressing students' misconceptions or errors were either '"reteaching' (Cooper, 2009) or showing students' how to do it correctly (Son, 2013)" (p. 14). Moreover, a large number of PSTs did not use student work at all in their response to the students, other than to say that the student's method is wrong or they briefly mentioned it as "a stepping-stone" to correct the student error (33 and 13 out of 75, respectively, Son, 2013, p. 62). Although these studies did not explicitly adopt the views and approaches towards misconceptions we discussed above, the approaches to student misconceptions that PSTs adopted that are in the literature are aligned with "reteaching" or "confront-and-replace" approaches to misconceptions, rather than with using student misconceptions as a resource for reorganizing and refining students' existing knowledge, and are connected to PSTs' difficulties with interpreting students' errors in terms of the students' thinking reflected in their written work.

Research Methods

Research Setting, Participants, and PD

We conducted our study in a single-variable calculus course, Calculus I for STEM majors, at a mid-sized public U.S. university. At the institution, Calculus I is offered as large coordinated lectures with 100-150 students taught by faculty instructors, and with discussion sections for 25-30 students taught by TAs. In the discussion sections, TAs mainly solved textbook problems based on questions from students. At the time of the study, 16 TAs at the institution taught discussion sections. We invited all 16 TAs to participate in our study. From these, five TAs volunteered. Those TAs were considered overall "strong" PhD students in the department, four were in their 3rd-4th year in the program with years of experience teaching calculus discussion sections and one was in the second semester of their first year in the program and also with experience teaching discussion sections. The TAs also participated in a bigger study that included classroom observation, collection of students' work, and interviews, as well as professional development (PD). However, we will only focus on the PD in this proposal. The authors (one mathematics educator, and one mathematician who has also conducted mathematics education research) conducted four 75-minute PD sessions that were one or two-weeks apart. Each session centered around one worksheet (with non-routine problems) and started with one of the TAs presenting their solutions to the problems as a prompt for discussion about how to make the solution better mathematically, or friendlier to students, or about their own difficulties when they tried to solve the problems. Then, the PD analyzed 2-4 student written solutions to the worksheet problems that we collected from the previous years. The student work samples we chose were examples of common misconceptions based on what is reported in the existing literature on students' thinking about calculus, and included evidence of student's reasoning (based on the researchers' determinations). Based on their analysis, TAs wrote responses to the following three prompts: "Please describe in detail what you think each student did in response to this problem," "Please explain what you learned about these students' understandings", and "Pretend that you are the TA of the student. What questions/problems might you pose next? Why?" There were also additional verbal prompts from the facilitators, such as "where may they be coming from?" and "why do you think the student answered that way?". Then, they discussed their noticing, interpreting, and planning with other TAs. These prompts were adopted from the noticing literature (Jacobs et al., 2010) to collect what TAs' were attending to, how they were interpreting student work, and their plan to address what they noticed and interpreted. In this proposal, we will focus on the observations we made about the TAs interpretations and plans.

Analysis

The data we used was the TAs' written responses to the three prompts and their discussions during the PD. We first transcribed the TAs discussions during the PD and typed out their written responses. Then, we classified chunks of written responses or verbal participation in the PD (divided by who spoke and the misconceptions or errors that were addressed) into noticing, interpreting, and planning. Then, the two authors categorized the chunks into several codes based on open coding (Strauss & Corbin, 1998). Once we came up with all the codes that were descriptive of the data, we again coded about 20% of the chunks, chosen randomly from each TA and from each PD session, to check inter-coder reliability. Our results matched except for a few chunks for which we discussed and revised the description of the codes until we reached agreement. Note that we analyzed all of the interpretations and plans proposed during PD, not only the ones that TAs eventually settled on by the end of the PD sessions.

Results

TAs' interpretation of student work

We analyzed TAs' interpretations in terms of the three views towards misconception discussed earlier: incorrect regardless of student reasoning behind it, reflection of student's flawed ideas or thinking that needs to be replaced with the right one, or reflection of student's existing knowledge that can be refined and reorganized to accommodate new concepts/context to learn. In our analysis, we first identified types of TAs' interpretations that indicated they considered students' work as a reflection of students' mathematical thinking. Our analysis indicated that in most chunks of interpreting student work (83 among 107), TAs interpretations indicate that they considered the students' work as a reflection of the students' mathematical thinking. Among these 83 chunks, 35 interpretations regarded a student's thinking about a concept, relation, theorem, or procedure as being different from what is mathematically accepted, 35 interpretations regarded a student's application of knowledge as in an inappropriate context, and 13 regarded a student's thinking as taking an approach deemed correct but with mistakes or as not carrying the approach out to its conclusion. The remaining 24 chunks, where TAs interpretations did not indicate that they considered the students' work as a reflection of their mathematical thinking, included interpretations such as that the student is just guessing, that they are computing without thinking, and that students' work is unclear, with no further interpretation.

The 35 interpretations regarding a student's application of knowledge in an inappropriate context are particularly important for promoting instruction that helps "students reflect on their present commitments, find new productive contexts for existing knowledge, and refine parts of their knowledge for specific scientific and mathematical purposes" (Smith et. al. 1993, p. 150) because of the straightforward way such interpretations can be translated into plans to help students understand the appropriate context for their approaches and how the present context is different. For example, three TAs interpreted the following student work (Figure 1) as an example of applying limit laws in an inappropriate context.

1. Find $f^{'}(0)$ if $f(x) = \{g(x)sin\left(\frac{1}{x}\right), \quad if\ x \neq 0\ 0, \qquad if\ x = 0$, and $g(0) = g^{'}(0) = 0.$

$$\lim_{x \to 0} f(x) = 0 \times \lim_{x \to 0} \sin\left(\frac{1}{x}\right) = 0 = f(0)$$

Figure 1. Part of Student Work Sample that TAs Interpreted as Unwarranted Generalization

One TA said, "the student tried to distribute the limit symbol over products while the limit does not exist for one of the factors. This also reflects reliance on memory rather than understanding," and another TA elaborated further by saying "they don't know limit when h goes to 0, that $sin(\frac{1}{h})$ does not exist. So, they assume that it exists and 0 times something that is 0." Here, the TA's interpretations indicates that the student's application of the limit law in this context was incorrect, but also notes contexts where their method would have worked (i.e., when the limit of each factors exists). Translating this interpretation into an instructional plan that uses the student's misconception as a resource for new learning to promote knowledge refinement could happen if the TA planned to discuss types of limit problems where the student method of distributing the limit worked, and specifying the difference between those problems and the current problem as a rationale for why the method is not applicable any more. This way, the student's existing knowledge will be refined with boundaries of where that particular method is "correct" and new ways of approaching a similar-looking but different context can be learned in comparison to the old way. In next section, we will discuss whether and how TAs' plans were in line with such a view of considering students' misconception as resource of new learning.

TAs' plans to address students' work

To understand what TAs planned to do to address student misconceptions, we first coded each TA's plan, and categorized them based on the conceptualization of how to use student work: reteaching without using it, confront the misconception that the error is showing and replace it with the correct counterpart, and using it as resource for learning. As we discussed, the most productive use of students' error in this regard would be using students' misconceptions as resource for student learning. Our analysis shows that most TAs' plans involved confronting students with their errors while none of the TAs plans used students' misconceptions as a resource for student learning. Three main instructional approaches were (a) giving students opportunities to realize their error (52 plans), (b) pointing out the error and have them fix it or teach them how to fix it without giving such opportunities (12 plans), and (c) solving the problem without addressing students' errors and starting with a simpler problem, giving hints, or scaffolding them with small steps (7 plans). Plans included in (b) and (c) are clearly aligned with "confront-and-replace" and "reteach" approaches. Plans included in (a) could potentially align with using students' misconceptions as resources for learning and knowledge refinement, but further investigation of these plans showed that they did not.

The plans included in (a) could be categorized into several further subcategories, the most prominent of which were (i) asking students to explain the part of their answer that includes an error (11 plans), (ii) providing students opportunities to check if their answer has the desired properties, hypotheses that their answer is assuming when students applied a theorem, or consistency within their answer (16 plans), and (iii) giving students counter examples to their methods/responses or providing examples or tasks for students to realize the error (13 plans). Seven plans involved a combination of these approaches, and the remaining plans combined these with some additional guidance on a next step assuming that the error was fixed by students.

A common feature of most of these plans is that they stop with the student realizing their error. Only a small number of plans included next steps after students realized their errors and most of these next steps were about restarting the problem by scaffolding students with small steps or having the student try a simpler problem than the originally given problems. Therefore, these plans confront a student with their error and either leave the student to try again or give steps to replace the student's incorrect method with a correct one, but do not use the student's misconception as a resource for refining their mathematical thinking. An example of this can be seen from the discussion in PD of the student work sample in Figure 2.

Suppose for some function $h(x)$, we know that $h(0) = 2$ and that $h'(x) = \left(1 + x^3\right)^{\frac{1}{3}}$. Draw the graph of $h(x)$.

$h'(x) = (1+x^3)^{1/3}$ $\quad f(x) = (x)^{1/3}$ $\quad g(x) = 1 + x^3$

$\quad\quad\quad\quad \downarrow \quad\quad\quad\quad\quad \downarrow$

$\quad\quad\quad \frac{3}{4}(x)^{4/3} \quad\quad x + \frac{x^4}{4}$

$$h(x) = \frac{3}{4}(1+x^3)^{4/3} \cdot x + \frac{x^4}{4}$$

$$\boxed{h(x) = \frac{3x(1+x^3)^{4/3} + x^4}{4}}$$

Figure 2. Student work sample TAs plan to address by asking the student to check if their answer has the desired property.

The following excerpt shows one TA's response to one of the researchers' questions about next steps, after the TA presented their plan of having the student differentiate their answer to show that it would not result in the derivative originally included in the problem (Figure 2):

Researcher 1: Let's say you know they…took the derivative and they got the wrong answer and they got confused, what would you do?

TA E: Both integration and differentiation to them is just a bunch of following. They don't understand really why the nuts and bolts of why this exists. So I would just be like if you have learned this, you have learned it the wrong way, this formula does not exist and one way to clearly see that is the derivative is not the derivative that we want it to be. But if you want to know why it's actually wrong then take an analysis course or something.

In this excerpt, the TA re-emphasized why the contradiction should tell them that what they did was wrong, and this excerpt exemplifies that TAs often do not have explicit plans to use students' misconceptions as a resource for learning. In fact, all plans to the problem in Figure 2 involved starting over by asking what they know about the original function given the derivative without attempting to "integrate". An approach that uses the student's misconception as a resource for learning could use some of the TAs' interpretations of the work in Figure 2 as an "unwarranted" generalization of a differentiation rule (i.e., chain rule) to integration. The plan could focus on how to properly implement the student's idea of "going backwards" with a differentiation rule (in this case, integration by u-substitution) and discuss the differences between contexts where that works and the given problem. Then, one could start navigating the different methods to accommodate the new, similar looking, but different context. However, we did not see any plans that take this type of approach for any student work samples in our data.

Discussion and Conclusion

Our analysis of TAs' plans to address students' misconceptions showed that their plans mainly focus on addressing what student did wrong in the solution by providing students opportunities to realize their errors and potentially fix them, or by directly teaching how to fix the errors. These results are more positive than just "reteaching" the concepts or procedures without addressing students' errors, which was one of the main approaches PSTs adopted in some of the existing studies in K-12 literature (Cooper, 2010; Son, 2013). However, TAs' plans were mainly aligned with the "confront-and-replace" view of misconceptions, which has been criticized in the existing literature. Moreover, in most of their plans, TAs did not include any plans to provide alternatives after giving students opportunities to realize their error, other than just starting over the problem in a mathematically correct way. With these approaches, students will have opportunities to learn that what they did is wrong, but are not helped to understand what problems in their thinking led to such a wrong answer. Thus, this way of addressing student errors is not supporting learning as expanding students' thinking (Smith et al., 1993).

In contrast to their planning, a significant number of TAs' interpretations of students' misconceptions were aligned with a desirable view of misconceptions based on the literature, which is as a resource of new learning. In these interpretations, TAs identified the methods that students may have adopted, and the contexts where their method might have worked in the past, although it did not work for the problems that were given to them at the time. Although we did not see any evidence from our data that this way of interpreting misconceptions was reflected in TAs' planning of addressing students' errors, we found that there is potential to expand this view to their planning. Since our TAs have ability to identify where the students' methods would have worked in the past, they could share that with their students to acknowledge that their attempt to solve the problem had a rationale that was supported by their learning in the past, and then help them realize the differences between the contexts where their methods worked and the current problem context. This way, students' previous knowledge about the concept or procedure would be refined and reorganized in their learning of new way of addressing the problem. We anticipate that this way of planning that is aligned with what is recommended by mathematics education literature on how to address student misconceptions as resource of learning (Smith et al., 1993; Li & Schoenfeld, 2019; Rake & Ronau, 2018) could be promoted in PD for TAs. The TAs in our study were capable of and showed a tendency to attribute potential reasons for students' "wrong" choice of methods to the problem to their past "successful" experiences of using them. PD could promote this way of interpreting when applicable and provide opportunities to think about how to incorporate such interpretations in their planning to address students' misconceptions. This type of PD has the benefit of helping TAs learn about an approach to teaching that is broadly applicable and could be used in a variety of courses regardless of their content and focus.

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A Design-Based Research Approach to Addressing Epistemological Obstacles in Introductory Proofs Courses

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As has been well-documented, the epistemological obstacles associated with teaching and learning mathematical proofs persist despite research-based instruction. We describe the ongoing design process of our NSF-funded project aimed at understanding and addressing those obstacles in introductory proofs courses, using proof by mathematical induction as an anchor. Our process is framed by two cycles of designed-based research. The first cycle corresponds to designing and implementing research-based instruction on mathematical induction, whereas the second cycle broadens the scope of our research to other introductory proofs topics. This paper reports on the outcomes of the first cycle, the transition between the first and second cycles, and the project's end products.

Keywords: designed-based research, epistemological obstacles, proofs, proof by mathematical induction, quantifiers.

There is a wide recognition among researchers and practitioners that proofs and proving are of vital importance in students' learning of mathematics at all levels of education (NCTM, 2000). At the same time, a growing body of research has identified numerous challenges associated with teaching and learning proofs (e.g., Brousseau, 1997; Brown, 2008; Dawkins & Weber, 2017; Hanna & de Villiers, 2012; Harel & Sowder, 2007; Shipman, 2016; Sierpińska, 1987; Stylianides, 2014, 2016). Sierpińska (1987) and Brousseau (1997) have conceptualized the necessary challenges in students' mathematical development in terms of epistemological obstacles (EOs). EOs in teaching and learning mathematical proofs are the primary focus of this project.

Here, we frame EOs as cognitive challenges that persist even in response to research-based instruction. Thus, EOs can be experienced both by students and teachers during instructional interactions. When instructors experience EOs, there is a tension between the desire to circumvent them and the need to provide students with opportunities to develop logical structures that are fundamental for proving. However, as it has been pointed out by Brousseau (2002), addressing an obstacle head on is essential for overcoming it because "it will resist being rejected and, as it must, it will try to adapt itself locally, to modify itself at the least cost, to optimize itself in a reduced field, following a well-known process of accommodation" (p. 85). As such, because EOs are persistent, students must internalize an intellectual need for the underlying concepts to motivate, persevere, and successfully overcome the obstacles.

Therefore, the overarching goals of our project are to 1) identify the EOs associated with teaching and learning mathematical proofs and 2) design instructional tools for evoking and addressing these obstacles. The research questions guiding our study are as follows:

1. What are the EOs associated with teaching and learning mathematical proofs?
2. What instructional tools are suitable for evoking and addressing these EOs?

In answering these questions, we employ a cyclic design-based research approach (Anderson & Shattuck, 2012; Bakker & Van Eerde, 2015; Cobb et al., 2003; Gravemeijer & Cobb, 2006). The first cycle was conducted in Spring 2018, when the third and fourth authors implemented research-based instruction on proof by mathematical induction (PMI) in their classrooms. The results are presented in Norton et al. (2022). We use PMI as an anchor for broadening the scope of our research. During the second cycle, the same teachers will use the designed instructional materials in their respective Introduction to Proofs classes – one in Fall 2022 and one in Spring 2023. This paper reports on the outcomes of the first cycle, the transition between the first and second cycles, and the project's end products (see Figure 1).

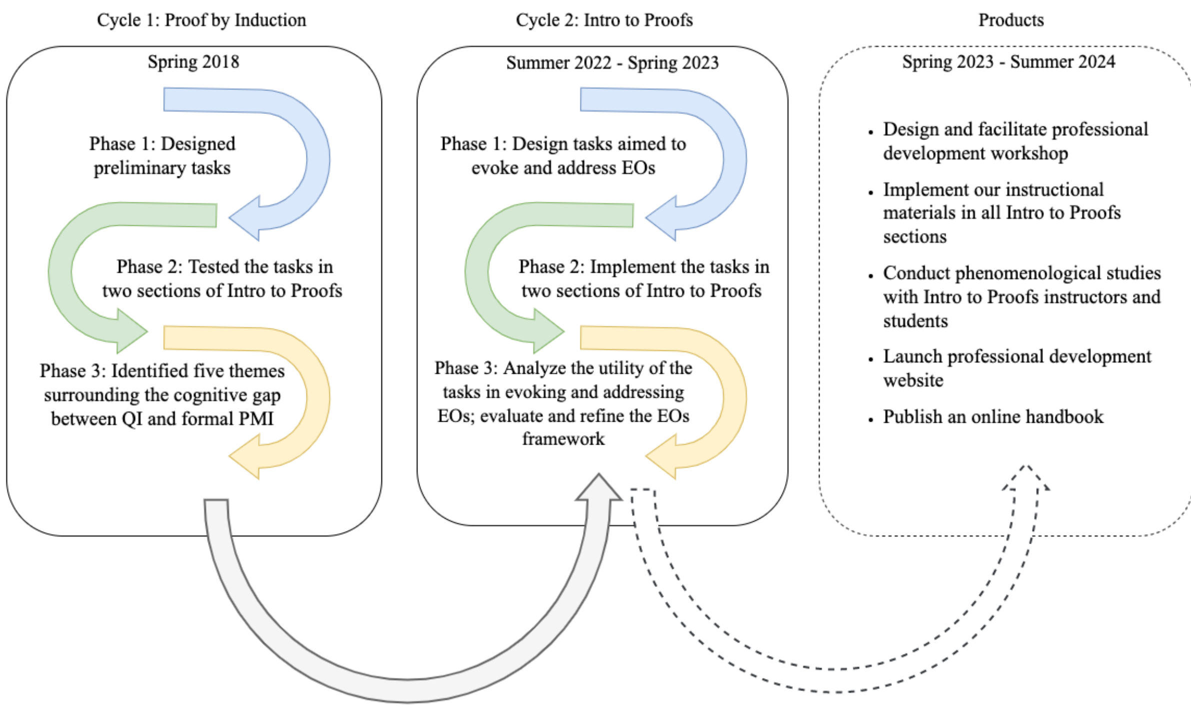

Figure 1: Two cycles and the end products of the project.

Literature Review and Theoretical Framework

Prior research has documented EOs on a variety of concepts studied in proofs-based courses. The following challenges have been associated with students' mastery of PMI: understanding the role of the base case (Baker, 1996; Ernst, 1984; Ron & Dreyfus, 2004; Stylianides et al., 2007), treating logical implication as an invariant relationship (Dubinsky, 1986, 1991; Norton & Arnold, 2017, 2019), discerning between the truth of the conjecture and the inductive hypothesis (Movshovirz-Hadar, 1993), attending to (hidden) quantifiers (Shipman, 2016) and the proper use of related language (Ernst, 1984; Movshovitz-Hadar, 1993; Stylianides et al., 2007), and having domain-specific knowledge particular to the conjecture (Dubinsky, 1991). Many of these challenges, especially quantification, extend well beyond PMI (Dawkins & Roh, 2020; Lew & Mejía-Ramos, 2019).

Instructional Approaches for Addressing EOs

Awareness of EOs should inform the design of instructional approaches to introductory proofs topics. Traditional instructional approaches may be insufficient in addressing these obstacles, necessitating the development of alternative instructional techniques that support student learning.

In the case of PMI, traditional instruction introduces the technique as a three-step procedure: (1) prove the base case, (2) assume the inductive hypothesis, and (3) prove the inductive step. However, this procedure can inadvertently cause learners to bypass the logic of PMI, ultimately circumventing experiencing necessary challenges (Harel, 2002).

An alternative approach that combines procedure with logical structure was first suggested by Avital and Libeskind (1978) and then further elaborated by Harel (2002) as "quasi-induction." A student who uses this method shows that $P(1)$ is true and that $P(1) \rightarrow P(2)$, $P(2) \rightarrow P(3)$, and so on. This leads to the plausible conclusion that eventually $P(n-1) \rightarrow P(n)$. Although formal PMI may be considered to be a natural generalization of quasi-induction, there is a significant cognitive gap between the two (Harel, 2002). Norton et al. (2022) implemented research-based instruction to address and better understand this gap.

Design-Based Research

Design-based research (DBR) is an interventionist approach aimed at weaving together educational practice and theory (Bakker & Van Eerde, 2015). DBR is typically used to develop educational materials and accompanying theoretical insights into how these materials can be used in practice.

Gravemeijer and Cobb (2006) framed DBR in terms of three interrelated phases: 1) preparing for the experiment, 2) experimenting in the classroom, and 3) conducting retrospective analysis. These phases occur in repeated cycles in which the last phase of the previous cycle informs the first phase of the following cycle, and so on. During the first phase, researchers scrutinize the problem of interest, synthesize the available research literature, curricula, and textbooks, and articulate the learning objectives and theoretical intents of the experiment. In the second phase, the experiment is conducted using the designed instructional materials as a guideline for teaching and observing. Data collection and preliminary data analysis also takes place during this phase. Once the experiment is complete and the data has been collected, the retrospective analysis begins in phase three. The comprehensive data sets must be analyzed systematically while simultaneously documenting the grounds for the subsequent cycle. As such, the retrospective analysis should examine the utility of the designed instructional materials and spark ideas for how they might be refined and complemented.

Cycle 1 (Spring 2018)

As aforementioned, prior research indicates a number of EOs experienced by introductory proofs students and suggests fruitful instructional approaches for supporting students in overcoming these obstacles. In particular, the method of quasi-induction (QI) has been validated as accessible to students and beneficial for their mastery of PMI (Cusi & Malara, 2008; Harel, 2002). However, a cognitive gap remains in transitioning between QI and formal PMI. Therefore, Cycle 1 centered around understanding the factors contributing to this gap and designing instructional materials that help students address it.

Preparing for the Experiment

Informed by prior research, we designed a set of scenarios about an unspecified proposition $P(n)$ (Table 1). For each scenario, students are tasked to decide whether the given information is sufficient to prove that $P(n)$ is true for all positive integers n. The tasks are independent of mathematical content. They are built from students' conceptualizations of logical implication, and they aim to bridge the gap between QI and PMI. We first validated these tasks in Arnold and Norton (2017) and Norton and Arnold (2017) via clinical interviews with students from an introductory proofs course, and we further refined them in a follow-up study (Norton & Arnold, 2019).

Table 1. Cycle 1 tasks.

Suppose $P(n)$ is a statement about a positive integer n, and we want to prove the claim that $P(n)$ is true for all positive integers n. For each scenario, decide whether the given information is enough to prove $P(n)$ for all positive integers n.	
A	$P(1)$ is true; for all integers $k \geq 1, P(k)$ is true.
B	$P(1)$ is true; there is an integer $k \geq 1$ such that $P(k)$ is true.
C	$P(1)$ is true; there is an integer $k \geq 1$ such that $P(k) \rightarrow P(k+1)$.
D	$P(1)$ is true; for all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
E	For all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
F	For all integers $k \geq 1$, $P(k)$ and $P(k+1)$ are true.
G	$P(1)$ is true; for all integers $k \geq 2$, $P(k) \rightarrow P(k+1)$.

Conducting the Experiment

We implemented research-based instruction in two sections of Introduction to Proofs at a large public university in the southeastern United States. The third author used the tasks within an informal pre-assessment. The fourth author used them to generate class discussion. Following these approaches, both instructors sought to promote the inductive implication as an invariant relationship between the inductive assumption and the inductive step (rather than treating the inductive assumption and step as separate components), and to explicitly address the issues of quantification. All classes associated with instruction on PMI were recorded. The third author had three class meetings, 50 minutes each. The fourth author held three meetings, 75 minutes each. Recordings captured the instructors' activity, the notes and PowerPoint slides displayed on the overhead projector, and students' interactions.

Retrospective Analysis

We performed two rounds of retrospective analysis of classroom data. During the first round, we coded the data using a set of codes informed by existing literature on students' struggles with PMI. They included base case (B), quantifiers (Q), inductive implication (I), domain knowledge (D), and conflation (C) between the inductive hypothesis, implication, or proposition. As new themes emerged for which we did not have an existing code, a new code was created. In this case, we reviewed and re-coded the prior data, using a constant comparative method (Glaser, 1965). This iterative process repeated until the codebook stabilized. The new codes included reducing the problem to computational setting (R_c), “why k, not n?” (K), effects of formal instruction (F), and the cognitive gap (G).[1]

We conducted the second round of analysis with an emphasis on the cognitive gap. Specifically, we reanalyzed the data to document how each code was related to the gap. As a byproduct of this analysis, two more codes emerged – intellectual need (N), and the use of language for communicating the intended logic (L).

The analysis revealed five themes describing the relationships between the cognitive gap and other challenges, depicted in Figure 2. Students must develop an intellectual need to generalize the logic involved in building the quasi-inductive chain of inferences (Theme 1). However, even when students have an intellectual need for formal PMI, issues with quantification arise (Theme 2). These challenges include careful quantification of the inductive implication and a shift in the language for quantifying the inductive assumption when proving the implication. As a consequence, students require new language to support the distinctions they make in quantifying the inductive assumption, the inductive implication, and the proposition they must prove (Theme 3). Finally, we found that a shift in notation from n to k when denoting the inductive implication seemed unnecessary to students. Students did not demonstrate the intellectual need for it in either class (Theme 5), nor for the use of proper language for quantifying n and k (Theme 4).

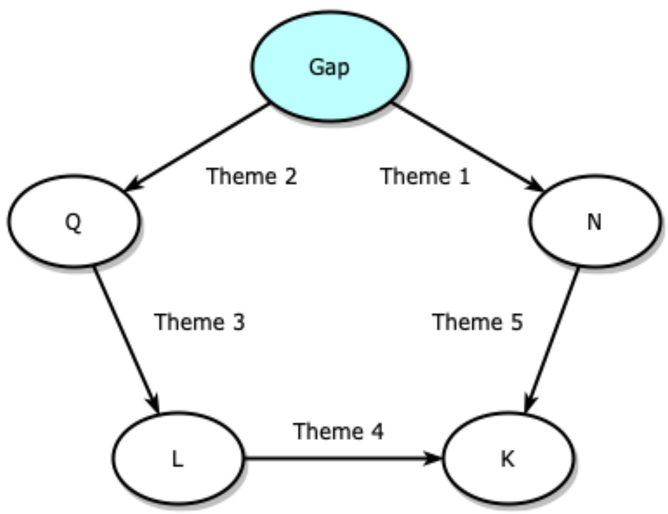

Figure 2. Five themes emerged from Cycle 1.

Cycle 2 (Summer 2022-Spring 2023)

Cycle 2 is ongoing. Its purpose is to broaden the scope of our study to the fundamental topics traditionally studied in introductory proofs courses, such as mathematical statements, logical implications and their transformations, quantifiers, and functions.

Preparing for the Experiment

Cycle 2 began with an advisory board meeting in June 2022. The advisory board members included three widely recognized experts in research on PMI, quantifiers, and logical reasoning. We relied on their expertise, on the previous research findings, and on the results from Cycle 1,

[1] A detailed description of the codes can be found in Norton et al. (2022).

to 1) identify the EOs associated with teaching and learning proofs, and 2) design instructional tasks aimed at evoking, addressing, and assessing the discussed EOs.

In particular, as one of the outcomes of Cycle 1, we found that students' apparent treatment of inductive implication is closely linked with a language issue inherent in quantifying an implication. On the one hand, the implication $P(k) \rightarrow P(k+1)$ must be proved *for all* values of k. On the other hand, the proof of the implication requires one to assume $P(k)$ is true *for some fixed but arbitrary* value of k. Ignoring the former quantification leads to an incomplete argument. Ignoring the latter quantification leads to circular reasoning. Similar issues with language in quantifying mathematical objects have been documented in prior research (Dawkins & Roh, 2020; Lew & Mejía-Ramos, 2019).

To evoke and address challenges related to teaching and learning quantifiers, we designed six multiply quantified statements about a linear equation $mx + b = 0$ (see Table 2). The statements exhaust all combinations of ordering explicit quantifiers and their attached variables that make sense geometrically. Statement 2 is the only false statement. The variety of statements will allow students to compare and contrast the quantifiers in a different order, and, as a result, understand the role of quantifiers in mathematical statements.

Table 2. A sample Cycle 2 task.

Below are six statements about real numbers *m, x,* and *b.*
TASK 1: give a geometric interpretation to each of these statements. One of them is not true. Which one? Explain why this statement is false.
TASK 2: Match the true statements with the following proofs:

Statement	Proof
The exist real numbers *m,b,* and *x,* such that $mx + b = 0$.	Let $b = 0, m = 1$, and $x = 0$. Therefore, $mx + b = 1 \cdot 0 + 0$.
For all real numbers *m,b,* and *x,* $mx + b = 0$.	N/A
There exists a real number *m,* such that for all real numbers *b,* there exists a real number *x,* such that $mx + b = 0$.	Let $m = 1$. Then, for any real number *b,* take $x = -b$. Therefore, $mx + b = 1 \cdot (-b) + b = -b + b = 0$.
There exists a real number *m,* such that for all real numbers x, there exists a real number *b,* such that $mx + b = 0$.	Let $m = 1$. Then, for any real number *x,* take $b = -x$. Therefore, $mx + b = 1 \cdot (-x) + x = -x + x = 0$.
There exists a real number *b,* and there exists a real number *m,* such that for all real numbers *x,* $mx + b = 0$.	Fix an arbitrary *x.* Let $b = 0$. Observe that, when $m = 0$, $mx + b = 0 \cdot (-x) + 0 = 0 + 0 = 0$.
There exists a real number *b,* and there exists a real number *x,* such that for all real numbers *m,* $mx + b = 0$.	Let $b = 0$. Observe that, when $x = 0$, $mx + b = m \cdot 0 + 0 = 0 + 0 = 0$ for any real number *m.*

Conducting the Experiment

As in Cycle 1, the third and the fourth authors will implement research-based instruction in their respective Introduction to Proofs classes: one in Fall 2022 and one in Spring 2023. The classroom interactions pertaining to the aforementioned topics will be video and audio recorded.

Retrospective Analysis

We will qualitatively analyze the complex student-teacher and student-student interactions. The analysis will be reminiscent of the procedures we used in Cycle 1, involving the elements of constant comparative method (Glaser, 1965). To present the data in a comprehensive and feasible fashion, we will create a graph of connected codes, in which the nodes and edges will respectively represent the key codes that pertain to answering the research questions and the relationships between these codes, respectively.

Implications and Products

We will use the results of the first two cycles of our project to deepen our understanding of the EOs and disseminate research findings. Specifically, throughout the research-based instruction, we will collect data for a phenomenographic study to gain more nuanced insights into how students experience EOs during instructional interactions. In parallel with Cycle 2 data analysis, we will recruit instructors teaching Introduction to Proofs courses in Fall 2023 who are willing to participate in a week-long workshop preceding the fall semester and implement the proposed instruction. The workshop will be organized by the authors and will include selected videos of instructional interactions from Cycles 1 and 2. We will also conduct a phenomenographic study to see how the instructors experience the proposed EOs.

On the basis of the data collected in Cycles 1 and 2, we will compile an online handbook documenting the instructional interactions surrounding the EOs. The handbook will include our modified instructional tasks designed for evoking these interactions, as well as video clips of classroom interactions from the Introduction to Proofs classes taught by the third and fourth authors. The handbook is intended to increase instructors' pedagogical content knowledge on foundational topics of the course. In addition to serving as an instructional guide for bridging research and practice, the handbook will also provide instructional videos that can be shared directly with students.

To bolster the broader impacts of our project, we will create a "Train-the-Trainers" style professional development website. This website will transform and extend the training activities of our workshop (organized in the summer of 2023) into professional development activities that can be hosted and facilitated by any university wishing to implement the instructional tools resulting from this project. Each activity will be accompanied by a facilitator's guide for empowering and equipping members of other universities to conduct their own in-house training workshop for the instructors of their introductory proofs courses.

Acknowledgements

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Exploring Geometric Reasoning with Function Composition

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Students learn about function composition, $(g \circ f)(x)$, in secondary school. From two given equations, one might identify the composite function algebraically via substitution, $g(f(x))$. But what about when functions are given as graphs? This study aims to explore how students were able to reason geometrically about composing functions. We identify common types of resultant graphs participants generated in trying to sketch the composition of various graphically-depicted functions, and the common types of reasoning we infer those participants engaged in while doing so. The results generate both questions, and hypotheses, about how best to support students' development of deep – and diverse – meanings for composing functions.

Keywords: function composition; graphs; geometric reasoning

Students are exposed to function composition in secondary school (cf., CCSSM, 2010). Primarily, they are taught that $(g \circ f)(x) = g(f(x))$ – meaning, for two functions defined by equations, one can procedurally determine the composite function via algebraic substitution. Additionally, in the context of inverse functions, secondary school students are taught that $(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$. Yet, such algebraic approaches are limited in their ability to develop sufficiently deep mathematical understandings of function composition. In this study, we try to explore the ways students' reason geometrically about the composition of two functions.

Background and Framework

In this study, we seek to answer the following research question: *In what ways do students graphically respond to function composition tasks?*

Function and Function Composition in Extant Literature

Much literature in mathematics education has been devoted to the function concept. This has included characterizing its essential abstract features (e.g., Freudenthal, 1983) and debating them (Mirin et al., 2021), considering its place in the school curriculum (e.g., McCallum, 2019; Paoletti & Moore, 2018), the notation we use for it (e.g., Paoletti et al., 2018; Thompson & Milner, 2019; Zazkis & Kontorovich, 2016), and studying how students and teachers understand and conceptualize it – including inverse functions (e.g., Breidenbach et al., 1992; Even, 1990; 1992). Scholars have also explored students' conceptions about real-valued functions (from $\mathbb{R}$ to $\mathbb{R}$) through the notion of covariational reasoning (e.g., Paoletti & Moore, 2017). In this context, a deep understanding of function demands students understand how quantities co-vary together – meaning, how changes in one quantity correspond to changes in the other quantity. However, studies have shown that students have difficulties in understanding the relationship between co-varying quantities and their graphs (e.g., Carlson et al., 2002; Schoenfeld, 1985). It can be concluded that there is a rich literature base for functions – including inverse functions.

Yet when it comes to function composition, very little exists. Much of the literature about composition of functions is in relation to calculating derivatives via the chain rule (e.g., Clark et al., 1997). In terms of covariational reasoning, function composition requires students understand how the quantity represented by the variable x covaries with the doubly-transformed output

$g(f(x))$, but this has been a minimal focus in terms of research (e.g., Moore & Bowling, 2008). Perhaps the study most closely related to function composition, on its own, was Ayers et al.'s (1988) exploration of using computer programming to help students develop the notion of function composition. In sum, there is a dearth of literature on function composition (especially in comparison to that about the function concept) – and none of it has considered it through a geometric lens. The study in this report aims to address this gap in the literature.

Geometric and Algebraic reasoning

A key premise underlying this research is that algebraic and geometric reasonings are different, where each provide distinct conceptions that are complementary for developing deep mathematical meanings. Broadly speaking, algebra is a mathematical field that examines particular structures, based on a set of objects (e.g., $\mathbb{R}$), and a binary operation(s) defined on them (e.g., $+$). Whereas geometry explores spaces that are related with distance, shape, and size (e.g., the Euclidean plane). To highlight this difference, a function, algebraically, is often characterized by its equation, e.g., $f(x) = 2x + 1$ (i.e., the structure of the set of points included in the relation); whereas, geometrically, we depict a function as its graph (typically in the Euclidean plane), e.g., $f(x)$ is a line. Moreover, not only do these fields explore different things, but key ways of reasoning differ between them. Driscoll et al. (1999; 2007), for example, differentiated Algebraic from Geometric "habits of mind." In our context, for simplistic purposes, we use *geometric reasoning* to mean reasoning that relies on figures and shapes in the plane and their properties; *algebraic reasoning*, by contrast, would rely on equations.

Beyond just the ways we might think about functions from school algebra, abstract algebra provides yet another point of connection to function composition. A primary connection for prospective teachers between abstract algebra and school mathematics is recognizing how functions and inverse functions are situated within their group structure (Wasserman, 2016). Notably, this structure is related through an operation on functions – namely, function composition. That is to say, in order for teachers to genuinely understand this mathematical connection between their university studies and the mathematics they will teach, they need to view composition as an "operation" in the same way that they view, for example, multiplication as an operation. They need intuitions, tools, ways of reasoning, and so forth, in order to be able to consider what the "result" of that operation of composition would be for two particular functions – even if those functions are given as graphs. Dreyfus and Eisenberg (1983) pointed out that without a formula, the graphical representation of function has very little meaning for most students entering calculus. This can be challenging, even for the mathematically adept.

Methodology and Data Sources

As a way to explore how students reason geometrically about function composition, we had (n = 143) university students in two mathematics and mathematics education programs (primarily from precalculus classes) sketch what they thought the composite function $(g \circ f)(x)$ would look like, primarily based on two given graphs of functions.

Task Design

On Desmos, participants were asked to sketch, using digital tools and within a limited timeframe (30 seconds, to capture their intuitions and reasonings), the composite function $(g \circ f)(x)$ for six pairs of functions. The majority of these pairs were represented only graphically (Figure 1 displays an example Desmos page; Figure 2 displays the other 5 tasks). The purpose in providing participants with somewhat unusual, and non-algebraic forms of the functions, was to deter them from immediately converting to an algebraic equation to complete the composition. Doing so increased the likelihood that students would have to try to reason about function composition geometrically based on the shapes of the graphs. In the last two tasks, functions were given as an equation, and as a table, as a point of comparison. In the complete study, participants: i) give an initial sketch within 30 seconds; ii) provide a written explanation for how, and why, they sketched the composite function the way they did; and iii) had a final opportunity to go back and modify any of their original sketches. In this paper, we analyze just their initial sketches. We do not include the "correct" composition $g(f(x))$ for the tasks in this paper, in part because it was uncommon for participants to sketch the correct graph but also to point out the genuine challenges of reasoning geometrically about function composition to the reader.

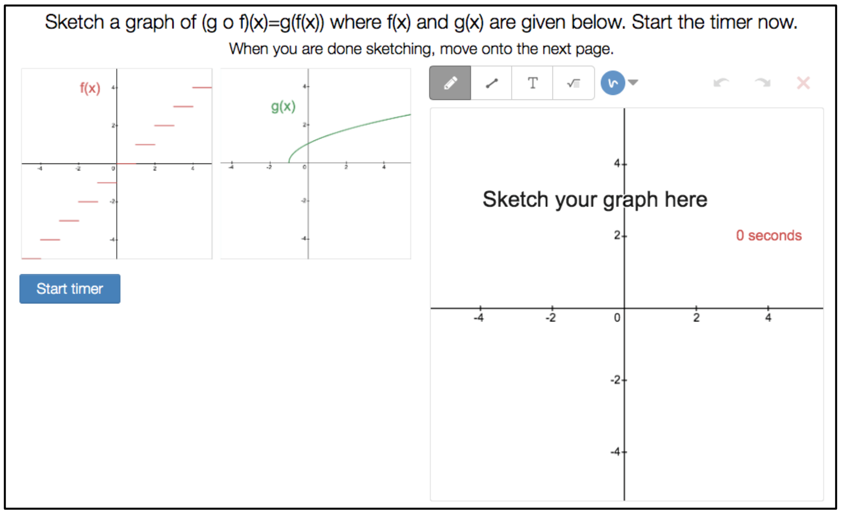

Figure 1. Task 2 in Desmos

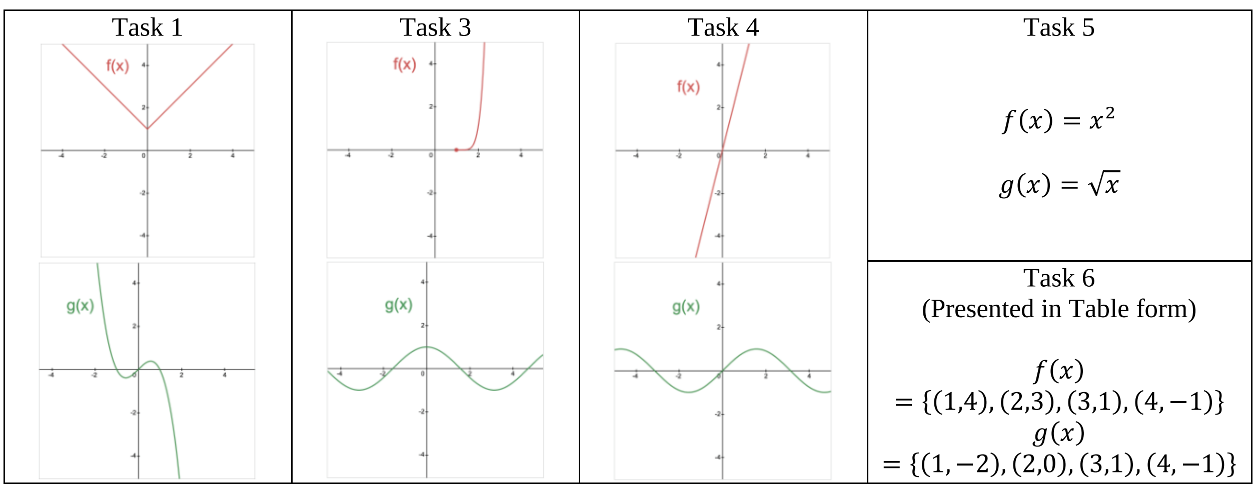

Figure 2. Given functions for Tasks 1, 3, 4, 5, 6

Analysis and Coding

Our analysis identified normative types of graphs, and processes for sketching those graphs, that were shared across participants. Notably, we aimed to create categories that could be applied across the entire set of tasks, although a few categories were specific to particular tasks. We anticipated these to include both incorrect and correct resultant graphs of composite functions, and a diverse range of reasoning students may have used to approach this sketching. We generated and refined all categories of coding as a group; when there were uncertainties in how to code a graph, we resolved such cases together and made refinements to categories as needed. As a rule of thumb, given that these were sketches, using a digital tool, we were reasonably lenient in trying to give students appropriate credit for their resultant graphs.

Due to space constraints, we do not include all codes used in our analysis. However, Table 1 provides codes and definitions for the majority of codes – and, in particular, those we report on in this paper. Clarifying examples are given in the results section.

Table 1. Codes for analyzing data

Code	Description of the composite function $g(f(x))$ sketched
BLANK	No graph
DOUBLE	Both graphs on top of each other
ONE	One of the graphs
ONE-T	One of the graphs, but with some basic transformation
MIXED HALF	Half of one graph put together with half of other graph
MIXED PROP	One graph put together with property of other graph
INVERTED	Correct graph of $f(g(x))$
CORRECT	Correct graph of $g(f(x))$
Y=X	Graphed the line $y = x$
OTHER	Some other graph

Results

Table 2 provides an overarching summary of students' initial ways of reasoning about sketching the composite function on the six tasks. Notably, we only report on the codes given in Table 1, so most, but not all (n = 143), of students' responses are present. Then, we look more carefully at results from the three types of tasks: (i) given graphs (Tasks 1-4), (ii) given equations (Task 5), and (iii) given tables (Task 6).

Table 2. Summary of data on Tasks 1-6

Code	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6
BLANK	20	24	26	25	24	33
DOUBLE	26	22	31	34	19	26
ONE	10	12	10	15	10	26
ONE-T	31	16	17	19	5	0
MIXED HALF	2	6	12	8	4	1
MIXED PROP	9	26	12	12	0	0
INVERTED	1	0	0	2	3	0
CORRECT	7	5	1	4	27	3
Y=X	0	0	0	0	25	0
OTHER	27	27	30	20	24	48

Given Graphs

In Tasks 1-4, students were given *graphs* of $f(x)$ and $g(x)$ and asked to sketch the composite function. Although there are some differences between the four tasks, for the most part, we observed consistent themes in the different ways students approached these tasks. As a primary, overall result, students had very limited success in reasoning geometrically to produce a correct composite function. Only 3% of the responses (17 of 572) were correct, whereas roughly 35% were left "blank" or given some "other" unusual response. (All percentages given in this section are across all responses to Tasks 1-4.)

More interesting are the results for the different ways students appeared to reason geometrically about the result of composing two given graphs. Although we saw both DOUBLE and ONE codes on the tasks with given equations and tables, these were prominent ways of reasoning geometrically as well; and the transformation of one (ONE-T), and the mixing of both graphs (MIXED HALF, and MIXED PROP) were almost entirely observed in the graphical context.

The DOUBLE category was reasonably common (20% of responses) – meaning, participants frequently understood how function composition "combines" two graphs to be simply "put both functions on the same graph." Participants also opted to just graph ONE of the graphs in about 10% of responses. Figure 3 gives a characteristic image for both of these from Task 2.

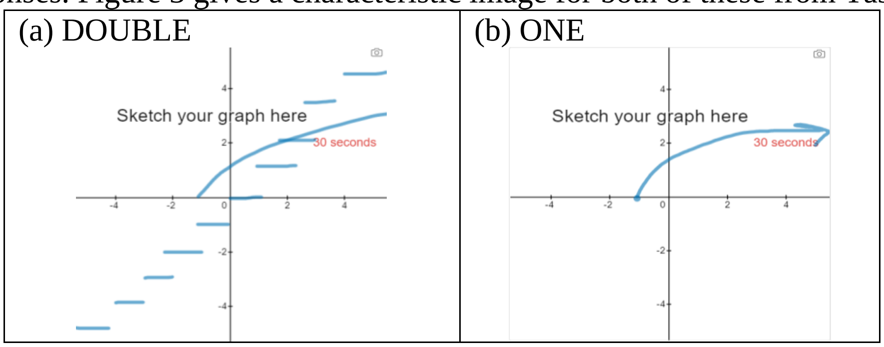

Figure 3. Examples from Task 2 of (a) "double" reasoning, and (b) "one" reasoning

Other common responses – and, in this case, ones there were mostly particular to the functions being given as graphs (Tasks 1-4) – had to do with modifying one or both functions in some way based on graphical features. This sort of geometric reasoning was how students interpreted "combining" two functions (via composition) given as graphs. For the ONE-T code (15% of responses), some students took one function as the base and then applied some transformation to it. Figure 4a shows one example, from Task 2, where a student took the initial $g(x)$ function (approximately square root) and then translated it by a vector of approximately $\langle 1,1 \rangle$, likely inspired by the step function going over one and up one. Although this is one example, there was quite a variety of different sorts of transformations included in this category – sometimes reflections, rotations, etc. For the MIXED HALF code (5% of responses), rather than graphing both functions at once, some students took half of one and combined it with half of the other to create a function – generally, a "left" and a "right" half. Figure 4b shows an example (also from Task 2). The MIXED PROP code (10% of responses) was one in which students applied a property of one function to the other function. As an example (from Task 2), Figure 4c shows one student who took the "disconnected dash" property of the step function $f(x)$, and applied it to the $g(x)$ function (approximately square root), to create a dashed square root graph. Again, there was a lot of diversity in the property being applied within this category – e.g., the domain of one was applied to the other, the pointiness of an absolute value function applied to the curves of the other, and so forth.

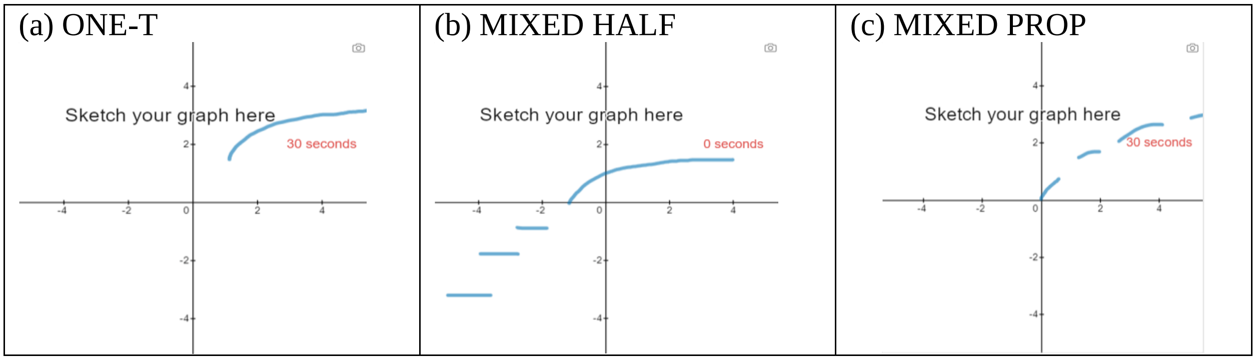

Figure 4. Examples from Task 2 of (a) "one-t" reasoning, (b) "mixed half" reasoning, and (c) "mixed property" reasoning

Given Equations

For Task 5, with two given *equations*, we point out two major results. Participants had the most success correctly plotting the composition of two functions given their equations (19%), which was significantly more than on any other task. One of the unique aspects of Task 5 was that the two functions would also have been known to be inverse functions. This may have simplified the task somewhat, as the output should be the function $i(x) = x$. But a particular nuance of this task is that the correct answer for the composite function is $g(f(x)) = |x|$ – it is $y = x$ only on a restricted domain ($x \geq 0$), and then with the reflective symmetry of the squaring function applied to it. Hence, these participants likely did more than just algebraic substitution to come up with their sketch. Another 18% of responses just graphed the line $y = x$, likely drawing on just their algebraic sensibilities. (This Y=X code was unique to this Task, as inverse functions were not given in any of the other tasks.)

Given Tables

We initially hypothesized that giving two functions in *table* form (Task 6) might facilitate students' ability to compute the composition of two functions; starting with one value, and then doubly-transforming it through f and then g can be readily accomplished with tables of values. This did not bear out in the data. Students struggled to find the composition, with about 33% of responses being some "other" unusual graph–almost twice as many of this category as on each of the other five tasks.

Conclusions and Implications

In this section, we summarize some of the important conclusions and discussion points based on the data. We highlight three important takeaways for the reader.

First, as an answer to the research question about the ways in which students reason geometrically about function composition, the data suggest students had significant difficulties sketching the correct composite function. We reiterate, on tasks where participants were given graphs (and tables), almost no students were able to give correct sketches. More were successful given equations, but even still, roughly the same number of participants gave correct responses as left it blank. This finding suggests that students are not being provided sufficient opportunities to learn about sketching composite functions overall, and especially from graphs and tables. Moreover, in the results, we described the five primary ways students tried to reason geometrically about what "combining" two functions (via composition) would look. These were: both graphs (DOUBLE), one graph (ONE), transforming one graph (ONE-T), half of one and half of the other (MIXED HALF), and applying a property of one to the other (MIXED PROP). Combined, these forms of reasoning accounted for approximately 70% of non-blank responses (58% of total responses).

Second, and perhaps unsurprisingly, students seem to be most familiar with composing functions in the context of two given equations – in particular, familiar functions, and whose composition they knew how to graph. This is the task in which students had the most success. Even still, we saw some of the challenges and limitations of algebraic reasoning in terms of producing a completely correct composite function – i.e., simplifying $\sqrt{x^2}$ to be x. This suggests the importance of having other ways to reason about function composition, specifically including geometric ways of reasoning from the graphs and their properties, which might be a useful way to avoid such oversimplifications. Although the input-output nature of functions may be on full display with tables, it was interesting to observe that students struggled with composing functions when given two tables. This generated some possible hypotheses as to why students struggled to make sense of the composite function being given tables: i) students experiences learning about composition were so strongly grounded in algebraic equations that they do not maintain ways of reasoning for composition in other contexts; ii) the discrete nature of the tables posed difficulties – because the collections of points, and their composition, would be disconnected (or possibly even undefined) on a graph; and iii) the quantified nature of the variable x in functions, where, unlike substituting the expression $f(x)$ into $g(x)$ *once*, the table form requires that students complete *multiple* substitutions – for *each* a in the domain of f, *each* of the $f(a)$ values needs to be substituted into $g(x)$ to obtain the composite collection of points.

Third, as a final point, we elaborate on why we think these particular findings are important. One reason is that the study data collected here document the genuine challenges students face in sketching the composition of two functions – in almost any form they are given, but especially graphically and tabularly. This finding provides a basis upon which to build future work. It suggests students have not been provided opportunities to develop the intuitions, ways of reasoning, and tools needed to develop a robust concept of function composition; in particular, a way to understand function composition as an operation – something that, in some way, "combines" two given functions. A second reason these findings are important is because they document the various ways that students might try to reason geometrically about function composition. Although most of these are non-normative, and would not result in a graph of the actual composite function, they provide a launching point on which instructors can build more normative conceptions. Notably, composite functions often do resemble their initial parent functions – taking, in some form, some aspect of the shape or properties of those initial functions. Third, the findings here point to both the challenges, as well as the unexplored opportunities, to identify ways to advance and enrich students' conceptions of function composition. More work is needed to identify productive intuitions and way of reasoning that might be helpful to build on, and to explore representations and digital tools that might afford learners an opportunity to develop a rich conception of function composition. In the same way that students develop meaningful ways to think about arithmetic operations like addition and multiplication, developing ways to recognize function composition as an "operation" in a similar manner is an important aim in mathematics education – in large part because of the universality of the function concept, and the utility of this operation of composing them.

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Using Latent Profile Analysis to Assess Teaching Change

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Teaching observations can be used in multiple ways to describe and assess instruction. We addressed the challenge of measuring instructional change with observational protocols, data that often do not lend themselves easily to statistical comparisons. We first grouped 790 mathematics classes using Latent Profile Analysis and found four reliable categories of classes. Based on the grouping we proposed a proportional measure called Proportion Non-Didactic Lecture (PND). The measure is the proportion of interactive to lecture classes for each instructor. The PND worked in simple hypothesis tests but lacked some statistical power due to possible scaler ceiling effects. The measure correlated highly with a dependent measure derived from the Reformed Teaching Observation Protocol (RTOP), a holistic observational measure. The PND also provided effective descriptions and visualizations of instructional approaches and how these changed from pre to post.

Keywords: Structured Observations, Undergraduate STEM Teaching,

Introduction

Numerous studies show that active, engaging, and collaborative classrooms help students learn and persist in college, but adoption of new teaching practices has been slow (American Association for the Advancement of Science, 2013; Laursen et al., 2019; Matz et al., 2018). In a recent study, observations of 2008 STEM classes at 24 institutions found that most courses were primarily lecture-based, with only a small proportion of classes incorporating significant amounts of student-centered learning (Stains et al., 2018). Professional development programs are one tool intended to help instructors implement new teaching methods and change the status quo in STEM undergraduate teaching (Laursen et al., 2019; Manduca et al., 2017). But learning whether or not these programs change teaching practices is challenging because typical means of measurement, such as surveys, student testing, and classroom observations, all have methodological shortcomings and may be difficult to implement (AAAS, 2013; Ebert-May et al., 2011; Weston et al., 2021).

While observation data are often perceived as more objective than self-report data from surveys or interviews (AAAS, 2013), data derived from observational studies pose particular challenges when used in statistical tests, thus complicating the ability to make claims about the efficacy of professional development and other interventions (Bell et al., 2012). Some observational systems also may lack clarity in their descriptions of teacher and student activities, making it difficult to learn how instruction has changed over time and what exactly changed in the teaching practices of participants (Lund et al., 2015). Because observation is resource-intensive, investigators often observe only a small number of sessions, which may not provide a representative sample of teaching practices across an entire course (Weston et al., 2021).

Shortcomings of Segmented Observational Protocols as Dependent Measures

Segmented observational protocols such as the COPUS and TDOP are employed in comparative research designs but pose measurement challenges. Typically, these instruments code each 2-minute segment of class time for instructor and student behaviors such as lecture or group work. Difficulties arise in using segmented observational protocols in research studies for several reasons. First, the use of single observation codes (such as the proportion of class time devoted to lecture) can result in poor and incomplete representation of the complex underlying instructional styles occurring in the classroom (Bell et al., 2012). In effect, this can oversimplify what is occurring the classroom. Data drawn from a segmented protocol may also have unwieldy distributional characteristics. The distributions of many relatively low-frequency codes are dramatically skewed, with high numbers of zero observations for any given classroom, and skewed distributions are also common when aggregated over multiple classrooms and instructors (Tomkin et al., 2019). The distributional properties of segmented observational data may necessitate the use of non-parametric tests, which in turn cause possible loss of statistical power (Dwivedi et al., 2017). Another concern is the high number of codes generated by segmented protocols compared to a holistic protocol's single aggregate score or few sub-scale scores. When multiple hypothesis tests (e.g., multiple t-tests) are made in the same study, the true probability of making Type-I errors (saying there is a difference when one doesn't exist) increases substantially (Abdi, 2007), which can lead to false claims about the efficacy of an intervention.

Shortcomings of Holistic Observational Protocols as Dependent Measures

Many studies that employ observational data to assess change use the Research Teaching Observation Protocol (RTOP), a holistic observational measure (Sawada et al., 2002). Holistic instruments ask observers to rate elements of a class such as "The lesson promoted strongly coherent conceptual understanding." These types of instruments often ask for more expert judgments of teaching quality versus observations of behaviors (Hora& Ferrare, 2013). While the measures derived from the RTOP have high internal reliability and some criterion validity, the measure seemed to lack structural score validity in that its proposed sub-scales did not form separate factors in the original validity study (Piburn et al., 2000). Those using the measure also seem limited in their ability to extrapolate from scores to more concrete descriptions of teaching. This is partly caused by the somewhat vague wording of some score range categories that are presented in early RTOP validity documents (Sawada et al., 2003) and studies using the RTOP for outcome comparisons (Ebert-May et al., 2011). An example would be the score range category "46-60 Significant student engagement with some minds-on as well as hands-on involvement," which provides little guidance on what instructors and students are doing in the classroom. This lack of descriptive utility for the RTOP was discussed by Lund et al. (2015), who noted that the same score ranges can describe classes with very different instructional practices, and teaching descriptions varied even more widely from study to study.

Rationale for Study & Research Questions

In the current study, we consider two protocols, TAMI-OP and RTOP, evaluating their characteristics as measures on their own merits while also recognizing them as typical examples of segmented and holistic protocols. These protocols are also distinguished by their descriptive and evaluative approaches. In our current study, we worked from a large dataset that included observations scored with both the TAMI-OP and the RTOP. We asked if a simplified measure formed from a segmented observational protocol, TAMI-OP, could be used with common statistical tests and avoid multiple comparisons while maintaining score validity. Research questions include:

1) What are the characteristics of profile groups for classes that can be derived from our TAMI-OP observational dataset of mathematics instructors?

2) What dependent measures can be derived from the TAMI-OP?
3) How do the RTOP aggregate dependent measure and the segmented TAMI-OP dependent measure function with statistical tests?
4) How can the segmented TAMI-OP dependent measure be extrapolated to provide descriptions of teaching and teaching change?

Methods

Instruments

We developed segmented observational protocol called the Toolkit for Assessing Mathematics Instruction-Observation Protocol (TAMI-OP) (Hayward et al., 2018). At two-minute intervals during the class, observers coded for the presence (yes/no) of 11 student behaviors and 9 instructor behaviors. We called these categories *activity codes* or more generally, observation *items*, including codes for Lecture, Student Questions, Group Work and Student Presentation among other activities. We also completed the RTOP for a subset of 484 of the same classes observed with the TAMI-OP. Both the TAMI-OP and RTOP had adequate interrater reliability, generalizability and internal reliability.

Sample

Our full dataset contained 790 observations of full classes by 74 teachers, gathered from three different research studies related to professional development in mathematics teaching. The observation sample from this study includes 15 instructors who taught 278 classes, some pre- and some post-intervention. The results for these instructors are used as an example of how these measures characterize teaching change but are not meant to offer a formal assessment of that program. All data were collected with human subjects approval.

The instructors in the combined data set taught a range of mathematics courses at different undergraduate levels. Classes included Calculus 1 and 2, Geometry, general education mathematics, statistics, and upper division courses for math majors (see Table 3 for full description). Class sizes ranged from 30 or less (65%), 31 to 75 (25%) to over 100 (10%). The instructors included women and men, experienced and early-career instructors; they taught at a variety of types of institutions distributed across the US and used a variety of teaching practices.

Latent Profile Analysis

Latent Profile Analysis (LPA) is a statistical classification technique that identifies subpopulations or groups within a population based on a set of continuous variables (Spurk et al., 2020). LPA is similar but preferable to traditional cluster analysis because it offers the ability to assess the ideal number of groups in a solution and generate probabilities of group membership, which provide estimates of how close any given case is to a profile exemplar (Ferguson et al., 2020).

The software R-Studio 3.5.0 was used to conduct a Latent Profile Analysis of the 790 classes in our database. The component variables for analysis all used class-level proportions of activity codes. While these variables are continuous, most did not form normal univariate distributions. We used a Maximum Likelihood (ME) estimation, and tested models with different constraints on variance and covariance. Best fitting models used estimation with equal variances and covariance equal to zero. No outliers were found or removed from the data, and there were no missing data.

Results

We found four reliable profiles that characterized the 790 mathematics classes in our sample. We determined the ideal number of profiles through a balance of quantitative fit indexes and the logical coherence of the resulting groupings. We named profiles for the variables that best differentiated between groups, resulting in profiles named *Didactic Lecture, Student Presentation and Review, Interactive Lecture,* and *Group Work*. Figure 1 presents the individual averages for each observation code for each profile.

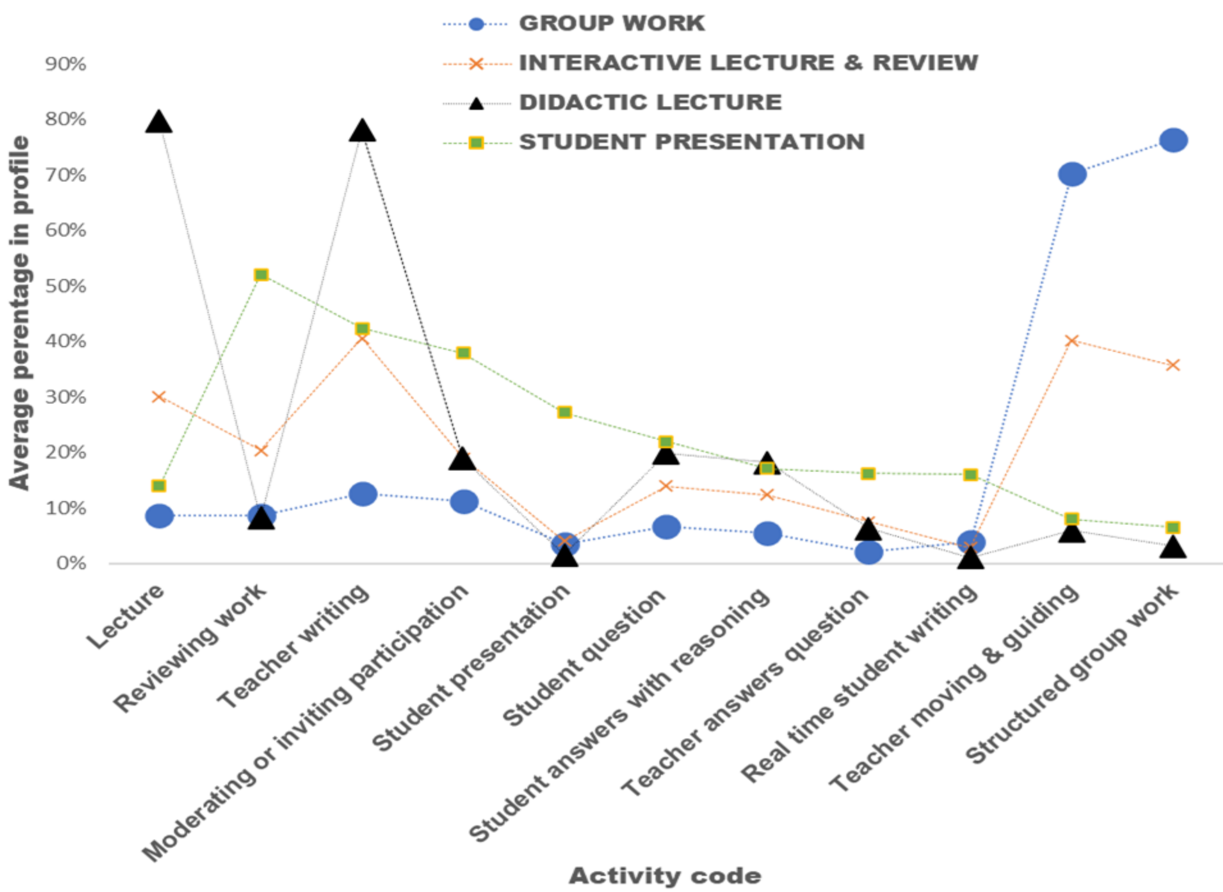
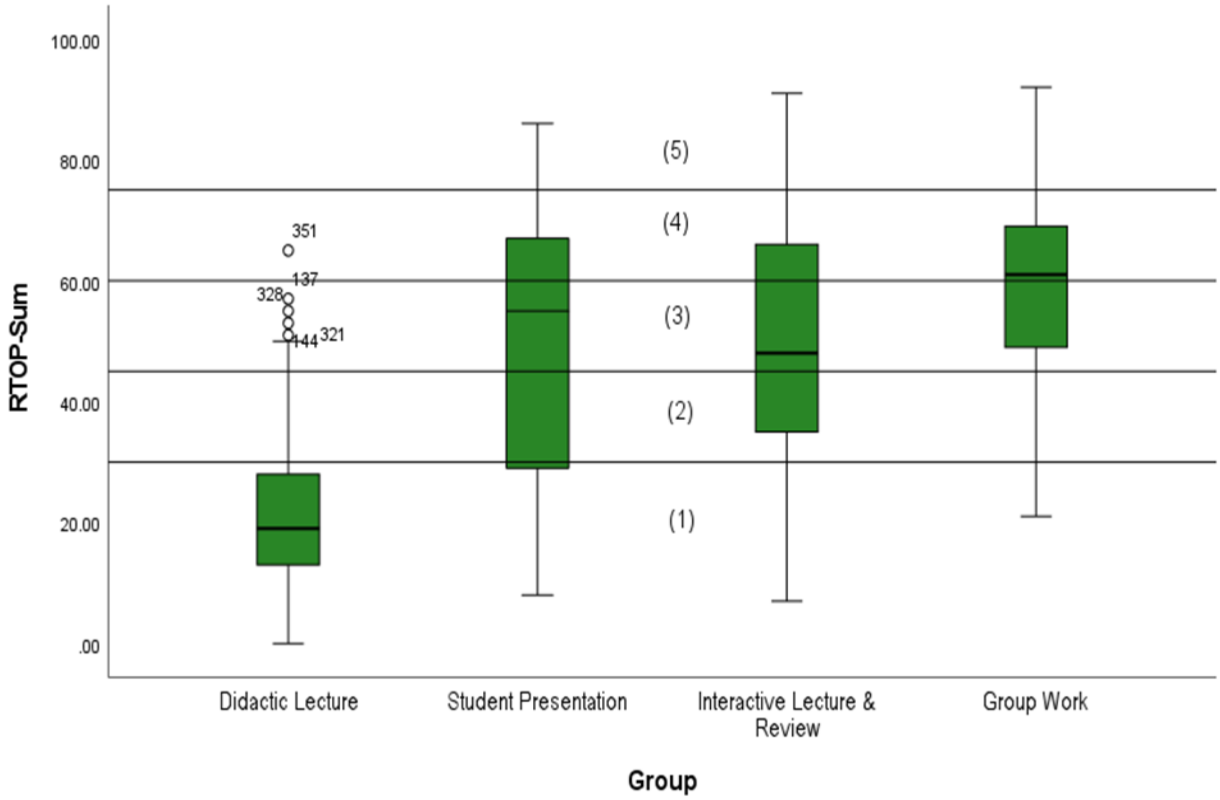


Figure 1. Individual averages for each observation code for each profile.

We first attempted to derive outcome measures based on the TAMI-OP with factor analysis but found resulting dependent measures were not reliable enough to use in analyses. A viable outcome measure derived from the LPA was the simple proportion of non-didactic lecture classes used by each teacher: *Proportion Non-Didactic Lecture* (PND). This is a teacher level measure that is the number of Non-Didactic classes divided by total class observed for the each instructor. For example, the observation data set for a particular teacher may have six out of eight classes that fit the profile for the *Didactic Lecture* profile and two that do not, resulting in a proportion of non-didactic classes of PND = 0.25.

We also examined some of the psychometric qualities of the RTOP-Sum, the dependent measure derived from a total of 25 RTOP numerical ratings. The resulting measure showed high

internal reliability ($\alpha = 0.97$), and the RTOP-Sum and the PND had a very high correlation at r=0.81. Attempted Exploratory and Confirmatory Factor Analyses did not find that proposed RTOP subscales presented as item blocks in the instrument formed separate factors.
The relationship between the four latent profiles found with LPA and RTOP-Sum scores can be seen in figure 2.



Note: Numbers in parentheses correspond to RTOP categories: (1) straight lecture, (2) lecture with some demonstration and minor student participation, (3) significant student engagement with some minds-on as well as hands-on involvement, (4) active student participation in the critique as well as carrying out of experiments, (5) active student involvement in open-ended inquiry, resulting in alternative hypotheses and critical reflection. Boxplot lines mark the mean RTOP score for each profile.

Figure 2. individual averages for each observation code for each profile.

We applied these measures to a sample data set of 15 teachers and 278 classes that included both pre and post observations for the same group of teachers, each of whom provided data from the same or similar courses taught before and after a professional development intervention. To learn how the PND functioned with basic statistical tests, we conducted a parametric Paired Sample t-test and a non-parametric Marginal Homogeneity test comparing pre and post values for the PND and RTOP-Sum. We also calculated effect sizes for pre/post gains. The results presented in Table 1 show statistically significant results for change in both the RTOP and PND measures. While both measures detect significant differences in a pre/post comparison study, the RTOP-Sum has a bigger effect size and lower p-value than the PND, indicating that the RTOP-Sum has greater statistical power in this study.

Table 1: Test statistics for the RTOP-Sum and PND measures for pre/post comparison

Test	RTOP-Sum (Scale 0 – 100)	PND (Scale 0 – 1)
Paired t-test (one-sided)	Mean difference = 17 SD = 14.5 Correlation pre/post = 0.63 Standard error = 3.76 t = 4.56, df = 14, p <0.001***	Mean difference = 0.22 SD = 0.27 Correlation pre/post = 0.58 Standard error = 0.07 t = 3.1, df = 14, p <0.004***
Related samples Wilcoxon Signed Rank Test (two-sided)	Test statistic = 119 N = 15 Standard error = 17.6 t-statistic = 3.35 Asymptotic Sig < 0.001***	Test statistic = 82 N = 15 Standard error = 14.3 t-statistic = 2.5 Asymptotic Sig = 0.01*
Effect size	Cohen's d = 1.17	Cohen's d = 0.81

Note: significance levels are indicated by p< 0.05*, p< 0.01 **, p< 0.001***

The descriptive utility of the PND is linked to its derivation from component Latent Profile Analysis groups. The separate activity codes and global variables used to form groups were also graphed to learn which activities changed from pre to post (not shown here due to space considerations). Most codes changed in ways consistent with the goals of the professional development in which they participated, with lecture and teacher writing decreasing, and group work and student presentation increasing. The average number of activities and balance among activities also increased.

Discussion

Profiles of classes created from Latent Profile Analysis provided four groups, which we labeled *Didactic Lecture, Interactive Lecture and Review*, *Student Presentation* and *Group Work*. The grouping method was reliable, and we believe these groups represent different underlying styles of teaching and learning present in our observations of 790 mathematics classrooms. In the *Didactic Lecture* group, instructors averaged 80% of their time lecturing, usually with little question and answer. This contrasted with the three non-lecture groups where students participated in more interactive activities such as group work (usually working though problem sets), presenting problems on the board, or participating in more back-and-forth dialogue with the instructor during lecture and review. Instructors for classes in the three non-didactic lecture groups also engaged in more activities in their classrooms and tended to have more balance in time devoted to each activity.

From the LPA results we created a measure called the Proportion of Non-Didactic Lecture (PND) that represented the proportion of more interactive classes, contrasted to didactic lecture classes, for each instructor. The value of a measure lies in its ability to summarize data from multiple activity codes and other variables into one measure while avoiding the pitfalls of poor construct representation, strict reliance on non-parametric tests, and multiple comparisons found in many studies that use segmented data (Tomkin et al., 2019). We found that the PND measure had some shortcomings caused by its reliance on proportional frequency data. In our wider dataset the PND had a significant number of “1” values, which created the possibility of ceiling effects and lacked distributional normality. While most statistical tests are robust to non-normality (Glass and Hopkins, 1996), comparisons made with small numbers like ours (i.e., the pre/post subset of 15 instructors) have less statistical power. In fact, the pre/post statistical comparison conducted with the measure showed less statistical power than did comparison with the RTOP-Sum, but in our case provided similar statistical inferences as the RTOP about pre-post change.

There are several other critical caveats to the use of a measure based on LPA or any other clustering technique. The final categorization of classes is dependent on both the sample used and the variables included in the model. The ultimate category where classes end up can vary given the characteristics of the initial pool of classes and the specification of the model (Williams and Kibowski, 2016). Any project also needs a relatively large pool of classes to make cluster or profile methods viable. In their overview of LPA studies, Spurk and coauthors (2020) found a median sample size near 500; in our study we were fortunate to have a collection of nearly 800 classes. It is possible to leverage the earlier work of others; those using the COPUS can take advantage of the COPUS Analyzer (Harshman and Stains, 2020) an online method for profiling observational data. We also can categorize new classes based on the original clustering algorithm. While it may seem obvious, pre and post or participant/comparison groupings (for any clustering technique) must be made at the same time and from the same model. Also, the creation of an LPA model should be done independently from, and before any type of statistical comparison is made. Shopping for the model that creates the largest effect for a comparison would constitute a breach of research ethics.

Deriving a proportional measure from segmented observational data is also limited by several important assumptions. First, there must be enough classes observed for each teacher to form a reliable measure, a number that is usually higher than is found in most research studies (Weston et al., 2021), and observing enough classes for a reliable measure is resource intensive. Related to this are possible interactions between the number of classes sampled for each instructor and the probability that rarer classes will show up in the classes sampled. If greater or fewer classes for each teacher are sampled from pre to post this can create bias in estimates of teaching change. Unequal sampling occurred in our small study because of logistical concerns, ideally pre and post samples should be balanced. Second, profiling or clustering solutions must conform to a continuum from didactic to interactive instruction. This seems to be a common finding for profile studies where a large proportion of classes are didactic lecture (Denaro et al., 2021; Lund et al., 2015; Stains et al., 2018). The main limiting factor for some studies may be the small number of truly interactive classes observed; in Stains et al. (2018) approximately 25% of classes were student-centered, although mathematics classes had the highest percentage of these courses (~35%).

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Lessons Learned using *FullProof*, a Digital Proof Platform, in a Geometry for Teachers Course

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Proof-writing is a core practice of mathematicians and a crucial skill every future teacher and mathematics major needs to acquire. A proof-based geometry course is a fruitful ground for developing proof-writing skills. Despite recent pedagogical and technological advances, proof remains challenging to learn and teach. We conducted a pilot study to explore potential advantages of an innovative digital proof platform, FullProof, which aims to advance students' proof-writing skills in Euclidean Geometry. We integrated FullProof in three Geometry for Teachers (GeT) courses at three universities. We report on the findings pertaining to students' interactions with FullProof, their perceptions of the usefulness of this tool to their proof-writing skills, and the lessons learned from the FullProof integration from an instructor perspective.

Keywords: Geometry, Proof, Prospective Secondary Teachers, Automated Proof Writing

Introduction and Theoretical Perspectives

Proof-writing is a core disciplinary practice of mathematicians, and a crucial skill all mathematics majors and future mathematics teachers need to acquire. Developing undergraduate students' facility and comfort with proofs are among the main objectives of many mathematics courses, including geometry. While geometry courses may be either required or elected for mathematics and other STEM majors, they are usually required for mathematics education majors. Hereafter we refer to such courses as GeT- Geometry for Teachers (An et al., in press; Grover & Connor, 2000). GeT courses are essential for cultivating robust and flexible knowledge and skills such as writing, analyzing, and critiquing proofs that will allow prospective secondary teachers (PSTs) to support their students' learning of geometry (AMTE, 2017; González & Herbst, 2006; NCTM, 2000; NGA & CCSSO, 2010).

Technology plays an important role in supporting PSTs' engagement with proof. Dynamic Geometry Environments (DGE) like *GeoGebra, Geometer's Sketchpad* and others have been essential in providing opportunities to explore geometrical properties, make and test conjectures (Jones, 2000; Mariotti & Baccaglini-Frank, 2018). These tools have also been beneficial for fostering PSTs' attitudes towards proof and comfort with proof and teaching it to secondary students (Abdelfatah, 2011; Kardelen & Menekse, 2017). However, DGEs offer little support for the most critical component of mathematics exploration - the *writing* of a deductive proof of a conjecture or a theorem. Recent advancements in artificial intelligence and machine learning offer new tools that support automated and interactive proof writing and verification (e.g., Lodder et al., 2021; Matsuda & VanLehn, 2005; Wang & Su, 2017), and give new opportunities for innovative research in the educational applications of these tools (Hanna et al., 2019).

FullProof is a digital platform that was developed to advance students' proof-writing skills in Euclidean Geometry. In this tool, students write a complete step-by-step proof, using an interactive diagram, an equation editor, and a library of theorems and definitions. Students can choose their own proof paths. The software checks the proof and provides feedback and/or hints to improve their work. Our project aimed to explore the potential advantages of the *FullProof* digital tool for GeT students. Our overarching research question was: *How and to what extent does engaging GeT students with the FullProof software affect their confidence with and attitudes toward geometry proofs?*

Research Setting and Methods

In Fall 2021, we conducted an exploratory study in which we integrated *FullProof* in three GeT courses in three US public universities in the Southeast, Northeast, and Midwest (See Table 1 for course and participant info). In each course, a large portion of the curriculum was devoted to Euclidean Geometry from a synthetic perspective. All instructors had experience with teaching the GeT course. To familiarize students with the platform, the instructors introduced *FullProof* during class and worked together with students to solve several proofs.

Table 1. Course and Participant Information

University Type	University A Southeast, large public	University B Northeast, large public	University C Midwest, medium public
Student population	~ 20 students: PSTs and STEM majors	~15 students: PSTs and STEM majors	~ 6-18 students: PSTs only
Course focus and modality	Content focused Hybrid	Content focused Face-to-face	Mixed content and pedagogy. Face-to-face
No. of students	14	10	6

The instructors integrated the *FullProof* platform into the courses as a support system for writing proofs, both in class and in homework. Each instructor used about 20 proof problems, from *FullProof* 's vast collection of problems on various topics from Euclidean geometry, at three levels of difficulty: high, medium, and low. Figure 1 shows a screen capture of a sample problem (medium difficulty) about triangle midsegments. Given DE is a midsegment of the triangle ABC. F is a point of BC such that BF=(1/2)FC. The segment AF intersects DE at point G and the segment BE at point O. Prove that BO≅EO.

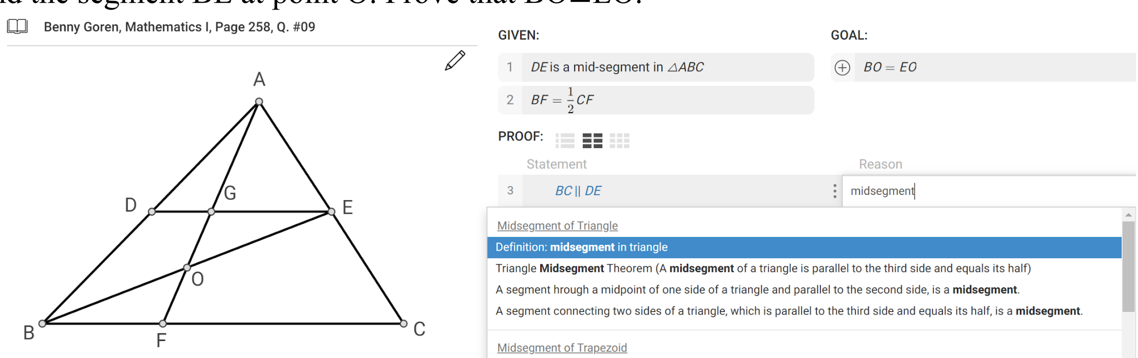


Figure 1. Sample proof problem

While the figure itself cannot be dynamically manipulated, each segment, point, and angle in the figure become highlighted when pointed to. Angles and segments can be marked, color-coded, and named as shown in Fig. 2. The right side of the screen shows the givens and the statement to be proven. The students write the proof by typing statements in the numbered lines and justifying them. The justifications can be searched by keywords, as shown in Fig. 1, or by browsing through the *FullProof* justification library. Despite the two-column format, *FullProof* allows for variation in proof strategies and the order of steps. For example, the problem in Fig. 1 can be solved with congruent triangles or properties of centroid.

The students may ask for a hint by clicking the hint button. The system will produce hints in order of increased specificity from a vague "try using triangle congruence," to suggesting a certain theorem to use, or even proposing a particular step, like "try proving ΔBFO ≅ ΔEGO". Once finished, the students click the "check" button, and in a few seconds, the algorithm checks the proof and provides feedback. Correct proof lines get a green check mark. Incorrect or partially correct lines are marked down with an explanation of the mistake. Fig. 2 shows a solution with two mistakes – a missing proof step and a missing justification for the last step. Clicking on these notifications will show a student what the missing step was (here ΔBFO ≅ ΔEGO by angle-side-angle theorem).

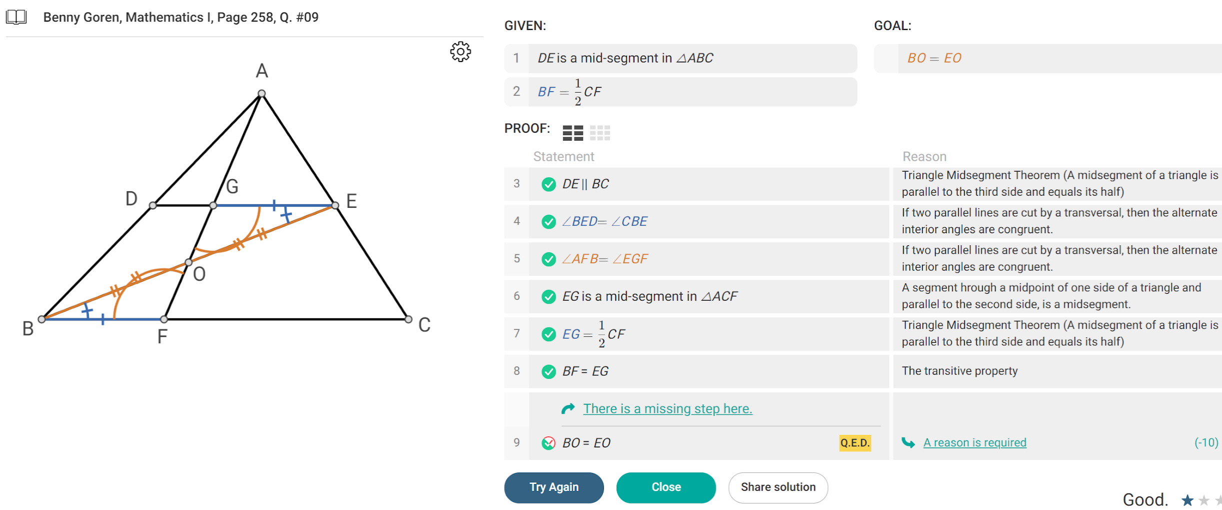


Figure 2: Sample solution in FullProof and the feedback provided by the software.

Clicking the "try again" button allows a student to improve their work, submit a corrected solution and receive more feedback. The number of hints and submissions is unlimited, allowing the students to achieve a perfect score eventually. The information about the number of hints and submission attempts is stored in the system. The instructor's interface contains information about all students in the class – which of the assigned problems they solved, how many hints and attempts they used, and can see the last submitted solution (Fig. 3).

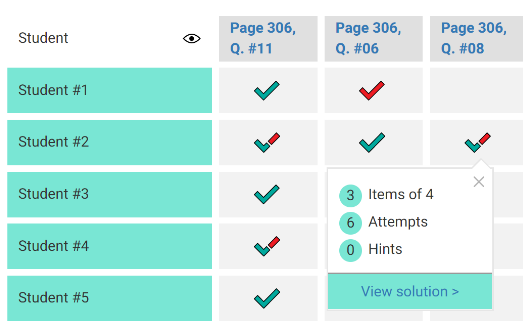


Figure 3: Excerpt of instructor interface in FullProof.

Data Collection and Analysis

To respond to our research question, we administered pre- and post-survey via Qualtrics. We focused on the results of two sets of Likert-type questions about attitudes toward proof, and comfort level with writing proofs (see Fig. 4 for sample items). The items were adapted from the relevant instruments in the literature (e.g., Kaspersen & Ytterhaug, 2020; Stylianou et al., 2015). The Likert-scale for proof-related attitudes was: *Never/Almost never (1), Sometimes (2), Often (3), Always/almost always (4)*; and for comfort-level was: *Strongly Disagree (1), Disagree (2), Agree (3), Strongly Agree (4).*

Attitudes towards Proof questions (20 items). Examples:

1. I take the initiative to learn more about proof writing than what is required in this course.
3. **When I learn a new proof, I try to think of situations when it wouldn't work.**
13. **When I face a proof problem, I consider different possible ways I can prove it.**
18. **I can explain why my solutions are correct.**
20. If I immediately do not understand what to do, I keep trying.

Comfort with proof (10 items). Examples:

1. I feel comfortable when I prove mathematical theorems in the *FullProof* platform.
2. **I feel confident in my ability to prove mathematical results using the *FullProof* platform.**
3. **I am familiar with different proof techniques using the *FullProof* platform.**
4. **I am confident about my ability to teach proof using the *FullProof* platform.**
6. I believe I can learn to read and write proofs if I put enough effort into it.

Figure 4. Selected Likert Items from the FullProof *Survey.*

Twenty-nine students responded to the pre-survey while only 20 responded to the post-survey. To analyze Likert-scale data, we initially used the chi-squared test of independence, which is appropriate for categorical data, but since some of the categories had less than five data points, the chi-square test is unreliable. To address this challenge and to avoid discarding unpaired observations by using a paired sample *t*-test, we used a new statistical test–developed by Ben Derrick, University of the West of England, which allows analyzing observations taken at two points in time, where the population membership changes over time but retains some common members (Derrick, 2020).

The post-survey also contained seven open-ended questions, developed by the researchers, about student perceptions of the *FullProof* platform (Fig. 5). We analyzed the data qualitatively, using open coding and thematic analysis (Patton, 2002) to reveal recurring themes in students' responses, specifically the types of positive and negative appraisals of *FullProof*. Each appraisal is defined as a statement that has a relatively complete and independent meaning. To capture all themes, comments that included multiple ideas were coded multiple times.

1. How much experience did you have writing proofs before using this tool?
2. **Has this tool changed the way you write proofs? If yes, explain in what way.**
3. **What are some of the successes/challenges you have had in proof writing when using this tool?**
4. **How has this tool changed your understanding of reasoning and proof?**
5. **Would you use this tool in your future geometry classroom? Why? How? If not, explain why not.**
6. What other features would you like to see in this tool?
7. **What is your overall impression of the FullProof tool?**

Figure 5. Open-ended Items from the FullProof *Survey.*

In addition to the survey, we analyzed student solutions to a few problems assigned in all three GeT courses to examine the types of proof approaches utilized by the students, and the types of mistakes they made. Due to space constraints, we do not report on this analysis here.

Results

Survey Results

Attitudes toward proof. Fig. 6 shows the statistical test results for three of the survey questions related to proof attitudes. These questions were selected since their Derrick test *p*-values were relatively close to 0.05, indicating that the results are statistically significant.

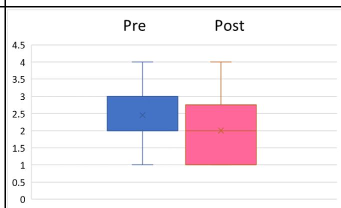
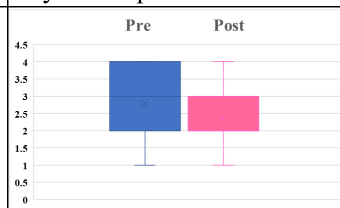
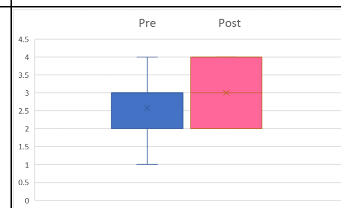
Question	When I learn a new proof, I try to think of situations when it wouldn't work.	When I face a proof problem, I consider different possible ways I can prove it.	I can explain why my solutions are correct.
Box plot			
Derrick test	0.0431	0.0713	0.0438

Figure 6. Sample Survey Results about Mathematical Identity.

While the pre-post changes for the questions in Fig. 6 are significant, the results are not necessarily in the intended direction. For example, the boxplots for the first two questions show a decrease in participants' tendency to think about situations in which a proof wouldn't work or consider different proof paths. With respect to the first question, the results may be reflective of the nature of geometry proof problems, like the one in Fig. 1, as opposed to proofs of theorems; and the static nature of the FullProof figures, as opposed to DGE. These features of *FullProof* are indeed less likely to invoke one's thinking about situations where the proof is inapplicable. The decrease in considering various solution methods was surprising to us since *FullProof* allows for much flexibility in proof approaches that instructors always encouraged. However, the hints from the software may lead students to a certain approach. One explanation for this result is that students' responses at the end of the semester reflect a pragmatic preference for one correct solution that guarantees a good grade, rather than exploring multiple solutions. It is also possible that students' confidence at the beginning of the semester decreased due to the course being harder than they anticipated. The results for the third question are encouraging, with students' increased confidence in their ability to explain their correct solutions at the end of the semester.

Comfort-level with proof. Fig. 7 shows results related to using the *FullProof* platform.

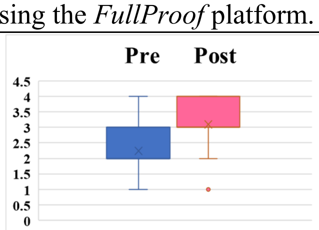
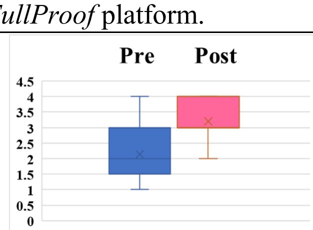
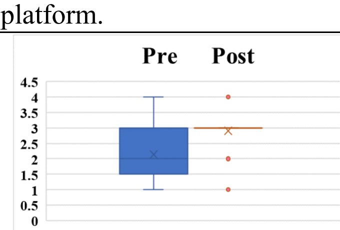
Question	I feel confident in my ability to prove mathematical results using the *FullProof* platform.	I am familiar with different proof techniques using the *FullProof* platform.	I am confident in my ability to teach proof using the *FullProof* platform.
Box plot			
Derrick test:	0.0009	0.000136	0.003014

Figure 7. Sample Survey Results about Comfort Level with Proof.

These questions had the lowest p-values, indicating significant pre-to-post changes, which is not surprising since the students had never used *FullProof* prior to the course. Still, we are encouraged by the positive direction and amount of change. For example, in the first two questions in Fig. 7, 75% of the post-survey responses were higher than the pre-survey responses. These results indicate significant increases in students' familiarity with various proof techniques and increased confidence in their ability to write proofs and even teach proofs with the *FullProof* platform. The increased familiarity with different proof techniques in *FullProof* (question 2, Fig. 7) suggests that students' lack of consideration of multiple proving approaches (question 2, Fig. 6) occurred despite increased knowledge of these techniques. This may support our assumption about pragmatic reasons for not considering multiple ways of proving. The analysis of open-ended questions sheds additional light on these results.

Student Appraisals of the *FullProof* Platform

Nineteen students responded to the six open questions about *FullProof*. There was a total of 84 appraisals, 61 (73%) positive and 23 (27%) negative. "Appraisal" is a thematic unit, such that a single written comment may contain both positive and negative appraisals, or more than one appraisal of the same type. Six out of 19 students provided positive appraisals only, others shared a combination of both. One interesting pattern in the data was: when asked to describe some of the successes and challenges they had with *FullProof* (#3, Fig. 5), 65% of comments described challenges as opposed to 35% of successes. However, the overall impressions of *FullProof* (# 7, Fig. 5) were very positive with 73% of comments containing positive reviews.

Three main themes emerged among the positive appraisals. The largest category (41%) described the *affordances of FullProof for supporting students' writing and understanding of proofs*. These included the searchability of reasons for proof steps, clear structure that supports communication and comprehension, interactive feedback, and the ability to pursue multiple solution paths. One student wrote, "*FullProof* has helped me a lot when writing proofs. I like how I can search for reasons if I am not completely sure about a reason/theorem." Another student wrote: "*FullProof* made writing proofs easier not only to write, but also to understand."

The second theme (26%) described the *advantageous technical features of FullProof,* such as hints which help them move forward if one is stuck, interactive feedback pointing to errors, visual clarity, and color-coded elements of a diagram. The third theme (33%) described the affordances of *FullProof* as a pedagogical tool for teaching others. In this theme, the participants highlighted the elements of the software that would support them as teachers in a geometry classroom. Students wrote that *FullProof* presents "a good instructional strategy to implement in the classroom," and the platform "Makes it easier for the students to see what their errors were."

The 23 negative appraisals were distributed rather uniformly across five themes. The main critique (7 out of 23 appraisals, 30%) concerned the discrepancy between the wording of the theorems in the *FullProof* platform and in GeT course. For example, *FullProof* does not have the angle-angle-side triangle congruence theorem, so students need to complete an extra proof step to use the angle-side-angle congruence theorem. Another category of critiques (22%) described the variation in standards of rigor employed by *FullProof* vs. GeT instructors. The students wrote that the software allowed them to skip steps and earn full points for their solution, while the students were aware that their instructor would have likely deducted points. One student wrote: "It allows some things to slide, that should be wrong." Other types of critiques concerned some minor technical issues like the lack of an undo button (17%). Three students admitted having personal struggles with technology, in general (13%). Finally, some students wished to break out from the two-column structure altogether to write paragraph-style proofs (17%).

Additional patterns emerged from the analysis of specific questions. When asked "Has *FullProof* changed the way you write proofs?" the responses split almost evenly between *yes* and *no*. But 83% of the accompanying written comments for this question were positive regarding *FullProof*. A similar pattern was observed in a question: "How has *FullProof* changed your understanding of reasoning and proof?" Sixty-one percent of students wrote that *FullProof* positively influenced their understanding, but interestingly, 91% of their comments contained some specific description of the positive effect of *FullProof*. In response to the question "Would you use *FullProof* in your future classrooms," 14 out of 17 students who answered this question responded positively, citing the advantages of the platform.

Discussion

Overall, the quantitative analysis shows little impact of the use of *FullProof* on student attitude towards proof and comfort level with proof writing, since pre-post changes on most survey questions were not statistically significant at the 0.05 level (as indicated by large p-values on Derrick's test). Yet, there were some questions showing significant changes using the Derrick test. While there is still some risk of a Type I error when using the partially overlapping sample t-test, Derrick's research indicates that this is "the recommended test for analyzing data recorded on an ordinal scale when partially overlapping samples are present and is particularly useful when sample sizes are small" (Derrick, p. 147).

We are also aware of the limitation of our quasi-experimental research design. Due to the lack of randomization and a single experimental group in the research setting, potential confounding variables were not controlled (Johnson & Christensen, 2012). Each course was taught by a different instructor, with no common curriculum or textbook across the institutions. There was also some variation in how instructors used *FullProof* in their courses. Finally, because of the Covid-19 pandemic, one institution required the instructor to accommodate hybrid teaching modality, which affected course pedagogy, including the use of the *FullProof* platform.

Because of these variations, our study design emulates natural conditions of how instructors might use any technological tool within the unique constraints of their institutional environments. Against this backdrop, we find it encouraging that the qualitative analysis showed more positive appraisals than negative appraisals about using the platform, which indicates the overall positive impact of using *FullProof* in GeT courses. The significance of this result is due to our analyses not distinguishing between students of different majors - mathematics education or not - meaning that potential advantages of *FullProof* apply to all GeT students.

As instructors, we note several advantages to using *FullProof* in our GeT courses. Due to the availability of multiple proof problems at various difficulty levels, and the feedback provided by the *FullProof* platform, we were able to assign a greater number of problems and more challenging problems, compared to the past. Since *FullProof* offers hints and multiple submissions, students were able to get assistance from the software itself. However, this also presented a challenge, when some students abused this functionality to get hints on every step of the proof. In the future, we plan to limit the number of hints and submission attempts. Another limitation of *FullProof,* also noted by the students, is that there are some discrepancies in the standards of rigor with the GeT courses. However, this can be leveraged to engage students in analyzing and critiquing proofs, which is an important learning objective of GeT courses (An, et al., in press). Based on the results of our exploratory study, we revised our research instruments and protocols, to use in the next round of the study. Future studies should expand the scope of research questions to include an examination of cognitive and meta-cognitive processes afforded by *FullProof*.

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Graphical Resources: Different Types of Knowledge Elements Used in Graphical Reasoning

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In broad terms, much of the research on graphical reasoning can be characterized as focusing on misconceptions, covariational and quantitative reasoning, and graphing as a social practice. In contrast, other research has focused on graphing as a cognitive process, emphasizing the fine-grained knowledge elements related to graphing, with a focus on characterizing ideas students associate with graphical patterns (i.e., graphical forms). This paper moves beyond graphical forms to characterize other categories of fine-grained knowledge – "graphical resources" – that are activated and used in concert when constructing and interpreting graphs. In this study, we identified six categories of graphical resources: graphical forms resources, framing resources, ontological resources, convention resources, quantitative resources, and function resources. We posit that holistically considering different categories of fine-grained graph-related knowledge resources can connect various bodies of research on graphing.

Keywords: graphs and graphing, knowledge resources, student reasoning, graphical forms

Graphical reasoning is a central and critical competency within mathematics and across many disciplinary fields (National Council of Teachers of Mathematics, 2006, 2009; National Research Council, 2012). Much of the education research work to date on graphing has focused on taking one large-scale item at a time and examining it, such as examining misconceptions (Aspinwall et al., 1997; Beichner, 1994; McDermott et al., 1987), covariational and quantitative reasoning (Carlson et al., 2002; Thompson & Carlson, 2017), and viewing graphing as a social practice (Roth & Bowen, 2001, 2003) with community-specific conventions (Moore & Silverman, 2015; Moore et al., 2019). More recently, some researchers have begun to take a finer-grained approach to student graphical reasoning, framing graphical activity as a cognitive process (Elby, 2000; Eshach, 2013; Rodriguez et al., 2019a; Rodriguez et al., 2019b). Fine-grained approaches place emphasis on the productive, intuitive ideas students draw on to guide many aspects of their graphing activity (Rodriguez & Jones, in preparation). In fact, a fine-grained view of knowledge structure may provide a more predictive and explanatory account of students' reasoning.

Rodriguez et al. (2019a) examined one key type of intuitive knowledge, called "graphical forms," which are simple ideas associated with a graphical pattern. As a basic example, a student might associate a graph's steepness with the idea of rate, a graphical form aptly named *steepness as rate*. However, this work has focused only on a sub-category of graphical knowledge elements, namely intuitive associations with the graph (curve) itself. In our work, we began to realize there were other types of such knowledge, beyond only graphical forms, making up a significant portion of students' graphing activity. These resources also seemed related to other phenomena identified in the literature, such as misconceptions, covariation, and conventions, suggesting that such fine-grained knowledge plays a significant role across all of students' graphical reasoning. We consequently directed our attention to the research question: What are various types of knowledge resources students use when constructing or interpreting graphs?

Theoretical Framing

We align with the *knowledge in pieces* (KiP) paradigm (diSessa, 1988; diSessa & Sherin, 1998), which moves away from viewing knowledge as consisting of large unitary objects toward viewing knowledge as a complex system of individual elements (diSessa et al., 2016, p. 30). We call such individual knowledge elements "resources" (Hammer, 2000). Resources can be very simple and intuitive, such as diSessa's examples from physics of *force as mover* or *more effort begets more result* (1988, p. 53). These resources can work together to direct thinking, and the context directly influences which resources are employed (diSessa, 2004). That is, not all resources associated with a given idea are activated in all situations (diSessa & Wagner, 2005). In fact, KiP would view an important part of gaining expertise as reorganizing resources and learning to appropriately apply them in the right contexts (diSessa, 2002).

Sherin (2001) applied the KiP paradigm to students' understanding of equations and mathematical expressions. He called resources that pair a conceptual idea with a specific symbol structure a "symbolic form." Some examples are pairing the structure [] + [] with the idea of *parts of a whole*, or the structure [] – [] with the idea of *competition*. Rodriguez et al. (2019a) then took the idea of symbolic forms and extended it to graphs, called "graphical forms", with a similar idea of pairing a conceptual idea with a specific aspect of a graph. Some graphical forms identified in the literature are *steepness as rate*, *straight means constant rate*, *curve means changing rate*, and *intersection means same* (Elby, 2000; Rodriguez et al., 2019a; Rodriguez et al., 2019b; Rodriguez et al., 2020). A longer library of 26 graphical forms can be found in (Rodriguez & Jones, 2021), and include forms related to points (e.g., *point as instance*), cardinal directions (e.g., *horizontal as constant value*), trends (e.g., *plateau as leveling off*), smoothness (e.g., *curved means realistic*), local features (e.g., *jump discontinuity means sudden*), position (e.g., *displacement as difference*), and pairs of graphs (e.g., *transformation as same*).

Yet, graphical forms may not be the only type of fine-grained knowledge resource students use. In fact, we believe various types of small-scale resources may relate to many of the larger-scale phenomena seen in prior research. In our work documenting graphical forms, we began to identify other types of resources that seemed important for how students constructed or interpreted graphs. This paper contributes by presenting these other types of resources (which we generally call "graphical resources") and attending to how our graphical resources categories relate to other research on graphing, including misconceptions, covariation, and conventions.

Methods

To examine students as they constructed and interpreted graphs, we created two separate task-based interviews (Goldin, 1997). In the first interview, students were given four different contexts involving real-world quantities and were asked to construct a graph of one quantity as a function of another. In the second interview, students were shown various graphs representing mostly real-world contexts and were asked to discuss what the graphs indicated about the contexts. Figure 1 (top of the next page) contains very brief summaries of the tasks used in the two interviews. Some graphs were taken from or based on a common calculus textbook (Stewart et al., 2021), and others were taken from or based on graphs used in past research (Carlson et al., 2002). Two of the "construction" tasks used time-based graphs, while two used non-time-based graphs because of the research showing students' potential overreliance on time (Bowen et al., 1999; Jones, 2017; Rodriguez et al., 2020).

We interviewed students in pairs to permit discussion between them and to help make their thinking visible. We recruited 12 students from two different universities, resulting in six pairs. All pairs participated in both interviews, with the two interviews separated by a few days. We recruited students from first-semester calculus courses, to ensure they had familiarity with graphs and graphing (Patton, 2002). We gave the students pseudonyms that suggest their pairings: Anna/Aria, Berto/Blaine, Cindy/Caleb, Donato/Demyan, Ellie/Eric, and Fiona/Felicity.

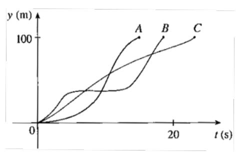
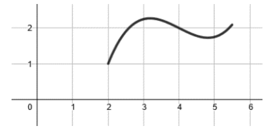
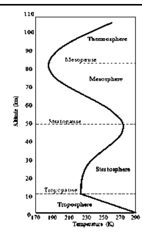
Brief Summaries of Interview 1 Tasks (Construction)			
1. Graph temperature of a frozen pizza that is thawed and baked as a function of time.	2. Graph the grass height of a lawn as a function of time over the summer.	3. Graph the volume of soda in a bottle as a function of the soda height.	4. Graph the distance a puck would travel as a function of the surface's friction.
Brief Summaries of Interview 2 Tasks (Interpretation)			
1. Graph of runners in a race over time. 	2. Graph of position of a remote controlled car over time. 	3. Graph representing temperature at different elevations. 	4. [Various mathematical graphs from a textbook were given. Students were asked to compare and contrast them.]

Figure 1. Brief summaries of the tasks given to students in the two interviews

We analyzed the data as follows. We first divided the interviews into "units of meaning" (Campbell et al., 2013) based on the focus of the students' statements, writing, gestures. This is distinct from dividing interviews based on speaking turns or pauses because it retains the focus on keeping a student's explanation intact. These units were bounded based on discussion turning to a different idea or moving in a different direction. Within each bounded episode, we identified candidate knowledge resources. We inferred these resources by what the student said, gestured, and wrote, while keeping the context of that bounded episode in mind. Our focus was on inductive thematic saturation (Saunders et al., 2018), which emphasizes the generation of new codes. Thus, we were initially very inclusive regarding the list of resources we created. Once we had generated this large list of potential resources, our next analytic action was to refine this list by identifying resources that were largely the same (and combining them) or occasionally recognizing that a candidate resource actually contained two separate ideas (and separating them). We then took our refined list and created larger-scale themes (Braun & Clarke, 2006) that categorized the final list of resources into different graphical resource types. Because the category *graphical forms* already existed, we used that category as an a priori category, but five other categories emerged inductively through our analysis, for a total of six categories.

Results

The six categories of graphical resources we identified were: (1) *graphical forms resources*, (2) *framing resources*, (3) *convention resources*, (4) *ontological resources*, (5) *quantitative resources*, and (6) *function resources*. Because *graphical forms* have been discussed in detail in the literature (see Theoretical Framing section), we do not focus on them in this paper.

Framing Resources

The first category, called *framing resources*, involved resources that paired a simple idea with any aspect of a visual graphical image that was not the graph curve itself, such as the axes, the origin, or blank sections. *Framing resources* are important because they reflect ideas related to aspects of the representation that cue the reader into the context of interest, while also physically encasing (or literally "framing") the graphical curve. To illustrate a few resources in this category, consider Cindy and Caleb, who had started the pizza temperature context by drawing axes. Caleb at first labelled the major waypoints of being thawed, baked, cooled off, and placed back in the fridge (Figure 2). However, Cindy countered with the following:

Cindy: I don't know if this is the best way to represent it, but I feel like you should have some way to represent that the intervals of time aren't the same in between thawing [Th], baking [B], and other stuff like that [C and F].

Figure 2. Caleb's uniform spacing that Cindy felt indicated equal times

Here, Cindy expressed that the equal spacings on an axis indicated equal amounts of time. We call this resource *spacing means duration*. To illustrate another framing resources that, by contrast, dealt with the origin, consider Anna and Aria, working with the same pizza context, who were discussing where to start their graph.

Aria: I just feel, I like starting at the corner [i.e., the origin] … When I start at zero [i.e., the origin], I know that I can like, just go from there. I see it as a starting point.

Aria appeared to draw on a resource we call *origin as starting point*. Interestingly, she and Anna later decided they did not know enough about the Celsius scale (which has freezing point at 0ºC), and chose to use a Fahrenheit scale instead. In this case, rather than starting at the origin, they started at a point on the *y*-axis, in a similar resource we call *y-axis as starting point*.

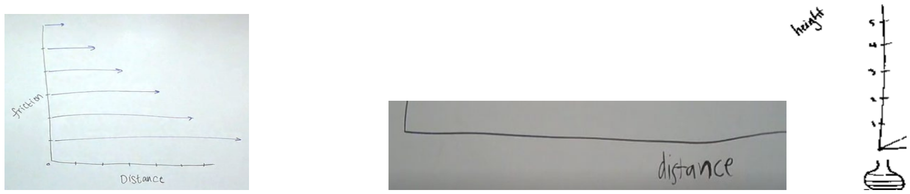
Overall, we found many of these *framing* graphical resources, including some we called *axes need labels, axes need scaling, squiggle means skip*, *y-axis maps to key moment, negative means backward, "+/–" as opposites,* and *axis arrows as continuation*.

Two Types of "Belief" Resources

In our analysis looking for fine-grained resources students used, we identified two separate categories of intuitive "belief-based" resources, regarding implicit assumptions about graphs.

Ontological Resources. The first category of belief-based graphical resources dealt with intuitive beliefs about the very nature of graphs themselves. Thus, we chose to call these *ontological resources*. In fact, one major item in the graphing literature, *graphs as pictures* (Beichner, 1994; Elby, 2000; McDermott et al., 1987), could be reinterpreted as a particularly common and oft-used ontological resource. Here, students believe a graphical image should reflect a picture of the situation. Relevant to our work, we assert that characterizing this as a stable misconception does not adequately capture nuances in students' reasoning and the ways in which this could reflect productive decisions in certain contexts for constructing graphs. Figure 6a below, which is the graph Anna and Aria initially produced for the friction-distance task, shows an example of this ontological resource in action. They claimed that each arrow was one sliding puck at a given friction level. For these students, this reflected an accessible graphing approach that visually resembles the physical scenario while illustrating the expected observation that higher friction (vertical axes) is associated with a shorter distance (horizontal axes).

However, our analysis also revealed a softer version of this ontological resource, in which a graph preserves only a specific visual stimulus. For example, Figure 6b shows the placement of "distance" in the friction task on the horizontal axis and Figure 6c shows the placement of "height" in the bottle task on the vertical axis. While these are certainly fine, mathematically, it did conflict with other student beliefs (see next section) that the independent variable needs to be on the horizontal axis. This result suggests that maintaining visual similarity between the graph and the physical scenario may have a particularly high cueing priority (see diSessa, 1993).



(a) (b) (c)

Figure 6. (a) graph as picture; (b) and (c) preservation of key visual stimulus

While *graphs as pictures* and the softer *graphs preserve visual features* were common, they were not the only ontological resources. For example, when Cindy was interpreting the graph in the atmosphere task, she explained,

> *Cindy*: You can kind of narrate the graph. So, you could be like, as you go up in the troposphere then the temperature… steadily decreases … And then once you get to the stratosphere, then it stays relatively the same for about 10 kilometers…

Cindy drew on a ontological resource we call *graph as a story*. Students held other ontological resources, too, such as *graphs as collections of points*, *graphs as connect-the-dots*, or *graph continuation* (which dealt with the assumption that graphs persist outside the viewing window).

Convention resources. The second type of "beliefs-based" resource were more about fine-grained beliefs about what students considered appropriate graphical practice. That is, rather than being about the *nature* of a graph, these dealt with common social practices (i.e., conventions) (see Moore et al., 2019). Thus, we call these *convention resources*. In fact, based on our work, we believe that much of the convention literature could be reframed to fit within a KiP paradigm. In the following excerpt, Ellie and Eric were working on the bottle task and explained their choice to put volume on the vertical axis and height on the horizontal axis.

> *Eric*: From like the reading of it, 'cause it says as a function of the height. That makes me assume that the height is your independent variable, which is on your *x*-axis [points to horizontal axis] and your volume is on your *y*-axis, because it's the dependent variable.

This excerpt contains evidence of three different convention resources. The first is a very common one already described in the literature, where it is believed the independent variable must be on the horizontal axis (Moore et al., 2014). A second related resource is that the *other* variable (ostensibly, the "dependent" variable) must be on the vertical axis. A third, and new, convention resource seen in this excerpt is that the horizontal axis *is* the "*x*-axis" and the vertical axis *is* the "*y*-axis." That is, "*x*" and "*y*" are essentially the names of the two axes.

Other convention resources dealt with axis direction and scaling. In the pizza task, Anna said,

> *Anna*: Time's going to increase this way [slides hand rightward along horizontal axis]. Temperature, I'm assuming we're going to say it increases [slides hand upward along vertical axis].

Here we can see the simple intuitive idea that *right means increase* along the horizontal axis and *up means increase* along the vertical axes. Note that this is not a graphical form, because the intuitive idea is not coupled with the graph itself, but is rather about the axes. Also, we call these convention resources, because it is not necessary for those directions to mean "increase." In fact, during the friction task, Caleb decided to have the horizontal axis represent *decreasing* friction:

> *Caleb*: 'All' friction to me is zero (i.e., at the origin) … It's weird, I know. But regardless … 'all' friction to me is zero. To me, the x-axis is the amount of friction that is *not* present.

The convention resource Caleb appeared to draw on was *direction is relative*, allowing him to decide that he could control which direction meant "increasing." Thus, convention resources can at times involve a recognition of the malleability of conventions (Thompson, 1992).

Overall, we identified many convention resources, including *π implies trig*, *origin is relative,* and *graphs read left-to-right*. Again, we propose that the convention literature might be considered a subset of graphical resources, in that these conventions are often simple, intuitive, and even implicit, ideas students associate with a graphical system.

Quantitative Resources

Another category of graphical resources related to quantities' relationships with contextualized graphs, which we call *quantitative resources*. These resources may be specific to graphs that represent real-world situations. While there were fewer resources in this category, they seemed quite impactful to students' thinking. To illustrate, students had particular ideas associated with what counted as "independent," influencing how they set up their graphical axes. When explaining why time was the independent variable in the pizza context, Anna explained,

> *Anna*: Like, time is going to march on no matter what happens.

We call this *independent as marching on*. The students' work often suggested this resource's use. For example, the bottle task stated height as the independent variable, leading both Eric and Caleb to inaccurately claim that height increase was consistent (or "constant" in their words).

> *Eric*: The height is the constant increase.

> *Caleb*: The height will always increase at the same pace.

In this context, volume might actually be considered the more steadily increasing quantity. This resource actually led several students to want to put the volume on the horizontal axis, even though it conflicted with their resource of "function output" being on the vertical axis.

This resource was strong enough that some students wanted certain graphs to be with respect to time, even when they were not. We call this *time as natural independent*. For example, In the bottle context, Anna and Aria were trying to make sense of height as an independent variable.

> *Anna*: It's weird. 'Cause it's not over a function of time like we've had in the last two graphs … Time feels way easier to me, but that's just me.
> *Aria*: …Time's pretty good and constant, in a sense … time goes constant.

Even when students did use non-time independent variables, they often spoke of it as "marching on" in real time, such as Blaine saying in the bottle context, "We're pouring in a constant volume at each interval of time."

Function Resources

Due to space constraints, we end our results by simply stating there was an additional category of graphical resources, *function resources*, though these figured into the students' reasoning less significantly than the others. Function resources were evidenced when students used explicitly function-based ideas to shape their graphical thinking. Some resources here were *input-output pair*, or *recall of graph shape*.

Discussion

Our contribution has been to expand the work on the fine-grained knowledge resources that students use when constructing or interpreting graphs. We identified six categories of resources students employed in their graphical thinking. This work is important, because these small-scale ideas tended to heavily influence how the students operated graphically, including where to start, how to put pieces of the graph together, how to put surrounding features together (e.g., axes), and what a graph is meant to convey. Collectively, an emergent combination of these graphical resources accounts for a lot of observed graphical activity. In fact, we claim these fine-grained graphical resources may underlie a much what has been discussed in the graphing literature.

For example, consider the work on misconceptions (e.g., Beichner, 1994; McDermott et al., 1987), in which the inferences students draw from graphs are often seen as larger, unitary mental objects. This perspective provides little explanatory and predictive power related to students' reasoning and how instruction can support students (Elby, 2000). Suppose a student sees a straight-line graph with axes that use a logarithmic scale. If a student thinks that the quantities vary linearly, a misconception perspective might posit the student does not understand logarithmic scales. However, it is possible this student is simply invoking the intuitive notion of *spacing means duration*, an assumption that has "worked fine" in the students' past experience (diSessa, 1988). There may be no stable misconception to replace; rather, the student may simply need to be redirected to the fact that *spacing* means something different in this context.

Next, consider the work on covariation (e.g., Carlson et al., 2002; Johnson, 2015; Moore et al., 2013; Thompson & Carlson, 2017). We believe our fine-grained approach can add clarity on how students go from covariational reasoning to a produced graph. That is, much of what is described in the covariational literature is fundamentally *quantitative*. The ideas of one variable increasing as another variable increases, or of tracking the changes in their values, are not inherently graphical. We posit that it takes these fine-grained graphical resources to translate the *quantitative* nature of covariational reasoning into a graphical inscription.

Finally, some important work has been done framing graphing as a situated social practice (Roth & Bowen, 2001, 2003) with community conventions (Moore & Silverman, 2015; Moore et al., 2019). We believe social practices are comprised of many small-scale beliefs that can be flexibly invoked, or not, given a certain context. For example, a student can hold ontological assumptions such as *graph preserves visual stimulus* and *graph as a story* simultaneously. Rather than one of these defining how a student operates for all graphs, they are both fine-grained beliefs that might be activated at different times, depending on the context. Furthermore, we note that some of the graphical resources related to features like the axes (e.g., *axes need labels*) reflect assumptions or beliefs about the nature of graphs or conventions related to graphical activity. This is important because of the way in which these simple ideas influence students' choices about how to proceed with a graphing task (e.g., convention resources such as *independent on horizontal* or *up means increase*).

Together with the previously described graphical forms, the various categories of graphical resources we have described in this paper can greatly enhance our ability to identify the knowledge and ideas students are using when creating or interpreting graphs. We assert this has the potential to serve as a foundation and a unifying perspective that incorporate the many different perspectives used in the graphing literature.

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How Students Reason about Compound Unit Structures: m/s^2, ft-lbs, and (kg·m)/s

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Intensive quantities result from quantitative operations on two or more extensive quantities. As such, their units of measure consist of "compound units." Students regularly encounter symbolically-written compound unit structures that are directly given to them, rather than constructed or developed, such as m/s^2, ft-lbs, or kg·m/s. It is consequently important to understand how students might try to reason about such symbolically-presented compound unit structures, which is the focus of this study. We examined "ways of reasoning" students used to make sense of such units, and describe in this paper five themes that emerged during analysis: (1) decomposing into separate units, (2) treating units as variables, (3) using covariational/ multivariation reasoning, (4) posing a quantification, and (5) bringing in pure math concepts.

Keywords: units, quantities, intensive quantities, compound units, ways of reasoning

Introduction and Brief Literature Review

In recent years, mathematics education has taken seriously the need to attend to students' quantitative reasoning (Johnson, 2016; Smith & Thompson, 2007; Thompson, 2011). While quantities might at first seem to be outside the realm of pure mathematics, such reasoning can be important for mathematical sense-making, deeper understandings, and being able to productively use mathematics outside of the mathematics classroom (Ellis, 2011; Jones, 2015, 2022; Moore, 2014; Moore et al., 2020; Thompson, 1994, 2011). Some quantities, called *extensive* quantities, have been defined as being directly measurable using a simple unit, such as length or volume. Other quantities, called *intensive* quantities, have been defined as not being directly measurable, such as speed or work (Kertil et al., 2019; Schwartz, 1988). In this definition, intensive quantities' units of measure consist of *relationships* between other quantities (Lesh et al., 1988). The way we symbolically write unit structures for intensive quantities, called *compound units*, usually suggests the quantitative operations done to achieve the measure, such as *m/s* suggesting a proportion (and hence division) between distance and time quantities (Thompson, 1994).

Much of the work on intensive quantities has focused on the construction of fairly simple intensive quantities and their units of measure (Bowers & Doerr, 2001; Johnson, 2012; Kertil et al., 2019; Thompson, 1994, 2011), or on how reasoning about rates and ratios is beneficial for mathematical learning (Ärlebäck et al., 2013; Boyce et al., 2020; Byerley, 2019). Yet, students regularly encounter more sophisticated intensive quantities and their symbolic compound unit structures, such as m/s^2, *ft-lbs*, or (*kg·m*)/*s*. Further, because context is often in the service of other mathematical learning goals, these compound units are typically given directly, rather than constructed. To use context and quantities effectively, it becomes critical to understand how students might think about these symbolically-presented compound unit structures. This paper contributes by explicitly examining how students might approach reasoning about these compound units. That is, we did not focus on the process of quantity construction, but rather on how students reason about given compound unit structures. Our research question was: What *ways of reasoning* do students use when presented with symbolic compound unit structures?

Theoretical Perspectives: Quantities, Reasoning, and Quantitative Reasoning

Our examination of students' *ways of reasoning about compound unit structures* is based on our theoretical perspectives on (a) quantities, (b) reasoning, and (c) quantitative reasoning. For quantities, we used Thompson's (1990, 1994, 2011) definition: a quantity is a cognitive entity that consists of an object or system, some attribute of that object or system, and a conceivable way to measure that attribute. The last of these defines the "unit" of measure, and it is common practice, for intensive quantities, to symbolically represent that unit in a way that suggests the relationships and measurement process (e.g., *m/s*). In this way, the symbolic unit structure is not necessarily an intrinsic part of a quantity, but is a widespread way to refer to it. However, more complicated compound units might obscure those relationships, such as using a "square" in m/s^2.

We define a way of reasoning as the "activities that students engage in while solving, explaining, justifying, identifying, and so on" (Hohensee, 2014, p. 136; see also Gravemeijer, 2004; McClain et al., 2000). Thompson (1990, 1994, 2011) defines *quantitative reasoning* as a special type of reasoning, specifically "the analysis of a situation into a quantitative structure" (1990, p. 12). Yet, this definition requires that the person start with some situation and then proceed toward a structure, and perhaps a symbolic representation of that structure (e.g., a formula, a unit, or an equation). We believe that students often might be in positions of needing to reason in the opposite direction, from a more complicated symbolic unit to an idea of what that compound unit means. It is this reverse act of quantitative reasoning we focused on in our study. Thus, we define a *way of reasoning about a compound unit structure* as the mental activities one employs when attempting to make sense of a given symbolic unit expression.

Methods

The data used for this paper comes from a 30-minute task-based interview (Goldin, 1997), in which students were shown symbolic compound unit structures and were asked to describe what they meant. This interview protocol thus mimics how we believe students often encounter unit structures in normal educational settings – without specific direction. Of course, the interview contained the artificial directive to make sense of that unit, meaning students' responses were very likely *in-the-moment* rather than *stable* (Thompson et al., 2014). The tasks were chosen to include compound units for different types of intensive quantities: (a) a rate within a rate, (b) a double-proportion (Thompson & Saldanha, 2003), and (c) a double-proportion within a rate:

- You have probably learned about acceleration, which can have this unit in the metric system [m/s^2 written on paper]. What does this unit mean?
- In science, "work" is defined as a product of force and distance, $W = F{\cdot}d$. If F is measured in pounds and d is measured in feet, what does this unit for work [*ft-lbs*] mean?
- In metric, "momentum" has this unit [$\frac{kg{\cdot}m}{s}$ written on paper]. What does that unit mean?

10 students (5 female, 5 male) participated in the interview, recruited from four different first-semester calculus classes at a single large university. The recruitment cite was purposeful (Patton, 2002), so that the students would have had exposure to different types of units in math or non-math classes (confirmed by most students' own statements). Our data analysis focused on identifying students' *ways of reasoning*. Analysis began simply by having each of us independently summarize the students' work on each task. We used these summaries to partition the interviews into bounded "reasoning" episodes, beginning when a student started to reason about the unit in a particular way and ending once that train of thought concluded. The students often had several "reasoning episodes" on a single task. Within each reasoning episode, we qualitatively described the actions the students seemed to be taking, based on what they said,

wrote, and gestured. We then grouped related ways of reasoning into distinct themes (Braun & Clarke, 2006), but also tracked variations within each theme. We were less concerned about normative correctness or robustness, and were mainly focused on identifying themes of *how* students approached reasoning about these symbolic unit structures, when asked to.

Results

We organize our results around the five different *way of reasoning* themes we identified.

Way of Reasoning Theme #1: Decomposing into Separate Units

The first theme involved ways of reasoning based on mental actions that split the compound unit structure into separate units, and imagining quantities associated with each part.

Decomposing m/s². For m/s^2, the students employed this reasoning in several different ways:

> *Student 1*: I think of it more as meters per second [writes *m/s*], speed. And an increase in meters per second, per second. This [points to *m/s*] is a rate, and then this [points to m/s^2] is a rate over time, so a change in rate.

> *Student 6*: Acceleration is a velocity over a given amount of time. And, I'm trying to think if that velocity, which is a distance over time, needs to be the same unit of time. [Later:] My natural thinking would put it [i.e., velocity] in meters over seconds… Then I would say it's [i.e., acceleration] over this amount of seconds as well [i.e., the same time value as velocity].

> *Student 10*: So the difference [i.e., between velocity and acceleration] would be the seconds being squared… Say your velocity is you're traveling 1 meter in 2 seconds [writes "$\frac{1\,m}{2\,seconds}$"]. Then, it [i.e., acceleration] would be 1 meter per every 2 seconds squared or 4 seconds [writes "$\frac{1\,m}{4\,seconds}$"].

Student 1 parsed the unit m/s^2 into a well-defined rate (velocity, *m/s*) and a separate well-defined extensive quantity (time, *s*). While Student 6 parsed it similarly, his issue was wondering if the two *s*'s in *m/s* and *s* represented the same time value. The quantities associated with *m, s,* and *s* seemed to retain extensive features for him (as opposed to combining distance and time into a well-defined rate quantity), such that both *s*'s were seen as the same extensive quantity. Student 10 broke up the unit quite differently, into two extensive quantities associated with *m* and s^2. The s^2 seemed to just signify "time," where its value was simply squared.

Decomposing ft-lbs. While *ft-lbs* can only be decomposed into *ft* and *lbs*, students still had different ways of reasoning about the decomposition, as seen in these two excerpts:

> *Student 2*: Maybe in units of, like, pounds-feet, like, you have this many pounds and you have to bring it 1 foot or 2 feet, or 1-foot or 2-foot or 3-foot increments.

> *Student 9*: It is quite literally the pounds of force times the amount of distance it is. So, if you're pushing on something with 20 pounds of force over 10 feet, that's 200 pound-feet.

The main difference was the relationship imagined between the quantities force and distance. Student 2 fixed one quantity (force) at a certain amount, and thought of it being iterated over the other quantity (distance). By contrast, Student 9 seemed to reason about pounds and feet as, essentially, separate quantities that did not interact other than through a multiplication procedure.

Decomposing $\frac{kg\cdot m}{s}$. There was even greater variety to how students decomposed $(kg\cdot m)/s$:

> *Student 1*: You could figure out, with an object's mass and its speed, you would have its momentum. And that would be, you can compare objects of different sizes and different speeds to each other.

> *Student 8*: So, a kilogram is the weight of something, I guess, right. And meters is how far it can travel, given an amount of time. So, like, if we have something that weighs 100 kilograms, and it goes 10 meters, and let's say 10 seconds, then that would mean momentum is 100.
>
> *Student 5*: So, weight over time times distance over time [writes "$\frac{kg}{s} \cdot \frac{m}{s}$"].
>
> *Student 10*: So, this top part [circles "*kg·m*"] is work. I know work equals, because that would be a weight times, or a force times a distance. So, we have work per second.

Student 1 broke the unit into an extensive mass (*kg*) and an intensive velocity (*m/s*). Student 8 broke up the unit into three separate extensive quantities (*kg, m, s*). Student 5 mistakenly broke it into two separate intensive quantities (*kg/s* and *m/s*). Student 10 grouped the unit by its numerator (*kg·m*) and its denominator (*s*), incorrectly associating (as other students did, too) "*kg*" with weight or force. This led him to associate this unit with the rate of change of work in time.

Way of Reasoning Theme #2: Treating Units as Variables

The second theme involved ways of reasoning that essentially interpreted the symbols in the unit structure as though they were algebraic variables instead.

Units as variables, m/s^2. Several students justified for themselves the appearance of m/s^2 using reasoning that cast "*m*" and "*s*" as if they were mathematical variables, such as *x* or *y*.

> *Student 8*: If it's the change in speed over time, then it's change in *m s* over *s* [writes "$\frac{m/s}{s}$"]. So then it's like *s* over 1 [writes "*s*/1" in denominator], and then if we flipped it up like that, then it would be, it'd just be meters divided by *s* times *s*. Or meters times 1 divided by *s* times *s* [writes "$\frac{m \cdot 1}{s \cdot s}$"], so it would be meters divided by seconds squared [adds "= m/s^2"]. That makes more sense.

Student 8's reasoning reflected algebra work. If we swapped the symbols *m* and *s* for *x* and *y*, the reasoning follows the operation of: $\frac{x/y}{y} = \frac{x/y}{y/1} = \frac{x}{y^2}$. The units' symbols, at this moment, were not being treated as units of measure for quantities, but rather as if they were variables. This result suggests that students may blur the line between symbolic units and quantities' values.

Units as variables, ft-lbs and $\frac{kg \cdot m}{s}$. The students also treated symbolic units as variables to justify to themselves why work had the unit of *ft-lbs* and momentum had the unit (*kg·m*)/*s*.

> *Student 10*: Force is measured in pounds. And distance is measured in feet. Pounds times feet.
>
> *Student 3*: If you take away kilograms, it's just meters per second [writes "*m/s*"]. And then you're multiplying by kilograms. You multiply by kilograms over 1 [writes "$\frac{m}{s} \cdot \frac{kg}{1}$"], because that's kilograms. And because of math, kilograms times meters over seconds [adds "= $\frac{kg \cdot m}{s}$"].

As before, these symbolic units are described as though being algebraically operated on. Student 10's use of pounds × feet is similar to the operation $x \times y = xy$, and Student 3's reasoning about mass × speed follows the algebraic steps of $(x/y) \cdot z = (x/y) \cdot (z/1) = xz/y$.

Way of Reasoning Theme #3: Using Covariational or Multivariational Reasoning

The third way of reasoning theme appeared to be based on a student having already *decomposed the units* and treated them *as variables*. After doing so, some students employed covariational or multivariational reasoning on the unit structures themselves (see Carlson et al., 2002; Jones, 2022; Thompson & Carlson, 2017). In short, covariational reasoning is the coordination of (usually) two variables changing in relation to each other, and multivariational reasoning is the coordination of more than two variables changing in relation to each other.

Covariational Reasoning, m/s^2. In conjunction with how the students decomposed the unit m/s^2, they used that decomposition to covary the quantities imagined as associated with the parts:

> *Student 7*: You have 1 second, 2 seconds, 3 seconds [draws a line and marks these on it]. And right here [points to interval prior to 1 second], the guy's bookin' it, he's going 10 meters per second. And then right here [points between 1 and 2 seconds] he's bookin', he's going 12 meters per second… Whenever you have meters per second squared, you're changing every time. Like every second, this number [points to 10 *m/s*] is changing.

> *Student 4*: If you were doing *s* is equal to 2, that's [i.e., s^2] going to be 4, because 2 squared is 4. And meters was 6. And then if we say seconds was 3, so 3 squared is 9. But then it [meters] was 20.

Student 7 had broken the unit up as *m/s* and *s* and described how the velocity value would change for different times. Student 4 had split up the unit as *m* and s^2, and thus covaried between time and distance. As s^2 changed from a value of 4 to 9, the value of *m* changed from 6 to 20.

Covariational/Multivariational Reasoning, ft-lbs. For the *ft-lbs* task, the students struggled to coordinate force (*lbs*), distance (*ft*), and work (*ft-lbs*) all together. Instead, they usually ignored the work value while only coordinating force (*lbs*) and distance (*ft*):

> *Student 1*: I think of it as, if you use a certain amount of force, then it's going to go a certain distance. And if you use less force, it going to go less distance. And if there's a greater force that's necessary to have it moved, then it's going to move less.

However, a couple students did attempt to coordinate all three quantities, as in this excerpt:

> *Student 2*: Because, like your work's going to obviously be less if you have to carry it, like, or move it a shorter distance versus a longer distance. But also it's going to increase with your pounds.

A final important part of the students using covariational reasoning with the *ft-lbs* unit structure is that some of them had a strong tendency to want the unit to reflect a *rate* rather than a double proportion. For example, Student 6 began this task by explaining:

> *Student 6*: I was just thinking that it'd be like, pounds over a given amount of distance [i.e., *lb/ft*]. But that's literally saying the opposite. Because this [$W = F \cdot d$] is multiply those two together, while putting it in that fashion of saying pounds over a certain amount of feet is in a sense a fraction.

Covariational/Multivariational Reasoning, $\frac{kg \cdot m}{s}$. The momentum context has several potential quantities: mass, distance, time, momentum itself, and possibly velocity or even (nonnormatively) "work", depending on the quantities the students envision. The students similarly used the mental action of fixing certain quantities in order to say how others varied.

> *Student 1*: If we're talking about something that goes, let's say 20 kilogram, going 1 meter per second. That's kind of a heavier object going very slowly. And that's going to have the same momentum as a 1 kilogram object going 20 meters per second.

> *Student 3*: Momentum is completely different for two different masses. Because of the difference in masses. The momentum of an entire planet is way different than the momentum of a marble.

Student 2: So, it's in like units of, to move 1 kilogram moving 1 meter per 1 second. And then that's your momentum. And your increments of momentum you'd be working with… So whatever your number means, that's how many times the momentum, that 1 kilogram over 1 meter over 1 second kind of a thing.

Student 1 had broken the unit into *kg* and *m/s*, and then held momentum constant to see the covariational relationship between mass and speed for a fixed momentum. By contrast, Student 3 held parts of the unit-quantities constant (distance and time) to imagine covariation between mass and momentum. Student 2 fixed each of the individual quantities (mass, distance, time) to "1", and then imagined them being iterated in order to create more units of "momentum."

Way of Reasoning Theme #4: Posing a Measurable Attribute

The fourth general way of reasoning involved imagining some attribute the unit might be associated with measuring. All of the students did so on their own, even though they had not been expressly asked to engage in a quantification process, indicating that imagining a quantity might be a common way for students to try to reason about a symbolic unit structure. Yet, they did not seem to think in terms of *unit → quantity → attribute*, but rather directly from *unit → attribute*, suggesting that students may blur the line between a quantity and its symbolic unit.

Posing an Attribute, m/s^2. The students knew that m/s^2 was a unit for acceleration, but when asked what the unit meant, they associated several "attributes" with slight nuances. These included: (a) speed change, (b) going faster or slower, (c) a distance-time relationship, (d) speed at a specific moment, and (e) speed duration over time. We note that (c), (d), and (e) are normatively incorrect, and (b) is ambiguous, as moving fast does not guarantee acceleration.

Posing an attribute, ft-lbs. When asked what "*ft-lbs*" meant, the students similarly associated several potential "attributes" with the unit. These included: (a) an acting force, (b) a sense of effort, (c) a torque (rotational), and (d) a change in kinetic energy. We note that (a) is normatively incorrect and that (b) is ambiguous, as effort and work are not necessarily equivalent. Also, while Student 9 was correct that *ft-lb* was a unit of measure for a change in kinetic energy, he was incorrect to also equate it to the change in velocity of the object.

Posing an attribute, $\frac{kg \cdot m}{s}$. The students created an even wider variety of attributes that they associated with *kg·m/s*. These included: (a) the strength of a collision, (b) the difficulty in stopping an object, (c) the damage or "hurt" done by an object, (d) the effort to move an object, (e) how far an object wants to move, (f) a mass in motion, and (g) an amount of time to cover a distance. Several of these attributes are well-connected to "momentum," though others are problematic, including an amount of time, a distance, and an effort to move an object.

Way of Reasoning Theme #5: Bringing in Related Mathematical Concepts

The final way of reasoning theme involved the activation and usage of related "pure" math knowledge, by which we mean knowledge pertaining to the abstract world of mathematics.

Using related mathematical knowledge, m/s^2. Several students invoked the derivative concept when reasoning about m/s^2. For example, Student 2 began this task by saying:

Student 2: Acceleration is the derivative of your velocity, which is how fast you're going. How many meters over a second. So, it's how much you're changing your velocity at any given point.

Bringing in the derivative helped Student 2 think about "speed over seconds," which then propelled her toward decomposing m/s^2 into *m/s* and *s* (see Theme #1).

Another pure math concept students invoked for m/s^2 was graphs. Student 9 used graphs to provide a reasonable justification for a velocity/time measure leading to the units $(m/s)/s = m/s^2$

> *Student 9*: [Draws a position-time graph:] So, as the slope is the change in position over time, which is exactly what velocity is… By the same vein, if we were to graph a velocity function [draws another graph labelled *v* and *t*]… Finding the slope at any given point of the velocity function gives the change in velocity over time at that moment, which is our acceleration.

Using related mathematical knowledge, ft-lbs and $\frac{kg \cdot m}{s}$. One pure math concept that the students used on occasion in reasoning about the *ft-lbs* and *kg·m/s* units was *factoring*:

> *Student 4*: Let's say there's 6 [*ft-lbs*]. There's some options like 2 and 3, or 6 and 1. So, then, if I travelled 3 feet and for each foot I traveled I applied 2 pounds, then overall the amount of work that was put in, is 6 foot-pounds.

Discussion

This study contributes by examining student reasoning in the "opposite" direction of how it is usually studied in the quantitative reasoning literature. That is, rather than examining the construction of a quantity and its unit (Bowers & Doerr, 2001; Johnson, 2012; Kertil et al., 2019; Thompson, 1994, 2011), we examined students' ways of reasoning when tasked with starting with the symbolically represented unit structure and reasoning from there. We posit that this is a common reasoning activity students need to do if we expect them to couple mathematical understanding with context and quantitative reasoning.

One important implication of our study is to problematize the frequent use of symbolic units for intensive quantities as though they are self-evident. A quantity like acceleration is frequently used in mathematics education to enrich understanding of mathematical concepts (Hitier & González-Martín, 2022; Roschelle et al., 2000; Schwalbach & Dosemagen, 2000; Taşar, 2010), but our study shows we cannot assume students have certain ways of reasoning about the unit structure, m/s^2. Our students exhibited many ways to reason, sometimes leading to normative interpretations and other times not. For example, while some students imagined m/s^2 to suggest a rate within a rate, $(m/s)/s$, others imagined it as suggesting a comparison between distance and time, m and s^2. Further, students struggled to make sense of *ft-lbs*, even though such double proportion units show up in algebra, trigonometry, and calculus (Stewart et al., 2021; Sullivan & Sullivan, 2009; Thompson & Saldanha, 2003), including "*person-hour*" tasks. Some students wanted *ft-lbs* to be a rate instead, *lbs/ft*, while others did not maintain a multivariational image between force, distance, *and* work simultaneously. Some students were also unsure of how to imagine $(kg \cdot m)/s$, either incorrectly associating *kg* with a force quantity, or treating all three as distinct extensive quantities (rather than a product of an extensive quantity and an intensive rate).

Our study sheds other useful light on how students think of symbolic units, as well. Students likely blur the line between a quantity, its value, and the symbolic unit used to represent a unit of its measure. The students regularly treated unit symbols, such as *m*, *s*, or *kg*, as though they were algebraic variables, even performing algebraic operations on them. Students also imagined potential attributes that were directly associated with the unit. Thus, rather than an image of a quantity as consisting of separate-but-related object, attribute, unit of measure, and value, the students seemed to freely blend these all together (and unproblematically, from their standpoint). Understanding these ways that students reason can help instructors be more attentive to what students might think when encountering contextualized quantities and units in learning math.

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“The runner jumped back?”: Developing Productive Understandings of Symbolization Through Animations and Applets

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In this report, I discuss one potential role an applet plays in the context of a teaching experiment. Mathematics students are constantly tasked with understanding mathematical notation, which is often context-dependent or sometimes loosely defined. While the purpose of mathematical symbolization is to help mathematicians communicate concisely, researchers have indicated that students have difficulties understanding and utilizing mathematical notation to represent quantities. Therefore, examining the mechanisms by which students can refine their understandings into more conventional ones can be of value to the field of mathematics education. This study provides an example of using applets to help students engage in reflective discourse. The findings of this study indicate that giving students a medium to visualize their mathematics can support them in understanding what their mathematical symbolization actually represents.

Keywords: Symbolization, Applets, Desmos, Teaching Experiment, Average Rate of Change

Symbolization is an integral part in mathematics to represent quantities and operations. As mathematicians, we utilize numbers and symbols to encapsulate information (Pimm, 1991), such as $\frac{dy}{dx}$ to mean “the (instantaneous) rate at which the value of the variable, y, changes with respect to changes in the value of the variable x.” Comprehending and utilizing numbers and symbols is an essential skill in learning mathematics. However, researchers have indicated that students have difficulties interpreting and using mathematical notation to represent quantities (Tall & Vinner, 1981; Torigoe & Gladding, 2007; Chin & Pierce, 2019; Begg & Pierce, 2021). Therefore, investigating how students develop understandings of mathematical notation is an important area of research in mathematics education. Fonger et al.’s (2016) study is an example of the benefits of examining student learning with a focus on students’ representational fluency. Their study found that leveraging students’ ways of thinking supported their ability to reason quantitatively and utilize appropriate symbolization.

Theoretical Background

When a student engages in symbolizing or interpreting mathematical notation, they have an expectation of the calculations they envision and that the parts that compose an expression or formula represent a process of quantifying or representing quantities within a situation. This is what O’Bryan (2018) terms *emergent symbol meaning*. This perspective is important because symbols do not inherently carry meaning; instead, an individual interprets the notation based on their conception of the mathematical problem. For example, consider a situation involving an arithmetic sum of adding up the numbers 1 through 10 (inclusive). One student might utilize the expression $(1 + 10) * \frac{10}{2}$ [This would be of the form $\frac{n}{2}(a_1 + a_{10})$] to represent the process of adding up the first and last numbers and repeating this process $\frac{10}{2}$ times (i.e., 1 + 10, 2 + 9, 3 + 8, …). However, another student might write $\frac{1+10}{2} * 10$, to represent their conception that the average of this set of numbers is 5.5 (this holds true due to it being an arithmetic sequence), and they can imagine replacing all the numbers with 5.5 and adding those up 10 times ($5.5 * 10$). Despite using the same numbers and symbols, these students had different ways of conceiving the situation. Another student might incorrectly write $\frac{10}{2} * 10$ because 10 is the highest number of the sequence, and they want to find the middle number of the sequence (and thus write $\frac{10}{2}$) and then add that up 10 times. In this last case, the student can check that their result is incorrect but might not have the means to determine why their symbolization of the situation was incorrect (and how they might make refinements). This last case highlights the importance of determining how we can support students in reflecting on their symbolization and how we might assist students in making adjustments when they have an incorrect response.

One method of helping students reflect on their mathematics is the usage of animations and applets. According to researchers, animations can have an *enabling function* to help reduce the cognitive load (Mayer, 2001) and support reflective discourse by focusing the conversation on students' understanding (Cobb et al., 1997). One way to enable reflective discourse involves having students anticipate what they will see before the animation plays and then explain what they see in the animation (Schnotz & Rasch, 2005; Thompson, 2019).

In this study, I report on a teaching experiment that leveraged Desmos animations and applets. This report's research question is "What role does an animation or applet play in the context of reflective discourse?"

Methodology

This study was conducted by engaging 2 students in individual teaching experiments on the idea of rate of change. The teaching experiment involved six sessions that focused on characterizing and advancing students' ways of thinking about rate of change. In this paper, I discuss one of the sessions and how the students interacted with a Desmos applet.

Teaching experiment methodology (Steffe & Thompson, 2000) involves a sequence of teaching sessions that include a student, a teacher, a witness, and a camera to record each episode. A teaching experiment's purpose is to identify a start and ending point of student progress and how students construct knowledge as they progress through each teaching episode. The goal of a teaching experiment then is to hypothesize and test models of student thinking.

The Desmos Activity (a summary of the activity can be seen here: Desmos Activity)

In the teaching session, students were presented with a situation where a runner traveled 100 meters in 32 seconds at a varying speed (a non-constant speed). Students were tasked with

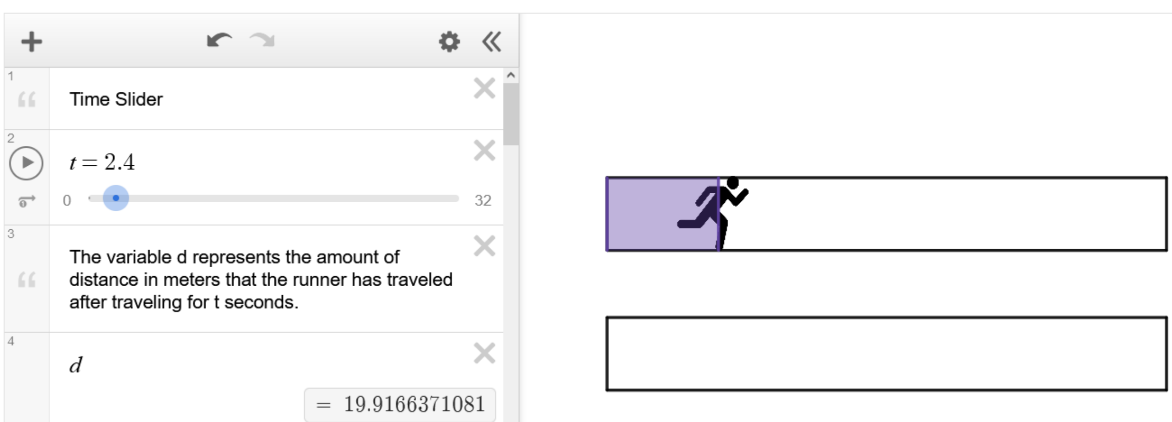

Figure 1: The Desmos Applet for the Runner Task

programming a function on Desmos to model the distance of a hypothetical runner that would run at the average speeds of the first runner in prespecified time intervals. On the Desmos applet (Figure 1) is a slider to vary the amount of time elapsed since the first runner started running (denoted by the variable, t). As students vary the value of t, the applet shows the value of d, the distance in meters that the first runner has traveled after traveling for t seconds.

Students are asked to program a second runner that runs at the average speed of the first runner over pre-defined time intervals. Initially, students are guided to split the 32-second interval (the total time the first runner took to travel 100 meters) into two equally sized time intervals and then utilize the first runner's average speed in each time interval as their programmed runners' constant speed. It is important to note that in the teaching session preceding this one, students developed a meaning for average rate of change as the constant rate of change that would achieve the same net change in the dependent quantity over the same change in the independent quantity. In this session, one goal was to support students in utilizing the value of a rate of change to determine how the related quantities of distance and time would covary. In this task, students were asked to define a function, b, where $b(x)$ represented the distance traveled by the programmed runner after traveling for x seconds. Some of the syntax needed to define a piece-wise function in Desmos was pre-made for the students to allow them to focus on defining a distance function by reducing the cognitive load (Mayer, 2001). As students progress through the activity, they are asked to program the runner using smaller and smaller time intervals with the eventual goal of making their programmed runner run at the same speed as the original runner (the goal is to form a basis for the idea instantaneous rate of change).

Results

This section discusses how the 2 students in the teaching experiment interacted with building the two-interval portion of the task and the role the Desmos applet played in supporting their ability to represent quantities with appropriate mathematical notation.

Scott

Scott was a first-year computer science major and had no trouble determining the appropriate average speeds for each interval. He used the time slider to determine the distance the first runner traveled by 16 seconds (67.098) and used this in his average rate of change expression, $\frac{67.098}{16}x$. However, for the next time interval, he wrote $\frac{100-67.098}{32-16}(x-16)$ (Figure 2). The interviewer then asked Scott what he attempted to represent with his expression. Scott replied that $\frac{100-67.098}{32-16}$ was the first runner's average speed in the second half of the race, and the expression $(x-16)$ represented an amount of time and together that would give you distance. He clarified that he wrote $(x-16)$ because "we have to worry about this interval and so total

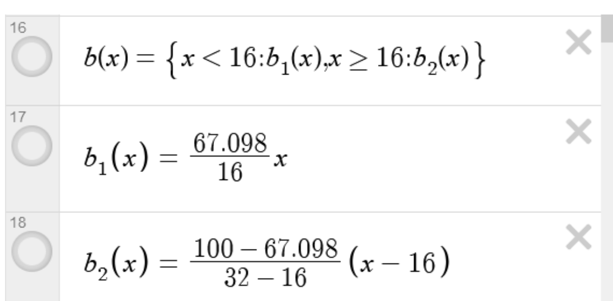

Figure 2: Scott's Initial Attempt at the Runner Task

time, x, that means we are taking away the first interval of time." His explanation suggested that he was using x to represent a number of seconds and that he conceptualized $(x - 16)$ to represent a new quantity, the number of seconds elapsed after the first 16 seconds of the race.

However, what was absent from Scott's second expression was the amount of distance traveled by the runner during the first 16 seconds of the race. The interviewer decided to press play on the animation to demonstrate what Scott's expression represented in this situation. When the time elapsed hit the 16-second mark, his runner jumped back to the starting position. Scott was surprised and stated "the runner jumped back?", and he paused to look at his expression, $\frac{100-67.098}{32-16}(x - 16)$. The interviewer then asked him what he attempted to represent and let the animation rerun. After watching the animation a second time, Scott remarked that he was missing the distance the runner had already traveled and proceeded to add 67.098 to his expression. He stated that he realized that his current expression only represented how his runner's distance would change, instead of his total distance traveled as time elapsed.

Throughout the rest of the teaching session, the Desmos applet played a pivotal role in supporting Scott in connecting his mathematical expressions with how he imagined distance accruing. The Desmos animation provided a visual representation of his mathematical expressions, and when the runner jumped back to the start of the race, Scott was perturbed by what he saw. Since Scott's expectation of what would happen did not align with what he observed, this prompted Scott to reconsider what quantities his symbols represented and then refined his function definition to accurately represent the quantitative relationships he intended. Having the Desmos animation display the runner's movement and distance from the starting line provided Scott with immediate feedback by displaying the quantities being represented by his expression. Seeing his programmed runner make an instantaneous jump back to the starting point prompted Scott to reconsider how to represent the distance he wanted to model (the distance of the programmed runner from the starting line as a function of time elapsed since the start of the race); in particular, he recognized that he needed both an expression to represent how that runner's distance would vary as time varied, $\frac{100-67.098}{32-16}(x - 16)$, and the initial distance already accrued (67.098 meters) after 16 seconds since the start of the race.

Hans

Hans was a first-year aerospace engineering major and in the previous teaching sessions, he struggled with associating quantities with a mathematical representation. Initially, during the 2-interval portion of the task, Hans only wrote the average speed, $\frac{67.098-0}{16-0}$, and then verbalized that he did not know what he needed to write to represent time passing. The interviewer used the time slider to show the first runner moving as time passed and asked Hans about how long the first runner had traveled for. Hans noted that it was whatever the value of the independent variable and then proceeded to "add time" to his previous expression by multiplying x to his $\frac{67.098-0}{16-0}$. He repeated this for the second interval by typing in $\frac{100-67.098}{32-16}x$, but he did not attend to x as representing the total amount of time elapsed; he also failed to include the distance the runner had traveled during the first 16 seconds (Figure 3a).

For the first issue, the interviewer hypothesized that Hans was thinking "rate multiplied with time equals distance" and was not yet distinguishing between total distance and distance varying within an interval (similar with total time and the amount of time elapsed in the interval). Before playing the animation, the interviewer asked Hans to explain what each portion of his expression represented. Hans identified the average speed for each interval and the amount of time elapsed and then noted that all together, it "represented a change in distance" (Figure 3b). Afterwards, the interviewer asked what the value would represent for specific values of time. Hans then noticed that he needed to change his second expression because "we need $(x-16)$ to account for the first 16 seconds of our race" and that"$(x-16)$ is our change in time from 16 seconds." Even though talking through what he was trying to represent helped Hans in making adjustments, he had not yet noticed his other error.

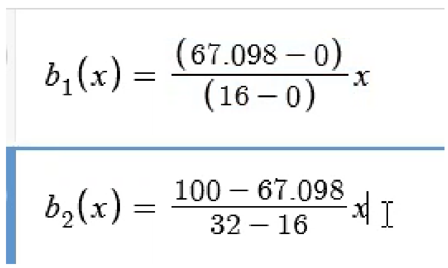

Figure 3a: Hans' mathematical expressions for the average speed runner

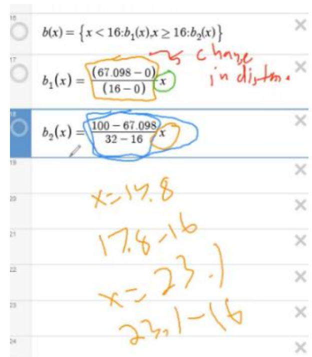

Figure 3b: Hans explaining his symbolization for specific time

To aid Hans in conceptualizing the distance traveled in the first 16 seconds, the interviewer played the animation, and the runner jumped back to the starting line after the runner had traveled for 16 seconds. Unlike Scott, Hans initial reaction was to add the expression $\frac{67.098-0}{16-0}x$ to $\frac{100-67.098}{32-16}x$. He appeared to conceptualize the expression $\frac{67.098-0}{16-0}x$ as a completed distance traveled that could be determined by multiplying a rate times a time (Table 1). The interviewer then prompted Hans to talk about the quantities he conceptualized and attempted to represent [Lines 4-6] while probing the conception of the starting point for a quantity's measurement [Lines 7-9]. Eventually, he noted that we still needed "the first distance" and proceeded to add 67.098 to his expression [Line 11]. Hans' statement that he needed "the first distance" suggested that he shifted to making a distinction between different distances in the situation since he chose to label the distance accrued in the first 16 seconds as the "first distance."

Table 1: Hans working through the Runner Task

1	Hans:	So would I… add the top?
2	Int:	So you want to add this? *Highlights $\frac{67.098-0}{16-0}x$*
3	Hans:	Yeah
4	Int:	Remember x is going to vary (Hans: oh yeah) so what quantity do you want to add?
5		So think about what you're trying to represent. This over here *highlights
6		$\frac{100-67.098}{32-16}(x-16)$*, what is this again?
7	Hans:	Um that's my change in distance.
8	Int:	From what?
9	Hans:	From $x-16$… like change in distance from… from 16 seconds
10	Int:	Okay so what part are we missing?
11	Hans:	The beginning part of the race… the first distance

The Desmos animation supported Hans in developing fluency in applying an average rate of change in a context and connecting his mathematical expressions with the quantities he attempted to represent. Similar to Scott, Hans was perturbed when the runner instantly jumped back to the starting line, which prompted him to reflect on what quantities his symbols represented. Upon seeing the runner jump back to the starting point, Hans realized that he needed the distance traveled by the runner during the first 16 seconds of the race, and this led to a conversation (Table 1) that supported Hans in making adjustments to accurately represent the quantitative relationships he conceptualized. Additionally, the Desmos applet's slider functionality aided Hans in understanding the role of a variable as representing the value of a varying quantity (Figure 3).

Discussion

Both students made similar errors in failing to distinguish between a change in distance versus total distance traveled. Additionally, before playing the animation, neither student indicated any uncertainty about what their mathematical expression would represent in the situation. In their mind, these students thought that what they symbolized would match what they expected (Figure 4a). However, when what they saw in the animation did not align with what they imagined, they experienced a perturbation that allowed the interviewer to facilitate a conversation to aid the students in reasoning about quantities and how they would represent them. This discussion led students to reconsider what they had symbolized and how to make refinements to represent the quantities they had in mind (Figure 4b).

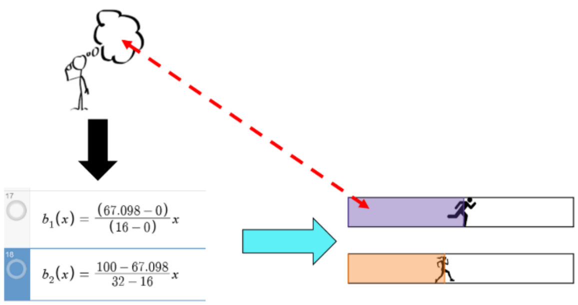

Figure 4a:A student anticipates what they might see

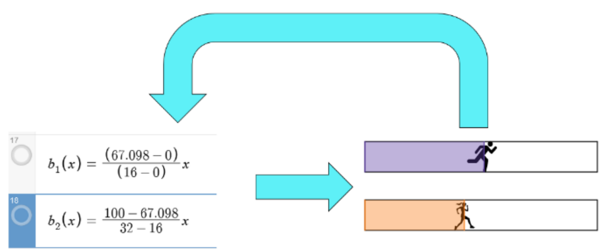

Figure 4b:Students reflect on what they initially symbolized

While the interviewer could have intervened and pointed out the errors in each student's notation, the Desmos animation provided a visualization of what their mathematics would represent and created opportunities for the students to experience perturbations in order to engage in self-reflection. This result suggests that students should have opportunities to explore and engage with what their mathematical symbolization might represent in a particular context.

Limitations and Future Directions

While the results of this study can only be known to be true for these 2 students, the findings of this study provide insight into how we may address the issue of students' difficulties with mathematical notation. Additionally, the tasks in this study utilized kinematic situations, but researchers have indicated that students often struggle to apply their understanding of mathematical ideas outside of kinematic situations (Rasmussen & King, 2000; Marrongelle, 2004).

Future research can further examine the role of applets and animations in a quantitative capacity through classroom studies. Further, more research on the usage of applets and animations should be conducted in non-kinematic situations.

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Nontraditional Pathways to Teaching Postsecondary Math: A Qualitative Study on Educators Between the Boundaries of High School and College

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Past research typically assumes that an instructor is a high school or college instructor, but not both. The mechanisms to obtain teaching credentials for each are also traditionally separate, but some instructors teach at both levels simultaneously or transition between them during their careers. To better understand them, we surveyed instructors with experience in both high school and college math teaching. For our qualitative study, we asked questions centered around the Pedagogical Content Knowledge domains within Mathematical Knowledge for Teaching (Ball et al., 2008; Shulman, 1986). In this paper, we discuss the survey, data collection, coding, and findings on teacher perceptions of their jobs that fall across institutional boundaries.

Keywords: Pedagogical Content Knowledge, high school, college, teacher education, professional development

The University of Tennessee's Master of Mathematics program targets current U.S. high school math teachers interested in transitioning to postsecondary teaching. Some are interested in full-time employment as a lecturer/instructor. Others are interested in continuing their full-time high school careers with additional part-time teaching at a postsecondary institution, including dual-enrollment teaching. In both cases, we consider them as part of a pool of prospective VITAL faculty, which is the MAA's acronym for Visitors, Instructors, Teaching assistants, Adjuncts, and Lecturers, a growing teaching force in the US (Blair et al., 2018; Levy, 2019).

These prospective VITAL faculty are taking a nontraditional route to becoming postsecondary educators. Rather than enrolling in courses that form foundational knowledge for research, they enroll in courses emphasizing applications to teaching. Due to their current high school employment, these educators cannot also work as Graduate Teaching Assistants (GTAs) to gain college teaching experience. Even if they had the time, a traditional GTA role often caters to novice instructors (Speer, et al., 2005) and is arguably inappropriate for these educators who have more experience and knowledge of pedagogy. As a group, they are understudied in research. One reason is that many are associated with two-year colleges, and the literature on two-year math instruction is already sparse in comparison to K-12 or university education (Mesa et al., 2014; Mesa, 2017). A second reason is that dual-enrollment literature tends to focus on policy and students rather than on the instructors' practices (An and Taylor, 2019; Barnett and Stamm, 2010; Gonzalez, 2018; Johnson, 2018; Karp et al., 2004; Mokher and McLendon, 2009).

To better understand and eventually support these educators, our research team started by considering the views of similar faculty. In our qualitative study, we surveyed educators with experience in both high school and college math teaching. Because these educators work between institutional boundaries, they have a unique perspective and can directly compare high school versus college teaching in a way that other math educators cannot. For this first phase, we focus on the research question: What are the perceptions of math teaching for educators with experience in both high school and college?

Framework: Pedagogical Content Knowledge

As identified by Shulman (1986), an instructor should be proficient both in the material they teach and in pedagogy. Although he identified these two types of knowledge for teaching in separation, Shulman claimed that instructors employ both types of knowledge in conjunction and that research should reflect this. Furthermore, he proposed the existence of PCK, the idea that instructors must possess knowledge of the subject they are teaching which other experts of that subject do not require. Stemming from interest sparked by Shulman's work, Ball et al. (2008) formalized an empirically supported and practice-based framework of MKT. Two overarching domains compose the framework: Subject Matter Knowledge and PCK, the latter of which we focus on in our work because some subdomains of Subject Matter Knowledge aren't as clearly defined for secondary and post-secondary teaching (Speer et al., 2015). In addition to Shulman's Knowledge of Content and Curriculum (KCC), Ball et al. identified two empirically differentiable subdomains of PCK: Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT).

In more detail, KCC refers to instructors being familiar with the materials and resources available to them to teach their topic, as well as the contexts when they should be used, such as horizontal/lateral and vertical curriculum knowledge (Shulman, 1986). KCS refers to teachers anticipating students' thoughts and misconceptions (Ball et al., 2008). Lastly, KCT refers to instructors knowing how best to teach their subject with illuminating examples and appropriate scaffolding. We used PCK as our framework to study teachers with experience in both high school and college math teaching; we describe implementation details in the next section.

Methods

Survey and Cognitive Interviews

We developed open-ended survey questions relevant to each subdomain to explore various facets of teaching. We asked two versions of each question: the first being a version about high school math teaching, and the second being about college. We asked participants to describe their teaching practices, a strategy that aligns with Ball et al.'s (2008) practice-based approach.

We first conducted semi-structured cognitive interviews (Karabenick et al., 2007) with five participants using video meetings (Zoom), and we made slight adjustments to our questions for clarity before we collected data from our target population.

As an example of our questions aligning to KCT, we asked instructors how they choose class activities to help their **high school** students learn, and then we followed with a question about how they choose class activities to help their **college** students learn. We developed each question with one specific subdomain of PCK in mind (see Table 1), but during the cognitive interviews, we encountered a type of "boundary problem" like what was originally described by Ball et al. (2008) as some of our interviewees blurred the line between subdomains of PCK. As Ball et al. had originally admitted, "It is not always easy to discern where one of our categories divides from the next, and this affects the precision (or lack thereof) of our definitions" (p. 403). Thus, rather than attempting a perfect one-to-one mapping of our survey questions with subdomains of PCK, we take the stance that this boundary problem is inherent to the subject.

We further recognize that we, ourselves, are components to our instruments in our research (Maxwell, 2012). We acknowledge that our research team consists of individuals who have been in or are currently in a traditional research-focused math graduate program and have thereby not taken the same career path as the individuals in our study. One of our research team members has completed a teacher education program and has taught in a secondary classroom in the UK (instead of in the US). Four of us were GTAs for most of the first phase of this project. Currently, two of us are full-time lecturers at a large, public, and research-focused university.

Table 1. A classification of survey questions according to the subdomains of pedagogical content knowledge.

Question Pairs	Intended PCK Subdomain
Q4/5: Describe how you choose class activities to help your high school/college students learn.	KCT
Q6/7: Describe how you make decisions about which topics to teach in a high school/college class session.	KCC
Q8/9: Do you use technology to support high school/college student understanding of particular topics? If so, how?	KCT
Q10/11: Describe how you scaffold content for a class of high school/college students with a range of backgrounds.	KCS
Q12/13: Describe how you support high school/college students in communicating and justifying their mathematical thinking.	KCS
Q14/15: What do you think is the relationship between high school/college math courses and the rest of a student's education?	KCC

Data Collection

We sent the online qualitative survey instrument via relevant email and professional society message boards. The first question asked participants if they had taught both high school and college math courses in the U.S. Participants in the target population self-reported their experience in further detail with information about the types of institutions they worked at, courses they taught, and years of experience. In this first dataset, we have 26 responses. If we think of the "traditional" pathway to teaching college math as starting from graduate math coursework and ending in a tenured professor role, then our participants presented a range of nontraditional pathways to college teaching. For instance, Participant 5 started as a tutor and adjunct at a community college, moved to a full-time two-year college role, went back to math graduate school, and then ended up teaching at a private high school while continuing to adjunct at a college. Most of the participants started in high school and added on college teaching later, but we also have people like Participant 11 who started college teaching as a GTA in a math graduate program, taught full-time at a college, and then switched to full-time high school teaching. Most of our participants did not disclose their tenure-track or non-tenure-track status, but many discussed experiences in temporary (n = 11) or adjunct (n = 7) college teaching roles, and the average number of years of teaching is 17 (including part-time and full-time teaching).

Coding and Analysis

We analyzed our data with thematic analysis (Braun and Clarke, 2006). Our initial phase was familiarizing ourselves with the data by reviewing it in totality, which included typed responses and transcriptions from audio recordings—our survey was developed on the Phonic platform, which allowed participants to choose between typing written responses and recording audio responses (Phonic). We employed an inductive approach to search for potential patterns and to compile our initial codebook. Next, we organized an updated, condensed codebook with clearer definitions for a total of sixteen codes. We re-coded each question independently with two researchers per question. Finally, we computed percent agreement for interrater reliability, discussed discrepancies, and re-coded one last time.

Our codes emerged from our reading of the data, but we also find clear relationships between our codes and facets of PCK. For instance, we have three codes on the reasons behind an instructor's usage of technology, which relates to both KCS and KCT. Participant answers were typically coded as using Technology for Scaffolding, using Technology for Understanding (including visualization of content), or thinking that Technology is Distracting. We primarily used these codes on our pair of questions about instructor use of technology. Interestingly, we also found that these codes emerged in other pairs of questions. For instance, when we asked participants about choosing class activities, Participant 19 reported that they "try to choose activities that have a strong visual representation and, when possible, a dynamic piece of technology that they can engage with to explore mathematical relationships" (coded as Technology for Understanding). On the other hand, some participants merely listed technologies that they use when we explicitly asked about technology usage. For example, Participant 12 simply said, "Yes. Graphing calculators and Geogebra." Their answer involves technology but does not present any information past that, so we left this response uncoded.

As with the technology codes, we had other codes that we expected to see in certain pairs of questions. In one pair of questions, we asked participants to "describe how you support high school/college students in communicating and justifying their mathematical thinking." Rather than focusing on the frequencies of our Justification code here, we considered the times that it showed up in other questions unprompted. We focus on the details of the unprompted codes in the next section.

Findings

Instructors Have Student-Centered Intentions

Student Interests. One of our most common codes involves descriptions of practices that cater to Student Interests (including career goals, motivating examples, applications, other courses, and explicit mention of student engagement), and we saw this code across all questions in the survey. The Student Interests code emerged most frequently in Q14/15 when we asked, "What do you think is the relationship between high school/college math courses and the rest of a student's education?" Participant 1 indicated that they adapt their college teaching practices based on student career goals. For non-STEM students, they said, "in the end what I want the student to remember 5 years down the road is not the properties of logarithms but rather how I might engage with critical thinking and reasoning to work my way through solving a problem that arises." For STEM students, this instructor said, "I need my STEM students to understand how they are going to be using the math we are currently doing later in their studies and in their possible future careers. They still need those critical thinking and problem-solving skills, BUT they also need to [be] able to think more mathematically and quantitatively." When asked about topics in specific college class sessions, Participant 1 consistently centered on Student Interests as they stated, "I generally pick topics that I know students will need for subsequent courses or that I feel apply the mathematics in creative ways." Many other instructors were also attentive to student interests across questions, and we applied this code seventy-six times across all twelve content questions in our survey, which was one of the most frequent of the codes we used.

Interestingly, when comparing pairs of questions, we saw more mention of Student Interests for high school students than for college students in Q4/5 and Q6/7. Participant 15 is one of the instructors who mentioned student interests in their response about high school but not in their response about college. For high school teaching, they said that they must consider topics on the standardized exam "but that also help students connect math to the real world." This differs from their practice when teaching college-level math, for which they said, "I look for activities that students can start in class but finish at home, since many students work at different speeds and college class time is quite limited. I favor online activities that can be automatically graded." In college, the time pressure to cover content is so great that this instructor changed their teaching practice, and they opted for the ease of automatically graded problem sets.

Social and Emotional Needs. Another relevant code is that of the instructor attention to students' Social and Emotional Needs (including connections to peers, collaborative skills, attitudes, confidence, and comfort). We also saw this code emerge in every pair of questions that we asked, though it showed up more frequently in our questions about student justification and communication (Q12/13). When asked to "describe how you support high school students in communicating and justifying their mathematical thinking," Participant 14 simply described what the peer-to-peer interaction is in their class: "Students explain to each other how they got their answers. Students think about other approaches to a problem and try to explain other's methods." Several other participants also utilized group work and discussion in "think-pair-share activities" (Participant 17) or by having students "explain their work to each other" (Participant 18). For college teaching, multiple instructors said that their classroom choices were the "same" (Participant 17) or "identical to high school" (Participant 24). Furthermore, it was also important to some instructors to "build enough trust and rapport between [the teacher] and among the students to allow students to contribute imprecise or tentative ideas" (Participant 8). Participant 11 also emphasized the creation of a "comfortable" classroom environment. Overall, we found that our survey respondents believed that students' communication and justification skills were inextricably linked to their social and emotional needs.

Instructors are Influenced by Test and Time Pressures

Test Pressures. As expected, we saw our Test Pressures code appear in all questions, although it did not explicitly show up as frequently as other codes. In Q14, many teachers discussed foundational knowledge, but Participant 25 talked about the practical side of college applications by saying, "As long as math ability is tested on [the] SAT and a basis for college acceptances, it is probably the most critical course selection in a high school student's transcript. It can affect science classes and the ability to schedule any other honors level courses." Participant 22 also reported test pressures but in reference to Advanced Placement (AP) tests. In Q6, they reported, "For all but one year teaching high school, the school I taught at followed the AP curriculum (for better or worse). So the topics for the course were fairly well set in advance."

For college teaching, local expectations for exams also influence instructor choices. In the case of Participant 17, test pressure came up in the question about utilizing technology in the classroom, and they said, "I don't use technology as much in my college classes because they won't be allowed to use the technology during the exams, which make up most of their grade." For their high school teaching, this instructor used graphing calculators and Desmos, but they change their teaching practice in college when it didn't align with the expectations for the high-stakes exams. Participant 8 had a similar sentiment and said their department also had a "no-calculator policy"—since they couldn't "assume or guarantee access to graphing technologies," they would "rarely engage students in using technology to explore or understand ideas."

Time Pressure. For college teaching, our participants also felt that the lack of class time (when compared to high school) affected their classroom decisions. Notably, our code of Time came up more frequently in the question about choosing class activities for college classes when compared to the parallel high school question. As we saw earlier, Participant 15 was attuned to Student Interests but was pressured by the lack of time and so they chose online activities with automated grading. Participant 20 also thought college class sessions were too short, and although they reported a mix of lecture and group activities in their high school classes, they said, “Due to the limit of a 55-minute [college] class, it is mostly a lecture style class,” a choice echoed by Participant 18. Additionally, Participant 22 described how they heavily utilized Inquiry-Based Learning (IBL) in their high school teaching and tried to incorporate it in college, but “this did not work out—the students were not used to this type of learning, and because I had to get students ready for the next calculus course, I felt I didn’t have the time to use IBL methods.” Even for instructors who used more active learning methods in their high school teaching, some felt so much time pressure in college teaching that they used lectures more or active learning methods less.

Instructors Provide Commentary on Their Working Environment

Predetermined Content. One of our codes is about Predetermined Content, meaning the content was determined by someone other than the instructor themselves. We expected to see this when asking teachers about high school course content decisions as we assumed that state standards would frequently be discussed. Unexpectedly, we saw that this code came up more in the college teaching answers than for the high school ones—in fact, 64% of the responses for Q7 were coded as Predetermined compared to 30% of the responses for Q6. Perhaps Participant 7 explained this by saying, “My understanding is there is a lot more leeway with the topics that are taught at the college level. However, I believe there should be a bit of a consensus if you are teaching a class that is a prerequisite for another class.” We also expected there to be more flexibility in college courses overall, but our participants reported that they primarily taught introductory non-major courses, and so they were more likely to be working with a course coordinator or a preset list of topics. For Q7, Participant 9 had a representative answer: “The topics to teach are again completely set by the [department] syllabus—I don’t really have a choice on this.” When reflecting on their lack of autonomy, many instructors maintained a matter-of-fact tone without a positive or negative connotation; those like Participant 20 simply stated, “I am told which topics will be covered in the course.” However, in both descriptions of high school and college teaching, some instructors had negative associations with their lack of choice. For example, Participant 21 said, “Unfortunately, the district curriculum prescribes my topics and associated pacing.” When talking about their college teaching, the same instructor said, “Again, topics are already decided. In general, I will avoid teaching content that I know is not on the test. For example, even though I may find some underlying concepts in the complex numbers rather interesting, I avoid teaching them because students generally only need to know the procedure.” Due to predetermined content that was procedurally focused and test-driven, this instructor actively avoided teaching concepts and topics that they find interesting.

Commentary on the System. Another notable code is the one regarding Commentary on the System. This is a code that cuts across and outside multiple dimensions of PCK, but we saw it appear in every question of our survey. In most cases, teachers were providing commentary on barriers. Some of these barriers related to prerequisite student knowledge, such as when Participant 9 talked about how they must “spend a long time ‘undoing’ the unintended consequences of prior instruction. Students have typically automated all kinds of procedures without fully understanding when or why they should be used.” Other instructors, like Participant 4, commented on the lack of technological resources to support student learning: “It was a poor school district. The extent of technology in my classroom was a very old computer, an overhead projector, and 5 TI-85 calculators," a complaint that other instructors, such as Participants 25 and 26, mentioned as well.

Finally, multiple instructors commented on math classes themselves being barriers to students. On Q14, Participant 4 said, "Unfortunately, I think high school math serves as a sort of gatekeeper for college. Students who come to the university with poor math understanding get stuck taking remedial courses - often more than once, which can lead to setting back graduation or even dropping out." Participant 8 described the ideal role of math versus the perceived reality:

> Ideally, high school courses would help students construct foundational knowledge and develop skills in reasoning and problem solving that would support further study. In practice, high school math courses often lead to well-practiced but poorly understood procedures (and sometimes poorly recalled procedures) for technical tasks, and lead students not to trust their own reasoning and not to trust instructors to treat that reasoning as valuable or respected.

For Q15, Participant 26 added that math courses were "just a hoop for [students] to get through." Lastly, Participant 17 warned, "For students who struggle in these gateway courses, it changes the trajectory of their lives."

Conclusion

As far as the research team knows, this dataset and analysis provide the first direct comparison between the teaching practices for high school and college math classes in the US by instructors with personal classroom experience in both. We primarily examined these with a lens focused on subdomains of PCK, and we began to answer the question: What are the perceptions of math teaching for educators with experience in both high school and college?

Currently, we are implementing initial findings into our capstone course for the master's program to provide relevant perspectives from teachers in a more similar career trajectory to those of our prospective VITAL faculty. Additionally, we will next consider the specific views of dual-enrollment instructors, such as full-time high school teachers with an adjunct appointment at a college, full-time college instructors who teach high school students, or other arrangements that complicate the traditional secondary and post-secondary divisions.

Although readers of this article are more likely to be engaged in research and less likely to have taken the same career path as those in our study, we suggest that their responses provide a means of reflection for our own practices and policies as college educators, course coordinators, administrators, and leaders of professional development sessions. These educators who work between institutional boundaries have a unique means of direct comparison for high school and college math teaching. Thus, we end with a series of questions prompted by our engagement with our dataset: When faced with pressures and local policies, why do some instructors change their instructional practices from active learning to lecture while others maintain their course? How can we alleviate those pressures or modify policies to better leverage these instructors' existing PCK? What are the reasons that instructors think their own math courses are merely gatekeepers, and how can we make changes to better align the reality that these instructors describe to the ideals of their student-centered practices?

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Values, Methods, and Facts: How Do Math and Science Align and Differ from the Perspectives of Mathematicians and Biologists?

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Despite the interrelated nature of math and science, limited work has characterized mathematicians' and scientists' beliefs about each other's disciplines or courses. In this study, we use nine interviews with mathematicians and biologists to characterize the interrelationship of community values, methods, and accepted facts in math and science and their relationship to introductory courses. Results include similar descriptions of biology and of what scientists do by all participants, but differing characterizations of what math is and what mathematicians do and claims that math courses do not teach students what they need. Implications include a need for the mathematical community to better align introductory course material with their values so that people outside the community better understand what mathematicians value and do.

Keywords: Interdisciplinary Education Research, Nature of Math/Science, Disciplinary Norms

Cross-disciplinary research connecting math and science education is a relatively new but burgeoning area of work. Much of this work has emphasized content connections, including how specific math-based content is learned and used in biology (e.g., Robeva et al., 2010; Williams et al., 2021), chemistry (e.g., Bergman & French, 2019; Jones, 2019), and physics (e.g., Jones, 2015; Schermerhorn et al., 2022; Serbin et al., 2020; Serbin & Wawro, 2022). Nevertheless, much work remains to be done at the interface of math and science education, especially as it relates to beliefs about teaching and the nature of math and science. In particular, we address the following research questions: (1) How do mathematicians and biologists characterize the nature of math, science, and biology? (2) What, if any, connection is there between how mathematicians and biologists view and want students to take away from their own and each other's disciplines?

Literature Review and Theoretical Perspective

Extensive work has characterized beliefs in math and science separately. Math educators have studied teachers' beliefs about math, teaching, and learning in K-12 and university contexts (e.g., Melhuish et al., 2022; Philipp, 2007). Much of the university-level work has focused on beliefs underpinning pedagogical approaches, such as influences on pedagogy (e.g., Johnson et al., 2018) and pedagogy in active-learning and lecture-based classrooms (e.g., Fukawa-Connelly, 2012; Woods & Weber, 2020). Moreover, work has examined perceptions of the nature of math (e.g., Ernest, 1989; Rupnow, 2021), including views of math as a disconnected set of tools, something discovered, or something created by mathematicians. Characterizations of science beliefs have focused on teaching influences (e.g., Al-Amoush et al., 2014; Rupnow et al., 2020); certainty of knowledge in science (Urhahne et al., 2011; whether views of science disciplines differ (Topcu, 2013); and descriptions of the nature of science (Southerland et al., 2003).

Studies attending to university math and science education simultaneously have been more limited. Apkarian and colleagues (2021) characterized instructional practices in math, chemistry, and physics, with special attention to active learning uptake. Reinholz and colleagues (2019) examined disciplinary cultures in biology, math, chemistry, and physics based on data from discipline-based education researchers at a cross-disciplinary education conference. From their work, they concluded that while STEM is often treated as one entity, meaningful differences exist between disciplinary cultures. Moreover, to our knowledge, research has not examined mathematicians' and scientists' beliefs about each other's courses and views of their disciplines, despite the impact understanding each other's perspectives might have on determining programs of study, who is best positioned to teach courses, and appropriate content in service courses.

Based on this background, we adopted Laudan's (1984) Triadic Network of Justification to frame our examination of faculty beliefs and their relation to introductory courses. The Triadic Network of Justification is grounded in philosophy of science and examines the interrelationship of axiology, methodology, and factual claims within a field's decision-making. In particular, a field's axiology (espoused values and principles) influences what counts as an acceptable methodology just as the feasibility of methods constrain what can be valued by the community. Similarly, use of acceptable methods impacts whether an assertion can be stated as fact while accepted facts within a community impact the methods that will be employed for exploring new areas. Finally, what is accepted as fact impacts the values of the community just as the values of the community impact what is acceptable as a fact. Laudan's work has been employed in math education research to examine the values (axiology) upheld through proof norms (methodology) (Dawkins & Weber, 2017). Here we will examine the values, methods, and accepted facts as taught in introductory courses in math and science and contrast their alignment in the two fields.

Methods

This research effort represents an interdisciplinary collaboration of math, statistics, and biology education researchers, and we believe our respective positionalities inform and enrich the research. Two members of the team are Assistant professors and two are graduate students. One member has extensive experience doing qualitative coding of interviews, and the other three have an intermediate level of experience. Our varied statuses and prior experiences impacted how we interpreted the data, and we worked to value and incorporate everyone's insights.

We audio-recorded interviews with five mathematicians and four biologists (nine total) at two R2 universities in the same state. One researcher, with a math education background, interviewed all participants for 60-90 minutes over Zoom or in-person. Interview questions focused on the nature of math, science, and biology; how these disciplines are learned; and how introductory courses in these disciplines are taught. Gender-neutral pseudonyms were assigned such that math faculty were given names starting with M and biology faculty with B.

The interviews were iteratively analyzed in accordance with thematic analysis (Braun & Clarke, 2006). First, two transcripts were open-coded by two authors who attended to prior work characterizing math and science but did not enter analysis with particular codes in mind. Those authors condensed and organized the open-codes into focused codes and applied them to the same transcripts. After refining, all authors independently coded the remaining transcripts, with at least two authors coding, discussing, and reaching consensus on each. This led to minor coding additions and refinements, and revisions were applied to earlier transcripts. Finally, all authors discussed clarity and themes, which led to adoption of the theoretical perspective. Final codes were aligned with the theoretical perspective and applied throughout the transcripts.

Results

Axiology: Cultural Values in Math and Science

Math knowledge, once gained, is definite and science is continually evolving. Participants noted many distinctions and similarities between math and science as fields. One distinction agreed on by mathematicians and biologists was that math knowledge, once gained, is definite and science is continually evolving. For example, Max said, regarding math and biology:

> It's my impression that biology as a discipline evolves at a so much faster pace than mathematics does. Mathematics absolutely expects that the calculus that you learn in calculus...is valid, will be valid, is state of the art....In biology...what was thought to be state of the art when you came in as a freshman might not be...by the time you leave.

Further, the idea that math is more certain than science, which thrives on constant evolution, was presented by many participants. Blake claimed, "If you put 10 mathematicians looking at the same data, all 10 should come up with the same answer, whereas in physics, chemistry, and biology I suspect that you'd get a lot more variation." Not only does this biologist see math results as more definite than results in science, but math results are also more widely accepted.

Math and Science are siloed/are separated by jargon. Several participants suggested silos or disconnections between disciplines might exist due to the need to specialize within one's own discipline. Brooks noted divisions even within disciplines: "even within biology, the medical biologists and the environmental biologists are completely different, and they don't communicate with each other." Further, several participants suggested differing cultures can create divisions between disciplines. Max noted the impact of being part of a culture on how one sees the world:

> It instills in one a discipline, whether it's history or math, or physics, or accounting, or whatever. Everybody who has been through a discipline has acquired a certain lens with which they see the world, and that lens is not unique to that individual...you share with others this way of thinking and seeing things that you, you can recognize as part of a community of thought...Each individual may bring their own little nuance to it. But...we recognize mathematics as a community of thought, and people in...a shared language.

Knowing the in-group approaches and language are part of becoming a member of a discipline, whereas not knowing them could make it difficult to communicate across disciplines.

Math is what mathematicians do/anyone can do science. Participants suggested that math is characterized by mathematicians' actions, but anyone can do science. Max, Morgan, and Marley claimed "math is what mathematicians do" when initially defining math. In contrast to math's community-membership focus, many participants reversed this in science, where anyone who acts like a scientist is part of the scientific community. Brooks stated:

> I think science is any investigation of something unknown. It could be anything we typically think of as physical sciences...but I also think it can just be trying to figure out something that you don't know in any sense. I know Adam Savage from Mythbusters says that everybody is a scientist, and I kind of agree with that mentality.

Similarly, Bailey noted "the scientist for me is someone that at least follow[s] scientific reasoning." These comments suggest that someone who engages in scientific processes is a scientist, whereas who does the activity determines whether something is math.

Science is a cross-disciplinary search for knowledge. Both math and biology participants emphasized the value and necessity of cross-disciplinary knowledge and interaction. Biologists discussed how different disciplines within science can work together to make each stronger, such as Blake's comment on students seeing methods in one area of science applied elsewhere:

> I think because of teaching and reiterating that scientific method and showing, for example, a physics student that the same scientific method you use over in physics can be applied in biology is really important for some of that connectivity....Anytime a biology student can broaden their knowledge base in other STEM fields, the more well equipped they are to ask less biased questions within biology.

Math is a tool. Relatedly, there was consensus that math acts as a necessary tool within science. All biology participants said math was an important tool within their discipline, many specifically using the word "tool". For instance, Blair said, "[Math is] providing some tools of how you describe biological processes quantitatively, in terms of models." Others, such as Blake, went further and did not see any applications for math beyond use in other fields:

> That sounds harsh, but I think that is how I think of [math]. So, it doesn't really have applicability on its own other than it's a tool. But it's like *the* tool all, all right to be used in all of these other fields....[italics are participant's emphasis]

Note this biologist sees applications of math in concert with other areas, just not on its own. Mathematicians, such as Mo, also referred to math's utility in problem-solving: "You use the mathematics to understand different parts of the world, different sciences, and also a lot of things that are not typically considered science, but you use it to analyze things."

Biology is the study of life. All participants characterized biology as the study of life. Max expanded that definition to include "the processes that make up life". Marley further included what biologists do, incorporating topics adjacent to life: "I mean, a biologist is not going to study something that's inanimate. It's going to be studying life or things that are very close to life, like viruses. They're not really alive, but they have an impact on life." Biologists described biology as the study of living things or organisms. Three of the four biologists provided an expanded definition of biology that incorporated biological systems, such as Bailey:

> So biology is a study of living things. And it can be as an organismal... for example, where the organism lives, their habitat. Then you zoom in animals by themselves, the behavior, then up to how a cell work[s]. And there's connection between all these.

While biologists were more precise and provided more nuance in their responses, mathematicians and biologists largely agreed on what biology is about.

Math is about patterns or numbers. Biologists largely described math as the study of numbers, such as Blake: "So thinking about how numbers may…relate to one another in new ways...coming up with new sorts of theories or laws or rules as to how numerical relationships correspond to one another or relate to one another." In contrast, mathematicians mostly focused on patterns and relationships with some references to numbers, such as Miller:

> If we have to add restaurant bill or divide restaurant bill, [my friends] say, "Oh, you're the mathematician, you do it," the joke is. And I say, "Guys, that's not mathematics. That's arithmetic." So, the definition of mathematics is not addition, subtraction. The definition of mathematics is to study how mathematical objects relate to each other, the rate of change, the tools and methods that are required.

While math includes numbers, Miller suggests math is about relationships among objects.

Methodology: Ways of Doing and Learning Math and Science

Science is done by the scientific method. All mathematicians and biologists described science as being done and/or connected by a systematic approach, especially the scientific method. For instance, Blake referred to the scientific method numerous times, including: "I guess [science] is a way of studying, a way of understanding…. when I think of science, I think of the scientific method." Mathematicians also focused on the systematic approach to data collection and analysis when defining science, such as Marley:

> Sort of the study of nature, but it's a little bit more than that. So if you allow me to give a definition that's as broad as my definition of mathematics, I think it's just...It's observation and analysis....There has to be observation. There has to be analysis to search for patterns.

While particulars varied, references to the scientific method or aspects of it, including hypotheses, data collection, analysis, and repetition consistently appeared across participants.

Math is done via proof or unknown. Mathematicians often characterized doing mathematics in terms of proof and logical deduction. For instance, Morgan observed:

> What do mathematicians look at? Well, they look at things that interest them. And it's usually not just, "Hey, where can I get—starting with these assumptions, where can I get just using rules of inference?" It's not so interesting work. You can take Peano arithmetic and say, "Okay. Well, what follows from that?" Well, no, look at things that interest us.

While mathematicians are expected to demonstrate results through a sequence of logical deductions, interests are also expected to play a role in mathematicians' approaches to research. In contrast, biologists expressed uncertainty about what mathematicians do. For instance, Bailey tentatively suggested mathematicians use the scientific method to focus on calculations:

> Maybe, I'm wrong, but I think you have a question. You want to demonstrate something. You have a hypothesis, so you will write down on top of your page what you—or, I don't know how you do it. And then you start to make equation and see if you can solve it.

Blake professed even broader uncertainty:

> Any time I've talked to someone who does theoretical mathematics it is, it is at its intro level of being explained to me so far over my head and so jargon filled that I can't pull from it the applicable aspects to my field. So I can't really explain to you what theoretical mathematics is other than it's, you know, developing theory in math.

Of note, while everyone seemed reasonably confident in characterizing what scientists do, biologists expressed uncertainty and/or differing characterizations of math than mathematicians.

How Learning is Approached. In addition to characterizations of how researchers do math and science, we also note approaches to learning math and science, as they could be considered ways to show appropriate methods for engagement in those fields. The most common description of how people learn math was through exposure and repetition. For example, Mo said, "A little bit like you acquire speaking skills. You see it used and mimic it and learn some things on your own." In contrast, the participants described how biology is learned through hands-on activity, including experiencing the natural world and research experiences. Brooks claimed:

> I think a lot of people learn biology growing up through experience...learning what animals are…these sorts of things people often learn before they've ever taken a biology course. And getting out into nature is often a way that many people get into environmental fields because they always loved going out to nature, going on hikes, watching birds...And then all of the human health pathways are a little bit more obvious because if you experience your own healthcare, you see how that works.

Further, the participants described how learning biology is done through research experiences. For example, Max noted students can do research even as undergraduates: "Biology, I think it's really possible for a undergraduate, you know, Junior, to be doing a real research project."

Factual Claims: Key Facts within Math and Science Taught in Introductory Courses

While accepted facts in math would include proven theorems or formulas and in biology might include widely accepted theories like cell theory (all organisms are made of cells), here we emphasize instead content students are likely to take away from their introductory courses.

Content in biology introductory courses. All the biologists made a distinction between what biology is taught at the introductory level for students majoring in biology (i.e., "majors") and those not (i.e., "non-majors" such as humanities majors, business majors). For example, Blake described how both versions of the introductory courses cover similar topics but emphasized forming a solid foundation of concepts and skills for majors to build on:

> I think the goal for non-majors is to provide them with background content on what biology is, so that it is the study of living organisms to provide some background content on the breadth of organisms that exist…and then do some introductory lessons on the scientific method, and some of those patterns and processes that we get into in later biology..... For our majors...teaching the same stuff that we're doing with non-majors but doing it at, with the expectation that they are going to build on that knowledge. So really driving home the scientific process, which they're going to use throughout the rest of their, their major, really driving home some of the core concepts in biology.

Of note, biologists claimed biology majors' first courses in college are meant to provide opportunities to engage with biology in the same manner as they would in subsequent courses, and non-majors' courses provide a similar experience, just with less detail.

Content in math introductory courses. In contrast, multiple math and biology participants noted math introductory courses tend to teach separated procedures and content that mismatches what is needed. For example, Brooks described the non-applicability of differential equations:

> Differential Equations, the only thing I took out of that class was learning how to do the math if I was filling up a bathtub that was emptying at the same time, right? That was all I saw in Differential Equations, so I think I do inherently just think of things in that sense and where is this coming from? Where's it useful for?

They later noted they appreciate math is "something behind the scenes… it's just a part of the whole system" but math is not taught in such an integrated way, instead "taught as a separate thing". The mathematicians agreed math is taught in a way that does not highlight utility, stating "for most people mathematics means algebra and algebra means meaningless symbol manipulation" (Max) and "they just sort of memorize algorithms and that's useless" (Morgan).

To remedy these issues with how math is taught, the participants suggested that introductory math should teach particular content that aligns with other disciplines and mathematical thinking. For example, when asked what biology majors should take away from introductory math courses, Marley proposed a calculus/ordinary differential equations (ODE) series with biology applications and gave an example about the rate of metabolizing tea:

> Students should take sort of a Calculus I and a Calculus II in preparation for biology, but it should be a biology-based Calculus I, Calculus II.…it's about related rates, rates of change.…I've got my cup of tea here, and I've learned how the body eliminates drugs from the bloodstream.…It's a very simple ODE. And now I kind of now understand why is it I drink my tea, I drink tea in the morning, and then I need more tea later, next day, why? Well, because it's natural that there should be some sort of an exponential.

In addition to applications, the participants also said that introductory math courses should teach connected mathematical thinking. For instance, Mo claimed:

> We want to give them an idea of where those formulas come from because I think that gives a better understanding of how everything fits together. So it's not just covering the topics that they need, but it is also making sure that it's not just 'formula is given', but increase their understanding of the mathematics behind the things.

Together, participants claimed math courses should be specific to the student audience and focus on why content works.

Discussion

This study aimed to examine views of math, science, and biology by mathematicians and biologists as well as consider how well aligned (or not) the values, approaches to research, and material taught in introductory courses are in math and biology. To the first point, there was general agreement that biology is focused on the study of life, science uses the scientific method, and math is a tool for solving problems. However, mathematicians characterized math in terms of who does it (mathematicians) and in terms of studying patterns among (varied) objects as well as treating proof as central to activity, whereas biologists largely focused on numbers or expressed uncertainty. All participants also largely agreed that introductory math courses often teach disconnected procedures and not what is needed for other disciplines. They also largely agreed that the scientific method and content about living things were taught in introductory biology courses, though biologists elaborated on depth differences for majors and non-majors.

To the second aim and returning to our theoretical perspective (Laudan, 1984), we see relatively close alignment when we consider the axiology, methodology, and factual claims (as represented in courses) in biology as portrayed by mathematicians and biologists. While some values are in tension (i.e., science as a cross-disciplinary search for knowledge, science separated by jargon/in silos), we can see that both values are attended to and incorporated in the scientific method. In particular, hypotheses are free to change as science evolves and the questions asked are free to be informed by broad research teams, though individuals may have expertise in only portions of the question being asked. The emphasis on teaching the scientific method as well as all participants' certainty of what biology is seems to suggest relatively clear alignment between methods taught and accepted facts in biology. Finally, the emphasis on methodology rather than particular facts reveals connections between accepted facts and the value that science changes.

In contrast, math's axiology, methodology, and factual claims represented through course choices are harder to connect. If we attend only to mathematicians' views, the values of math being definite and of math being about relationships among objects harmonize with the methodology of deductive reasoning in an axiomatic system. However, teaching disconnected procedures does not create a foundation for proof-production. Instead, they at best relate to math's role as a (potentially not understood) tool for doing science. Biologists seemed to view math as a tool focused on numbers but had limited ideas of what mathematicians do beyond vague references to "math theory" and computation. Because math methodology is unknown to them, they cannot connect it to values or facts. However, the focus on math as a tool related to numbers potentially connects to the individual procedures or particular content taught in courses. Thus, while some aspects are connected in each viewpoint on math, there are also reasons for biologists to wonder what mathematicians do and for mathematicians to wonder why introductory courses are taught as they are.

These disconnections in math's values and introductory courses highlight problems for the math and math education communities to address. It seems that math courses are trying to serve populations that do not fully agree on what math is, thus serving no one well. If other members of the STEM community do not incorporate mathematicians' definition of math into their own definition or know "what mathematicians do", it is not reasonable to expect those in less contact with math to know what math is about or what mathematicians do either. One way to address this is to encourage mathematicians to clearly articulate what math is about to people outside the mathematical community so that their definition becomes more accepted. Another route might be for math departments to co-teach distinct courses with and for different STEM disciplines so that "applications" are relevant to all STEM majors and align with their needs, while math majors separately have opportunities to learn more about the mathematical community and know what their final courses will be about before it is too late to change majors.

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Understanding Instructors' Beliefs and Obligations within Cultures of Exclusion

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We present a study of how five early-career college math instructors using active learning describe their role as instructors and situate these descriptions in two existing theoretical frameworks: Herbst and Chazan's theory of practical rationality and Louie's culture of exclusion. This overlayed theoretical perspective, grounded in our case studies, points to levers for faculty professional development and instructional change.

Keywords: Beliefs, Equity, Active Learning, Instructor Development, Professional Obligations

While the general pedagogical category of "active learning" in college mathematics classrooms is widely regarded to be better for student learning than traditional lecture (Conference Board of the Mathematical Sciences, 2016; Freeman et al., 2014; Theobald et al., 2020), many questions remain in how to best support college math instructors to adopt such practices, and carry them out to most successfully and equitably support student learning. We take it as given that instructor practices and beliefs have important and nuanced interactions in their effects on student learning. Thus, learning environments are improved by attending to both instructor beliefs and practices. The goal of this particular study is to better understand some of the beliefs held by instructors using active learning (in this case, beliefs related to how they view their role as an instructor), while recognizing that these beliefs are pieces of a more complex instructor ecosystem. Our research questions are the following:

RQ1: How do early-career math instructors using active learning describe their role as instructors?

RQ2: What professional obligations and aspects of the culture of exclusion are present in these descriptions?

Literature Review

Numerous studies across STEM fields have investigated instructor beliefs (Czajka & McConnell, 2019; Luft & Roehrig, 2007; Moore et al., 2015). Scholars working with post-secondary mathematics instructors have attempted to categorize instructor beliefs while establishing connections to their teaching practice, particularly those related to active learning (Päuler-Kuppinger & Jucks, 2017; Raychaudhuri & Hsu, 2012; Vishnubhotla et al., 2022). Päuler-Kuppinger and Jucks (2017) found that scholars in "hard disciplines," like mathematics, are more likely to have a static understanding of knowledge that corresponds with content- or teacher-centered teaching methods. The findings of Raychaudhuri and Hsu (2012) complicate those of Päuler-Kuppinger and Jucks (2017) in that instructors held ontological views of mathematics as a connected and deep discipline that conflicted with their teacher-centered, behaviorist pedagogical views. Vishnubhotla and colleagues (2022) found that in investigating instructors' practices when utilizing inquiry-based learning (IBL) approaches, instructors who believed that lecture positively contributed to learning were more likely to emphasize lecture in their IBL instruction. These findings reflect earlier claims regarding the relationship between beliefs and practice in mathematics teaching more broadly.

Previous findings, such as Leatham (2006), pushed for researchers to study beliefs in ways that aim to build connections between beliefs and view them as sensible systems. Ernest's earlier work (1989) enhances this notion as he asserts that connections or inconsistencies between beliefs and practice can be explained not only by how the instructor relates their beliefs to one another, but also the amount of awareness an instructor has of their beliefs and the institutional systems that surround the instructor. These claims are consistent with Thompson (1992), who argued that the relationship between beliefs and practice is complex and influenced by factors including instructors' views regarding the nature of mathematics as well as instructors' interaction with social and political contexts. These claims complement those of Speer (2005) who summarizes these conceptualizations of beliefs well: "Beliefs appear to be, in essence, factors shaping teachers' decisions about what knowledge is relevant, what teaching routines are appropriate, what goals should be accomplished, and what the important features are of the social context of the classroom" (p. 365).

While the previously referenced findings (Päuler-Kuppinger & Jucks, 2017; Raychaudhuri & Hsu, 2012; Vishnubhotla et al., 2022) focus on individual factors that shape teachers' decisions without taking into account much of the surrounding context, several scholars have emphasized the importance of *context* in understanding the relationship between beliefs and practice (Ernest, 1989; Speer, 2005; Thompson, 1992). Johnson and colleagues (2019) examined how individual and situational factors influenced the instructional practices of mathematics instructors. They found that individual factors, such as personal beliefs, were more likely to explain instructional differences across instructors than situational factors. Interestingly, Johnson and collaborators chose not to investigate how situational factors influenced the individual factors (beliefs, knowledge, and goals) for instructors. Several other studies investigate this problem through what Herbst and Chazan (2012) refer to as practical rationality, who argue that the justifications teachers use for their beliefs and practices can be understood by investigating their professional obligations to their content discipline, individual students, interpersonal interactions in the classroom, and institutions such as departments, schools, and districts. Scholars in post-secondary mathematics education have taken up this work, particularly as it pertains to active learning (Mesa et al., 2020; Shultz, 2022).

In the context of mathematics education, Louie (2017) argues that the dominant culture of exclusion is likely to influence the beliefs, knowledge, and skills of individual teachers. Across several studies, Louie (2017, 2018) and colleagues (2021) have shown that the culture of exclusion in mathematics, characterized by narrow definitions of mathematical activity and ability, prevents teachers and instructors from enacting instructional practices in ways that disrupt existing inequities. This applies to current efforts to understand instructor beliefs about active learning in two ways. First, it further supports the notion that focusing on individual beliefs without considering broader context is insufficient. Second, when examining contextual factors such as professional obligations, it is crucial to view these obligations as being embedded within the culture of exclusion. In doing so, professional obligations become potential reproducers or disrupters of the culture of exclusion. We will argue later that understanding the beliefs of instructors via their professional obligations within the culture of exclusion allows us to identify potential levers through which we can influence instructors' active learning practices in a direction that is inclusive and equitable.

Theoretical Framework

Our investigation assumes a complex interaction of instructor practices and beliefs, and also situates these interactions in an ecosystem that includes contexts with obligations, as well as the

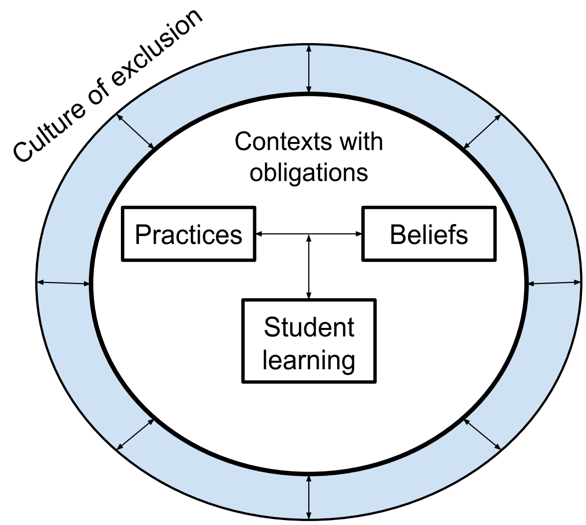

Figure 1. Illustration of theoretical framework

pervasive culture of exclusion (see Figure 1). From the extant literature, we conceptualize beliefs as factors influencing teachers' decisions regarding teaching and learning, consistent with Speer (2005). This conception of beliefs is in line with Leatham's (2006) in which the emphasis is not on the strength of the beliefs but rather their connections to the individuals' other beliefs and practices.

We also contend that an instructor's beliefs are informed by the professional obligations (Herbst & Chazan, 2012) with which they contend. Herbst and Chazan (2012) name four obligations: disciplinary, individual, interpersonal, and institutional. We see these four obligations as being situated within the culture of exclusion (Louie, 2017), which is characterized by narrow views of mathematical activity and ability. Each of the professional obligations has the opportunity to resist or reinforce the culture of exclusion. Figure 2 adapted from Louie (2017, p. 496) with obligation descriptions from Herbst and Chazan (2012), illustrates our proposal for how this might occur.

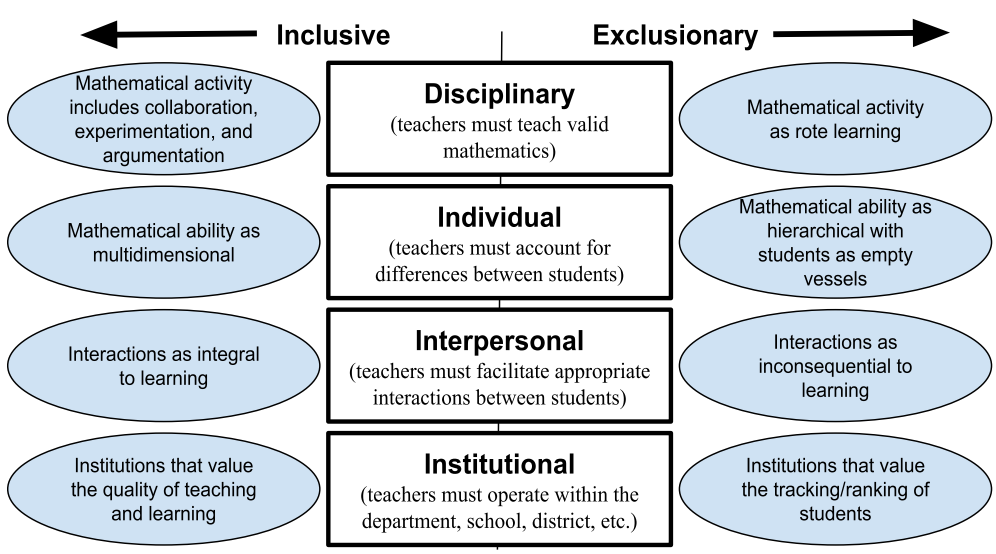

Figure 2. Relationships between professional obligations and the culture of exclusion

Data and Methods

Researcher Positionality

The first author is a white woman with a PhD in mathematics, who grew up in an income-secure household. These identities affect her worldview, often implicitly. Her personal goals as a college math instructor, faculty developer, and math education researcher are to create rich, supportive, and affirming learning experiences in college mathematics classrooms, especially for students who have been historically marginalized in mathematics; these goals explicitly affect her research questions and analyses. The second author is a white man who is a former secondary mathematics teacher and current PhD candidate in mathematics education. His commitments to equity and justice in education ground his approach to research, albeit from the limited perspective of a person with multiple privileged identities.

Participants

This study focuses on early career faculty (graduate students and postdoctoral fellows). The reasons for this are (1) this is a population with more malleable beliefs (Ellis, 2014); (2) this population is responsible for a meaningful percentage of college math courses taught (Ellis, 2014); (3) understanding beliefs of more novice instructors, in particular, yields insights into an important stage of belief development; (4) at the institution of this study, these instructors were overwhelmingly the population teaching with active learning. We invited eight postdocs, representing a wide range of pilot survey responses, to be interviewed; seven agreed to be interviewed. In contrast, the eight graduate student participants were selected on a volunteer basis via a departmental graduate student listserv. Out of those fifteen, we focus on the responses of five in this paper. Those five include four graduate students (two white women and two white men) and one post-doc (a white man), all of whom teach small sections (< 24 students) in multi-section coordinated courses (18 - 100 sections).

Interviews and Analysis

The interviews (15-30 minutes) included three questions adapted from the TBI instrument (Luft & Roehrig, 2007) designed to elicit important instructor beliefs: (Q1) How do you maximize student learning? (Q2) How do you describe your role as an instructor? (Q3) How do you know students are learning? Interview audio was transcribed using rev.ai, and cleaned for accuracy and anonymity. Among responses to Q2 was a surprisingly common response of "coach" (n=5) to describe their roles as an instructor. For the scope of this short paper, we restrict our attention to that recurring metaphor. The first author identified various sub-meanings of "coach" using open coding (Corbin & Strauss, 1990), then refined those meanings through discussions with the second author. The theoretical frameworks described above were not used to create these categories of meanings, but to understand their implications (see next section).

Results

In this section we characterize four different meanings underlying participants' use of the metaphor of coach to describe their role as instructors. We illustrate how each meaning alludes to particular obligations (Herbst & Chazan, 2012), while exploring whether those obligations are leveraged in a way that perpetuates or resists the culture of exclusion (Louie, 2017).

Responsive Supporter (n=4). *...providing responsive support to students; for example, by giving hints when needed or meeting with students one-on-one.* Our examples of this role draw primarily on the obligation to the individual because the interviewees described back and forth interactions with individual students, giving hints and asking questions; however, there is an implicit obligation to the discipline because the descriptions are focused on getting students to a mathematically correct answer. For example, one participant said: “When they are struggling on something, I try to, you know, give tips in some sense….But…sometimes it's more of leading questions, things like that.” Another said: “[I] really try to, you know, fill in gaps in knowledge, that kind of thing.” That is, instructors describe responsively interacting with individual students to correct misconceptions and fill in “gaps in knowledge”—disciplinary obligations. Although these are certainly caring instructors, we view both obligations implicit in this role—individual and disciplinary—as being leveraged in ways that perpetuate an exclusive (or dominant) framing of mathematics teaching. The *lack* of attention to the classroom community implies that interactions between students are relatively inconsequential to the instructor. Further, the goals of these interactions presuppose that the student is (mostly) an empty vessel, and the instructor is the knowledge-giver. While Responsive Supporters do acknowledge students as having some knowledge to be built upon, they do not describe their role as instructors as drawing out what students know, or making connections to students’ prior knowledge; rather, these instructors describe their role as to correct and redirect, highlighting students deficits rather than assets.

Skill Builder (n=3). *...helping students to build skills beyond the content, such as studying for exams, or approaching non-routine problems.* Our examples of this role draw primarily on two sources of obligations: individual and institutional. Descriptions refer to characteristics of students and their needs, not fostering a community or supporting interpersonal interactions, as seen in the following quotation: “I mean a lot of these students are freshmen … this is the first math class they're taking. So a lot of what they encourage us to do…is really trying to instill good, not just math study habits…but…taking exams in general.” We also see in the previous quotation two distinct references to institutional obligations: (1) the instructor is enacting what “they encourage us to do” and (2) what’s being encouraged is to help students succeed on the institutional measure of exams. Some respondents mention “problem-solving” or “learning from your mistakes”, which sound like a disciplinary obligation—how you *do* mathematics. However, when they expand more on what this means in their setting, we saw responses such as the following: “Cause you know, a lot of times, particularly in… that course when you're…solving problems that are about, like, justifying your answer, a student could have a correct numerical answer, but lose points because they didn't provide adequate justification for it. And so it's really important to look at where you've lost points on the other stuff...” That is, the “learning from mistakes” this instructor describes is less about actual mathematical learning, and more about learning to meet institutional expectations.

Preparer (n=3). *...helping students face a challenge on their own, that the instructor has no control over, such as a common exam.* This is distinct from the above role in that it is exam-preparation specific, and less about general skills. However, like Skill Builder above, this role leans heavily on both individual and institutional obligations. The institutional obligation is the more overt obligation in this case, since this role is explicitly about conforming to institutional expectations around exams. However, we also clearly see instructors’ obligations to the individual, wanting their students to succeed, in quotations like this one, “You are kind of like trying to get them to the point where they are prepared for this as much as possible...Like they are…being prepared by you in order to [perform] in some situation and you have some amount of, like, personal stakes in it and you really do want to see them succeed.” In the previous quotation as well as the following, we also see that the instructor and students are working as a team to overcome this hurdle: “There's going to be, there's going to be this exam...some adversary that we're all gonna work hard and practice together to get through together," though the consequence of low scores for the instructor are certainly less grave than for the students. Like the Skill Builder examples, it is caring and invested instructors who describe their role in this way. Nevertheless, in the Preparer meaning of the "coach" metaphor we see the iron grip of the institutional obligation to the common exam as supporting a narrow definition of mathematical activity and performance; this exclusionary frame is then also transferred to the individual obligation, by attending to the training and performance of individual students.

Cheerleader (n=2). *... a role of encouragement and emotional support.* Finally, our examples of Cheerleader most explicitly draw on obligations to individuals. However, when this instructor describes her role as "helping build people back up cause it's pretty common for people to have the first exam tear them down and then to, you know, help them get back on their feet and do well on the rest of the semester, which is totally possible," we also see an important underlying institutional obligation. This moral support is in the service of helping students navigate institutional structures and expectations. Rather than affectively supporting student joy around mathematics, we see her affective support as helping students recover from exam-induced trauma. That is, both obligations here are serving the culture of exclusion, rather than resisting it.

Discussion

Our first observation from the roles and obligations above is the absence of the interpersonal obligation. This is surprising given an obvious role of a "coach": helping a team learn to play together. While the role of helping students work together showed up in the larger dataset, it is not part of what instructors meant when they referred to themselves as a kind of "coach". Also notable is that only the role of Responsive Supporter seems particularly specific to an active learning environment, indicating that many components of how an instructor views their role in a college mathematics classroom transcend specific modalities.

Grounded in our observations from our case study of "coach" meanings above, we hypothesize that obligations to the individual, institution, and discipline may have a tendency to perpetuate the culture of exclusion. And, though missing from our sample, we hypothesize that beliefs derived from an interpersonal obligation may more naturally resist the culture of exclusion, since a defining characteristic of an *inclusive* frame towards mathematics learning is seeing "interactions and interpersonal relationships as integral to learning" (Louie et al., 2021, p. 101). These hypothesized tendencies are depicted in Figure 3. It's important to note that the obligations leveraged in our sample straddle this inclusive/exclusionary divide. However, despite some aspects of resisting the culture of exclusion—using groupwork at all, primarily using conceptual tasks, acknowledging that students come with some mathematical resources—the stronger forces (as described above) are in the exclusionary direction.

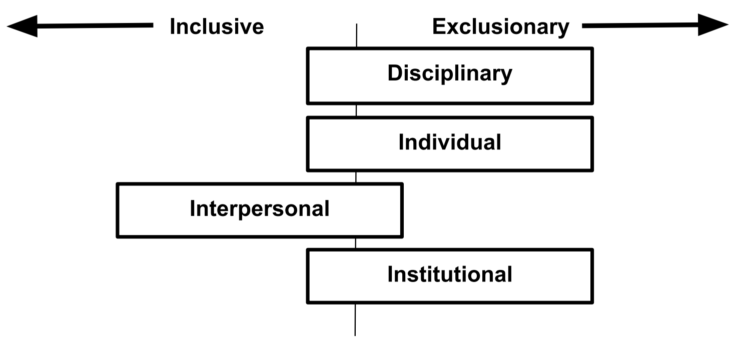

Figure 3. The tendencies of obligations to be inclusive or exclusionary based on case data

Suggestion 1: Reform the Obligation

Our suggestions refuse to view instructors as purely individual actors; rather they are embedded in cultures and institutions that must be considered as central to any change efforts. Our first suggestion follows from Webel's and Platt's (2015) strategy of leveraging existing instructor obligations (Herbst & Chazan, 2012) to effect positive change in practice. We layer onto their suggestion a need to reflect on how such obligations may readily sustain (or resist) the culture of exclusion (Louie, 2017, 2018). Take, for instance, Mesa and colleague's finding (2020) finding that coordinated courses may be an especially salient source of instructor obligations. Layering upon this finding a focus on framing surfaces a new question: how can features and structures of a coordinated course create obligations that *resist* the culture of exclusion, rather than perpetuate it? A requirement or expectation to use groupwork is one example of an institutional obligation that may help point instructors' beliefs and frames in an inclusive direction; but as we discuss in Suggestion 2, that expectation alone is not sufficient. Using coordinated course structures to resist the culture of exclusion may include both breaking down structures that perpetuate exclusive frames (e.g. curved grading schemes designed to rank students, timed exams that valorize correctness and speed) and simultaneously building more inclusive structures (e.g., course grade components that give credit for revising mistakes on exams and homework) and inclusive messaging (e.g., "exit tickets" at instructor meetings that ask instructors to reflect on the many different ways students were "good at math" in the last week).

Suggestion 2: Magnify the Interpersonal obligation

Another implication of our hypotheses is that a source of resisting the culture of exclusion may lie in magnifying instructors' obligations to interpersonal sources. For example, in coordinated courses using groupwork, instructors may settle on the sole purpose of groupwork as being "students do" rather than "students do together," unless course coordinators facilitate explicit, reflective conversations to illuminate the intended benefits of groupwork. A belief that groupwork is valuable simply for its hands-on nature (without respect to its collaborative nature) values student-instructor interactions, but fails to value student-student interactions.

Study Scope and Limitations

The sample size of the overall study (n=15) and the sub-sample presented here (n=5) can in no way characterize college math instructor beliefs across the United States; this is not our goal or claim. A second important limitation of this study is that we have no connection between these professed instructor beliefs and their classroom practices. Relatedly, we are applying an interpretive lens to this instructor discourse, that is colored by our experience as researchers and instructors.

Conclusion

By examining instructor beliefs through the frameworks of obligations and the culture of exclusion, we have pointed to two foci for instructional change: (1) examining the frames surrounding professional obligations with the ultimate goal of reforming obligations themselves to better resist the culture of exclusion; (2) magnifying professional obligations, such as the interpersonal obligation, that may more naturally and readily resist the culture of exclusion. We do not take it for granted that active learning modalities support equity simply by being hands-on; rather, in any pedagogical modality we must interrogate practices, beliefs, and obligations to best support a culture of inclusion.

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Exploring Fitness Trackers as a Research Method for
Developing Personas of Stress during Classroom Problem Solving

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In this report, I examine the capacity of fitness trackers to assist in differentiating students' experiences of stress during classroom problem solving. I cluster $N = 26$ *students' heart rate variability plots based on categories of description created using a phenomenographic analysis of their reported emotional experiences during a scripted lesson on roots of polynomial functions. My results reveal that fitness tracker data can be a valuable indicator of when a student will report classroom distress. However, the tracker data can give both "false-positive" and "false-negative" indications a student will report distress, reinforcing that biometric data cannot be treated as a proxy for students reported experiences of stress. Nonetheless, my results do support the use of fitness tracker data to guide deeper inquiry into student experiences. For example, fitness tracker data can assist in identifying classroom moments to discuss in a video stimulated recall interview.*

Keywords: math anxiety, academic distress, problem solving, fitness trackers, heart rate

Although active learning may generally have a positive impact on students' performances in science and mathematics courses (Freeman et al., 2014), not all groups have equal access to these benefits; for example, because student participation is essential to most styles of active learning, there is a higher frequency of peer-peer interaction, creating more opportunities for students with minoritized social identities to experience a marginalizing event (Reinholz et al., 2022). Classroom stressors can foster academic distress, or anxiety caused by university course environments. Academic distress has other known correlates: coursework demands, assessment pressures, instructor teaching style, and a student's sense of belonging (Larcombe et al., 2022). Additionally considering the high prevalence of math anxiety among undergraduate students in the United States (Lanius et al., 2022; Lanius, 2022; Hopko et al., 2003; Hopko, Crittendon, Grant, & Wilson, 2005; Hopko, Hunt, & Armento, 2005; Isiksal et al., 2009; Rancer et al., 2013), studies concerning psychological stress during classroom problem solving are particularly timely and important. Wearable technologies capable of collecting heart rate data, more succinctly called fitness trackers, provide an alluring prospective tool for *in situ* indication of student anxiety or distress during classroom problem solving (Singh et al., 2018; Kim et al., 2018). To explore the research potential and pitfalls of fitness trackers, this report compares clusters of student's fitness-tracker data plots to categories of description created using a phenomenographic analysis of their reported emotional experiences during a lesson on roots of polynomial functions.

Heart Rate Fitness Trackers & Distress Indication

A meta-analysis of 37 studies concluded that current evidence supports the use of Heart rate variability (HRV) for the indication of psychological stress in a particular situation (Kim et al., 2018)—for example, job-strain (De Bacquer et al., 2011) or computer-work stress (Hjortskov et al., 2004). Additionally, HRV has been shown to be a statistically significant indicator of junior high students reporting math anxiety while performing an arithmetic task such as $7 + 2$ (Tang et al., 2021). Heart rate variability (HRV) is a summative measurement of the variation in length of time between heart beats during a window of time. Slight variations in length of time between heart beats is normal in healthy individuals and is caused by the autonomic nervous system receiving conflicting feedback from the parasympathetic nervous system (informally, the rest-and-digest-system) and the sympathetic nervous system (or the fight-or-flight-system) (Singh et al., 2018). When the sympathetic and parasympathetic nervous systems are in balance, there is greater variation in the length of time between heart beats. On the other hand, when the body needs to react to an external stimulus, such as stress, messages from the sympathetic nervous system overpower messages from the parasympathetic nervous system, and there is less variation in the length of time between heart beats. Ultra short term HRV, which uses heart rate data from a couple of minutes, can be reliably computed for monitoring mental stress (Salahuddin et al., 2007), thus allowing us to synchronize changes in a student's HRV to tasks and stimuli they experience in their undergraduate math course.

Research Questions

What are the stress-related emotional experiences of undergraduate students during classroom problem solving? How does an undergraduate student's heart rate variability change if they report experiencing math anxiety or academic distress during classroom problem solving?

Theoretical Perspectives

Concerning Methodology. I utilized phenomenography (Marton, 1986). To clarify how phenomenography differs from the more familiar phenomenology, in a phenomenological study the phenomenon itself is investigated; in a phenomenographic study the variation in how people experience the phenomenon is investigated (Larsson & Holmström, 2007). Phenomenographic research is an iterative process where the researcher familiarizes themselves with the descriptions of experience, identifies distinctive or structurally significant characteristics, groups and classifies similar answers, and builds named categories; see Table 1 of Han & Ellis (2019). The final product of my analysis is categories of description that explain the variations of experience present in the data. Phenomenography can then be used to build personas, lifelike characters to assist in student-centered design in educational contexts (Huynh et al., 2021).

Concerning Fitness Tracker Data. I must avoid data-essentialism, or the position that students can be reduced to their biometric data or that the data collected by fitness trackers is "objective" reality (Svensson & Poveda Guillen, 2020). Research data are produced through a laborious and time-intensive process involving the decisions of a variety of people. Further, fitness tracker data is limited by the capacity of our locations and technologies (Ribes & Jackson, 2013). Consequently, I view biometric data as a human artifact that must be situated in its context.

Methods

Fitness Trackers & Heart Rate Variability

Photoplethysmography (PPG) uses light-emitting diodes (LED) and photodiodes to detect blood volume changes directly under the surface of the skin, with each peak in blood volume corresponding to a beat of the heart (Singh et al., 2018); see Figure 1. The PPG sensor records the length of time between successive peaks of blood volume, providing an accurate approximation of the length of time between heart beats. To process the collected heart rate data, I partitioned each students PP interval data into 3 time windows and utilized the HRV variable root mean square of successive differences (RMSSD) to compute an HRV value for each student in each time window of the lesson; see Figure 1 for the RMSSD formula. The most common cause of inaccuracy with PPG measurement is motion artifacts (Fine et al., 2021), for example rapid arm movement, which occurs during certain types of intense physical exercise, where the optical sensor loses contact with the skin. Although I used a procedure for detecting and cleaning optical-coupling-based errors within the heart rate data, I sought to avoid the issue entirely and encouraged research participants to secure the watch snugly to the forearm of their non-dominant hand. Further, participants remained sitting during the experiment, which minimized the amount of arm movement.

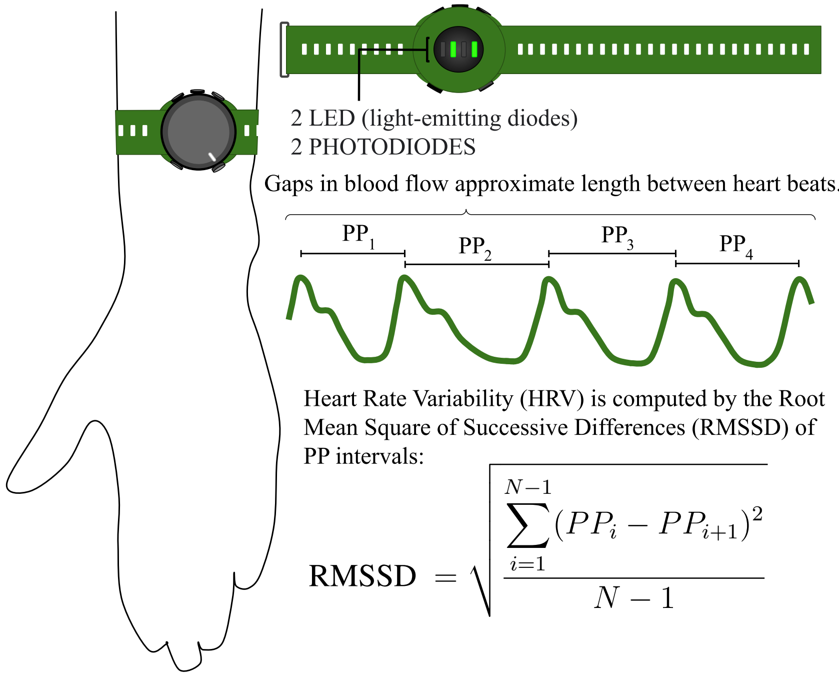

Figure 1. Photoplethysmography (PPG) sensor data to compute heart rate variability (HRV)

Scripted Math Lesson & Reflection Survey

After securing institutional review board approval from Auburn University's Office of Research Compliance, I invited undergraduate students enrolled in a lower-level math course to participate in my study. Students wore a fitness tracker during a 15 minute scripted math lesson. The lesson has a purposefully placed stressor called Cardano's formula (Harris & Stocker, 1998): Any cubic polynomial of the form $s(x) = x^3 + ax + b$ such that $4a^3 + 27b^2 > 0$ has real root

$$x = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} + \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}.$$

Lesson Window 0. The lesson began with an instructor introducing the topic *Section 3.1 Roots of Polynomials* on a chalkboard. After seeing a notation-heavy definition of polynomial function, the instructor presented an example of a linear and a quadratic polynomial. The students were then assigned *Problem 1. True or False: The function* $s(x) = 4$ *is a polynomial* to complete on a provided handout. Window 0 will be treated as an acclimation period for the study and will not be included within participant's heart rate variability plots below.

Lesson Window 1. After waiting for visual cues that all students had completed problem 1, the instructor provided the correct answer and reviewed the definition of root and provided an example. Students were then instructed to complete *Problem 2. Find the root of* $p(x) = 3x + 1$.

Lesson Window 2. Again, after the instructor received visual cues that it was time to continue the lesson, they provided the answer to problem 2 and presented the quadratic formula, using it to find the roots of an example.

Lesson Window 3. Following a presentation of Cardano's formula, the instructor assigned *Problem 3. Use Cardano's formula to find a root of* $p(x) = x^3 - 1$ stating "Don't forget to check that Cardano's formula applies by verifying $4a^3 + 27b^2 > 0$." After waiting 2 - 3 minutes, the instructor revealed that the correct answer was $x = 1$ and concluded the lesson.

Note that the partition of events into windows 0 - 4 is done to guarantee each window has sufficient time for an accurate and reliable HRV computation and is not based on the composition of events. Immediately following the lesson, students completed a reflection survey to report when they had experienced anxiety and to describe their experience in their own words. Many students also chose to contemporaneously report their emotional experience by writing on the provided lesson handout.

Results

The 29 participants included 18 women and 11 men. Five students were enrolled in College Algebra, Finite Math, or Pre-Calculus. Twelve students were enrolled in Calculus I or Calculus II. Eight were enrolled in Calculus III, Differential Equations, or Linear Algebra. One student was enrolled in an elementary math content course for pre-service teachers and three participants did not specify a current math course.

Clusters of Fitness-Tracker Data Plots

I clustered participants HRV plots together based on the categories of description created from my phenomenographic analysis of students' reflection data and lesson handouts. Note that there is great individual variation in the typical ranges one might see for HRV-RMSSD values, which can be as low as 21 ms and as great as 103 ms (Umetani et al., 1998). To make the HRV plots comparable between individuals, I normalized every plot by dividing a student's three RMSSD values by their maximum RMSSD value. Within each normalized HRV plot, a dashed line is given as a visual indicator of values that suggest the student will report feeling distressed while attempting an assigned problem. The placement of this dashed line was determined using a logistic regression model that estimates the probability a student will report distress given their HRV value is below this line is greater than 50%, with the odds increasing the further the value moves below the line. This statistical analysis is not the focus of this report and thus is omitted here, but will appear in other work. I provide the line here as a visual aid to assist readers in interpreting the plots. For each cluster of plots, I constructed an average normalized HRV plot by averaging the normalized HRV values of students in that cluster. Of the $N = 29$ profiles to analyze, there were 5 categories of description that were common to at least 3 students. Categories with fewer than 3 participants will be excluded from this discussion because there were too few data points to meaningfully cluster the normalized HRV plots and take an average.

Categories of Description

Labels. I utilize the word distressed in labeling the categories to signify disruptive unpleasant feelings or emotions that interrupted or inhibited the student's ability to adapt or cope with a lesson stressor. The other key label is abstaining, which indicates that these students chose not to attempt an assigned problem.

Category 1: Distressed. These 5 students first reported distress in Lesson Window 3 where Cardano's formula was introduced and they were instructed to use the formula to find the root of a cubic polynomial. One participant wrote that they felt overwhelmed *"by the sudden barrage of variables and unfamiliar terms"* with another explaining, *"Not having enough examples before having to try by myself gave anxiety."* One participant experienced academic distress because they were not able to finish the problem before the instructor moved on; they wanted to be sure that they could get the right answer on their own. The normalized HRV plots of the five students in this cluster can be found in Figure 2. All but one students normalized HRV values remained above the distress indicator line until the third window, where all HRV values lie below.

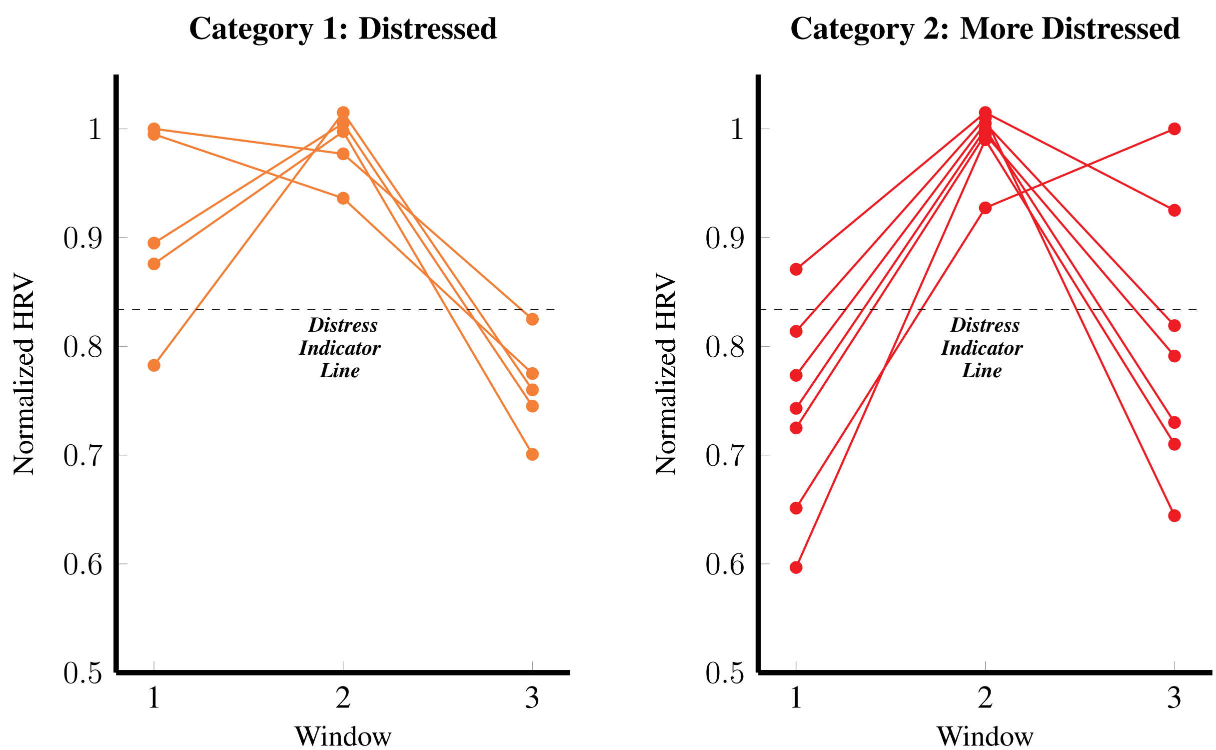

Figure 2. Clustered normalized HRV plots for Category 1 and Category 2

Category 2: More Distressed. For the 7 students in this cluster, the source of distress in Lesson Window 1 was the instructor announcing that problem 1 was true. Six students expressed that they felt bad or disappointed that they had gotten the answer wrong, with one student writing *"I felt like it was a trick question and felt badly that I didn't know it"*. Several students indicated that they did not understand why their answer was wrong, with one student explaining that their source of distress was that they thought the instructor lied or misspoke and that problem 1 was in fact false. Note that the lesson script does provide a brief explanation as to why problem 1 is true, but students were not given the opportunity to ask any questions of clarification during the lesson. None of these students reported distress during Lesson Window 2, which started with them receiving feedback that they answered problem 2 correctly. This group reported distress in Lesson Window 3, conveying feelings similar to those reported by the Category 1 group: anxiety, panic, overload, and "blanking out". The normalized HRV plots of the 7 students can be found in Figure 2. Note although one students plot makes the same inverted V shape as the majority of the other plots, their window 1 and window 2 values do not lie below the distress indicator line. Also note one participant's HRV lies well above the distress indicator line in window 3.

Category 3: Distressed & Abstaining. These 3 students first reported distress in Lesson Window 3, however - unlike the participants in the previous two categories - they chose not to attempt problem 3. One participant, rather than copying the problem from the board onto their handout, wrote *"I genuinely don't know how to do this one at all"* and another reflected after the lesson *"I simply did not know how to solve it."* Their 3 plots can be found in Figure 3.

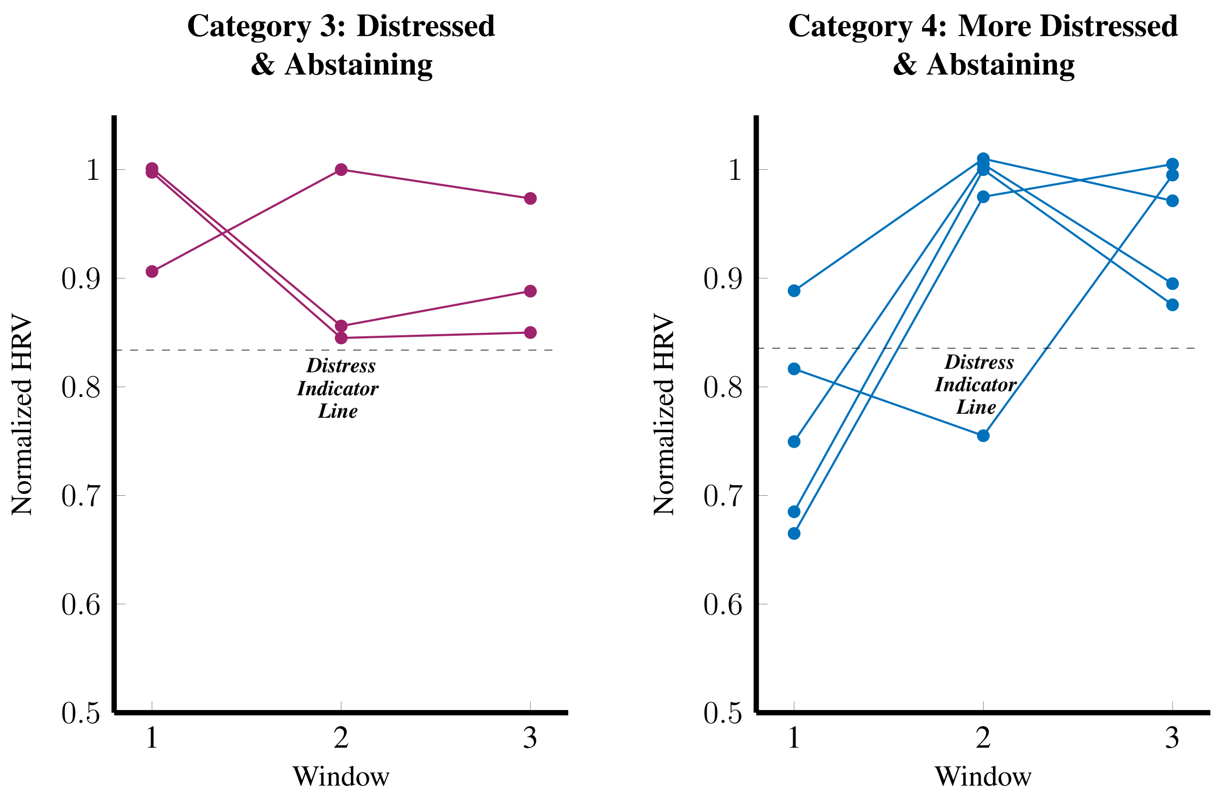

Figure 3. Clustered normalized HRV plots for Category 3 and Category 4

Category 4: More Distressed & Abstaining. For the 5 students in this cluster, Lesson Window 1 caused distress when the instructor provided the solution to Problem 1. One student wrote, *"I got it wrong, wasn't sure about my answer in the first place."* This group felt less stressed during Lesson Window 2, with one student explaining *"during the second problem I regained my confidence because I knew what to do"*. In Lesson Window 3, these students reported distress and chose not to attempt problem 3. One student wrote *"I literally would not know where to start"* in the space provided to copy the problem and do scratch work. Their normalized HRV plots are in Figure 3.

Category 5: Undistressed. The final category contains the 6 students who did not report feeling distressed during the lesson. Their normalized HRV plots appear in Figure 4. Note that at various times, participants normalized HRV values dropped below the distress indicator line.

Category Averages. The cluster average for each category of description appears in Figure 4. Note that, on average, student's normalized HRV values dropped below the distress indicator line if and only if they reported feeling distressed while actively attempting an assigned problem.

Discussion & Implications

Within the normalized HRV plots, there are 4 instances of a "false-negative", where the HRV plot suggests the student will not report feeling distressed while attempting to solve a problem and the student did report distress. There are 6 instances of a "false positive", or instance where the HRV plot suggests the student will report feeling distressed while attempting to solve a problem and the student did not report distress. Thus, the *overall alignment rate* across these 26 profiles was 87%. These results suggest that fitness tracker data can be valuable to guide deeper inquiry into the student experience; however, low HRV does not guarantee a student is distressed and high HRV does not guarantee a student is undistressed. Further, HRV cannot provide a direct measure of the *amount* of distress or give an instantaneous indication a person is experiencing distress.

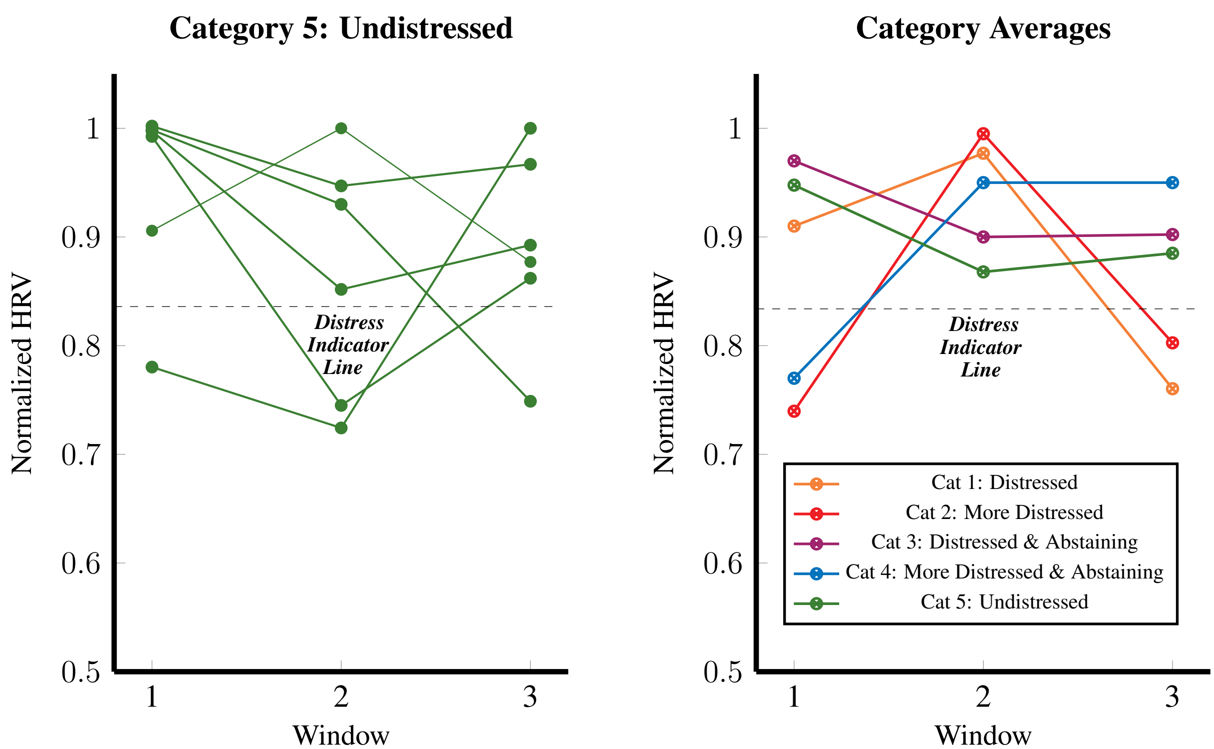

Figure 4. Clustered normalized HRV plots for Category 5 and Category Average plots

The role of HRV in Education Research

In Figure 4, we see that the plot average for Category 3: Distressed & Abstaining has the same overall shape and structure as the plot average for Category 5: Undistressed. This suggests that HRV can only discern when a student may report feeling distressed if the student is actively problem solving. I recommend that fitness tracker data primarily be used for triangulation, as a way to identify moments during classroom problem solving for video stimulated recall interview with research participants. With deliberate and careful use, fitness trackers provide an innovative research design tool that can integrate qualitative and quantitative measures in the study of psychological stress during classroom problem solving.

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Star Polygons: Symbolizing as Advancing Mathematical Activity in a Scripting Task

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Research indicates that investigating phenomena, rather than reproducing facts, should be a core experience in mathematics education. But despite its centrality to quality teacher education, it is still unclear how the effects of investigation tasks on teachers' mathematical development can be analyzed. In this study, we demonstrate the potential of scripting tasks as an avenue for capturing mathematical activity during an investigation. Participants, comprised of prospective and practicing teachers, explored the concept of a "star polygon" and recorded their deliberations, ideas, and conclusions as a scripted dialogue. We analyzed their submissions using the construct of advancing mathematical activity (Rasmussen et al., 2005; Rasmussen et al., 2015), focusing on how participants' interpretation and creation of symbols intersects other types of activity.

Keywords: Star Polygons, Scripting Tasks, Teacher Education, Advancing Mathematical Activity, Symbolizing

Introduction

Mathematics education research strongly advocates for learning environments centered on investigation, as opposed to techniques and procedures (e.g., Da Ponte, 2007, Yerushalmy, 2009; Leikin, 2014). Investigation tasks—also described as open problems or problem fields (Pehkomen, 2019)—require students to consider a particular object or a set of objects, collect data about them, identify their properties, and justify the discovered properties or relations. As such, conjecturing and then refining, proving, or refuting conjectures are the heart of investigation tasks.

There are multiple advantages for students engaged in mathematical investigations (e.g., Quinnell, 2010). For example, they develop a strong conceptual understanding, an awareness of the nature of mathematics as a discipline, and the ability to think creatively about mathematical ideas. Investigation-based learning environments also foster a supportive classroom atmosphere while simultaneously recreating the actual practice of mathematicians.

Numerous professional development efforts intend to prepare teachers for including problem solving and investigation tasks in their classroom and support them in this endeavor. However, for teachers to guide students' investigations, it is essential that teachers have experience in engaging in such activities themselves (e.g., Perdomo-Díaz et al, 2019). While mathematics educators agree on the importance of engaging teachers in mathematical investigations, there is no explicit guidance on how teachers' engagement in investigations can be studied.

Existing reports on teachers engaged in mathematical investigations employ close observations of teachers' work in class, video-recording of their work on a particular mathematical problem (e.g., Rott et al, 2021), and problem solving journals (Liljedahl, 2007) or portfolios (Gourdeau, 2019) in which research participants recorded their progress. In this study we present an additional approach—script-writing—for recording and analyzing teachers' engagement in a mathematical investigation. In doing so, we address the following research questions: *What aspects of participants' symbolizing activity are captured by scripted dialogues recreating their progress on an investigation task?*

Advancing Mathematical Activity

To answer this question, we turn to Rasmussen et al.'s (2005) alternative formulation of Tall's (1992) concept of advanced mathematical thinking that they call *advancing mathematical activity*. By focusing on mathematics that is "advancing," this construct allows researchers to study student activity that improves their understanding of mathematics, even if their work would not be considered "advanced" by some standard; by considering "activity" rather than "thinking," this construct attempts to capture a more holistic and diverse type of engagement with mathematics. Throughout their work, Rasmussen and colleagues identify four types of advancing mathematical activity that are socially or culturally situated practices: symbolizing, algorithmatizing, defining, and theoremizing (2005; 2015). In this report we focus on symbolizing.

Symbolizing "arises in part as a means to record reasoning" (Rasmussen et al., 2005; p. 57); that is, a symbol is a "notational device that [is] consistent with [a student's] mathematical reasoning," and symbolizing is the creation thereof (p. 58). The authors stress that they do not interpret symbolizing as necessarily an act of dissociation from context; designating a symbol as the representation of a quantity is not only a means of abstraction but also allows students to create, take ownership of, and share mathematical objects that characterize their thinking. Symbols, once refined, can act themselves as the input for future symbolizing activity. That is, symbols can manifest as a synthesis of multiple previous symbols.

Although we focus primarily on the presentation of symbolizing activity throughout this report, it is impossible to completely disentangle the types of advancing mathematical activity; with this in mind, we introduce the other types of activity as well. *Algorithmatizing* is the creation and discovery of (either local or global) algorithms. When engaged in *defining*, a student might construct a definition in light of examples and counterexamples; alternatively, they might leverage an existing definition in order to explore a related set of objects. Finally, theoremizing encompasses "both conjecturing and steps toward justifying the assertions" (Rasmussen et al., 2015; p. 264).

Methods

Participants and Setting

A total of 27 respondents participated in the study and, working in groups, produced 14 unique submissions; we refer to these as S-1 through S-14. The first seven submissions were generated by prospective teachers during a course on mathematical problem solving. This course, taken during the last term of the teacher certification program, leveraged problem solving activities to connect a variety of secondary and advanced topics in mathematics. The latter seven submissions were generated by practicing teachers who held at least a bachelor's degree in mathematics or science. The practicing teachers were enrolled in a professional development course that provided an overview of important topics and ideas in mathematics through the lens of pedagogy.

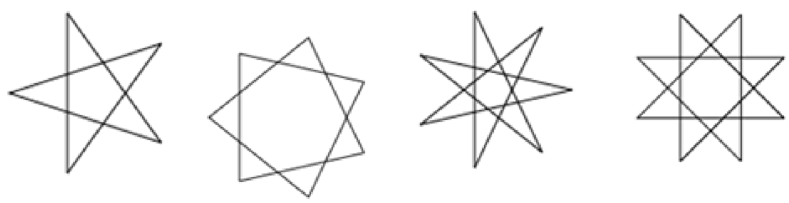
Participants' scripts often featured themselves as characters. To preserve their anonymity, throughout this report, named characters that appear in excerpts from submitted scripts have been given pseudonyms.

The Star Polygons Task

As part of their respective courses, both prospective and practicing teachers regularly engaged with *scripting tasks*; in such a task, participants are asked to write a script of a dialogue within a mathematical context. A scripting task might be used for lesson planning (where it is referred to as a *lesson play*, for example in Zazkis et al., 2009; Zazkis, et al., 2013). At other times, a scripting task is an activity that can be used to study how teachers respond to a student error (e.g., Zazkis et al., 2013), a student question (e.g., Bergman et al., in press; Marmur & Zazkis, 2018; Kontorovich & Zazkis, 2016), or an argument among the student characters (e.g., Marmur et al., 2020; Zazkis & Zazkis, 2014).

The *star polygons task* used in this study is a novel type of scripting task in that it prompted participants to record, as a script, their journey in a mathematical investigation. Preliminary pilot testing of a more open-ended version of the star polygons task showed that a certain amount of guidance was needed to support participants' initial engagement with the task. The revised task deployed in this study (Figure 1) provided a symbol system for naming and describing star polygons before asking participants to make sense of it. As we exhibit in the findings, this allowance facilitated communication and conjecturing without supplanting meaningful symbolizing activity.

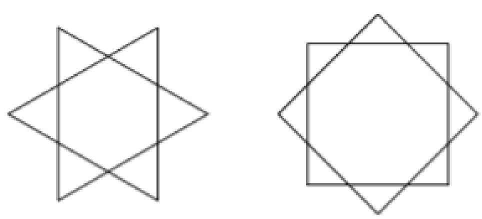
The figures below exemplify regular star polygons on 5, 7 and 8 vertices.



These are denoted – from left to right – as (5, 2), (7, 2), (7, 3) and (8,3). Explain the notation.

In (5,2) we refer to "2" as the number of "skips" or "steps" (skip to the second).

But these are NOT star polygons. Explain why.



Is there a star polygon on 10 vertices? On 12? On 30? On 31?

Explore different numbers. Describe your explorations and conclusions as a play, that is, in a form of a dialogue among problem solvers. Record your (your characters') conjectures and the ways of testing them.

Figure 1. The first part of the star polygons task.

The first part of the task introduced in Figure 1 engaged participants with investigating why select examples are not considered star polygons. Having justified this distinction, participants were instructed to explore whether a star polygon can be constructed on a given number of vertices.

A second part of the task, not pictured in Figure 1, prompted participants to describe a method for deciding if there exists a star polygon on N vertices and, if so, how many different star polygons on N vertices there are. Finally, participants exemplified their described strategy on a large number of vertices. Each part of the task was recorded as part of the scripted dialogues that comprised the collected submissions.

Data analysis

We initially analyzed participants' submissions using reflexive thematic analysis (Clarke & Braun, 2006, 2019; Nowell et al., 2017). That is, we hoped to allow the submissions themselves to guide our understanding of participants' mathematical activity as captured in their investigations. We began this process of immersion and interpretation by first reading and rereading the submitted scripts in order to thoroughly familiarize ourselves with the data. While doing so, we recognized that the most compelling aspect of the scripted dialogues was the description of the interaction between characters as they discussed and constructed a collective, social understanding and monitored each other's claims.

To emphasize the fundamentally communal nature of the activity reported by participants in their submissions, we leveraged Rasmussen and colleagues' (2005, 2015) four types of advancing mathematical activity. This decision was given credibility by the authors' emphasis that these are "acts of participation in a variety of different socially or culturally situated mathematical practices" (Rasmussen et al., 2005, p. 52). Furthermore, advancing mathematical activity has already been used to describe collective mathematical progress as part of an emergent framework (Rasmussen et al., 2015). We thus coded the data explicitly for instances of symbolizing, algorithmatizing, defining, and theoremizing. This process was guided by the definitions and examples of each practice presented in the advancing mathematical activity framework (Rasmussen et al., 2005; Rasmussen et al., 2015).

In the following section, we present instances of symbolizing activity that emerged during explorations of the star polygons task. Where appropriate, we point out where the symbolizing activity overlaps with other types of advancing mathematical activity. Finally, we pay particular attention to the ways in which participants' symbolizing activity was affected by their adoption of the prescribed (n, k) (n, k) notation within the task description.

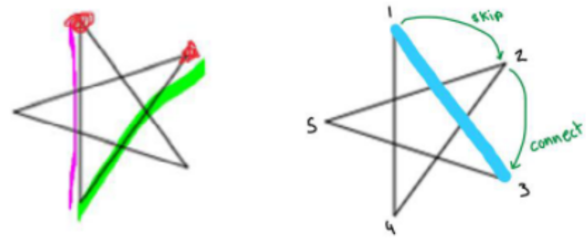
Findings

Evidence from student submissions suggests that the (n, k) (n, k) notation provided in the task description, despite providing a preconstructed symbol system to the participants, did not supplant participants' symbolizing activity. In fact, many submissions exhibited more robust symbolizing activity in direct response to the provided notation. For example, this occurred when participants used symbols to explain their intuitions about the notation to their groupmates. S-5 included the following exchange:

Martina: For the $(5,2)$ $(5,2)$ and $(7,2)$ $(7,2)$ stars, if you follow along and "draw" the lines, you are skipping 1 vertex to draw the next one. Maybe the 2 2 notates the number of steps you take as you draw the star polygon?

Bridget: Hmm let me know if I interpreted this the wrong way but here is what I got. To go from vertex Circle to vertex Triangle, I took the pink route then the green, so I counted those as two steps.

Joyce: Hmm, I was thinking about it differently. Here is a diagram of what I thought.



(a) Bridget's diagram (b) Joyce's diagram

Figure 2. Symbolizing activity in S-3.

Martina, Bridget, and Joyce demonstrate that there are multiple possible explanations for what constitutes a "skip" and use symbolizing activity to capture the underlying reasoning of each interpretation. Their discussion illustrates that participants still engaged in symbolizing activity in order to encapsulate and then compare their understanding of the (n, k) (n, k) notation within the context of the provided star polygon diagrams.

Other groups used symbolizing to hypothesize or reject alternative explanations for the provided notation. In S-13, a character proposes another interpretation of the k k in the (n, k) (n, k) symbol:

Abeni: I thought the 5 5 represented the number of vertices and the 2 2 represented how many times there are 5 5 vertices. So this one being $(5{,}2)$ $(5{,}2)$ means that it has 2 2 sets of 5 5 vertices. See in the picture below using the different colours. There are 5 5 blue vertices and 5 5 red vertices. Two different sets of vertices all together.

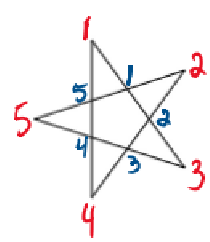


Figure 3. Abeni's suggestion for interpreting the (n, k) (n, k) notation in S-13.

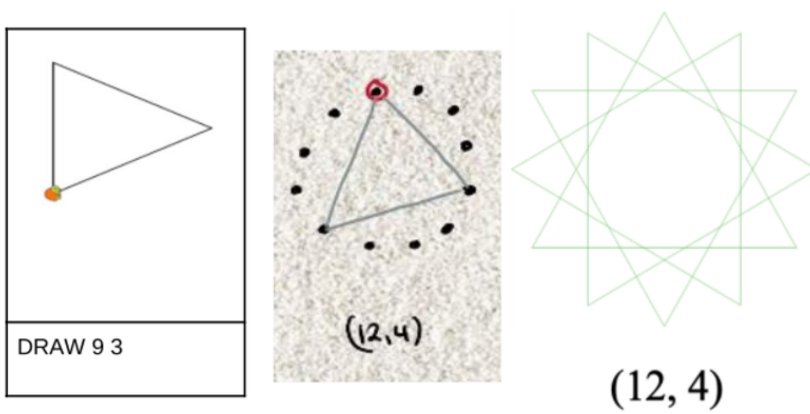
Abeni and her partner go on to draw and label multiple examples of star polygons, eventually concluding that both interpretations of the (n, k) (n, k) notation are valid. It is not until considering the provided non-examples that the characters recognize that Abeni's suggested interpretation cannot be used to differentiate these objects from the earlier examples. As Abeni observes, "My observation of 2 sets of 6 vertices holds true for this [the first non-example in Fig. 1]. So, I am not sure why it would not be a star polygon." In exploring this perceived contradiction, the characters of S-13 demonstrate how symbolizing can contribute to the act of defining.

Providing a symbolic notation had another unexpected benefit with respect to defining and theoremizing: many groups used the (n, k) (n, k) symbol as input for further symbolizing activity that facilitated the other types of advancing mathematical activity. For example, consider the following excerpt from S-5:

> *Matt*: I think that we can have two overlapping star polygons that generate irregular star polygons if we can find a number (x, y) (x, y) which satisfies the equation $(x, y) = 2 \cdot (a, b)$ $(x, y) = 2 \cdot (a, b)$ where (a, b) is a star polygon. So, for example $(14{,}6) = 2 \cdot (7{,}3)$ $(14{,}6) = 2 \cdot (7{,}3)$, and (7,3) is a star polygon. So, (14,6) should look like 2 2 star polygons layered on top of each other.

Here, Matt manipulates and generalizes the (n, k) (n, k) symbol through further symbolizing activity in order to apply that notation to a newly defined category of non-examples ("irregular star polygons"). After defining this object, Matt generates a conjecture about particular combinations of n n and k k and applies their symbol system to specific examples. In this way, Matt's symbolizing allowed him to better engage with defining and theoremizing activity.

Perhaps the most interesting symbolizing activity occurred when participants interpreted the (n, k) (n, k) symbol as instructions for executing an algorithm for constructing additional star polygons. The symbols that they generated as a result of this algorithm were simultaneously products of their algorithmatizing activity and symbols used to explain the rationale for their algorithms. Consider the following three symbols produced by S-14, S-9, and S-2 provided in Figure 4.



(a) S-14's diagram (b) S-9's diagram (c) S-2's diagram

Figure 4. Various applications of the (n, k) (n, k) *symbol as an algorithm.*

The characters in S-14 and S-2 used two different computer software systems to generate their symbols (Figure 4(a) and 4(c), respectively), whereas characters in S-9 produced their symbol (Figure 4(b)) by hand. Each final symbol is a composition of an essential symbolic component (the equilateral triangle) and various secondary symbolic elements (e.g., "unused" vertices, additional overlapping triangles, indications of a "starting vertex"). In total, each symbol in Figure 4 encapsulates its respective group's understanding of how the notational symbol (n, k) (n, k) can be interpreted and applied as an algorithm. The differences in their interpretations, manifested as the secondary symbolic elements, affected how each group would go on to define when two star polygons should be fundamentally "different."

Discussion

Overall, our findings illustrate a variety of ways in which symbolizing activity manifested while participants investigated the star polygons task. Participants' efforts to interpret the provided (n, k) (n, k) notation facilitated their future exploration of non-examples of star polygons; in this way, symbolizing led to more robust defining. Participants also created new definitions when they manipulated the (n, k) (n, k) notation and used these observations to conjecture relationships between different star polygons. These conjectures and their justification link symbolizing to theoremizing. Finally, most participants treated the (n, k) (n, k) notation as an algorithm that described the construction of a star polygon. Clearly, when this was the case, symbolizing and algorithmatizing were intimately connected.

These observations, in sum, contribute towards answering our research question: *What aspects of participants' symbolizing activity are captured by scripted dialogues recreating their progress on an investigation task?* That is, scripting tasks effectively capture symbolizing activity in a variety of modalities. Furthermore, these different manifestations of symbolizing connected to and supported the other types of advancing mathematical activity. Existing research often acknowledges this interconnection between advancing mathematical activities; this study contributes to mathematics education by highlighting specific instances of these relationships.

The Effect of Technology

In our findings, we observed that the provision of a symbolic notation for naming and describing star polygons did not detract from participants' advancing mathematical activity in general, or their symbolizing in particular; in fact, it often contributed to it. Interpreting, manipulating, and extending this notation was a productive focus of the submissions.

Just as the introduction of this notation had a positive effect on participants' engagement with the task, some kind of technology for generating star polygons was equally invaluable—it allowed participants to generate examples on a large number of vertices without constructing them tediously (and sometimes incorrectly) by hand. However, we also draw attention to the ways that technology removed opportunities to engage in advancing mathematical activity.

For example, Figure 4(c) is a star polygon that was generated by an online applet. We note that the default functionality of this applet is to draw "compound" star polygons when the number of vertices shares a common factor with the number of skips, rather than to leave some vertices "empty" (cf. Figure 4(b)). By supplanting participants' symbolizing with this particular representation of the notation, technology had a perceptible effect on participants' later mathematical activities. For instance, the symbol $(10,4)$ $(10,4)$ was interpreted by every group to be another non-example of overlapping polygons (in the vein of the provided non-examples in Figure 1) rather than a $(5,2)$ $(5,2)$ star polygon with more "empty" vertices. When this was the case, technology acted as a *reorganizer* (in the sense of Sherman, 2014) by shifting the focus of participants' thinking. Unfortunately, this reorganization appeared to remove an opportunity for theoremizing and defining.

Looking Forward

Another avenue for future research could be to investigate the didactic potential of the star polygons task itself for student guided reinvention of topics in either abstract algebra or number theory using the instructional design theory of realistic mathematics education (Gravemeijer & Doorman, 1999). Evidence in the star polygon submissions indicates that star polygons may act as an experientially real setting for recreating Euler's totient function, finite cyclic groups, their generators, and isomorphisms between finite cyclic groups of the same order.

Finally, we wonder what other ways this type of scripting task could be leveraged to observe mathematical activity in investigation group work. Future research may not be directly related to star polygons, but instead seek to apply scripting as a means of data collection for other mathematical investigations.

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Learning from Lesson Study in the College Geometry Classroom

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Authors of this proposal are members of an inter-institutional working group focused on the teaching and learning of transformations in college geometry courses taken by prospective secondary teachers. After exploring axioms and definitions for transformational geometry in our courses, we decided to shift to identifying not just what, but how students were learning about transformations in our courses. To explore this, we began a lesson study (Boyce et al., 2021). In this report, we discuss our engagement in the lesson study, its outcomes, and new directions.

Keywords: Geometry, Lesson Study, Professional Development, Transformations

The authors of this proposal are members of an inter-institutional working group focused on the teaching and learning of transformations in college geometry courses taken by prospective secondary teachers. As a newly formed group of educators with members representing a diversity of professional backgrounds in various higher education institutions from across the United States, it is not surprising that our preliminary work often focused on how different our courses and needs were. It was when we decided to move forward by focusing on our commonalities that the idea of conducting a lesson study took hold. We chose the goal of creating a lesson that could be used in all of our classrooms with a content focus on transformational geometry. As we refined that goal, we decided to craft a lesson that focused on definitions and axiomatic systems. We also wanted the lesson to engage students in learning from each other while comparing definitions in a way that was connected to secondary geometry objectives. Thus, we began our journey into a lesson study with a focus on transformational geometry.

Theoretical Framework

Lesson study, a teaching improvement and knowledge-building method with roots in Japanese elementary education, enables instructors to collaborate in small groups to prepare, teach, observe, and refine individual class lessons. It has been used extensively for professional development in K-12 mathematics settings; its use in tertiary settings is more limited but growing (Applegate et al., 2020; Kamen et al., 2011). The benefits of lesson study include building community and deepening mathematical and pedagogical knowledge (Fernandez, 2020). A guiding framework for the lesson study project can be found in the work of Cerbin (2011, p. 11).

The major phases of a lesson study cycle are:
1. finding a focus for the lesson study;
2. planning the research lesson;
3. planning the study;
4. teaching the lesson, observing, and gathering evidence;
5. debriefing, analyzing, and revising the lesson;
6. repeating the research cycle; and
7. documenting and disseminating lesson study findings.

In our case, the specific content focus for the lesson was the concept of isometry. In our planning, we chose to adapt a lesson successfully used by one of our group members, finding that it allowed us to embed all of our targeted goals for the lesson. As we focused on planning the study, we determined which of our group members would teach the lesson, whose classroom would be used, and how we would observe and record data related to student learning and their engagement with the lesson. Once the lesson was taught and the data and evidence were collected, we were able to debrief and organize the findings so that we could use what we learned to revise the lesson and begin again. It is important to point out that the data presented in this dissemination is from the debriefing sessions and represents the perceptions of the faculty involved. We used thematic coding (Gibbs, 2007) to categorize discussions in our debriefing sessions according to (1) our thoughts on the previous lesson and (2) what we planned to accomplish in future implementations of the lessons; and within each of these codes, we identified course content topics. In this report, we describe how engaging in lesson study has led us in new directions for attending to learning goals for our geometry students. We begin by providing some background on the lesson we created and implemented.

Lesson Background

At the onset we set some goals for the design of a lesson we could implement at our different institutions. We wanted students to learn from each other and talk about geometric concepts, definitions, and axiomatic systems in productive ways. We also wanted a lesson that would allow students to compare definitions and do so in a way that could be connected to secondary geometry from a transformation perspective. We explored several ideas for topics, such as symmetries and frieze patterns, and decided to design a lesson focused on Adinkra symbols. There are hundreds of Adinkra symbols, exhibiting a rich variety of symmetries and meanings (Adapo, 2010). Adinkra symbols originate in Ghana, and they have cultural significance, especially in Black urban communities within the United States (Oppong-Wadie, 2020). Introducing them to the prospective teachers in our geometry courses also had the potential to help them connect to this culture, which they could then build upon in their teaching.

The lesson we created consists of seven components. In the first component, students individually sort twelve Adinkra symbols and label their groupings. The second component involved students comparing and contrasting the ways their classmates created and labeled their groupings of the symbols (cf. Larsen, 2009). Did they identify symmetries or do something else? What mathematical language did they use in their labels of groupings? Here, the goals were to (a) establish a common vocabulary about symmetries and (b) provide opportunities for diverse mathematical thinking about the Adinkra symbols. During the planning stage of the lesson study, each member of the working group sorted the symbols themselves. They then compared and contrasted their groupings with the goal of anticipating the variety of student responses.

The third and fourth components of the lesson focused on a particular Adinkra symbol, Boa Me Na Me Mmoa Wo. Its name in English is, “Help me, and let me help you.” Boa Me Na

Me Mmoa Wo has reflectional symmetry (a vertical line of symmetry) and "nearly" has horizontal reflectional symmetry. For the third component, we asked students to identify the aesthetics of Boa Me Na Me Mmoa Wo and create another figure exhibiting those qualities. The fourth component involved them sharing and attempting to reach a consensus on a description of the aesthetic exhibited by Boa Me Na Me Mmoa Wo.

In the fifth component, the instructor provides additional foci for *mutuality*, which Eglash (2021) identified as the "near" symmetry aesthetic that Boa Me Na Me Mmoa Wo exhibits. Students read a one-page excerpt from Eglash's website that explains the concept of mutuality and calls for its definition as a mathematical property. The sixth lesson component involves students attempting to create such a definition. Working group members created and shared their own definitions in the lesson planning stage to help anticipate student responses. The final lesson component involves students reflecting on their learning experiences throughout the lesson.

The initial lesson was created with the intention of being repeated in a variety of settings. So far, it has been taught four times at three different universities. Due to differences in class time, some implementations moved initial or final lesson components to pre/post-class assignments. Each teaching session was delivered remotely, via Microsoft Teams or Zoom. Several (free-to-use) technologies were used to allow students to share their work (Flipgrid, Google Jamboard, Google Slides, and Padlet).

Research Questions

When considering the goals and results of a lesson study, there are two types of learning outcomes to investigate: the students' and the instructors'. Our purpose in this report is the latter; thus our analytical focus is on debriefing sessions of observers (including instructors) reflecting on lesson implementations. Our research questions are 1) to what did working group members attend in their debriefing sessions and 2) how did working group members' attentions shift over the course of the lesson study.

Methods

Data Sources

We primarily used Google Docs to create, document, and share our ideas, beginning with the group's initial lesson brainstorming and lesson planning. After the lesson was first taught, we incorporated (anonymized) versions of students' work as references for the observations that were written in the same shared document. In addition to screenshots of "final" student creations, we included screenshots of intermediate products (such as when students were working in breakout rooms). We incorporated transcriptions for all Flipgrid-completed video responses. Following the teaching of a lesson, we met over Zoom (recorded and transcribed) for an hour of debriefing, with at least one working group member taking notes. In addition to inputting their own observations, working group members added comments to the Google Doc to highlight others' ideas or to propose connections or questions for discussion. The initial lesson plan was modified between teaching sessions to accommodate differences in the class meeting duration as well as suggestions for modification from the working group participants. This updated copy of the lesson plan was used to enter copies of student work, observations and reflections, comments, and questions for the following teaching session.

Members of the working group were present as observers in two of the four teaching instances, the first and fourth. We focus on data from these two teaching sessions in the current study. The first observed setting was a 210 minute class taught in April 2021. The second observed setting was a 75-minute class taught in April 2022. We focus on the analysis of the post-teaching debriefing sessions for these two teaching sessions. The comparison of the two debriefing sessions aims to clarify how the teaching of the lesson has evolved from the observers' perspectives, how these outcomes inform lesson revisions, and how these changes reflect instructor learning.

Methodology

For the analysis of post-teaching debriefing sessions, the focus was on observers' (including the instructors') reflections on how the teaching went and what aspects could be continued or improved. The session transcriptions were open coded using qualitative data analysis software (NVivo 12). The second author (who was not involved in either debrief sessions or classroom observations) manually coded the data and compared it with the results of NVivo's automated insights to capture elements within the data that may have been overlooked. Multiple iterations of inductive/deductive thematic analysis (Braun & Clarke, 2006; Gibbs, 2007) were conducted to identify and refine the themes emerging from the data. The fifth author (who was involved in both debrief sessions and one of the observations) reviewed these themes and re-coded the data from each of the two debriefing sessions into two categories (thoughts on the implementation of the session and thoughts on next steps). The analysis shown in the results section represents the themes from the debrief sessions in a Sankey diagram (Figure 1), where the numbers in the diagram represent the occurrences of the themes in the conversations (Riehmann et al., 2005).

Results

The Session 1 [S1] debriefing was attended by six observers and the Session 4 [S4] debriefing was attended by five observers (including the instructors). Figure 1 depicts the eight coding themes that emerged from analysis across the debriefing sessions and their relationships with the two categories of reflection thoughts. We elaborate on these results by first identifying themes that were exhibited in one debrief but not the other, and then identifying themes that were present across the two debriefing sessions. In this way, we identify shifts in instructors' attentions over the course of the lesson study.

Divergent Themes Across Debrief Sessions

In the first debriefing session, *technology use and its issues*, *student self-evaluating, and student-instructor dialogue,* were three main points of discussion when talking about the effectiveness of the lesson. In one of these conversations about technology use, an observer mentioned, "....but they seem to get into precision and locked in on precision very quickly—they had to use GeoGebra. Because it wouldn't be precise, if they just sketched something." Another noted, "I also think sometimes the technology might have limited their own drawings." A co-instructor spoke about a norm of students self-evaluating their work in her class (the ones in this observation):

> And I mean, when it's really off that I would rather me say you're off task, I would give them the framework and tell me evaluate yourself: where do you see there's issues with what's happening and what's in relation to how effective this should be? And they're like, okay, yeah, this is off. And they kind of would make a note, but they would self-assess…because I hold them accountable.

Finally, when discussing the dialogue between students and teachers, the discussions focused on the necessity for these whole-class discussions to be more effective toward the goals of the lesson, as well as brainstorming ways to keep the class engaged. For example, a participant noted, “Because I think too often, we're always mesmerized by, you know, word choice. But is it mathematically sound? Is it accurate? What are they emphasizing? And if you're not careful, it’ll become non-productive.”

The fourth debriefing session highlights a shift away from identifying local issues with the lesson and towards more global issues, such as ideas for how the lesson fits more broadly with *meeting the goals of the course.* When it came to meeting the goals of the course, the debriefing participants discussed where to place the lesson in the course’s topic sequence: “I feel like this for me, would be a an initial lesson, as I don't see it as a heavy content-oriented lesson and I was thinking like where would I go after this if I took the class time?” and “[G]ood idea, because at the moment we don't have that follow up lesson and, yes, students might be left to think, what is it that I gained from this?”

Convergent Themes Across Debrief Sessions

A number of themes emerged across both debrief sessions. One example was how instructors could work towards aligning with other important mathematics education topics, such as *defining and precision*. When speaking about the role of the lesson and its relationship to defining and precision, an observer in S1 offered,

> I was thinking for the part where we would be offering a definition of mutuality and then looking at everyone else's sorting or their creation of their new symbol to see if that those fit the definitions, we could ask, ‘[W]hat makes a good mathematical definition—what is a mathematical definition?’

Another S1 observer commented that

> [I] thought that it was interesting that they were talking about precision, and in terms of like, being able to produce a very exact diagram. But, you know, you can be precise and have a very sketchy, inaccurate diagram. It's about like, how you're actually defining what's part of the diagram.

The other themes that were present across debriefing sessions, *student collaboration*, *technology and its issues*, *time management*, and *scaffolding*, were often intertwined with the *defining and precision* theme. For instance, an observer in S1 suggested,

> At [component three of the lesson], they are just trying to make something that has the same aesthetic as that figure. So [right now] it's still very open. So yeah, the idea that … it has to have ovals and triangles, and it has to have squares is valid. I mean, there's nothing to say that shouldn't be…. How soon do we want to tell them that that's wrong? Or guide them? … I think that if we did early, we might have been more apt to get more discussion about the transformations or the actual definitions.

Another S1 observer suggested,

> I’m wondering if we had gone into the groups, if we had been able to prompt them with a question. Like not, necessarily given it away, but had been able to ask them a question that could have, you know, prompted for their diagrams to get a little bit closer.

In S4, a different observer commented that,

> We provide only one example given to them and say this shape exhibits something we call mutuality. I’m wondering if we found a second shape that could be used, and after a certain point within the lesson said, ‘Okay, here’s another shape that exhibits mutuality,

how would you change your definition?’ If it’s a well-chosen second shape, it might start pushing them in the direction of using more of a transformational definition.

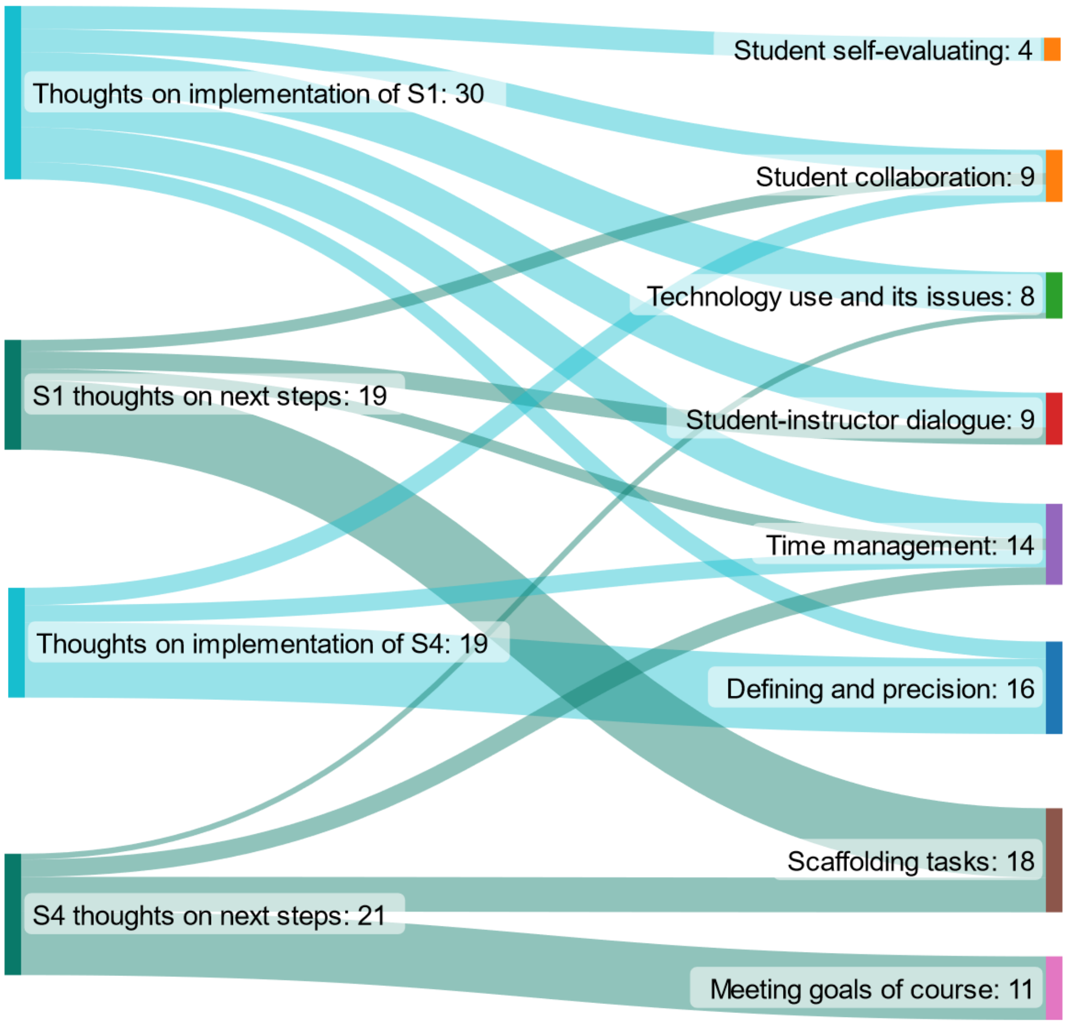

Figure 1: Sankey diagram of themes across debriefing sessions

Discussion

In this report, we have described how the foci of debriefing sessions changed over a one-year span within a lesson study. We have illustrated how observers’ attention regarding implementations and ideas for modifications evolved during debriefing sessions. These shifts accompany new ideas for lesson modifications. Our next steps in our lesson study include enhancing opportunities for instructors to make connections between the mathematical content found in the lesson we created to topics included in individual courses. An important outcome of these discussions is consideration of the placement of the lesson within a particular course. One instructor is planning for a next version of the lesson to be used as a springboard into a lesson about definitions in mathematics. The instructor wants to embed this topic into the course to better align the course offering with student learning outcomes developed by a group of mathematicians and mathematics educators. Those outcomes include that students will understand the role of definitions in mathematical discourse (GeT: A Pencil, 2022a). As a result, the next time the lesson is taught in this particular course, it will be followed by a lesson that will lead students to determine the need for precision in definitions by engaging them in an activity in which they will compare and contrast their constructed definitions during the lesson. In particular, they will examine if all constructed definitions of mutuality lead to the same symbols satisfying the definitions.

Furthermore, the next version of the lesson is planned to be modified to include more or alternative Adinkra symbols in order to give students sufficient opportunities to notice features that relate to transformational geometry topics of translations, reflections, rotations, and dilations. This is also a topic aligned to a targeted learning outcome whereby students will use transformations to explore definitions and theorems about congruence, similarity, and symmetry (GeT: A Pencil 2022b). In previous versions of the lesson, none of the symbols presented to students exhibited connections specifically to translations and dilations. Also, given that the change in Adinkra symbols will include more than one example that exhibits the concept of mutuality, the instructor will include time for students to reflect upon and discuss how the meaning of an Adinkra symbol might influence the existence of, or make more apparent, mutuality in the symbol itself.

As we continue to seek ways to make geometric concepts meaningful and engaging for our prospective teachers and their students, it behooves us to go beyond what has been done traditionally and venture into uncharted terrain. On the basis of his familiarity with Adinkra symbols, having conducted research on its uses in instruction and education by extension, the fourth author joined this project after the S4 debriefing to not only make meaningful contributions, but also emphasize that by designing and implementing a lesson study, students can be engaged in studying mathematics concepts from a non-western perspective. The task at hand includes working to provide a format for identification of geometric ideas and the development of working definitions in anticipation of student engagement with Adinkra symbols from the standpoint of its aesthetic qualities. The importance of the sorting activity to this lesson study will inform suggestions for the inclusion or deletion of specific Adinkra symbols when being revised. He will serve as a resource person for meanings, translations, and context of Adinkra symbols selected for use in this lesson study. By doing the aforementioned, we hope that prospective teachers will be exposed to different ways of teaching mathematics concepts and connecting to cultural knowledge for the benefit of learners.

Cerbin claims that "[a] lesson is a place where our instructional goals come to life" (2011, p. 3). Living that "life" with others serves to make the teaching experience deeper and more meaningful. A lesson study challenges us to not only individually decide what is important, but necessitates the communication of our beliefs to others. Through the process, we learn about ourselves and about others. We grow. Clearly, participating in this type of undertaking requires trust and respect. Without a doubt, the time spent on becoming a community and developing common goals and solutions has been time well spent for us. In the end, the work impacts our students. Focusing on a lesson as a shared experience allows us to nibble our way into broader changes with respect to what we teach and how we teach it. The experience also provides us a community where we can discuss student difficulties (in general) with others who care as much about the outcome as we do.

Acknowledgments

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Centering Connections and Collaboration in the Evaluation of STEM Education Transformation

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In this paper, we seek to understand the formation of a national network constructed for change in STEM education. To center the role of making connections in community and collaborations between network members, we use social network analysis (SNA) methods. We examine the impact of an in-person conference with the goals to bring together disparate groups around a common issue on network collaboration structure. We also examine the effect of post-conference virtual community building activities on network collaboration structure. A major limitation of SNA is response rate which, due to the pandemic, artificially drove a loss of collaboration. However, this loss was mitigated by new collaborations, particularly by influencers and boundary spanners. In addition, a sustained, year-long suite of virtual activities designed for community building and collaboration increased the collaboration activity in the network by 12.5%.

Keywords: evaluation, professional development, STEM education, networks, collaboration

A common form of professional development in STEM is the development of a learning community or community of practice. Originally designed to minimize professional development time while maximizing impact (Vescio et al. 2008), involvement in learning communities has been shown to impact postsecondary STEM classroom instructional practice (e.g. Manduca et al. 2017) and can also positively impacts persistence in STEM professional development activities (e.g. McCourt et al. 2017). The kinds of collegial interactions and support fostered in communities of practice have become recognized as a key factor in instructor adoption of cutting edge STEM classroom practices and can outweigh other barriers to reform (McConnell et al. 2020). McCourt et al. noted that participation in networks of individuals outside of one's department or institution provided participants with much needed space for pedagogical discussions and can transmit novel ideas across institutional boundaries (2017).

Educational innovation in postsecondary STEM education is both about infusing emerging technologies into classroom content, such as data science and course-based undergraduate research experiences, and about the broader human context, considering the social justice implications and the need to broaden participation (Rodenbusch et al. 2016, Holm and Saxe, 2016, Ballen et al. 2017, Alexander et al. 2022). This reflects a broader national dialogue on STEM education reform. In this context, an alternate target of intervention is the intentional building of networks and connections to support reform in undergraduate STEM education (e.g. Diaz Eaton et al. 2017). While the studies above focus on the efficacy of educator communities of practice in affecting classroom practice, a different set of literature driven by Kezar and colleagues strives to understand how these communities of practice can become communities of transformation (Gehrke and Kezar 2015, Kezar et al. 2018, Akman et al. 2020). However, researching programs that aim to act on a national reform level is challenging given the time scale of such change. In this paper we use network analysis as a tool to explore the structure of an emergent community of transformation. We ask:

- How does an in-person meeting affect collaboration network structure?
- Does participation in an in-person meeting affect engagement/persistence in the network?
- How does the availability of virtual professional development impact network collaboration structure and growth?

- Is network structure influenced by *boundary spanners* and/or *influencers*?

For the purposes of this paper, we define *boundary spanners* as individuals which broker knowledge between clusters of individuals and *influencers* as individuals who recruit a disproportionately large number of professional collaborations during the time of the study.

Theoretical Background

The co-authors come from a variety of educational training backgrounds in STEM and education including mathematics, education, biology, economics, and evaluation and include a variety of racial and ethnic points of view as Latinx, White, and Asian. Author 1, who is Latinx, draws their understanding of networks both from personal experience with support networks and from research in mathematical ecology and sustainability. However, both perspectives intersect in the writings of Rochelle Gutiérrez in "Living Mathematx: Towards a Vision for the Future" (2017). Gutiérrez outlines three Indigenous epistemologies, In Lak'ech, reciprocity, and Nepantla, which inform her framework for the teaching of mathematics, but which also deeply resonate with our theoretical perspectives on developing communities of practice and transformation. In Lak'ech describes a mutualistic community in which individuals can see the commonalities between each other. Reciprocity describes how communities of practice help one another achieve each other's goals through the sharing of experience and knowledge. Nepantla is the work of knowledge brokers and *boundary spanners*. *Boundary spanners* weave the community of transformation, moving across learning communities, negotiating tensions across contexts and goals, and moving the network towards transformation (Gutiérrez 2017, Masuda et al. 2018). We also draw from other frameworks which focus on the theory of change, or functional aspects of designing such learning communities and broader networks, and how they connect to change theory (Wenger 1998, LeMahieu et al. 2017, Gehrke and Kezar 2015). For example, an often first step of bringing together a community around a common goal, may be considered a way to foster In Lak'ech.

Social Network Analysis (SNA) is a mixed-methods approach that considers people (nodes) and the connections between them (edges) as important to the research question at hand and is often used in education research to examine learning effects due to peer interaction (Carolan 2013). In postsecondary STEM education, SNA research has focused on understanding classroom interactions and scientific mentoring (Aikens et al. 2016, Grunspan et al. 2017) and more recently has been used to examine adoption of evidence-based instructional practices in communities of practice (Ma et al 2019). However, SNA research in organizational theory examines community networks in the context of organizational change and the transformation of a field. We draw from mathematical work rethinking the influencer as someone who is well connected to someone which embodies reciprocity, making new active collaborations (Jun-Lan et al 2019). Work by Masuda et al. (2018) highlights the importance of Nepantla, examining the role of boundary spanning in innovation. Research by Frank et al. (2022) intervened in a network to promote boundary spanning. Still, many STEM education projects are designing networks as interventions, but lack techniques for evaluation of the network building aspects (Diaz Eaton et al. 2017). SNA promises an avenue that helps us understand both community formation and centers the connections and reciprocity important in a decolonized view of STEM education transformation. In this paper, we use data from a National Science Foundation (NSF)-funded research coordination network (RCN) in STEM education and use SNA techniques to explore the emergent community structure and collaboration.

Methods

Data Collection

The NSF RCN UBE program funds networks of individuals to advance undergraduate biology education and "activities across disciplinary, organizational, geographic, and international boundaries are encouraged" (NSF 2022, Diaz Eaton et al. 2016). The Sustainability Strategies for Open Resources to promote and Equitable Undergraduate Biology Education (SCORE-UBE) Summit was hosted by Bates College on October 17th through 19th, 2019. The goal of SCORE was to create a network for participants with various open education (OE) and STEM education affiliations to network, learn and address issues related to sustainability of open education and centering values of social justice, equity, diversity, and inclusion within open education. A total of 35 individuals representing more than 17 OE organizations participated in the SCORE Summit. The growth of the SCORE Network continued to grow through additional participation across new initiatives sponsored by additional funding grants and OE collaborations and via virtual activities hosted on QUBES Hub (an online community of math and biology educators who share education resources and methods to support instruction of undergraduate students to use quantitative approaches to tackle real, complex, biological problems). Over the 22 months of the study, activities included two virtual peer mentoring networks on equity and inclusion in open and STEM education and open educational practices, a virtual leadership program in open STEM education, a collaborative grant program which resulted in a virtual showcase of projects, and a working group on metadata justice for OER.

By June 2021, 72 individuals were current members of the SCORE group on QUBES Hub. SCORE members represented at least 67 different organizations/institutions – including 65 academic/higher ed institutions, nonprofits focused on open education, funding agencies/foundations, consulting firms, community of practice networks, professional societies, museums, and research labs. Members were geographically located across 28 US states and India. Disciplinary degrees of the steering committee included mathematics (2), biological sciences (2), STEM education (1), and library science (1).

All participants for the SCORE-UBE Summit and members who joined the SCORE group on QUBES Hub were asked to complete surveys using a multiple-criterion recognition question approach (Singleton and Asher, 1977). Members indicated the nature of their relationship with each participant in the network by selecting one of the following response options: (1) This is me; (2) I am not familiar with this person; (3) I am familiar with this person's work; (4) I have met this person; or (4) I have professionally collaborated with this person. For the survey, "professionally collaborated" was defined as actively engaging in cooperative work with another person, and this level is the focus of the social network analysis to follow. Response rates across survey administrations were 72.2% (26 of 36) of participants surveyed at pre, 30.5% (11 of 36) at mid, and 29.5% (18 of 61) at post. Over the course of the survey administration names were removed from the survey for those who did not participate in SCORE activities (i.e., registered but no attendance) and names were added to include new members who joined the SCORE QUBES Hub platform.

Network Analysis

Survey responses were de-identified by the evaluator in preparation for network analysis. For the purposes of the analysis below, we defined an edge between participant 1 to participant 2 if participant 1 reported that they had professionally collaborated with participant 2 (*i.e.* reported a level 5 connection in the survey). Therefore, the graphs constructed are directed graphs. For each pairwise comparison, we limited the adjacency matrix describing this network graph to those individuals who appeared in the both survey datasets, either as a respondent or as someone who was named as a collaborator. Comparisons of networks are visualized in a way that shows the transformation in collaboration during specific time intervals.

Connectance is calculated as the number of edges realized in the total network out of the possible connections. An individual is considered as an *influencer* if they gained a meaningful share of new edges during a particular time interval. An individual is considered a *boundary spanner* if multiple new collaborations are with individuals who are not already collaborating with each other.

Communities are detected using algorithm 'cluster_edge_betweenness' which calculates the *edge betweenness* (number of shortest path through the edge), removes the edge with highest value, and recalculates it, etc. Each color indicates a specific community in visualizations below.

All analyses and visualizations were conducted using the 'igraph' package in R (2022) and all code can be found on GitHub (https://github.com/yzha0/RUME_SNA).

Results

Summit's Effect on Collaborations

To gain a better understanding of how the Summit affected the network, we look at the general transformation of the network in Figure 1 over a time internal in which the SCORE Summit was held. Recall that connections in these graphs are those reported as "active collaborations." Connectance among individuals is somewhat low in pre-Summit survey (0.160), which is to be expected as the goal of this program is to bring together individuals across multiple contexts who may not have encountered each other before. From Figure 1, green links represent the new edge growth after the Summit and dotted links represent the loss of active collaboration in the network.

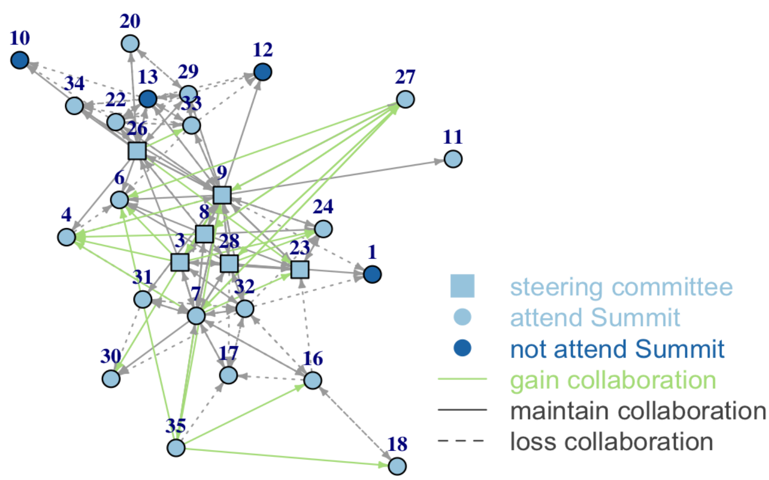

Figure 1. Change in collaboration network from prior to the SCORE Summit in October 2019 to after the SCORE Summit in February 2020. No activities other than the Summit were planned during this time interval.

Connectance decreases over this Summit time period (to 0.112). *A priori,* given the "have collaborated" language in the survey, we would not have expected any loss because historical collaboration would have continued to be reported. Indeed, very little of this loss (4 edges) were due to an individual reporting a lower collaborative status with another network member.

Instead, the majority of collaborative loss seemed to be driven by the response rate differentials. Individuals not attending the Summit (Figure 1, dark blue) suffered more collaborative loss than those who attended (Figure 1, blue), though most of these individuals did not respond to the February survey, which likely amplified this signal.

A few individuals were responsible for adding the majority of new connections, mitigating collaborative loss. Individual 3 seemed to report many new active collaborations, though the collaborators did not in all cases report them. It could be that Individual 3 was initiating collaborative discussions, whereas others saw them as more exploratory in nature. A clear influencer was Individual 27, who prior to the Summit, had weak connections to other SCORE Network participants - no active collaborations, but in some cases a familiarity with the person's work, but had never met. As a PI of a math education project, this person was an "outsider" to the "in-network" of a biology-heavy STEM education group, which brought new expertise and experience from outside the usual connections.

Promoting Collaboration via Virtual Activities

Between February 2020 and June 2021, the SCORE Network received additional funding from the Hewlett Foundation and launched multiple virtual activities. This included learning communities, a collaborative webinar series on Inclusive STEM Education, a newsletter, and a mini-grants program to foster collaborative projects within the network. While the network itself increased in size, in Figure 2 we limit ourselves to examining the change among those who were in the network in February 2020.

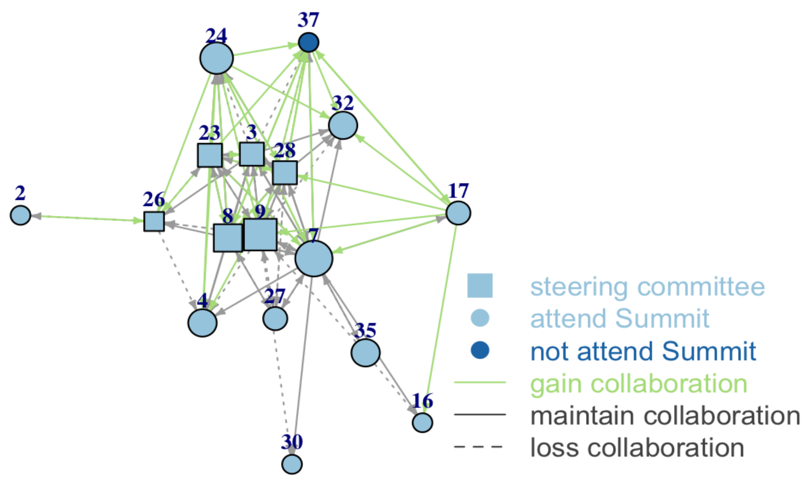

Figure 2. Change in collaboration network from just after the SCORE Summit in February 2020 to June 2021 at the end of the NSF SCORE-UBE grant period. A number of virtual and community building activities were planned during this time interval. Node size is scaled to the number of learning communities and mini-grants activities in which the individual actively participated during this time.

Despite another significant response rate drop among initially surveyed members, which should artificially drive connectance down, professional collaboration connectance in the network among survey respondents increased from 0.235 to 0.265, an effect size increase of 12.5%. By cross referencing the collaborations list to activity participation list, we can attribute a portion of this increase to a SCORE Network mini-grants program and joint authorship on an article about the Summit themes. The mini-grants program supported intentional collaborations across members to pursue and implement additional funding to support centering equity within

STEM Open Education as well as collaborations of members across multiple open education organizations to combine resources to support sustainability of their own initiatives.

At this point, SCORE Network members who were included in the initial survey, but ultimately did not attend the Summit are now no longer connected - ultimately they did not actively persist in this professional development network. However, we do see influencers and boundary spanners acting in this network. Individual 37 is a new member introduced to the network through virtual activities. They become an influencer in this network, gaining a significant number of active collaborations. Individual 17, who attended the Summit, becomes an influencer in the network and also a boundary spanner in the network, as their collaborations span across individuals that do not already have connections between each other.

In Figure 2, individuals that took part in multiple SCORE activities are represented with a larger node size. One can visually see that SCORE-sponsored activity participation does not necessarily correlate with increased collaborations.

Building an Interdisciplinary Community

Prior to the Summit, among the invitees, there were distinct two clusters of active collaboration in the network (see Figure 3, left), which together included approximately 43% of all nodes. The green cluster was primarily related to life and environmental science education and strongly tied to the QUBES community. The yellow cluster was related to the hosting location, Bates College. All other individuals did not have a strong clustering to each other and affiliation areas included Open Education, math, libraries, and non-profit foundations. By June 2021, the clustering has changed, so that approximately 82% of all nodes are now part of one cluster (see Figure 3, right) with open education, mathematics, and biology clustered together.

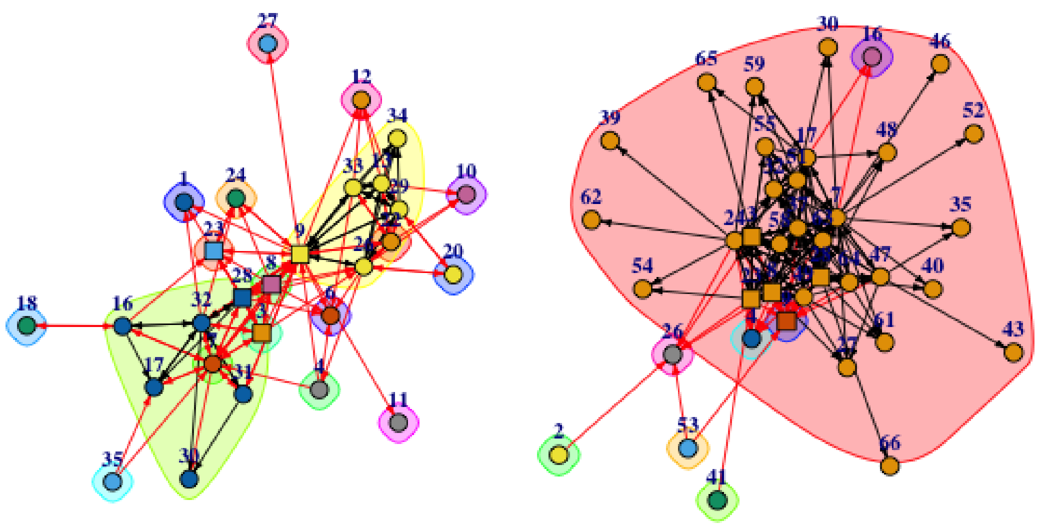

Figure 3. (Left) Clustering of the collaboration network in 2019 before SCORE activities. Navy nodes are included in the green-shaded cluster and yellow nodes are included in the yellow-shaded cluster. (Right) Clustering of the collaboration network after Summit and Virtual activities in June 2021. Orange nodes are included in the red-shaded cluster.

Discussion

The goals of the SCORE Summit were to help the community see commonalities, form relationships, and spark conversations about new directions and collaboration. The Summit itself did not provide a vehicle for active collaboration, yet individuals with new and boundary spanning interests provided an opportunity for new collaborations to launch. Attending the in-person summit seemed to affect persistence in the network for those that were present in the network early into the formation of SCORE. However, many more members joined during the virtual activities period, indicating that virtual activities are important to continued network growth. In addition, the vast majority of new collaborations were gained during virtual activities, and many virtual activities were led or co-led by SUMMIT participants. These virtual programs were designed to foster new and continued discussions on equity and inclusion and collaborations and spur new collaborations among Network members. This supports prior work pointing to the effectiveness of hybrid approaches (Brooks 2010, Hayward and Laursen, 2018). A network participant reflects on influence of this network in mathematics education reform:

> *"The [redacted mathematics society committee] has expressed an interest making considerations of diversity, equity, and inclusion more intentional and systematic in the way we think about the mathematics curriculum…Thanks in good part to the SCORE network and our summer study group, I feel more equipped to contribute to these discussions than I did a year or two ago."*

SNA allows us to quantify the increased collaboration and visualize the way that the structure of collaborations in the network has changed after a combination of in-person and virtual programming. We see a more unified community, possible indications of In Lak'ech - seeing commonalities. We can also see how particular *boundary spanners* and *influencers* aid in Nepantla, modifying the weave of the network to achieve this change.

Joining this structural data with qualitative surveys of learning due to the activities would help us better evaluate reciprocity - whether and how these structural bridges might reflect collaborative learning across the network. This would build on and connect to other work in STEM education studying communities of practice, such as Hayward and Laursen's paper (2018) which used SNA alongside qualitative techniques to understand the importance of receiving peer support and encouragement in implementing new ideas.

The determination of actors and relationships included in the pre-survey study were bounded by those who registered to attend the SCORE Summit, and while subsequent survey administrations sought to capture the relational data of new members to the network at these different time periods, this multiple-criterion recognition question approach (Singleton and Asher, 1977) did not capture changes in relationships for these additional members over time since new members were not included in previous survey administrations. Another limitation to examining network growth was the low response rates at mid- and final-survey administration resulting in an underrepresentation of network growth. While many factors are found to contribute to low response rates, it is uncertain how the increased use of survey-based research activities during the COVID-19 pandemic might have resulted in survey fatigue and decreased response rates (de Koning et al. 2021, Grandstaff and Webber 2021). Future network analysis which examines growth for communities of practice which evolve across time may consider different wording for multiple recognition questions or utilize an alternative approach such as the multiple name generator approach (Kogovsek et al., 2002). Despite these limitations with data collection, SNA provided valuable insights into structural changes in the network which may help support SCORE in a move towards a community of transformation.

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APOS Levels, Stages, and Coordinated Topics as Transitional Points Case of Range of a Linear Transformation

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This paper discusses findings from an ongoing study investigating mental mechanisms involved in the conceptualization of linear transformations from the perspective of APOS theory. Data reported in this paper came from 44-first year linear algebra students' responses on a task regarding the range of a linear transformation. Our analysis revealed differences in APOS Levels, Stages, and Coordinated topics. Moreover, our findings pointed to connections between Levels/Stages of the range concept and mental mechanisms of representations of matrix multiplications. That is, our findings revealed that the absence of these representations from ones conceptualization process may result in irreversible misconceptions. More importantly, we identified more transitions occurring among Levels/Stages when linear combination expressions were considered as representations of matrix multiplications.

Keywords: Linear Transformations, APOS theory, Mental Mechanisms, and Transitional Points.

In linear algebra education, many topics, such as vector space and eigen-theory, have been studied extensively. To this point, linear transformations received very little attention. In the Literature, we found three studies that investigated connections between transformations and function ideas (Andrews-Larson, et al. 2017; Zandieh et al. 2017; Turgut, 2019), and only one by Oktac, et al. (2021) that investigated mental mechanisms of the null space of a linear transformation from the perspective of APOS theory. Our ongoing work is too investigating various features of linear transformations from the perspective of APOS theory. In this paper, we discuss APOS Stages, Levels and Coordinated topics our participants displayed in their responses to a question regarding the range concept. Our main goal was to examine potential transitional points in the learning of the range concept.

Literature and Theoretical Perspective

The Literature on Linear Algebra Education

Two recent publications, the ICME-13 Monographs and Journal of ZDM 2019 special issue on linear algebra, compiled most up-to-date research on linear algebra education. Between the two, some studies investigated general topics such as instructional approaches (Trigueros, 2018) and visualization (Harel, 2019) and others looked at specific topics such as vector spaces (Caglayan, 2019), eigen-theory (Karakok, 2019), and linear independence (Dogan, et al., 2022; Dogan, 2019). Many of them included, in their analysis, theoretical perspectives; for example APOS (Oktac, 2021), Tall's three worlds of mathematical thinking (Stewart et al, 2019), and Harel's DNR concepts (Harel, 2018).

Up to this point, we found only four studies concerning linear transformations. Three of which are by Andrews-Larson, et. al. (2017), Zandieh, et. al. (2017), and Turgut (2019). The forth is by Oktac, et. al. (2021). The first three studied the conceptualization of matrices as linear transformations. These authors discussed how the knowledge of the function concept had implications for one's identification of linear transformations. The fourth study by Oktac et. al. (2021), borrowing from APOS theory, discussed APOS Levels as displayed in their student responses to a question regarding the null space of a linear transformation, specifically, focusing on the pre-image notion. Their findings pointed to the role of reversal of linear transformations in the conceptualization of pre-image and null space ideas. Our work reported here, also borrowing from APOS, studied our participant responses to a question concerning the range of a linear transformation.

Theoretical Perspective-APOS

APOS theory describes features of mental mechanisms applied in the formation of the knowledge of advanced mathematical topics. The theory describes the mental mechanisms through Stages, namely, Action (A), Process (P), Object (O), and Schema (S). In our analysis, we considered APOS Stages as mental mechanisms our participants were working with at the time of the data collection. The theory considers the Stages (A, P, O, S), interchangeably, to stand for conceptions, structures, mechanisms, and stages (Arnon et. al., 2014).

According to the theory, learners are functioning at an Action Stage if they are acting on entities via external stimuli. For example, one may observe a learner functioning at an Action Stage if the learner feels the need to solve the system Ax=0 before determining the linear independence of a set of vectors. Here, solving the system Ax=0 provides an external stimulus for the learner.

APOS considers a learner to be working at a Process Stage if the learner determines the linear independence through a mental recognition (an internal stimulus) of certain features of vectors such as the ones with the same or different component values. Here, the mental recognition of the features provides an internal stimulus for the learner.

According to Oktac, et. al. (2021), "When Processes are encapsulated they become Objects to which Actions or Processes related to other concepts can be applied." (p. 3). Thus, a learner identifying linear independence via the comparison of number of vectors and the dimension of a space is considered to have encapsulated Processes. That is, the learner is considered to be acting at an Object Stage, having already objectified vectors as objects of a space (set).

According to APOS, there are additional conceptual entities that can facilitate the development of and transitions between Stages. The ones that are relevant to our work are named as Coordination, Reversal and Levels. Coordination is described as involving mental mechanisms where two or more mechanisms united to give rise to a new Stage. Reversal is described as mental processes being reverted. Levels are defined as partial mental mechanisms whose features fall in-between Stages. An example of a Level, between Process and Object Stages, is when a learner considers the linear independence concept in the context of a set of three vectors with only one having a zero component value. Here, the learner is clearly functioning with an internal stimulus, mentally recognizing a pattern among component values of vectors. That is, the learner is applying a mental mechanism acting at a Level between Process and Object Stages.

APOS, furthermore, considers Stages as invariant constructs and asserts the necessity of each Stage for the development of successive Stages (Arnon et. al. 2014).That is, Stages are considered to be sequential. Levels, on the other hand, are considered non-invariant and may vary. The theory further elaborates that "a subject may be able to move to the next Level or Stage rapidly so that a Level is skipped, done very quickly, or is not observable in the already acquired higher Level or Stage" (Arnon et. al. 2014, p. 139).

Methodology

Our data came from 44 first-year matrix algebra students' responses to a question on the nullspace and range of a linear transformation (see fig.1) at a US institution from spring 2020 semester. Student came from a matrix algebra course met twice a week at noon via Blackboard, an online course delivery platform, for an hour and 20 minutes. The question was administered before the end of a class meeting through a problem solving session lasting about 20 minutes. Groups of three to four were formed and sent to designated online rooms for groups to discuss the question among themselves. Instructor was absent from the rooms and discussions. Before leaving the main class meeting room to their designated rooms, students were instructed to, individually, submit online, by 11:59 pm the same day, their own responses with explanations in detail. The particular class meeting ended by groups finishing their discussions in the designated rooms. The student population comprised of engineering, science, and mathematics majors. Linear algebra was the first theory-oriented course with high abstract level that the students took. The students came with very little or no maturity for the high abstraction required by this course.

We analyzed student responses to the question from the perspective of APOS theory. Specifically, we, first, categorized the responses based on student descriptions of the range concept (or its vectors) of the transformation given in Figure 1. Next, we identified APOS Stages and Levels displayed in responses of each category. In this paper, we discuss only the findings that came from part ii of the question seen in Figure 1.

Consider T:R^2->R^2 with $T([a\ b\ ]) = [1\ 1\ 1\ 1\ ][a\ b\ ]$. State at least three vectors that are:
i) In Null space (T) ii) In Range (T). Explain your reasoning in detail.

Figure 1. Mathematical task considered during group discussions and individual responses submitted online.

Results

We identified five distinct categories. Table 1 summarizes categories, Levels, Stages, and Coordinated topics. Overall, categories 1 and 2 had the highest tally of responses (12). The highest variability among Levels/Stages came in category 3 with three different mental mechanisms and the least variability came in category 5 with one Stage. Each of the categories 1-3 considered a representation of the matrix multiplication given in Figure 1. By means of these representations, some responses in these categories applied external stimuli in determining range vectors. One external stimulus was the evaluation of the matrix multiplication, A$[a\ b\ ]$, mainly in category 1; the general form, $[a + b\ a + b\ ]$, in category 2; and the linear combination form, $a[1\ 1\ ] + b[1\ 1\ ]$, in category 3. Each of these evaluations were done for specific values of *a* and *b*. The values of *a* and *b* took varying roles (for the participants) from *(a, b)* pairs being vectors of the domain of the transformation to *a*, *b* values being scalars for the linear combination form, $a[1\ 1\ ] + b[1\ 1\ ]$.

Mental mechanisms applying internal stimuli were also present, in these three categories, in an increasing trend from category 1 to 3. That is, category 1 (discussed in detail below) included about 8% of responses (1 out of 12 responses in the category) with the application of internal stimuli. This presented itself when students considered a general structure, out of the matrix multiplication given in Figure 1, for range vectors making statements similar to: "the multiplication of the matrix and *a, b* values gives top and bottom the same." Category 2 included about 58% of responses (7 out of 12 responses in the category) where the expression $[a + b\ a + b\ ]$ was interpreted as range vectors having the same component values without any external calculations. Category 3 included about 70% of responses (7 of 10 responses in the category) mentally connecting the linear combination form, $a[1\ 1\,] + b[1\ 1\,]$, to a spanning set or a basis set of the range, and considering range vectors as the object of these sets.

In comparison to those in categories 1-3, category 4 (total 6) displayed drastically different mental mechanisms. Responses in this category determined range vectors based upon the structure of vectors of the null space. They did this, for the most part, acting on internal stimuli. This category is discussed in detail later in the paper.

Responses (total 4) in category 5 were either incomplete or irrelevant to the task on hand. One response, for instance, gave 2x2 matrices as range vectors. Three of them calculated externally the reduce row echelon form (Johnson et. al. 2002) of amalgam of the matrix in Figure 1 with specific values of *a* and *b*. They, however, fell short in, correctly, identifying range vectors by using the rref form. Two of these left their work at the rref form without any further explanations. The third one made a statement along the lines of: “no range since the rref implies no solution.” In class, the rref process was introduced to identify vectors of a basis/a spanning set for the range of linear transformations. The three students appeared to have recognized the rref process having something to do with linear transformations but fell short in displaying any knowledge of connections between the rref forms and bases/spanning sets.

Let’s us now look at categories 1 and 4 with little more details in order to give readers further insights into the nature of responses and mental mechanisms.

Table 1. Category, Level, Stage, Coordinated Topic, and Tally.

Range	Tally	#/Level/Stage	Coordinated Topics
1.MM*	12	2/A-P*	MM/Pi*/NS*
2. CV*	12	2/A-P	MM/Pi.
3. LC*	10	3/A-O*	LC/B*/S*
4. NS*	6	2/P-O	NS
5. IR*	4	1/A*	RREF/SS*

**A-Action Stage; P-Process Stage; O-Object Stage; *on prior-knowledge (not on Range); MM-Matrix Multiplication; Pi-Pre-image; NS-Null Space; LC-Linear Combination; S-Spanning set; B-Basis; SS-Solution set; CV-Component values; IR-irrelevant.*

Category 1: Range as the Result of Matrix Multiplications

There are 12 responses in this category. In order to determine range vectors, majority of participants evaluated matrix multiplications for specific values of *(a, b)* pairs, which represented (for the participants) domain vectors of the transformation. Moreover, some considered any values for *a* and *b* while others considered values with the condition, *a=b*. The response in Figure 2 exemplifies the latter behavior. This response is carrying out the multiplication of the matrix in fig 1 with *a, b* values where *a=b* i.e $[2\ 2\,]$. Figure 3 gives a case for the former behavior. This response is using different values for the pair *(a, b)*. For example, the participant is multiplying the pair $[10\ 7\,]$. with the matrix in fig 1 and obtaining a range vector $[17\ 17\,]$. Mental mechanisms displayed in both responses in Figures 2 and 3 needed external stimuli in determining range vectors. Thus, the mental mechanisms were acting at Action Stages.

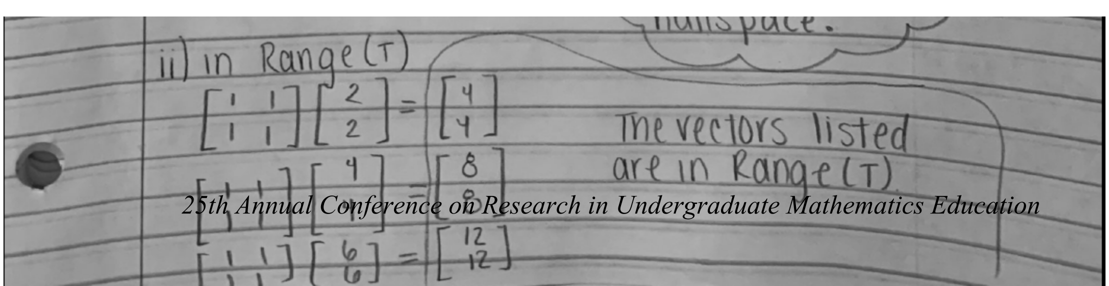

Figure 2. Domain vectors with a=b -externally calculating matrix multiplication.

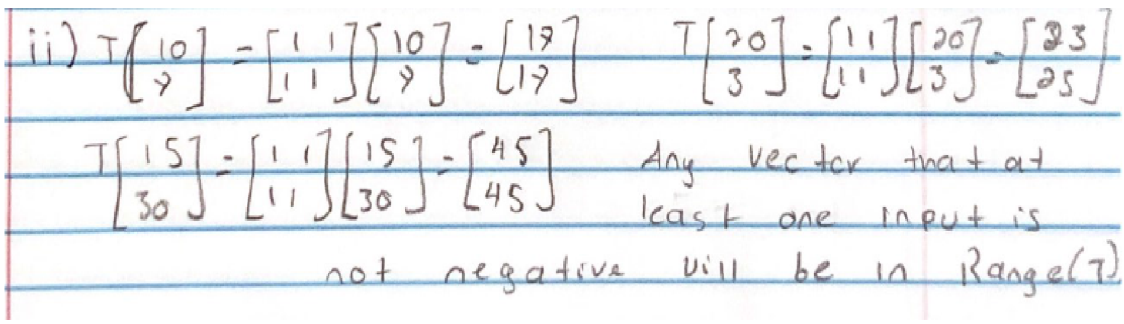

Figure 3. Domain vectors with any a, b values -externally calculating matrix multiplication.

In addition to the two behaviors described above, there were also a few responses revealing mental mechanisms acting at Process Stages. That is, they revealed participants considering matrix multiplication processes internally describing what would be obtained if a multiplication was carried out without any external computations. Figure 4 portrays one such tendency. This participant is considering generic pairs *(a, b)* and asserting that "all these vectors are the answer of $T([a\ b\ ]) = [1\ 1\ 1\ 1\ ][a\ b\ ]$" with no external calculations. The mental mechanism revealed in this response is clearly the one acting at a Level closer to a Process Stage.

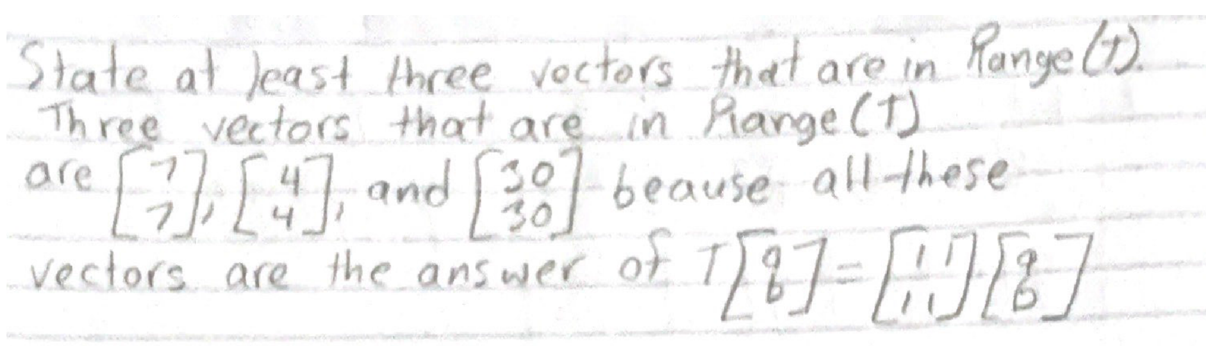

Figure 4. Consideration of matrix multiplication processes internally (mentally).

Category 4: Range Compared to Null Space

Responses in category 4 differed noticeably from those of categories 1-3. As far as we know, this category reveals mental mechanisms that have not yet been reported in the Literature. There are 6 responses in the category. This category determined range vectors based upon the structure of vectors of the null space of the transformation in Figure 1. We identified two distinct mental mechanisms in this category. One considered range vectors as those (from the domain of the transformation) with characteristics dissimilar to the ones in the null space of the transformation. Response in Figure 5 is displaying this tendency. This participant is considering vectors with same component values as vectors of the range. The participant's comparison of range vectors to vectors of the null space is clear in the explanation, “product with $[1\ 1\ 1\ 1\ ]$ will not give a $[0\ 0\ ]$ matrix.” Mental mechanisms displayed in similar responses are classified as acting at Levels closer to Process stages.

There were also responses considering the range and null spaces as complementary sets making up the domain of the transformation. Figure 6 is revealing this behaviour clearly. This participant is declaring that any vector that is not in the form $[x\ \ -x\ ]$ (identified as the structure of vectors of the null space) as the vectors of the range. In light of the following statement “any vector of the form [x;-x] <u>exists</u> within the null space of T,” (Emphasis is on the term “exists”) given for part i of the question, one can identify mental mechanisms, displayed in this response, as the ones acting at Levels closer to Object Stages. That is, in this response, vectors are considered as objects of the null space, subsequently objects of the range set.

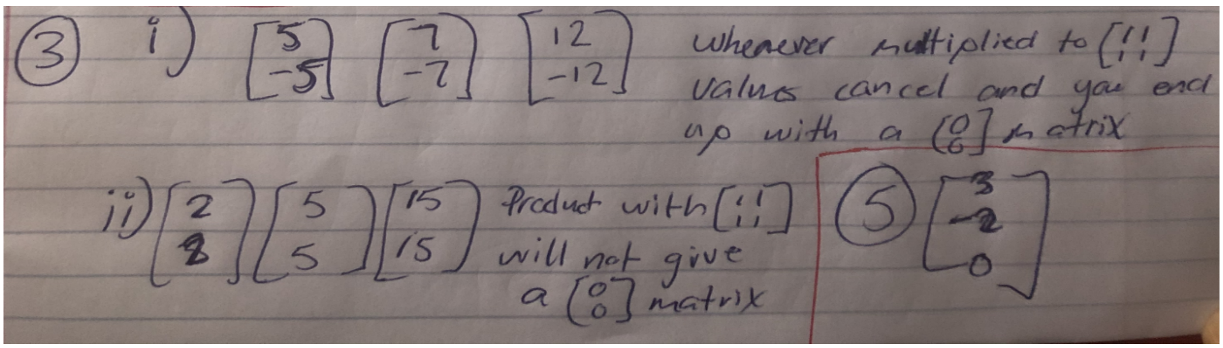

Figure 5. Range vectors in comparison to the structure of vectors of the nullspace.

3i.) Three vectors in the Nullspace of T are [1;-1], [2;-2], and [-3;3]. In fact, any vector of the form [x;-x] exists within the nullspace of T.
3ii.) Three vectors in the range of T are any vectors NOT of the form [x;-x], which creates an infinite list. Some examples are [1;-4], [5;2], and [x; y] such that y is not equal to x.

Figure 6. Vectors other than those in the nulspace (Response was typed by the student).

Unfortunately, even though mental mechanisms applied in the responses of this category were acting at higher Levels/Stages, inaccurate knowledge structures were displayed. This seemed to have been the result of, somehow, participants’ attention being directed to the null space concept, escaping any ideas from categories 1-3 (matrix multiplication representations).

Discussion and Conclusion

In this paper, we discussed findings from an ongoing study investigating mental mechanisms involved in the conceptualization of linear transformations from the perspective of APOS theory, specifically, findings from responses of 44-first year linear algebra students on a task concerning the range of the linear transformation given Figure 1. Our main goal was to identify transitional points in the learning of range ideas.

Overall, our findings show that responses employing various representations of the matrix multiplication, A$[a\ b\ ]$, in Figure 1, presented evolving Levels of mental mechanisms. That is, from category 1 to category 3, we documented mental mechanisms acting at evolving Levels from Action Stages to Object stages.

The highest variability among Levels occurred in category 3 when participants considered linear combination forms, $a[1\ 1\ ] +\ b[1\ 1\ ]$, in place of matrix multiplications. In this category, we documented mental mechanisms acting at all three APOS Stages, Action, Process and Object. Object Stage mental constructs were absent from categories 1 and 2. Indeed, category 3 and category 4 were the only categories whose responses revealed mental mechanisms considering range vectors as objects of sets (thus, acting at Object Stages). This notion was conducted accurately in category 3, but not so in category 4. Even though, in category 4, mental mechanisms were considering vectors as objects of a range set, acting at Levels closer to Object Stages, they were faulty mechanisms incorrectly recognizing range vectors as those of the domain vectors not in the null space (or not similar to the structure of vectors of the null space). In fact, participants, in this category, appeared to have skipped entirely the ideas included in categories 1-3, specifically, the consideration of representations of matrix multiplications.

We also found that approaches employing the rref process (a very common approach in many matrix algebra textbooks in the US) may not be very efficient in helping learners form an accurate understanding of the range concept. In fact, responses (total 6) in category 5 are testimonials to the inefficiency of these approaches.

In summary, our findings point to mental mechanisms reflected in categories 1-3 as being the more effective ones toward the accurate formation and advancement of the knowledge of range ideas. Our findings, furthermore, point to desired (in the context of accurate knowledge structures displayed in participant responses) transitions between APOS Stages to be occurring with the consideration of representations of matrix multiplications, especially, in the case of linear combination representations of matrix multiplications. Even though the notion of evolving representations is not specifically addressed in this paper (a topic of future studies), we conjecture that changing characteristics of the representation of matrix multiplications may be a sign of evolving mental mechanisms acting on the representations.

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“Looking Outside of my Bubble”: Whiteness-at-Work in Mathematics Faculty Sensemaking about Serving Latin* Students

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Professional development (PD) is often recommended to equip faculty to serve racially minoritized students through instruction. However, limited work has examined equity-oriented PD for mathematics faculty, who often hold views of instruction as race-neutral. This contributed report explores the influence of a two-year PD for faculty in a mathematics department engaged in equity-oriented reform at a Hispanic-Serving Institution. We present two cases of white faculty members who demonstrated a limited ability to interrogate their white racial identities in relation to their instructional impact, despite their engagement in a sustained PD designed to promote racial equity. Implications are provided for equity-oriented PD for mathematics faculty.

Keywords: whiteness, faculty, pedagogical reasoning, Hispanic-Serving Institutions

Study Purpose & Background

Faculty sensemaking about race directly influences instructional practices in postsecondary environments. For example, white faculty with high levels of racial consciousness often disrupt inequitable educational structures to expand educational opportunities for racially minoritized students, while white faculty with low levels of racial consciousness may incorporate instructional practices that uphold white supremacy (Haynes, 2021). Faculty’s critical reflection about their own identity is a component of this sensemaking that is vital to enact anti-racist pedagogy (Haynes & Bazner, 2019; Kishimoto, 2018). For example, white faculty who have interrogated their own racial identity demonstrated less concern for protecting white interests and assumed more risks to teach in racially equitable ways (Haynes, 2017). Similarly, faculty of color often draw on their experiences of race and racism to inform inclusive and humanizing teaching practices (e.g., Williams, 2016) despite the risks to their legitimacy that they assume when adopting unconventional teaching practices as minoritized faculty (e.g., Sulé, 2011).

Within STEM fields, ideologies related to social neutrality impede recognition of inequities (Gutiérrez, 2013; Leyva et al., 2022; McNeill, Leyva, White, & Mitchell, 2022). Colorblindness is a dominant ideology among STEM faculty as a group (Russo-Tait, 2021), with some documented race-consciousness especially among STEM faculty of color (Bensimon et al., 2019; Ching, 2022). For example, Haynes and Patton (2017) report on a white computer science instructor whose course content had racial relevance (e.g., a computer game using a migrant farm worker avatar) but who did not address race due to his perception of computer science as race-neutral. Mathematics reflects these trends with many instructors holding colorblind views of mathematics instruction, with exceptions found largely among faculty of color (McNeill & Jefferson, 2022; McNeill, Leyva, & Marshall, 2022).

Studies call for faculty professional development (PD) to improve instructional capacity to serve racially minoritized students and alleviate pressures on faculty of color to carry out institutional equity aims (e.g., Leyva et al., 2021; Casado Pérez, 2019). PD for mathematics instructors is often integrated through collaboration, apprenticeship, and guidance from course coordinators (Ellis, 2015; Rämö et al., 2019). However, such PD designs have not focused on equity. In a study of community college mathematics faculty in an isolated equity-oriented PD workshop, Ching (2018) found that faculty demonstrated conceptual change initially, but failed to sustain equitable instructional viewpoints over time. Such findings indicate a need for research that explores the efficacy of integrated and long-term mathematics faculty PD in promoting critical self-reflection about faculty identity and instructional practices.

To address this need, our report presents an analysis of racial sensemaking and identity development among two white mathematics faculty at a Hispanic-Serving Institution (HSI). We draw from a larger study of the department's reform of instruction and organizational practices to better serve Latin*[1] students, which involved faculty participation in a two-year, integrated PD. Our research questions explore how the PD influenced faculty's critical self-reflection: (1) How do faculty understand their white identities in relation to serving Latin* students through mathematics instruction?; and (2) In what ways does this change over the course of the PD? We raise implications for equity-oriented PD for mathematics faculty based on our findings.

Theoretical Perspectives

We engage two theories to guide our analysis of faculty sensemaking. To examine the ways that participants understand their identity in relation to serving Latin* students, we adopt emerging mathematical and racial identity constructions (EMRICs; Oppland-Cordell, 2014). Originally used to explore Latin* mathematics students' identity development, EMRICs characterize the multifaceted ways in which participants make meaning of their own racialized identities in mathematics spaces, individually, socially, and politically. For example, a Latin* student changing peer groups to enable math collaboration in Spanish can signify a shift in their EMRIC (Oppland-Cordell, 2014). Addressing inequity in mathematics requires white faculty to reckon with the tensions inherent in fostering Latin* student success within a historically white discipline (Hottinger, 2016) as white authority figures. Our uptake of EMRICs thus, facilitates exploration of faculty's reconciliation of their mathematics and racial identities.

Previous work has depicted mathematics faculty shifting between colorblind and race-conscious views of instruction (McNeill, Leyva, & Marshall, 2022). This reflects broader societal patterns in which whiteness demonstrates elasticity to maintain dominance in a continuously changing U.S. racial context (e.g. Bonilla-Silva, 2006). Whiteness flexibly adapts to embody contradictions and paradoxes that maintain racial inequity while seeming race-neutral or progressive at face value. To examine these contradictions in mathematics faculty sensemaking, we engage Yoon's (2012) construct of whiteness-at-work that explores how paradoxes are enacted to reinscribe whiteness as neutral and invisible while reinforcing the marginalization of people of color. For example, white teachers expressing a desire to call out colleagues on racially problematic viewpoints while also avoiding workplace conflict exemplifies whiteness-at-work (Yoon, 2012).

Methods

Study Context and Participants

Our present analysis comes from a larger study exploring the effectiveness of an ongoing, equity-oriented PD in the mathematics department at Sonoma State University (SSU), conducted by two SSU faculty involved in the PD, three SSU students, and educational researchers at

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[1] The asterisk in Latin* creates space for fluidity in gender identities among Latin American people. Latin* responds to (mis)use of Latinx, a term reserved for Latin* gender-nonconforming peoples (Salinas & Lozano, 2019).

Vanderbilt University. SSU is a medium-sized, public HSI in the western United States. In 2021, undergraduate students at SSU were approximately 45% white, 35% Latin*, 7% two or more races, 5% Asian, 2% Black or African American, and 6% some other race. The two-year PD aims to develop culturally responsive instruction and student support practices to better serve Latin* students. PD began in summer 2021 and data collection started in fall 2021. This report presents an analysis of data collected over the first year of PD.

The SSU mathematics department holds a widespread commitment to advance equity, as evidenced by nearly complete mathematics faculty participation in the PD, and by approximately 75% of the PD faculty participating in the study. Active and inquiry-based learning is the norm in SSU mathematics instruction, and faculty regularly voice teaching philosophies related to normalizing mistakes, encouraging multiple approaches, building students' confidence with mathematics, and fostering positive instructional and peer relationships.

Data Collection

Educational researchers at Vanderbilt (1 faculty, 2 Ph.D. students, 1 master's student, and 1 undergraduate) assumed all data collection responsibilities to maintain participant privacy. Data from faculty participants consisted of journaling and interviews. Faculty journaled about events specific to instruction and support that they perceived as marginalizing or supportive for Latin* students, including a description of the event and a reflection on their interpretation. Faculty journaled throughout the 2021-2022 academic year without a required number of entries.

Near the end of each semester, participants completed a 90-minute, semi-structured individual interview on Zoom. Interviews were audiotaped and transcribed. The first interview explored how participants characterized serving Latin* students and solicited participants' interpretations of two instructional events that reflected emergent themes from journaling. The second interview explored participants' experiences of the PD, and asked participants how thinking about Latin* students as a group or as individuals supported their aims related to serving Latin* students. Both interviews prompted faculty to reflect on how their own identities shaped their perspectives, practices, and instructional efficacy in serving Latin* students.

Data Analysis & Positionality

Vanderbilt research team members de-identified data prior to sharing with SSU research team members for analysis. Information that explicitly or implicitly revealed participants' identities (e.g., names, professional histories) was redacted. For the present research report, we completed an analysis of data specific to two white participants, Tina and David, who regularly invoked ideologies tied to whiteness (e.g. universalism[2]). Our focus on white faculty responds to the reality that even at HSIs, most Latin* mathematics students will be taught by white faculty. Thus, understanding how white faculty develop capacity to serve Latin* students is essential. Our findings begin with profiles of the focal participants to provide context for interpreting their narratives. We omit details about their professional roles to maintain confidentiality.

A research team member from each university coded David's data. Two Vanderbilt team members coded Tina's data since de-identification was still in progress. Team members independently and inductively coded data to flag instances when participants grappled with their own identity in relation to professional practices. One coder from each pair synthesized codes for each participant's data to identify whiteness-at-work in their perspectives and described instructional practices. Themes were exchanged and discussed during weekly team meetings.

[2] In relation to whiteness, universalism describes the framing of white experiences as universal (Bonilla-Silva, 2006)

Our research team approached the present analysis with critical reflexivity. The team consists of a nonbinary Latinx person, a Latina cisgender woman, two Latino cisgender men, a Black cisgender woman, a Black and white mixed cisgender woman, a white transmasculine person, two white cisgender women, and a white cisgender man. We brought awareness of how our varying forms of privilege and oppression influence our inquiry on whiteness in mathematics. The team resisted deficit engagement with participants' reflections and constantly recognized how mathematics instruction is situated in broader systems of social power. Interviewers and coders bracketed their lived experiences when engaging with participants' reflections to avoid analytically distorting their perspectives, all while approaching the study with a lens of criticality to interrogate whiteness in mathematical contexts across HSIs.

Findings

David and Tina hold a combined 40 years of experience as mathematics faculty. They each entered interviews with questions prepared and took notes throughout, showing a desire to learn from the interviews. Of all faculty, David submitted the most journaled events and reported that he "spent a couple hours on every one of those" each week in addition to his PD participation. Tina discussed PD topics at home when struggling to integrate her learning about racism into her professional practices. Both shared a deep concern for students' comfort and wellbeing. As examples, David shared a time in which his offer of instructional support to a Latina student resulted in the student leaving the class crying for reasons he didn't understand. David sent multiple emails to the student afterwards in an effort to re-establish rapport and after the semester passed was "still losing sleep over that student." Tina shared an experience in which a Latino student sought her support after experiencing sexual violence, which she characterized as "one of the successes of [her] career… that a student in crisis felt comfortable coming to [her]." This context importantly illustrates that their ideological engagement in whiteness cannot be attributed to lack of effort or care.

In what follows we illustrate how David's and Tina's EMRICs depict whiteness-at-work, maintaining the invisibility of whiteness to impeding their focus on Latin* students, addressing our first research question about faculty's perception of their racial identities. Tina cited gender as an analog to support her (in)attention to race, using experiences of gender marginalization to claim insider insight into Latin* marginalization and alternately appealing to math as gender-neutral to justify her inattention to students' race. David demonstrates awareness of systemic racism, including inequity in education, but uses race-neutral frames to reason about teaching and learning in mathematics. In both cases, the construction of mathematics as socially neutral supports colorblindness in instruction. In each case, we discuss how their perspective changed over the first year of the PD, addressing our second research question.

Tina

Tina described "foster[ing] females who are going through the program" (Interview 1) as central to her professional role. This shaped her orientation to the PD, viewing her developing support of Latin* students as analogous to her support of women: "I like this idea of broadening my horizons of how to not just help females, but all students of all ethnicities" (Interview 1). Her characterization of helping "students of all ethnicities," however, illustrates incongruities in her analogy of gender to race; when discussing women, she explicitly names this group, while her reference to students of all ethnicities illustrates a discursive avoidance of Latin* identity. This incongruity signals whiteness-at-work; although Tina sees the PD as an opportunity to develop advocacy for Latin* students as she had for women, she does not center Latin* students when discussing issues of race as she does for females when discussing gender, as we further elaborate.

When asked how she saw herself serving Latin* students, Tina shared an experience of coordinating sections of calculus with all-male colleagues as an example of how she leveraged her own experiences of marginalization in mathematics to relate to Latin* underrepresentation.

> I was… the only woman in the room… I remember one of the males looking at me and saying – We're putting together a calculus final, you got to understand that – He looks at me and goes, "As a woman, what questions do you think should be on the calculus final?" And I thought, "As a woman?"... It wasn't an issue that should depend on gender at all… I think I may be able to understand a little better than maybe some people about how odd it is to be sitting in a room and to be, say, the only woman, or the only Hispanic student, or the only African American student, or the only…Muslim student… I taught a class once, I had a nun… She's obviously going to feel singled out. (Interview 1)

Although Tina cites her gendered experience as a way to relate to Latin* students, she doesn't discuss how Latin* students could experience marginalization in unique ways. Instead, Tina de-centers Latin* servingness by universalizing the experience of being singled out. Her framing of final exam questions as gender-neutral demonstrates how the construction of mathematics as socially neutral supports her generalizations about underrepresentation in mathematics.

Tina saw her gender influencing the effect of her professional practices on students, but did not identify her racial identity as playing a role. For example, when asked if there were any racial trends among students who seek her support, Tina described establishing mentoring roles specifically with female students, but characterized her student support as race-neutral.

> Math… you picture the male professor. And I think…females really associate with me just because they're seeing a female doing math… I really don't really have a perspective on ethnicity. I'm not sitting in my office keeping track of ethnicity of students, and anybody who comes to my office is always welcome in my office. (Interview 1)

Despite Tina's previous assertions that her experiences of marginalization sensitized her to Latin* underrepresentation in mathematics, she does not voice an understanding that Latin* students, like female students, may experience mentorship differently from faculty of different racial identities. Tina's statement that anyone is welcome in her office suggests that her white racial identity would play a role in student support only if she were to discriminate against Latin* students. She further elaborated this view when asked if she perceived her white racial identity as playing a role in Latin* students' comfort with class participation.

> The area we lived in [growing up] there was a lot of Hispanic students when I went to high school… maybe I'm a little different because I grew up constantly around them, constantly seeing them doing as well in their classes as I did, so I never had any stereotypes, any ideas of, they can't do this or they can't do that… My identity, maybe I just was more colorblind because that was just who I was with all the time. (Interview 2)

Tina discusses colorblindness as a desirable personality trait that negates the role of her racial identity in her teaching practices, rather than as one that reinforces systemic racism. Explaining her racial identity as formed from adolescent experiences suggests a stagnant EMRIC.

Tina's portrayal of her identity remained largely unchanged after a year of PD. However, at the end of the second interview she reoriented to focus on Latin* students. She shared, "I've spent a lot of time understanding what issues women deal with, I think it's important that I try to understand better what Latin students…specifically face that may or may not be the same challenges that other students face" (Interview 2). This pivot allowed her to mine her teaching experiences for insights on serving Latin* students. She shared, “From what I've seen with the Latin culture, I think it's harder for them to ask for help” (Interview 2). Such remarks still evade recognition of how her white identity influences serving Latin* students; she didn’t consider how Latin* students’ may seek help differently from white faculty or Latin* faculty, for example. However, this shift represents a disruption of her colorblindness that centers Latin* students.

David

David described himself as newly exploring issues of equity through the PD: “I've recently learned the difference between equality and equity… over summer during my reading on anti-racism, preparing for my workshop... My struggle is looking outside of my bubble to make sure that I can embrace those ideas” (Interview 1). As part of this struggle, David had trouble characterizing how race played into students’ experiences of mathematics instruction.

Interviewer: How do you see race or ethnicity playing a role in… students' concern about [speaking in class]?

David: Just that, if a student's not comfortable. And so how would race or ethnicity play a part of that? It's just, it would be common that they're not as comfortable as other students possibly. And yeah, probably just a comfort level and maybe they're shy for a number of reasons, but that's what I would think it is.

David’s response suggests race enters math instruction only to mediate students’ comfort. His understanding of how inequity arises in everyday mathematics instruction lacked specificity. For example, when asked what his goals were with respect to understanding and relating to Latin* students, David responded, “Let's zoom out, big picture, there's inequities in education. And now, understanding that's important, but when it comes down to in the classroom, helping students, I can't think of doing something specific directed towards Latinx” (Interview 2). Such responses demonstrate that David’s understanding of racial inequality was decontextualized from his professional practice, reflecting ideologies of mathematics as a race-neutral discipline.

David held an oversimplified understanding of whiteness as structural advantage that left him unclear about how to acknowledge his identity in classroom teaching, “I've yet to stand up… in front of the class and tell them that I'm a white male and that I apologize… I haven't figured out how to do that” (Interview 1). Similarly to his difficulty grasping the role of race in instructional contexts, David struggled to articulate specifics related to his white identity. When asked how his identity influenced his teaching practice, he initially characterized his identity as tasking him with the responsibility to help those with less privilege “I have to recognize that I'm white and that I do have white privilege, and I, just to be able to help other people and recognize that it's a real thing” (Interview 1). He further characterized his goals related to serving Latin* students as “trying to be educated on the subject and sympathetic” (Interview 2). Frequently, David referenced students’ “disadvantaged backgrounds” (Interview 1) as motivating his desire to help. These motivators suggest that David may be subject to the common pitfall of using pity to guide his attempts to serve Latin* students (Dowd & Bensimon, 2015). This orientation allowed for an EMRIC as a helper, fostering Latin* students’ assimilation for success in white disciplinary structures. Such orientations, combined with his limited ability to recognize inequity occurring in his classroom, leave the influence of his white identity, as well as the hegemonic role of whiteness in postsecondary mathematics, uninterrogated. Whiteness-at-work can be seen in David’s helping attempts that reinforce whiteness in mathematics.

While David articulated the role of his white identity in reductive ways during the first interview, he later characterized himself as not knowing the influence of his identity on his professional practices.

Interviewer: How you see your own identity… as a white person… as a man… shaping your perspective on how effective your teaching practices are in serving Latinx students?

David: How is my identity? I don't know. I don't know. How is my own identity? It's a really hard question. I'm not sure. One of the first things I was supposed to do in this [PD] workshop before the semester started was to come up with my own racial identity… It was the hardest thing. And how my identity relates to how I'm supporting Latinx students? I don't know. I'm not sure that being a… white male, what that does…Am I not as good because of my identity? I don't know. It's really challenging for me. (Interview 2)

Such recognition could be seen as progress in his developing white identity, as David shifted from a position of "helper" that reflects paternalism (Jones & Okun, 2001), to a position that reflects more openness to learn about his whiteness. His receptivity to learning was further underscored by his verbalized commitments to growth, despite the emotional strain he felt when discussing topics of race, "Just to be 100% honest, our last interview, it was really hard for me, but I just believe that I can't walk away from the conversation. I have to try" (Interview 2).

Despite David's expressed commitment, he reported reducing actions intended to develop capacity to serve Latin* students. In the fall, David described being "focus[ed]... [on] really paying attention to Latinx students" (Interview 1) despite feeling "uncomfortable" (Interview 1) in doing so. However, by the second interview, David had lessened his practice of paying particular attention to Latin* students, "This semester, when I tried to put myself in the headspace of like, 'Am I helping a Latinx student?' That just felt awkward" (Interview 2). David's shift in attention demonstrates whiteness-at-work. Although David's stated intention is to engage in equity-related learning despite the discomfort that may cause, he relinquished his practice of specifically attending to the needs of Latin* students because it "felt awkward." In this way, David's commitments had limited efficacy in advancing his EMRIC.

Discussion & Implications

Although David's and Tina's cases differed in many respects, they shared a struggle to recognize how their white identities influenced their capacity for serving Latin* students through instruction. Relatedly, both faculty demonstrated limited awareness of how race influenced the mathematics discipline or institutional structures. Tina's claim that test questions are gender-neutral, and her framing of race and gender as analogous, indicates that the construction of mathematics as socially neutral shaped her inattention to racialized features of mathematics practice. Although David made no explicit statements about mathematics as neutral, his appeal solely to decontextualized social phenomena, like student comfort, when explaining classroom inequities illustrates that he does not perceive mathematics itself as a racialized feature that should be attended to when serving Latin* students. The invisibility of their own white identities and of whiteness in mathematics served as mutually reinforcing to impede critical EMRICs.

Our findings indicate further need for PDs to integrate activities that scaffold examination of the hegemonic role of whiteness. Although, as David's reflection indicated, understanding one's own racial identity was an activity in the PD, this was not central to PD activities. We argue, contrary to David's articulation of his obstacle as "struggling to look outside of [his] bubble," that struggling to look *inside* their bubble was the primary obstacle in advancing equity. Equity-oriented PDs can address this by extending reflections on racial identity to explore characteristics of white culture (e.g., valuing white segregated environments; DiAngelo, 2016), white ideologies (e.g., meritocracy; Bonilla-Silva, 2006), and whiteness embedded in disciplinary epistemologies and values (e.g., proof; Hottinger, 2016; McNeill & Jefferson, accepted). Future research can explore the efficacy of PDs that integrate such methods.

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Research-Informed Practice to Improve Inquiry-Based Calculus with Learning Assistants

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Despite the promise of inquiry approaches, student resistance can occur because active learning opposes their expectations of classroom norms and responsibilities. This study reports on students' experiences in inquiry-based learning (IBL) Calculus supported by learning assistants (LAs). We used an iterative process adapted from Hine (2013) to make changes to our study across three semesters to improve student experiences in IBL Calculus. Survey data of 285 students across 7 sections of IBL Calculus was supported by our observations of IBL Calculus and weekly LA seminars with 12 LAs to determine changes needed each semester. Quantitative data were analyzed using descriptive statistics and z-tests, and qualitative data were open-coded to evaluate the effectiveness of changes. We report changes made to LA preparation and course structure to support LAs' instructional practices and increase student buy-in to IBL Calculus. Over the three semesters, the overall student experience in IBL Calculus improved.

Keywords: research informed practice, inquiry-based learning, learning assistants

As practitioners, we strive to improve the experience, learning, and understanding of students in our own courses. As researchers, we strive to push the bounds of our knowledge of student experience, student learning, and student understanding in *all* mathematics classrooms. Our research should inform our practice so that we systematically use our new knowledge to directly and positively impact our students. While researching the implementation of inquiry-based learning (IBL) in Calculus supported by learning assistants (LAs), we determined aspects of LAs' work preparation and Calculus students' classroom experiences that were ineffective. We sough to answer the question, "How can an action research cycle be used to improve students' experiences in Calculus IBL classrooms with LAs?" Based on our data analysis, we made changes each semester to develop practices that support positive experiences for both students and LAs. In this paper, we describe the data-supported, iterative revision process we used over three semesters to redesign the Calculus course structure and LA preparation and report the results of these changes on student perceptions of the course.

Literature Review

One of the goals of research is to inform practice, but it can be challenging to translate research into practical strategies (Carlson & Rasmussen, 2008). Through action research, educators conduct research in their own classrooms with the practice-based goals of improving their instruction and the experiences of their students (Gibbs et al., 2017; Hine, 2013). Action research is commonly used in higher education to obtain "evidence of how practice can be improved and its impact on the learning of students" (Gibbs et al., 2017, p. 17). The cyclic nature of action research allows for refinement of current issues, a systematic approach to inquiry, and continuous learning for the educator (Hine, 2013). We used an iterative process adapted from Hine (2013) (Figure 1) to study our implementation of IBL in Calculus over three semesters. Whereas Hine's model moves directly from reporting outcomes to taking action, we added a critical intermediate step, “determining future changes” to capture the work of interpreting the results to hypothesize what changes to our IBL structure might improve student outcomes.

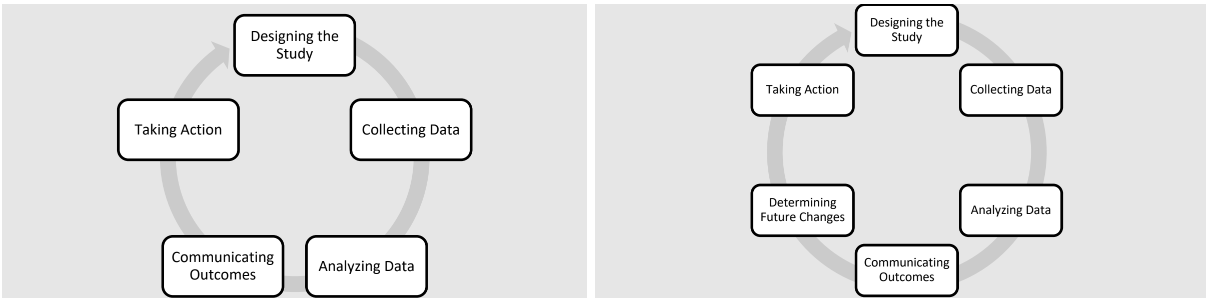

Figure 1: Action Research Cycle (Hine, 2013) and Modified Action Research Cycle (adapted from Hine, 2013)

Inquiry approaches to learning continue to grow in mathematics education (Laursen & Rasmussen, 2019). In IBL, students work through a “carefully scaffolded sequence of mathematical tasks” (Ernst et al., 2017, p. 570) under the guidance of an instructor to construct their own understanding of a given concept. Research has found that students in inquiry classes have reported improvements in learning specific content, developing affective skills, and communicating and collaborating with peers (Laursen et al., 2011). Despite the promise of IBL, students can be resistant because its structure contradicts their expectations of classroom norms and student/teacher responsibilities (Deslauriers et al., 2019). Students must develop autonomy in problem solving to be successful in an IBL setting, where instructors support student work instead of providing direct instruction on how to solve problems.

Student resistance to the change in student/teacher responsibilities has been associated with poorer student course performance (Cavanagh et al., 2016), and successful implementation of any inquiry approach to teaching and learning must account for student resistance to changed norms. Some ways instructors might combat student resistance to IBL include discussing reasons for the course structure (Deslauriers et al., 2019; Yoshinobu, 2019), presenting students with videos outlining the importance of working in groups (Clinton & Kelly, 2020), and explaining the importance of having a growth mindset (Cavanagh et al., 2018). In our implementation of IBL Calculus, we used LAs to help alleviate student resistance, by providing students more available access to help. LAs act as apprentice instructors, facilitate group work alongside the instructor, and have been shown to improve student outcomes in STEM courses. Barrasso and Spilios’ (2021) meta-analysis indicates that students in LA-supported STEM courses have better attitudes, course satisfaction, and learning gains than students in STEM courses without LAs.

Method of Inquiry

Context

This study was conducted within IBL sections of Calculus I courses in a large, public, midwestern university across three semesters. The IBL sections used the textbook *Active Calculus: Single Variable* (Boelkins et al., 2018) which was designed for use with an inquiry-based approach to instruction. Students spend the majority of class time working in small groups facilitated by the instructor and LAs to solve problems that emphasize key concepts and relationships of calculus. While groups worked on tasks, the instructor and LAs circulated to troubleshoot student difficulties, pose questions, check student understanding, encourage group discussions, and generally encourage student thinking and autonomy in problem solving. Of 303 students enrolled in IBL Calculus across three semesters of this study, 285 agreed to participate.

LAs attended IBL Calculus class three times per week and met with the course instructor once a week to discuss both their experiences as an LA in IBL Calculus and readings on active learning. This study incorporated 12 LAs in 7 sections of IBL Calculus. LAs were a mix of undergraduate and graduate students, males and females, international and domestic students, and all had an interest in active learning in mathematics.

Data Collection and Analysis

We collected LA weekly written reflections, field notes of LA seminar meetings, and observations of IBL Calculus. Two weeks prior to the end of the semester, we used a survey to gather student perceptions of the course. The surveys consisted of Likert scale questions and open-ended questions, all of which explored various aspects of students' IBL Calculus experiences. To analyze qualitative data, we open-coded free response questions, which were answered by all respondents. To analyze quantitative survey data, we calculated descriptive statistics and performed independent samples z-tests to determine the significance of changes in student responses from semester to semester. Employing our modified action research cycle (adapted from Hine, 2013) we interpreted the results to determine future changes needed to our enactment of IBL Calculus, which we implemented in subsequent semesters to continue our cyclical process. Here, we report the changes we identified from our analysis each semester and the impact of those changes on students' experiences in IBL Calculus (for additional details of the study see Bubp et al., 2020).

Results

Semester 1 Key Findings

After our first semester, we used student survey responses to determine students' assessment of their interactions with LAs. In particular, students reported the LAs were less helpful in class than the instructor and their peers, with 36% of students responding that LAs were 'little to no help'. Further, they identified the following LA actions to be 'little to no help': ask questions about your work (33%), ask you to explain your work (38%), give feedback on your work (24%), and help you but do not directly answer your question (50%). In the open-ended response questions, 20% of students indicated negative interactions with LAs. One student described their interactions as: "TA's coming over and staring over your shoulder, correcting your work only to give you a more confusing explanation." Similarly, from observations we noticed LAs struggled to employ effective questioning strategies and facilitate discussions in small groups.

Using student surveys, LA reflections, and LA seminar field notes we identified two problems with our LA preparation that contributed to their struggles interacting with students. First, by choosing LAs with strong content knowledge and some prior experience with active learning, we assumed that minimal preparation for facilitating student group work in advance of the semester start would be needed. However, LAs' content backgrounds had some gaps, and since IBL Calculus relied on objects-in-motion and graphing approaches with which some LAs were unfamiliar, many had difficulty helping students understand these approaches. Second, our weekly LA seminar did not sufficiently support the LAs in developing practical aspects of implementing and supporting IBL in the classroom. Instead, we read and discussed research literature on IBL to support our graduate students in developing their understanding of the current state of IBL research and consider how they might analyze data of IBL classrooms.

Semester 2 Revisions

To address these issues, we made strategic changes for the second semester. First, to better prepare LAs for their role in the classroom, we provided pre-semester preparation and communicated clear expectations for their role. A first-semester LA drafted an *LA Manual*, which we reviewed and required LAs to read prior to the start of the semester. We set the expectation that LAs had to complete for themselves all IBL Calculus classwork in advance of each class meeting where they would be assisting students on that work. The *LA Manual* also emphasized the importance of asking questions to move students forward in their work without giving them direct answers to problems. We shared ways to connect with students such as by learning names and ways to build trust by admitting mistakes and listening carefully to students.

Second, we improved during-the-semester preparation by making LA weekly seminars less focused on theory and research and more focused on classroom activity and problems of practice: what concepts students struggled with or excelled at, how groups worked together, instances of inequitable participation, and instances of LAs not knowing how to respond to a student. We specifically discussed strategies to help LAs develop an awareness of student understanding and an ability to improve their interactions with students and students' interactions with each other.

Third, in addition to improving LA questioning techniques, we changed some structural elements of how LAs functioned in the class to improve students' perceptions of their interactions with them. In the first semester we had three LAs to a section for a total of four teachers. This led to students being asked the same question repeatedly by different LAs and hearing too many different explanations from LAs. In the second semester we only used two LAs per section, assigned each LA to specific groups, and established a norm of LAs asking groups if they had already been questioned by someone about their work.

Semester 2 Key Findings

Second semester students found LAs to be more helpful than their instructor and peers, with 67% of students saying LAs 'helped a lot or the most' and only 13% saying LAs 'helped little or not at all' as compared to 46% and 35%, respectively, from the previous semester. Students who reported specific types of interactions with LAs (Table 1) as 'usually or very helpful' statistically significantly increased from the first semester and those reporting the interactions as 'little or not helpful' statistically significantly decreased.

Table 1. Percent of students reporting these interactions with LAs were or were not helpful.

Interactions	Semester 1		Semester 2	
	% little or not helpful	**% usually or very helpful**	**% little or not helpful**	**% usually or very helpful**
Question students' work	32.50	38.75	13.10*	61.90*
Ask students to explain work	37.50	36.25	8.33*	61.90*
Give feedback on students' work	23.75	51.25	6.02*	81.93*
Help but no direct answers	50.00	29.48	24.39*	52.44*

*Indicates statistically significant ($\alpha = .05$)

Although we saw important improvements in student-LA interactions in the second semester, we yet identified areas in need of improvement for the following semesters. Students indicated these interactions with the LAs occurred 'every day or nearly every day': waiting too long for help from LA (11%), LAs solving problems directly for students (32%), LAs providing an incorrect answer (6%), and LAs contradicting one another (6%). Students waiting too long for help indicated we needed to attend more carefully to how LAs monitored students. Though few students stated that LAs provided an incorrect answer or contradicted one another, that this was happening at all was cause for concern. While students might have considered LAs directly solving problems for them as beneficial, this was a practice we wanted to discourage.

Second semester findings showed that only 39% of students reported that the class structure 'helped a lot or most'. Additionally, we noted issues in the effective functioning of groups as 87% of students reported negative experiences working with groups. Specifically, 30% felt their group members were not knowledgeable, 11% believed their group members were on a different level, 14% indicated their group members did not regularly attend class, and 13% did not want to rely on group members over their instructor. We observed that student attendance issues impacted the groups through time lost to groups catching up absent students, inequitable contributions, and groups feeling responsible for carrying weak or non-contributing members. We hypothesized that students' struggles with group work influenced their appreciation of the IBL structure. The 67% of students who made negative comments about IBL referenced a need for more direct help from the instructor and less reliance on their groups: 26% wanted more lecture, 21% did not want to figure things out on their own, 21% wanted lecture for new material, 13% wanted more examples, and 13% did not want to depend on their peers.

Semester 3 Revisions

We made additional changes to the weekly LA seminar, shifting the focus from how to research IBL to understanding student thinking and facilitating group interactions. We hypothesized that students' continued poor reviews of classroom interactions were due to LAs responding to student questions and difficulties on-the-fly. Therefore, we oriented the seminar to prepare for what was coming up in class each week and not only what had previously happened. In addition to completing course materials in advance, we asked LAs to consider possible student misconceptions and questions they could pose to students in the upcoming classes. To further improve their pedagogical skills, LAs also read articles about pedagogical techniques such as questioning techniques, ways of providing feedback, and cognitive demand. LAs reflected on their readings and identified ways to apply them to their classroom practice.

Based on open-ended survey results, we hypothesized that student reaction to their and the instructors' changed responsibilities contributed to their dissatisfaction with the course structure. To address this student resistance, we developed four "Learning about Learning" (LaL) assignments that students completed early in the semester and were intended to help them understand the value of IBL and feel more comfortable as independent problem solvers who collaborated with their peers and communicated their mathematical ideas. The LaL assignments included readings (Anderton, 2019; Dunlosky et al., 2013) and the online course 'How to Learn Math for Students' (Boaler, n.d.) which addressed ideas such as: common misconceptions about math, good study habits, growth mindset, productive struggle and mistakes, communicating mathematical ideas, and collaboration. Each LaL assignment required a written reflection in which students discussed what they learned and how that related to their success in Calculus.

To address students' dislike of group work we used more purposeful groupings with similar majors grouped together and balanced by gender. To minimize group difficulties due to attendance, we deducted points for missing more than 4 classes per semester.

Semester 3 Key Findings

In the third semester, we observed further improvements in student interactions with LAs, a more positive response to the IBL structure, and more effective group work. Students had to wait for help statistically significantly less often, and additionally, although not statistically significant, LAs were solving problems for students, giving incorrect answers, and contradicting other LAs less often (see Table 2). In the open-survey questions, a student described, "They [LAs] were able to guide you in the right direction without giving you the answer outright... [and] we never had to wait long if we had a question." Another responded, "They provided help quickly when it was needed and helped us to understand concepts to solve a problem rather than telling us exactly how to solve the problem." This supports the quantitative analysis that student wait time was shorter and that LAs were no longer directly solving problems for students.

Table 2. Percent of students reporting the following frequencies of these actions occurring with LAs.

Interactions	Semester 2 **% occur every or nearly every day**	Semester 3 **% occur every or nearly every day**
Wait too long for help	10.71	1.33*
Solve problem for students	32.14	21.33
Give an incorrect answer	5.95	1.33
Contradict each other	5.95	1.33

*Indicates statistically significant ($\alpha = .05$)

The response to the IBL structure statistically significantly improved by the third semester, with the proportion of students reporting the course structure 'helped a lot' or 'helped most' increasing from 39% to 58%. In addition, statistically significantly fewer students responded that they needed lecture for new material and that they did not want to figure things out on their own. Although not statistically significant, fewer students responded that they needed more lecture in general and that they wanted more examples (Table 3). We note here that the percentage of students who reported needing more lecture remains high and we attribute this to the difficulty of overcoming students' expectations of what teaching and learning mathematics should be. We expect this issue to be an on-going focus of future revisions.

Table 3. Percent of students' open-ended responses coded with the following codes.

Code	% Semester 2	% Semester 3
Lecture needed for new material	20.24	5.93*
Want more examples	13.10	7.63
Need more lecture	26.19	22.88
Had to figure out on own	21.43	10.17*
Didn't want to depend on peers instead of professor	13.10	5.08*
Group members not knowledgeable	29.76	27.97
Group members move on without me/at a different level	10.71	14.41
Group members not attending	14.29	5.93*

*Indicates statistically significant ($\alpha = .05$)

With the changes to group structures, statistically significant more students valued their peers both in- and out-of-class (Table 4). Furthermore, statistically significantly fewer students responded that they did not want to depend on their peers instead of their instructor and that their group members were not attending class (Table 3). Qualitative data supported this finding; one student described, "My group is really amazing and we do a good job of collaborating and helping each other when one of us is stuck in or out of the classroom. They are all pretty good at explaining why they got a certain answer." We also note that some students still report group members are not knowledgeable, and hence not considered a good classroom resource, and that too much variability in groups made students feel left behind. This points to our need to continue addressing supporting students in working in groups.

Changes Across All Semesters

Across all three semesters of this iterative process, when we focused on understanding reasons behind our results, we were better positioned to determine changes for the future and take action. In determining changes from the first semester, we recognized the need to improve the LAs' interactions with students by better preparing LAs for their role in an inquiry-based classroom. And, in the second semester, we observed statistically significant improvements in students' responses to the LAs; the percent of students who found them unhelpful decreased and the percent of students who found them helpful increased (Table 4). Results also pointed to a need to focus on IBL structure, students' perceptions of that structure, and group interactions. And in the third semester we saw statistically significant improvements in students finding their peers more helpful (Table 4). On the other hand, there was no significant change in the student responses to the LAs being helpful and available in class. We interpret this reduced perception of LA helpfulness as the result of students becoming more comfortable with productive struggle with their peers. One interpretation of this finding is that as students' value for their peers increased, they became more autonomous in solving problems and less dependent on the LAs. Reduced need for assistance from LAs could in turn explain the decrease in the proportion of students who responded that the LAs 'helped a lot' or 'helped the most' with their learning.

Table 4. Percent of students reporting the following class features as helpful or not helpful.

Class Feature	Semester 1		Semester 2		Semester 3	
	% little/ no help	**% helped lot/most**	**% little/ no help**	**% helped lot/most**	**% little/ no help**	**% helped lot/most**
IBL Structure	38.55	31.32	39.29	39.29	21.56**	57.76**
LAs in Class	34.94	45.78	13.41*	67.07*	12.28	59.65
Peers in Class	25.30	56.62	21.69	46.99	9.57**	66.96**
Peers out of Class	30.16	39.10	46.99	27.12	25.00**	45.00**

*Indicates statistically significant from Semester 1 to Semester 2 ($\alpha = .05$)
**Indicates statistically significant from Semester 2 to Semester 3 ($\alpha = .05$)

Implications for Practice

In our work, we have again confirmed the power of using research to improve practice. We sought to use an iterative systematic process to improve the implementation of an innovative pedagogical structure and over three semesters were able to make that happen. Drilling down to identify specific difficulties students were having with our courses was essential for making appropriate revisions. The changes we made in our enactment of IBL Calculus were small, yet they often had a statistically significant positive impact on students' experience in the classroom. Through two sets of revisions, we improved LA preparation, in-class group interactions, and student appreciation for IBL. From this study we have learned some critical aspects of introducing novel pedagogical approaches: the importance of instructor preparation, of fostering student buy-in and understanding of the rationale for the pedagogical approach, and a systematic approach to understanding and reacting to students' experiences.

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One Multilingual International Student's Experience in Collegiate Mathematics Classrooms

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Although the number of multilingual international students is increasing in undergraduate mathematics classrooms, there is little research about this population at the collegiate level. To better serve this growing population, it is important to understand their experiences using multiple languages in their mathematics learning. In this study, I constructed a narrative of one multilingual international student, Jia. I used narrative inquiry as a research method and collected data from synchronous- and email interviews and observations. The narrative demonstrates how Jia used her Chinese and English and how she dealt with her challenges. The study found that engaging in groupwork was particularly challenging for her as a multilingual student. I focus on not presenting her multilingualness as a deficit feature and discuss how to mitigate challenges multilingual students can face in their mathematics learning.

Keywords: Multilingual, Active Learning, Groupwork, Linguistic Diversity, Narrative Inquiry

In the increasing globalizing world, the number of international students in collegiate classrooms worldwide has likewise been increasing, especially in undergraduate mathematics classrooms in the United States. When international students move from their home countries to the U.S, they bring much diversity into the classroom: racial, cultural, and linguistic, to name a few. I mainly focus on linguistic diversity in a mathematics classroom because having a different first language than the language of instruction (LOI) is directly related to communication, which can impact one's access to learning opportunities and mathematics identity. Despite a substantial portion of international students in undergraduate mathematics classrooms, such students' experiences have not received enough attention in the field. Researchers have focused on multilingual students' experiences in K-12 mathematics education (e.g., Huitzilopochtli et al., 2021; Moschkovich, 1999; Takeuchi, 2016; Turner et al., 2013), but there has not been much research at the undergraduate level, especially in multilingual international students in the U.S. Collegiate mathematics classrooms are likely to be more diverse in terms of students' home countries and languages that they bring compared to K-12 mathematics, where *bilingual* classrooms–classrooms that have two main languages such as Spanish and English–are more common for K-12 mathematics education study (e.g., de Araujo et al., 2018; Domínguez, 2011; Moschkovich, 2007).

In this study, I present a narrative of Jia's (pseudonym) experience answering the following research questions: *What is Jia's experience as a multilingual international student in college mathematics classrooms? How does she use her languages in college mathematics classrooms?* Jia's narrative reveals what challenges some multilingual international students experience, how they might feel or be positioned in mathematics classrooms, and how multilinguals use multiple languages in their mathematics learning. From this study, I call for more attention to multilingual international students in undergraduate mathematics classrooms from the mathematics education field so that the field can better understand the population and serve them more equitably.

Relevant Literature and Theoretical Perspectives

In this study, by multilingual, I mean people who use one language in some situations and another in other situations (Planas & Setati, 2009). This definition does not imply that multilinguals need to speak both languages fluently as if both are their native language, which is a common understanding of the term multilingual (Planas & Setati, 2009). According to this, international students who are traditionally described by terms such as "English as a Second Language," "English as Another Language," or "non-native English speakers" are "multilingual" as they use English during the official instruction of the course as well as some other contexts and often use their first language in other settings.

When multilingual international students move to the U.S. from their home country, their pattern of language use changes: the language that they use in their home country loses its position as the LOI and English–which is usually not a significant portion of their language use–takes the position as the LOI. The switch of the LOI is one of the challenges international students report (Brenner, 1998; Macintyre et al., 1998). Although their language usage changes regardless of various disciplines, mathematics becomes one of the contexts that require a specific discourse (Sfard, 2008). One of the common myths about mathematics is that "mathematics is language-free" because it uses its own symbols and equations (e.g., Schleppegrell, 2007; Takeuchi, 2016). However, even though students share the same mathematical symbols, they need to translate when they want to verbalize their interpretations because mathematical sentences with symbols are dense. Sfard (2008) construed mathematics objects as discursive constructs and hence argued that doing mathematics is having a discourse of (mathematical) discourse, which requires intense language use. Therefore, mathematics needs significant language use to perform, which means multilingual international students whose first language is not the LOI can struggle with mathematical languages. Language is a primary and unavoidable medium when people in classrooms communicate; hence, it can influence students' learning opportunities (Meyer et al., 2016).

Moschkovich (2002) reported that in the past, people had focused on multilingual students' experiences from the perspectives of monolinguals who use the LOI, such as focusing on acquiring vocabulary in LOI or building different meanings of words and grammar in the mathematics. It was not difficult to find when researchers focused on what multilingual international students do not do well, such as concentrating on not speaking English as fluently as native English speaker students do. Some researchers have framed or assumed that multilingual students have linguistic deficiency implicitly or explicitly and then studied how that deficiency influences their learning and experiences (Campbell et al., 2007; Hwang et al., 2022; Neville-Barton & Barton, 2005). I encourage mathematics educators to avoid using such deficit perspectives to understand multilingual students' experiences. Rather than focusing on what they cannot do, researchers need to focus on how students use their multiple languages in their learning and what their current experiences are in their mathematics education (Barwell, 2018; Moschkovich, 2002). Therefore, in this study, I focus on the mathematics identity of one multilingual international student, Jia, specifically focusing on how she used her repertoires of Chinese and English in proof-based undergraduate mathematics classrooms.

Language as Sources of Meaning

In this study, I adapt the view of Barwell (2018), who viewed language as sources of meaning. This perspective implies that language is fluid, agentive, and stratified. Barwell claimed that meaning-making is situated locally through utterance, which can evolve depending on the new utterance. Also, language is agentive as spoken language has the speaker's intentions connecting the past- and current speaker's intentions. Lastly, language is stratified because there were cases when people valued specific languages more than other languages, such as mathematical language and daily language in mathematics classrooms or English and other languages in U.S. classrooms. According to this perspective, language is social and political

(Barwell et al., 2007). It provides a lens to study the role of languages in students' interactions and experiences and their learning of contents.

Method

Part of the data used in this paper was collected as a part of a larger project that investigates students' agency and autonomy in the Introduction-to-Proof (ITP) course and some subsequent proof-based mathematics courses at a large public university in the Midwestern U.S. Jia volunteered to participate in the study in Fall 2019 while she was taking the ITP course. During the ITP course, two in-person interviews and seven activity logs about students' experiences related to the course were collected each week. I observed the course roughly once a week. The two in-person interviews were transcribed from the audio recording, and then I open-coded the data to find an initial list of the codes from her interviews and observations. After that, I found the themes in the initial list and then wrote a profile for her experience. In Fall 2021, follow-up questions specific about her experiences as a multilingual were asked via email interviews, given her preference. I received 12 email responses to the interview questions from Jia in the semester and the following semester. Each email response was analyzed to find initial codes of events (e.g., her group work experience in one ITP class or her use of Chinese in mathematics learning) and themes (e.g., her challenges related to languages) and to create the next set of questions. It was a repetitive and cyclic process. Once the data collection with her was finished, I found the themes (e.g., her challenges related to languages, or her experiences related to gendered bias) across the first and second rounds of data collection and built a narrative of her experiences. To deliver her story more vivid, I decided to use the first-person perspective for her narrative. After I wrote a narrative, I did member checking with Jia twice to have her agree on our narrative of her. Even with member checking, I acknowledge that there is no only "right" or "valid" way of her narratives, and it can evolve over time.

In the following section, I provide only a part of Jia's narrative that I wrote as our finding because of the constraints of the space. The whole narrative consisted of four parts 1) introducing herself, 2) challenges that she experienced in the ITP course and subsequent courses as a multilingual, 3) her ways to utilize her languages in mathematics learning, and 4) her motivations to pursue mathematics even with these challenges. For the sake of this paper, I wanted to highlight and call for attention to the challenges one multilingual student was experiencing, and hence first two parts were included.

Narrative of Jia

My name is Jia. I am a female Chinese, was born in China, and lived there until I came to the U.S for my undergraduate education. So, Chinese, or Mandarin, is my first language and English is my second language. As I am studying in the United States, the language "officially" used in the classroom is English, which I had some concerns with. As my major is statistics and minors are Computational Mathematics Science and Engineering (CMSE), data science, and mathematics, I have taken a lot of mathematics courses. Of course, languages matter in other disciplines as well, but today I am telling you about my experiences in mathematics courses.

Challenges in the Introduction-to-Proof Course

One of the mathematics courses that I remember is the Introductions-to-Proof course. I took the course in the fall semester of my sophomore year, and, as its name indicates, it was a transition course between Calculus courses and more proof-based mathematics courses. Actually, before the course started, I did not have a lot of ideas about the course. I asked some of my

Chinese friends and they said that the course is not difficult, but hard to do perfectly. They said that sometimes it asks you to explain your thoughts, but it can be quite difficult for international students, like us, to express thoughts clearly. And, I started to agree with them as the course was going on. There were several points that I was worried about my English while taking the course although I did well in this course. Sometimes I could not understand what the problem was asking to me or had difficulties understanding the definition, and other times were difficult because of the interactions with other people, like with the instructor or groupmates. I think these challenges were both from my language and my education, which are different here content-wise and teaching method-wise.

One of my biggest challenges was that the ITP course was based on groupwork. We had a lot of groupwork in the course, about two-thirds of the class each week was set aside for us to solve problems in groups, which I enjoyed and found helpful in general because groupwork helps me come up with some ideas or understand the material that I could not understand from my instructor's explanation... as the ITP course changed groups somewhat regularly, my experiences were diverse. The first week, we formed a group based on where we were sitting…. I wanted to have some familiar people in the class, so when I stepped into the classroom, I went to a Chinese female student. And then, there was another Chinese female student, so we became a group. We added each other on WeChat (a SNS that mainly Chinese use) and became good friends, and we sometimes asked questions about the class. (Actually, I became a good friend with one of them and we still hang out sometimes.) As we sat close to each other, they became my groupmates for the first two weeks in the class, which was a good experience for me. I liked it because I could use Chinese with them, which I think is efficient.

After that, my instructor and TA switched groups roughly every two weeks during the first half of the semester. The TA somehow generated the groups and wrote them on the board. And then, I started realizing that who I am assigned to work with in a group influences my groupwork experience. None of my second groupmates were Chinese and all of them looked like their first language was English. I think communication between groupmates is important. But, in this group, I felt isolated. No, let me rephrase that: I was not very engaged. The other people in the group were discussing enthusiastically, but I was like in an, "Ahh, yeah- (indifferent)" kind of mode. I felt my language was the biggest reason. I felt like I did not speak that good.

I always had ideas about how to solve the problems, but I could not explain my thinking succinctly and effectively, or I was not sure whether the words that I wrote were expressing my thinking in English. It happened several times in that semester. For example, something like this: Once, we were asked to compare a variable, x, with a number. In those processes, our group missed that x is a negative number and that led us to make a mistake by not changing the direction of less than or greater than sign (e.g., $>$ or $<$). I said, "x is negative, so we need to change the signal" and they did not understand what I was telling them. I was expecting some positive reactions from them, like "Oh, yes, that's the point," but they did not get it and so could not give me such a reaction. Because I was not sure how to say it any other way, I just wrote it down on my note and showed it to them. Then, one of the groupmates understood and he explained that to the other people. It was a little bit frustrating, but this kind of experience was common at that time. After some of those experiences, I started staying quiet because I knew other people would usually say the same idea that I would have if I waited. Of course, if it took too long, I spoke about my idea, but sometimes I felt I was wasting others' time because other people just could explain the idea in one sentence.

I did not think my experience was challenging only because I did not know how to explain mathematics in English. One other reason that I remember was that, sometimes, I could not be sure which ways or methods they learned to approach the same type of questions in their mathematics in their K-12 education. I thought we all would have learned similar mathematics, but it seemed not. I think it made the situation harder because I think that it would have been easier if we all had similar background knowledge about mathematics. Another part of my experience that I think added more difficulty was that I could not easily jump in on their small talk. This could also be related to my personality; I don't think I am a very social person. But sometimes they speak fast or say things about things from outside the classroom that I did not understand, and then I felt excluded. But I still thought that it was better than doing all of the work alone.

After another round of group rotations, I again was in a group with my friends from the first week as well as another Chinese female student. About middle of the way through the semester, our instructor sent us a survey asking if we had a preference on whom to work with and if so, who we wanted to work with. I put the three Chinese female students on my list, and so did the other three. So, we became a permanent group for the rest of the semester. I was very happy with the group because I felt our communication was so efficient; being able to speak Chinese reduced the time we needed to explain what we were not understanding in English. Compared to when I was in groups where there was the pressure that I needed to explain things precisely, which made me stutter and feel even more nervous, it was so easy to communicate. I became a lot more talkative. We could do small talk as well, and I enjoyed talking with them about my life outside the classroom.

Thinking back, I should have encouraged myself to step out of my comfort zone more. Now, I just try to express my thoughts as clearly as I can. And I will encourage others, as well, if they are in a similar situation. I thought about what I would have done if I were in China and some of my classmates were international students who were not speaking Chinese perfectly. Actually, I had an experience similar to this situation although it is not in school. I once went to a shopping mall in China, and one international traveler came to me and asked a question. But, his Chinese was not very good and so he expressed with some Chinese words with body languages. I answered both in Chinese and English, but he did not understand. I was not mad but patiently explained one more time with gestures. I did not think he bothered me, I knew he needed help, and it was not a waste of my time. Then, I realized that I would have been patient and willing to listen to those international students who do not speak perfect Chinese. I think that it should be the same in the U.S. What do you think?

Jia's Ways to Leverage Multiple Languages

When I was stuck, as I said just before, I search online or ask my Chinese friends. For instance, I struggled a lot in my Abstract Algebra I course. As the course was so challenging, I needed to re-learn the material from resources I found online. When I searched for the course materials online, I could find some parts of some textbooks as well as lecture notes about the course in Chinese. I read those and tried to learn the course material by myself. Although there was no perfect substitute for the course textbook in Chinese, it was still helpful. Sometimes I used videos on YouTube or Bilibili (video sharing platform like YouTube but in Chinese usually) as additional resources. When I search online, I use both of my languages. I usually start with English, but when it seems it will take a lot of time for me to understand, then I switch to Chinese.

In addition to searching the course material to help my understanding of the material, I sometimes search for the terms that appear on the question. If I was stuck on some problem, then the main problem was usually that I misunderstood what the question was asking. Although I struggle sometimes because I need to communicate mainly with my second language in the classroom, I see some benefits of being able to use both languages and have figured out how to deal with the challenge. In addition to what I already told you, I know I can ask some friends who speak English and other friends who speak Chinese when I have questions. Also, I use Chinese in my note-taking as well. I use it as an index so that I can quickly check where I need to look because I can notice Chinese easily and read Chinese a lot faster. Another time I use Chinese is for the definitions. I write some definitions or some keywords in Chinese so that I can remember them better. I switch between the two languages… I also write some memos or how I feel in Chinese, such as "don't fall asleep, you can do it!" "why is it so hard?" or "Confused, review after the class," so that I can focus more on the class and encourage myself.

Although I have been figuring out how to survive in mathematics classes, COVID-19 added another challenge. Because of COVID, all of my mathematics courses were changed to online, which led professors to explain mathematics lectures on zoom and write them on a sheet of paper. Compared to when they are writing on the board and explaining in front of physical students, they wrote a lot faster as they did not need to write it as big as they did on the board. Also, for some reason, they spoke faster and somehow it felt to me that they were reading the book rather than explaining things to us. I sometimes could not write everything in my notes and could not completely understand what they were explaining. Also, the courses did not have a lot of groupwork during the class time, so I could not even ask questions to peers during the groupwork. So, I tended to review the videos instead of going to the synchronous class because I could rewind and stop whenever I wanted to. …

So far, I told you a lot of the challenges that I experienced. But, all of these challenges neither discouraged me from studying mathematics nor led me to lose my confidence. I think my challenges in mathematics courses related to language mean that I just need more time than native English speakers, but they do not mean that I am not good at mathematics. … I think I just need to take more steps moving between the three languages, Mathematics, English, and Chinese. I think mathematics would have been hard even if I studied it in Chinese, but mathematics is charming so I would work hard anyway to understand the material. I am confident that I will definitely learn well if I have enough time (although time just becomes an issue).

Discussion

Jia's experience is one possible shape of many different experiences that multilingual international students have. This study documented how one multilingual student uses her multiple languages in various mathematical contexts, such as doing homework, taking class, or participating in groupwork. The narrative shows that it can be challenging for multilingual international students to participate in groupwork because the situation is not easy to use multiple languages to take advantage of multiple languages. Jia could use Chinese in various ways in other contexts, such as listening to lectures, taking notes, or doing homework. However, it was hard to use Chinese when working in groups, especially when she was with all native English speakers who could not speak Chinese (Hwang et al., 2022).

Jia's challenge in groupwork as a multilingual has significance, considering the emphasis on active learning in collegiate mathematics education, which is often connected to equitable learning for all (Freeman et al., 2014; Johnson et al., 2020; Prince, 2004). I am not suggesting that groupwork harms students but arguing that mathematics educators should be aware of possible challenges students can face. Although Jia attributed her challenging experiences during groupwork to herself, I claim we should focus on external reasons for her actions. Jia said that when she was in a group with at least one other familiar student, usually Chinese, she felt more comfortable sharing her thoughts, which has been found in existing research (Takeuchi, 2016). While she felt connected to having a peer from the same culture, she mentioned that she felt excluded when other groupmates were having conversations about their lives outside the classroom. Given that sometimes conversations that are not relevant to course material create changes in power dynamics between students (Esmonde & Langer-Osuna, 2013), not being able to join small talk because of cultural and linguistic differences can influence the power dynamics in the classroom, which connects to language as stratified and stratifying in mathematics classrooms (Barwell, 2018).

Further, I suggest reflecting on whether people are indirectly implying the notion that students should use English in the classroom and whether multilingual students are discouraged from using their other languages. In her interview, Jia noted that she asked a question to a Chinese teaching assistant, but he responded in English, which Jia thought might be his job. However, I question why the teaching assistant felt he needed to answer in Chinese even though both he and Jia probably knew that their conversation would be clearer and more efficient when they spoke in Chinese. This decision again might have come from the notion that because English is the LOI, it should be more valued than other languages (Barwell, 2018). English is the most reliable common language in U.S. classrooms, making it a good resource to use, but it does not mean other languages should be devalued. Instructors should be aware of and value different ways of communication among students and between instructors and students. This is not to develop "otherness" but for students to use all the assets that they have for their learning.

The current study calls for attention from mathematics educators on how instructors can help multilingual international students utilize the multiple languages that they have. For example, this finding implies that mathematics instructors need to think about how to form a group when they compose a group. In addition, from Jia's narrative, I also found that she used written discourse as an important source of learning, either when she communicated with her groupmates or when she was learning materials. Jia implied that writing on the board could serve as a visual recording and also slow the pace of the instructor's lecture, which provides room for all students to think about the materials.

In this paper, I introduced Jia's perspective as one of the numerous experiences of multilingual international students in mathematics classrooms. However, mathematics educators need to consider how multilingual international students are positioned or viewed in the classroom. In the larger study for which this data was originally collected, our research team has seen some participants describe "linguistic barriers" with international students, which might indirectly imply how general students consider international students. As classrooms in the U.S. are becoming more diverse, I believe how to deal with the perpetuated notion that multilingual international students have "linguistic barriers" in society is an important aspect to consider when mathematics educators teach students.

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Quantitative Reasoning Augments Scaffolding for Mathematical Model Construction

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In this report, we begin to explore the value of leveraging constructs from quantitative reasoning for informing scaffolding moves that would assist modelers in overcoming blockages to their mathematization of canonical real-world problems from a first course on differential equations. Our claims are based in qualitative retrospective analysis of a set of 7 design cycles used to develop a task trajectory and resulting learning environment responsive to STEM students' mathematical reasoning during mathematical modeling. We present a set of four scaffolding moves contingent upon and responsive to participants' in-the-moment quantitative reasoning that guided them towards a meaningful model for a predator-prey scenario.

Keywords: mathematical modeling, differential equations, quantitative reasoning, scaffolding

Mathematical modeling and quantitative reasoning are both key skills for successful interdisciplinary collaboration (National Research Council, 2012). In many undergraduate STEM programs, a course on differential equations serves the need for developing these skills. Doing so requires a consistent emphasis on how mathematical relationships are tied to conditions in and assumptions about the real world (Czocher, 2017). Thus, one of the most-studied research problems in the mathematical modeling (hereafter: modeling) genre regards mathematizing – the process of expressing important aspects of a real-world scenario in conventional mathematical notation. Mathematizing is critical, but it is difficult for learners to do and difficult for researchers to operationalize (Cevikbas et al., 2021; Stillman & Brown, 2014). These dual difficulties have challenged educators' capacity to effectively scaffold students' reasoning during modeling. Stender and Kaiser (2015) have called for educators to enact scaffolding based on a "student's current thinking in terms of the problem and the student's possible cognitive operations" (p. 1256). Meeting this challenge requires a cognitive account of mathematizing.

Recently, scholars have characterized mathematization during modeling in terms of constructs from quantitative reasoning. Several scholars have taken the position that constructs from quantitative reasoning afford an explanation of how modelers imbue real-world meanings into their mathematical inscriptions, thereby ensuring their inscriptions are representations of the modeler's conception of the real-world scenario (Czocher et al., 2022; Larson, 2013; Thompson, 2011). The open question regards uncovering how familiarity with students' quantitative operations during modeling may underpin effective scaffolding for extending students' capacity to mathematize. In this paper, we begin to address the question of how constructs from quantitative reasoning could be leveraged for designing effective, contingent scaffolding.

Theoretical Lenses

We adopt a Vygotskian view of scaffolding as an interactive process between educator and learner that supports the learner in performing a task she might not otherwise be able to accomplish (van de Pol et al., 2010, p. 274). One of the key characteristics of effective scaffolding is contingency, or adapting to the learner's current level of performance (van de Pol et al., 2010). Specifically relevant to mathematical modeling, Stender and Kaiser (2015) concluded that the moves constituting effective scaffolding may depend on which phase of the modeling process the learner is engaged in.

From a cognitive perspective, modeling is conceptualized as an iterative process transforming a phenomenon in the real world into a mathematical problem to solve. Within this perspective, it is common to accentuate the duality of a model as a person's conceptual system and as a representation using graphs, tables, or equations (Lesh et al., 2003). A mathematical model as a conceptual system refers to the constellation of knowledge the modeler has about how the world works (e.g., cats breed very quickly) and ideas about mathematical concepts (e.g., a derivative is the slope of a line tangent to a curve at a given point). In the case of equations, an expression of a mathematical model refers to a written representation that uses conventional notation like symbols for variables and arithmetic operations.

For an equation to carry meaning as a model of a scenario, it needs to hold mathematical meanings for the modeler and those mathematical meanings need to have counterparts in the real-world situation at hand. Theories of quantitative and covariational reasoning – how individuals conceive of quantities and relationships among them (Thompson & Carlson, 2017) – explain how meaning is given to the variables and arithmetic operations that constitute an equation (Czocher & Hardison, 2019). Quantities are created in the mind of the modeler when she conceptualizes a specified object in the real world as having a specified attribute that can be measured. Quantities can vary independently or interdependently with other quantities. For example, a modeler could imagine that the number (measure of an attribute) of a population of birds (object) in a backyard habitat waxes and wanes. She could also imagine that the number of birds increases as the number of feral cats in the habitat declines. Quantitative operations produce new quantities from already-conceived ones, usually through combination or composition (Ellis, 2007; Thompson, 2011). For example, a modeler could conceive change in bird population between two points in time by subtracting the latter population from the former.

Methods

Our data are drawn from the Cats & Birds task, which appeared as part of a larger study of effective scaffolding for promoting modeling competencies.

Cats, our most popular pet, are becoming our most embattled. A national debate has simmered since a 2013 study by the Smithsonian's Migratory Bird Center and the U.S. Fish and Wildlife Service concluded that cats kill up to 3.7 billion birds and 20.7 billion small mammals annually in the United States. The study blamed feral "unowned" cats but noted that their domestic peers "still cause substantial wildlife mortality." In this problem, we will build a model (step-by-step) that predicts the species' population dynamics, considering the interaction of the two species.

We intended for participants to arrive at the following model (using their own symbol choices), which are a version of the Lotka-Volterra equations:

$$\frac{dB}{dt} = \gamma B(t) - [\lambda B(t) + \alpha\beta B(t)C(t)]$$

$$\frac{dC}{dt} = \kappa[\alpha\beta B(t)C(t)] - \rho C(t)$$

We worked with 19 undergraduate STEM majors who were enrolled in or had completed a course on differential equations. They participated in a series of 10 hour-long clinical task-based interviews. Cats & Birds featured structural scaffolding in the form of Subtasks to help participants learn to justify the quantities and relationships chosen for inclusion in a model. Seven iterative rounds of design research (beyond the scope of this report) aided in determining which conceptual steps needed to be addressed as subtasks and how big those steps needed to be. Relevant to this report, participants regularly exhibited difficulty handling the complexity surrounding the facts that the number of cats and birds can vary, not all birds meet all cats, and cats are imperfect hunters. Thus, the first three Subtasks were developed to support participants' thinking about how to mathematize the number of cat-bird interactions at time t $(B(t) \times C(t))$ and accounting for encounter rates (α) and successful hunting rates (β). We then followed with Subtask #4:

4. Consider the decrease in magnitude of bird population due only to cat predation during a short interval of time Δt. Write an expression modeling this decrease, in terms of the size of cat and bird populations present at time t.

Goal: Produce $\left[\alpha \times \beta \times \left(B(t) \times C(t)\right)\right] \times \Delta t$ and interpret it as the change in bird population due only to cat predation during a small, fixed, but arbitrary duration of time

The contingent interview protocol targeting mathematizing was designed to explicitly focus on aspects of quantities and quantitative reasoning. For example, follow-ups requested justification for choices of arithmetic operations (e.g., Can you say why you used × here?) or to explain how they were thinking of specific quantities (e.g., do you see ΔB as a change in bird population or as the number of surviving birds after some are killed? What units does B(t)×C(t) have?) in order to inform our interviewer-proffered scaffolding. We used cross fertilization (Brown, 1992) to adapt what we learned about student reasoning from one participant for use in another participant's sessions. Our retrospective analysis of the design cycles focused on participants' quantifications and quantitative reasoning that aided in mathematization.

Results

Among those who did not autonomously produce the target expression for Subtask #4, we found four scaffolding moves that effectively moved the participant towards the Lotka-Volterra equations as a meaningful model for the interaction of species in the backyard habitat.

Contingent scaffolding 1: attend to the object and/or attribute represented by the symbol for a variable. For example, some participants conceived *decrease in population as time passes* to mean a record of the population size at different points in time. Sensibly, they produced a model representing the remaining population, $B\left(t_f\right) = B(t_i) -$ Birds Killed. We asked the participant to reason through what quantity each symbol represented and then followed up with clarifying questions like: *Great! Do you see the computation on the right-hand side as yielding the number of birds remaining or the number of birds that died?* We then asked the participant to identify (and then focus on) the "change-by" amount and used small numerical values such as $B(t_i) = 10$ and Birds Killed $= 3$ to solidify the distinction. When answering the former, we asked the participant to identify (and then focus on) the "change-by" amount.

Guiding the participant to attend to object/attribute/symbol alignment also meant suggesting changes to their notation to better align their intended meanings with conventionally correct meanings. For example, some participants represented decrease in population as time passes using the symbols $B(\Delta t) = B\left(t_f\right) - B(t_i)$. We found that participants conceived the value of the change in population as dependent upon the length of time interval and strongly desired their model to reflect that. Introducing the notation $\Delta B(t, \Delta t)$ to signify the fact that population change depends on duration of time as well as when the duration began resolved the conflict between student meaning and conventional meaning for the symbols.

This scaffolding move was also supportive for participants who used t to represent an arbitrary duration of time (rather than Δt). For these participants, we found that explicitly appealing to convention as a rationale for swapping the symbol t for Δt endorsed the participant's reasoning while making a correction in a way that did not disrupt their thinking.

Contingent scaffolding 2: Use precise and detailed language to refer to referents. Returning to the example of participants conceiving *decrease in population as time passes* to mean a record of the population size at different points in time, we found that keeping careful distinction in language between *change* (for them, the result of a change) and *change-by* (the amount by which a quantity changes) was effective. Introducing the additional vocabulary was fruitful for these students who already associated a quantitative meaning with *change as result*. Another important area for precision of language was when discussing time, in particular, carefully and consistently using phrases like *at time* t and *during a fixed but arbitrary duration of time* Δt. This precision aided participants who inconsistently used $B(t)$ to refer to *population since time 0*. We purposefully avoided the phrase *over time* and instead used *as time passes*, to avoid signaling division in cases where it would be inappropriate. In all cases, we avoided referring to quantities by symbols, instead speaking what quantity the symbol represented.

Contingent scaffolding 3: Support participant in quantification. Some participants conceived *decrease in population as time passes* to mean *accumulation of dead birds since the beginning of time*. These participants intentionally used t to measure time lapsed from 0. We took care to acknowledge their model was correct *under the assumption* that intervals of time were always measured from time 0. Building on their own understanding, we asked: "Now what if we wanted to consider a duration of time that did not begin at 0?" We built on participants' demonstrated knowledge that $\Delta t = t_f - t_i$ to develop a brief intervention using a numberline and concrete values for an initial time t_i and a final time t_f. We emphasized that given Δt as an unknown, but fixed value, it could correspond to infinitely many specific time intervals with different starting (t_i) or ending (t_f) points, illustrated in the exchange below:

Int: Now what is the delta t between t equals 2 and t equals 5?

Peet: It would be 3.

Int: How did you get that?

Peet: Because I just subtracted 5 minus 2.

Int: So you found the distance between, in terms of time anyways. But you found the length of time, which is represented by a distance on the page, between 2 and 5. And you got delta t equals to 3. Can you find another pair of points corresponding to delta t equals 3?

Peet: Ok, yeah. I would do something along the lines of 0 and 3. Or 1 and 4. 2 and 5. 3 and 6.

In this excerpt, the interviewer aided Peet in *quantification*, the mental act of conceiving a measurement process for a specified attribute associated with a specific object (Thompson, 2011). The exchange continued:

Int: I'm going to say initial is 1.

Peet: OK.

Int: Delta t is 2.

Peet: OK.

Int: Where is t initial plus delta t?

Peet: It would be t final, which is 3.

Int: 3. Very good. And if t initial is 1 and delta t is 1.5, where is t initial plus delta t?

Peet: It's going to be 2.5.

Int: It would be at 2.5. So of independently we can set how long the duration of time is that we're looking at. And independently from that, we can set where does the duration of time begin.

Peet: OK.

Int: So going back up to your equation, we can think about tau as being the place where delta tau begins-- the moment that delta tau begins.

Peet: Right. OK. So in other words, you're saying the tau-- these two taus are the exact same.

Int: They don't have the same value. They have the same role.

Peet: Yes, yes. I understand.

Int: So tau 0 minus tau-- so just subtracting those two times-- should give us the delta tau.

Peet: OK.

Int: Did that help?

Peet: Right. Yes, it did. I'm trying to cement it into my mind

As a result of the above exchanges, Peet conceived Δt as a quantity that could vary independently of its composition from two values for the quantity *time*.

Some participants needed help conceiving a quantity as agnostic to usual units. For example, some participants arrived at Subtask #4 with the idea that $[\alpha \times \beta \times (B(t) \times C(t))]$ represented a count for bird deaths. For these participants, because the output units did not include a "per time" element, it did not make sense to them to multiply the expression by Δt. We found two approaches to support these students into developing a quantification of time that was agnostic to usual units like hours, months, years, etc.

1. All these participants agreed that "for something to have happened, time must have passed." We then noted that the birds must have been killed by cats during *some* finite block of time where the exact duration was unimportant. We asked them to consider $[\alpha \times \beta \times (B(t) \times C(t))]$ as the number of successful hunts (or birds killed) during one Block of time.
2. We leveraged participants' natural inclination to justify multiplication through repeated addition. To do this, we asked participants to successively iterate the expression: if $[\alpha \times \beta \times (B(t) \times C(t))]$ is the number of birds killed by cats during one Block of time, how many birds would be killed during two Blocks? During three Blocks? During five Blocks? During half a Block? During 2.7 Blocks? During a fixed, but arbitrary length of time Δt?

This approach had two advantages. First, it allowed participants to generalize to $\times \Delta t$ by building on a simple, observable pattern with concrete numbers. The exchange followed a pattern similar to development of units coordination among children learning proportional reasoning (e.g., Hackenberg, 2007). Second, it built for participants a conceptual object called a *multiplicative object* (Thompson & Carlson, 2017). A multiplicative object is the result of a mental act that couples quantities and their attributes, much like an ordered tuple, where each component of the tuple can vary independently and the resulting tuple is recognized as a quantity as well. Here, the participants formed a conception of "birds killed during an arbitrary time interval" by coupling $[\alpha \times \beta \times (B(t) \times C(t))]$ with Δt. For these participants, the units of the coupling were more similar to "light-years" than to "kills per unit time times time".

Contingent scaffolding 4: Guiding participant to re-quantify a quantity. Some participants realized that the cat and bird populations would vary as time progresses. These participants interpreted $\alpha \times \beta \times (\mathrm{B(t)} \times \mathrm{C(t)}$ as a per-unit-time rate that also varied as time progresses. They formulated an integral expression for the number of birds killed during $\Delta t = t_f - t_i$, $\int_{t_i}^{t_f} [\alpha \times \beta \times (\mathrm{B(t)} \times \mathrm{C(t)})] \cdot dt$. While this is a correct model for the question asked, it is not useful for moving toward a *differential equation* to model the interaction of the species.

We found two ways to help participants conceive $\alpha \times \beta \times (B(t) \times C(t)$ as (nearly) constant during a small Δt, a critical conceptual step for passing from $\Delta B/\Delta t$ to dB/dt. The first way was to suggest that $B(t)$ was large enough such that the fluctuation of a few birds would not matter so much. Many participants were surprised that "it was okay" to make this kind of simplifying assumption, since students often have strong impulses to preserve precision in their models (Czocher, 2019). We then directed the participants to consider a small duration of time, on the order of a day or an hour, rather than a month or a season. With these conditions in place, the participants were able to write $[\alpha \times \beta \times \left(B(t) \times C(t)\right) \times \Delta t]$. The second way was to encourage participants to consider an average value for $B(t)$ and $C(t)$ during Δt. Some students naturally thought of an arithmetic mean using the time interval's end points $B(t)$ and $B(t+\Delta t)$, some took the value $B\left(t+\frac{\Delta t}{2}\right)$, and others introduced concrete numbers to communicate the idea of a weighted daily average. For example, Niali used the notation $B_{\bar{x}}(\Delta t)$ to signify the average valule of the bird population during the interval of time Δt and obtained the following formula to represent the quantity "bird decline during Δt": $P_k \times \left[P_e \times [B_{\bar{x}}(\Delta t) \times C_{\bar{x}}(\Delta t)]\right] \times \Delta t$, where P_k and P_e notated the percent of encounters that end in a bird's death and the percent of encounters that happened per unit time, respectively. Through replication with other participants, we found that Niali's method was viable for those who had completed a course in probability or statistics, and the first method was sensible to a broader range of participants.

In Subtask #4, the same set of symbols can represent multiple aspects of the scenario. The expression $[\alpha \times \beta \times \left(B(t) \times C(t)\right) \times \Delta t]$ represented "successful hunts," "birds killed," and "change in bird population." Shifting interpretations for this expression is instrumental to making progress in the model's development, because eventually the expression needs to be associated with ΔB as its output. These equivalences are interchangeable for instructors, and also to many students, but for some undergraduate STEM majors, shifting interpretations of this expression is nontrivial. For these participants, these three interpretations correspond to three distinct quantities (mental constructs) because their referents in the scenario (objects and attributes) are distinct, as shown in Table 1. To scaffold these participants, we appealed to the tripartite nature of quantities to guide participants' attention towards the object attribute pairings forming the referent for each interpretation of the expression.

Table 1 Three interpretations for the expression$[\alpha \times \beta \times \left(B(t) \times C(t)\right) \times \Delta t]$ *with explicit reference to objects and their attributers in the scenario.*

Object	Attribute	Unit
Set of encounters	Successful hunts	Count
Bird population	Dead birds	Count
Bird population	Change during Δt	Count

Discussion & Conclusions

In this paper we reported four contingent scaffolding moves with quantities and quantitative reasoning at their foundation that supported undergraduate STEM majors to mathematize aspects of a canonical modeling problem from differential equations The interviewer questioning, which contemporaneously focused on students' quantitative meanings for their equations simultaneously revealed participants' conceptual difficulties and provided avenues for overcoming them. Contingent scaffolding move 1 targeted participants meanings for their notation. Guiding participants to attend to the object and/or attribute their notation represented helped them communicate their intended meaning with conventional notation, and in some cases, helped them modify their notation to correspond with their intended meaning. For example, asking the participant what quantity each symbol in $B(t_f) = B(t_i) - \text{Birds Killed}$ represented typically led to participants realizing $B(t_i) - \text{Birds Killed}$ represented "number of birds remaining" instead of the intended "decrease in the bird population." An appreciable subset of these participants then modified their model to $B(t_f) - B(t_i) = -\text{Birds Killed}$. It was then a natural next step for participants to associate $B(t_f) - B(t_i)$ with "change in bird population" via the expression $\Delta B = -\text{Birds Killed}$. Contingent scaffolding move 2 targeted clear and concise communication between participant and interviewer. This aided in mathematization by clarifying which aspect of the real-world situation the modeler intended to model. For example, the language "change in birds over time" is imprecise. It is reasonably interpreted as either "absolute change in the bird population during an arbitrary duration of time" or "change in the bird population per change in time." Contingent scaffolding move 3 aided in mathematization by helping participants quantify an important aspect of the real-world scenario that could reasonably (from their perspective) further operated upon. For example, Peet conceived Δt as a quantity that could vary independently of two specific values of time. Only then could he combine it with the already-constructed quantity *frequency of encounters that result in death* to mathematize *decrease in the bird population*. Similarly, scaffolding move 4 aided in mathematization by helping a participant quantify an important aspect of the real-world scenario so that they could combine that quantity with other quantities. However, the goal of move 4 is not to help them construct a new quantity, but to guide them into re-quantifying an already constructed quantity. In both examples, we helped participants conceive $\alpha \times \beta \times (\mathrm{B(t)} \times \mathrm{C}(t))$ in such a way that the modeler found it reasonable to combine $\alpha \times \beta \times (\mathrm{B(t)} \times \mathrm{C(t)})$ and Δt.

Some of the moves we implemented may seem counterintuitive. For example, typically curricular materials across grade bands assume that learners need to comprehensively study one variable functions before encountering multi-variable functions. The evidence we considered here suggests that some cognitive conflicts encountered during modeling could be resolved if learners had access to notational conventions and increased capacity for analyzing multivariable relationships. Participants clearly recognized complex multivariable interdependences, an observation echoed in the covariational reasoning literature (Jones, 2020). One troublesome implication is that the standard curriculum may be depriving learners of important mathematical tools for use in modeling that would allow them to identify and represent mathematical aspects of real-world considerations.

We wish to emphasize that at any moment during the interview sessions, we could have directly remediated errors or misconceptions, shown the correct equation, and explained why it was correct. However, our purpose was to understand how the participants were thinking and how educators could respond in student-centered ways that would build on or change their thinking. Hence, the nature of our claims take the form: "When a participant exhibited X under Y conditions, doing Z was supportive." While the numbers in our sample were small and the number of participants ripe for each contingency even smaller, the examples we shared are emblematic of the ways that quantitative reasoning, quantification, and quantitative operations can be both a source of cognitive dissonance for students as well as the basis for effective scaffolding in modeling that still respects learners' intuitive ways of reasoning.

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Interactive Tutorials in Undergraduate Mathematics: What Are They Good For?

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In-person tutorials present a prime opportunity for students to work with others to engage their problem-solving and critical-thinking skills. We investigate the use of interactive tutorials in a large, second-year service mathematics course as part of a quasi-experiment whereby students could either attend in-person tutorials (for a participation mark) or complete the tutorials individually without attending the live sessions and submit solutions for credit. Our analyses of student data and self-efficacy measures suggest that consistent attendance at in-person tutorials may maintain and improve students' self-efficacy. Median changes to self-efficacy measures for students who attended in-person tutorials were higher than those who completed tutorials individually without attending the live sessions. The difference between the changes to the Emotional Regulation aspect of tutorial self-efficacy was statistically significant ($p = .031$)*, highlighting the role in-person tutorials play in supporting students' mathematical learning beyond academic outcomes.*

Keywords: self-efficacy, tutorials, in-person learning, online learning

The integration of online learning and technology into higher education continues to induce a paradigm shift in the way we view teaching and learning in mathematics. Recently, the Covid-19 pandemic has perturbed the norms of teaching and learning. Despite setbacks, it has been a chance for institutions to realize the potential and necessity for online delivery methods, not only as means to continue activities during a crisis but also to stay relevant in an increasingly competitive tertiary market. It is difficult to compete with the low-cost and flexible nature of online education. As a result, we can expect to see the number of online components within traditional courses continue to grow. The use of lecture capture technology and learning management systems have made it possible for lectures and resources to be readily accessible. However, finding an online substitute that can deliver the same level of engagement and retain the interactive components is an issue yet to be resolved. One component of undergraduate mathematics that cannot be easily replaced is the opportunity for collaborative face-to-face learning – in New Zealand, this usually happens during tutorials, which are weekly 1-hour classes scheduled in addition to lectures.

Tutorials, like those conducted within our course of interest, have not received as much attention from researchers, especially in comparison to lectures. Hence, we must gain a deeper understanding of the contribution of in-person tutorials to one's experience of learning mathematics. This study aims to quantify the effects of in-person undergraduate mathematics tutorials on self-efficacy and address the benefits that tutorials can provide that simply cannot be matched through online means.

Theoretical framework

Self-efficacy

Theoretical framework Self-efficacy is a psychological construct that allows researchers to understand how an individual's action can shape an outcome. Used extensively within educational psychology, Bandura (1997) defines perceived self-efficacy as "beliefs in one's capabilities to organize and execute the courses of action required to produce given attainments" (p.3). Self-efficacy may have a potentially greater initial impact on performance than other related constructs, such as outcome expectations (Sexton & Tuckman, 1991).

Self-efficacy, first proposed by Bandura (1977), is influenced by four main sources: mastery experience, vicarious experience, social persuasion, and psychological states (1997). Mastery experience is considered the most significant source (Bandura, 1997; Usher & Pajares, 2008) as it provides the most realistic indication that students can succeed in a given task. All four sources have a significant contribution to self-efficacy beliefs.

Self-efficacy, the measure of one's belief in their ability to successfully perform a given task, has been well-established as a strong predictor of performance and intrinsic factors in mathematics. Early research into mathematics self-efficacy showed it had a much stronger direct effect on performance than variables such as prior experience, gender, or self-concept (Pajares & Miller, 1994). Similar positive relationships between self-efficacy and performance have been reported in schools (Pajares & Kranzler, 1995; Pantziara & Philippou, 2015) and at the undergraduate level (Pajares & Miller, 1994; Peters, 2013). As a significant predictor of achievement, analyzing measures of self-efficacy can help us to better understand their impact on students and inform the way they are implemented into higher education courses.

Motivation

Using a quasi-experimental design, we sought to examine the potential impacts of tutorials in two ways. First, we were interested to see if there was a significant difference in student performance (final exam grades and overall grades) between students that primarily attended the in-person tutorial sessions compared to those that submitted their work online. Previous studies in similar settings report the use of live components to be associated with higher achievement (Howard et al., 2018; Inglis et al., 2011). However, these studies focused on the difference between attending live lectures versus watching lecture recordings. Students often justify their use of asynchronous video resources by praising the control of pace it provides and the flexibility it affords (Howard et al., 2018). However, the negative association between achievement and higher use of lecture recordings has primarily been consistent across the undergraduate mathematics education literature reported in a recent systematic review (Lindsay & Evans, 2021). It is thought that lecture recordings promote surface-level approaches, thus degrading the quality of learning. However, considering the interactive nature of tutorials, we expect to see a similar relationship between in-person tutorial attendance and performance.

Second, we set out to investigate tutorials' impact on a significant intrinsic construct: self-efficacy. As a significant predictor of success, among other variables, understanding this impact would allow researchers and course designers to utilize this understudied aspect of undergraduate courses best, whether it be to justify policies in favor of more fully online mathematics education at undergraduate levels or as evidence to consolidate the use of in-person tutorials for the near future in a delicate time of change in education.

Study setting

This study investigated the tutorials within a large, second-year service mathematics course at the University of Auckland. The course, General Mathematics 2, is split into three main topics that cover the content required across many disciplines: calculus, linear algebra, and differential equations. Assessments in the course are divided into four sections: coursework (including engagement with tutorials) – 15%, mid-semester test – 20%, quizzes – 15%, and a final exam – 50%.

While the Covid-19 pandemic disrupted many on-campus activities throughout 2020 and led to a shift to remote delivery methods, the first semester of 2021 presented itself as a relatively unperturbed semester, with most university activities returning to campus. However, with the risk of Covid-19 still looming over the community and international students being located outside of the country, remote options to complete coursework were made available for students.

Tutorials

This format of interactive tutorials is standard at the undergraduate level at the University of Auckland and is often met with positive feedback from students that note the usefulness of learning from others in a supportive space (Oates et al., 2016). In the tutorials, students are given opportunities to collaborate with their peers while problem-solving. Each tutorial session is one hour long and occurs weekly. A tutor is available during the session to help address questions and monitor each group's progress. While attendance or completion of the tutorials in our course was not mandatory, it was highly encouraged for all students and had a small contribution to overall grades. Students could receive marks contributing to their coursework grade by attending an in-person session or completing the questions and uploading their work to a learning management system (Canvas).

The term 'tutorial' is often used in two contrasting ways in the literature. One refers to learning opportunities where attendance is optional and is directed at supporting students, often serving a remedial purpose. In many institutions, mathematics support centers subsume this role (Cronin & Meehan, 2021; Mullen et al., 2022). The other, which we utilize in this work, refers to a core component of undergraduate courses that, like lectures, aim to provide another medium to engage learners with the content. This shares many aspects of a lab session in chemistry courses, where students apply knowledge from lectures to engage their problem-solving and critical-thinking skills.

Data

Student Data

Student data was collected during the first semester of 2021 via Canvas. In total, there were 354 students enrolled in the course. Student data included tutorial attendance/completion and performance in the course, including final exams. Of the 354 students enrolled,186 students attended or completed all 9 tutorials. Due to Covid-19 health restrictions, the first scheduled tutorial session (in the second week of the semester) occurred online rather than in-person. To best capture the potential impact of consistent engagement of tutorials, we have not included the data of students that did not complete all the tutorials or were limited to one option in our analyses (e.g., offshore students that could not attend the on-campus sessions). Data from these 129 students (36% of the course) will be analyzed further to capture the impact of consistent tutorial engagement over the semester.

Self-efficacy Measures

Student self-efficacy was measured using the Measure of Assessment Self-Efficacy for Mathematics Tutorials (MASE-T) (Evans & Jeong, 2023; Riegel et al., 2022). The instrument measures assessment-related self-efficacy across two factors: *Comprehension and Execution* (CE), and *Emotional Regulation* (ER). The instrument has been validated and shown to be reliable at measuring student assessment self-efficacy across different contexts, including tutorials. Of the students enrolled in the course, 219 students provided measures of their self-efficacy at the start and the end of the semester. In total, data from 100 students who completed all tutorials and provided self-efficacy measures will be used for the analysis.

Attendance Groups

Students were classified into two groups by the number of tutorials they attended over the course. Table 1 shows the distributions of the 129 students by the number of in-person tutorials they attended. Students were grouped into two groups: those that attended less than half of the total in-person tutorials (i.e., 0-4 in-person tutorials attended, $n = 23$) and those that attended more than half of the total in-person tutorials (i.e., 5-9 in-person tutorials attended, $n = 106$). Herein, the two groups will be referred to as *online* and *in-person*, respectively.

Table 1. Number of in-person tutorials attended across the semester.

In-person tutorials attended	*n*	%
0	11	8.5
1	3	2.3
2	5	3.9
3	2	1.6
4	2	1.6
5	1	0.8
6	6	4.7
7	15	11.6
8	25	19.4
9	59	45.7

The attendance behavior of students throughout the semester suggests that students showed a strong preference for one of the learning environments over the other, as seen by the large proportion of students that attended either 0-2 or 7-9 in-person tutorials. The number of students opting for each method of tutorial completion (in-person/online) remained relatively consistent across the semester (Table 2). One possible explanation for the slight decrease in the number of in-person attendees for tutorials 4 and 5 could be that some students may have prioritized their time to prepare for the mid-semester test and completed the tutorials individually for those weeks.

Table 2. Tutorial attendance across the semester by attendance groups.

Tutorial	2	3	4	5	6	7	8	9	10
In-person	107	106	92	90	101	102	102	104	100
Online	22	23	37	39	28	27	27	25	29

Results

Performance

An independent-sample *t*-test was run to determine if there were differences in the mean final exam score and overall grade of online and in-person students. An initial assessment for outliers, normality (as assessed by Shapiro-Wilk's test ($p > .05$), and homogeneity of variances (Levene's test ($p = .405$, $p = .325$)) was conducted for both the final exam grade and overall grade data with no violations being detected in either set of data.

The mean final exam score was higher for online students (*M* = 73.8, *SD* = 13.3) than in-person students (*M* = 61.3, *SD* = 16.8), a statistically significant difference with medium effect size, *M* = 12.5, 95% CI [5.1, 19.9], $t(127) = 3.349$, $p = .001$, $d = .77$. The online group also had a higher mean overall grade (*M* = 78.2, *SD* = 9.9) than the in-person group (*M* = 70.8, *SD* = 12.6). This difference was also statistically significant with medium effect size , *M* = 7.4, 95% CI [1.8, 12.9], $t(127) = 2.637$, $p = .009$, $d = .61$.

Self-efficacy

Due to violations in the normality of the self-efficacy data, Mann-Whitney *U* tests were undertaken to determine whether there were any significant differences between groups. First, the median and mean ranks for all measures of initial self-efficacy were higher in the online group than in the in-person group. By the end of the semester, this was not the case. In particular, the median and mean rank for all tutorial self-efficacy measures were higher in the in-person group than in the online group. The online group still had a higher median and mean rank for all exam self-efficacy measures by the end of the semester. However, these results were not statistically significant ($p > .05$).

Second, the analysis of the changes in participant self-efficacy between the start and the end of the semester revealed the following. A significant difference was observed in the *Emotional Regulation* factor of tutorial self-efficacy, which was statistically significantly higher at the end of the semester for the in-person group (mean rank = 53.12) than for the online group (mean rank = 35.63), $U = 860.5$, $z = 2.153$, $p = .031$ (Figure 1).

Similarly, change in the *Comprehension and Execution* factor of tutorial self-efficacy (U = 792, $z = 1.491$, $p = .136$) and overall change in tutorial self-efficacy ($U = 835$, $z = 1.907$, p = .057) was higher for the in-person group with these differences almost reaching a level of statistical significance.

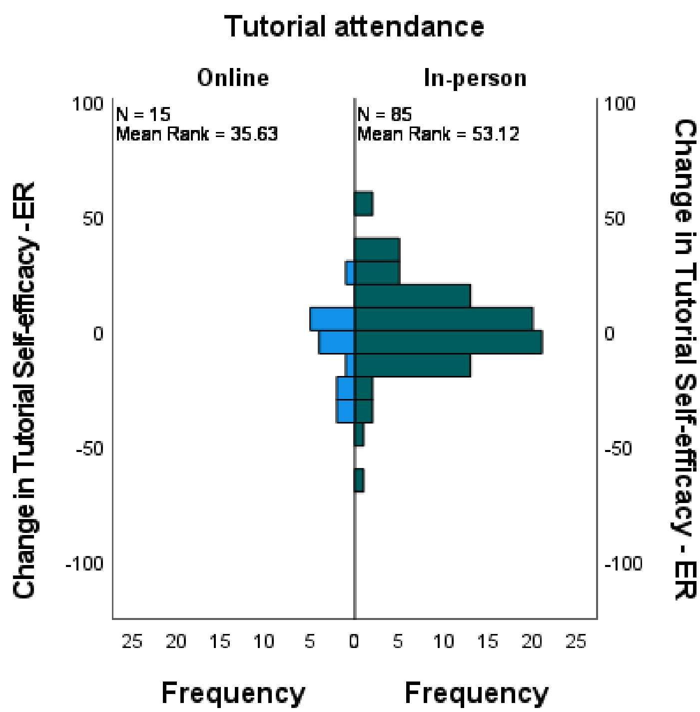

Figure 1: Histograms of the change in the Emotional Regulation (ER) factor of tutorial self-efficacy. The in-person group saw a greater increase than the online group, p = .031.

Discussion

Performance

Our findings showed a statistically significant between-group difference in performance in the final exam and overall grades in the course. A significant difference in performance between groups of students that utilize different study methods is not a unique finding. Inglis et al. (2011) reported that students that relied more on online recordings (lecture capture) were associated with lower grades in the course, while live lecture attendance was met with improved grades. Similar findings were reported by Howard et al. (2018).

While we report a correlation within our data pertaining to learning activities within a large undergraduate course, our study differs from the existing literature in two ways. The first is that our study investigated the use of tutorials, which, as an integral part of tertiary mathematics, has not been fully explored using quantitative analyses. This study analyses the value of undergraduate mathematics tutorials through the lens of self-efficacy. Secondly, our correlation found that using a live study component, that is, in-person tutorials, was associated with lower performance grades, unlike the studies mentioned previously. While we cannot say whether students with a higher aptitude for mathematics chose to study a particular way or that the use of tutorials may not foster mathematical learning, we speculate that there is more to this phenomenon that warrants further investigation. With both forms of tutorial engagement involving the same problem sets, it seems unlikely that this difference is due to differences in content coverage.

One possible cause for this difference could lie in prior knowledge. In a recent study, Zakariya et al. (2021) consider prior knowledge when examining students' approaches to learning and their performance. Their analysis suggests that low prior knowledge may be conducive to surface-level approaches to learning which can have a significant, negative effect on performance (Zakariya et al., 2021). Within our learning context, we can expect students enrolling into a service mathematics course to have a wide range of mathematical abilities that may have influenced how they choose to learn. Possibly, students with low prior knowledge perceived in-person tutorials as an easier option to score 100% by simply showing up, whereas the online option required a written submission for marking.

The presence of high- and low-achieving students in both groups could indicate that students are making more conscious decisions about how they choose to study. This may be driven by a heightened appreciation of the benefits of each study option. Research from the pandemic shows that many students desired opportunities for social interaction, immediate feedback, and wanted to return to campus, noting difficulties in staying focused and motivated at home (Mullen et al., 2022; Radmehr & Goodchild, 2022). However, the oft-praised flexibility provided by remote study options could very well be a priority for many individuals that prefer to study online (Thompson & McDowell, 2019). Thus, we cannot make any concrete claims about students' motivation behind their preferred study method without using qualitative data.

Self-efficacy

First, the median initial self-efficacy measures were higher for online students than the in-person students. While this difference was not significant, it is worth noting as a potential factor in student decision-making about how to study. This hypothesis is in line with the literature reporting that students find enrolling in online courses for subjects that are perceived to be easier for them to be more favorable (Jaggars, 2014).

When comparing the changes to self-efficacy, the in-person students saw no negative changes to the median exam and tutorial self-efficacy measures. For the online group, the change to median self-efficacy measures remained unchanged for the *Comprehension and Execution* factor of their exam self-efficacy, while it dropped for all others. The greatest median changes for online students were seen in their tutorial self-efficacy, with significant changes in the *Emotional Regulation* factor of tutorial self-efficacy. The localization of this significant change is consistent with the domain-specific nature of self-efficacy (Pajares & Miller, 1995). Additionally, it strongly suggests that the benefits of interactive tutorials at the undergraduate level can branch into affective factors. It may help to maintain positive beliefs within individuals and foster a greater sense of community between learners.

The in-person tutorials with a group-work component promote a high degree of social engagement between peers and experienced tutors, seen in the form of immediate feedback and words of encouragement and motivation. At the undergraduate level, these are conducive to active learning and position tutorials as positively contributing to achievement (Freeman et al., 2014). The negative correlation between achievement and in-person tutorial attendance, while significant, cannot be used to make a claim for or against one form of tutorial engagement. In order to do so, we need to control for factors that influence performance, such as prior knowledge. We can also reflect on the differences offered by each delivery mode.

One key difference between the two groups in our study is the learning environment where student active (cognitive) engagement occurs. In-person tutorials provide an opportunity for collaborative learning through the presence of peers and tutors. From the perspective of cognitive psychology, this could be beneficial, as evidenced by the *collective working memory effect* (Kirschner et al., 2011; Kirschner et al., 2018). This effect suggests that learning as a group is more effective than individual learning if the material is sufficiently complex, exceeding the limits of each individual student's working memory. "In this situation, the cognitive load of processing this complex material is shared among the members of a collaborative learning team enabling more effective processing and easier comprehension of the material." (Kirschner et al., 2018, p. 222). However, the researchers caution that bringing together a group of learners is no guarantee that the learning would happen efficiently as many other factors are involved.

Our findings show that, on average, the trajectory of students' *Emotional Regulation* component of tutorial self-efficacy is improved through their participation in in-person tutorials. Thus, shifting our perspective to what is uniquely offered by in-person tutorials could be worthwhile to better understand their importance in undergraduate mathematics learning.

Limitations

It would be ill-considered to generalize the finding to different courses or contexts or to completely refute the effects tutorials may have on other aspects of self-efficacy. Considering many of the differences were borderline statistically significant, it would be worth investigating different courses, including those specific to mathematics majors.

We recognize the importance of collecting more measures of learning, including qualitative data, such as students' perspectives on tutorials as a learning resource. This could improve our understanding of students' preferences, provide us with a different lens to analyze our findings, and allow us to explore the more nuanced differences between the delivery modes.

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How Do We Disentangle Equality from Equivalence? Well, It Depends

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In this paper, we report on interviews with mathematicians exploring the ways in which they think about the relationship between equality and equivalence. Given sometimes unclear and conflicting presentations of equality and equivalence in the literature, we are motivated to understand subtleties about how these constructs interact; doing so can have pedagogical implications, which we explore in this paper. We present three major themes that emerged from our analysis of the mathematicians' discussions of equality: 1) that equality represents a well-specified equivalence relation, 2) that audience matters when specifying equivalence and equality, and 3) that context is imperative when discussing equivalence and equality.

Keywords: equivalence, equality, equivalence relations, mathematicians

In this paper, we unpack the nuances of multiple mathematicians' understandings and uses of equivalence and equality, and the potential implications of these nuances for K-16 pedagogy. Rather than bringing new mathematical insights to light (see Mazur (2008)), we seek to consider pedagogical implications pertaining to relationships among and distinctions between equivalence and equality as expressed by mathematicians. We suggest that such insights are valuable for researchers who want to better understand the nature of equivalence, and for instructors who seek general principles and guidelines for supporting students in productive ways of reasoning about both equivalence and equality. We address the following research question: *How do mathematicians view the relationship between equivalence and equality?*

Relevant Literature and Theoretical Perspectives

In this section, we introduce relevant literature and situate our work within theoretical perspectives. This includes framing our working definitions of equivalence and some mathematical discussion about equivalence and equivalence relations.

Equivalence and Equality

There has been a considerable amount of research exploring equivalence and equality; much of this work has been conducted in K-12 settings (e.g., Knuth et al., 2006; McNeil & Alibali, 2005; Solares & Kieran, 2013), although some has explored these ideas with older students as well (e.g., Chin & Tall, 2001; Godfrey & Thomas, 2008; Stephens, 2006). We focus here on work that has explored the relationship between these two foundational mathematical notions, which are broadly synonymous but subtly different: while the equivalence concept is well-specified (via the definition of an equivalence relation), we observe that explicating its relationship to and distinction from equality is important but not yet well articulated or understood. Despite its ubiquity, for instance, equality has been referred to as "slippery" (Mazur, 2008, p. 222) and "precarious" (Saldanha & Kieran, 2005, p. 5). Hersch (1997) encapsulated this idea well when he noted that equality "is used freely, from kindergarten to postgraduate. It's never defined or explained" (p. 50).

The K-16 curriculum provides a multitude of examples that highlight the potential benefits of being able to disentangle equality from equivalence. For example, Saldanha & Kieran (2005) found that "many students' emerging thinking about equivalence was rightly bound up – indeed, arguably confounded – with notions of numerical equality" (p. 5). We infer that, here, *equality* refers to the equivalence of numerical expressions, whereas *equivalence* refers to the equivalence of algebraic expressions. Note that, in this situation, it would not be inappropriate to use equality to refer to *both* the equivalence of numerical expressions and the equivalence of algebraic expressions, as is in fact normative.

Also consider the notion of equation solving, which not only involves "a grasp of the notion that [the] right and left sides of the equation are equivalent expressions, but also that each equation can be replaced by an equivalent equation (i.e., one having the same solution set)" (Kieran, 1981, p. 323). That is, two forms of equivalence are at play. We observe that, in such instances, the algebraic expressions in question could be reasonably said to be equal (and thus also equivalent). Though while the equations are equivalent because they share the same solution set, they are *not* considered to be equal (we consider it normative to say, for example, that $2x + 1 = 5$ is *equivalent* to $2x = 4$, rather than to say that $2x + 1 = 5$ is *equal* to $2x = 4$). Thus, in contrast to the first example we discussed, equality applies to one instance (expressions) but not the other (equations).

The topics featured in these examples – the equivalence of expressions and equations – are foundational in the mathematics curriculum, yet equality does not uniformly apply in each situation. Indeed, these peculiarities illustrate that "[m]ath lingo sometimes says 'equal,' sometimes 'equivalent'" (Hersch, 1997, p. 50). We observe that the rules governing *when* one might use one or the other are implicit and underspecified. This spurs us to ask: in what situations might we use equality, and why?

The literature does contain some attempts to answer this question. Mazur (2008) conducted a deep and mathematically rigorous exploration, using category theory to explicate the relationship between equivalence and equality. However, we note that his argument – while undoubtedly useful to a research mathematician and generalizable across many context – is far beyond the scope of mathematics with which K-16 students (and most K-16 teachers) are familiar. What is needed, we propose, is an explanation of the relationship between these ideas that, in addition to being plausible and generalizable, is also accessible to K-16 students and instructors. In this vein, McNeil and colleagues (2012), for example, explained that numerical equivalence is "formally represented by the equal sign" (p. 1109). Similarly, Fyfe and Brown (2018) framed numerical equivalence as "the relation indicating that two quantities are equal and interchangeable" (p. 158). The example of equivalent equations, for instance, illustrates that equivalence is not always "formally represented by the equal sign" and certainly can – but need not always – indicate that "two quantities are equal." While these descriptions of the equivalence-equality relationship are certainly appropriate for the contexts from which they were drawn, we note that they do not frame the relationship in a way that generalizes to other contexts, and thus are of limited use for our purposes here.

Others have highlighted equivalence as a more general and sophisticated version of equality, but have pointed out that equality is nevertheless used differently (e.g., Rupnow & Sassman, 2022). Kieran's (1981) seminal paper on equivalence begins with Gattegno's (1974) observation that, with respect to equality, "equivalence is concerned with a wider relationship" (p. 83). Additionally, Fischbein (1999) argued that "the concept of equivalence is more general and much more subtle than the concept of equality. What seems to be trivial for equality, is not necessarily trivial for equivalence" (p. 23). Our initial impression is that framing the well-specified notion of equivalence as more general than the "slippery" and "precarious" notion of equality is both useful as a starting point for our analysis and consistent across mathematical contexts. A contribution of our work, therefore, is to further explore and elaborate these conceptual statements regarding the specificity of equality and the generality of equivalence as a result of analyzing the perspectives of multiple mathematicians.

Framing Equivalence

Here we define and characterize equivalence, which connects to our interview design and our analysis. We broadly draw on Cook, Reed, and Lockwood's (2022) framework, which employed conceptual analysis to identify three interpretations of equivalence taken on by students across mathematical domains and scholastic settings (these included *common characteristic*, *descriptive*, and *transformational* interpretations). We note that each of these focus on the features of students' possible interpretations of the equivalence of specific objects; that is, these interpretations involve what Hamdan (2006) calls a *local*, element-wise view of equivalence.

While we inferred the presence of these *local* interpretations when analyzing our interviews with mathematicians, we determined that the mathematicians' perspectives on the relationship between equivalence and equality can be better characterized in terms of a *global* view of equivalence, which "requires going beyond the element-wise conception of a relation" (Hamdan, 2006, pp. 130-131) in order to focus on the structure of a set in terms of the equivalence classes based on the relation that partitions the set. One key global interpretation of equivalence taken on by the mathematicians, which theoretically grounds our discussion of equivalence and equality here, is that objects "are identified as 'the same' if they lie in the same equivalence class" (Rupnow & Sassman, 2022, p. 119). A key aspect of this interpretation is that it is by nature a stipulated attribution of equivalence. Indeed, given any set, an extreme use of this interpretation might entail an arbitrary partitioning of the set and then attribution of equivalence to the members of each individual partition. While employing this interpretation is usually much more purposeful (and less arbitrary), we find that the key subtleties of attributing equality or equivalence to mathematical objects are grounded in one's ability to assign equivalence based on membership in an equivalence class. In fact, our exploration of when to use equality largely centers on the conditions by which the mathematicians might not just attribute equivalence to members of each class, but instead might determine it appropriate to ignore differences amongst elements in the same class altogether.

Methods

Data Collection

The data on which we report is taken from a larger study examining the ways that both mathematicians and students view equivalence across undergraduate mathematical domains. Early data collection targeted research mathematicians to, among other goals, generate hypotheses about how students might productively understand equivalence at high levels of mathematics. One outcome of these interviews was an early awareness that the relationship between equality and equivalence was relevant to these mathematicians' understandings and uses of equivalence, and that there were many subtleties to the mathematicians' views on equality that have key implications for mathematics instruction in general.

As part of the larger study, we interviewed research mathematicians specializing either in abstract algebra or in combinatorics. Our primary objective was to build an initial theory that accounted for the key ways in which algebraists and combinatorists might think about equivalence. This report is based upon individual semi-structured interviews (Fylan, 2005) we conducted with eight mathematicians, all of whom were tenured mathematics faculty at large universities across the United States. The primary data for this report were the video records of the 90-minute interviews conducted with the mathematicians (referred to here by the pseudonyms Dr. A, Dr. B, etc.). Though we asked a variety of questions about equivalence and how it manifests in their research and instruction, here we focus primarily on their responses to the following question: "Do you think that there is a difference between equivalence and equality? If so, please explain in what kinds of situations you would use equality and in what kinds of situations you would use equivalence. If not, please explain why."

Data Analysis

The video records were transcribed in full, and the transcripts were initially analyzed according to Cook et al.'s (2022) framework for interpreting local, element-to-element instances of equivalence. After inferring the various ways in which the mathematicians were interpreting equivalence, and determining that a global, equivalence-class based perspective on equivalence (Hamdan, 2006) would be more productive for analyzing the mathematicians' conceptions of equality, we isolated the segments of the interviews in which the mathematicians specifically discussed the relationship between equivalence and equality. We then employed thematic analysis (Braun & Clarke, 2012) on this subset of the data. Thematic analysis was useful for our purposes because it provided a mechanism by which we could identify themes related to the mathematicians' views of the relationship between equivalence and equality and also make sense of these themes with respect to our research question. Our analysis was primarily data-driven and exploratory, though we did use Hamdan's (2006) global perspective on equivalence and equivalence classes to guide our identification and description of themes. We followed Braun and Clarke's (2012) stages of thematic analysis (familiarizing oneself with the data, developing initial codes, searching for themes, reviewing themes, and characterizing and naming themes), then met to compare and negotiate our interpretations. From this process we identified the three themes that we discuss below.

Results

We now detail the overarching themes we identified regarding the nature of the relationship between equality and equivalence. Generally, the mathematicians we interviewed framed equality as a relationship between objects that results from declaring and imposing a well-specified equivalence relation. Their criteria for when they would comfortably use equality (instead of the more general notion of equivalence) to describe the relationship depended on our two major themes of *context* and *audience*. Though we observed instances of each of these themes across a multitude of excerpts and interviews in our data set, due to space constraints we present each theme only with prototypical, illustrative responses from our data.

Equality Indicates a Well-Specified Equivalence Relation

Our first theme is that equality indicates a well-specified equivalence relation. That is, equality is used to relate elements in the same equivalence class to each other once an equivalence relation has been clearly defined and enacted on a given set. To illustrate this point, we focus on responses from Dr. A, a combinatorist who explained the nuances of the equality-equivalence relationship by pointing out the differences in the role of equality before and after a well-specified equivalence relation is imposed. For Dr. A, before an equivalence relation is imposed, equality is “kind of the trivial case where every block of the partition is a single element.” In such cases, he said, “there’s one element that’s equal, you know, the equivalence classes have size one. There’s no variation.” In other words, prior to the implementation of a well-specified equivalence relation, equality means *identity*. That is, equality only applies to elements that are represented in exactly the same way with no variation in representation (e.g., prior to establishing what it means for two numerical expressions to be equivalent, 3 is only equal to 3 and is not, for example, equal to 1+2). Dr. A then explained equality in the context of using modular congruence as an equivalence relation:

> *Dr. A*: But if I’m willing to just say, “No, I’m not interested in all the integers anymore, I now just want one element from these classes,” I feel, like, okay committing to that. Then it’s okay. That's sort of my, that's what it means to me to now say, “Okay, when I say equals … I'm not thinking of three as three, I’m thinking of three as a … representative of this set [of integers congruent to 3 modulo 7],” or whatever. But it’s kind of like you’re changing the idea of what [the equal sign] even means. And now the definition of equality, I guess, is okay in my mind now […]. It’s just that that equality has to be defined.

We highlight two key features of this theme that we infer from this comment. First, Dr. A emphasizes that “equality has to be defined.” He made a similar comment at another point in the interview, noting that “I’m okay with [equality] once you declare it and clean it up.” We therefore infer that, for Dr. A, an essential element of declaring objects to be equal is that equality is well-specified. First, notice that Dr. A’s language distinguishes the meaning of equality *before* and *after* the introduction of modular congruence as an equivalence relation. Before, equality meant only identity (e.g., 3 is only equal to 3). After, the introduction of the equivalence relation and collapsing of the integers into equivalence class representatives establishes a new form of equality (e.g., 3 can now be considered equal to 10, 17, etc.). Bringing these two features together, we infer that equality indicates a relationship between elements that results after one declares and imposes a well-specified equivalence relation.

Audience Matters When Specifying Equivalence and Equality

Our second theme is that one’s use of equality is also conditioned by one’s image of the audience. For example, Dr. A noted that equality “has to be either explicitly there or at least understood by all the people in the conversation as an okay use of equality.” Also consider the following comment from Dr. B, who had previously indicated the importance of specifying what equality means:

> *Dr. B*: So, what’s probably going on in my head, the rule of thumb is if I, as the speaker, I’m confident that everybody in the conversation really understands what’s going on, then we can get careless. In [an introductory abstract algebra class], I don’t think it’s ever appropriate to have that confidence. […] So yeah, if I’m doing modular arithmetic in [class] $\overline{1} = \overline{8}$ instead of $1 = 8$. And I'm gonna be very clear that I'm always using bars […] because with that collection of students, I don’t know, there’s probably a significant fraction where the notational abuse is fine and honestly helpful, but there's another bunch

who are just gonna be so completely lost if you start to do it that I wouldn't try it. On the other hand, if I'm teaching […] the master's level class, yeah, I'm not putting those bars on after the first day that I introduced the quotient definition.

Dr. C offered a very similar explanation:

Dr. C: The students may not be comfortable with equivalence and equality being the same, like, and this probably comes up […] in modular. You know, if I'm in Z_5 and I'm saying, "Oh, six is the same as one," […] I think for mathematicians that's, we're very casual with our language, you know? So, […] that could be okay. But maybe for an undergrad that would be not so much okay to say 6=1.

Overall, the mathematicians we interviewed indicated that, with respect to a particular topic, equality might be allowable when in discourse with some populations (e.g., graduate students and mathematicians) but not advisable when in discourse about the same topic in other populations (e.g., undergraduates in their first abstract algebra course).

The Mathematical Context Is Imperative When Discussing Equivalence And Equality

Our final theme is that nuances related to the mathematical contexts under discussion also influence whether or not equality is used. For example, above we discussed a comment from Dr. B in which he discussed how he would feel free to write statements like $1 = 8$ to connote the equivalence of elements in the quotient Z_7 while amongst mathematicians or in a graduate course. He went on to add, however, that this might not be the case if he has "got multiple quotients interacting or something." We infer here that, in such situations, the mathematical context at hand might shape one's use of equality. Also consider the following comment from Dr. D:

Dr. D: So, equality is an equivalence relation, […] so if I'm looking at a quotient ring – so I'm defining, you know, ring mod ideal – then $Q = R$ mod i. Then I can really say that $a + i$ really *equals* [his emphasis] $b + i$ in Q because we have defined equality to be that way. […] So, in terms of discrete math, I will *define* $\{a, b, c, d\}$ to *equal* $\{c, b, a, d\}$. I will define those to be equal if I put the curly brackets on and I understand that these things are sets. But I will say (a, b, c, d) is not equal to (c, b, a, d). […] equality depends on the context that we've agreed upon.

Similarly, Dr. C explained that, to him, consideration of mathematical context also influences when equality might be most appropriately used:

Dr. C: It's very situational to the problem being asked. […] I might be going off on a tangent now, but I'm just thinking when I say equals it depends on where I'm working, you know? An integer, if I'm saying six and one are integers, then no, they can't be equal. They're different. But if I'm working in Z_5, then yeah, they're totally equal.

We view these considerations of the mathematical context as complementary (and not disjoint from) considerations of the audience.

Discussion

In response to our research question (*How do mathematicians view the relationship between equivalence and equality?*), our analysis indicates that – in alignment with the literature – equivalence is indeed the more general, comprehensive notion, and that mathematicians' use of equality is conditioned by (1) whether or not the form of equivalence in question has been well-specified, (2) the mathematical context, and (3) their images of the intended audience for the mathematical discourse in question.

A major goal of this report is to make a case that more care is needed when conceptualizing equality in educational contexts, and that there are subtle aspects of how we use equality in both research and teaching contexts that can have important implications for research and instruction. Returning to the literature, when it is said that equivalence is "formally represented by the equal sign" (McNeil et al., 2012, p. 1109), our findings suggest that there is more specificity needed to situate *when* we use equality to formally represent equivalence relations. This functionally brings clarity to the theme in the literature about how equivalence is more general than equality: for the mathematicians, equality represented a well-specified equivalence relation that was appropriate for the intended audience and mathematical context.

We believe our analysis builds upon other pieces of the discussion of equivalence and equality in the literature as well. Particularly, we propose that a reason that equality has been called "slippery" and "precarious" is perhaps because its use depends not solely on mathematical rigor and precision, but on the user's image of the audience and mathematical context. That is, to a certain extent, this is an *epistemological* – and not solely a *mathematical* – distinction. These themes support the idea that there is not just one definition of equality, underscoring the importance of our first theme (as well as many calls by researchers and curriculum designers): it is critical to specify exactly the scope of situations in a given context to which we apply notions of equality. As a practical takeaway, then, our findings highlight that we should be much more explicit about equivalence in our teaching of mathematics across the spectrum. Currently, there is a danger that equivalence is being brushed under the rug, under the guise of equality, and there are not opportunities for broader conversation about how and why we attribute equivalence. Further, it may be useful to emphasize that our use of the equal sign expands in complexity and layering as we progress throughout the K-12 curriculum (and then into the undergraduate curriculum). Our findings may raise additional suggestions for certain populations; for example, for pre-service secondary teachers this is an opportunity to draw potentially meaningful connections between advanced mathematics (e.g., equivalence relations and equivalence classes) and the secondary mathematics they will soon be teaching (e.g., equivalence in the context of rational numbers, expressions, and equations).

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Exploring How Status Influences Relational and Participatory Equity in Inquiry-Oriented Small Group Interactions

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While inquiry approaches have the potential to open up more equitable spaces in the classroom, research has shown this is not guaranteed. In this study we focus on the role that status can play in creating (in)equities during small group work. In particular, we compare one low-status student's interaction with two high-status students in two small group episodes. We found that the two episodes differed in terms of participatory and relational equity and our results describe how status and certain patterns of interaction may account for this difference.

Keywords: Status, equity, small group interactions, inquiry-oriented instruction

There is some consensus that research on inquiry approaches in undergraduate mathematics classrooms needs to consider equity issues (Adiredja & Andrews-Larson, 2017; Laursen & Rasmussen, 2019; Melhuish et al., 2022). Scholars have suggested that group work – a common feature of inquiry instruction – may be a potential factor contributing to inequities in learning mathematics at the undergraduate level, particularly along gender lines (Brown, 2018; Ernest et al., 2019; Hicks et al., 2020; Johnson et al., 2020). Johnson et al. (2020) established that men achieved higher scores than women on a conceptual assessment after taking inquiry-oriented abstract algebra. The authors speculated that group work may open up opportunities for microaggressions and implicit bias to occur, which could negatively impact women's confidence in their own mathematical ability, resulting in lower performance. Ernest et al. (2019) characterized several student interactions during group work in an inquiry setting as overtly sexist and implicitly exclusionary towards women. This small body of research points to a need to better understand social processes during group work in inquiry classrooms that may be able to explain such inequitable findings.

Scholars at the K-12 level have argued that status and positioning during group work may amplify inequities in inquiry learning environments (Battey & McMichael, 2021; Shah & Lewis, 2019). Yet, how status influences (in)equitable interactions during group work has been underexplored in research at the undergraduate level. Therefore, this study set out to explore the role of status during group work in an inquiry setting from participatory equity and relational equity perspectives.

Theoretical Framework

In this study, equity is defined as situations in which all students can access resources they need for learning, and *in*equity as situations that *limit* access to resources needed for learning (Shah & Lewis, 2019). Resources needed for learning mathematics in small groups can include physical materials (e.g., notes, definitions, technology), skills to complete the task (e.g., problem-solving strategies, intuition behind a proof), and a safe learning environment (e.g., learn from mistakes, space to make sense). Two interrelated concepts further indicate how situations are more or less inequitable in small groups. First, participatory equity describes situations in which "opportunities to participate—and participation itself—are fairly distributed among all students involved in a learning interaction" (Shah & Lewis, 2019, p. 428). Second, relational equity (Boaler, 2008) describes situations in which the *quality* of interactions provides each student "access to opportunities to develop identities as capable learners" (p. 428). As human interactions naturally fluctuate, a situation will never be completely equitable or inequitable. For instance, in one situation a student may have more opportunities to participate and influence decision making at one moment, then limited access to such opportunities at another moment.

Unbalanced power dynamics due to differences in status characteristics may amplify inequities in small group work (Esmonde, 2009a; Shah & Lewis, 2019). A status characteristic is known as an "agreed-upon social ranking where everyone feels that it is better to have a high rank than a low rank" (Cohen et al., 1999, p. 84). In the classroom, academic status characteristics such as perceived ability are extremely powerful status markers (Cohen & Lotan, 2014), which can be impacted by diffuse status characteristics (Ridgeway, 2018) such as gender, race, ethnicity, socioeconomic class, etc. (Cohen et al., 1999). For example, it is a widely held belief that "Asian men are good at math" (Shah, 2017) which raises their perceived ability in math, thus increasing their status. Students with higher perceived status typically contribute more in class and small groups, gaining more opportunities to access resources needed for learning. Status hierarchies form quickly when people regularly interact with one another (Berger & Webster, 2018; Ridgeway, 2018) – a key characteristic of inquiry classrooms (Laursen & Rasmussen, 2019). While status hierarchies may become stable over time, it is possible to disrupt this process to promote more equal status positions between students (Cohen et al., 1999). We argue that exploring status positions while students work in small groups can help us understand the role status might play in shaping situations that are more or less (in)equitable.

To explore how status shapes group work interactions with respect to participatory and relational equity, we leveraged research using status and positioning theories to analyze small group interactions (e.g., Adams-Wiggins et al., 2020; Alexander et al., 2009; Chizhik, 2001; Esmonde, 2009b; Shah & Lewis, 2019). Table 1 provides the analytic constructs used in our analysis with low- and high-status indicators and examples from our data. The observable group activities in interaction come from status construction theory (Berger & Webster, 2018; Wagner & Berger, 1993), and the low- and high-status indicators were identified from several sources drawing on status theory (e.g., Chizhik, 1999, 2001; Cohen et al., 1999; O'Donnell, 1999).

Table 1. Status Constructs Used to Analyze Small Group Interactions

Observable Activities in Interaction	Low-Status Indicator	High-Status Indicator	Example from Small Group Data
<u>Opportunities to Contribute</u>	Grants opportunities to act (seeks help from others)	Receives opportunities to act (answers questions)	Justin: So now that we have this, how does this relate? (granting opportunities to act)
<u>Communicated Evaluations of Contributions</u>	Negative evaluations ("That is wrong")	Positive evaluations ("That is a good idea")	Alison: So, we have to define Q and W. They're just symmetries. Abigail: Yeah that's smart.
<u>Influence Over Group Decision Making</u>	Ideas rejected or ignored by the group (less ability	Ideas accepted by the group (influences the	Abigail asks if they should start on part two. This idea is not taken up as Justin

	to influence disagreement outcomes)	outcome of disagreements)	suggests they are not done with part one yet.

Methods

Data Collection and Episode Selection

Data comes from a project developing inquiry-oriented introduction to proof curricular materials designed to introduce students to various proof-related activities (e.g., proof construction and comprehension, defining, conjecturing, etc.) using real analysis and group theory concepts. We focused on a curriculum implementation in an Introduction to Proofs course that was taught remotely over Zoom (due to COVID-19) at a large public university. Table 2 contains the 14 participating students' ethnoracial background by gender (some students self-identified as more than one race/ethnicity). Students engaged in group work at least once per day with groups changing every day. The second author attended class each day, entering breakout rooms during group work as a silent observer to capture the students' activity. Screen recordings captured activity over Zoom as well as activity on Google Docs which became a virtual space for students to work together on tasks.

Table 2. Self-Reported Student Ethnoracial Backgrounds and Gender

Gender	White	East Asian	Southeast Asian	Hispanic/Latinx
Woman (*n*=6)	5	0	0	2
Non-Binary (*n*=1)	1	0	0	0
Man (*n*=7)	5	3	2	1
Total (*n*=14)	11	3	2	3

Two group work episodes were selected as the main data source for this study. A group work episode was defined as the time period in which students worked in groups on a specified task. The two episodes were selected based on the second author's extensive work studying the students' collective mathematical activity (see Alzaga Elizondo, 2022). First, in episode 1, Lee (Southeast Asian male) and Alison (white female) were working to prove that group isomorphisms preserve inverses. Lee took a clear lead in proposing and developing the mathematical ideas while Alison followed along, regularly providing evaluations. From the second author's classroom and group observations, Lee had higher perceived status among the classroom community. Contrastingly, in episode 2, Alison, Abigail (white female) and Justin (white male) worked to prove that a group element can appear at most once in a row of the group's Cayley Table. Alison contributed several key ideas to her group's proof such as the overall structure of a contradiction argument and several line-by-line justifications. While Alison was an active participant in both episodes, she appeared to contribute to the co-construction of mathematics in qualitatively different ways. Given the more collaborative activity in episode 2, we conjectured that empirically analyzing status in relation to participatory and relational equity while focusing on Alison would show episode 2 as being more equitable compared to episode 1.

Analysis

Students in remote environments communicate using different modalities when working together to solve a problem (Hoffman 2019). The first step in our analysis was to create multimodal transcripts that captured both verbal communication and any activity across different modes (i.e., on Google Doc and Zoom). To map any nonlinear forms of communication, we used descriptive text (i.e., "as she talks, she writes:" or "while everyone else works they edit the text") and/or time stamps. Together, the transcripts provided a complete image of the students' multimodal discourse, which we used with the video data for subsequent analysis.

We leveraged equity analytics (Reinholz & Shah, 2018) to quantify the number of talk turns, the duration of talk, and proportion of talk in each episode. Specifically, we used the multimodal transcripts to count the number of both verbal talk turns and non-verbal participation. Students often used the Google Doc to communicate and illustrate ideas (Alzaga Elizondo, 2022), thus we defined a non-verbal participation turn as any time a student added something to the shared document. Then, we used the constructs in Table 1 to analyze status relations. Specifically, at each talk turn in the multimodal transcripts, the first author attributed who granted/accepted opportunities to contribute, evaluated specific contributions, and whose ideas were taken up, ignored or rejected and how this impacted influence over group decision making. Additional analytic notes were recorded to describe how the students treated and interacted with one another as well as how status differences emerged. The second author went through all analytic notes and transcripts to add anything new or challenge the first author's interpretations. The two authors then met to discuss and compare analytic notes and reconcile any differences.

As we moved into an interpretive phase of analysis, we wrote analytic memos (Creswell & Poth, 2016) that centered on interactions between low-status and high-status pairings: Alison and Lee (Episode 1) and Alison and Justin (Episode 2). The purpose of creating the memos was to systematically find evidence and any counter-evidence for emerging interpretations of the interactions and compare across the two episodes. For example, in episode 2, we began to see an interaction pattern emerge between Alison and Justin that was different from interactions between Alison and Lee and seemed connected to Justin maintaining his high status position (which he implicitly established throughout the episode). The first author wrote memos from re-watching the video data with the transcripts and status analysis, and the second author served to challenge interpretations in the memos. Any disagreements were resolved through discussion.

Results

In the following, we first demonstrate how participation was distributed differently among the students in each episode, with Alison participating less overall in episode 2 compared to episode 1. Then, from a relational equity perspective, we found that the quality of interactions also differed between the episodes. We argue that the students' status relations provides insight into *how* the quality of interactions differed. Specifically, in episode 1, Alison was invited to co-construct a proof with Lee (a high status student) by evaluating lines in the proof, while in episode 2, Alison had to defend her mathematical contributions repeatedly to Justin, a higher status student who established himself as an expert on what formal proofs should look like throughout the episode.

Participatory Equity

Table 3 describes the verbal and non-verbal participation distributions across participants in each episode, which provides insight into how fairly distributed their participation was (Reinholz & Shah, 2018). Meaning, if participation is distributed equally across three people working in a group, each person should have about one-third of the whole amount of talk-time allotted to them. We caution that equitable or fair participation is not synonymous with equal participation (see Reinholz & Shah, 2018). With that in mind, in episode 1, Alison and Lee each participated about a proportionate 50% of the time, with Alison talking for slightly longer durations and Lee participating more non-verbally. In comparison, Alison participated less than her allotted one-third and for shorter lengths of time in episode 2, and Alison and Justin contributed about the same non-verbally (i.e., adding to the shared google doc). While Alison did not participate a fair share amount in episode 2, we would not claim that this alone implies inequity occurred. Therefore, to further compare the two episodes, we turn to our status analysis and interpretations from a relational equity perspective.

Table 3. Verbal and Nonverbal Participation Distributions.

Episode 1					
Participation	Talk Turns	Duration	Percentage	Proportion	Nonverbal
Alison	50	8.1 min	49.49%	0.99	9
Lee	49	6.6 min	50.51%	1.01	24
Total	**99**	**24 min**	**100%**		

Episode 2					
Participation	Talk Turns	Duration	Percentage	Proportion	Nonverbal
Alison	51	5.6 min	28.65%	0.86	7
Abigail	51	7.0 min	28.65%	0.86	11
Justin	76	9.7 min	42.70%	1.28	7
Total	**178**	**24.5 min**	**100%**		

Status and Relational Equity: Episode 1

The mathematical proof activity between Alison and Lee was characterized as less collaborative compared to the group's activity in episode 2 because Lee contributed most of the mathematical ideas used to complete the task and how Alison responded to those ideas had little impact on the mathematical direction of their work (Alzaga Elizondo, 2022). Our analysis suggests that Lee's higher status (relative to Alison) may have provided him with more opportunities to contribute a substantial amount of the mathematical work. However, throughout the episode, Lee regularly made moves to grant Alison opportunities to contribute to the task by inviting her to evaluate the work. We argue that the ways Lee invited Alison to check and evaluate lines of the proof and incorporated her suggestions provided evidence that Alison was trusted and viewed as a peer with expertise, which may have equalized their status differences to some degree. The following exchange exemplifies Alison and Lee's interactions:

Lee: From here can I just jump to like, therefore e_2- therefore phi(e_1) is the identity by definition or is that skipping some steps? (pause)

Alison: Hold on I'm thinking (pause) yeah I think that's good. (reads) "By definition of identity." So phi(e_1) must be the identity in-

Lee: Oh wait, but we're not saying for all H, we have to prove that that's all H.

Alison: the identity- What do you mean all h? Oh for the identity for all h? But we've already proved that there is only one identity. Isn't that in the definition of identity?

Lee: Yeah, but like this is showing the identity for all these, some elements of H until we show- I guess we can use onto right?

Alison: Yeah. (nods affirmatively) Yeah, we probably have to use onto.

Lee granted Alison an opportunity to contribute an evaluation by asking her if he could “just jump” to the conclusion “therefore phi(e_1) is the identity by definition” or if there were missing steps (which there were). Inviting Alison to check and evaluate their work in this way was a typical interaction pattern between the pair. Alison responded by asking for time to think, then accepted Lee’s invitation by saying she thinks the proof is “good” (positive evaluation). She continued reading the proof until Lee realized they did not prove that phi(e_1) is the identity “for all” elements in H. Alison offered the idea that the class had “already proved that there is only one identity” and asked if that’s in the definition of identity (granting Lee an opportunity to contribute). Lee accepted by first positively evaluating Alison’s contribution that there is only one identity (“yeah”), then suggesting that their proof is showing the identity property for “all these, some elements of H” (meaning all the elements in the image of phi, which may only be a subset of H). Lee granted Alison another opportunity to contribute by asking “I guess we can use onto right?” which she accepted and positively evaluated (“yeah”), further agreeing by reiterating they “probably have to use onto.”

While Lee (a high status student) contributed more substantial mathematical ideas than Alison (a lower status student), the way he granted her opportunities to evaluate their work before moving forward may have balanced out the potential status differences between them. From a relational equity standpoint, their interactions showed mutual respect for each other’s expertise and potentially provided opportunities to access a positive mathematical identity (Esmonde, 2009b).

Status and Relational Equity: Episode 2

In a prior study, this group’s mathematical proof activity was characterized as highly collaborative because all students contributed to the mathematical direction of their work (Alzaga Elizondo, 2022). However, their interactions viewed through a status and relational equity lens paints a slightly different picture. Specifically, one interaction pattern between Alison and Justin that emerged was Alison contributing a (valid) direction to make progress on their proof which Justin would positively evaluate only to then immediately question whether her idea was formal enough. The following exchange exemplifies this interaction pattern. Prior to the exchange, Abigail was defending her claim: by the nature of the (Cayley) tables, each entry in the first row and first column are unique (referring to the row and column headers). Justin and Abigail had a back and forth about whether she thought the identity symmetry was different than three rotation symmetries (for an equilateral triangle). Alison entered the conversation by contributing “it’s different actions” and suggesting an idea for their proof.

Alison: Yeah it's different actions but I think for the sake of our proof we need to somehow say we're limited to you know (pause) just four, well in this case we don't have- however many symmetries there are. (pause) I might be articulating that wrong. (pause)

Justin: I think the original route we’re going down is right, where we have this, I think this is definitely the right way. I’m just trying to make sure that we have the proper way saying that proper, like-

Alison: Yeah, I agree, we have to find the way to set it up before we can just say. (pause)

Justin: Well, (cross talk with Abigail) I just want to make sure there’s no holes, I guess-

Abigail: (cross talking) Q is identical to W then

Alison: (responding to Justin) I understand that.

Alison stated they need to argue there are a limited number of symmetries to choose from “for the sake of our proof.” She then made a self-doubting comment (“I might be articulating that wrong”), which we interpreted as a low-status indicator (see Adams-Wiggins et al., 2020). Justin positively evaluated Alison's contribution and her original idea to prove by contradiction ("I think the route we're going down is right") and countered by stating that he wanted to make sure it was formal enough (the "proper way"). Alison agreed with him that they "have to find the way to set it up." After a brief pause, Justin reiterated that he had to "make sure there's no holes" implicitly suggesting that Alison's logic may be faulty. Alison reasserted that she understood his concern ("I understand that").

We interpreted this repeated interaction pattern as potentially increasing rather than balancing status differences between Alison and Justin. Alison (a lower status student) contributed more substantial mathematical ideas than Justin (a high status student), which could have equalized their status. While her contributions were regularly met with initial positive evaluations, they were often immediately followed by repeated questioning from Justin who viewed himself as the expert on the "proper way" of proving (meaning, what is acceptable as a formal proof by the mathematical community). From a relational equity perspective, while their interactions were mostly friendly, Alison was treated as a peer with less expertise in formal proof compared to Justin, which could have potentially limited the opportunity to access a positive mathematical identity.

Discussion

Our results show how small group interactions that were deemed highly collaborative in terms of mathematical proving activity (episode 2) were potentially more *in*equitable compared to small group interactions that were characterized as less collaborative (episode 1). First, participation was distributed differently among the students in each episode, with the focal student, Alison, participating less overall in episode 2 compared to episode 1. Then, from a status and relational equity viewpoint, the quality of interactions also differed between the episodes. In episode 1, while Alison (a lower status student) contributed less substantial mathematical ideas than Lee (higher status), an interaction pattern between them may have balanced their status differences, promoting mutual respect for each other's expertise (attenuating inequity; Shah & Lewis, 2019). Contrastingly, in episode 2, Alison contributed more substantial mathematical ideas than Justin (higher status), yet an interaction pattern between them potentially increased their status differences by maintaining Justin's high status as an expert in formal proofs while implicitly lowering Alison's status (amplifying inequity).

We see two primary implications from this study. First, our work shows that studying small group collaboration in inquiry classrooms solely from a mathematical perspective does not paint a complete picture in terms of how (in)equities might occur. Second, if differences in status are not intentionally disrupted (particularly regarding gender and perceived mathematical ability), women's opportunities to access positive mathematical identities may be limited, which could lead to negative outcomes (e.g., Ernest et al., 2019; Johnson et al., 2020). Therefore, inquiry instructors may need specific training on how to enact group work in ways that mitigate such status differences (Melhuish et al., 2022). A direction for future work could be to adapt research at the K-12 level that has developed and implemented group work principles to reduce status hierarchies (i.e., Complex Instruction, Cohen et al., 1999) for undergraduate inquiry settings.

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Conceptions of Active Learning, Meaningful Applications, and Academic Success Skills Held by Undergraduate Mathematics Instructors Participating in a Statewide Faculty Development Project

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The Mathematical Inquiry Project engages mathematics faculty from all Oklahoma's public institutions of higher education in collaborative learning and resource development guided by three components of learning mathematics through inquiry: active learning, meaningful applications, and academic success skills. We report our analysis of participants' interpretations of these components and the social and cognitive mechanisms they entail. Our results (1) provide a descriptive analysis of beliefs and priorities about learning mathematics through inquiry held by post-secondary mathematics faculty and (2) contribute to research and outreach on faculty change using a model incorporating implicit and explicit learning theories, within which the professional activity and collaboration develop a community that supports productive aspects of learning mathematics.

Keywords: inquiry-based learning, entry-level math, professional development

The Mathematical Inquiry Project (MIP) and its Definition of Inquiry-Based Learning

The MIP is a collaboration among mathematics faculty from all 27 of Oklahoma's public higher education institutions. The project aims to improve learning in entry-level mathematics courses by engaging faculty in collaboration regarding inquiry-based learning, conceptualized to entail three components: *active learning*, *meaningful applications*, and *academic success skills*. The MIP defines these components through the lens of constructivist epistemology. The definitions were developed to guide project implementation and encourage participants to adopt a critical stance toward the nature of students' conceptions in their design of curricular materials that seek to engage students in the precise mental actions required to construct targeted mathematical meanings. The MIP defines the three components as follows:

> *Students engage in active learning* when they work to solve a problem whose resolution requires them to select, perform, and evaluate actions whose structures are equivalent to the structures of the concepts to be learned.
>
> *Applications are meaningfully incorporated* in a mathematics class to the extent that they support students in identifying mathematical relationships, making and justifying claims, and generalizing across contexts to extract common mathematical structure.
>
> *Academic success skills* foster students' construction of their identity as learners in ways that enable productive engagement in their education and the associated academic community.

In the summers of 2019-2021, the MIP leadership team (the authors) led five different week-long workshops during which faculty discussed key conceptual threads of entry-level mathematics courses and academic success skills in those courses. The leadership team aimed to support the participants' thinking about the social and cognitive mechanisms involved in developing productive ways of understanding foundational concepts in entry-level undergraduate mathematics, and how to design curricular materials that promote students' construction of those ways of understanding. For example, participants discussed the ideas of *conceptual analysis*

(Thompson, 2008) and *hypothetical learning trajectories* (Simon & Tzur, 2012), and critically examined curricular materials developed with these in mind. Participants also unpacked the MIP's definition of inquiry and its operationalizion in the design of sample curriculum materials. In later stages of the MIP, currently ongoing, participants are creating their own course materials and will share them via workshops and peer mentoring. The goal of engaging participants in curriculum development and dissemination is to build capacity through broad professional participation and leadership in an emerging community that values and supports reflective practice in learning mathematics through inquiry.

Active learning, meaningful applications, and academic success skills are of interest to the research in undergraduate mathematics education (RUME) community. In a review of the RUME 2022 proceedings (Karunakaran & Higgins, 2022), 12 of 82 contributed reports[1] contained the phrase 'active learning' in the abstract, keywords, or body. These reports tended to define or discuss active learning terms of pedagogical strategies (e.g., "think-pair-share," Cervello Rogers et al., 2022), its characteristics (e.g., student engagement and collaboration, Johnson et al., 2022), or affordances (e.g., equity; Rios, 2022). Two papers framed active learning with reference to cognitive theories (e.g., Kerrigan et al., 2022; Rogers et al., 2022). Some definitions of active learning in the proceedings mention engaging students in meaningful mathematical tasks or projects (e.g., Broley et al., 2022; Chowdhury et al., 2022; Fukawa-Connelly et al., 2022; Johnson et al., 2022). Moreover, the proceedings also indicate interest in academic success skills such as students' mathematical identity (e.g., Guglielmo et al., 2022; Voigt et al., 2022) and problem-solving (e.g., Corey, Weinberg, & Tallman, 2022). We generally observe, however, that much extant literature focuses on students' activity when engaged in such tasks and less on instructors' views of what meaningful mathematics might entail and how it might be productively realized in the classroom. We seek to address this gap.

Research on faculty change highlights that supporting strategies should seek to alter individual's beliefs as opposed to mandating "effective" curricular resources or enacting top-down policy initiatives to impact teaching quality (Henderson et al., 2011). The literature establishes a need to characterize faculty's conceptions of mathematical learning through inquiry and describe how particular professional development experiences contributed to their evolution (Williams et al., 2022). The MIP is one such professional development project. This paper contributes to research on faculty change by constructing models of participants' explicit and implicit learning theories in relation to their conceptions of the three components of learning through mathematical inquiry. We focus on the following research question: Following engagement in a professional development initiative centered on the components of mathematical inquiry detailed above, what are participants' images of (a) active learning, (b) meaningful applications, and (c) academic success skills?

Theoretical Perspective

The MIP definitions of active learning and meaningful applications are based on Piaget's genetic epistemology. We find this a useful way to characterize mathematical inquiry because (1) it focuses on an individual's cognition, which provides a means for MIP participants to conceptualize and think carefully about students' thinking and learning, and (2) the field, as noted in the section above, has typically not dealt with issues of active learning and inquiry from an explicit epistemological perspective. In Piaget's view, conceptual development is a process of constructing and modifying cognitive structures to enable individuals to establish a fit, or

[1] These numbers exclude our 2022 conference proceedings regarding the MIP.

equilibrium, between their conceptual model of reality and the reality they experience (von Glasersfeld, 1995). Piaget regarded cognitive development as a process that inclines toward a balance, or equilibrium, via *assimilation* and *accommodation*. Assimilation is the process whereby a subject incorporates experiences into existing cognitive structures, and accommodation entails the modification of an individual's cognitive structures to enable their assimilation of novel experiences. Assimilation and accommodation, and thus equilibration, rely heavily on the notion of *abstraction*, of which Piaget distinguished four varieties (*empirical*, *pseudo-empirical*, *reflecting*, and *reflected*). Piaget (2001) explained that higher forms of knowledge (such as logico-mathematical structures) derive from abstractions of the subject's actions and the results of applying them in specific contexts. The nature of mathematics knowledge structures, or *operative schemes*, are organizations of internalized (mental) actions constructed through the process of pseudo-empirical or reflecting abstraction. The *action of the subject* is thus the basis from which mathematical knowledge structures are constructed.

Commensurate with this epistemological framing, the MIP definitions of active learning and meaningful applications are established on the premise that supporting students' construction of mathematical knowledge is a process of engaging them in the specific *actions*—and engendering abstraction of either the products of these actions (*pseudo-empirical abstraction*) or the actions themselves (*reflecting abstraction*)—that reflect the structure of the internalized actions organized within the schemes that an instructor expects students to construct. Notice that the MIP definitions of active learning and meaningful applications specify the conceptual mechanisms engendered. The relation of our definitions to an explicit learning *process* differs from characterizations of learning mathematics through inquiry that specify the environmental, pedagogical, or social conditions that must be established for students to be actively learning or for a teacher to incorporate meaningful applications in their instruction.

The MIP definition of academic success skills is similarly informed by constructivist epistemology, while being supplemented by theories of emotion (Ortony, Clore, and Collins, 1988), identity (Blumer, 1986), and goal structures (Middleton, Jansen, & Goldin, 2017). See Tallman and Uscanga (2020) for a synthesis of these theories and their relation to students' affective experiences.

The MIP afforded opportunities for faculty members to engage with the definitions of the three components. Hence faculty experienced opportunities to engage in reflecting and reflected abstractions (Piaget, 2001) regarding the characteristics and mechanisms entailed in active learning, meaningful applications, and academic success skills. Our study was designed to investigate their images of these three components following this professional development.

Methods

We conducted semi-structured interviews on Zoom with 15 MIP participants in spring 2020. Interviewees had participated in at least one workshop. Interview questions included:

1. Please describe your image of active learning in entry-level college math courses.
   a. Why is this important for entry-level math courses?
   b. Can you describe a specific example of active learning in an entry-level math course, yours, or someone else's? What made this example effective? What could have been better?
   c. Has your participation in the MIP activities changed your thinking about active learning? How?
2. Here is the MIP's definition of active learning. [Participants were shown the definition].
   a. Are there parts of this that you think are important but haven't discussed yet?

b. Do you particularly agree or disagree with emphasizing any aspect of the MIP definition for improving instruction in entry-level college mathematics?

The questions were repeated twice more with ‘active learning’ replaced by ‘meaningful applications’ and ‘academic success skills.’ The interviews were audio recorded and transcribed for analysis.

We employed the constant comparative method (Strauss & Corbin, 1994) to identify themes in the data. For each of the three components of inquiry, one author read all the transcripts, highlighting phrases that characterized participants’ images of that component. When a new phrase was added to the list, the author reread previous transcripts seeking instances of it. The author then grouped similar items into themes and described each theme using the list of phrases. That author applied the themes to re-code all the data, adjusting the themes’ descriptions until they adequately described the span of participants’ images. To ensure consistency amongst coders, a different author independently coded the data, employing the first coder’s themes and seeking excerpts that were not described by the themes. The two authors discussed differences until they agreed on a final set of themes. The final themes for each component are described in the results. The themes for active learning refine those presented in Ireland et al. (2022). Themes for a component are not mutually exclusive; for instance, ‘motivating students with real-world examples’ was coded as both ‘real world examples’ and ‘affective affordance.’

Results

We present the answers to our research question in terms of themes we identified in participants’ interpretations of (a) active learning (b) meaningful applications and (c) academic success skills. Participants focused primarily on characteristics, affordances, and requirements of the three components with some attention to learning mechanisms. While each of the themes we explore below emerged in our analysis from a multitude of supporting transcript excerpts, here we illustrate a proper subset of the themes we identified and support them with one or two illustrative, prototypical excerpts.

Active Learning

Student engagement. All participants described student engagement as part of active learning. Jude, for example, defined active learning as “involvement of [students’] brains” in contrast to instruction in which only the “instructor’s brain that is running.” Generally, participants’ comments associating active learning to student engagement suggested that they viewed engagement (and related ideas of interest and involvement) as the mechanism by which mathematics learning occurs.

Format in which students engage with the content. All participants discussed active learning in terms of the format in which students would engage with the content, such as small-group work, projects, class discussions, guided or scaffolded problem sets, or using manipulatives or calculators. For example, Jack said, “I incorporate a lot of collaborative project learning so that they can see the math in action… the less I talk… the better.”

Participants’ descriptions within this theme focus on characteristics of active learning, that is, what active learning might “look like” to a classroom observer. This contrasts with our view of the MIP’s three components of inquiry, which all involve descriptions of cognitive activity that is not observable in the same way. From our theoretical perspective, active learning might occur when students work in small groups, but working in small groups is not a sufficient condition for students’ construction of a desired mathematical meaning.

Nature of the mathematical tasks. Some participants described characteristics of the mathematical tasks they present to students, such as novel problems (not exercises) that would support understanding over simple memorization. Participants believed active learning occurred when problems required students to struggle productively and problem-solve. Some participants believed real-world examples were useful for engaging students in active learning. The attention to these characteristics is notable because it indicates attention to a key consideration for curriculum design: what students are engaged in and how they are engaged in it is consequential for the mathematical meanings they construct.

Participants expressed their belief that productive struggle is a mechanism of learning. For example, Quinn described,

> I have come to see [active learning] as asking students to truly problem-solve in a way that, that they may or may not have seen before…. Instead of having me say "here are the tools that we're going to use, here is exactly how we're going to use them and exactly the order that I want you to use them… it's set up more like "okay here are the tools, now here's a problem and maybe here are some guiding questions."

Quinn was unique among participants in that she not only described productive struggle, but hinted at the idea of students needing to engage in particular mental activity (selecting "tools").

Reveals student thinking. Two participants valued active learning for revealing student thinking, allowing the instructor to modify instruction and address individual or critical issues. Reagan[2] said:

> active learning [makes it] easier for the instructor to know… where the students struggle… the instructor can… feed additional example or deep explain certain concept to help them better retain the information… my hope is through repetition they get the structure.

We infer that Reagan thought repetition was a learning mechanism that helped students "retain" information and "get" general structure.

Affective requirements and benefits of active learning. Participants' images of active learning included what they saw as affective requirements for it and affective benefits of it. Per the former, participants described that active learning would only be successful if students were interested in the problems, motivated to work on them, possessed perseverance, and if the classroom community felt safe. Participants felt students' having growth mindsets was important, and that a positive classroom community could help minimize students' math anxiety. For example, Finn said, "you have to have an environment where everyone's safe to throw out answers and get responses back." Participants also believed active learning could help students develop all these characteristics, noting that active learning sometimes served to increase students' interest, motivation, and perseverance. Eden, for instance, described active learning as "making them do the work, it's [building] self-efficacy."

Meaningful Applications

Format in which students engage with the content. Like their definition of academic success skills, participants described their images of meaningful applications as problems embedded in formats like small-group work or class discussions. One difference is that participants felt presenting a real-world example in a lecture constituted a meaningful application.

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[2] English is not Reagan's first language.

Nature of the mathematical tasks. While the MIP characterization of meaningful applications hinges on applications that are mathematically meaningful, participants seemed to think of a meaningful application as one that was relevant to students' daily lives. Some participants mentioned that an application would be meaningful if it were relevant to other mathematics. For instance, Quinn said

> I also think of meaningful applications as in what can, what can they do with a definition within mathematics…And if it, and if it helps them with their problem-solving skill then it is extremely meaningful.

In addition to real-world examples, participants described that meaningful tasks were usually formed when students were given open-ended questions or novel problems. One participant noted that a task that supported students in generalizing would be a meaningful task, and several participants said tasks that connected to other mathematics were meaningful applications.

Affective requirements and benefits of meaningful applications. Participants' images of meaningful applications included the affective requirement of managing students' fear and dislike of applications. Affective benefits included piquing students' interest (which in turn supported student engagement and motivation) and increasing their mathematics self-efficacy.

Academic Success Skills

Instructor agency. Several participants described academic success skills in terms of their own agency as instructors in supporting students' development of academic success skills. This theme is notable because it indicates that some participants saw the intended connection between the three definitions: that active engagement with meaningful applications will build students' academic success skills. Participants mentioned that it was part of their duty as instructors to help students develop academic success skills (e.g., Quinn gave an example of developing time-management multiple times in the semester; Reagan described training students to include detail in their solutions). Others described how they used content to support students in developing affective academic success skills. For example, Ellison said,

> we want to be presenting students with problems that are at the right level… an interesting problem to the student but they still feel like they've got what it takes to be able to solve it, then it's going to keep them engaged.

Part of what participants saw as their agency as instructors was the necessity of their developing classroom community. Anna saw this as important because it enabled students to engage and participate productively:

> I can provide them opportunities to fail until they feel safe… [know] that they're not going to be laughed at… it's developing trust.

Characteristics of students. Participants' images of academic success skills included many skills they felt students needed to possess or develop. Participants' images of academic success skills included perseverance, knowing how to learn, knowing how to manage math anxiety, having an identity as a learner, having basic math knowledge, and having behavioral skills. For example, Ellison's response discusses many of these characteristics:

> If you have [academic success skills] it means that you have some knowledge about how to tackle different problems that you're given. You also have some positive motivation happening that's contributing to your desire and ability to be able to complete problems… Has an inherent sense that they can do it. And, and they also need to have the, the content knowledge skills to be able to, to solve a problem as well.

Most participants also mentioned that academic success skills also entailed behavioral skills such as time management, doing homework, taking notes, attending class, and doing homework.

Discussion

Participants' characterizations of the components of inquiry primarily emphasized environmental conditions or pedagogical strategies that would need to be in place for students to be engaged in active learning. Others have noted these sorts of emphases as common in faculty members' conceptions of active learning (e.g., Williams et al., 2022). The MIP participants attended somewhat to mechanisms of learning, seeming to believe that student engagement, repetition, generalizing, and productive struggle were learning mechanisms. These appeared to be rational, stable beliefs that the participants had developed over the course of their teaching careers. That faculty members hold these beliefs is not inherently problematic, but what students are engaged in (or repeating, or struggling with) is consequential for the mathematical meanings they develop. The participants' images of meaningful applications centered primarily around real-life relevance over mathematical meaning. Taking the images of active learning and meaningful applications together, we see a need for perturbing participants' images of these components in ways which go beyond students engaging with problems with relevant contexts. How to perturb participants such that they engage in the reflective abstraction that their current images of the components may be inadequate for supporting the development of mathematical meaning is an area for future research. We note work with provided definitions may perturb participants somewhat, and our evidence for this is Gemma's inclusion of generalizing as a learning mechanism because generalizing was specifically mentioned in the definitions. More broadly, we see participants' mention of engagement, repetition productive struggle as learning mechanisms as evidence of some attention to the idea of what a learning mechanism is, and both offer inroads to discussing how what students engage or struggle with as consequential. Thinking of the three pillars in terms of characteristics, rather than mechanisms, can be problematic because focusing on characteristics makes it harder to design and implement resources to support effective active learning. Without an understanding of why those characteristics may produce the conditions for learning and what concepts may be supported, an instructor cannot make instructional decisions to optimize the potential for students to construct desired concepts and reasoning patterns.

Our results broadly reinforce the literature on instructional change, which emphasizes that instructors' initial conceptions and beliefs about teaching are resistant to change and require continual, ongoing professional development. We propose that the themes we have identified nevertheless paint a clear image of how undergraduate mathematics instructors conceive of active learning, how they attempt to realize it in their classrooms, and what they value about it. These findings therefore provide important information for other researchers and change agents. More specifically, our results build on this prior research by describing the conceptions of undergraduate instructors across several institutions and after professional development has taken place.

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How a Graduate Mathematics for Teachers Course
"Helped Me See Things from a Different Perspective"

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Secondary teachers and college instructors are required to take several graduate-level mathematics courses to advance in their careers. However, there is a common disconnect between graduate mathematics courses and the teaching of secondary mathematics. We developed a graduate-level mathematics for teachers course that was designed to help teachers connect abstract algebra, real analysis, and statistics to the content and teaching of secondary/ early college mathematics. In this study, we examined changes in graduate students' confidence and perceptions of the relevance of advanced mathematics to teaching secondary and college mathematics. Students attributed changes in their perceptions of the connections between advanced and secondary mathematics to the course's implementation of pedagogical scenario tasks. We provide implications for the mathematical preparation of teachers.

Keywords: Graduate Courses, Mathematic Knowledge for Teaching, Perceptions of Relevance of Advanced Mathematics, Confidence

Mathematics teachers enroll in graduate-level mathematics courses for professional growth, graduate degrees, teaching certifications, and mobilities in a future career (e.g., academic specialist, college instructor). They are required to complete a significant number of credit hours, e.g., 18-credit hours to teach dual-enrollment courses. However, teachers are disconnected from the studies of advanced mathematics in graduate courses not only because they took their advanced mathematics courses several years ago, but also because the content and pedagogy of such graduate-level mathematics are not connected to their profession in teaching (Zazkis & Leikin, 2010). These graduate mathematics courses have acted as gatekeepers to professionals in teaching careers, inhibiting them from pursuing graduate studies in mathematics or dual-enrollment teaching certificates. The discontinuity between content courses and classroom teaching has been a long-lasting challenge in teacher preparation. Researchers have been addressing this challenge in their innovative approaches to improve curriculum and instruction for undergraduate-level mathematics courses and in their collaborative efforts to connect content and methods courses (e.g., Wasserman & McGuffey, 2021; Weber et. al., 2020). Graduate-level courses, however, have not been highlighted in the literature as much as undergraduate courses.

One way to address those discontinuities in graduate mathematics courses for teachers is to implement innovative course design and instructional practices that employ and empower teachers' mathematical and pedagogical practices in their learning. This will also provide a positive standpoint of *mathematics for teaching* within the discipline of mathematics by tackling the existing deficit view—inferior to and less rigorous or formal than real mathematics—and by embracing an alternative view—advanced mathematics for the teaching of mathematics.

In Spring 2022, we developed a course for students in the Master's program in mathematics with a teaching concentration at a large public university. The course design emphasized fundamental knowledge and skills in advanced mathematics and their applications to the teaching of secondary school mathematics and lower-level college mathematics including college algebra, calculus, and introductory statistics. The courses adopted student-centered activities in class to facilitate the mathematical development of secondary teachers and their reflection on the pedagogical implications for their teaching practices. In this paper, we present how the participating preservice and in-service teachers in this course perceived (a) changes in their views on connections between advanced mathematics and secondary/college mathematics, (b) changes in their confidence in advanced mathematics and teaching secondary/college mathematics, and (c) effects of the course design on those changes in their views and confidence.

Literature Review and Theoretical Background

Knowledge of advanced mathematics is perceived as an essential component of teachers' content knowledge (e.g., Conference Board of the Mathematical Sciences, 2012), especially when it connects to teachers' understanding of school mathematics. Constructs related to teachers' knowledge of advanced mathematics and its connections to school mathematics can ubiquitously be found in frameworks of mathematical knowledge for teaching (e.g., Ball et al., 2008; Ball & Bass, 2009; McCrory et al., 2012; Wasserman, 2018; Zazkis & Mamolo, 2011). Developing knowledge of advanced mathematics can lead teachers to understand relations among mathematical content and structures, hone mathematical practices, and hold core mathematical values, which have been shown to support teachers in various teaching practices (e.g., Baldinger, 2018; Serbin, 2021; Zazkis & Kontorovich, 2016; Zazkis & Marmur, 2018; Zbiek & Heid, 2018). Taking undergraduate and graduate-level mathematics courses thus has the potential for fostering the development of prospective and in-service teachers' mathematical knowledge, practices, and values. However, these potential outcomes might not come to fruition if the prospective or in-service teachers are not explicitly given opportunities to make connections among the content and practices used in both advanced and secondary mathematics.

To address the commonly documented disconnect between teachers' studies of advanced mathematics and teachers' teaching practices, teacher educators have implemented mathematics for teachers courses with innovative course designs and instructional practices that leverage teachers' mathematical and pedagogical practices in their learning (Goar & Lai, 2021; Wasserman et al., 2019). Researchers have argued for alternative curricula and instruction to explicitly address the mathematical pedagogical knowledge and practices for teachers. The design of innovative mathematics for teachers courses should include an emphasis on content connections and pedagogical mathematical practice connections. Teachers must be given opportunities to make connections among advanced and secondary mathematical content that help them better understand the content they teach. This is the premise of Wasserman's (2018) theory of Knowledge of Nonlocal Mathematics for Teaching. Wasserman (2018) claimed, "knowledge of nonlocal [advanced] mathematics becomes potentially productive for teaching at the moment that such knowledge alters teachers' perceptions of or ontological understandings about the local content they teach" (p. 121). Thus, advanced mathematics courses for teachers should be designed to guide teachers to make connections that change their understanding of the content they teach and thus have the potential to be useful in their teaching. These courses should also be designed to provide opportunities for teachers to develop *Pedagogical Mathematical Practices* (PMPs), which Wasserman and McGuffey (2021) defined by drawing on the extant literature on mathematical practices (e.g., Rasmussen et. al., 2005) and pedagogical practices (NCTM, 2014). PMPs lie in an intersection of two domains of mathematical practices and pedagogical practices and are the kinds of practices that are common across both mathematicians and mathematics teachers in terms of their actions, habits, and lines of reasoning. Examples of PMPs include attending to assumptions and mathematical constraints or limitations, using special cases to test and illustrate mathematical ideas, exposing logic as underpinning mathematical interpretation, using simpler objects to study more complex objects, and not giving rules without accompanying mathematical explanation. The conceptual framework informing the design of the course under investigation in the current study draws on Wasserman's (2018) theory of Knowledge of Nonlocal Mathematics for Teaching and Wasserman and McGuffey's (2021) framework of PMPs. These theories provide the rationale for the graduate-level Mathematics for Teachers course design, which is described further below.

Course Design

This graduate-level Mathematics for Teachers course aimed to provide mathematics teachers and college instructors with an opportunity to study key topics in advanced mathematics and to apply the topics to their mathematical knowledge and practice for teaching secondary school mathematics and lower-level college mathematics, such as college algebra, pre-calculus, calculus, and introductory statistics. This course consisted of three modules based on subject areas in mathematics at the graduate level: Abstract Algebra, Analysis, and Probability/ Statistics. Each module included a survey of definitions and basic properties of fundamental mathematical concepts in the corresponding area. The classroom instruction emphasized the role of in-class activities and projects to build students' conceptual understanding and develop their advanced mathematical reasoning and practices.

The Abstract Algebra module was designed using Wasserman's (2018) theory of Knowledge of Nonlocal Mathematics for Teaching, along with the design heuristics of guided reinvention from Realistic Mathematics Education (Gravemeijer, 1998) and pedagogical mathematical practices (Wasserman & McGuffey, 2021). The module was designed to guide students to develop an intuitive understanding of the structure of groups, rings, integral domains, fields, unique factorization domains, and isomorphisms. It implemented Larsen's (2013) task sequence for the reinvention of groups and group isomorphism, as well as an adaptation of Cook's (2012; 2015) task sequence for reinventing rings. The curriculum then guided students to connect their understanding of the Abstract Algebra content to content covered in secondary and early college mathematics. The course curriculum led students to apply their understanding of the connections between the advanced and secondary content through the implementation of pedagogical scenario tasks. These tasks typically gave samples of hypothetical student work and asked the graduate students to interpret the hypothetical student's understanding and decide how to respond if they were that student's teacher. The pedagogical scenario tasks also were presented in the form of Lesson Play scripting tasks (Zazkis et al., 2013), in which the graduate students were given a script of classroom dialogue between a teacher and students and were prompted to finish a script of the conversation. These tasks were designed to elicit evidence of students' use of mathematical knowledge and PMPs (Wasserman & McGuffey, 2021).

The Analysis module of this course adopted an instructional model, ULTRA, for teaching in real analysis courses (Wasserman et al., 2017) that emphasizes explicit connections between secondary and advanced mathematics by introducing pedagogical situations in secondary classrooms, stepping up to studies on relevant topics in real analysis, and stepping down to the same situation. This module used selected chapters of *Understanding Analysis and its Connections to Secondary Mathematics Teaching* (Wasserman et al., 2022) written for implementing the ULTRA model. Students participated in group discussions on classroom scenarios given in the textbook and collectively worked on mathematical tasks to review relevant real analysis concepts including epsilon arguments, limits, continuity, and differentiability.

Methods

In this study, we conducted a pre-survey, post-surveys, and semi-structured interviews designed to examine changes in their confidence in high school and advanced mathematics and their perceived content connections and relevance of advanced mathematics for teaching secondary or early college mathematics. Six graduate students enrolled in the course participated in this study and completed a pre-survey at the beginning of the semester and two post-surveys, one in the middle of the semester about the Abstract Algebra and Analysis modules, and the other after the semester about the Probability & Statistics module and the overall course experience. The surveys include 7-point Likert scale items about their confidence and views on connections between secondary and advanced level mathematics. We compared the average scores of students' responses to each of the items to determine changes in their confidence and perceptions about advanced mathematics for teaching. The participants also participated in the interview about their course experiences and changes in their confidence and perceptions after completing the second post-survey. The interviews were audio recorded in Zoom and transcribed for analysis. We used an open coding process (Miles et al., 2013) to examine the participants' perceived changes in their confidence and perceptions of connections between learning advanced mathematics and teaching secondary and early college mathematics. One author independently generated codes in the first round, and the other author confirmed the codes. Then both authors discussed codes, created axial codes, and identified themes explaining how the course experiences affected the ways in which students' confidence and perceptions changed.

Results

Table 1. Mean scores on pre-survey and post-survey items.

Survey Item	Pre-survey Mean Score (N=5)	Post-survey Mean Score (N=6)
1. I am confident with explaining concepts in high school mathematics.	5.80	6.00
2. I am confident with explaining concepts in undergraduate level mathematics.	4.40	5.67
3. I am confident with explaining concepts in graduate level mathematics.	2.20	4.50
4. Understanding advanced mathematics will help me better understand secondary-school level mathematics.	6.60	6.83
5. Understanding advanced mathematics will help me better perform secondary-school level mathematics teaching.	6.60	6.67

Changes in Confidence with Explaining Mathematical Concepts at Different Levels

The comparison between students' responses to the pre and post-surveys indicated that their confidence in explaining mathematical concepts at the high school, undergraduate, and graduate levels increased overall after taking this course. The average scores on the survey items related to their confidence increased from 5.80 to 6.00 at the high school level, 4.40 to 5.67 at the undergraduate level, and 2.20 to 4.50 at the graduate level (See Table 1). Their confidence increased more at the higher levels. The students reported lower confidence in more advanced levels in the beginning but made greater increases at the end (e.g., +0.20 at the graduate level, +1.27 at the undergraduate level, +2.30 at the high school level). Their interview responses indicated that their course experience helped increase their confidence in advanced mathematics and prepared them to take further advanced mathematics courses in their graduate program. Student E said, "I definitely feel more competent in doing certain types of mathematics". Student L said, "I think this class will be perfect for that, for future courses, and I do feel more prepared to go into those [further graduate] classes." However, a student reported that their confidence in advanced mathematics had not changed a lot because the course emphasized less on writing mathematical proofs and a deep investigation of advanced topics in this course than what they had expected from a traditional graduate-level mathematics course. Student R said the course helped increase his confidence not as much as he expected because of less emphasis on proof. He said, "I want to say [my confidence increased] a little bit, not that much since it wasn't really a class where I was taught how to do proofs again because I forgot it, it's been a while. And then I was just expected to do the proofs, not like that. So just a little bit, but not as much as I wanted to." Overall, this course helped students increase their confidence in all levels of mathematics, especially for higher levels in all three subjects in analysis, algebra, and probability & Statistics.

Changes in Perceived Content Connections and Relevance of Advanced Mathematics for Teaching Secondary or Early College Mathematics

This graduate-level Mathematics for Teachers course helped graduate students recognize content connections among advanced and secondary mathematics, and it helped change their perceptions regarding the relevance of advanced mathematics for the teaching of secondary mathematics. *Some students reported that they did not have an opportunity to reflect on connections in other graduate mathematics courses.* For example, Student S explained, "I don't think I've thought before about how they related… now I think it would be very helpful in teaching secondary math." This course gave students an opportunity to connect advanced and secondary mathematics, which they said they did not have in other graduate math courses.

Some students reported that after taking this course, they could *better understand advanced mathematical content and secondary mathematical content.* In the survey data, we identified a slight increase in the mean score of the survey item 4 (see Table 1) from 6.60 to 6.83 (on a 7-point scale). The students' responses to open-ended survey items and their interviews gave further insight into how they perceived a change in their understanding of mathematics. For example, one student wrote on a post-survey, "I had always had a bit of trouble understanding groups, rings, and fields, but seeing the connection of this topic with high school content and relating it to higher level thinking made me understand it more." Thus, they acknowledged that relating the advanced mathematics with secondary mathematics during the course helped them better understand the advanced mathematical content. Several students reported a change in their understanding of secondary or early college mathematics. For example, Student L explained,

> "Before, I used to see advanced mathematics as something different from high school…advanced mathematics is really connected rather than just a continuation of one and another. So really seeing the theory behind it helps, like I think I can teach this in a better way, and explain to the students a better way, rather than just saying 'that's just the way it is,' and giving them a little bit of theory behind it."

Student L previously perceived advanced and secondary mathematics as disparate. After the course, she perceived more connections among the content. She explained how understanding the theory from advanced mathematics helped her better understand the content and explain the content to students. Student A similarly acknowledged how studying advanced mathematics helped her make sense of often rote secondary mathematics procedures. She explained,

> "You can do, for example, exponents or logarithms very rotely just by following the procedure, but it doesn't make sense, I think, until you've studied it from a more advanced math point of view, and you can see why it's being done that way."

This illustrates how after taking this course, the students reported perceiving more connections among previously disparate content and better understanding both advanced and secondary math.

The students also acknowledged *a change in their perceptions of the relevance of advanced mathematics for teaching secondary mathematics*. In the survey data, we identified a slight increase in the mean score of the survey item 5 (see table 1) from 6.60 to 6.67 (on a 7-point scale). The students' responses to interview questions provided additional insight into how they perceived advanced mathematics as relevant to and helpful for their teaching. For example, Student E claimed, "It helped me see things from a different perspective. I didn't really ever think that some of the things that we did in the module were relevant to high school teaching or students' thinking at those levels." Students expressed a *wide variety of ways in which they thought advanced mathematics was relevant to and useful for their teaching*. Some of these included using their knowledge of advanced mathematics in their teaching practices of facilitating class discourse, providing visualizations of concepts, attending to precision in the mathematical terms used in definitions, and making sense of students' mathematical thinking. Furthermore, several students acknowledged that advanced mathematics would be helpful in their teaching practices of designing lessons, making content understandable to students, and giving explanations in response to questions. Particularly, Student R explained:

> "There are some kind of students, I call them to be scary… who ask you why…I never really asked myself why. So after having a student like that and taking this class where I see where things come, I can talk like, 'this is a group, this is what is happening.' Yes, this theory helped me out in the way I'm going to be now introducing lectures."

Student R recalled experiences in which he did not know how to respond to students who ask "why." He explained that understanding the theory from advanced mathematics (abstract algebra in this context) helped him understand and explain why certain procedures work in equation solving. Student R illustrated a change in his perception of how advanced mathematics could be useful for him in his teaching practices of responding to students and explaining content.

The *students attributed these changes in their perceptions of the relevance of advanced mathematics for teaching to both the mathematical content covered in the course and the pedagogical scenario tasks used in the course*. First, when asked if anything from the course might have affected a change in their views on the relevance of advanced mathematics for teaching, some acknowledged the mathematical content in the course. Furthermore, several students attributed this change to the course's use of pedagogical scenarios. Student A explained:

> "The scenarios were helpful in that regard, like where we actually were asked to think about that very question of, in a classroom… how to explain something to a student who had some kind of misconception. So how to explain it in a way that would make sense to them… I think that that's what helped, was to actually confront that in a scenario."

These students reported that the pedagogical tasks helped them relate secondary and early college mathematics to advanced mathematics, understand students' mathematical thinking, and learn "how to handle those classroom discussions" (Student L). Student S explained how these tasks helped her become aware of different ways that students can think about mathematics:

> "The scenarios…opened up the panorama about how our students could think outside the box. A lot of kids would think differently about just one topic…. Seeing how that relates to so much more than just basic math…helped me change my point of view."

Overall, these students self-reported changes in their perceptions about the connections between advanced and secondary mathematics and its utility in their teaching of secondary or early college mathematics. The students attributed these changes in perceptions to the mathematical content covered in the course and the pedagogical scenarios used in the course.

Discussion and Conclusion

In this study, we examined how a graduate-level Mathematics for Teachers course supported improving students' confidence in secondary and advanced level mathematics and their perceptions of content connections and relevance of advanced mathematics for teaching secondary and early college mathematics. This course helped students increase their confidence in explaining mathematical concepts at high school, college, and graduate levels, and it was more helpful at the higher levels. More importantly, this course helped them recognize content connections between advanced and secondary mathematics which they had not experienced in the past. Students reported the pedagogical tasks using classroom scenarios helped them to connect the content in advanced mathematics to the teaching of secondary and college mathematics. Students perceived the relevance of learning advanced mathematics for the teaching of secondary mathematics in ways that are meaningful for them, e.g., understanding problematized secondary mathematics content in the given situations from advanced standpoints, and using advanced knowledge of mathematics for reflecting on their teaching practices.

Our course design leveraged Wasserman's (2018) theory of Knowledge of Nonlocal Mathematics for Teaching, which broadly suggests that for a teacher's knowledge of advanced mathematics to impact their teaching, the advanced mathematics must first change the teacher's ontological understanding of the content they teach. This theory informed our course design heuristics of both (a) guiding students to connect advanced and secondary mathematics to better understand the secondary content they might teach and (b) incorporating pedagogical scenario tasks to connect the advanced mathematics to the *teaching* of secondary mathematics via pedagogical mathematical practices (Wasserman & McGuffey, 2021). Several of the graduate students enrolled in the course attributed the changes in their perceptions of the relevance of advanced mathematics to their teaching to the course's implementation of these pedagogical scenario tasks. Thus, these tasks helped the students recognize how their knowledge of advanced mathematics could be useful in their teaching of secondary or early college mathematics.

The utility of these pedagogical scenario tasks presents fruitful avenues for future research and implications for the preparation of mathematics teachers. We plan to continue our course design efforts by designing curricula that can support prospective and in-service teachers in connecting advanced mathematics to the teaching of secondary mathematics through these pedagogical scenario tasks. We will also further investigate how graduate students use their knowledge of advanced mathematics and their pedagogical mathematical practices (Wasserman & McGuffey, 2021) as they perform these pedagogical scenario tasks. We suggest that both undergraduate and graduate mathematics courses for prospective or in-service teachers should be designed to implement pedagogical scenario tasks, because they provide opportunities for students to identify points of connection between advanced and secondary mathematics.

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Prioritizing Mathematical Topics for General Education Requirements

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With increased availability of technology and data, the mathematical and statistical needs for many academic disciplines have likely changed over the past twenty years. In order to create appropriate general education courses in the mathematical sciences to meet these needs, a survey was conducted of individuals representing various academic programs in Alabama to determine the mathematical needs of students in their programs. The survey results indicate that traditional college algebra courses are not needed outside of the disciplines that require calculus. The needs of the non-calculus-based programs include linear and exponential functions; mathematical practices; statistical literacy; and the use of spreadsheets for data analysis.

Keywords: general education, data analysis, mathematical practices

Nearly all colleges and universities in the United States include at least one course in the mathematical sciences as part of the general education requirements for their bachelor's degree. Nearly half of the students enrolled in a post-secondary mathematics course in the United States are in courses focusing on polynomial, rational, exponential, logarithmic, and/or trigonometric functions (Blair et al., 2018, p. 213). While such topics are essential for students needing to understand calculus for their academic program, they may not be the best choices of topics for courses in the mathematical sciences for students who do not need an understanding of calculus.

A common alternative to the traditional algebra and function based undergraduate course is a course focusing on quantitative literacy. Beginning in the 1980's, the Consortium for Mathematics and Its Applications (COMAP) created curriculum for such courses (Campbell, 2007) and in the 1990's, the Committee on the Undergraduate Program in Mathematics of the Mathematical Association of America (MAA) created a sub-committee to explore how quantitative literacy could be included as part of undergraduate general studies programs (Sons, 1995).

To improve all lower-division mathematics courses the MAA Committee on Curriculum Renewal Across the First Two Years (CRAFTY) started the Curriculum Foundations Project to better understand the mathematics needed by students in partner disciplines. This involved a series of workshops with faculty members from mathematics-intensive disciplines (1999-2001) and less mathematics-intensive disciplines (2005-2007) (Ganter & Haver, 2020) resulting in two documents (Ganter & Barker, 2004; Ganter & Haver, 2011) describing the mathematical needs of students in partner disciplines. This interdisciplinary work continues with projects like the Synergistic Undergraduate Mathematics via Multi-institutional Interdisciplinary Teaching Partnerships (SUMMIT-P) consortium (Ellington, 2020; Ganter et al., 2019a, 2019b).

In order to more holistically determine the mathematical needs of students from various non-STEM disciplines, the Arkansas Department of Higher Education partnered with the Dana Center to form the Arkansas Math Pathways Task Force (AMPT) to create a survey for departments to provide feedback about which topics in the mathematical sciences would be beneficial for their students (Korth et al., 2018) in order to provide guidance on what general education mathematics courses would be most useful for students in different majors. In considering these results, a similar working group in Alabama felt less confident that the current mathematics courses included in the general education program were appropriate. Thus, the current study expands upon the work of AMPT by creating a survey of academic programs at public post-secondary institutions in Alabama at a finer grain size and for a larger number of mathematical and statistical topics and practices than the AMPT survey in order to guide the redesign of mathematics courses included in the general education requirements for different majors.

Methods

A working group of faculty members from the mathematical sciences and institutional assessment from multiple higher education institutions commissioned by the Alabama Commission on Higher Education created a survey of mathematical and statistical topics and practices based upon syllabi from introductory post-secondary mathematics and statistics courses, standard secondary mathematics curriculum (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), and post-secondary professional organization standards (American Association of Two-Year Colleges, 1995, 2006; GAISE College Report ASA Revision Committee, 2016; Mathematical Association of America, 2004, 2015). After the creation of the survey items, they were reviewed for content accuracy by 10 mathematics faculty and readability by four non-mathematics faculty that work in assessment divisions of their institutions.

Based on information provided by the chief academic officers from each of the public higher education institutions in the state, the survey was sent to individuals who oversee each academic program in the state. Calculus courses satisfy the general education courses for a range of academic programs, including Mathematics, Engineering, Physics, Chemistry, and Computer Science; these programs were not included in the study as they were outside its scope. Of the 548 eligible programs, 174 submitted complete responses for a 32% response rate.

Due to the sample size from our survey responses we are not able to draw conclusions at the level of a single major. However, by combining majors into meta-majors based upon assumed mathematical needs, we can attain more generalizable conclusions. Academic programs that include a significant focus on using large data sets and/or computer programming as a key component of their field are likely to have a significant need for mathematical and statistical knowledge outside of calculus. For this reason, we group together majors such as geographic information systems, computer forensics, econometrics, and computer and information sciences into a meta-major that we will call **Data Sciences**. We group academic programs such as biology, geography, environmental science, and nursing together into a meta-major that we will call **Natural Sciences**, although this grouping does not include physics or chemistry since they require calculus. Majors related to education, psychology, human development, political science, and social work all incorporate the use of data and interpretation of statistical studies in their fields, so we incorporate these disciplines into a meta-major that we will call **Social Sciences**. Academic programs with a focus on business and/or management are grouped into a **Business** meta-major. Although such programs often include "business calculus" for the general education requirement, calculus is not inherent to the work of those fields and so we included this in this study. Finally, the **Humanities** meta-major includes majors that focus on studying and/or contributing to human cultures, including languages, arts, communication, philosophy, religion, and anthropology. Number of academic programs and response rates in each of these meta-majors is included in Table 1.

Table 1. Response rates by meta-major.

Meta-major	Eligible Programs	Responses	Response Rate
Data Sciences	34	13	38%
Natural Sciences	115	38	33%
Social Sciences	164	48	29%
Business	106	29	27%
Humanities	129	46	36%
Totals	548	174	32%

Mathematical topics were arranged into five major categories commonly included in general education mathematics courses, while statistical topics were organized into eight general categories. In order to facilitate the completion of the survey while still getting the desired level of detail, the survey included items addressing broader topics of mathematics within each of these categories, for which respondents denoted that their students "definitely needed", "somewhat needed", or "not needed". The "definitely needed" and "somewhat needed" responses were compressed into a "needed" category and these respondents were asked one or two follow-up questions to determine the need of students at a finer grain analysis. For example, respondents were asked "What best describes the importance of the following statement about students in your degree (or program)?" with the statement regarding functions being, "Students will understand properties of functions (domain, range, compositions, inverses) and use algebraic, symbolic, graphical, and numerical techniques to model related contexts and solve related equations." The respondents who denoted this statement as needed for their students received follow-up questions about the types of functions needed and the properties of those functions that their students would need to understand.

For each of the items in the survey, the need for that topic or practice for each meta-major was determined by the proportion of programs denoting a need for the topic. If the proportion was above half to a statistically significant level ($p<0.05$), we considered that topic as needed for the meta-major. If the proportion was below 50% at a statistically significant level ($p<0.05$), we considered the topic as not needed for the meta-major. If there was no significant difference from 50%, we did not draw a conclusion about that topic for that meta-major.

Results

The primary result of the study is that most of the meta-majors did not need most of the mathematical content traditionally included in introductory college mathematics courses. However, all the meta-majors, other than Humanities, reported a need for all the general statistical topics and practices from introductory college statistics courses, although not all the subtopics.

Mathematical Topics and Practices

Of the students enrolled in introductory mathematics courses in the fall of 2015, 20% were enrolled in precollege level (courses not receiving college credit), 40% were enrolled in a course focusing on functions and equations that involve polynomial, exponential, logarithmic, and/or trigonometric functions, and 20% were enrolled in liberal arts or finite math courses focusing on some logic, set theory, financial mathematics, and probability (Blaire et al., 2018, p. 213). The respondents to our survey reported that these topics are not the ones that are most needed for students in their majors (Table 2). None of the meta-majors denoted a need for polynomial and trigonometric functions, core components of the most common introductory courses, and instead denoted a need for linear functions, a topic usually included in the precollege level course. Furthermore, the topics related to functions denoted as needed by the largest number of programs was the ability of students to find numerical approximations of solutions using technology, a topic often not a major focus of college courses.

Table 2. Mathematical needs by meta-major.

Mathematical Topics	Data Sciences	Natural Sciences	Social Sciences	Business	Humanities
Function	Linear and Exponential	Linear and Exponential	Linear		Not Needed
Equations and Inequalities	Linear	Linear			Not Needed
Geometry and Trigonometry			Not Needed	Not Needed	Not Needed
Logic and Set Theory		Not Needed	Not Needed	Not Needed	Not Needed
Financial Mathematics		Not Needed	Not Needed		Not Needed

The respondents did denote that their students need experience with the mathematical practices that come from general education mathematics courses:

1. Interpreting quantitative information (needed by all but Humanities),
2. Strategically evaluating, inferring and reasoning (needed by all),
3. Manipulating mathematical expressions and computing quantities (needed by Data Science; not needed by Social Sciences or Humanities),
4. Communicating mathematical ideas in various forms (needed by Business, not needed by Humanities),
5. Make sense of quantitative problems and persevere in solving them (needed by Data Science and Business; not needed by Humanities),
6. Apply the mathematics they know to solve problems arising in everyday life, society, and the workplace (needed by all), and
7. Look for patterns and relationships and make generalizations (needed by all).

Combining the information of topics and practices, the survey communicates a need for introductory mathematics in the general education curriculum with a course focusing on the mathematical practices within the context of linear and exponential functions and equations. Such content could easily be incorporated into a data analysis course that includes linear and exponential regression.

Statistical Topics and Practices

Using the goals for introductory statistics courses established by the American Statistical Association (GAISE College Report ASA Revision Committee, 2016), the survey asked the respondents about the following statistical topics and practices:

1. Students should become critical consumers of statistically-based results reported (needed by all),
2. Students should be able to recognize statistical questions and design appropriate statistical studies (needed by all but Humanities),
3. Students should be able to produce and interpret data visualizations, numerical summaries, and statistical models (needed by all but Humanities),
4. Students should recognize and be able to explain the central role of variability and randomness in the field of statistics (needed by all but Humanities),
5. Students should demonstrate an understanding of, and ability to use, basic ideas of statistical inference, both hypothesis tests and interval estimation, in a variety of settings (needed by all but Humanities),
6. Students should be able to interpret and draw conclusions from standard output from mathematical and statistical software packages (needed by all but Humanities, not needed by Humanities),
7. Students should demonstrate an awareness of ethical issues associated with sound statistical practice (needed by all), and
8. Students can use tools and techniques involving the theory of probability to understand the nature of chance and to quantify variation (needed by all but Humanities, not needed by Humanities).

We note that outside of the Humanities meta-major, all these topics and practices were denoted as necessary for students as part of their academic program. For the Natural Sciences, Social Sciences, and Business meta-majors, the only topics denoted as not needed by there students are creating and computing statistical representations by hand; using any technology other than spreadsheets, SPSS, or R; and details of the theory of probability involving probability distribution functions and the Central Limit Theorem. All the other sub-topics were either denoted as needed or it was inconclusive about the need.

Conclusions and Implications

The primary conclusion from the survey results is that there is a much greater need among undergraduate students for courses in statistics and data analysis than the traditional courses of college algebra and precalculus. The results also corroborate the goals for students in introductory statistics courses of the ASA GAISE Report (GAISE College Report ASA Revision Committee, 2016, p. 8), and its recommendations of what to teach in such a course: “Teach statistical thinking” and “Focus on conceptual understanding” (p. 6).

One implication is for colleges and universities to consider the role of the mathematical sciences in the general studies curriculum at their institution. Mathematical ways of thinking have always been a core component of a liberal arts education, and the current study confirms that academic disciplines across the spectrum still value students having those ways of thinking. What they do not seem to value is the current menu of topics that have been standard for introductory mathematics courses. This should cause departments in the mathematical sciences to reflect on the goals of their current course offerings and to perhaps create new courses for this broad constituency of students. As a result of the survey, the Alabama Commission on Higher Education that initiated the study changed the corresponding category from a Mathematics requirement to a Mathematical Sciences requirement and changed from not allowing statistics courses to satisfy the requirement to encouraging such courses to be used to satisfy the new requirement.

Many of the academic programs surveyed in this study have program requirements that include mathematics courses based on topics that they denoted as not needed by their students, either explicitly or implicitly as a prerequisite. The results of this study should motivate such departments to collaborate with the mathematical sciences departments to determine the most appropriate courses for their students: if such courses do not exist, they should work together to develop new courses.

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Student Quantitative Understanding of Single Variable Calculus

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Student understanding of key concepts from Single Variable Calculus and the role of infinitesimals was investigated. Analysis was done using a framework which highlights quantitative reasoning. Data showed robust understanding of instantaneous rate and change across multiple representations. I examined the rich student connections between those concepts expressed discretely in Algebra and expressed instantaneously in Calculus, and how conceptions of infinitesimal quantities were formed in order to support those connections. Implications for instruction are considered.

Keywords: Calculus, quantitative reasoning, infinitesimals, derivative

Introduction & Literature Review

First semester Calculus is a critical course for all STEM majors, a required course containing foundational ideas for future study. Student success has been below desired levels for a long time, and efforts toward improving student success are extensive and ongoing (Bressoud, Mesa & Rasmussen, 2015). There has been much investigation into student understanding of Single Variable Calculus (SVC), exploring conceptions of limit (Oehrtman, 2009; Tall, 1992), derivative (Zandieh, 2000; Park, 2013), integral (Jones, 2015; Sealey, 2014), and the Fundamental Theorem of Calculus (Carlson, Smith & Persson, 2003; Radmehr & Drake, 2017). Some work has been done to connect those conceptions to a student's related prior conceptions (Thompson, 1994; Pustejovsky, 1999; Samuels, 2011). However, none of these studies investigated these questions for students learning Calculus using infinitesimals.

Recently there has been a growing call to revise Calculus instruction away from the Analysis-based approach using limits and toward a more quantitative approach (Augusto-Milner, Jimenez-Rodriguez, 2021). There have been a handful of documented attempts to do so with infinitesimals (Ely, 2021). Some research has theorized the underlying student thinking in such an approach (Ely & Ellis, 2018). None of these studies of infinitesimal-based Calculus instruction has examined how students conceive of important Calculus ideas. These intersecting gaps in the literature suggest the following research questions. (1) What is a characterization of student understanding of key Calculus concepts after one semester of Calculus taught with infinitesimals? (2) How are understandings of Algebra concepts in discrete form used to generate understandings of related Calculus concepts in instantaneous form?

Framework

To analyze student understanding of Calculus, an approach oriented towards quantitative reasoning (Thompson & Carlson, 2017) was used. This mode of reasoning entails "conceptualizing a situation in terms of quantities and relationships among quantities" (Thompson & Carlson, 2017, p425), where a quantity is a measurable attribute combined with a way to measure that attribute. The quantities of SVC and the relationships between them can be described using the ACRA Framework (Samuels, 2022). The framework is summarized here briefly with a delineation of the quantities and some of their relationships. An *amount* is a real value, a *change* is a difference between two amounts, a *rate* is a ratio of two changes, and an *accumulation* is a sum of consecutive changes. One important relationship is that the product of two rates is another rate. (The infinitesimal version is the chain rule, see Table 1.) Changes and rates over real intervals are encountered in a typical Algebra course; henceforth this scenario is referred to as *standard* (Keisler, 1976). Changes and rates over arbitrarily small intervals first arise in SVC. A positive infinitesimal is a positive number which is smaller than all positive real numbers; the usual rules of arithmetic apply. Instead of using limits, one can interpret differentials as infinitesimals to describe the arbitrarily small quantities in Calculus. (For further details, see (Samuels, 2022).) This study was targeted specifically at student conceptions after taking Calculus I. Because

Table 1. Part of the ACRA Framework for Quantities in Calculus

	Real	**Infinitesimal**
Change	$\Delta y = y_2 - y_1 = f(x_2) - f(x_1)$	$dx,\ dy$
Rate	$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$	$\frac{dy}{dx}$
Rate Equation	$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$	$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

amount has no separate Calculus version, and because accumulation/integration is covered only briefly in the course, those topics were not explored in this study. Thus the part of the framework used, and the key Calculus concepts on which data was collected, included change and rate, and their relationships.

Methodology

In 2022 Spring the author taught a Calculus I course at a northeastern urban college designed to use infinitesimals instead of limits. Two months after the end of the course, a clinical semi-structured interview (Hunting, 1997) was conducted with one of the students, who had volunteered. A script was prepared beforehand with questions on rate and change, and the relationship between them, in both precalculus and calculus contexts, and in multiple representations (verbal, numerical, graphical, symbolic). Flexibility was allowed for follow-up questions which might be suggested in-the-moment by student responses. The interview was video recorded, transcribed, and coded using the framework.

Results

Ari [a pseudonym] was first asked to “talk about rate in a way that does not involve calculus”. He explained that rate is “rise over run… [it] could be expressed as ratio” and that a linear graph has a constant rate. Using numbers in an example he created, “the rate, the amount of inches after every 30 seconds is 0.5, it goes up by 0.5”. He then illustrated this with the graph in Figure 1, explaining that “it shows the change for x, which is 30 seconds, and the change in y, which is 0.5”.

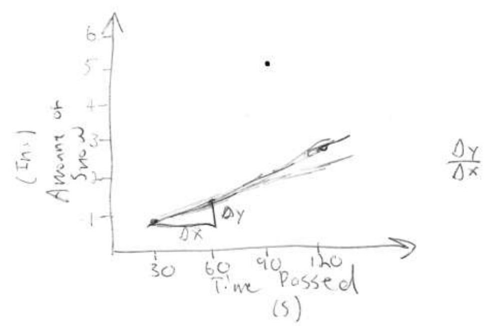

Figure 1. Ari’s inscriptions describing rate for a linear graph

In another example, Ari correctly explained how to calculate rate from two points. He used the standard formula, referring to it both as “delta y over delta x” and as “rise over run”, and referred to the result as both “rate” and “slope”. When asked for contextual examples of rates, Ari gave two: inches per second for snow falling, and miles per hour for a runner. Thus, Ari demonstrated a flexible knowledge of standard change and rate, including how to calculate them and to connect them, in multiple notations and representations.

Next, Ari was asked “can you give me an example where you would need Calculus if you want to talk about a rate?” From the example $y=x^2$, Ari observed that “the rate is always changing...it's not a constant rate. You would have to zoom in infinitely at a certain point on the graph to get, like, a slope. And then it gets straighter... that would represent the instantaneous rate.”

The interviewer gave Ari the graph of $y=x^2$ on Desmos with a touch screen, so the graph could be manipulated with fingers. He used a reverse pinch to zoom in at a particular point. He then observed that "when you get really far, it looks like the rise is like – you see how the line is much straighter, so it literally looks like a line – you would get like, the rate on y equals four." Subsequently he explained that this could also be depicted on the original (non-zoomed) graph as the tangent line at the point (2,4) with a slope of 4.

During a discussion of the slope of the magnified section of the graph, Ari wrestled with the magnitudes of the changes. He initially said that delta y was 4 and delta x was 1. After considering the graph at several levels of zoom, he corrected himself and identified the measuring unit of the grid lines of the graph (automatically drawn by the software) as the side of one square. He noted that the length, "when you zoom in, it always changes… it gets smaller". In the following excerpt, Ari starts by saying that infinitely small values do not exist, and by the end, says they do exist and exhibits a multifaceted conception of them. (His inscriptions during this exchange appear in Figure 2.)

Interviewer: We're going to imagine that we did what you said, which is imagine that we zoomed in infinitely far. We're not going to compare it to anything.

Ari: Okay, so if you zoom in it infinitely far, then it's going to keep getting smaller. Okay, so it's like to zero.

I: So once you zoom in infinitely and you say, I've done it, I've zoomed in infinitely, let's imagine that we accomplished that, then what would be the size of that?

Ari: You can't conceive of it. Yeah. Because it's infinite. It's not defined by any value.

…

I: Is there a positive number which is smaller than all positive real numbers?

Ari: No. Well, I mean, zero point 0 0 0 infinite zeros 1…

I: ... So if the length of this [horizontal] bit was point infinite zeroes 1, then what would be the length of this bit, the vertical bit?

Ari: It would be four times that, but it would be infinite still, the zeros would be infinite still, but it's like that same amount times 4. Which creates a change. Which causes a change because it's like 4…

I: Do we have any kind of name for these things? So, like, this horizontal bit of infinite zeros one, or this horizontal bit, which is also infinite zeros, but four times bigger.

Ari: So this [horizontal] bit would be like *dx*… and then the vertical bit is *dy*…

I: And then if you wanted to talk about the rate, how could you write that out? The rate for this specific example.

Ari: So *dy* is like the amount that it goes up an instantaneous rate. So at that point, when you do that a bit, that's how much it's changing at that point in the graph. So that's what *dy* represents. And then *dx* represents, *dx* represents when it changes by one to the right when you zoom out of the original function. So it's like you're relating the linear rate, like, how it would look linearly. This is like just one line, and that was like the rate then. Yeah, that's how it is. Right. But then at a different point, it would be different.

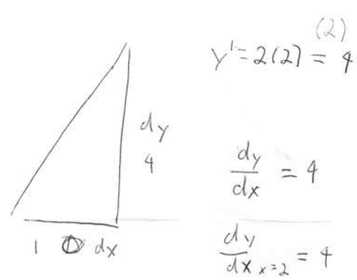

Figure 2. Ari's inscriptions when explaining that the instantaneous rate at a point is 4

In the next excerpt, Ari compared "very small" and "infinitely small" quantities from finite and infinite zooming, and their roles.

Interviewer: You started this by saying you have to zoom in infinitely in order to talk about the [instantaneous] rate. So if you were to zoom in a lot, so you get just keep on going until you get two point 100 zeros and then one, because you zoomed in a lot but not infinitely, is that enough to talk about rate?

Ari: Well, not if it's not infinite.

I: Okay. Why is that not good enough? If I zoom in a lot, why is that not good?

Ari: Because it is an exponential function and at every different point, the rate is always changing because 1.5 to the second power is, say like, okay, two squared is like four, and then three squared is nine, and then four squared is 16. So the difference between those is different. It would be the same the smaller the numbers are. So that's why you can't do that. So that's why it has to be infinite, because then you'll be able to figure out the instantaneous rate.

Ari also described a differential as an instantaneous change, and a derivative as a ratio of two instantaneous changes.

Interviewer: On this graph [in Figure 3], can you indicate for me, can you indicate *dy* and can you indicate *dt* on the graph?

Ari: So when you zoom in infinitely, vertically it would be *dy* [marking *dy*], this would be *dt* [marking *dt*], and then together [drawing the tangent line], that line right there is *dy*/*dt*.

[In response to a question, Ari used an equation he created to calculate *dy/dt* and *dy*.]

I: [What is the difference between] writing *dy* as opposed to writing *dy/dt*.

Ari: Oh, okay, yeah… *dy*/*dt* is like the ratio for the instantaneous change for both variables, for *t* and *y* in relation to each other, like this one [pointing to his earlier calculation for *dy/dt*]. And then when it's just *dy*, it's just the instantaneous change for *y*, which is how much vertically or how much the *y*-value is changing instantaneously at a point.

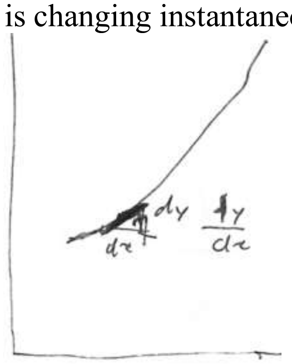

Figure 3. Ari's graph, used to indicate dy, dt, dy/dt, and the tangent line

Ari discussed derivative in context. When asked for other scenarios in which it would arise, he suggested the context of maximizing profit. He proposed a scenario relating profit in dollars, and quantity in bicycles. When asked for the units of the derivative he noted that it is "dollars per bicycle… Because the output, the value is represented by dollars."

Ari calculated a derivative using the chain rule (with substitution). For $y = \ln\left(\cos\left(e^{x^2+1}\right)\right)$, he used substitutions $g=x^2+1$, $j=e^g$, $h=\cos(j)$, $y=\ln(h)$, then used $\frac{dy}{dx} = \frac{dy}{dh}\cdot\frac{dh}{dj}\cdot\frac{dj}{dg}\cdot\frac{dg}{dx}$ before expressing the final answer correctly in terms of x. He then described his reasoning about it.

Interviewer: I noticed you made some cancel marks here. Can you talk about that?

Ari: Okay, so pretty much because when you take the derivative of g with respect to x and then j with respect to g, for instance, like, cross out dg because (pause). Well, I just see it as when there's a denominator and numerator, you cross it out because they, like, cancel each other…

I: Are you stating with confidence that's what you did or you stating with uncertainty that's what you did?

Ari: Well, I'm stating with confidence, but I think the problem is that because it's in derivative form. Because I remember when I took Calculus before somewhere, they said that you shouldn't see it as fractions. Like, you shouldn't see it as numbers. Like, it involves infinite number. And it's just kind of like, conceptually, it's not something you should do or see it that way. But it's just kind of like, in your class, I kind of just, like, memorized it this way because I've always hated the chain rule. And I was like, this is a good way to just do it.

When discussing the chain rule, Ari commented on proving derivative properties with limits and how unhelpful that was. "They showed the proof for the chain rule and for the product rule and all the rules, and it was very complicated. It was unlike anything like the calculus class I took, AP Calculus, it was just, like, very different, it was so much more complicated. I was like, what is this?"

Ari talked about which of his understandings improved from studying Calculus with infinitesimals.

Interviewer: Are there things feel that you understand better now after taking this class?

Ari: I guess the concept of the instantaneous rate of change, I think you showed me, like, there was some Greek letter [epsilon for infinitesimals], and then you multiply it with the number, and that would represent, like, the infinite value, the infinitesimal value, when you showed it to me that way. And that kind of like it helped me… Other classes introduce calculus with limits and that's very complicated, but that's the first thing that they do. Whereas [in this class there was] the delta x delta y thing first with the linear functions and then the rate function or change equation, rate equation. So, yeah, I kind of was able to grasp the concept of instantaneous change better. … When I don't understand something quick enough, sometimes it makes me learn it, because I want to understand it deeply. And then when I don't feel like I can, I just resort to, like with this [chain rule], memorizing and just putting it on the back burner. And other things were much easier. So I was just, like, that's what happened [with limits]. And then that's what kind of just held me back when it came to feeling confident in doing calculus. … I just think [previous Calculus instruction] was really focused on limits… it was your class when I first heard the term infinitesimals.

Discussion

Ari exhibited a rich and connected understanding of key concepts of Calculus. He was able to describe change and rate and the connection between them, both at a standard and infinitesimal level. He exhibited these conceptions in multiple problems across the four mathematical representations: numerical, graphical, symbolic, and verbal. Within the graphical representation, he was able to explain, for both change and rate, the connection between the standard presentation and the infinitesimal presentation. He did this using a verbal explanation, by drawing graphs and by manipulating a graph rendered by a computer grapher. He was very clear about the distinction between "very small" and "infinitesimal" magnitudes, and how the latter was necessary to describe an instantaneous rate, i.e. the derivative.

A significant part of Ari's concept image (Tall & Vinner, 1981) for instantaneous rate relied on the graphical representation, using infinite zooming to generate a straight graph. This conception of local linearity has been shown to be a productive cognitive root (Tall, 1991) in Calculus; several studies have shown that first introducing the derivative in this way leads to robust understanding and performance in Calculus (Tall, 1992; Samuels, 2012), and this study aligns with those results.

Ari described both the connection and the distinction between standard rate and infinitesimal rate. He utilized, first, a conception with finite real quantities, and second, a conscious introduction of the concept of infinity (in this case, infinitely small) to form a new scenario – in this case, quantities on the infinitesimal scale which are the context for infinitesimal change and rate. To characterize these two notions acting in concert, I introduce the term *transfinite thinking*. Thus, it can be said that Ari exhibited transfinite thinking in his conception of the derivative. Learner connections between finite and infinite scenarios have been investigated previously, both in Calculus and non-Calculus contexts (Ely & Samuels, 2019; Oehrtman, 2009; Mamolo & Zazkis, 2008). Here for the first time I provide a rigorous definition of transfinite thinking.

His conception of infinitesimals was given, not immediately, but after a progressions of statements. First, he conceived only of very small real values. Next, he said it would "keep getting smaller", indicating he was considering the *process* rather than the *result*. Then he said that an infinitely small quantity was something "you can't conceive of...because its infinite, its not defined by any value". Next he gave a numerical representation of an infinitesimal as "point zero zero zero infinitely many zeroes, one". Finally, he was able to perform arithmetic operations and make quantitative statements with infinitesimals, as when he referred to one (dy) being four times as large as another (dx). For Ari, the transfinite reasoning to extend his conception of rate between two points to rate at a single point was very much a multistage process which arose organically during the interview. This could form the basis for a hypothetical learning trajectory (Simon, 1995) for instantaneous rate with infinitesimals.

It is important to note that the idea of representing an infinitesimal as 0.00...1, a decimal with infinitely many zeroes, was never discussed in the Calculus class. This was a construction created by Ari in the moment. (Since he stated that he had never heard of infinitesimals before the Calculus course, I assume he did not hear of this somewhere else.) Research has previously shown that students have both spontaneous and intuitive notions of infinitesimals (Ely, 2010) which sometimes conflict with their classroom instruction. For Ari, it conflicted with his previous Calculus instruction but agreed with his experience in the current course. It is argued, both here and in the literature, that these intuitive notions should be encouraged, since they can form the basis of correct intuitions about Calculus which can be formalized into rigorous mathematics.

In constructing the infinite decimal, Ari gave quantitative meaning to the infinitesimal in the numerical representation. He was able to extend his quantitative reasoning further, making a *quantitative relationship between two infinitesimals*. He stated this relationship both multiplicatively and as a quotient. He described dy saying "it would be 4 times that [dx], but it would be infinite[simal] still…that same [infinitesimal] amount times 4". Initially he thought that $|dy|=4$, but he realized that $dy = 4$ units, where $dx = 1$ unit. He elaborated on the nature of that unit, describing it as infinitesimal and only apparent on the graph after an infinite zoom. Through unitizing and coordinating infinitesimals, he constructed a coherent quantitative conception of instantaneous rate.

Ari noted that his previous Calculus class began with limits and was "really focused on limits". He tried to understand them, but could not, calling them "complicated" and feeling limits "held me back". He resorted to simply memorizing procedures. Calculus students being forced to abandon understanding for memorization is unfortunate, and also consistent with previous results (Carlson, 1998). Particular difficulty with limits in SVC is also a common finding (Liang, 2016; Cornu, 1991).

An interesting aspect of Ari's situation was that he had previously taken a Calculus course. Students taking Calculus more than once is not only common, it is the majority experience among college Calculus students (Bressoud, Mesa & Rasmussen, 2015). How students negotiate this path is thus highly relevant. He was able to talk about the difference in his understandings in the two courses. Regarding calculating with differentials as in the chain rule, in his first course differentials were not thought of as canceling (i.e. arithmetic operations were not valid) even though those operations resembled the solving procedure, and he "hated the chain rule". In his last course, arithmetic with differentials was allowed, and in the interview he demonstrated the ability to calculate the derivative of a multiply nested function, calling the chain rule "a good way to just do it". Previously, proofs and explanations with limits created confusion, leaving him wondering "what is this?", and not understanding instantaneous rate. The presentation with infinitesimals allowed him to create personal understanding. After first discussing rate with the "delta x delta y … rate equation" in the standard case and then extending it using infinitesimals to instantaneous rate, he developed the rich understanding as previously described. This is evidence in favor of the calls to redesign Calculus to focus on the quantitative reasoning of infinitesimals and eliminate the Analysis of limits (Augusto-Milner & Jimenez-Rodriguez, 2021). The debate about relying on infinitesimals or limits for the foundation of Calculus has raged for centuries (Ely, 2021), and is likely to continue.

Ari exhibited two types of weakness in his understanding. One, some understandings were temporarily forgotten before being recalled. A likely factor was the researcher's intentional choice to conduct the interview two months after the course ended. Conceptions are likely only to be robust with either recency or reinforcement; without support in subsequent Calculus courses, meaningful and productive Calculus conceptions involving differentials can go away quite quickly and be replaced by unproductive and incorrect ones (Simmons et al., 2022). In the interview, Ari had space to (re)develop his conceptions, and in multiple cases did so with success.

Two, some topics were simply not understood at all. For example, while Ari was able to successfully calculate a derivative using the chain rule, he felt as though he only knew *how* to complete the procedure and he didn't understand *why* it worked. One possible explanation for his conceptual difficulty with this rate relationship was indirectly suggested by Ari. The conflicting instruction in previous Calculus classes may have made him wonder whether or not the cancellation of differentials in the fractions was actually legitimate. If students are taught explicitly that quantitative reasoning and arithmetic manipulation with differentials are incorrect and not allowed (a common practice), it may be challenging to get students subsequently to accept them. That is unfortunate, since those conceptions are mathematically valid and pedagogically productive.

It is important to point out that data in this study was drawn from a single student. I cannot claim that any observations generalize to the entire population. Rather, the analysis sheds light on how a student might possibly form conceptions about Calculus using infinitesimals. Due to the relative novelty of the context, this signifies a contribution to the literature.

Conclusion

In this report, I investigated an understudied area of student understanding of Single Variable Calculus, conceptions of rate and change after a one semester course on Calculus with infinitesimals. The data showed robust student conceptions of instantaneous rate and change across multiple representations. Graphically, change was conceived as a displacement, and a rate was the slope of a straight line, which could be presented as the ratio of two changes. There was an array of calculations with formulas and descriptions of applications. This conception was applied in both the standard and instantaneous contexts, demonstrating a strong link between prior and Calculus knowledge, connected using transfinite thinking. The student was able to articulate the role of infinitesimals in generating a locally constant rate to make meaning out of instantaneous rate. Graphically, this was described with an infinite zoom.

Calculus instruction using infinitesimals is not typical practice. There is a growing body of evidence that it is a propitious approach for students to make quantitative meaning in Calculus, and this study is a contribution in that direction. It also demonstrated how student knowledge of the quantities and relationships within Calculus, and their growth from prior knowledge by transfinite thinking, can be exhibited by students, and productively analyzed using the ACRA Framework.

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In Proving Attempts that Generate Stuck Points, What Characterizes Undergraduate Students' Overall Proving Process?

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Learning to prove mathematical propositions is a cornerstone of mathematics as a discipline (de Villiers, 1990). While the field has generated research that has analyzed the final products of proof (Selden & Selden, 2009) and there are frameworks for analyzing problem-solving processes (e.g., Carlson & Bloom, 2005; Schoenfeld, 1992), much remains to be known about analyzing undergraduate students' proving processes. With a focus on impasses in the proving processes, this study provides a more fine-grained account by characterizing students' overall proving process and navigating actions. The analysis lays the foundation for discussing the relative productivity processes students may take as they are engaging in proving.

Keywords: Proving Processes, Stuck Points, Navigating Actions, Productive Struggles

Since proving is a different mathematical activity as compared to students' prior experience, research has also shown that many undergraduate students struggle to learn to prove, including those who major in mathematics (Moore, 1994; Selden, 2012). Within the proof literature that focuses on undergraduate students, much work has concerned students' final proof products (such as written work or exams). However, such products cannot reveal the sequence of mental and/or physical actions that resulted in the final proof that students present in their written work (Selden & Selden, 2009). In particular, the proving process might entail several places where students get stuck, only some of which ultimately contribute to the final proof.

While the idea of struggle has been examined in problem solving in the middle grades (e.g., Warshauer, 2011), there has been sparse research on this idea in proving at the collegiate level. With a focus on impasses in the proving processes and the research questions "In proving attempts that generate stuck points, what characterizes undergraduate students' overall proving process?", this study provides a more fine-grained account by characterizing both undergraduate students' overall proving process and their navigating actions. Given the difficulty undergraduate students face in higher-level math courses, understanding the ways in which they struggle is important for building more inclusive classroom environments.

Literature Review and Theoretical Framework

View Proving as Problem Solving

A number of authors have remarked on the close relationship between problem solving and proving (e.g., Furinghetti & Morselli, 2009; Moore, 1994). In fact, Weber (2005) suggested focusing on problem-solving aspects of proof because this "allows insight into some important themes that other perspectives on proving do not address, including the heuristics that mathematicians use to construct proofs" (p. 352). Thus, in this study, I regard proving as problem solving and used Carlson and Bloom's (2005) Multidimensional Problem-Solving Framework to inform my research design. Expanding from Schoenfeld's (1985) framework, Carlson and Bloom (2005) developed four phases for problem-solving processes (Orienting, Planning, Executing, and Checking), each phase with the same four associated problem-solving attributes from Schoenfeld (resources, heuristics, affect, and monitoring). However, the original context of the work of Carlson and Bloom (2005) considered only how experts (like mathematicians) behave to successfully solve problems. Thus, their Multidimensional Problem-

Solving Framework does not characterize where a novice prover may seem to get stuck and how they may behave if they get stuck.

Immersion, Stuck Point and Productive Struggles

Wallas (1926) argued that creative problem-solving begins by first, immersing oneself in the problem to better understand it and exhaust conventional ideas. After a period of *immersion*, one likely reaches an impasse, or a mental block. The immersion phase of creativity shares some commonalities with *productive struggle* (Hiebert & Grouws, 2007), *productive failure* (Kapur, 2014) and other constructivist approaches to teaching in math education in which a student expends great effort during an initial period to explore and try to solve a math problem before receiving direct instruction. Involving in productive struggle in problem-solving is an invaluable mathematical practice which can help learners overcome uncertain obstacles to make sense of mathematics (Middleton et al., 2015). This is essential as students make meaning through productive struggle, or as they grapple with mathematical ideas that are within reach, but not yet well formed (Hiebert & Grouws, 2007; Kapur, 2008; Warshauer, 2014). From the student point of view, these *stuck points* have been described as perceived impasses, or moments when a student feels substantially stuck and unsure how to continue (VanLehn et al., 2003).

To date, research on stuck point in proving in the mathematics education literature has been sparse. Identifying a "stuck point" can be hard since it requires finding observable behaviors to serve as indicators of a person's internal mental state. Savic (2012) conducted a study of how mathematicians recover from proving impasses. In his study, all of the mathematicians were able to recover from the impasses, thus his focus was on their *Incubation Period*: where they temporarily shift their attention away from the problem and do something else. Methodologically, Savic used Livescribe pens that were given to the mathematicians for days to gather real-time data. Although a mathematician may have had a key insight while walking around (for example) that allowed them to get unstuck, from this study, we still know little about the actual cognitive process of getting unstuck. Thus, Savic (2016) had called for future study of proving processes involved several impasses (stuck points) with participants that are less skilled in math (different from mathematicians), since their incubation period might not help them to move forward. In this study, I thus began by characterizing the major actions that were evident in students' proof construction through "stuck points" in their proving processes.

Method

My study of undergraduate students drew participants from the Introduction to Proof course at a large research university in the Midwest. The Introduction to Proof (ITP) course is required for undergraduates majoring in STEM and mathematics education. The course focuses on developing mathematical argumentation techniques and proof strategies. The data for this study consisted of semi-structured task-based interviews with 10 undergraduates enrolled in a transition-to-proof course. Each participant was asked to prove or disprove two mathematical statements. The majority of the interviews took around 70 minutes, which provided enough time for the participants to have at least 30 minutes to engage with each task. In order to capture students' proving processes with their specific strategies and steps, a think-aloud protocol was used (Schoenfeld, 1985). Interviews were video-recorded, and students' real-time proof work was captured using a Livescribe™ pen.

As I have discussed in the literature review, in order to account for the proving processes of undergraduate students, the existing problem-solving frameworks need to be expanded. In particular, the role of students' stuck points in their process is important, as is understanding more about what students do when they get stuck. To do that, I utilize a combination of my own partition of students' process around "stuck point" and a modification of Boero (1999)'s proving phases and Carlson and Bloom (2015)'s Problem Solving framework. My partition was developed from the flowing three steps:

Step 1: Operationalizing and Identifying "Stuck Points". Like I have pointed out, identifying a "stuck point" can be hard since it requires finding observable behaviors to serve as indicators of a person's internal mental state. In this sense, body and facial language become more important in telling whether a person has gotten stuck. For my purposes, stuck meant a period of time during the proving process when a prover felt or recognized that their argument had not been progressing fruitfully and that they had no new ideas. Whether the participant discovered an error was not important; the prover's awareness that the argument had not been progressing, but they were *hesitant about what to do next* was the key focus for me. For the purposes of my study, an impasse (an episode of "stuckness") consisted of interruption in the proving processes that was initiated by a student struggle that was in some way visible, whether voiced, gestured, or written. An impasse ended when the student overcame a stuck point and continued attempting or finishing the task, or the student gave a sign of no resolution and indicated wanting to switch to a different task or end the interview. In total, there were 41 episodes of such "stuck points" and majority of them are in the last four phases according to Boero (1999): Exploring of the Content (identify appropriate argumentations and envisage possible links), Selecting of arguments into deductive chain, Organizing the chain of argument into proof, and finally Write a Formal Proof.

Step 2: Categorizing Major Actions Around Stuck Points. I used the existing Warshauer (2011)'s framework for the four categories of students' stuck point (getting started, carrying out a process, uncertainty in explaining and sense-making, and expressing misconceptions and errors) to enhance my initial coding scheme by developing sub-categories to capture students' actions. Different from Warshauer (2011)'s study that mainly aim to examine the categories of students' struggles in K-12 problem-solving environment with teachers' interaction, my study focuses more on undergraduate students' proving processes and their own navigating actions.

Through this process, several major categories were identified. Each student's proving processes were captured in three major categories initially: their arguments, their stuck point, and their actions. After identifying those bigger categories, smaller categories were then identified inside of each major category. Since I was primarily interested in students' arguments and actions around their stuck points, I refined categories related to proving activity to the following four: initial argument, stuck points, navigating actions, and related outcomes. Once the initial coding scheme with those themes was developed, I completed the first round of data analysis to test and refine those themes. Particularly, I reorganized my coding scheme to highlight students' responses to "stuck points" or impasses. Besides different types of stuck points that were identified by the students, their actions for attempting to resolve those stuck points, monitoring, and persistent actions were also classified and captured.

Step 3: Capturing Each Students' Proving Process with Map. To capture each student's stuck points for each individual task, I used a flow chart to map out each student's proving processes with their major navigating actions. I split the proving process map into two dimensions. Vertically, the process map captured the four categories as described in Step 2. Horizontally, the map captured the order of each individual argument and action. The arrows in between indicate the directions of the movements and the relationship between the arguments and the actions. Instead of capturing only arguments, I also added a category that captures students' actions to help bridge the gap of the relationship between the argument and the actions, as described above. In *Figure 1* below, I provide an example of one student's proving process map for Task 2. From this process map, one can clearly see the student's arguments, stuck points, navigating actions, and outcomes related to those actions, as well as how one moved from one to the other. This proving process map provides a bird's-eye view of the proving process for each task for each individual student.

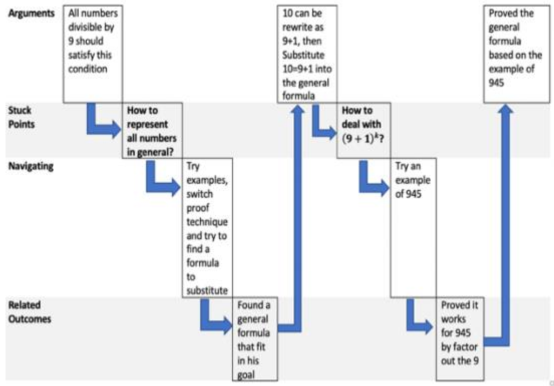

Figure 1. Example of Student's Proving Process Map for Task 2

Result and Discussion

In my subsequent work, the detail analysis of each student's proving process will be provided with map. In this proposal, I turn to focusing on the qualities of navigating actions students used, situated in the context of their entire proving attempts. My analysis resulted in three major types of navigating actions that were generated based on each participant's proving process map and the major categories of my analytical framework. For each task, each participant's proving process was characterized by their process map based on shifts between (1) initial argumentation (Argument), (2) major stuck points (Stuck Points), (3) actions they took to try to navigate out of being stuck (Navigating Actions), and (4) related results (Related Outcome). If they are able to use the related outcomes (i.e., new insight, useful calculation, or new connection) that they generated in a given cycle, then the proving process progresses, either to completion or until they reach a new impasse. In the next sections, I will describe the three major types of proving process around stuck points based on these shifts in the order of increasing complexity. Because of the limited space for this proposal, I will only discuss one student for each type for Task 2 (Prove or disprove: An integer is divisible by 9 if the sum of its digits is divisible by 9).

Type 1: No Related Outcome Produced

Trajectory: Arguments⟶ Stuck Points⟷Navigating Actions

For this process type (as in all the types I consider), the student started with some sort of argument and got stuck when trying to construct a proof. The student then took several actions or attempts to try to navigate out of the stuck point but did not produce any related results, thus going back to their navigating actions. In summary, this type can be characterized as going back and forth between stuck points and navigating actions without any related outcome to help one move forward (see *Figure 2*). I will use Nikki's process to explain this type.

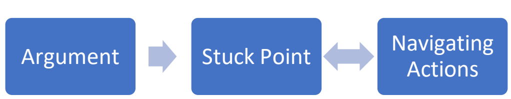

Figure 2. Process Flow for Type 1 Process Around Stuck Point

As soon as Nikki saw the task, she remembered that she had done a similar problem before. She quickly wrote down the problem and figured out what were the "P" and "Q" in this statement to set up for a proof by contrapositive. She then rewrote the statement as "If the sum $x_1 + x_2 + \cdots + x_k$ equals 9, then the number *n* is divisible by 9" (Nikki, Task 2). But Nikki got stuck after she had decided that was the argument that she wanted to prove. "How are they related?" she questioned. At this point she went back to the beginning of the problem and double-checked which statement should be thought of as "P" and which statement as "Q," then told me that she didn't know where to go next. She then started trying some examples such as 27, then thought about what happened if the sum was 9 or the sum of a multiple of 9 (like 18). But these navigating actions didn't help her to go forward. She asked aloud, "How can I represent this example in a more general form?" and she got stuck again with her attempts to try examples. Nikki then tried to look for the general equation for expressing the sum of the digits. She found the expression on the formula sheet and tried to understand the expression again by computing some examples and by rewriting 27 as $2 * 10^1 + 7$. But she was soon stuck again with the k^{th} term: "What does *k* here mean in the expression of 10^k? Is *k* the number of digits or something else?" (Nikki, Task 2). She told me this general form she found (on the provided formula sheet) didn't really help her, because she didn't even understand the expression itself. After several attempts, Nikki didn't produce any related outcome from her navigating actions, and she remained stuck in the end of the episode.

As I have noticed from Nikki's example, this type of process doesn't involve the production of related outcomes from navigating actions. Students with this process tend to go back and forth between stuck points and navigating actions since no significant related outcome is produced to help them move forward. Thus, in the end of the process, students don't overcome the stuck point and they are unable to make progress on their argument.

Type 2: Related Outcome Produced but Not Link to the Argument

Trajectory: Arguments⟶Stuck Point 1⟶Navigating Actions⟶Related Outcomes ⟶Stuck Point 2 (⟷Navigating Actions)

The majority of the participants' proving processes can be characterized as this type. For this process type, like before, a student starts with some sort of argument and gets stuck when trying to prove the argument. The student then takes several actions or attempts to try to navigate out of the stuck point (so far, as in Type 1). However, the student successfully overcomes one stuck point by producing a new idea or connection (a "related outcome"). But they soon run into another stuck point when they try to expand the related outcome into an argument. This process may also include some navigating actions to try to overcome the second stuck point, but the student is unsuccessful in doing so (see *Figure 3*).

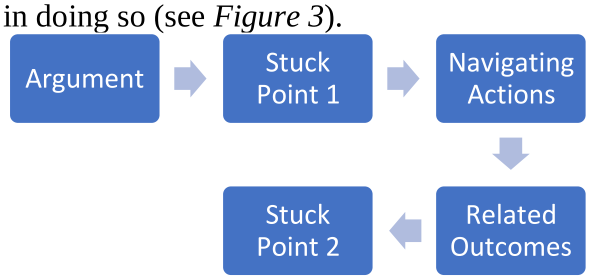

Figure 3. Process Flow for Type 2 Process Around Stuck Point

In Task 2, Devin conjectured the statement works for all multiples of 9 after trying examples like 9, 18, 27. But he then got stuck with how to show the statement in general. He tried to represent a given number in terms of the sum of its digits in a general form and assumed $9/N$ where N is a number such that $ones + tens + \cdots =$ multiple of 9 and $m1, m2, m3$ are the place values of each of the digits. Thus, any number can be represented as $N = 1 * m1 + 10 * m2 + 100 * m3 + \cdots$ But with this generalized formula in hand, how to use this to prove the statement became her new problem. He then thought about whether a number that is a multiple of 9 could be factored differently. Devin took the examples of 36 and 135 to try to factor them out and see whether he can have some insights. He rewrote them as $36 = 4 * 9, 135 = 1 * 100 + 3 * 10 + 5 * 1 = 100 + 30 + 5 = 2 * 5 * 5 + 6 * 5 + 5 = 5 * 27 = 5 * 3 * 9$. But "What do these factorizations tell me?" became his new question, a question he still had in the end.

The examples of Devin illustrate what Type 2 processes look like and how they differ from Type 1 processes in terms of the role of related outcomes. Devin had several related outcomes produced by overcoming some of their stuck points. This helped them to move forward, and, in fact, they were just a few steps away from fully overcoming all the stuck points and successfully proving the statement. Thus, at this point, it will be insightful to compare their processes with at least one related outcome linked with argument.

Type 3: At Least One Related Outcome Linked with Argument

Trajectory: Arguments⟶Stuck Points⟶Navigating Actions⟶Related Outcomes ⟶Arguments

Similar to the previous two types, for this type of process, a student also starts out with an initial argument, but they also get stuck. The student then takes several actions or attempts to try to navigate out of the stuck point (as in Types 1 and 2 process flows). However, the Type 3 process advances to proving the statement using the related outcomes generated in the process and ultimately results in having at least one related outcome linked with the argument. Since there could be multiple stuck points resulting in multiple processes, the student will be categorized in this type as long as at least one productive cycle is complete. The cycle can repeat if a second or third stuck point is encountered. To summarize, this type of process always involves a related outcome each time and the participant is able to leverage these related outcomes to move forward with developing their argument (see *Figure 4*). Note here that students are considered as making productive progress (Type 3) as long as at least one productive cycle is complete, which means they don't need to successfully prove the task in order to be considered as making productive progress.

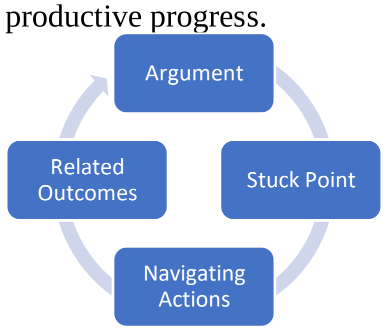

Figure 4. Process Flow for Type 3 Process Around a Stuck Point

Soni successfully overcame several stuck points in both Task 1 and Task 2. For Task 2, Soni started with outlining some of his ideas for finding the differences between counting by 9s and counting by 10s and rewrote 10 as $9 + 1$. Having this general goal in mind about what he would do, he felt the need for a generalized formula to represent any integer with its sum of the digits. Soni looked into the formula sheet and found the formula that any integer can be represented as $a_k(10)^k + a_{k-1}(10)^{k-1} + \cdots + a_0(10)^0$. But then how to use the general formula to prove the statement became his problem. Soni then thought about induction to help him achieve his goal. However, he soon realized that it didn't work here. He decided to go back to his initial thinking about representing 10 as $9 + 1$ and substituting all 10s in the formula as $9 + 1$. However, he got stuck again on how to expand $a_k(9 + 1)^k + a_{k-1}(9 + 1)^{k-1} + \cdots + a_0(9 + 1)^0$. Soni then decided to try a specific example (*Figure 5*) to show it could be factor out by 9, then follow the same way to show the generalized cases. Following this way of thinking, he successfully navigated out of his stuck points and proved the statement in the end.

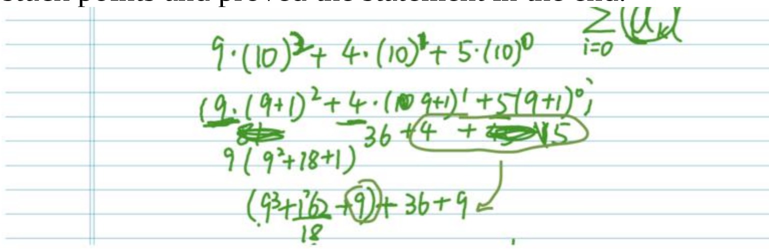

Figure 5. Soni's work of representing 945 using counting by 9s

From Soni, we see what could be described as a more productive struggle engaged in their proving processes since they completed at least one cycle of link back to the argument. Looking at Soni's case, we would see that he was clear about the goals for each of his navigating actions and had a clear understanding about where he got stuck. He also successfully produced several related outcomes from his navigating actions to help him move forward.

Conclusions and Implication

As I discussed in the literature review, although students' and mathematicians' proof practices have been examined in several ways, there are still important aspects that have not been examined in the existing literature. Previous proof construction research, building largely upon the work of Toulmin (1958), focused only on the structure of the argument but not the actual action and the actor (e.g., Pedemonte, 2007). To address those issues and characterize undergraduate students' proving processes, through our analytical process (conceptualizing stuck point, characterizing actions, and capturing general processes using map), I classified three different types of overall processes when undergraduate students face several stuck points and try to navigate out. In general, we can see that even though some of the students' navigating actions might look similar (e.g., look for a generalized formula to represent digits), students in Type 3 seems to make more productive progress compared to the other types. The reason could vary between students, but in general, students in Type 3 are able to clearly identify their stuck points, have a goal for each of their attempts, and able to produce related outcome from their attempt and then link that back to their initial goal/argument.

To conclude, students need to engage in a way that they can make some progress in order to be productive. In general, most of the students interviewed in this study had their process maps and actions categorized as Type 2. If we know where students get stuck in their overall proving process when trying to navigate out stuck points, we will be able to better support their productive struggles and help them to move to Type 3. In the subsequent study, I will describe individual cases in detail to discuss the relative productivity of different possible navigating actions students may take as they are engaging in proving. The different proving processes and navigation actions characterized in this study can help instructors to have a better understanding of students' proving processes when navigating out stuck points and to be more explicit in their own modeling of their own actions for their students.

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Contextuality in Reasoning about the Exponential Function: A Microgenetic Learning Analysis

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This paper explores how two undergraduate students come to perceive exponential structure in a series of word problems. This study investigates whether features of task situations can create opportunities for supporting students in recognizing situations as appropriate for using exponential models. The analysis is guided by coordination classes theory (diSessa & Sherin, 1998) and Transfer-in-Pieces perspective (Wagner, 2006; 2010), which emphasizes the role of multiple instantiations of a concept across multiple contexts as learners construct their understanding. The results suggest that contexts that embody the notion of recursive dependency may help students activate and use appropriate knowledge resources related to exponential models.

Keywords: Exponential models, Context-dependence, Knowledge in Pieces

Introduction and Purpose of the Study

There is an essential need to explore students' understanding of different topics at the college level, especially those topics that students find difficult. Exponential functions are a difficult, yet essential, mathematical concept that plays an important role in the study of advanced mathematics (Ellis et al. 2016; Weber 2002). Several researchers illuminated the importance of deep understanding of exponential function as an important topic in the mathematics education field (Brendefur, Bunning, Secada, 2014). Weber (2002) focused on students' initial understandings of the exponential function. The main result from Weber's (2002) work was that although all students could compute exponents in simple questions, only few students could reason about the process of exponentiation. The result stimulates some essential questions regarding understanding the exponential function: What features of contextual tasks do students notice and how does this impact their reasoning patterns? In particular, what features of tasks meant to cue the use of exponential models are salient to students? How do students' reasoning patterns shift as they engage with a sequence of tasks designed to help them link their prior experience with linear modeling to exponential modeling?

Theoretical Framework

This work requires a perspective that brings into focus on the role of contextuality of a task in students' thinking and reasoning. To this end, I turned to Wagner (2006; 2010) who addressed the role of knowledge in transfer processes. Wagner (2006) demonstrated in "transfer-in-pieces" that prioritizing context sensitivity of knowledge supports transfer. Wagner (2006) framed his analysis using the construct of *coordination classes* which is rooted in diSessa's (1993) Knowledge-in-Pieces (*KiP*) heuristic epistemological framework. Coordination classes give a model for conceptual understanding that involves how individuals extract concept relevant information from a context. These constructs can help to explain how learners initially interact with mathematics content in a wide range of contextual tasks. In describing the knowledge that emerged in this study, the coordination class theory construct of *concept projection* proved to be useful. A *concept projection* is, "the particular set of strategies and cognitive operations that are used by an individual in applying his or her concept (coordination class) in a particular situation…" (diSessa & Wagner, 2005, p. 128).

Wagner (2010) provided an extension of transfer- in-pieces in which the alignment with the Piagetian frame is explicitly pointed out to create a framework for transfer either in similar situations or accommodation of a new concept. "Concept projections are thus collections of assimilatory and interpretive knowledge elements associated with some concept, and the construction of new concept projections reflects a process of conceptual accommodation to new contextual situations" (Wagner, 2010, p.451). Building on Wagner's (2006, 2010) perspective, I explored how learners can build new knowledge elements —exponential understanding—which is the result of passing through multiple disequilibrium and accommodating particular context. In this work, another goal is to investigate the affordance of context for such accommodations. In other words, I seek to explore from the novice's perspective what is the feature of contexts that exponential understanding can be accommodated to? The affordances of a context are a fundamental aspect of investigation of coordination classes and transfer. "Transfer requires that the structure perceived in a new situation be matched or closely matched to the structure perceived in a previously encountered situation" (Wagner, 2006, p. 5). A major difficulty is the explanation of contextual situation which can be matched to the ultimate, desired abstraction (Wagner, 2006).

In this work, I investigate the affordances of *recursive dependency* implicit in very common exponential contexts for instigating productive conceptual conflict. I define recursive dependency as a feature of a context task in which reasoning about each step depends on the previous step. As an example, the molding process of the population of a city, increasing by a particular rate every year, needs learners to realize that the number of people of the city in each year *depend on the previous* year, rather than the initial year. I argue that the *recursive dependency* is the heart of, specifically, identification of exponential function as it is shown in two undergraduate students' reasoning in this study.

Data Collection

Data for this study were taken from semi-structured task-based interviews of four undergraduate students who enrolled in a developmental math course—Algebra II—at a large, public, Midwestern university. The interviews were conducted in an online format consisting of 60-100 minutes of problem solving. The interview involved screen sharing a specially designed set of slides I created through Desmos Activity Builder (DAB). Participants agreed to meet with me for 60-90 minutes one-on-one in an online private session. They worked in DAB and their thoughts as he described as they worked on a contextual problem involving exponential function was audio and video recorded. Screenshots of all written work in DAB were saved and some are represented in appendix A. All 60-100 minutes of the interviews were transcribed for detailed analysis.

The main interest to this research was moments in which participants' patterns of reasoning appeared to be in transition or if it appeared that they came to notice a conceptual conflict. Among four participants there was one student, Leia, who understood the context qualitatively in City A problem and calculated the population after 10 days and 20 days, however, introduced a linear model and stuck with a linear model. Another student, Nick, who already knew exponentiation in different contexts and models exponentially. Two students, Tom, and Andrew, who underwent conceptual change and came to discern exponential function in different contexts. This study focuses on how Tom and Andrew underwent some conceptual change to understand exponential function.

Among the problems presented to students were sets of common contexts for exponential situations, like population growth and the amount of a drug remaining in a patient's body. However, there is a special effort for raising the possibility of more engagement from each student. For example, city A and City B problems, presented in Table 1, is an explanation of the population of people who are infected by Covid 19. This study happened during the peak of the pandemic, when checking the number of Covid 19 outbreaks was a concern. the set of problems that students attempted to solve in the 60-100 minutes interview. I sometimes asked probing questions, however, I avoided taking an evaluative role and emphasized my appreciation for valuable sharing of their thinking with me. An analysis of explanatory language that Tom and Andrew used during the interviews enabled me to recognize the importance of well-matched recursive dependency in the exponential context to generate productive disequilibrium and support transfer.

Table 1. Exponential problems that participants worked on in the 60-100 minutes interviews.

Exponential problems
<u>Baseline problem</u> What does the function $f(x) = b^x$ mean to you? What do you think of when you see this function?
<u>People Diagnosed with Covid-19 in City A</u> In City A the number of people who have Covid-19 increases by 11% every 10 days. The initial population of people who are diagnosed with Covid-19 in City A is 1500. Create an equation that models this situation.
<u>People Diagnosed with Covid-19 in City B</u> In City B the number of people who have Covid-19 decreases by 11% every 10 days. The initial population of people who are diagnosed with Covid-19 in City B is 1500. Create an equation that models this situation.
<u>Drug Remaining in the Body</u> Imagine that a patient is taking 100 milligrams of a drug. Suppose only 27% of the drug remains in the patient's body after an hour. How long does it take before only 10% remains?

Analysis

Microgenetic Learning Analysis (*MLA*) is an analytic methodology for progressing from observations in video data of moment-by-moment reasoning processes to systematically describing the nature and form of knowledge growth and change. (Parnafes & diSessa, 2013). My study also can be summarized as MLA of two undergraduate students' (Tom and Andrew) reasoning, coming to understand and discern the appropriateness of contexts for modeling exponentially. In this work, the MLA process includes transcribing the episodes, enriching them with details about what was written and pointed to in Desmos by participants, and then summarizing the work on each task and the discussion around it. Describing learners' first reaction for a modeling system and how they both modeled linearly; then playing genuinely just with context helped them to go through an assimilatory activation and, ultimately, accommodate a new reality which was concept projection of exponential function. I defined this process of, first, reacting proportionally to, eventually, accommodating exponentiation as the process of "changing". As I observed, both learners could experience this "changing" process either in City A problem (Tom's experience) or in City B problem (Andrew's experience) with assistance of productive conceptual conflicts in which a well-imbedded recursive dependency appeared as a highly influential factor for change.

Analysis of Tom's reasoning

City A problem: Facing conceptual conflicts (disequilibrium). Coming to solve the problem of city A is the heart of this study, in which we can see the prominent pattern of logical readout of the situation, coordinating the context and his pattern of reasoning, experiencing a productive disequilibrium, accommodating new understanding of the situation, and finally, generalizing the problem to make an abstract model. To me, the key observation across the session was the moment that Tom in the middle of making sense of the structure leaves all his findings (including the population of Covid 19 after 20 days: getting 11% of 1665 and adding to 1665, not 1500, which had the result of 1848) and faces a huge *disequilibrium* with satisfying a systematic and already established proportional perspective.

Interviewer: How did you find that number?
Tom: I divided 1848 by 1500 and found 1.321 which does not make any sense
Interviewer: what about first 10 days
Tom: That works really well. 1665/1500 is 1.11 which makes a lot of sense. I thought 1848.15/1500 should be 1.22 because it increases 11% every 10 days

The *recursive dependency*, well-imbedded in the context task, appeared to help Tom to prefer working with his understanding of context rather than sticking with his proportional expectation. He eventually could coordinate his perception of the situation with his strategy for calculating an increasing Covid19 population in each step and to resolve an apparent contradiction in his thinking.

Tom: Ok! So, oh! Wow! That's a lot! Hah! Ok! So, in the first step we are taking 11% of 1500 and then plus 1500 which is 1.11×1500 and then in the second step we do $1.11^2 \times 1500$ and in the third step it should be $1.11^2 \times 1500$ right? Is that it?
Tom: SO! f (x) = $1.11^x \times 1500$ would be: Right? Cool.
Interviewer: Do you want to calculate what you find in the second step? I guess it would be interesting for you
Tom: Sure! It is 1.321! What? What is that? ***Is it magic?***

City B Problem: Testing Near Transfer. In Tom's work on the City B problem, he showed transfer more explicitly in a similar context. I asked him to calculate the population of people who have covid 19 after 10 days. While he was calculating it seemed that he already had the exponential model in his mind.

Tom: Ok! I found it! If I take this (1.11 from the previous model), because it is one point eleven! If I want to decrease (Looking ups and down)! hundred mines 11… like 89 percent? If

I do 0.89 (More confidently and surely say the last words) that gives me the model here, the 11 percent decrease (laughing happily!)

Interviewer: Do you want to write the model?

Tom: Yes, sure! I think it should be f (x) $= 0.89^{x} \times 1500$, right?

Remaining Drug Problem: Fidelity to Proportional Modeling when Context Lacking Recursive Dependency. After 73 minutes of productive challenge to understand exponential function, Tom confidently initiates reasoning proportionally. In the end, he modeled this problem with a linear and solved a linear equation to figure out how long it would take for 10% of the drug to remain. Figure 3 in appendix is a screenshot of Tom's work when he was sharing his problem solving on the WebEx. I tried to have a conversation to see why the first reaction to this question involves linear thinking. I asked him why he thinks that a linear model can work in this problem, and he answered that "I am not an expert, but our body must work linearly. It cannot work first slowly and then [become] hardworking!" At this moment in Tom's interview, I could have asked "why?" (Why can the body not work with different rates of the drug?) However, it did come to my mind as I was experiencing the first observation of how learning is highly contextual. In the interviews with Andrew which were conducted after Tom's, I was able to get more insight.

Analysis of Andrew's reasoning

City A Problem: Appropriate Use of Exponential Model. Andrew found a strategy to calculate the population of Covid19 immediately, however, just like Tom, after seeing some steps in the exponential model, he checked himself with a proportional model. Facing a contradiction between understanding of the problem and the linear model, I asked him what his decision is.

Interviewer: If I want you to calculate after 30 days, how do you go, are you plugging 3 in your linear model or you are calculating from the first strategy and why?

Andrew: I would probably do the first one, because the way that I understand the increases, it is not that taking the old percentage and taking 330, it is taking the ***11% from the new*** **number.**

I observed that a well-matched recursive dependency emerged to help Andrew to decide his strategy and confidently leave proportional thinking. Like Tom, Andrew eventually modeled exponentially this problem.

City B Problem: Facing a Multi-dimensional Disequilibrium. Andrew recognized what decreased means as he could see 0.89×1500 for the first set of 10 days and $0.89 \times 0.89 \times 1500$ for the second set of 10 days. I thought he was ready to make the model, just easily in a similar context; however, he was still in disrepair with the linear model.

Andrew: I think I could! it would be 1500 times (X-0.11)? so for the second step it would be 1500 times 2-0.11, but no! that would be an increase! No?! It cannot be correct, because it is two thousand and something which is not even a decrease.

Interviewer: Ok let's talk about your strategy, not the model… so far you found 0.89 … how about after 20 days?

There was an implicit point here in Andrew's reaction, in this episode. Although the similarity of City A and City B problem should have resulted in the transfer for abstraction more immediately, there was a tacit avoidance in Andrew's reaction in which, I believe, he was hesitating to connect a decreasing process to an exponential model. I think there was an intuition that if a number comes to the power of something, it should be an increasing process. Andrew was facing multiple conflicts simultaneously; that is why I made up the "multi-dimensional disequilibrium" for his reasoning pattern in the City B problem. Eventually, he could see the model after working on the population after 30 days.

Remaining drug problem: Fidelity to Proportional Modeling when Context Lacking Recursive Dependency. Like Tom, after 63 minutes of productive struggle for understanding the exponential model, Andrew returned to proportional thinking very confidently. He made the proportion for finding the slope and then a linear model. This time I had that crucial "why" to ask, moreover, involving more of learner's voice in the study:

Andrew: I was assuming a consistent rate
Interviewer: *Why* do you think it has a consistent rate?
Andrew: Based on kind of information that I get
Interviewer: So, imagine in the question we have this assumption that the drug is working exponentially in the body, how would your answer change?
Andrew: Exponentially cannot work here, because it just says this much left, or this much remains after the one hour.
Interviewer: What do you expect from a problem for model exponentially
Andrew: I think, special kind of increase or decrease

It turns out that "special kind of increase and decrease" is related to dependency to the previous step that I defined as recursive dependency.

Discussion

The analysis of Tom and Andrew revealed prominent patterns of logical readouts in the situation, coordinating context to reasoning, experiencing productive disequilibrium, accommodating new understanding of the situation, and finally, generalizing problems to make abstract models. During their reasoning about the contextual problems, both learners attempted to create a coordination between the context he perceived and a strategic pattern of thinking to solve the problem.

In the city A problem, there was a moment, in the first interview, when Tom could have made a generalization, but instead he faced a huge **conceptual conflict (disequilibrium)** with a previous normative and settled way of thinking: proportional reasoning. Andrew, however, experienced this productive disequilibrium majorly in the city B problem. Eventually, playing genuinely just with context helped Andrew to face a multiple disequilibrium to go through an **assimilatory activation** and, ultimately, accommodate a new reality which was **concept projection of exponential function.** Here, the strength of the context can be illuminated as the most important key, which can shadow the importance of similarity.

I conjecture based on the results of both interviews that if the context could have provided enough support for exponential thinking, perhaps students also could have moved through a possible conceptual conflict and accommodate the desired model to the context.

However, when the context was weak in establishing a recursive dependency, learners strongly coordinated the proportional reasoning and modeled the situation linearly.

To elaborate, the learners' mathematical behavior demonstrated a natural product of amplified understanding in the analysis, which is the core of transfer. Understanding a concept is, indeed, inseparable from developing logical readout strategies and **coordination of knowledge resources**. The analysis of Tom's and Andrew's exponential understanding is thus consistent with Wagner's (2006; 2010) Transfer-in-Pieces, where readout strategies and coordination of knowledge resources were matched with affordances of the contexts. After analyzing the interviews, I argue that the Remaining-Drug problem failed to afford recursive thinking. That is, the focus of the situation on the quantity "remaining," "staying," or even "half-life" could not create understanding of the amount of the remaining drug in each hour depending on the amount of the remaining drug in the previous hour. However, in contrast, the growing population of City A demonstrated a well-imbedded recursive dependency and afforded a dependence of the increased population to the previous increased population.

Implications

Such revelations about the interplay between contexts and the knowledge resources participants activate and use, is an important line of work that has practical relevance with respect to teaching and learning. Such analyses underscore the need for more epistemological investigations across disciplinary topics that can lend insight into the organization of learners' perceptions and inferences.

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Students' Enumerations of Tiles within a Rotation-Based Tiling of the Plane: Juxtaposing Units Coordination and Spatial Structuring Perspectives

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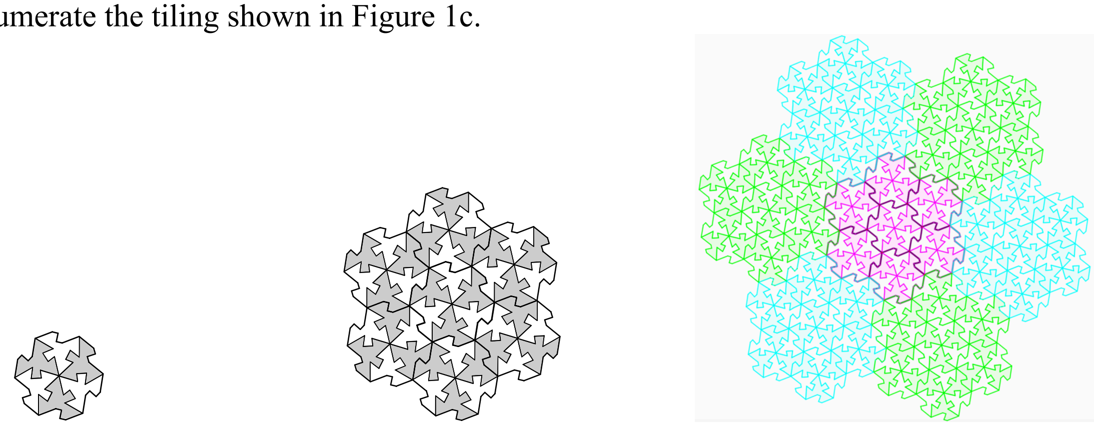
This report contributes novel insights into how undergraduate students think and reason about the number of tiles within a given tiling of the plane. We juxtapose two theoretical perspectives—units coordination and spatial-temporal-enactive (or S-) structuring—to provide an analytic account of how two undergraduate students reasoned about the tiling. In particular, this work suggests potential qualitative differences in how undergraduate students might engage in multiplicative reasoning in an unfamiliar spatial context.*

A tiling (or tessellation) is a collection of spatial structures that have been arranged in such a way that (a) there are no gaps or overlaps between adjacent structures, and (b) if the collection were repeated indefinitely, it would cover the entire plane. This report focuses on the tiling shown in Figure 1c. Few empirical studies have examined how students make sense of and reason about numerical and spatial units within tilings of the plane. Prior research has examined elementary school students' constructions of spatial tiling units in relation to their constructions of numerical units (e.g., Reynolds & Wheatley, 1996; Wheatley & Reynolds, 1996). Other research has focused on tilings from the perspective of mathematical aesthetics (e.g., Eberle, 2014, 2015). However, to date, the research literature has not yet captured how students, particularly those operating at higher levels of units coordination, form spatial and numerical units to enumerate tiles within tilings of the plane. In this report, we contribute initial insights toward this direction. Specifically, using the perspectives of units coordination and S*-structuring, we aimed to understand what multiplicative reasoning undergraduate students might use to enumerate the tiling shown in Figure 1c.



(a) (b) (c)

Figure 1. (a) Six-tile structure (6TS), (b) 42-tile structure (42TS), and (c) 294-tile structure (294TS)

Conceptual Framework

Reasoning about Numerical Units

The theory of units coordination constitutes the first component of our conceptual framework. Grounded in Steffe's work (e.g., Steffe, 1992, 1994), and subsequently developed by

Hackenberg, Norton, and colleagues into a three-stage[1] theory of students' multiplicative concepts (Hackenberg, 2010; Hackenberg & Tillema, 2009; Norton et al., 2016) and related forms of additive reasoning (Ulrich, 2015, 2016), units coordination posits that students gradually develop increasingly sophisticated ways of approaching (i.e., assimilating) additive and multiplicative situations. Stage 1 students can assimilate problem situations using units of one, and they can use units of one to actively build a unit of units (i.e., a level 2 unit) out of perceptual or imagined material. Stage 2 students can assimilate problem situations using both units of one and level 2 units; they can use these unit structures to actively build a unit of units of units (i.e., a level 3 unit) out of perceptual or imagined material. One indicator of Stage 2 reasoning is a tenuous or fleeting awareness of a three-levels-of-units structure. Stage 3 students can assimilate problem situations using three levels of units. That is, they can anticipate, at assimilation, the multiplicative structure of a situation, and they can reliably and flexibly use three levels of units in reasoning.

To illustrate how students at each stage might reason within a multiplicative context, consider the Bars Task (adapted from Norton et al., 2015). This task was presented to the students discussed in this report. Each student was shown a large bar, a medium bar, and a small bar. Placed end-to-end, three medium bars were of the same length as one long bar, and four small bars were of the same length as one medium bar (all bars had equal width). Stage 1 students can characteristically iterate the small bar (using number word utterances or motor-kinesthetic pulses) to determine how many small bars make up one large bar (12). Stage 2 students can iterate the medium bar, with each iteration representing a unit of four units (small bars). Stage 3 students can immediately determine that 12 small bars make up one large bar, with the understanding that the large bar consists of four medium bars, and each medium bar is equivalent to three small bars.

Reasoning about Spatial and Combinatorial Units

To date, little research has examined students' units coordinating structures within spatial/geometric contexts, with some notable exceptions. Wheatley and Reynolds (1996), for instance, found strong parallels between elementary school students' constructions and coordinations of numerical and spatial units. While prior research identified the unitization action as fundamental to the construction of *numerical* concepts (Steffe, 1992; Steffe et al., 1983; von Glasersfeld, 1982), Wheatley and Reynolds also found this action, in addition to mental imagery, to be critical within *spatial* contexts.

Outside of but related to the literature on units coordination, substantial research has aimed at understanding the mental operations and strategies that students use when engaged in tasks for enumerating spatial structures in contexts of geometric measurement (e.g., Barrett et al., 2017; Battista, 2004, 2007; Battista & Clements, 1996; Battista et al., 1998; Clements, 2004; Cullen et al., 2018; Smith & Barrett, 2018). We define the term *spatial unit* to mean a perceived or conceived object in space that is conceptualized as a singular entity (through an act of unitization). How one reasons about spatial units depends on how the individual mentally organizes, or structures, those units. Battista and Clements (1996) defined *spatial structuring* as "the mental act of constructing an organization or form for an object or set of objects" (p. 282). Without a viable spatial structuring to guide their enumerations of spatial units, students may experience difficulty keeping track of spatial units, which can lead to double-counting, or missing, some spatial units. Battista et al. (2018) defined *spatial-numerical linked structuring*

[1] Hackenberg's usage of the term "stage" is consistent with von Glasersfeld and Kelley's (1982) definition.

(SNLS) as a form of numerical reasoning that is linked to one's spatial structuring. From the perspective of units coordination, Zwanch et al. (in press) found that students with the second multiplicative concept (comparable to Stage 2; see Ulrich, 2015) may experience difficulty maintaining an awareness of a three-levels-of-units structure in geometric contexts, such as conceptualizing a strip of paper as a unit, each containing 7 units (7 inches), which each contain 2.5 units (roughly 2.5 cm per inch). Thus, enumerating spatial units depends on one's spatial structuring of the relevant spatial information *and* on one's available units coordinating structures. We hypothesize that the former may depend on the latter—that is, how one spatially organizes units depends on their current additive and multiplicative concepts.

In our research on students' combinatorial reasoning (Antonides & Battista, 2022), we elaborated Battista and Clements' (1996) construct of spatial structuring, defining a new construct called *spatial-temporal-enactive structuring* (or *S*-structuring*). We also defined S*NLS as the analog to SNLS, using S*-structuring in place of spatial structuring. S*-structuring focuses on how students construct, organize, and operate on composite units (level 2 units) and composites of composite units (level 3 units). Antonides and Battista (2022) identified and illustrated two specific forms of S*-structuring. *Intra-composite structuring* entails organizing units into individual composite units (such as combinatorial outcomes)—that is, actively constructing level 2 units out of level 1 units. For instance, within our framework, students engage in intra-composite structuring when they apply a pairing operation (Tillema, 2013) to append objects together to form a combinatorial outcome (e.g., a shirt-pant "outfit," a two-card hand, or a three-cube tower). *Inter-composite structuring* entails organizing composite units into composites of composite units—that is, actively structuring level 3 units out of level 2 units. For instance, students might construct all six 3-object permutations without repetition, organized into three groups with two permutations (i.e., two level 2 composites) in each group.

Methodology

The data discussed in this report come from a larger study (Antonides, 2022) that examined five undergraduate students' reasoning about enumeration. Each student participated in one-on-one clinical interviews and teaching episodes. Pre- and post-assessments took the form of semi-structured interviews (Clement, 2000), and the remaining sessions with students took the form of teaching episodes. During teaching episodes, direct instruction was not the general pedagogical approach; rather, each student was posed with tasks, questions, and scenarios (e.g., digital environments) that we hypothesized, based on our current models of their mathematical understandings, would provoke an accommodation to their current ways of operating. Two tasks are relevant to the current report.

> *Rotational Tiling Task*: Write an expression for the number of tiles in the tiling shown here [students were shown the image in Figure 1c]. One tile is a shape like this [they were shown an example of one tile].
>
> *Extension Task*: If this tiling pattern were to continue, how many tiles would be in the next step of the tiling?

Data were collected remotely, via Zoom, in the Spring 2020 semester. Students used a laptop and an iPad in all video conference calls: the former to capture facial expressions and gestures, and the latter to capture students' inscriptions. Sessions were recorded to capture audio/video data.

This report focuses on the cases of Claire and Kira (pseudonyms). Both students were in their first year, were 18 years of age, and used she/her pronouns. Claire was an elementary education major enrolled in a first-semester content course for future teachers (focusing on number and operation), and Kira was a psychology major enrolled in an intermediate algebra course (a prerequisite for college algebra). All data pertaining to students' solutions to the Bars Task (see the Conceptual Framework) and to the Rotational Tiling Task were transcribed. Analytic memos and codes (Maxwell, 2013) were used to make inferences about the data. Particular attention was given to the types of numerical units that students seemed to construct, and on which they seemed to operate in order to build higher-level unit structures. We further identified moments in our data where students seemed to engage in S*-structuring, and we made inferences about the nature of students' S*-structuring.

Findings

Our analyses of the Bars Task suggested that all students were operating at units coordination Stage 3. Knowing that three medium bars make one large bar, and that four small bars make one medium bar, each student quickly reasoned that 12 small bars would make up one large bar, with the understanding that the large bar could be seen as three groups with four small bars in each group. It should be noted, however, that the Bars Tasks have been validated as a written instrument for assessing the units coordinating structures of middle school students (Norton et al., 2015), but not undergraduate students (including preservice teachers). To date, no such written instrument has been validated for the latter population.

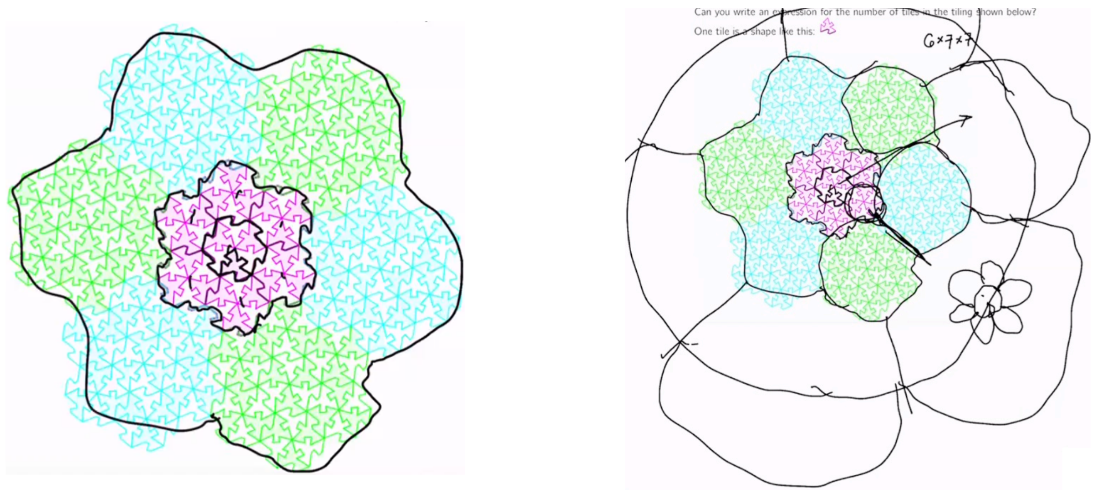
In describing our results and inferences, we use the terms 6TS to refer to a tiling structure consisting of six tiles (e.g., Figure 1a), 42TS to refer to a tiling structure consisting of 42 tiles (e.g., Figure 1b), and 294TS to refer to a tiling structure consisting of 294 tiles (e.g., Figure 1c), all with six-fold rotational symmetry.

The Case of Claire

Excerpt 1: Rotational Tiling Task. The interviewer read the Rotational Tiling Task to Claire, who immediately identified several embedded multiplicative group structures.

C: So, one of these. [She outlined a tile in the center 6TS; see Figure 2a.] The first time I would group those would be in this shape. [She outlined the entire 6TS.] And it looks like there's six of them. And then, the next grouping, I would say, is probably this shape. [She outlined the center 42TS.] And, in this one, it looks like there's one, two, three, four, five, six, seven [6TS structures]. So I'd probably say six times seven. … And then, I would do the same thing, but with this. [She outlined the entire 294TS.] And, um, I would define groups by the different colors. So this one [the center 42TS], in general, looks pretty pink. This one green, blue, and they all appear to be the same shape. And this also has seven [copies of 42TS in 294TS; see Figure 2a].

Claire's final answer to the Rotational Tiling Task was the written expression $6 \times 7 \times 7$.



(a) (b)

Figure 2. (a) Claire's illustration of four levels of units, and (b) her extension of the tiling pattern

Excerpt 2: Extension. The interviewer then read the Extension Task to Claire.

C: So, it would expand outward, I think. … I think there would be another ring, like this. So since this has, like, a *seam*, almost, with each, like, circle here [she drew a line segment separating two exterior 42TS in the original figure; see Figure 2b], I would probably say the same thing. It'd be like that… and another seam on each sort of thing.

Claire drew six large round figures along the exterior of the original tiling figure.

C: And then, I guess within each one, it would be all the levels before it. So, let's see. They would all start with an individual, and then a circle [6TS], and then this [42TS]. And then, like, this part [294TS]. … I guess I could probably copy it.

She then modified her multiplicative expression to $(6 \times 7 \times 7) \times 7$.

Analysis. In Excerpt 1, Claire established a multiplicative relationship between single tiles and one 6TS, constructing a level 2 unit (an instance of intra-composite structuring). She realized one 42TS could be made by iterating 6TS seven times, meaning she constructed 42TS as a level 3 unit. We interpret this as an instance of intra- and inter-composite structuring since she established a new unit as an organization of composite units. She expressed the number of tiles within one 42TS as 6×7. Finally, she established the entire image (one 294TS) as a level 4 unit structure (another instance of intra-/inter-composite structuring). To count the total number of tiles, she operated on her prior multiplicative expression, producing $(6 \times 7) \times 7$.

Claire's robust spatial/multiplicative reasoning becomes especially salient in the Extension Task. In this, she introduced the notion of a "seam" and used this concept to spatially structure units of 42TS within the 294TS. She then imagined, in mental activity, iterating units of 294TS along each seam, producing a new (level 5) unit structure—an instance of intra-/inter-composite structuring. Further, made clear in her verbal explanation and drawing in Figure 2b, she maintained the unit structures nested within each iteration of the 294TS.

The Case of Kira

Excerpt 3: Rotational Tiling Task. Kira approached the Rotational Tiling Task by first focusing on and counting the number of larger groups (42TS) in the figure.

K: So, I'm thinking I could count the number of tiles in each group. And then, um, multiply that by seven. And it would tell you the total number of tiles altogether.

To count the number of tiles in a larger grouping, she focused on the center 42TS. She counted six tiles in one smaller grouping (6TS), and she counted seven 6TS in the central 42TS.

K: So, six times seven, is 42. So, it looks like there's 42 tiles in one group. So… seven times 42, I feel like… would tell the number of tiles… tiles altogether? But just to be sure, I'm going to count the number of tiles in one of them, on the outside, because I don't want to just make that assumption about all of them.

To count the number of tiles in another 42TS, Kira shifted her focus to a 42TS on the right side of the tiling figure. She counted six tiles within one 6TS, and she counted seven 6TS within the grouping. She said $6 \times 7 = 42$, which she said was the same number of tiles that she found before. Feeling more confident that there are 42 tiles in each larger grouping, she concluded there are 42×7 tiles in the entire tiling. Using a calculator, she found the product to be 294.

Excerpt 4: Extension. Kira was then posed with the Extension Task.

K: So, like, would there be six more on the outside?

Int: Six more what?

K: Of the groups I've created, like the one, two—it looks like there's a group of 42. So, would there be six more of those groups on the outside, kind of?

Kira drew a picture, showing six 42TS structures added along the outside of the tiling figure, wedged between pairs of green and blue 42TS structures (see Figure 3).

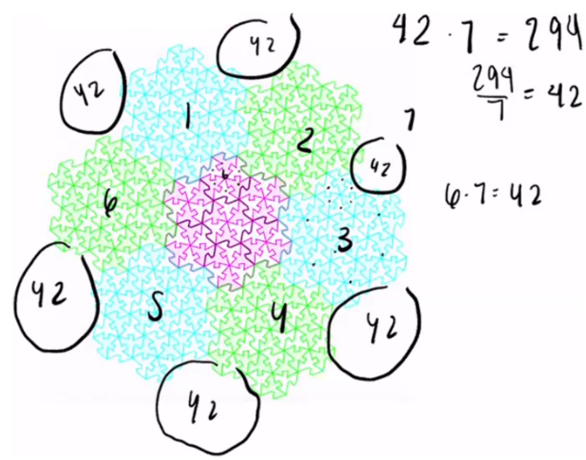


Figure 3. Kira's illustration of how the tiling pattern would continue

K: Now that I'm looking at the image again, I notice that it's like, odd, which is bothering me. So I don't know if there would be another circle somewhere else. But, I'm guessing since it's the same… number? Like the pattern of seven would continue.

Kira seemed to notice that there were seven 42TS in the original tiling, with seven 6TS in one 42TS, and from this she seemed to expect the "pattern of seven would continue." However, unlike Claire who conceptualized the original tiling image as a unit, and who iterated that unit so that seven copies were represented on the screen, Kira anticipated a seventh additional 42TS would need to be drawn, though she was unsure of where to draw it.

K: So I don't know where that one would go when it would continue after that. But, not looking at the image, I would just, um, take my original problem, 42 times 7, and make it 42 times 14? And I feel like that would give the number of tiles in total.

Int: Okay. Where's the 14?

K: Um. So the 14 came from seven plus seven, the number of tiles, if it would be—like this is me not looking at the image. Just, in my head. Since there were seven, I guess it would be, um, another seven. So 14. And there's 42 tiles in each group … 588.

Analysis. In Excerpt 3, Kira conceptualized the entire tiling as a level 3 unit structure. This was indicated by her strategy: count all tiles in one larger grouping (42TS), then multiply by the number of larger groupings (7). In counting the tiles in 42TS, Kira realized it consisted of seven smaller groupings (6TS), and so she multiplied 6×7, and thus she established 42TS as a level 3 unit. Unlike Claire who seemed to immediately anticipate a level 3 unit structure, Kira seemed to establish this unit structure in activity. She planned to operate on this level 3 unit by multiplying it by seven, but she expressed uncertainty. She became more confident only after performing 6×7 to count tiles in another 42TS. Even after she multiplied $42 \times 7 = 294$, finding the number of tiles in the original tiling image, she felt the need to check her answer by applying the inverse operation, $294 \div 7 = 42$, shown in Figure 3.

In Excerpt 4, Kira first extended the tiling pattern by constructing a spatial structuring with six additional 42TS. However—presumably noticing that there were seven 42TS in the original image—Kira anticipated that seven, rather than six, additional 42TS should be added in the extension, though she could not determine where to place the seventh structure. From this, she suggested there would be 42×14 tiles in the extended pattern. Kira, unlike Claire, did not seem to imagine a level 4 or level 5 unit structure. Rather, she created additional level 2 units (42TS) to create a larger level 3 unit.

Discussion and Conclusions

In this report, we set out to examine two undergraduate students' reasoning about the enumeration of tiles within a tiling of the plane using two analytical perspectives: units coordination and S*-structuring. While much work still needs to be done toward elaborating the relationship between these two perspectives (an avenue for future research), we believe this report offers initial steps toward this goal.

Claire established the tiling image in Figure 1c as a level 4 unit, clearly articulating and illustrating the multiple levels of units that she abstracted. She made several acts of intra-composite structuring and linked intra-/inter-composite structuring along the way, establishing 6TS as a level 2 unit, 42TS as a level 3 unit, and ultimately 294TS as a level 4 unit. Note that each level of unit is described as such because Claire could think about the four levels of multiplicative relationships that she built up to establish 294TS. She extended the tiling pattern by taking the 294TS as an object that she could iterate, guided by her spatial structuring of the original figure that was mediated by her notion of a "seam." Overall, Claire demonstrated S*NLS reasoning of a student who could establish, interrelate, and iterate four levels of units to construct a level 5 unit structure.

Kira enumerated tiles in the original tiling by establishing it as a level 3 unit—a unit of seven units, each containing 42 units. She separately counted each 42TS as a unit of six units, but she did not maintain this relationship, and it was not used in her enumeration of tiles in the tiling. In the Extension Task, Kira's S*NLS was distinguished from Claire's since her S*-structuring of the extended tiling was a level 3 unit—a unit of 13 (or 14) level 2 units—rather than an iterated level 4 unit.

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“Just Get Rid of It:” Students’ Symbolic Forms for Cancellation

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Students use symbolic manipulations when performing cancellation as they solve equations. Rather than performing rote manipulation of symbols, we want students to understand the meanings that these symbols convey. In this paper, we examine three pre-service mathematics teachers’ symbolic forms for additive, multiplicative, and compositional cancellation. Students used symbol templates where they either wrote both the identity and inverses, one but not the other, or neither. We further explore students’ varied conceptual schemas associated with these templates, as well as the relationships that exist among these symbolic forms.

Keywords: Abstract Algebra, Cancellation, Symbolic Forms, Prospective Teachers

Symbols are ubiquitous throughout all mathematics. Instructors typically want their students to use symbols with an understanding of the meanings conveyed by the symbols (e.g., Sherin, 2001). Students’ reasoning about symbols can be analyzed using Sherin’s (2001) symbolic form construct, which is the association of the symbols one writes and their understanding of the meanings those symbols convey. Students’ symbolic forms are present in a variety of contexts, particularly when they reason with symbols while performing cancellation in secondary algebra and abstract algebra contexts. When students cancel terms, they might conceptualize this procedure as “getting rid of terms” or “moving terms to the other side.” This process of cancellation is much more complex than this, as it involves the use of the inverse group axiom, defined as for all elements $a \in G$ where $(G,*)$ is a group, there exists an inverse element $b \in G$, such that $a * b = b * a = e$, where e is the identity element in G. For example, when solving the equation $x + 3 = 1$, one adds the additive inverse of 3 to both sides of the equation to obtain $x + (3 + -3) = 1 + (-3)$, which implies that $x + 0 = -2$. This point of connection that prospective secondary mathematics teachers (PSMTs) can make between abstract algebra and secondary algebra has been the focus of several research studies (e.g., Murray & Baldinger, 2018; Serbin, 2021; Serbin, 2022; Wasserman, 2014). Wasserman (2014) suggested that teachers’ understanding of algebraic structures could support them in “unpacking the interconnected role of inverse and identity elements in the ‘cancellation’ or ‘simplification’ process” (p. 204). We refer to the process of adding an element’s additive inverse to that element to yield the additive identity 0 (e.g., $3 + -3 = 0$) as additive cancellation. We similarly refer to the process of multiplying an element by its multiplicative inverse to yield the multiplicative identity 1 (e.g., $3 \cdot 1/3 = 1$) as multiplicative cancellation. We also refer to the process of composing a function with its inverse to yield the identity function (e.g., $e^{\ln(x)} = i(x) = x$) as compositional cancellation. We hypothesized that students might have similar symbolic forms for these different types of cancellation, as they all involve the explicit or implicit use of inverse and identity elements. We explore the following research question: What symbolic forms do students have for cancellation involving additive cancellation, multiplicative cancellation, and compositional cancellation, and what relationships exist between these symbolic forms?

Literature Review and Theoretical Background

Symbols are used for a variety of purposes in mathematics, some of which include communicating meaning, notating concepts and processes, and illustrating relationships between mathematical objects (Zandieh et al., 2017). Researchers have focused on students' symbol sense (Arcavi, 1994) and their practice of symbolizing, which is broadly defined as reasoning with inscriptions (Zandieh & Andrews-Larson, 2019). Some of this work has addressed students' symbolization in calculus (e.g., Jones, 2013) and linear algebra contexts (e.g., Henderson et al., 2010; Smith et al., 2022; Zandieh & Andrews-Larson, 2019). Reasoning with symbols involves both creating symbols to convey meaning and interpreting the meaning in inscriptions. Sherin (2001) suggested, "The particular arrangement of symbols in an equation expresses a meaning that can be understood" (p. 480). Sherin claimed that students should understand the meanings that the symbols and manipulations convey: "We do not want meaningless symbol manipulation; if students use symbolic expressions, we want them to use the symbols with understanding" (Sherin, 2001, p. 479). However, this is a nontrivial task for students. Furthermore, students often have different interpretations of symbols' meanings (e.g., Jones, 2013). Reasoning with symbols is a central activity in mathematics, so students should have productive understandings of the meanings symbols convey and draw on them as they perform symbolic manipulations.

Students use symbolic manipulations while performing algebraic cancellation as they simplify expressions and solve equations. Instead of performing rote cancellation without attending to the meaning of the symbols or manipulations, students should understand the properties used to manipulate symbols in equations. The properties that allow one to cancel terms while solving equations are the inverse and identity axioms of groups and rings. Wasserman (2014) suggested that understanding these algebraic structures could support teachers in "unpacking the interconnected role of inverse and identity elements in the 'cancellation' or 'simplification' process" (p. 204). Serbin (2021; 2022) found that reasoning about group and ring axioms helped PSMTs understand the shared structure of additive, multiplicative, and compositional identities and inverses. Interpreting the symbols and symbolic manipulations used while canceling terms may involve making sense of how the inverse elements are operated together to yield the identity. In this study, we focus on students' understandings associated with their symbols used in additive, multiplicative, and compositional cancellation. To do so, we leverage Sherin's (2001) construct of symbolic forms.

Symbolic forms (Sherin, 2001) are cognitive resources (Hammer, 2000) that students can activate as they do mathematics. A *symbolic form* is an association of a conceptual schema with a symbol template. A *conceptual schema* is the meaning that symbols convey, and the *symbol template* is the structure or arrangement of written symbols that expresses that meaning. Sherin asserted students develop symbolic forms by associating meaning with the structures symbolized in equations. This association can occur as students write symbolic expressions or interpret them. One symbol template can be associated with several conceptual schemas, and one conceptual schema may be associated with multiple symbol templates (e.g., Jones, 2013). The template-schema pair that students choose to activate while solving a problem depends on the framing or context of the situation (Hammer et al., 2005). We investigate the relationships between the symbolic forms that students activate for the three different types of cancellation.

Methods

Three PSMTs, Amelia, Christina, and Derek, who majored in Mathematics, participated in this study. Each was enrolled in a senior-level Mathematics for Secondary Teachers course at a large university in the US. During the course, the students learned how the properties of groups and rings were used to justify the procedures used to solve equations. Two individual 60-minute task-based clinical interviews (Clement, 2000) were conducted with each of the PSMTs. These two interviews were designed to elicit evidence of the PSMTs' understanding of the content covered in the two course instructional units. The first interview was conducted after the unit on ring properties and equation solving. The second was conducted after the unit on functions and inverses. Some of the interview tasks prompted students to solve equations and describe the properties used and show that two functions are inverses. Given the semi-structured nature of these interviews, the PSMTs were all given the same tasks but were asked different unplanned follow-up questions based on their responses. Both interviews were conducted virtually and transcribed. Copies of the PSMTs' written work from their task responses were also collected.

We analyzed the transcripts and task responses to identify the symbolic forms the PSMTs used to perform cancellation while solving equations. We first classified the type of cancellation according to the operation involved: additive, multiplicative, or compositional. We then inductively coded (Miles et al., 2013) the transcripts by assigning each instance of cancellation present a code that captured the essence of the symbol template and a code that captured the essence of the conceptual schema evident in the data. Each task response contained multiple codes. Furthermore, there were cases in which multiple conceptual schemas were associated with one symbol template, so each of those symbol template-conceptual schema pairs was coded separately. We coded the symbol templates and conceptual schemas together to reach an agreement on the assignment of each code. After, we performed axial coding to group the symbolic forms into broader categories. We identified four axial codes that captured the essence of every symbol template used in these PSMTs' written work of canceling terms: the participant wrote (a) both the identity and inverses, (b) the inverses but not the identity, (c) the identity but not the inverses, and (d) neither the identity nor the inverses. We lastly examined which relationships exist between these symbolic forms. For each participant, we identified the different conceptual schemas that were associated with each of the four symbol template categories. We wrote analytic memos (Maxwell, 2013) to describe patterns and inconsistencies that were present in each participant's use of symbolic forms representing the different types of cancellation and used these to compare the symbolic forms used by each student.

Results

As follows, we describe the four types of symbol templates for cancellation. The symbolic forms for each type of template are presented in Figures 1-4. In the symbol templates, { } refers to an expression with multiple terms, [ ] refers to a monomial, c_i refers to constants, a, b, c refer to set elements, f, f^{-1}, g refer to functions, and $\Longrightarrow$ refers to an implication.

No Attention to Inverses or Identity

All students used symbol templates for multiplicative and additive cancellations that did not contain symbols for the inverses and identity (see Figure 1). Amelia's conceptual schemas were focused on algebraic operations rather than on the structure of operating inverses together to get an identity. She used her conceptual schemas to justify "getting rid of terms" but did not focus on the inverse-identity structure. Christina also seemed to have a computational focus. An exception was when she referred to using multiplicative inverses, however, she only used the terminology to justify removing or "getting rid of" a coefficient. She was the only student who did not write the inverse or identity elements when performing compositional cancellation. Her conceptual schema in this case was still focused on the procedure of doing something to both sides of the equation. Derek demonstrated an understanding of the structure of operating inverse elements together to obtain the identity, even when he did not write out inverse or identity elements. Overall, students tended to ignore inverses and identity when getting the answer or simplifying expressions, moving terms over, getting rid of terms, or operating on both sides of the equation.

Attention to Inverses but Not Identity

Symbol templates containing inverses but not the identity occurred in all cancellations. Students attended to inverses but ignored the identity when doing something to both sides of the equation or getting rid of terms (see Figure 2). Amelia's conceptual schemas focused primarily on the operations that needed to be performed, rather than on the structure of operating inverses together to get an identity. However, when performing compositional cancellation, she used the "operate inverse elements together to get identity" conceptual schema to explain why e and ln cancel. Christina's conceptual schemas were focused on doing something to both sides of the equation to get rid of terms. Derek's conceptual schemas focused on the inverse-identity structure even when his symbol template did not contain the identity. This was how he justified simplifying, getting rid of terms, or doing something to both sides of the equation.

Attention to Identity but Not Inverses

Students used symbol templates that contained the identity but not inverses while performing additive cancellation (see Figure 3) when the additive identity was the only term left on one side of the equation. Amelia used this template when doing something to both sides of the equation.

	Symbol Template	Conceptual Schema	Student who Used Form
Multiplicative Cancellation	1. $c_1\{\ \} = c_2\{\ \} \Rightarrow \{\ \} = c_3\{\ \}$	Move over	Amelia
	3. $c_1\{\ \} = c_2 \Rightarrow \{\ \} = c_3$	Do something to both sides	Derek
		Use inverses (no attention to identity)	Derek
	4. $\frac{[\ ]}{\{\ \}} = [\ ] \Rightarrow [\ ] = [\ ]\{\ \}$	Do something to both sides	Amelia
		Get rid of something	Amelia
		Get the answer/simplify	Amelia
	8. $\frac{[\ ]}{[\ ]\{\ \}} + \frac{c_1\{\ \}}{[\ ]\{\ \}} = \frac{c_2}{[\ ]\{\ \}} \Rightarrow [\ ] + c_1\{\ \} = c_2$	Do something to both sides	Amelia
		Get rid of something	Amelia
		Get the answer/simplify	Amelia
		Use inverses (no attention to identity)	Amelia
	11. $c_1\{\ \} = c_2 \Rightarrow \{\ \} = \frac{c_2}{c_1}$	Get rid of something	Christina
		Use inverses (no attention to identity)	Christina
	15. $c_1[\ ] = c_2 \Rightarrow [\ ] = c_3$	Do something to both sides	Derek
		Get the answer/simplify	Amelia, Christina
		Use inverses (no attention to identity)	Derek
		Operate inverses together to get identity	Derek
Additive Cancellation	2. $[\ ] + c_1 = [\ ] + c_2 \Rightarrow [\ ] + [\ ] = c_2 - c_1$	Move over	Christina
	4. $[\ ] + c_1 = c_2 \Rightarrow [\ ] = c_3$	Move over	Derek
		Do something to both sides	Derek
		Get the answer/simplify	Amelia, Christina, Derek
		Use inverses (no attention to identity)	Derek
	14. $\{\ \} + c_1 = c_2 \Rightarrow \{\ \} = c_3$	Move over	Christina, Derek
		Get the answer/simplify	Amelia
		Uses inverses (no attention to identity)	Derek
Compositional Cancellation	2. $\{\ \}^k = c_1 \Rightarrow \pm\{\ \} = c_1$	Do something to both sides	Christina

Figure 1. Symbol Templates Containing No Inverses and No Identity

	Symbol Template	Conceptual Schema	Student who Used Form
Multiplicative Cancellation	2. $c_1\{\ \} = c_2 \Rightarrow \frac{c_1\{\ \}}{c_1} = \frac{c_2}{c_1} \Rightarrow \{\ \} = c_3$	Move over	Amelia
	6. $\{\ \} \cdot \frac{[\ ]}{\{\ \}} = \{\ \} \cdot [\ ] \Rightarrow [\ ] = \{\ \} \cdot [\ ]$	Do something to both sides	Derek
		Get rid of something	Derek
		Operate inverses together to get identity	Derek
	7. $c_1[\ ] = c_2 \Rightarrow \frac{1}{c_1} \cdot c_1[\ ] = c_2 \cdot \frac{1}{c_1} \Rightarrow [\ ] = c_3$	Use inverses (no attention to identity)	Derek
	9. $\left(\frac{c_1}{\{\ \}} + \frac{c_2}{[\ ]} = \frac{c_3}{[\ ]\{\ \}}\right) \cdot ([\ ]\{\ \}) \Rightarrow [\ ] + c_2\{\ \} = c_3$	Do something to both sides	Christina
		Get rid of something	Christina
	10. $[\ ]\{\ \} \cdot \frac{\{\ \}}{[\ ]\{\ \}} = \frac{c_1}{[\ ]\{\ \}} \cdot [\ ]\{\ \} \Rightarrow \{\ \} = c_1$	Do something to both sides	Derek
		Get rid of something	Derek
	16. $c_1[\ ] = c_2 \Rightarrow \frac{c_1[\ ]}{c_1} = \frac{c_2}{c_1} \Rightarrow [\ ] = c_3$	Do something to both sides	Amelia
Additive Cancellation	1. $[\ ] + c_1 = [\ ] + c_2 \Rightarrow [\ ] + c_1 \pm [\ ] \pm c_1 = [\ ] + c_2 \pm [\ ] \pm c_1$	Move over	Amelia
		Do something to both sides	Amelia
		Get rid of something	Amelia
	3. $\{\ \} = [\ ] + c_1 \pm [\ ] \Rightarrow \{\ \} = c_1$	Move over	Derek
		Do something to both sides	Derek
		Use inverses (no attention to identity)	Derek
		Operate inverses together to get identity	Derek
	6. $[\ ] + c_1 = 0 \Rightarrow [\ ] + c_1 \pm c_1 = 0 \pm c_1$	Do something to both sides	Derek
		Use inverses (no attention to identity)	Derek
	9. $[\ ] + c_1 \pm c_1 = c_2 \pm c_1 \Rightarrow [\ ] = c_3$	Move over	Christina
	10. $[\ ] + c_1 \pm c_1 \Rightarrow [\ ]$	Get the answer/simplify	Amelia, Derek
	12. $\{\ \} + c_1 \pm c_1 = c_2 \pm c_1 \Rightarrow \{\ \} = c_3$	Use inverses (no attention to identity)	Christina
	15. $\{\ \} + c_1 \pm c_1 \Rightarrow \{\ \}$	Get the answer/simplify	Amelia
	16. $[\ ] + c_1 = c_2 \Rightarrow [\ ] + c_1 \pm c_1 = c_2 \pm c_1$	Move over	Amelia
		Do something to both sides	Amelia
	17. $\{\ \} + c_1 = c_2 \Rightarrow \{\ \} + c_1 \pm c_1 = c_2 \pm c_1$	Move over	Amelia
		Get rid of something	Christina
Compositional Cancellation	1. $\{\ \}^k = \left(\{\ \}^{\frac{1}{k}}\right)^k \Rightarrow \{\ \} = \{\ \}$	Do something to both sides	Christina
		Get rid of something	Christina
	3. $\{\ \}^k = \left(\sqrt[k]{\{\ \}}\right)^k \Rightarrow \{\ \} \cdots \{\ \}$ (k times) $= \{\ \}$	Do something to both sides	Derek
	4. $\left(\sqrt[k]{\{\ \}}\right)^k = (c_1)^k \Rightarrow \{\ \} = c_2$	Do something to both sides	Amelia, Derek
		Get rid of something	Derek
		Get the answer/simplify	Derek
		Use inverses (no attention to identity)	Amelia
	5. $\left(\{\ \}^{\frac{1}{k}}\right)^k = (c_1)^k \Rightarrow \{\ \} = c_2$	Get rid of something	Christina
		Get the answer/simplify	Christina
	6. $\ln\{\ \} = c_1 \Rightarrow e^{\ln\{\ \}} = e^{c_1}$	Do something to both sides	Amelia
		Operate inverses together to get identity	Amelia
	7. $\{\ \}^k = c_1 \Rightarrow \sqrt[k]{\{\ \}^k} = \sqrt[k]{c_1}$	Do something to both sides	Amelia
		Get rid of something	Amelia
	8. $e^{\ln(c_1)} = c_1$	Get rid of something	Derek
		Use inverses (no attention to identity)	Amelia
	11. $\left(\sqrt[k]{[\ ]}\right)^k = [\ ]$	Get the answer/simplify	Amelia, Derek
	12. $\sqrt[k]{\{\ \}^k} = \{\ \}$	Get the answer/simplify	Amelia, Derek

Figure 2. Symbol Templates Containing Inverses but Not Identity

However, her conceptual schema did not involve any attention to the identity. Christina and Derek did not attend to the identity when canceling except when they wrote 0 on one side of the equation to not leave it blank. However, even when Derek did not write out inverses, his conceptual schemas referenced the structure of operating inverses together to get the identity.

	Symbol Template	Conceptual Schema	Student who Used Form
Additive Cancellation	7. $\{\} = c_1 \Rightarrow \{\} = 0$	Do something to both sides	Amelia
	11. $[\,] = \{\} \Rightarrow 0 = \{\}$	Operate inverses together to get identity	Derek
	13. $\{\} = \{\} \Rightarrow \{\} = 0$	Get the answer/simplify	Derek
		Use inverses (no attention to identity)	Derek
	18. $\{\} = \{\} \Rightarrow \{\} - \{\} = 0$	Get the answer/simplify	Christina

Figure 3. Symbol Templates Containing Identity but Not Inverses

	Symbol Template	Conceptual Schema	Student who Used Form
Multiplicative Cancellation	5. $\frac{[\,]}{\{\}}\{\} = [\,]\{\} \Rightarrow [\,] \cdot 1 = [\,]\{\}$	Do something to both sides	Christina
		Get rid of something	Christina
		Operate inverses together to get identity	Christina
	12. $aa^{-1} = \left(\frac{b}{c}\right) \cdot \left(\frac{c}{b}\right) = 1$	Operate inverses together to get identity	Amelia
	13. $(ba)a^{-1} = (ca)a^{-1} \Rightarrow be = ce \rightarrow b = c$	Operate inverses together to get identity	Amelia, Derek
	14. $a \cdot a^{-1} = e$	Operate inverses together to get identity	Derek
Additive Cancellation	5. $\{\} \pm c_1 = c_1 \pm c_1 \Rightarrow \{\} = 0$	Move over	Christina
		Use inverses (no attention to identity)	Derek
		Operate inverses together to get identity	Christina
	8. $[\,] \pm [\,] = \{\} \pm [\,] \Rightarrow 0 = \{\}$	Move over	Amelia, Christina
		Operate inverses together to get identity	Christina
Compositional Cancellation	9. $f(g(x)) = \cdots = x$	Operate inverses together to get identity	Amelia, Derek
	10. $f(f^{-1}(x)) = x = f^{-1}(f(x))$	Operate inverses together to get identity	Amelia

Figure 4. Symbol Templates Containing Both Inverses and Identity

Attention to Both Inverses and Identity

When the students used cancellation symbol templates that included symbolizations of both inverses and the identity, their associated conceptual schemas focused on the structure of operating inverse elements together to yield the identity element (see Figure 4). Amelia exhibited the "operate inverse elements together to get identity" conceptual schema for multiplicative cancellation only when she was working on abstract algebra tasks. She did not attend to the inverses and identity while solving equations. Furthermore, her symbol template for compositional cancellation contained symbols for the inverses and identity only when verifying that two given functions are inverses of each other. On the other hand, when Christina's symbol templates for additive and multiplicative cancellation contained symbols for the inverses and identity, her conceptual schemas were focused on the inverse-identity structure, i.e., that operating inverse elements together yields the identity. In regard to Derek, when his symbol templates for multiplicative and compositional cancellation included both the inverses and identity, his conceptual schema also referred to the inverse-identity structure of operating inverses together to get the identity. When Derek's symbol template for additive cancellation contained symbols for the inverses and identity, his demonstrated conceptual schema only focused on using inverses, without explicit attention to the identity.

Discussion and Conclusion

We examined three students' symbolic forms for additive, multiplicative, and compositional cancellation. We found that the students used four general types of symbol templates, in which they wrote (a) both the identity and inverses, (b) the inverses but not the identity, (c) the identity but not the inverses, and (d) neither the identity nor the inverses. We then examined the varied conceptual schemas associated with these templates. Amelia only explicitly attended to both multiplicative inverse and identity in the context of abstract algebra tasks and functional inverse and identity only in tasks that required her to determine if two given functions are inverse pairs. However, she did not attend to the structure of both inverse and identity as she performed cancellation while equation-solving. Amelia's evoked conceptual schema seemed to depend on her framing of the task. Ticknor (2012) had a similar finding that teachers' understandings of commutativity, associativity, and inverses were only situated in abstract algebra settings. Christina's symbolic forms were mainly procedural in nature. She attended to the structure of both inverse and identity only for multiplicative and additive cancellation. Her conceptual schemas focused on the equation solving process, as evident in her explanations of moving terms over, doing something to both sides, or getting rid of a term. She did not use any symbol templates that contained symbols for both inverse functions and the identity function when she performed compositional cancellation. She thus did not focus on the structure of the composition of inverse functions yielding the identity function while performing that cancellation. Finally, Derek's symbolic forms always demonstrated an understanding of the structure of operating inverses together to get the identity. Even in cases where he did not write out the identity or inverses, it was clear that he was aware of the inverse-identity structure in the cancellation.

As Amelia's and Christina's conceptual schemas demonstrate, having an understanding of abstract algebra concepts does not necessarily imply students can automatically use it to make sense of secondary mathematics content like equation solving. In Amelia's case, she only attended to both multiplicative inverses and identity in the context of abstract algebra where she is more likely used to that terminology and reverted back to procedural equation-solving, without attention to the inverse or identity, in most other situations. Ideally, teacher educators want PSMTs to understand that operating an element with its inverse yields the identity and be able to use that in other situations. We hypothesize that solving equations while attending to the structure of the inverse, identity, and binary operation will allow students and PSMTs to develop meaningful understandings of the symbolic manipulations involved in equation solving.

Our findings have some implications for teaching and future research. Mathematics instructors who prepare PSMTs should prompt PSMTs to explicitly write out the symbols used to represent inverses and identities while solving equations. Instructors should also guide students to identify which group or ring properties are used while solving equations. This has been found to be a fruitful way for undergraduate students and PSMTs to connect abstract algebra and secondary algebra (e.g., Cook, 2012; Cook, 2015; Murray & Baldinger, 2018; Wasserman, 2014). We intend to extend this analysis of the relationships that exist among students' symbolic forms for different types of cancellation. Future research can explore how mathematics teacher educators can guide PSMTs to make connections across abstract and secondary algebra that are not only situated in abstract algebra contexts.

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“The Shape Will Have No Volume”: Relationships Students Observed about Determinants in a Dynamic Geometric Applet

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We discuss two Dynamic Geometry Software applets designed as part of an Inquiry-Oriented instructional unit on determinants and share students' generalizations based on using the applet. Using the instructional design theory of Realistic Mathematics Education, our team developed a task sequence supporting students' guided reinvention of determinants. This unit leverages students' understanding of matrix transformations as distortion of space to meaningfully connect determinants to the transformation as the signed multiplicative change in area that objects in the domain undergo from the linear transformation. The applets are intended to provide students with feedback to help connect changes in the matrix to changes in the visualization of the linear transformation and, so, to changes in the determinant. Critically, the materials ask students to make generalizations while reflecting on their experiences using the applets. We discuss patterns among these generalizations and implications they have on the applets' design.

Keywords: Determinants, dynamic geometry software, inquiry, student reasoning

Matrix determinants are often taught formulaically, obscuring geometric connections between determinants and linear transformations. The Inquiry-Oriented Linear Algebra (IOLA) curricular materials (Wawro et al., 2013) build from experientially real task settings that allow for active student engagement in the guided reinvention of mathematics through student and instructor inquiry (Gravemeijer, 1999). The IOLA determinants task sequence uses distortion of space as an experientially real starting point to support students’ development of determinant as a measure of (signed) multiplicative change in the area. The sequence uses GeoGebra applets that allow students to actively explore the geometric effects of changing 2x2 and 3x3 matrix transformations and note their effect on the determinant. Through exploration of the applet, students observe links between the determinant and concepts such as linear (in)dependence, span, and invertibility, as well as how changes in matrix entries impact the determinant. In this study, we investigate the research question: Based on interacting with our curriculum's dynamic geometry applets, what relationships do students observe between matrices, determinants, and geometric objects transformed by the associated linear transformation?

Theoretical Background and Literature Review

According to Axler (1995), “determinants are difficult, non-intuitive, and often defined without motivation” (p. 139). Aygor and Ozdag (2012) found that students have difficulty identifying what happens to the determinant when switching rows, multiplying a column by k, row-reducing a matrix, and performing column operations. This might be because of difficulties relating the numeric, algebraic, and geometric aspects of the determinant (Durkaya et al., 2011) or the reliance of the determinant on bilinearity (Donevska-Todorova, 2016). These difficulties would be mitigated by connecting to students’ existing reasoning and understanding, as is consistent with the curriculum design theory of Realistic Mathematics Education (RME; Freudenthal, 1991). Adopting this theoretical framing for our curriculum design, we identified students’ understanding of matrix transformations as distortions of space as an experientially real starting point for developing notions of determinant. We sought to design an activity that could support students in establishing an understanding of determinants as a measure of how matrix transformations distort space and then leverage these connections toward more general properties of determinants. We situate this work within our research group's design research (NSF DUE 1915156, 1914841, 1914793), referred to as a Design Research Spiral, in which the design team iteratively drafts, implements, reflects, and refines the task sequence as guided by multiple design research theories at each cycle of refinement (Wawro et al., 2022). This paper presents materials and results from the penultimate cycle of refinement for this unit.

In order to support students' generalizing from their situated activity, we turned to Dynamic Geometric Software (DGS), which synchronously display dynamic representations of numeric, geometric, and algebraic aspects of the determinant. This makes DGS a suitable tool for supporting students' exploration of determinants and connecting this to geometric and algebraic knowledge. research has shown that DGS applets allow students to investigate, visualize, make predictions, calculate, simulate, and generalize certain situations, all of which are critical practices for inquiry (Gol Tabaghi & Sinclair, 2013; Gol Tabaghi, 2014; Greefrath et al., 2018; Hollenbrands, 2007; Paoletti et al., 2020; Zandieh et al., 2018). In linear algebra, applets, games, and simulations have been used to explore key topics such as linear combinations (Mauntel et al., 2021), eigentheory (Gol Tabaghi & Sinclair, 2013; Gol Tabaghi, 2014), and determinants (Donevska-Todorova, 2012, 2016; Donevska-Todorova & Turgut, 2022).

Donevska-Todorova (2012) used DGS to support students' exploration of determinants in a teaching experiment setting and found that DGS can help build meaningful connections between the determinant and conceptions of area in 2D. The author attributes the DGS's ability to simultaneously display numeric and geometric feedback as important for connecting to students' prior geometric understanding. However, Donevska-Todorova's setting is more focused on direct instruction and approaches the determinant as a way to calculate the area of a parallelogram at the origin whose sides correspond to the column vectors of the matrix, rather than connecting the geometry to the linear transformation defined by the matrix. In our materials, and consistent with the above cited literature, we designed the materials so that students contextualize the DGS relative to their existing notions of linear transformations. The goal is that this additional context will provide students with a foundation for the connections they make while using our applet.

The Determinants Task Sequence

The main goal of this unit is to build from students' knowledge of matrices as representations of linear transformations towards a conceptualization of the determinant of a 2x2 or 3x3 matrix as a measure of (signed) multiplicative change in area or volume, respectively. Task 1 is designed to support students toward suggesting *change in area* as a way of quantifying distortion that objects undergo from specific matrix transformations. In Task 2, students reinvent the 2x2 determinant formula so that by the end of Task 2, when the term *determinant* is introduced, the class has developed the notion of the determinant of a 2x2 matrix A as: (1) the signed area scaling factor of the linear transformation defined by A [i.e., the multiplicative change in area]; (2) the signed area of the unit square's image under A; (3) the signed area of the parallelogram created by the columns of A; and (4) $det(A) = ad - bc$, for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The main goal of Task 3 is for students to reinvent $det(A^{-1}) = 1/det(A)$ for invertible matrices by coordinating their knowledge of invertible linear transformations with their developing understanding of determinants as a measure of change in area. Tasks 1 and 3 provide space for students to use their understanding of composition of functions to reinvent $det(AB) = det(A)det(B)$ for matrices A and B and justify them via change in area lines of reasoning.

In Task 4, students further explore the geometric interpretation of matrix transformations and their determinants via GeoGebra applets for 2x2 and 3x3 matrices, which can be found at https://www.geogebra.org/u/iolalinearalgebra. Each applet consists of a matrix, sliders to control each entry in the matrix, a real-time calculation of the matrix determinant, and a real-time dynamic parallelogram or parallelepiped showing the image of the unit square or cube under the matrix transformation. As shown in Figure 1, both applets have sliders (a) & (h) that students use to change the matrix entries. The applets also calculate the determinant of the matrix in real-time as the user changes the entries of the matrix (b) & (i). In the 2x2 applet, users can see (and alter) a geometric representation of a preimage object (c) composed of a yellow vector, blue vector, the parallelogram they form, and the inscribed ellipse (defaulted to the standard basis vectors, unit square, and its inscribed circle). Users can change this shape by moving the terminal endpoints of the yellow and blue vectors (d). These color-coded vectors correspond with their images under the transformation (e). In the 3x3 applet, the preimage is fixed as the unit cube (defined by the standard basis vectors) and is not shown or alterable because of the complexity of programming such an object in GeoGebra. In both applets, the image of the respective image is shown on the right side of the screen (f) & (j) and changes color from green to red when the matrix determinant is negative. Users can reset the applet (g) & (k), which will revert the matrix back to the identity matrix and, in the 2x2 applet, will reset the preimage to the unit square.

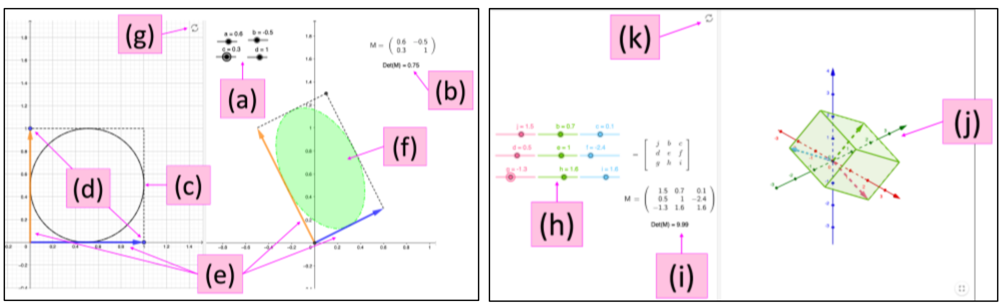

Figure 1. Screenshots of the two determinant applets, with markers (a) - (k) to help explain the applet components.

With the applet components, students are able to create any matrix within the constraints of the parameters (between -4 and 4, in increments of 0.1) and see how the changes affect the distortion of the image of the unit square or cube. Consistent with Rumack and Huinker (2019), Task 4 explicitly champions the role of mathematical curiosity as students explore, observe, conjecture, and justify; this allows students' mathematics to be what is leveraged and furthered. Adopting the RME design heuristic of guided reinvention, the instructor’s role includes facilitating students' open-ended activity toward more organized and focused statements or generalizations connecting changes in matrices to changes in the distortion of the space and how they relate to determinants.

Methods

The data for this study come from in an in-person introductory linear algebra class at a large, public, research university in the Mid-Atlantic region of the US. There were 27 students in the course, 13 of whom gave consent and completed the assignment analyzed here. In the university system, eight of these students chose he/him/his pronouns (pseudonyms begin with “M”), four chose she/her/hers pronouns (pseudonyms begin with “W”), and one did not choose pronouns (pseudonym P1). Most were second-year students by credit hours and were general engineering majors. The prerequisite was a B or higher in Calculus I or a passing grade in Calculus II.

The data analyzed and present herein are student responses to a written reflection. After most class sessions, students were asked to complete a reflection by the end of the day and submit their work via an online learning management system. Students were asked to spend 5-10 minutes on a reflection, for which full credit was awarded based on effort rather than correctness. The following is the reflection prompt analyzed in this paper: "We covered a lot of ground in class today! Let's focus on the applets and what they helped you understand. Give at least two observations you made from using either/both applet(s). State it in words / mathematical symbols and illustrate with a snapshot from the applet(s) to help convey it. If you have started to think about "why" for the observation, go ahead and include that, too." The prompt also included the URLs for the applets and a reminder that the class slides were on the course cloud-based folder.

To analyze the data in service of our research question, we aimed to understand what observations students made from their interactions with the determinant applets. Extending our previous preliminary analysis with a similar data set (Kerrigan et al., 2012), we first focused on three aspects of students' responses that align with features of the applets that users attend to: a user can notice (changes in) the entries of the matrix, (changes in) the value of the determinant of the matrix, or (changes in) the graphical appearance of the transformed object. For brevity, we label these as pertaining to the matrix [M], the determinant value [D], or the graphical imagery [G]. For each student response, we coded phrases as M, D, G, or "other." Distinctions between phrases were indicated by transition words such as "if," "then," "and," and "because," as well as by punctuation use. To establish viability of the coding scheme, two of the authors independently analyzed about half of the data according to these codes, and obtained a high level of agreement. These authors explained their codes and rationales to the third author and then all authors coded the remainder of the data set, again reaching near-total agreement. We expected and thought it natural that an observation from the applet would likely involve at least two phrases (e.g., "when I change this entry in the matrix" [M], "the determinant stays the same" [D]). Thus, we examined the data for relationships between phrases, looking for any causality or logical structure in the relationships (e.g., the aforementioned example has an implied if-then structure, so it would be coded as [M⇒D]). We referred to the phrases paired with their logical structure and additional supportive justifications (if they existed), as observations. Such an analysis is important to us as instructors and curriculum designers because the structure in these initial student observations is foundational in setting the stage for the co-development of key theorems (such as "Suppose A is a nxn matrix; A is not invertible $\Leftrightarrow det(A) = 0$"), and they grow out of students inquiring into the mathematics and the instructor inquiring into student reasoning (Rasmussen & Kwon, 2007).

Results

From the phrases within the 13 student responses, we found 43 phrases pertaining to the matrix [M], 41 phrases pertaining to the determinant [D], and 24 phrases pertaining to the graphical appearance of the transformed object [G]. Overall, we found there to be 42 observations. We organize the remainder of this section according to summaries of each of the three main phrase categories and then a summary of the various student applet observations.

Summary of results at the individual phrase level

First, 43 of the phrases pertained to aspects that students noticed regarding the matrix [M] in the 2x2 or 3x3 applet. 16 of these phrases related to the columns of the matrix forming a linearly dependent set. Although only 5 of the phrases say linear dependence explicitly, the other phrases give specific examples of linear dependence: the matrix entries being the same number (2), at least two columns being scalar multiples (2), and the matrix containing a column of zeros (7). As researchers, we grouped the latter three as types of linear dependence, but we cannot know if the students who wrote them were considering that concept as they wrote their phrases. The bulk of the remaining phrases coded with [M] had the common trait of focusing on changes in the matrix entry. Students wrote phrases about individual matrix entries or multiple entries (e.g., the values of the diagonal components), as well as about interchanging two columns, and multiplying row(s) or column(s) by a constant. As previously mentioned, the identity matrix is what first appears when opening or refreshing the applet, and the students are able to manipulate each individual matrix component. To reason about the ideas related to linear dependence and the ones related to multiplying a row, column, or the whole matrix by a constant, students had to set the entries to be the values they desired. On the other hand, for phrases that focus on "as an entry changes," the phrase has a more dynamic character. These are further detailed in the next section.

Second, 41 of the phrases pertained to observations the students made regarding the value of the determinant [D] that appeared in either the 2x2 or 3x3 applet or its computation. 15 of the phrases focused on a zero determinant value, 6 phrases focused on noticing a negative determinant value or a change in its sign, and 2 phrases highlighted an unchanging determinant value. Third, 24 phrases pertained to the graphical appearance of the transformed object [G] in either applet. Of these, 8 mention the resulting parallelepiped's volume (one also mentioned the resulting parallelogram's area in the 2x2 applet); more specifically, all but one mention a volume (or area) of zero. We group another 7 of the phrases together because they relate to the object losing dimension (e.g., the vectors are on the same line), and 5 phrases were other observations about the resulting image (such as flipping orientation). Finally, 2 students included snapshots of the 2x2 applet, 7 included snapshots of the 3x3 applet, and 4 did not include any snapshots. We did not code the images with [M], [D], or [G] unless a student explicitly referenced them in their written explanation. This occurred 4 times (twice for two students), all of which were coded [G].

Summary of results at the observation level

The prompt that students responded to stated "Give at least two observations you made from using either/both applet(s)." We referred to the phrases paired with their logical structure and any supportive justifications as observations. Consider student M1's response as an exemplar:

> "If a column has all zeros, the determinant is 0, and if the determinant is zero, the shape has no volume in the third dimension. Thinking about the determinant as (new volume)/(old volume), we can algebraically see that the numerator must be zero for the quotient to be zero. Also, if one of the column vectors is the zero vector, the shape does not span all three dimensions so it cannot possibly contain a volume"

We coded "If a column has all zeros" as [M], "the determinant is zero" as [D], and "the shape has no volume in the third dimension" as [G]. In the first part of M1's response, we coded two observations with these three phrases: [M⇒D] and [D⇒G]. The middle portion of his response, in which he brings in knowledge about the determinant as "(new volume)/(old volume)" and "the numerator must be zero for the quotient to be zero" was coded as "other" because it is something that the student could not directly observe in the applet itself. We coded "one of the column vectors is the zero vector" as [M], "the shape does not span all three dimensions" as [G], "it cannot possibly contain a volume" as [G], and paired those as observations [M⇒G] and [G⇒G].

Overall, our analysis shows that a vast majority of the observations that the students made began with a matrix statement [M]. In fact, only 5 of the 42 observations started with a statement about the graph [G] or the determinant [D] (see Figure 2). Relating back to the previous section, the [G] phrases mentioned were about the shape or the volume of the resulting parallelepiped.

Observation		
Phrase One	*implies*	*Phrase Two*
The shape does not span all three dimensions [G]		It cannot possibly contain a volume [G]
At least 1 dimension of the image would be 0 [G]		To the whole area/volume to be 0 [G]
When a matrix creates just a line or a plane [G]		The determinant would be zero [D]
The determinant is zero [D]		The shape has no volume in the third dimension. [G]
The determinant is zero [D]		Image was either flat or a straight line [G]

Figure 2. The 5 observations that started with a graphical statement [G] or determinant statement [D].

The most common type of observation was what we refer to as [MDG], by which we mean the student observation was of the form "[M] implies [D], which makes sense because of [G]." For example, M8 wrote, "If any of the columns in the 3x3 are LD with another, the det(M) will be zero due to there being no volume, only area or a line." He noticed when a certain matrix characteristic [M] occurred that a certain determinant value occurred [D], which he explained as sensible because of the corresponding graphical imagery [G]. We posit this likely corresponds to the nature of student activity while interacting with the applet: first manipulate the matrix, notice things about the determinant, and explain that in terms of what you see geometrically. Another example of an [MDG] is W4's observation (coding in the data): "One observation I made was that if the set is linearly dependent [M] or has two or more vectors that are scalar multiples [M], the determinant will be zero [D]. I found out after our discussion that this is because the vectors will be all on the same line [G] and therefore the shape will have no volume [G]."

As stated above, some observations conveyed a dynamic character, meaning that the student was most likely actively changing something in the applet while noticing the associated impact of that dynamic action. For example, M3 stated, "When observing the 3x3 applet in class today, I noticed that entries a_{11}, a_{22}, and a_{33}, as well as sometimes a_{13} and a_{31} were the only entries that changes the volume of the image," and M9 stated, "when moving the last column the 'footprint' didn't move, it just wobbled around like jelly." Both these observations are of the type [M⇒G]. We hypothesize that, for instance, M3 was dragging the slide for entry a_{11} and noticing the simultaneous change in the parallelepiped's volume. Or that M9 was, for instance, dragging the slider for entry a_{13} (called "*c*" in the applet) and noticing the simultaneous movement in the "top" of the parallelepiped. We note that the applet allows only one slider to be dragged at a time. As previously mentioned, 7 phrases were about a matrix with a column of zeros; each of these were part of an observation about the resulting determinant being zero. Because the default matrix in the applet is the identity matrix, it is likely that the observation was able to be made by only dragging one slider because each column of the identity matrix only has one nonzero entry.

In comparison to the dynamically oriented observations, some required multiple changes in the matrix entries and thus may have required many separate actions within the applet before an observation was made. For example, M6 wrote, "When I change two columns in the 3x3 applet, the determinant flips sign. It works the same on the 2x2 applet. I think it's because it flips the shape," which we coded as [MDG]. W6 wrote, "Another observation from the applet is that if you multiply one row by a constant the determine [sic] increases by that constant, and if you multiply the entire matrix by a constant, the determinant is multiplied by k to the power of n. For example, in a 2x2 matrix the determinant would be multiplied by k^2" which we coded as three instances of [M⇒D]. The actions needed to carry out both M6's and W6's second observation – switching one column with another and multiplying a matrix by a constant – require dragging multiple applet sliders *and* comparing an existing determinant with a resulting determinant. W6's first observation – multiplying a row by a constant increases the determinant by a factor of that constant – may have been made from only changing a row of the identity matrix or other diagonal matrix, in which only dragging one slider would have been needed for the observation.

Discussion

As design researchers, part of our work involves reflecting on student observations with respect to our curriculum design goals. Within the design research spiral (Wawro et al., 2022), considering student reasoning during tasks in light of the overall learning goals informs revisions that we make to the task sequence and supplementary materials, such as the applets. The first overarching learning goal include of this task sequence is (1) having students deepen their understanding of the geometric interpretation of linear transformations in $\mathbb{R}^2$ and in $\mathbb{R}^3$ through observing the impact that varying the matrix entries has on the determinant and on the transformed image. The second overarching learning goal is (2) for students to actively engage in the guided reinvention of key properties of nxn matrices A and B, such as: (2a) $det(A) = 0 \Leftrightarrow$ columns of A are linearly dependent, (2b) $det(A)$ is positive or negative according to whether the linear transformation defined by A preserves or reverses the orientation of the transformed objects ⇔If B is obtained by interchanging two rows (or two columns) of A, then $det(B) = -\,det(A)$, (2c) If B is obtained by multiplying a row (column) of A by scalar k, then $det(B) = kdet(A)$, and (2d) If $B = kA$ for a scalar k, then $det(B) = k^n det(A)$. Through reflection on data collected through various parts of the design research cycle, including the data analyzed in this paper, we feel confident that the applets have strong potential in helping an instructor achieve (1) and (2a) with their students. In particular, the results here show (2a) as a strong connection students make while working with the applet. These results also indicate that, when students are provided specific ideas to explore in the applet (namely, switching two columns or rows, multiplying a row by a constant k, or multiplying the whole matrix by a constant k), they can use the applet to make conjectures equivalent to (2b) - (2d). We would, however, like to consider ways that refining the applet might facilitate students' exploration of (2b) - (2d). For example, including features that can swap columns, swap rows, scale columns, or scale rows (rather than requiring individual entries to change one at a time), might help students develop (2b) - (2d) entirely on their own, rather than requiring targeted exploration prompts.

Our results showed that most student observations began with statements about the matrix [M] in either the 2x2 or 3x3 applet. We believe this is sensible, given the current design of the applets. The current aspect that is the most interactive is adjusting the matrix component values. For instance, students can rotate or zoom in on the transformed parallelepiped in the 3x3 applet but cannot manipulate, for instance, the image of the basis vectors that define the parallelepiped; if they could, we posit there may be more instances of students making [G⇒M] or [G⇒D] observations. One reason this is important to us as curriculum designers is the biconditionality of determinant-related properties and theorems. This informs applet revisions to better facilitate student understanding of and role in reinventing the biconditionality of the generalizations.

One limitation of the analysis presented here is that we analyzed post-hoc data of students' generalizations after interacting with the applet in class. We did not collect data of the students exploring the applet and conjecturing in real time. Analyzing real-time interactions would allow us to make stronger inferences about students' generalizations and the examples they explored to support them. We also had limited access to student thinking about why their observations were true because it was not required that they share justifications. Finally, students were informed all semester to spend 5-10 min on reflections and that they were graded on effort. Thus, we cannot know if students would have answered differently in a different setting. We look forward to our future work revising the applets, such as designing aspects to further facilitate specific learning goals. This work will also allow us to explore and theorize the alignment between specific aspects of RME, such as emergent models, and student exploration and engagement in DGS.

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Structuring and Connecting 2D and 3D Space Using Linear Combinations

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Developing a rich understanding of linear combinations is key to understanding linear algebra. In this paper, I explore the rich connections students make between the geometric and numeric representations of linear combinations through playing and analyzing a video game. I look at population of students who have never taken linear algebra before and analyze how they structure space using the video game, Vector Unknown, as a realistic starting point. I detail and analyze this activity including the activities that transition them from 2D to 3D space.

Keywords: Linear Algebra, Linear Combinations, Game-based Learning

Linear combinations are the heart of linear algebra and involve understanding variety of different representations and connections between those representations (Hillel, 2000; Sierpinska, 2000; Larson & Zandieh, 2013). Video games that are well-designed can help students engage with a variety of representations (Ke & Clark, 2019) and build connections between these representations resulting learning complex systems. (Gee, 2003). The game *Vector Unknown* (Author, 2021; Author, 2019; Zandieh et al., 2018) is a well- designed game created to connect different representations of linear combinations by combining the ideas of Realistic Mathematics Education (RME), Inquiry-Oriented Instruction (IOI), and Game-Based Learning (GBL). In this paper, I will look at how the game can be used as a realistic basis for establishing a student structuring of space built upon taking linear combinations and explore how students connect 2D and 3D space.

Literature Review

The Magic Carpet Ride task (Wawro et al., 2013) was the basis for the game *Vector Unknown*. The task begins with a sequence that initially introduces linear combinations and then develops the ideas of span and linear independence. The task leverages the idea of the travel metaphor, which is one of the keyways students understand linear combinations (Plaxco & Wawro, 2015). Dogan-Dunlap (2010) explored linear combinations using Sierpinska's modes of reasoning (2000) to analyze student reasoning about linear independence and found that students' geometric reasoning could provide a gateway to analytic thinking about linear independence indicating the importance of developing geometric reasoning about linear combinations. Paraguez and Oktac (2010) found that a lack of geometric representation resulted in the students having difficulty forming a schema of vector spaces, a keyway of organizing space. Zandieh et al. (2019) analyzed student metaphors of basis and found that when students changed their metaphors when describing basis in 2D and 3D space indicating a shift in understanding. This indicates a need to explore both how students connect geometric representations of space with more analytic elements and how they transition from 2D to 3D.

The Game Vector Unknown and Prior Game Research

Vector Unknown (https://tinyurl.com/linearbunny) is a game developed by capstone students at Arizona State University (Author, 2022; Author, 2021, Author, 2019). Participants are presented with four vectors in 2D, which they can drag into a vector equation and scale. The goal of the game is to reach the goal location, which is represented by a basket on a geometric grid. The game has three levels that were used for this interview. During the first level, the game provides the player with a geometric representation of the linear combination of two selected scaled vectors. The second level removes the geometric representation to force them to reflect more deeply upon the vector equation. Finally, the third level requires the player to visit several keys before reaching the goal. This forces the player to look at the geometric aspects of the game as the coordinates of the lock are not provided to the students.

Author (2021) found four core strategies that players used to play the game. The first strategy, quadrant-based reasoning, focuses on matching the quadrant of the vectors in the linear combination with the quadrant of the goal. The second strategy focuses on one vector involved and the player selects a vector and scales it to be as close to the goal as possible and then uses another vector to reach the goal. Thus, the player focuses on one coordinate involved, chooses either the x- or y-coordinate of the goal vector, and adjusts the scalars to match that one coordinate. The final strategy, button-pushing, involves players choosing vectors and rapidly adjusting the scalars to find a path to the goal. Each of these four strategies had a more geometric version in which players focused on the graphical display and a more numeric version where the focus was more on the vector equation.

Theoretical Framing

In this study, mathematical learning has been theorized as student's mathematical activity (Freudenthal, 1971, as cited by Gravemeijer & Terwell, 2000) that relates to the topic of linear combinations. The goal of this study is to describe the emergent mathematical activity relevant to topics in linear algebra such as linear combinations, span, and linear dependence developed from playing *Vector Unknown,* analyzing the gameplay, and designing a new 3D game based on the core choices made in *Vector Unknown*. I call this emerging activity structuring space. The sequence was designed to have students engage in at least three levels of activity from Realistic Mathematics Education (RME) (Gravemeijer, 1999). First, students engage in the situation of the game, which involves taking linear combinations. Next, they have to consider how other players might play the game referring to their own experiences. Finally, they have to develop design ideas for the 3D game which is intended to move them towards making generalizations about the linear combinations that are relevant beyond the 2D game. This paper presents an overview of the student's structuring of space from as situated in the dissertation work (Author, 2022) while providing examples to the core research questions:

1. How did students who have never taken linear algebra structure two-dimensional space with respect to linear combinations in relation to the game *Vector Unknown*?
2. How did students adapt their structure of two-dimensional space to a three-dimensional setting when designing a three-dimensional game based upon *Vector Unknown*?

Structure of the Interview

The interview was conducted over the course of four days. During the first day, the participants played the game individually. For the second day, I paired the participants into three groups according to their gaming experience (Gamer/Gamer (GG), Gamer/Non-Gamer (GNG), Non-Gamer/Non-Gamer(NGNG)). This was done because, at the time, it was thought that gaming experience might impact their activity with the game. First, each pair was asked to describe all possible goal locations that could be obtained from a set of vectors. Second, they were asked to create a set of easy, medium, and hard vectors (since the game had an easy, medium, and hard level) to reach a fixed goal of <-3,5>. Each group was shown how to use

GeoGebra during the second day for geometric exploration of all possible goals. On the third day, each pair was asked to design a 3D game by either creating easy, medium, and hard vectors and illustrating all the available vectors or creating easy, medium, or hard vectors to reach the goal <3,-4,5>. During the third day, students were again provided access to GeoGebra 3D to explore the geometric aspects of the linear combinations in 3D. During the final day, each participant played the game individually. This was done to explore if the participants' strategies for playing the game changed from the initial playthrough. This paper focuses on days 2 and 3 of the interview to analyze how they structure space.

Participants, Data Sources, and Methods of Analysis

For this paper, I looked at six students recruited from Calculus 1 and Calculus 2 classes from a large Southern University. None of the students had taken linear algebra before participating in the interview. The groups consisted of Alex(G)/Angel(G), Marti(G)/Betty(NG), and Gabby(NG)/Dee(NG) (demographic information provided in Table 1). The gaming experience was determined by the number of hours students spent per week gaming, with gamers being those who played 5 or more hours per week.

Table 1. Participant Demographic Information

Pseudonym	Year	Major	Gaming Experience	Gender Identity	Hispanic or Latino	Race	English Primary Language
Marti	Soph	Environment al science	G	Gender Variant/Non-conforming	Yes	White	No
Betty	Soph	Biology and Education	NG	Woman	Yes	White	Yes
Alex	Fresh	Civil Engineering	G	Man	No	White	Yes
Angel	Fresh	Mechanical Engineering	G	Man	Yes	White	Yes
Gabby	Fresh	Mechanical Engineering	NG	Woman	No	White	Yes
Dee	Junior	Mathematics and Education	NG	Woman	No	Other	Yes

I conducted the interviews over Zoom. They were recorded on a secure server and auto-transcribed using the Zoom software. The videos and transcripts were reviewed for instances of structuring space for each group and open coded (Strauss & Corbin, 1990). These codes were then condensed across all three groups to produce categories of structuring space. Finally, a round of axial coding was conducted to find themes of structuring space across the categories: structuring by design, structuring by generating, and structuring across dimensions.

Findings

In this section, I present four different types of structuring that are meant to capture the student's activity with examples of each from the student's work. Emergent categories are bolded throughout this section, and the emerging themes are the types of structuring. Structuring by design contains different design choices that were made by participants as they created different elements of the game that were based upon the participant's assumptions about the game and/or its difficulty. Structuring by generating captures process by which students generated example vectors which they utilized to generalize. Structuring all possible goals presents how students describe their structuring of all possible goal locations (or linear combinations). Structuring across dimensions describes how students made transitions and connections between 2D and 3D space. Together these four themes are meant to capture how students structure space in the context of analyzing the 2D game and designing a 3D game about linear combinations of vectors.

Structuring by Generating

This section presents different methods the participants utilized to create new vectors either to reach a provided goal or to list all possible goals given a set of vectors. **Pairing** vectors refers to selecting two vectors from a set and adding them together to generate new vectors. **Scaling** refers to scaling one or more vectors in pairs to generate a new goal location. Pairing and scaling [Figure 1] were used systematically for all three groups to generate goal locations spanned by a set of vectors. Here Betty pairs <-3,2> with each of the vectors <-2,4>, <1,2>, <-6,4>. She then pairs <-2,-4> with the remaining vectors <1,2> and <-6,4>, and finally <1,2> and <-6,4>. She then begins to scale the first vector in the first pair she generated. This progression of scaling was also seen in GeoGebra when students scaled according to difficulty. Gabby describes the activity of generating vectors to reach the goal of <-3,5> by **adding up/subtracting down** as looking "for two numbers that would add up to –3, –2, and –1. Fairly simple numbers, because it's medium difficulty, and then two other numbers that would add up to 5." Here she focused on the *x*-coordinates first and found two numbers that *add up* to -3, and then looking at the *y*-coordinate and she found two numbers that add up to 5. Adding up/subtracting down is the process of finding pairs that add up or subtract to goal x- or y-coordinate.

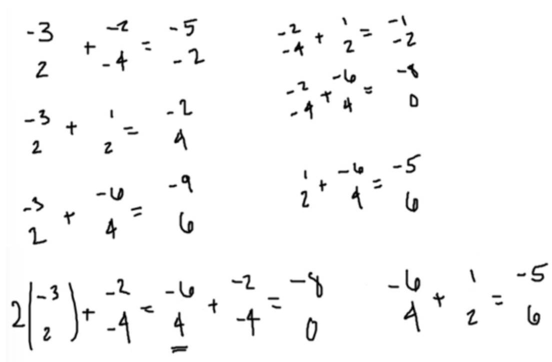


Figure 1. Example of pairing and scaling by Betty to generate possible goal locations

Structuring Across Dimensions

In this section, I investigate activities that students engaged in both 2D and 3D or activities that allowed them to connect the 2D and 3D worlds. Betty illustrates several of these strategies while looking at the linear combination of $v(-2, -4, -3) + o(3,0,0)$ and trying to determine all the possible goal locations [linear combinations] of these two vectors. Betty stated that she looked at where each of the combinations fell in the different planes. First of all, the 2D plane and then the 3D plane. This combination here [points to $v(-2, -4, -3) + o(3,0,0)$] has at least the capacity to go through all four 2D planes. Oh I mean quadrants [she manipulates the sliders to illustrate how it can go into all four planes from a top down view in Figure 2a]…and it can go into the upper level of the z-plane [she adjusts her view, Figure 2b, and adjusts the scalars to illustrate the point moving up and down along the blue axis].

First, Betty adjusted the camera to obtain a top-down viewpoint [Figure 2a]. From this viewpoint, she adjusted the scalars in the linear combinations to illustrate that the linear combination $v(-2, -4, -3) + o(3,0,0)$ visited all four quadrants in xy-plane, indicating that **reasoning about quadrants**, describing vectors using the signs or quadrants that they are located. After this, she rotated the camera and adjusted the scalars to illustrate that the point also went to the "upper level of the z-plane." I call this **traveling along an axis**, which is using checking to see if a point travels outside a particular. Essential to Betty's reasoning is here **adjusting the viewpoint,** which consists of rotating the camera to obtain an optimal view for checking a particular property or technique.

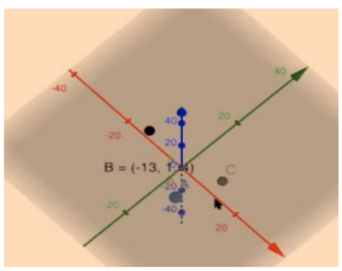
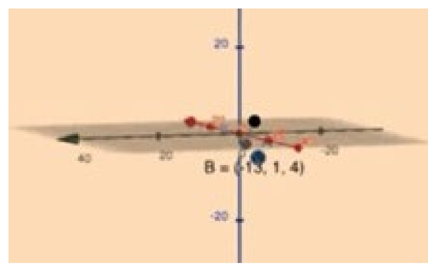
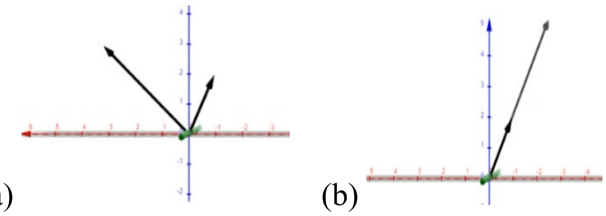
(a)  (b) 

Figure 2. Betty's analysis of the linear $v(-2, -4, -3) + o(3,0,0)$]

Marti, Betty's partner, also employed adjusting the viewpoint to see connections between the 2D and 3D worlds. In 2D, Marti was looking at the linear combination $f<-2,-4> + g<1,2>$ and noticed that it does "not matter how you multiply it seems to be moving along the same line. You can move it along both of the sliders, and it still moves in the same direction." This was in contrast to the linear combination $a<-3,2> + b<1,2>$ which "seems to cover the entire 360 [degrees] of the graph depending on what you multiply them by." Marti then began to classify other linear combinations according to whether or not they were in the **same line** or **rotated around 360 degrees**. This became an important design consideration when Marti started to design her 3D game. When creating vectors to reach a goal she wanted vectors that "had as much of a reach around the graph as possible." Marti tested linear combinations by rotating the camera to viewpoint that projected onto a plane and adjusted the scalars to ensure they traveled around 360 degrees [Figure 3a]. When two linear combinations aligned she changed the vectors in one of the linear combinations [Figure 3b]. This indicates that Marti was reflecting upon Marti's 2D work when creating vectors in the 3D game, in particular the delineation between linear combinations that traveled in the **same line** or rotated **360 degrees**.



(a) (b)

Figure 3. Marti looking for linear combinations that are in the same line or rotate 360 degrees

Structuring across dimensions is important because it captures activities that participants thought worked in both 2D and 3D dimensions, such as quadrants and same line/rotations. Also, adjusting the camera view was an important activity that allowed participants to build connections between 2D and 3D. Betty was able to see the connections using different viewpoints, and coordinate these viewpoints which occurred. She coordinated a planar projection viewpoint with an isometric viewpoint to deduce that the goal locations traveled off the projected plane. This category answers the research question 2, how students structure space across dimensions.

Structuring by Design

This section presents design choices students made while creating their own vectors for different difficulties in the 2D and 3D game. First, students would **attend to coordinate properties** which encompasses designing vectors by selecting coordinates with specific numerical properties such as being big numbers (such as 6 through 9), small numbers (such as 1 through 3), or zeroes for individual x-, y-, and z-coordinates. Since the game only allowed x- and y-coordinates that were between -20 and 20, several groups only considered possible goals locations that Marti described as "**within the grid**." This category was important because it provided an activity for students like Betty to generate goal locations and noting that $<-3,2> + 3<-6,4> = <-21, 14>$ and "found it would be going off the 20 and it would be going off the mark so it wouldn't count" as a possible goal location. While designing vectors to reach the goal of <-3,5>, Alex and Angel decided that they wanted to design a "**bluff vector**" which was intentionally designed to not help the player reach the goal. For example, they used the vector <5,-3> as a bluff vector. Finally, scaling **vectors according to difficulty** refers to the design decision to have more vectors to be scaled with the difficulty of the game. For example, Gabby and Dee present easy goal locations as (2,0,1) + (-1,2,3), medium difficulty as b(2,0,1) + (-1,2,3) or (2,0,1) + b(-1,2,3), and hard difficulty as b(2,0,1) + c(-1,2,3). Structuring by design was important because it presented goals created by students in the design of the game that generated activity allowing them to notice patterns.

Discussion and Limitations

I initially set out to categorize the different types of activities that students engage in after playing, reflecting, and designing a 3D version of the game *Vector Unknown* in regards to structuring space and linear combinations. Now that I have presented these results, I want to reflect on the ways in which my findings can inform a more general definition of structuring space. Examining my data, I found that students engaged in the following activities:

1. Worked with a variety of objects, including coordinate vector representations such as $\begin{bmatrix}1\\2\end{bmatrix}$; points and arrows on a 2D or 3D plane; symbolic representations of vectors; representations of paths in 3D space.

2. Took actions such as **pairing** and **scaling** vectors both on paper and in GeoGebra (through adjusting the sliders), **changing the viewpoint**, and creating vectors using **adding up/subtracting down**.
3. Set design goals that shaped their structuring activity such as structuring by design with **fitting the grid**, **having bluff vectors**, or wanting vectors with specific **coordinate properties**.
4. Made observations such as recognizing if **vectors span all quadrants** or lie along the same line in **structuring across dimensions** and observing **coordinate properties** about individual vectors.

Taking actions, setting goals, creating objects, and making observations form the skeleton of structuring as its key components. The act of structuring involves doing one or more of these activities and re-evaluating the connections between the remaining elements. In this way, introducing an action like changing the viewpoint might have the students set new goals such as finding the best viewpoint, work with different objects such as projections down to 2D space, and make new observations such as being able to classify 3D vectors as in the same line or a rotation. Now the foundation of a particular structuring is the starting activity that sets these parts in motion with example actions, goals, objects, and observations. Playing the game *Vector Unknown* provides the initial set of actions, goals, objects, and observations for structuring space under linear combinations for this study. Here the actions are the moves by the player (scaling and choosing vectors), the goal is to reach the basket, the initial objects are the vectors and their geometric representation, and the observations are highlighted by feedback provided to the player from the game. Playing the game and asking questions about the game introduce or modify each of these components resulting in a student structuring of space.

I observed that the idea of linear independence was observed through the idea of same line/visiting all quadrants/rotating 360 degrees. Quadrants appearing in both this analysis and the video game strategies (Author, 2020) indicate that they might be a fruitful starting point for understanding linear dependence. Additionally, I found that understanding viewpoint and, in particular, coordinating viewpoints was crucial to the participants transitioning between 2D and 3D space. This is important because looking at viewpoints is a paradigm that is often relegated to dynamic geometry software (such as GeoGebra) and video games and may not be covered in classes without these tools. Future work will investigate this data more thoroughly doing an in-depth analysis of the different viewpoints utilized and linking them to the mathematical activity for each viewpoint. I see view this as important to developing the geometric intuition that is considered valuable by Dogan-Dunlap (2010) and Paraguez and Oktac (2010). Finally, I want to stress the importance of the game in the student's structuring by contributing elements such as fitting the grid and designing bluff vectors. These design ideas from the game provided students with goals and a reason to generate vectors which they then used to make generalizations and explore advanced mathematical concepts creating a complex network of ideas (Gee, 2003).

The project was conducted with only three groups of students and thus does not capture the entirety of structuring space. Additionally, the students were introduced to GeoGebra at different times during the interview and in different versions. Some students utilized a GeoGebra applet in 3D, while others chose to use the default program. Because of this, their structuring when working in GeoGebra was likely impacted by the tool and when it was introduced.

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Providers of Professional Development for Novice College Mathematics Instructors: Perspectives and Values About Teaching and Learning

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Those who lead the preparation and assessment of novice college mathematics instructors for teaching (Providers) do their work in many ways (e.g., course coordination, seminars, workshops). Using data from a large national survey, this study examined reporting among 95 Providers about the structures of their departments, their goals for the professional development work they do, and their relative valuation among goals. Respondents completed a sorting and ranking activity about professional development goals and answered an open-ended question describing their sorting decisions. Qualitative coding identified six main themes for the respondents' 285 descriptions. Quantitative analysis used the rankings of goals within respondents' sorting categories to examine how Providers describe and value professional development goals related to professional community, classroom and department culture, and instructor response to students within their classrooms.

Keywords: Mathematics Graduate Student Instructors, Teaching Assistants, Professional Development, Pedagogy, Community, Q-Sort Methodology

Introduction

A significant and continuing challenge to undergraduate student success is the nature of instruction in college mathematics courses (Freeman et al., 2014; Laursen & Rasmussen, 2019). The last decade has seen several national efforts aimed at addressing this well-known problem, including online repositories of activities for use in the preparation of graduate students for teaching (i.e., CoMInDS), online professional learning opportunities for faculty to focus on student learning, and the release of student-centered policy documents like the Mathematical Association of America's (MAA) *Instructional Practices Guide* (IPG). Higher education leaders are now looking to departments to improve the preparation of college mathematics instructors (CMIs); particularly novice CMIs who are graduate students beginning their teaching journey as mathematics graduate teaching assistants (teaching in sessions adjacent to a primary course) and graduate teaching associates (teaching as instructor-of-record). Here, we use **GTA** to refer to both groups collectively and, as needed, distinguish between them with Lab-TA for teaching assistant and GSI-TA for teaching associate. These new CMIs are a critical subgroup because: (1) they teach thousands of undergraduates (Belnap & Allred, 2009), (2) many start as LabTAs, and (3) most have little or no preparation for teaching. In addition, there are many variations in type of institution (e.g., master's granting, doctoral granting) and structure within departments around professional development (PD). Depending on location, GTAs will be offered learning opportunities that may include short orientation workshops, teaching seminars, pedagogy courses, mentoring, or course coordination. Thus, mathematics departments have reached the point where they are asking which resources they should use, and how, to help their graduate students' learn to teach. They know the ingredients exist and are looking for the recipe for their local departmental use.

Context of the Study

This study is part of the CMI Preparation Project. The overarching goal for the project is to document and extend the use of effective professional learning about teaching for a variety of institutional and instructional contexts by generating a CMI Design Tool for use by departments. This study focuses on the findings from a national Needs and Uses Survey with the aim to understand how communities are described and discussed by providers. Analyses from the survey are informing the development of the tool.

Within this survey, respondents were *Providers*: those who work with novice CMIs to support learning about teaching undergraduate mathematics. In the survey, Providers answered questions about their department structure and the design of the professional development available locally to novice CMIs. The survey used a variant of the *Q-sort methodology* (Willig & Stanton-Rogers, 2017) where **Goals** for professional development were *sorted* and *ranked* into up to three **buckets** which were then *titled*. The Goals offered in each activity were developed in the project's earlier work by eight expert Providers from diverse backgrounds (e.g., from private and public, masters and doctoral granting institutions) who merged and distilled critical topics of CMI PD from their experience and research.

Each of the 15 Goals (Table 1) had an associated exemplar (when a respondent hovered over the Goal). Each respondent was asked to sort the 15 Goals into three buckets, rank each Goal within that bucket, and then provide a title and justification for that bucket's title.

Table 1. Goals Sorted by Providers

A: learn how to notice and manage challenges to equity, access, and success among undergraduates
B: learn how to initiate and sustain a productive classroom culture
C: learn about ways for students to build their math learner identities
D: learn strategies to promote and facilitate collaborative learning
E: get feedback on their teaching from peers
F: learn about students thinking and analyze how it reflects instructional decision-making
G: learn how to foster student in-class engagement
H: create a community of instructors and are supported in its creation
I: build knowledge on how students learn
J: learn methods for promoting whole-class discussion
K: learn how to recognize teaching practices that create a sense of belonging among students
L: learn how to implement self-assessment in teaching
M: plan and prepare lessons
N: learn how to develop teaching portfolios
O: learn about and promote student use of outside-of-class learning resources

This qualitative sorting methodology was chosen because prior research in ranking of such PD goals had illustrated that it was difficult to disassociate the judgment of value of teaching method from the logistical expectations of a department (Yee et al., 2018). To address this, for this survey, respondents had choice in how to assign goals to buckets, could freely rank within buckets, and were asked to give and explain bucket titles. There were three of these bucket activities (each having the participant sort Goals into three buckets) in the survey, each on its own page. This report is focused on the bucket activity that emphasized community and how PD

Providers view communities within teaching (Gobstein, 2016; CoMInDS, 2020; Yee et al., 2022). This sub-study addressed the questions

RQ1: How do Providers describe Goals that focus on community?

RQ2: How do Providers value Goals focusing on community?

Background and Theoretical Perspectives

Multiple calls for action to include inquiry-based mathematics education (Laursen & Rasmussen, 2019) and active-learning methods (Braun et al., 2017) have been on the forefront of suggestions for the professional development of CMIs. Recent research and development in collegiate mathematics education has attended to professional communities in several ways. The *Student Engagement in Mathematics through an Institutional Network for Active Learning* (SEMINAL) project (Gobstein, 2016) is expanding use of collaborative learning methods through active-learning. Additionally, the *Promoting Success in Undergraduate Mathematics through Graduate Teaching Assistant Training* (PSUM-GTT) project (Haddock et al., 2018) is diversifying multiple uses of peer-mentoring, training, and seminar courses for CMI professional development. With all of these resources and suggestions, it is easy for a mathematics department to be overwhelmed (Pengelley & Sinha, 2019).

In fact, although 70% of departments in a recent national study indicated they provide some kind of instructional development for novice CMIs, the nature of it varies widely (Ellis et al., 2016; Speer et al., 2017). Novice CMIs today are the very instructors who will play key roles in addressing the nation's need to improve student enrollment, retention and persistence in science, technology, engineering, and mathematics (STEM) courses and majors (Holdren & Lander, 2012; Laursen & Rasmussen, 2019). The opportunity and need coalesce in the first two years – where undergraduates first encounter college mathematics and novice CMIs are most likely to be teaching (Belnap & Allred, 2009). Clear from the proposers' and colleagues' work in developing, offering, researching, and disseminating ideas about professional learning in college mathematics instruction, central to success is strengthening novice CMIs' understanding of student thinking (Roach et al., 2013; Speer & Wagner, 2009; Wagner et al., 2007). Specifically, GTAs are a critical focus of novice CMI professional development.

Bragdon and colleagues (2017) categorized approaches in 120 different master's and doctoral-granting departments into nine models for PD focusing on GTAs. However, a challenge still remains: how to leverage and communicate about these characteristics with and in mathematics departments. To be able to do this we need to understand (1) how department members, including Providers, describe their Goals and (2) how they value their Goals. Research into CMI preparation has demonstrated how Providers' goals for novice CMIs significantly overlap with topics of secondary and primary mathematics methods courses (Yee et al., 2022). Yee et al., (2018) analyzed touchstones in methods courses from two perspectives: those of teacher educators and those of in-service teachers. The focus of this national study was phrased in terms of perceived importance: "Please tell us how important you feel it is for each of the following content items to be valued and addressed by secondary mathematics methods courses for preservice teachers." Although a very valuable study, what respondents valued was conflated with departmental expectations. To get around that, this project used a Q-sort methodology that more clearly isolated respondents' values from their departmental expectations by not only having respondents sort and rank the Goals, but also create and justify a *title* for each bucket, so that we can see the respondents' language and values directly.

Methods

This study used embedded mixed methods where qualitative thematic coding of bucket titles was embedded within the quantitative analysis of the ranking and frequency of Goals within bucket descriptive themes (Creswell, 2017). Respondents for the survey were recruited because they provided professional development about teaching to novices in their departments. The survey was sent to 242 people, 95 of whom completed the sorting activity. These 95 responses, with three descriptive bucket titles each, formed the data pool of 285 bucket titles used for the analysis shared here. Respondents came from a variety of academic positions (see Table 2).

Descriptive analysis along with central tendencies provided a context for interpreting the Community bucket activity data (Table 2). To answer RQ1, thematic coding was used on the descriptive titles created by respondents for their buckets. The goal for the thematic coding was to examine and characterize the language that was used by the respondents (see Table 3). For example, respondents' language suggested one theme be described as "learn how to foster student in-class engagement" – this *in vivo* title represents Provider language (alternatively, the researchers could have abbreviated this as "active learning"). However, we *purposefully* capture respondent language so that we can make sure the content is accessible by people who are not educational researchers, such as those in mathematics departments who will (one day) use the CMI Design Tool for which the survey is a first step.

Researchers went through four stages of thematic coding of the 285 bucket titles by (1) generating a modified word cloud of all titles to generate a preliminary set of themes, (2) collaboratively coding 30 bucket titles into themes and revising themes together, (3) choosing 30 new bucket titles of the 255 uncoded buckets and coding them individually then comparing the codes and revising themes and coding, (4) each researcher then coded each of the remaining 225 bucket titles, then met to discuss revisions until consensus was reached (100% agreement) amongst the bucket titles being coded and the wording used for the themes. It is important to note that for this study the Goals within each bucket were not considered when identifying the themes of the titles, only the titles.

To answer RQ2, we embedded the themes into the quantitative analysis of Goals' within-bucket and weighted rank (see Table 4). Thus, with each theme, we could see what value (weighted rank) was given to that Goal within that theme. Because few buckets had more than six ranked Goals, we wanted to weight the Goals accordingly. We chose to use harmonic progression (reciprocal progression) so that rank 1 of a Goal within a theme was weighted as 1, rank 2 was weighted as ½, rank 3 was weighted at 1/3, and so on. These weighted rankings were then added together for each Goal within each theme to generate a "heat map" (Table 4).

Findings

We first describe the contexts for Providers who did the survey, then how the bucket titles were themed, and finally how the Goals were valued within each theme. Providers identified, on average, 38 novice CMIs out of an average of 84 mathematics instructors for each of the departments. Those novice CMIs accounted for (on average) 45% of the teaching done within a department (median 49%). Additionally, Providers stated, on average, that graduate students were 30 of the 38 novice CMIs, suggesting 78% of novice CMIs within their departments were graduate students (median 74%). We also asked about a Provider's position and the target group for their PD. Among respondents, 40 were teaching faculty, 33 tenure-track research faculty, 3 tenure-track teaching faculty, and 16 "other" (described as former/current department chair, directors in Math department, non-tenure track teaching/research faculty, lecturer, and postdoc).

This suggests the largest group taking the survey were teaching faculty, followed by research faculty. Together these two groups constituted 80% of all respondents. The novice CMIs target groups were graduate students who were either LabTAs (e.g., recitation instructors, *N*=36) or GSI-TAs (i.e., instructors of record, *N*=35). Together graduate students made up 76% of the target groups for the respondents. Table 2 illustrates the respondents' departmental information relative to the teaching workforce.

Table 2. Structure of Departmental Teaching Workforce and CMI PD (out of 95 responses)

		Median	**Mean**	**SD**	
# of people teaching mathematics for the department		70	84	55	
# of above group who are novice CMIs		34	38	31	
# of novice CMIs who are graduate students		25	30	26	
Role within the CMI Professional Development	Teaching Faculty	Research Faculty		Adjunct Faculty	Other
	43	33		2	16
Target group for CMI Professional Development	LabTA	GSI-TA	NFI*	ULA**	Other
	36	35	7	1	14

**NFI=non-faculty instructor and **ULA=undergraduate learning assistant.*

Bucket Title Themes

To answer RQ1, themes were refined three more times as two researchers expanded, revised, and refined the initial themes to cover the most bucket titles and to best reflect respondents' choice of language. For example, the word "Knowledge" in Th3: (Teaching and Classroom Skills, Strategies, and Knowledge) had similar uses within the respondents' title justifications as use of the word "skills" when referencing teaching. This is important to note: "knowledge" and "skills" for teaching were not distinguished by Provider respondents. The six themes reported here covered 92% of all Goals that were sorted into buckets. Altogether, buckets with titles that fell under Th1, Teaching Community and Culture, contained 106 Goals, 9% of all Goals sorted by all Providers while Th2 bucket titles contained 240 (19%) of all sorted Goals (see Table 3).

Table 3. Bucket Title Themes, Frequencies, and Relative Frequencies

Theme Code	Theme Title	Goal Count	% Sorted Goals
Th1	Teaching Community and Culture	106	9%
Th2	Individual Teacher Development such as Assessment, Feedback, and Reflection on Teaching	240	19%
Th3	Teaching and Classroom Skills, Strategies, and Knowledge	263	21%
Th4	Student Learning, Support, Engagement, and Identity	256	21%
Th5	Classroom Identity, Community, Environment, and Culture	168	14%
Th6	Pedagogical Diversity, Equity, and Inclusion/Belonging Especially in the Classroom	100	8%

The survey asked respondents to briefly justify their bucket titles. Bucket justifications were only used when struggling to thematically code a bucket title. Overall, only 23 of the 285 bucket titles had more than one identified theme.

Goals Valued within Themes

With respect to RQ2, weighted rankings were assigned relative to the Provider's chosen rank for each Goal within each bucket. Table 4 provides a heat map coloring the Goals within each theme (row) – the highest weighted value in green and the lowest rated value in red. For example, for all respondents who had a bucket title that was coded under Th3, the most highly weighted value was 19 on Goal M: plan and prepare lessons. Goals (column headings) were ordered across the top to group greens and reds (see Yee, 2022 for more on the method).

Table 4. Heat Map of Goals within Each Theme According to Weighted Ranking

Goal: Theme	E	N	L	H	A	B	K	M	I	F	O	G	D	J	C	Total
Th1	11	8.6	7.6	18	0.3	1	1.2	2.4	0.5	0.6	1	0	0	0	0	52
Th2	25	16	20	20	2.1	1.3	3.1	7.8	8.3	7.6	2	1.3	0.5	0.5	1	117
Th3	4.4	3.4	2.9	5.5	11	8.6	3.1	19	7.7	4.7	5.1	15	8.8	7.5	3.9	110
Th4	1.3	0.4	0.8	0.8	7.8	7.5	5.9	9.9	17	11	8.5	13	12	7.7	9.9	113
Th5	0	0	0.1	1.1	13	13	7.9	3.5	1.7	2.2	2	6.7	5.9	7.7	4.1	69
Th6	0	0.2	0.2	0.4	12	9.4	6.3	1	1.6	1.4	1.4	1.5	1.6	1	6.4	44
Total	41	29	32	46	45	41	27	43	36	28	20	38	29	24	25	505

In the Total Row, adding up all weighted rankings of single columns, we see Goals H (46), A(45), M(43), B(41), and E(41) were the highest valued across all six themes. Goals A and B focused on classroom community among students while H and E focused on community around instructors. We also see that Goal O: learn about and promote student use of outside-of-class learning resources, had the lowest weighted ranking total (20). When looking across all rows, we see that every Goal but one, O, was in some themes' top five highest valued Goals. This suggests that Goal O was not highly valued for the participating Providers.

When looking at the Goals within each theme, we see some have higher values than others. Goal A "learn how to notice and manage challenges to equity, access, and success among undergraduates" was the highest weighted rank in Th5: Classroom Identity, Community, Environment, and Culture. This is interesting because Providers used the words that define Th5 and then associated this with Goal A. Specifically, equity, access, and success were encapsulated by Providers using the words Identity, Community, Environment, and Culture. These latter words may illustrate Providers seeing the outcomes of managing challenges around equity, access, and success as focusing on classroom environment and culture rather than the individual student. Goal H is a focus on the community of instructors and was the second highest valued in Th2 which focused on feedback and reflection. This may suggest Providers see a strong connection between teacher community, personal reflection, and assessment of teaching by others.

When comparing Th1 and Th2 in Table 4, we see Th2 had higher weighted ranks with Goal I and F, where the focus is on knowledge of how students learn, student thinking and how it affects instructional decisions. Th1 focused on community and culture while Th2 focused on individual development. Thus, Providers seemed to view Goals around student learning and thinking as less impactful to community development than to individual teacher growth.

When comparing and contrasting Th5 and Th6, we see these themes did not highly value Goals E, N, L, and H yet did value Goals A, B, and K highly. This aligns with the focus of Th5 and Th6 around the classroom. Goals D, G, and J had higher valued weighted ranking in Th5 over Th6. This suggests Providers who title buckets with Diversity, Equity, and Inclusion did not value engagement, collaboration, and whole-class conversation as much as those who titled their buckets with Identity, Community, Culture and Environment. More work is needed to understand this finding given that research in active learning has suggested that collaboration and engagement seem to encourage diversity, equity, and especially inclusion (Laursen et al., 2014).

Discussion

Overall, we found that Providers responding to the survey were primarily teaching faculty, followed closely by research faculty within mathematics departments. The Providers' departments had a workforce where on average 45% of all instructors were novice CMIs. These Providers focused primarily on the 75% of novice CMIs that were GTAs. In answering RQ1, we found that there were six dominant themes that captured the 285 bucket titles. Language chosen by the Providers was preserved within these themes illustrating important differences between how educational researchers use terms (such as knowledge and skills) and how Providers use these terms. These differences are important as we move forward in creating a Design Tool with language that is meaningful to its users. When looking across the themes, at how the Goals were sorted and ranked, certain distinctions about how themes and values varied illuminated additional challenges and opportunities for offering mathematics departments' guidance on Provider resources.

When comparing and contrasting weighted ranking of Goals within Th5 and Th6, Goals D, G, and J were all slightly higher with Th5 suggesting responding Providers associated student engagement, collaboration, and whole class discussion (Goals D, G, & J) within classroom identity, community, environment and culture (Th5), rather than within what Providers identified as "diversity, equity, and inclusion" (Th6). More investigation is needed on this. For example, it could be an artifact of assumptions about the nature of CMI influence on classroom culture, environment, and community as different or easier than that what Providers associate with diversity, equity, and inclusion (i.e., Providers were less inclined to connect engagement, collaboration, and whole-class discussion goals with Th6: Diversity, Equity, and Inclusion).

One of the largest implications is the need for clear distinction in the Design Tool between departmental values related to communities of undergraduates (e.g., a classroom community) and community among novice instructors (e.g., in a comparison of Th1 and Th2 versus Th5 and Th6 in Table 4). We see those Goals valued in Th1 and Th2 are not valued in Th5 and Th6. Moreover, the highest valued Goals overall (Table 4) were Goals H (46), A(45), M(43), B(41), and E (41). Goals H and E focus on the teacher community while Goals A and B focus on classroom culture and equity, access, and success with students (classroom community). Goal M, plan and prepare lessons, seemed to be the only Goal that was moderately valued across nearly all themes (Table 4) and had a high overall value. This may suggest that Providers can leverage planning and preparing lessons to bridge discussions among these communities. For example, when planning a lesson, the discussion can revolve around the student community, while discussion on how the lesson plan is used, by whom, and how, could connecte conversation back to instructors' community.

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A Measurement Hat Trick: Evidence from a Survey and Two Observation Protocols about Instructional Change after Intensive Professional Development

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Professional development about teaching is an important lever for change toward evidence-based instructional practices. Yet few studies in higher education offer robust evidence about whether and how instructors' teaching practices actually change as a result of professional development. We studied instructors who attended a four-day intensive workshop on inquiry-based learning (IBL) in college mathematics and who provided data about their teaching before and after. Of 293 instructors who attended, 136 provided complete survey data and 15 video-recorded their teaching, which we coded using two observation protocols. Comparing pre-workshop and follow-up teaching data, we find significant and well-corroborated changes toward IBL methods across all three methods, with effect sizes near 1, generally viewed as large in education research. The findings demonstrate that well-designed professional development can have a substantial impact on instruction even within the first year of implementation.

Keywords: IBL, professional development, TAMI, RTOP, measurement of teaching

Teaching-related professional development (TPD) is widely seen as important to encourage the uptake of evidence-based instructional practices (EBIPs) by college STEM instructors. TPD is attractive as a lever for change because research has demonstrated that EBIPs can make a big difference in students' experiences and success, and tested classroom resources are widely available to instructors in all STEM disciplines. Proponents argue that TPD is impactful because most college instructors have ample room to improve: while deeply grounded in their discipline, most have little formal knowledge of learning and teaching. Moreover, the effects of TPD will be amplified as each instructor teaches many courses and thousands of students across a career.

However, these arguments are not currently well backed by research showing that TPD does have the desired effect on instructor practice. At present, professional developers and funders can point to examples of impactful TPD as proof of concept (e.g., Khatri et al., 2013; Laursen et al., 2019), but the formal literature relies much on case studies of small programs: large, high-quality studies are scarce. Exceptions include recent work on TPD in mathematics (Archie et al., 2022) and studies in several science fields (Bathgate et al., 2019; Borda et al., 2020; Chasteen & Chattergoon, 2020; Ebert-May et al., 2011, 2015; Manduca et al., 2017) that measure instructors' implementation of active approaches after TPD, and the barriers and supports that shape this.

Building knowledge about TPD in higher education has been slow and challenging. Simply measuring teaching and change in teaching is difficult (AAAS, 2013); common tools such as surveys and observations both offer pitfalls. Moreover, studies of TPD outcomes cannot be done as experiments: researchers must find large, well-designed, and stable TPD programs to serve as study sites to provide adequate sample sizes and a consistent, high-quality intervention. TPD is neither required nor routinely offered by institutions of higher education, unlike K-12 settings. Finally, teachers need time to learn and enact new ideas and practices, then more time to become proficient enough to improve student learning, a reality that further extends the research timeline.

Thus, it is no surprise that rigorous studies examining the outcomes of TPD are needed. Our study of workshops on inquiry-based learning (IBL) in college mathematics addresses many of these challenges. The workshops reached nearly 300 instructors, yielding samples large enough for quantitative analyses. Built on a decade of prior work, the workshop model was informed by learning from K12 and college TPD experience, evaluation feedback, and education research on professional development. Workshops were led by teams of skillful IBL educators who worked from a common general structure and schedule, drew from and contributed to a shared set of workshop and course materials, and engaged in TPD leadership training. Together these features ensured sound program design and consistent, high-quality TPD delivery.

To address the measurement challenges, we used three instruments—a survey and two observation protocols—and compared the results. Funding constraints meant we had to measure implementation just one year after workshop participation, likely before teachers had mastered new approaches. We thus chose two descriptive, behavior-focused measures (one survey, one observation protocol), positing that new IBL users would first adjust how they use class time—e.g., using less lecture and more of the student-centered activities that are common in established IBL classrooms (Laursen et al., 2014; Hayward et al., 2016). Even if instructors were not yet skilled in facilitating these activities, we could detect initial changes in their choices about the use of class time. To address quality of instruction, we chose a second observation protocol that offers an evaluative perspective via holistic ratings of each lesson. With thorough video sampling we could reliably characterize instruction for the course as a whole, not just in selected episodes (Weston et al., 2021). In this analysis, we consider: *What do three independent measures of teaching tell us about whether and how instruction changed after professional development?*

Context for the Study

To situate the study, we provide brief descriptions of inquiry-based learning in mathematics, in the general form that was practiced by workshop leaders and taught in workshops, and of the IBL workshop design. For more detail, we refer readers to the references cited below.

Inquiry-based learning (IBL) is a form of instruction that has become increasingly common as an approach to active and collaborative learning in college mathematics. Laursen and Rasmussen (2019) identified four pillars, or core principles that IBL shares with other forms of inquiry-based mathematics education (IBME). In effective IBL classrooms, students engage deeply with meaningful mathematics and make sense of these ideas in collaborative work with their peers, which may take a variety of forms. Instructors support these student experiences by inquiring into student thinking—drawing out student ideas and helping them toward further sense-making—and by making design and facilitation choices that involve all in rigorous mathematical learning and identity-building. The “big tent” philosophy of IBL recognizes that instructors will implement these principles in different ways, depending on their setting, adjusting to suit their course material, class size, student audience, and personal preferences.

Consistent with other research on EBIPs (Freeman et al., 2014), IBL offers cognitive and affective benefits and supports student persistence and success in later courses (Kogan & Laursen, 2014; Laursen et al., 2014; Laursen & Rasmussen, 2019, & references cited therein). On average, IBL approaches were found to close equity gaps in student experience compared to lecture-heavy approaches (Laursen et al., 2014), again consistent with general findings in the literature (Theobald et al., 2020). Of course, IBME is no silver bullet (Ernest, Reinholz & Shah, 2019; Johnson et al., 2020; Reinholz et al., 2022): instructors must take proactive steps to develop self-awareness and empathy and to select pedagogical approaches that foster mathematical identity and enhance classroom climate (Dewsbury & Brame, 2019).

The design of the IBL workshops was purposeful, research-aligned, and iteratively refined (Yoshinobu et al., 2022). Briefly, four interwoven strands of workshop activity sought to foster IBL-supportive knowledge, skills, and beliefs and prepare instructors to implement IBL in their own teaching contexts. Video lesson study sessions helped instructors develop a mental model for an IBL classroom, while literature-to-practice sessions used provocative readings and discussion to help instructors internalize the four pillars and reflect on their own beliefs. "Nuts and bolts" sessions shared facilitation and task-design tactics to build instructor skills, and course planning sessions provided planning exercises and personal work time to help instructors consider how to prepare their own IBL course. Through an active e-mail mentoring program, peers and facilitators provided follow-up support—resources, ideas, trouble-shooting, encouragement—as instructors implemented their IBL course after the workshop (Hayward & Laursen, 2018). Deliberate attention to equitable instruction and assessment practices was increasingly incorporated into the workshops across the 2016-2019 period that we studied.

Study Methods

The study design was approved by the university's IRB and all instructors gave informed consent. Students in video-recorded classes were given information explaining that the focus of the study was their instructor, not them. They could move out of camera view if they wished.

Study Population and Survey Sample

The study examined participants in 11 IBL workshops conducted in 2016-2019 (Archie, Hayward & Laursen, 2021). Online workshops in 2020 were omitted from outcomes analysis. They were surveyed at two times: All 293 answered the pre-workshop survey, and 199 (68%) the follow-up survey about 17 months post-workshop. Of these, 136 (68%) identified a target course before the workshop and then described implementing IBL (or not) in their target course on the follow-up survey; they form our sample to study instructional change.

Overall, the implementing respondents were very similar to the workshop attendees. Half were women. Most (87%) were US citizens, nationals or residents; of these, 80% were white and 5% were Asian. About 3% described themselves as Hispanic or Latino/a/X. Participation was well balanced among people in tenured (30%), untenured (30%), and non-tenurable (30%) positions, from departments granting four-year (42%), masters (23%), and PhD (21%) degrees, and some from two-year (13%) colleges. About 23% taught at a minority-serving institution. Overall, they were modestly more diverse than the U.S. mathematics professoriate as a whole, and they came from institutions that broadly represent teaching settings in U.S. higher education.

Survey Methods

Survey instrument. The survey included closed- and open-ended items previously described (Hayward & Laursen, 2014; Archie, Hayward & Laursen, 2021; Archie et al., 2022; Hayward et al., 2016). Some items were used to provide formative feedback to workshop leaders, to characterize workshop features, and to measure participants' knowledge, skills and attitudes. Participants self-reported personal and professional demographic information. Responses across time points were matched via unique identifiers. Here we focus on items used to describe teaching strategies at both times, from the TAMI-IS (Toolkit for Assessing Mathematics Instruction-Instructor Survey; Hayward, Weston & Laursen, 2018). These descriptive, behavior-focused items ask "*what did you do*?" rather than "*how well did you do it?* Taking the course as a whole, instructors estimated the frequency of use for each of 11 teaching practices often seen in college math courses (e.g., group work, whole class discussion, formal lecture, short explanations, student presentations), coded using the scale: 0 = never, 1 = once or twice during the term, 2 = about once a month, 3 = about twice a month, 4 = weekly, 5 = more than once a week, or 6 = every class. Open-ended items probed patterns in practices and special events and elicited text descriptions of 'lecture,' 'presentations,' and 'group work' as practiced in courses.

Survey analysis. We created a composite variable to measure instructors' overall use of IBL methods. This variable, IBL frequency score, includes five of the 11 practices classified as 'core IBL' practices because they measure all variations of IBL that were emphasized or de-emphasized in workshops and was calculated as follows: *IBL frequency = student group work + student presentation + class discussion - lecture - instructor problem-solving*. IBL frequency scores were calculated for pre-workshop and follow-up time points; scores could range from -12 to +18. Lower scores indicate less frequent use of IBL teaching practices and more frequent use of lecture and instructor problem-solving, while higher scores indicate more frequent use of IBL methods and less frequent use of lecture and instructor problem-solving. We conducted paired samples t-tests to measure change in IBL frequency between pre-workshop and follow-up times for instructors who did (n=15) and did not (n=136) participate in classroom observation.

Observation Methods

Observation sample. The observation sample was a subset of the survey sample: 15 workshop participants who registered early and volunteered to collect video data in their target course in the term before their summer workshop, and for the same or similar course the next academic year. Their personal and institutional characteristics are broadly similar to the survey sample. Instructors placed a video camera in their classroom several times a term and completed the survey at the end of the same term. We asked them to record about 1/3 of class time. In all, we coded 278 class sessions, averaging 9 classes per course and totaling over 300 hr class time.

Observation instruments. We applied two observation protocols that emphasize different dimensions (Hora & Ferrare, 2013a). The TAMI-OP (TAMI Observation Protocol) is a segmented, descriptive (behavior-focused) protocol that aligns with the TAMI-IS (Hayward, Weston & Laursen, 2018). It is adapted from the TDOP (Hora & Ferrare, 2013b) and COPUS protocols (Smith, Jones, Gilbert, & Wieman, 2013) to reflect practices we saw in college mathematics. Twenty activity codes for instructor and student behaviors were coded within each 2-minute segment. Fourteen end-of-class holistic items, designed by our team, query features of classroom atmosphere and student-teacher interaction (Laursen et al., 2011; Appendix 2).

The RTOP (Reformed Teaching Observation Protocol) is a holistic, evaluative protocol developed for K12 science instruction and increasingly used in higher education (Sawada et al., 2002). Raters score the lesson on 25 items in 5 categories: lesson design, teaching of content and procedural knowledge, communication, and student-teacher relationship. For each lesson, items are scored 0-4 to describe how often each feature is seen: "never occurred" to "very descriptive" of the lesson. Summed RTOP scores of 0-100 indicate the degree to which a course is "reformed" toward student-centered, evidence-based instruction emphasizing inquiry.

Observation analysis. Using data from the TAMI-OP and RTOP instruments, two global measures were used to assess change in teaching:

1. the RTOP-Sum (RTOP), a composite of 25 observational survey questions. Here ratings are summed over all questions and averaged over observed classes in a course.
2. the Proportion of Non-Didactic lecture classes (PND) across a term of observation using the TAMI-OP. PND provides an estimate of the proportion of classes, for one instructor, that use non-lecture teaching methods.

The analytical method for TAMI-OP is based on Latent Profile Analysis (LPA), a statistical clustering method applied to a large observation data set (790 observations of 74 teachers) from this and other projects (Weston, Hayward & Laursen, 2023). The LPA yielded four reliable class types (Didactic Lecture, DL; Interactive Lecture and Review, IL; Student Presentation, SP; Group Work, GW) based on the proportions of various teaching activities present in an individual class session. From the LPA results, we created a simple dependent measure to characterize a teacher's course based on the proportion of all observed classes for each instructor (over an academic term) that were *not* classified as didactic lecture (PND, proportion non-didactic). A paired-sample t-test was used to assess pre-post change in the observational indicator. We used the Cohen's D effect size for the analyses, which is the difference in pre/post means divided by the standard deviation of the difference.

Results

We analyzed the data sets for pre- to post-workshop changes for all three measures and summarize the results in Table 1.

Table 1: Pre- to post-workshop changes in instruction using three measures from survey (S) & observation (O)

S/O	Measure	N	Mean, pre	Mean, follow-up	Effect size	t	df	p
S	IBL frequency from TAMI-IS	136	0.03	6.34	0.88	10.31	135	< 0.001
S	IBL frequency from TAMI-IS, observation subset only	15	2.27	8.80	0.96	3.71	14	0.002
O	PND from TAMI-OP	15	0.58	0.81	0.80	3.13	14	0.004
O	RTOP-Sum	15	37.89	55.06	1.17	4.58	14	<0.001

The range for each measure is IBL frequency -12 to +18; PND 0-1; RTOP-Sum 0-100.

Instructional Change Based on Surveys

Figure 1 shows the distribution of IBL frequency scores at pre-workshop and follow-up, for both the full survey sample and the observation subgroup. A paired samples t-test showed a statistically significant increase in IBL frequency from pre-workshop to follow-up (Table 1).

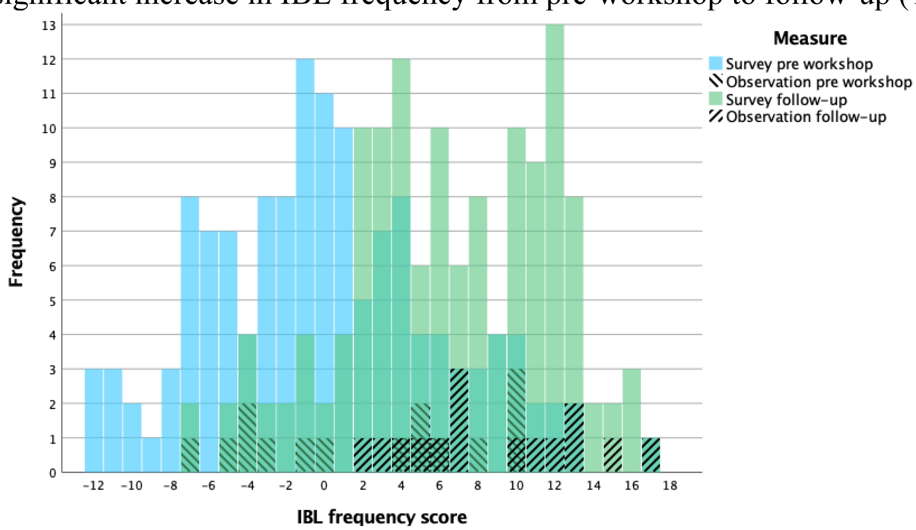

Figure 1. IBL frequency score from surveys, for full sample and observation subgroup at two times.

For video study participants, the change in IBL frequency score from pre-workshop to follow-up was also statistically significant, and comparable in magnitude to the average shift among all survey respondents (see Table 1). Video study participants used slightly more IBL core practices both before the workshop and subsequent to it, but these differences between the main group and video subgroup were not significant.

Half the main sample, and all 15 video participants, self-described as implementing at least “some IBL methods;” one third of the main sample and half of video participants described their implementation as a “full” IBL course. These are lower limits to the overall implementation rate, as 32% of all workshop attendees did not report on implementation. Their target courses served first-years to seniors; math majors (28%), mixed STEM majors (42%), non-STEM majors (13%) and pre-service teachers (4%). About 90% implemented IBL in a class of 35 students or fewer.

Instructional Change Based on Observations.

In general we saw gains in the average scores for both observation measures from pre to follow-up. Comparing these using paired samples t-tests, the changes in both the RTOP-Sum and the PND were statistically significant. For both measures, negative or no change was observed only for small numbers of instructors who had initial high scores. Figure 2 illustrates pre-post changes in PND for the 15 instructors. Each observed class session fits a latent profile (one colored cell) and PND for each course is based on the set of such profiles (one row of cells).

We also used LPA to examine changes in teaching approach. In post-TPD classes, instructors used less continuous lecture and more interactive lecture and group work, but also made less use of student presentations. Participants also adjusted the sequencing of activities. For example, changes in the timing of lecture and group work suggest a more structured and strategic approach to group work, with more consistent use of short lectures to summarize or signpost.

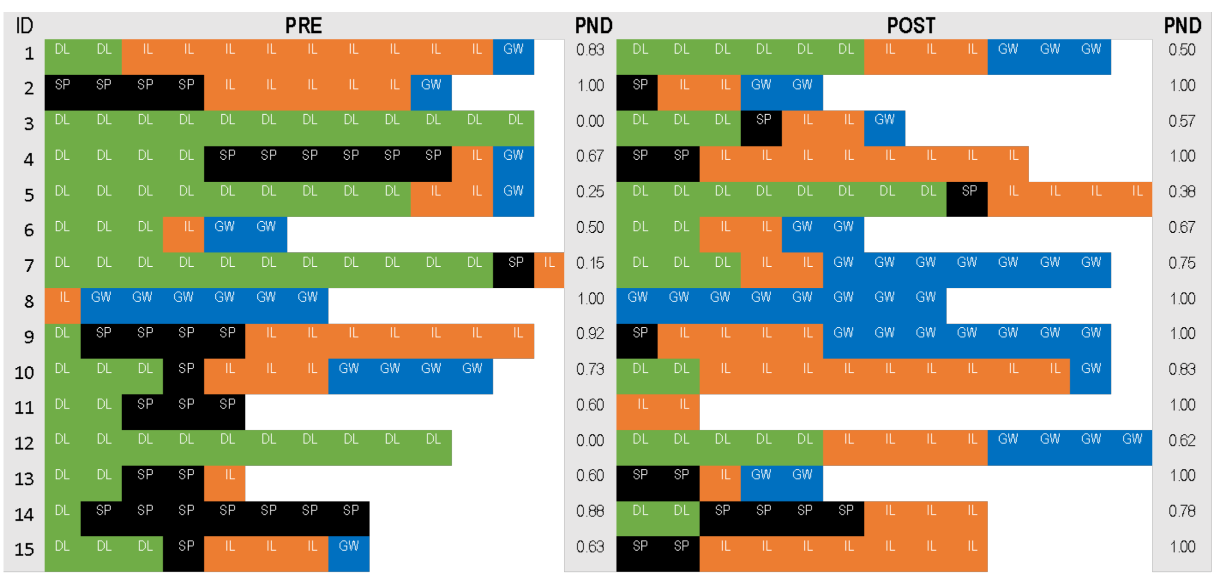

Figure 2. Latent profiles and PND for 15 observed courses for the same instructors (IDs 1-15) at pre-workshop and follow-up (“post”) times. Cells represent 1 observed class fitting a profile from LPA. DL (green) = didactic lecture; SP (black) = student presentation; IL (orange) = interactive lecture & review; GW (blue) = group work.

Discussion

This study identifies significant changes in the teaching practices of instructors after PD, using a rigorous approach to measurement. We found significant changes using three distinct measures. The effect sizes are impressively large (0.8 to 1.2) for an educational intervention. We used well characterized tools and large samples, and drew on different methods (survey vs. observation) with different foci (behavioral vs. evaluative). The TAMI-IS and TAMI-OP offer evidence that instructors' classroom practice shifted toward inquiry-based methods after TPD, and the RTOP-Sum offers evidence that post-TPD classes offered more student-centered, engaging and encouraging environments that were better aligned with evidence-based practice.

Survey responses from the subset of 15 instructors who were observed reflect a significant pre/post change similar to that seen in the larger survey sample. This indicates that such changes in teaching are not outliers, due to self-selection for video observation, but typical of the workshop's impact on instructors' teaching. Robust time sampling ensured that the coded observations were not "dog and pony shows" showing only the most active teaching days.

Other survey data enable us to attribute these changes to the workshop. Following the workshop, we observed robust and sustained growth in participants' knowledge and skills to use IBL; their motivation to use IBL and belief in its effectiveness remained strong and stable over time. In this sample and a larger group of IBL workshop participants, such growth in knowledge and skills, or "IBL capacity," was a good predictor of participants' later implementation of IBL (Archie et al., 2022). That is, changes in teaching behaviors are linked to the nature and strength of the gains that participants reported and sustained. Thus, the observed changes can be ascribed to participants learning and applying ideas they gained at the workshop.

Other studies of substantial TPD interventions for college instructors offer context for interpreting these results. Two studies, like ours, compared individuals to themselves before and after a TPD intervention. Chasteen and Chattergoon (2020) used the PIPS survey (Walter et al., 2016) to measure pre/post change in teaching practices of early-career physics faculty after a multi-day workshop, with an effect size of 0.8. In contrast, Ebert-May et al. (2011) reported no significant changes in RTOP scores after their TPD program for early-career biologists.

Two more studies compared teaching practices of TPD participants to samples of people who did not undergo TPD, and provided the data needed to compute effect sizes for mean differences between treated and untreated groups. After refining their biology TPD workshop, Ebert-May et al. (2015) measured differences in RTOP scores for participants vs. non-participants with an effect size of 0.68. Manduca et al. (2017) also used RTOP scores to compare treated (TPD) and untreated groups of Earth science instructors, with an effect size of 1.06. Both types of studies suggest that the changes seen here are impressive but not implausible—in line with what others have measured for STEM instructors who have taken part in thoughtfully designed, well supported, multi-day programs offered by skilled faculty developers in their discipline.

Conclusions

We report a "hat trick" of TPD measurement: three independent measurements of instructional change after a well-designed TPD intervention. The changes are large, statistically significant and well corroborated, and can be ascribed to the workshop. We propose that the workshop has this substantial impact due to its strategic approach (Kennedy, 2016), providing instructors with a set of principles and tools to achieve a desired teaching goal, thus empowering them to adapt their practices in ways sensitive to their own context and preferences (Yoshinobu et al., 2022).

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Modifying symbolic forms to study probability expressions in quantum mechanics

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As part of an effort to examine student understanding of expressions for probability in an upper-division spins-first quantum mechanics (QM) context, clinical think-aloud interviews were conducted with students following relevant instruction. Students were given various tasks to showcase their conceptual understanding of the mathematics and physics underpinning these expressions. The symbolic forms framework was used as an analytical lens. Various symbol templates and conceptual schemata were identified, in Dirac and function notations, with multiple schemata paired with different templates. The overlapping linking suggests that defining strict template-schema pairs may not be feasible or productive for studying student interpretations of expressions for probability in upper-division QM courses.

Keywords: Quantum mechanics, probability, notation, representation

Introduction and Background

Mathematics is used in physics for far more than simple computation. Physicists make use of mathematical expressions and relationships to help them understand and reason about the world (Uhden et al., 2012). This is more true than ever in quantum mechanics (QM), where physical intuition developed in everyday life has little application and one must rely on mathematical reasoning to help intuit, understand, and predict systems on the quantum scale. The level of abstraction and mathematical sophistication required in upper-division QM courses has been extensively shown to be difficult for students in many ways (Singh, 2001; Singh, 2008; Singh & Marshman, 2015; Emigh et al., 2015; Passante et al., 2020). Some of the reasons for this difficulty are the number of different notations on offer, the vast differences in how they each appear, and the variety of mathematics they each require (Gire & Price 2015).

In QM, particles and systems of particles are described as existing in "quantum states." These states are defined as having some set associated, measurable quantity or quantities, such as specific energy values, positions, or momenta. Often, these quantities are incapable of being simultaneously determined together, and the most we can know about a particle with one definite quantity is the likelihood of measuring the various possible values of a separate quantity. Physicists primarily use linear algebra to express these probabilistic relationships, and use Dirac notation to express these states as vectors: $|\psi\rangle = \sum_{n} c_{n}|n\rangle$. Here, $|\psi\rangle$ is a vector representation (called a "ket") of the state our particle is in, and the various $|n\rangle$'s are representative of the various states the particle could be measured in – an eigenbasis – each corresponding to a different value of whichever quantity we are measuring. The probability of measuring each of those $|n\rangle$ states is given by $\left|c_{n}\right|^{2}$, or by taking the complex square of the inner product of the initial state (vector) with the particular possible state (vector) that we're interested in. In Dirac notation, this is written as: $P_{n} = |\langle\psi\rangle|^{2}$. This type of analysis is only applicable when the quantities in question are discretized, but is extended to include defining states with wave functions: $|\psi\rangle = \int_{-\infty}^{\infty} \psi(x)|x\rangle dx$. This can then be used to find probabilities for measuring that quantity within a range of values: $P_{a\rightarrow b} = \int_a^b |\psi(x)|^2 dx$. The discrete probability written before can also be written with these wave functions: $|\langle\psi\rangle|^2 = \int_{-\infty}^{\infty} \varphi_n^*(x)\psi(x)dx$, where $\psi(x) = \sum_n c_n \varphi_n(x)$. The difference in appearance and the type of mathematics required for these different probabilities is one of the reasons for the aforementioned difficulties for students learning and using the requisite mathematics in QM.

One framework that has been used to explain student reasoning with mathematical expressions is that of symbolic forms (Sherin, 2001), which was proposed by Sherin to explain how students could generate symbolic expressions for different physical scenarios. Largely based on the knowledge-in-pieces model (diSessa, 1993), symbolic forms is predicated on the notion that students learn to develop a sort of grammar and syntax for different mathematical operations in physics, and subsequently apply that logic to make sense of physical relationships. Sherin found that there were some universal forms that expressions take that students grow to recognize, which he called symbol templates, and that students learn to associate these templates with a conceptual understanding known as a conceptual schema. Symbol templates, when combined with their associated conceptual schemata, become symbolic forms, which Sherin argues form the building blocks for student interpretations of mathematical expressions used in physics. For example, the *parts-of-a-whole* symbolic form has "[□ + □ + □ + ...]" as its symbol template, and its conceptual schema contains the idea of multiple parts of a larger entity being summed together. This framework has since been extended to study the structure of expressions in more advanced topics in physics (Dreyfus et al., 2017; Schermerhorn & Thompson, 2019; Ryan & Schermerhorn, 2020) and has been looked at as a means to describe students' blending of conceptual and formal mathematical reasoning (Kuo et al., 2013).

Previous work has also looked at the affordances and limitations of the different notations used in QM (Gire & Price, 2015; Schermerhorn et al., 2019; Wawro et al., 2020). Dreyfus et al. (2017) posited a number of potential symbolic forms within Dirac notation that they suspect students likely develop through the course of an upper-division QM course, though little work has been done to address how students interpret and work with expressions across and within the different representations that are commonly used in upper-division QM.

As part of a study on how students reason about expressions in the various notations used in QM and the ways in which they translate between them, we conducted clinical think-aloud interviews with a number of students following a one-semester, spins-first, upper-division QM course. In a spins-first course, students begin working with Dirac notation immediately in the context of spin-1/2 systems, before eventually including wave function notation when position is introduced as a continuous observable. The mathematical connections between Dirac state vectors and wave functions are shown in an effort to help deepen students' understanding of Dirac notation, as well as to make the transition to wave functions as smooth as possible. We asked the students to both generate and relate different expressions in both Dirac notation and wave function notation. Due to our focus on student understanding and interpretation of symbolic expressions, Sherin's symbolic forms framework (Sherin, 2001) was used as a starting point for analysis. Our analysis suggests a modest expansion of the symbolic forms framework to address multiple overlapping sets of templates and schemata. This paper describes our analysis of student interpretations of expressions for probability in both Dirac and wave function notations, as well as the ways in which students translate between the two.

Study Design and Methodology

In order to elicit students' understanding of expressions commonly used to represent probability in QM, both in-person (paired, N=2) and virtual (individual, N=2) clinical think-aloud interviews were conducted following a one-semester upper-division QM course in two separate academic years. Students in both interview formats were asked to generate expressions for probabilities and to reason about both the expressions they generated as well as the processes by which they were generated. The virtual interviewees were provided with a number of symbolic building blocks with which to construct their expressions (see Figure 1), while the in-person interviewees were given a whiteboard and markers. In both cases, subjects were asked to generate expressions for probabilities based on scenarios similar to those studied in class. Some in-person interview questions gave the participants an expression describing a state and asked them to use it to find specific probabilities. The virtual interviewees were also given a card-sorting task where they were asked to sort a variety of expressions commonly seen in QM coursework, as well as generic vector expressions such as $\vec{v}$, $\hat{j}$, and $\vec{u} \bullet \vec{v}$.

The students' responses were transcribed and analyzed to determine which expressions they either grouped or generated, any intermediary expressions they used, and the language they used to explain their expressions. Excerpts of interest contained episodes of students interpreting specific expressions, explicitly connecting templates and schemata.

$$\boxed{\hat{S}_z} \ \boxed{|x\rangle} \ \boxed{\varphi_n(x)} \ \boxed{\langle E_n|} \ \boxed{|} \ \boxed{|^2} \ \boxed{\hat{x}} \ \boxed{|\psi\rangle} \ \boxed{\varphi_n^*(x)} \ \boxed{\hat{H}} \ \boxed{\int_a^b} \ \boxed{x} \ \boxed{\psi(x)} \ \boxed{\langle x|} \ \boxed{dx} \ \boxed{|E_n\rangle} \ \boxed{\int_{-\infty}^{\infty}} \ \boxed{\psi^*(x)} \ \boxed{\langle\psi|}$$

Figure 1. The symbolic building blocks of expressions that virtual participants had to work with when constructing their expressions.

Results and Discussion

Our analysis has revealed several different conceptual schemata, as well as a number of symbol templates with which students tended to associate them (Table 1). Despite differences in the tasks given between the two interview formats, our analysis did not suggest any modality-based distinctions. We propose that the focus on template-schema pairs common to symbolic form analysis is limiting when multiple symbol templates share the same conceptual schema.

Table 1. Identified conceptual schemata with their associated symbol templates. Some schemata are associated with multiple templates and vice versa.

Conceptual Schema	Symbol Templates
Vector in a space	$|\rangle$, $\langle|$, $c_1|1\rangle + c_2|2\rangle + \ldots$
Function in a space	$c_1 f_1(a) + c_2 f_2(a) + \ldots$
Quantum state	$|\rangle$, $\langle|$, $f(a)$, $f^*(a)$, $f_n(a)$, $f_n^*(a)$
Projection/dot product	$\langle\rangle$, $\int f(a)g(a)da$

Probability density	$\langle\rangle$, $\lvert f(a)\rvert^2$
Probability	$\lvert c_n\rvert^2$, $\langle\rangle$, $\lvert\langle\rangle\rvert^2$, $\int f(a)g(a)da$, $\int \lvert f(a)\rvert^2$

Dirac notation paired with vector ideas

Students across all three interviews in this study (Aliyah and Bilbo in the virtual interviews and Castor and Delilah in-person) explicitly referred to both kets and bras as representing vector-like quantities, and Dirac brackets as being akin to dot products between them:

Bilbo (discussing sorted category containing $\vec{v}$*,* $\hat{j}$*,* $|\psi\rangle$*, and* $|E_n\rangle$*)*: to me, all vectors. Unit vector [$\hat{j}$], generic vector [$\vec{v}$], state- ... wave vector [$|\psi\rangle$], eigenstate vector [$|E_n\rangle$]

Bilbo (discussing operating $\hat{S}_z$ *on* $|\psi\rangle$*)*: certainly changes the state ... if the state is purely in Z [meaning expressed as a superposition state in the $\hat{S}_z$ spin-1/2 basis], I believe it'll still change it, but I think by only lengthwise stretching ... rather than rotating.

Aliyah (discussing $\langle\psi\rangle$*)*: why would I do ψ of ψ? Because physically like I'm thinking in terms of vectors it represents ψ along ψ

Aliyah (explaining what they mean by "__ along __"): it's a traditional way to think about vectors like because our dot product represents- like $\vec{a} \bullet \vec{b}$ represents, basically, the projection of $\vec{a}$ along $\vec{b}$ or projection of $\vec{b}$ along $\vec{a}$

Castor (explaining why for $n{\neq}m,\ \langle E_m\rangle = 0$*)*: Because of like orthonormality, the eigenstates are perpendicular in a space

Castor (explaining why $\langle E_2\rangle = 1$*)*: Because, like 100% of E2 [gestures at the bra] is in the direction of E2 [gestures at the ket]

These responses largely match up with what one would expect, as a large focus of a spins-first QM course is to help students reason geometrically with Dirac notation.

Through the lens of symbolic forms, two conceptual schemata appear to be expressed here: a "vector in a space" and the "projection" idea that often arises with dot products. The former conceptual schema is linked to "$|\rangle$" and "$\langle|$", while the latter is linked to "$\langle\rangle$".

Dirac bras and kets paired with "quantum state"

While interesting in its own right, a vector-like understanding is not the only way that students worked with and thought about these Dirac expressions. All four students described bras and kets as stand-ins for quantum states as well:

Aliyah: This [$|E_n\rangle$] represents a ket energy eigenstate, and this [$\langle E_n|$] represents a bra energy eigenstate. So these [$|\psi\rangle$ and $\langle\psi|$] are general ones, these [$|E_n\rangle$ and $\langle E_n|$] are specific energy eigenstates

Bilbo (discussing $|x\rangle$*)*: You could make x an eigenstate, you could make it a spin state ... put anything in there ... I just need it to be a ket

Castor: So typically when I write [$|\psi\rangle$] in terms of the energy eigenstates [points to $|E_2\rangle$]...

Delilah (discussing a superposition state written $|\psi\rangle = \frac{1}{2\sqrt{2}}\left(\sqrt{3}|E_1\rangle + |E_2\rangle + 2|E_3\rangle\right)$): We just represent it as the probab- the square root of the probability times the first state [points to $|E_1\rangle$] plus the square root of probability times the second state [points to $|E_2\rangle$] plus the square root of the probability times the third state [points to $|E_3\rangle$]

These students appear to be using some of the same symbol templates as in the previous section ("$|\rangle$" and "$\langle|$"), but with a "quantum state" conceptual schema. This is unsurprising, as the first half of a spins-first course uses bras and kets as representations for a quantum state.

Other Dirac pairings

An additional common use of Dirac brackets was as a means of describing and calculating probabilities or probability amplitudes:

Aliyah: This [$\langle\psi\rangle$] will also represent the probability of finding x - sorry, the probability of finding the general state ψ in the eigenstate x

Bilbo: That [$\langle\psi\rangle$] is just an inner product, though, I had been saying the inner product squared is a probability and that this [$\langle\psi\rangle$] is... just a density

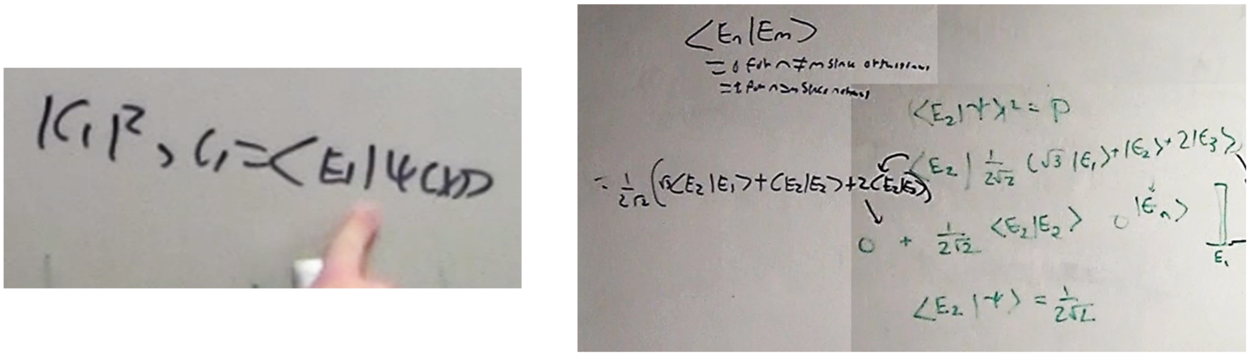
Again we see a symbol template from before ("$\langle\rangle$") being associated with a different conceptual schema: in this case, that of a probability-like concept. We note that "probability," "probability amplitude," and "probability density" are used somewhat interchangeably by the students throughout these interviews. Every student also eventually squares many of these Dirac brackets, which suggests yet another symbol template, "$|\langle\rangle|^2$" also with an associated "probability" conceptual schema.

While Castor and Delilah also related Dirac brackets to probability concepts, they almost exclusively did so by first claiming that the Dirac bracket gives "the coefficient," and that "the coefficient squared" then gives the probability:

Castor (explaining why a number they found was a probability): Because it's the coefficient for the first energy state. ... because we do, we do the same thing as... [writes $\langle\psi(x)\rangle$]

Delilah: Our probability for energy is the coefficient squared. And the coefficient is, E sub one times psi [writes $\left|c_1\right|^2,\ c_1 = \langle\psi(x)\rangle$] (see Figure 2a).

Castor and Delilah's focus on "the coefficient" as a sort of requisite step to allow a Dirac bracket to describe a probability was not seen in Aliyah or Bilbo's virtual interviews. This may be a case of Aliyah and Bilbo glossing over a step that they have since automated, or it could be evidence of Castor and Delilah using the Dirac bracket's "dot product" conceptual schema to reason about a larger process of "pulling out a coefficient" from an expression for a superposition state. This process was shown explicitly by Castor and Delilah, as can be seen in Figure 2b. This suggests that these students developed another symbol template during this course: "$c_1|1\rangle + c_2|2\rangle + ...$". The associated conceptual schema appears to be that of a vector being described with components along various basis vectors, in which case it is likely a compound form as discussed by Dreyfus et al. (2017), potentially of the "parts-of-a-whole" and "magnitude direction" symbolic forms, the former from Sherin (2001) and the latter from Schermerhorn and Thompson (2019). It also seems clear that Castor and Delilah make use of another symbol template, that of one of these coefficients squared, "$\left|c_n\right|^2$", and that they associate with it the conceptual schema for a "probability" as well.



(a) (b)

Figure 2. (a) Delilah expressing how the Dirac bracket gives the coefficient, which can then be squared to find a probability, and (b) Castor's (right) and Delilah's (left) work explaining how ⟨ψ⟩ yields the coefficient, which is then squared to find the probability. Delilah's work supplements Castor's by explicitly distributing the bra throughout the sum.

Functions paired with state ideas

Moving away from Dirac notation, students in all interviews also worked with functions, often describing them as representing quantum states:

Aliyah: Those [$\psi(x)$, $\psi^*(x)$, $\varphi_n(x)$, and $\varphi_n^*(x)$] represent states ... some of them represent general states [$\psi(x)$ and $\psi^*(x)$], some of them represent specific energy states [$\varphi_n(x)$, and $\varphi_n^*(x)$], but they represent states

Bilbo: This [$\psi(x)$] is just another function, so what I'm thinking of is like an eigenstate [$\varphi_n(x)$] and just a generic state [$\psi(x)$]

Castor: These [φ_1 and φ_2 in $|\psi(x)\rangle = c_1\varphi_1 + c_2\varphi_2 +$] are the position eigenstates

Delilah: $\psi(x)$ ... is, like c_1 times $\varphi(x)$ [writes $\psi(x) = c_1\varphi_1(x) +$] ... I think these [points to $\varphi_1(x)$] are the energy eigens- the energy eigenstates written in the position basis

Castor is being somewhat loose with their notation, mixing Dirac notation together with wave function notation in the same expression, but seems to have "translated" the kets they wrote previously into functions directly below them (see Figure 3). At this point the "quantum state" conceptual schema is being used to describe a very different-looking symbol template (that of a function, perhaps "$f(a)$" or "$f_n(a)$"). These excerpts potentially suggest a distinction between "$f(a)$" and "$f_n(a)$" as representing specifically "generic states" and "eigenstates," respectively. Figure 3 also suggests another symbol template analogous to that discussed in the previous section, but in this case in wave function notation: "$c_1f_1(a) + c_2f_2(a) + ...$". The paired schema is that of a "function in a space," described as a sum of component functions—analogous to the compound form discussed above.

This shared schema is a reasonable instructional outcome as, once students begin working with wave function notation, they are often told that kets can be translated to functions and use $\psi(x)$ and $\varphi_n(x)$ to describe generic states and energy eigenstates, respectively.

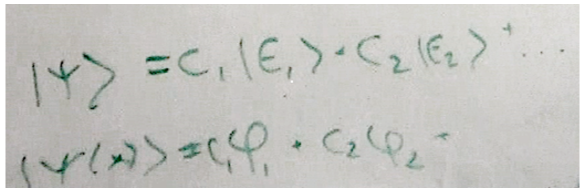


Figure 3. Castor's expressions for a superposition state. They wrote the top line first, then the bottom beneath it.

Other function pairings

Students would also often write integrals of these functions when asked to generate expressions for probabilities. Below is an example of Bilbo's response after being asked how they would find the probability of measuring a given energy:

Bilbo: in this case [$\int \varphi_n^*(x)\psi(x)dx$] here ... I'm thinking okay, you have this state [points to $\psi(x)$] ... and you want to ask the question of, you know, "what about that state [$\psi(x)$] being in this [$\varphi_n^*(x)$] eigenstate." And, or what is- to me I'm looking at this thing I'm thinking, "what is the projection of this eigenstate onto this wave function," or maybe vice versa, but I don't think it should matter- dot products are ... commutative

Bilbo generated an integral when asked for a probability, and discussed it with very similar language as was used for Dirac brackets above—that of a "projection" of one state onto another, both being represented as different functions within the integral. This supports a potential symbol template of "$\int f(a)g(a)da$" being paired with a conceptual schema encapsulating ideas of "projection" or "dot product," providing the functional analog to the Dirac inner product expression.

Castor and Delilah were asked to find the probability of measuring a particle to be in the left half of an infinite square well, and used very different language to discuss their work:

Delilah: [writes $\int_0^{\frac{L}{2}} \frac{1}{8}\left(\sqrt{3}\varphi_{E_1} + \varphi_{E_2} + 2\varphi_{E_3}\right)^2 dx$] So we're- at every position we're computing [points to integrand]- like every infinitesimally small position we're computing the probability [again points to integrand]-

Castor: You're taking the probability, kind of at like an instant, and taking that like, infinitely small sum to get your, like, probability distribution.

Delilah: [writes "$|\psi(x)|^2 = Probability\ Density$"] So yeah, so it just yeah, every- every infinitesimally- ... dx, yeah, essentially. We're finding the probability of it being there ... [it's] just the summation of all the values in the left half of the well.

It is worth pointing out that Castor and Delilah are, on the surface, doing a very different problem than Bilbo was above. Bilbo was finding a probability for an energy value, while Castor and Delilah were finding the probability of a range of position measurements. While a mathematical equivalence can be shown between these two problems, they appear to be different problems to the students, as they reasoned about the two integrals very differently. First, even though Castor and Delilah generated an integral, the symbol template is very different. Rather than two different functions (representing two separate quantum states) as the integrand, they instead used $|f(a)|^2$ (which may perhaps be thought of as $f^*(a)f(a)$, or the complex conjugation of the same function with itself). Second, Bilbo used very similar "projection" reasoning as was paired with the Dirac brackets in all three interviews, even explicitly bringing up dot products. Castor and Delilah instead reasoned with probability distributions and Riemann sums, invoking an "adding up pieces" model for integration (Hu & Rebello, 2013; Jones, 2015).

The difference in how these integrals are written and interpreted suggests Castor and Delilah are utilizing a different symbol template here: "$\int |f(a)|^2 da$" or perhaps "$\int f^*(a) f(a)$". It also seems evident that there may be a further symbol template here: "$|f(a)|^2$", which Castor and Delilah associate with a schema of "probability density."

Conclusions

The symbolic forms framework gives us a language with which to discuss student interpretations of symbolic mathematical expressions. In our study we have seen that, in a spins-first context, students develop a multitude of conceptual schemata and numerous symbol templates, each often associated with multiple of the other. While we could take each identified template/schema pair and declare them all as separate symbolic forms, the level of overlap between many of them suggests it may be more productive to instead treat them as laid out in Table 1: as conceptual schemata that can be expressed in various ways by using various symbol templates, depending on context. This could be seen as a complication of an otherwise simple framework, but we believe that it is a more accurate description of the reasoning needed in upper-division QM coursework. Students need to be able to start from either a Dirac state vector expressed as a ket or a wave function and work their way to an expression for a probability, sometimes expressed in Dirac notation and sometimes in wave function notation. We posit that it is these shared conceptual schemata among different symbol templates that allow for students to understand how and where to make mathematical decisions to arrive at an answer—and that they may even partially explain how students reason and translate across notations. Developing this interwoven mathematical vocabulary is one challenging aspect of learning quantum mechanics.

Acknowledgements

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Mathematicians' Characterizations of Equitable and Inclusive Teaching

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With an increased call to attend to diversity, more postsecondary mathematics instructors are interested in using equitable and inclusive teaching (EIT) practices. Yet, ambiguity persists on what EIT practices look like in mathematics classrooms. Our study investigates the characterizations of EIT from 13 participants, each involved in a year-long equity focused workshop, with the purpose of describing what topics arose in their characterizations. Our analysis shows characterizations of EIT included both the student and teacher role in EIT, and focused heavily on awareness of students' experiences and backgrounds, student engagement, and teacher self-reflection. We provide specific details from participants' characterizations for each category and connect them to established EIT literature. These findings can help researchers and department leaders engaged in change for EIT see what topics in EIT are being consumed and what topics might need emphasis or introduction.

Keywords: Equitable and inclusive teaching, instructor beliefs, postsecondary mathematics

The past two decades have seen enhanced calls to promote diversity, equity, and inclusion (DEI) in postsecondary education broadly and mathematics specifically (e.g., Inniss et al., 2021.; MAA, n.d.; U.S. Department of Education, 2016). Such efforts are often tied to the increasing gaps in Bachelor's degree attainment between white students and members of some racial minorities (National Academy of Science, 2010; Riegle-Crumb, 2019; U.S. Department of Education, 2016). In particular, Black and Latina/o students who enter college as a STEM major are more likely than white students to switch to a non-STEM major or leave college altogether (Riegle-Crumb, 2019). Participation in the "mathematics pipeline" plays a particularly significant role in shaping the opportunities for economic and social advancement afforded to students (Gutiérrez, 2012a, p.19), yet school mathematics traditionally excludes people of color, women, and individuals who have disabilities. Many educators agree students need opportunities to engage in mathematics that valorizes their own cultural experiences, and gives them a tool for challenging the status quo in mathematics and Western society broadly (Gutiérrez, 2012a). However, research has shown instructors may struggle to provide such opportunities for their students (Rubel, 2017), and more work is needed to better understand how instructors conceptualize equitable and inclusive practices.

Postsecondary mathematics instructors play a key role in both determining the mathematics students are exposed to, as well as in shaping cultural norms around teaching within mathematics departments (Reinholz & Apkarian, 2018; Smith et al., 2021). Their beliefs and attitudes toward equitable and inclusive teaching (EIT) determine their decisions in the classroom, and further their ability to advance change in their departments (Johnson et al., 2022). There has been a shift in the field toward the use of active learning teaching methods as a way to address inequities in mathematics participation, but these methods do not automatically create an equitable classroom

(Brown, 2018; Johnson et al., 2020, Reinholz et al., 2022). Williams et. al (2022) analyzed participants' conceptualizations of active learning and found that equity was rarely connected. Given the close tie between beliefs and practices (e.g., Speer, 2008), it is important to understand how mathematics instructors conceptualize equity. An analysis of how "inclusion" has been used in research articles from 2010 to 2016 shows this term is used as a single ideology and as a way of teaching (Roos, 2018). Thus, the phrase "equitable teaching and inclusion" has the potential to be used and understood in many different ways, possibly diluting its intention and power. Therefore, it is necessary to monitor how postsecondary instructors are interpreting and influenced by these concepts. To this end, we studied how postsecondary mathematics instructors committed to active learning and teaching more equitably conceptualize EIT. Our overarching research question is: *What characteristics arise when mathematics instructors are asked to define EIT?* We intentionally left our research question broad to capture the varied and rich responses shared by our participants; in our discussion we draw connections between our findings and the research literature.

Theoretical Background

Learning is a sociocultural experience involving a complex interaction among instructor, students, content, and the underlying social and cultural systems in which these components operate (Hawkins, 1974; Lampert, 2001). Mathematical knowledge is co-constructed in the classroom with students through a focus on establishing sociomathematical norms and the development of a mathematical identity (Brown, Collins, & Duguid, 1989; Cobb & Bowers, 1999; Cobb, Yackel, & Wood, 1992; Lave & Wenger, 1991). Yet, Gutiérrez (2012b) argues even those who acknowledge a sociocultural perspective on learning and see knowledge as constructed in negotiation with others in a community of practice (Wenger, 1998; Cobb & Yackel, 1998) often fail to take into consideration students' identities or issues of power in a class community's construction of mathematical knowledge (Gutiérrez, 2010).

Gutiérrez (2002) defines equity as "the inability to predict mathematics achievement and participation based solely on student characteristics such as race, class, ethnicity, sex, beliefs, and proficiency in the dominant language" (p. 153). In later work, she expands this definition to include four dimensions: access, achievement, identity, and power (Gutiérrez, 2012a). Briefly, access refers to the resources students have access to for learning, and achievement references educational student outcomes. Both dimensions make up the dominant axis of equity, as they focus on helping students succeed according to the dominant culture's values. Identity and power make up the critical axis. Gutiérrez uses a mirrors and windows metaphor to describe identity, which focuses on opportunities for students to see mathematics as meaningful in their everyday lives: there should be places in the mathematics curriculum both for students to see their identity reflected back at them (mirrors) as well as be given access to the broader world of mathematics (windows). Power refers to issues of social transformation, including which voices are highlighted in education, using mathematics for social justice, etc.

There are several pedagogical traditions which prioritize inclusivity (e.g., culturally relevant pedagogy, critical pedagogy, humanizing pedagogy, reality pedagogy). While each tradition has different foci, collectively, inclusive pedagogies highlight the importance of empowering students by using students' rich cultural backgrounds and identities to design curriculum, providing students the tools to enact social and political change, positioning students as experts in the learning process, fostering caring and trusting relationships between instructors and students, and embracing students' voice and lived experiences (including their social, political, psychological, and emotional selves) in the classroom (Carter, Andrews, & Castillo, 2016; Emdin, 2016; Howard, 2003; Macrine, 2009; Tuitt, Haines, & Stewert, 2018).

Methods and Participants

This qualitative study investigates how 13 undergraduate mathematics educators from eight institutions involved in SEMINAL's (see Smith et al., 2021 for more information) equity-focused professional learning community (PLC) conceptualize EIT. Each participant was paid a $1,000 stipend to participate, which met every two weeks from September 2020 to May 2021. Participants were interviewed for approximately 60 minutes at the beginning, middle, and end of this timespan. They shared their racial and gender identities in the interview and names were pseudonymized to support participants' confidentiality. Participants include three women of color (Aadaya, Camila, and Cassandra), one man of color (Robert), five white women (Crystal, Emma, Kathleen, Lacy, and Shea), and four white men (Bill, Collin, Mark, and Thomas).

For this study we focused on responses to the question "How would you define or characterize equitable and inclusive teaching (EIT)?" from their first and last interviews. We analyzed responses using rounds of first and second cycle coding (Miles et al., 2014). For the first cycle of analysis, we developed 13 participant summaries to identify key ideas within the participants' responses. These key ideas were highlighted within the transcript, summarized below, then agreed upon by at least three of us. In pairs we then engaged in a second cycle of coding to sort the summarized key ideas for one-third of the 13 participants into different categories. Each pair looked at a different third of the participants. We then met to compare categories. The commonalities across groups led to a condensed structure that focused on awareness of students, student engagement, teacher reflection, and specific actions for EIT. The first three authors drafted a codebook (see Table 1) of these overall categories with defined subcategories, independently coded three participant responses (by coding each of the highlighted key ideas within the response) and met to reconcile and subsequently refined the codebook. Afterwards, they independently coded and reconciled the remaining 10 participants.

Findings

We generated three major codes of how participants characterize EIT: *Awareness and Consideration of Students, Student Engagement,* and *Reflective Practice* with relevant subcodes (see Table 1). All participant's conceptualizations of EIT included each of these three codes. Table 1 includes counts for each subcode. Below we describe each of these codes in detail using examples from participants' responses.

Table 1. Characterization of EIT and Number of Responses

Code	Description	# of responses (n=13)
Awareness and Consideration of Students: Awareness and consideration of how students' environments and lived experiences impacts their identity and capacity; this one is about the more general classroom environment		

Background and Learning Experiences	Mention of the student's broader background or environment: prior academic/learning experiences, "meeting them where they are," etc. E.g., "They bring different experiences to the classroom"	11
Access to Material Resources	Mention of student access to materials, finances, resources E.g., "Some students can't attend office hours"	6
Emotions	Mention of students' feelings, emotional health, "social belonging" (which includes a nod to gender, identity, disabilities, bringing whole self to the classroom) E.g., "Students should feel safe"	11
Student Engagement: Emphasizes the student experiences and academic culture of the classroom; ensuring each student possesses a sense of academic belonging		
Participation	Focus on student participation in the classroom E.g., "Students should engage with peers and content"	13
Ownership/ Investment	Focus on student investment and ownership of participation (ownership of their learning), usually more specific than general participation E.g., "Students should feel like the material is for them"	3
Reflective Practice: Acknowledges influence of pedagogical or personal views on the classroom environment as well as specific actions for EIT		
Self-reflection	Acknowledging and/or reflecting on personal biases and teaching practices; may include broad statements of what an instructor should believe or do E.g., "I had all these preconceived notions"	12
Actions	Specific actions or teaching practices teachers ascribe to themselves as beneficial for the individual student experience as well as the classroom as a whole E.g., "Bring in examples that bring up topics of racism"	9

Awareness and Consideration of Students

All thirteen participants discussed being aware of students' experiences and considering how these experiences connect with issues of equity and inclusivity in the classroom. Three subcodes emerged: Background and Learning Experiences, Access to Material Resources, and Emotions.

The Background and Learning Experiences subcode refers to a broad consideration of students' individual backgrounds and characteristics (e.g., race, culture, socioeconomic status, first generation status, marginalization, prior academic experiences, etc.). For example, Cassandra asserted no one should have an "advantage or disadvantage based on something they have no control over" referring to students' background. Emma also discussed this idea, saying part of EIT is “accepting that your students are coming into class with different lived experiences and that affects how they interact in the classroom.” Mark’s definition of equity in the classroom included his recognition of “different levels of preparation, [...]comfort, or different prior experiences.” Some participants went further, suggesting consideration of students' backgrounds is necessary but not sufficient for EIT. Crystal explicitly noted it is not enough to “recognize” students – EIT involves being “proactive” although she did not elaborate on what this means.

Access to Material Resources captured explicit references to financial restraints or situations possibly impacting student access to the resources necessary to succeed in class. Responses ranged from minimizing financial strain on students (Thomas) to acknowledging the impact priorities outside the classroom (e.g., having to take care of siblings) have on student capacity to learn in and outside the classroom (Cassandra and Shea). There was significant overlap between the Access to Material Resources subcode and other subcodes under Awareness and Consideration of Students; five of the six participants whose excerpts were coded the former were double-coded with another subcode in this code. For example, Cassandra described how “you shouldn't be made to feel that your questions or answers or situation is somehow your fault or an indicator that you do not belong because of your financial background, because of what you look like, because of any physical or emotional issues that you may be dealing with," which involves both students’ backgrounds and their access to material resources. Participants in this category are making connections between a student’s background experiences and how they affect student access to materials.

Eleven out of 13 participants considered students’ emotional health and comfort level in class. The following excerpts describe the major ideas of this subcode: that students “shouldn’t be made to feel like [their] questions or answers or situation is somehow [their] fault or an indicator that [they] do not belong because … of any physical or emotional issues” (Cassandra) and that when in the classroom, students “feel like they can be their whole selves” (Shea). Seven participants explicitly used the keywords “safe”, “welcoming”, and “belonging” when describing their ideal classroom environment. In addition, several participants also made the distinction between “for all” and “for each” instruction, implicitly linking their responses to equity. Thomas observed that students with certain characteristics connect with teachers and peers “if they’re lucky”, while Crystal wanted “everyone in [her] classroom to feel valued and feel heard.”

Student Engagement

All participants mentioned providing student opportunities to engage and participate in the classroom as part of EIT, with a common theme of making sure certain groups of students were not “overlooked” in the classroom. The Participation subcode focuses on response excerpts focused on student participation and additions to the classroom; as Aadaya stated, “equitable and inclusive means that I'm not, you know, overlooking some students or kind of leaving them out of a discussion. So this is the participation aspect.” Thomas related student participation to his own experiences: “So I would want to involve students like me who, you know, who don't mind working on their own. I want to make sure that the quiet ones, which is, I guess, how I characterize myself, are not overlooked." In addition, Camila described inclusive teaching as “reaching out to the students in class, making sure that they participate in the class and again that it's not dominated by one group versus another." Crystal also discussed giving students who are “marginalized” opportunities to participate:

> One aspect is like recognizing everybody that's in the room, like the people who are marginalized. I want to be able to recognize and give them a chance to participate and have

them feel valued and seen as well as the more confident students who sit in the front of the room and participate all the time.

Some participants addressed wanting students to feel safe in the classroom, but Crystal's excerpt goes beyond this, illustrating she also wanted students to feel safe to "participate all the time".

The Engagement subcode shifts perspective to excerpts where participants described wanting to ensure student access to the mathematical content and students' sense of ownership in the classroom. Robert discussed creating a classroom where "each student has not only access to whatever we're talking about through whichever entry point, but then also them feeling empowered and safe to speak up in class." Similarly, Mark described wanting students to "feel like the mathematics is there meeting some need or meeting some interest, or like it's something that is presented for them … that they have some sense of ownership." Mark then elaborated: "I would say that they're not kind of feeling alienated from [the mathematical content], or that they're like trespassing somehow on a domain that the property belongs to someone else. That [the material], it's for them."

Reflective Practice

All 13 participants mentioned something about the role of the teacher, specifically a self-reflection and/or a specific teacher action for EIT. We use this section to discuss categories.

Twelve out of 13 of participants' responses were coded with Self-reflection. This subcode captured responses that indicated the importance of self-reflection to teach equitably and inclusively, as well as responses in which participants engaged in self-reflection about their own teaching practices. For example, Shea shared that as an instructor, she needs to "sit with the discomfort of knowing" that even if she thinks an activity is going well "that doesn't necessarily mean that it is inclusive or equitable," suggesting that EIT requires reflection on how students and teachers may experience the classroom differently. Other participants further discussed how reflections about student experiences should inform practice. Ryan shared the importance of "meeting students where they are and not having students meet you where you are" in EIT. In this example, Ryan acknowledged that his practice of teaching should change even if he did not include a specific way to enact that change. Of note is the specificity of some reflections. Aadaya discussed her realization that despite self-characterizing as "approachable", she possesses many "preconceived notions" regarding "this is how all students are." She concluded by linking a specific focus on being "vulnerable to understanding" students and their backgrounds as part of her definition of EIT.

We used the subcode Actions when participants described something specific that a teacher did or could do in connection with EIT. If the action was general, like "be more open," then it was not included. Nine out of the 13 participants' responses included a specific task or practice including pointing students (especially students with less access) to specific resources, purposefully including topics of racism or environmental change into the classroom, coordinating sections to be more comparable to one another, making sure that "everyday life problems" are inclusive, making an effort to pronounce names correctly, intentionally asking identity-related questions (e.g., "What pronouns do you use?"), making sure that the grading system aligns with letting students grow and be their whole selves, not reacting negatively to students who do not follow academic social norms because they might not have access to them (e.g., "not being insulted when students communicate with me in a way that is not academic"), and learning and changing teaching practices to help with student needs and disabilities (e.g., try to not talk while facing the board for students who read lips). Most of these specific actions were described as EIT practices instructors should do.

Discussion & Implications

Each of the participants' responses is categorized by at least one subcode from the overarching codes: Awareness and Consideration of Students, Student Engagement, and Reflective Practice. This suggests participants view EIT in terms of who students are, how that impacts their engagement in the classroom, and what the instructor can do to affect that engagement. Connecting to Gutiérrez's (2002) definition of equity, several participants expressed the belief that students should not be disadvantaged based on their backgrounds, including characteristics such as race, as well as students' academic experiences and access to material resources. This includes a belief that it is the instructor's responsibility to make sure students are not disadvantaged, or feeling alienated, by the dominant, sociomathematical norms of postsecondary mathematics classrooms. This reflects Gutiérrez's (2012a) notion of the importance of addressing students' access to resources for learning.

Many participants also reflected on their actions as teachers, specifically on their personal biases and practices. In particular, responses indicate that identifying areas of growth, maintaining flexibility, and leaning into the discomfort of change are necessary to create equitable classrooms. As Tuitt et al's (2018) state, instructors engaging in EIT practices must be resilient, as EIT involves a significant amount of discomfort and emotional labor. Additionally, this reflective practice (Schön, 1983) is central in all teaching, but in EIT this reflection involves thinking about how students experience the classroom differently than oneself and acting accordingly. Some participants mentioned efforts in their teaching to consider how students experienced their classrooms and recognize that experience differs from their own.

While some discussed the importance of including what Gutiérrez's (2012a) calls mirrors in the curriculum, or, as Tuitt et al. (2018) describe, opportunities for students to personalize content by seeing examples from their cultural or lived experiences, as well as the importance of including topics such as racism, most participants did not stress EIT as a tool for social and political change. This is unsurprising given how difficult instructors find it to create mirrors for their students in the classroom (e.g., Rubel, 2017). Yet, most participants view EIT as important in addressing students' emotional wellbeing, aligning with more humanizing pedagogies which encourage students to embrace their whole selves in the classroom, as well as desire to have students take ownership over their learning.

Our findings can help researchers and department leaders engaged in change for EIT see what topics in EIT are being consumed and what topics might need emphasis or introduction. The participants were connecting EIT with being aware of a variety of aspects about students, but we did not see many responses explicitly mentioning how these aspects relate to power. Yet power is an integral part of EIT and needs to be better understood. Who has the power in EIT? How do you distribute power in EIT? What role do departments have in distributing power within and outside classrooms? Additionally, participants often connected student participation with EIT, but there is an opportunity for emphasizing how students are feeling when they are participating. How are teachers supposed to learn how to attend to student learning, student feelings, and their own learning about EIT? How can departments help support their teachers learning this level of attention?

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Adapting the Argumentative Knowledge Construction Framework to Asynchronous Mathematical Discussions

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Weinberger and Fischer (2006) designed a framework for analyzing learning the context of asynchronous discussion activities. Operationalizing this framework, we analyzed the social and cognitive aspects of a discussion activity in an asynchronous calculus course. From this analysis, we identified aspects of Weinberger and Fischer's (2006) framework lacking in explanatory power for mathematics-specific discourse and developed an amended framework. We propose that this amended framework may enable in-depth analysis of major dimensions of students' mathematics knowledge construction as they engage in activities in an online asynchronous modality. This framework may also support the curriculum development for online asynchronous mathematics coursework.

Keywords: Calculus, Knowledge Co-Construction, Asynchronous Instruction, Design Research

Introduction

Classroom discourse can provide a powerful opportunity for students to gain mathematical knowledge. Though many students learn mathematics in asynchronous formats, the kinds of learning taking place in asynchronous formats remains severely underexamined. We are motivated to explore in more detail the learning within asynchronous discussions; specifically, we study discussion activities as we contend they are a primary point of contact in asynchronous formats among students, their peers, and their instructor.

In prior work (Keene, et al., 2016), we proposed use of Weinberger and Fischer's (2006) *Argumentative Knowledge Construction* framework for design research. We now present the results of a study operationalizing and extending Weinberger and Fischer's framework to examine the knowledge construction taking place in asynchronous calculus discussions. Our analysis of multiple discussion boards revealed that aspects of Weinberger and Fischer's framework required alteration to increase the explanatory power of the framework and enable an accounting of students' contributions to the discussion from an anti-deficit perspective. In this report, we present key aspects of our updated framework, highlighting its ability to gain insights into the learning taking place in discussions across social and cognitive dimensions. Accordingly, we propose to answer the following research questions:

1. How do the social, epistemic, and argumentative dimensions of knowledge construction interact with each other in the context of a mathematics classroom?
2. How does the *Argumentative Knowledge Construction* (AKC) Framework change when contextualized in a mathematics class?
3. What alterations to the AKC Framework increase our understanding of student learning in online calculus courses with discussions?

Theoretical Framework – Argumentative Knowledge Construction

Weinberger and Fischer proposed that computer-supported collaborative learning could be analyzed according to four dimensions: *participation, epistemic, argument,* and *social modes of co-construction* (Weinberger and Fischer, 2006).

The *participation dimension* examines the quantity and heterogeneity of students' contributions to the discussion board for each discussion activity. The *epistemic dimension* focuses on the content of students' contributions, attending particularly to the degree to which students' contributions adequately relate the particulars of a problem with the intended concepts that the problem engages. The *argument dimension* derives from Toulmin's (1964) model of arguments to qualify the types of micro- (single line) and macro- (multi-line) argumentative moves put forth by students in pursuit of a solution. Finally, the *dimensions of social modes of co-construction* "describe to what extent learners refer to contributions of their learning partners" (Weinberger & Fischer, 2006, p. 77). In an asynchronous modality, participants' textual, imagistic, and video submissions can be retroactively analyzed to build group-by-group comprehensive accounts of the knowledge construction associated with a particularly designed prompt. We consider the dimensions to productively account for knowledge construction in the online setting and so our work attempts to refine these dimensions in the mathematics context.

Table 1: Summary of Argumentative Knowledge Construction framework (Weinberger and Fischer, 2006).

Social Modes ● Externalization ● Elicitation ● Quick Consensus Building ● Integration-oriented Consensus Building ● Conflict-oriented Consensus Building	**Epistemic** *Construction of…* ● Problem Space ● Conceptual Space ● Problem ↔ Conceptual Space ● Problem ↔ Prior Knowledge ● Non-Epistemic Activities
Argument ● Single Arguments (micro-) o Simple, Qualified, Grounded, Grounded & Qualified, Non-Argumentative ● Line of Argumentation (macro-) o Argument, Counterargument, Integration, Non-Argumentative	**Participation** ● Quantity ● Heterogeneity

Literature Review

Despite the prevalence of online mathematics courses, we still know very little about the ways that students make mathematical meaning in these formats (Trenholm et al., 2019). From what has been explored thus far, some have considered mathematics courses among the more difficult to teach in an online format (Engelbrecht & Harding, 2005).

While discussion activities are common in face-to-face mathematics courses, discussions are not widely used in asynchronous formats. For instance, in a survey targeting the format and content of asynchronous courses, 39% of instructors surveyed used at least 1 discussion (Trenholm et al., 2015). Thus far, research on discussion activities focusing on Weinberger and Fischer's (2016) AKC framework have studied face-to-face mathematics learning (Keene, et al., 2016) and asynchronous learning in nonmathematical courses (Schrire, 2006; Clark & Sampson, 2008; Dubovi & Tabak, 2020). We contribute to this literature by examining the learning taking place within an asynchronous mathematics discussion activity, exploring how the AKC framework can be operationalized specifically in a mathematics setting.

As noted in the framework above, one part of the AKC framework is supported by the work of Toulmin (1958). In Toulmin's model, he provides a structure and function for argumentation in learning. This work involves the identification and connections between data used in argumentation, claims that participants make, and warrants they provide. Toulmin's model has been used in the analysis of in person mathematical discussions in active learning classrooms (Giannakoulias et al., 2010; Groth & Follmer, 2021; Mariotti & Pedemonte, 2019) but not in asynchronous learning. Nevertheless, this work supports our modifications of the AKC framework to contribute to our understanding and support design of new asynchronous learning activities.

Methodology

The data analyzed for this study were comprised of the textual records of two small-group discussion activities assigned to a fully asynchronous calculus course. Participants for this study were Calculus 1 students at an online, primarily undergraduate university during a 9-week term in spring 2022. The student population at this university is non-traditional: the average age is 31, 13.7% of the population is female, 60.3% of students are active military, and 19.5% of students are veterans. The data consisted of five discussions activities given to four assigned groups and moderated by one of the authors.

One researcher (the instructor of the course) initially reviewed each group's work according to the *participation* dimension. For our initial theory-building purposes, we sought to analyze groups for which there was potential to inform the interactions of the argumentative, social, and epistemic dimensions. As such, for the first analysis, we selected one group that we determined to have high participation, both in quantity and in heterogeneity. This group contained 3 male students that we refer to as BK, LG, and DG.

We began by examining this group's work on two discussion activities. For the first discussion activity, we individually coded each line of the discussion according to a single dimension of the AKC framework. We then met to compare and negotiate the codes for the selected dimension. After the discussion was coded for each dimension in the AKC framework, we discussed whether any lines in the discussion were not captured by the AKC framework or would benefit from a modification to the AKC framework. From this analysis, we proposed changes to that framework so that the epistemic, argumentative, and social dimensions yielded more robust insights for math-specific knowledge construction. We then analyzed the same group's discussion on a second discussion task according to this new framework in a similar iterative manner: individual coding for a single dimension, meeting to compare and negotiate codes, repeating these two steps until all dimensions were analyzed, and ending with a discussion of what was not captured by the modified AKC framework.

Due to space limitations, we will present results of the analysis on the second discussion prompt:

> *Your overall goal is to draw a random curve representing a function f defined on the interval $[2,\ 12]$, and then construct the graph of a second function g such that the following requirement is satisfied: For the composite function $h(x) = g(f(x))$ (meaning g composed with f), $h'(x) = 2$ at each x-value on $[0,\ 10]$.*

Data Analysis

Posts and replies to the discussion prompt were collected and manually entered to an Excel sheet. Posts were split by sentence and any equations/graphs were linked to a separate page. The researchers then individually coded each sentence by dimension according to the altered AKC framework. After each dimension was individually coded, the researchers negotiated the final coding for that dimension and discussed how effective the coding captured knowledge construction. Once all posts/replies were coded by each dimension, the researchers met to analyze the codes to gain cross-dimensional insights into the knowledge construction occurring during the discussion. This cross-dimensional analysis originally included tallies of the various codes and attention to the density or sparsity of certain interactions considering the progress made by each student in the epistemic dimension. While this provided general insights into how the students' interactions contributed to their epistemic progression, we determined that a more refined account of the social interactions would yield more robust insights. We present the results of our analysis considering this refined account of the social dimension.

Results

Due to space limitations for this proposal, we present one noteworthy result to appear from our initial analysis: conceptualizing the social dimension as a network of argumentative interactions. This alteration of the social and argumentative dimensions allowed us to analyze the interaction between macro- and micro-argumentative codes and provided insights that allowed for more readily seen inter-dimensional insights. What follows are two examples from the three-person group discussion of BK, LG, and DG. These examples will highlight what we learned by using the revised AKC framework. We will then present our revised framework in the discussion.

Interactions within the Argumentative Dimension

Figure 1 presents the visualization of 4 different chains of posts for the chosen discussion and group. The codes are chronological as they move down the page. Each column represents the analysis of one post and its replies. The three student participants were assigned different shades of grey: black (BK), light grey (LG), and dark grey (DG). Arrows represent replies to either the original post or another reply in the post chain. Visualizing these chains of arguments allowed us to identify patterns that were not easily identifiable in our original coarse analysis. Macro-argumentative codes are visualized by the shape of the post/reply as described in the legend.

We found that including Toulmin's model allowed for a more robust depiction of the social contributions of the students throughout the discussion than the original depiction of the social dimension. In these discussions, and more generally in mathematics discussions, social interactions largely (or most productively) entail argumentation. As such, including Toulmin's scheme in conjunction with the macro-argumentative codes allowed our new Argumentative dimension to account both for the argumentation taking place and for the social interactions contributing to knowledge construction. Moreover, the inclusion of Toulmin's Scheme for Argumentation allowed us to identify the type of data being used to make mathematical claims.

The visualization of the argumentative dimension also allowed for identifying social moves students made. For example, consider chain 1. We coded the first reply as a macro-counterargument and yet as a non-argumentative move. This captured a social counterargument that was not grounded in mathematics and was coded as conflict-oriented consensus building in the social dimension in the original AKC framework. We see another example in chain 2 with a circle X pairing, which suggests an integration-oriented consensus building in the social dimension in the original AKC framework. These results suggest that interactions within the argumentative dimension can account for codes in the original AKC social dimension, as we hypothesized by our alteration of the original framework.

Legend

Macro-Argumentative

Cloud	Argument
Star	Counterargument
Circle	Integration
Square	Instructor
Triangle	Non-Argumentative Move

Micro-Argumentative

C	Claim
D-S D-G D-T	Data represented as Symbols, Graph, or Table
W	Warrant
Q	Modal Qualifier
R	Rebuttal
X	Non-Argumentative Move

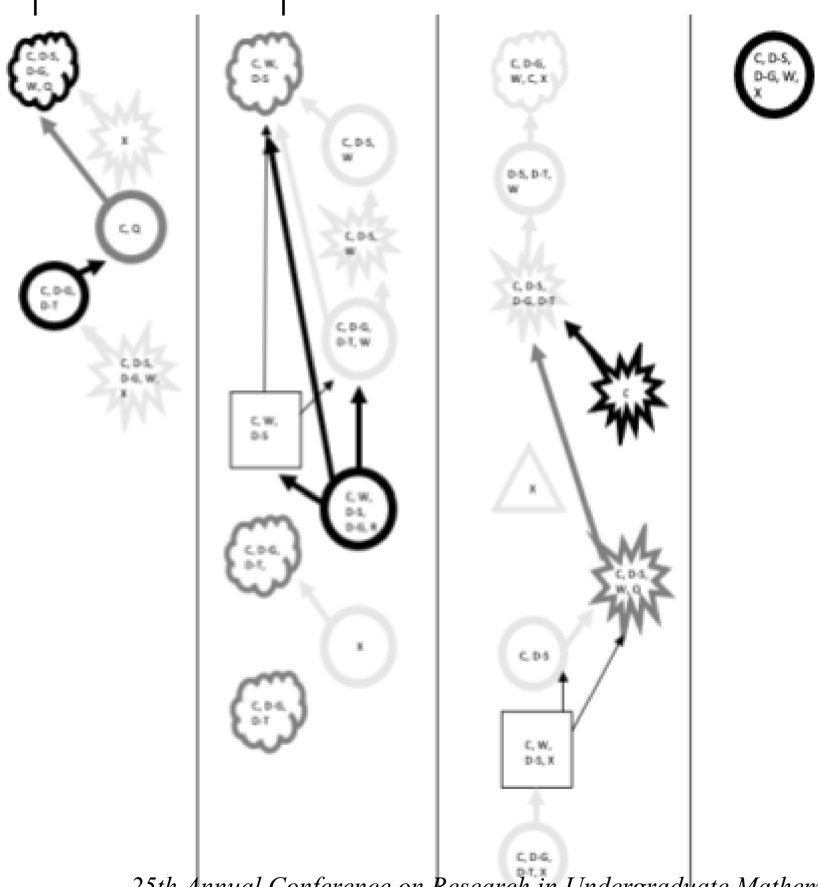


*Figure SEQ Figure * ARABIC1: Visualization of Macro- and Micro-Argumentative codes*

Interactions between the Argumentative and Epistemic Dimensions

Figure 2 presents the visualization of the same 4 chains of posts with the epistemic codes placed within the macro-argument codes. The major result from our using our modified framework for the analysis is greater insight into the interactions between the epistemic and argumentative dimensions. Specifically, our original analysis flagged the original discussion as largely argument-driven (rather than incorporating counterarguments and integrations heavily), and that much of the epistemic change was independent of the social interactions in the discussion. With this altered framework, we drew clearer links between epistemic progression and counterarguments or integrations. For example, in chain 1 we see LG integrate BK's argument including composition as both a product (as BK did) and as inputs/outputs. We then see BK integrate LG's integration and include composition as inputs/outputs. Another example occurs in chain 2 where the instructor introduces the idea that derivatives describe where a function is increasing/decreasing and then BK uses this idea to integrate three different arguments. These examples illustrate that the inclusion of a new epistemic code can cause students to either counter this new idea or integrate it into their own argument. Moreover, tracking epistemic codes allows us to see that interactions are not restricted to explicit replies. For example, the conceptualization of derivative as increasing/decreasing, as presented by the instructor in chain 2, was present by BK in chain 4. This is important to note given the asynchronous nature of the discussion and the ability of students to read over what others have said without directly interacting.

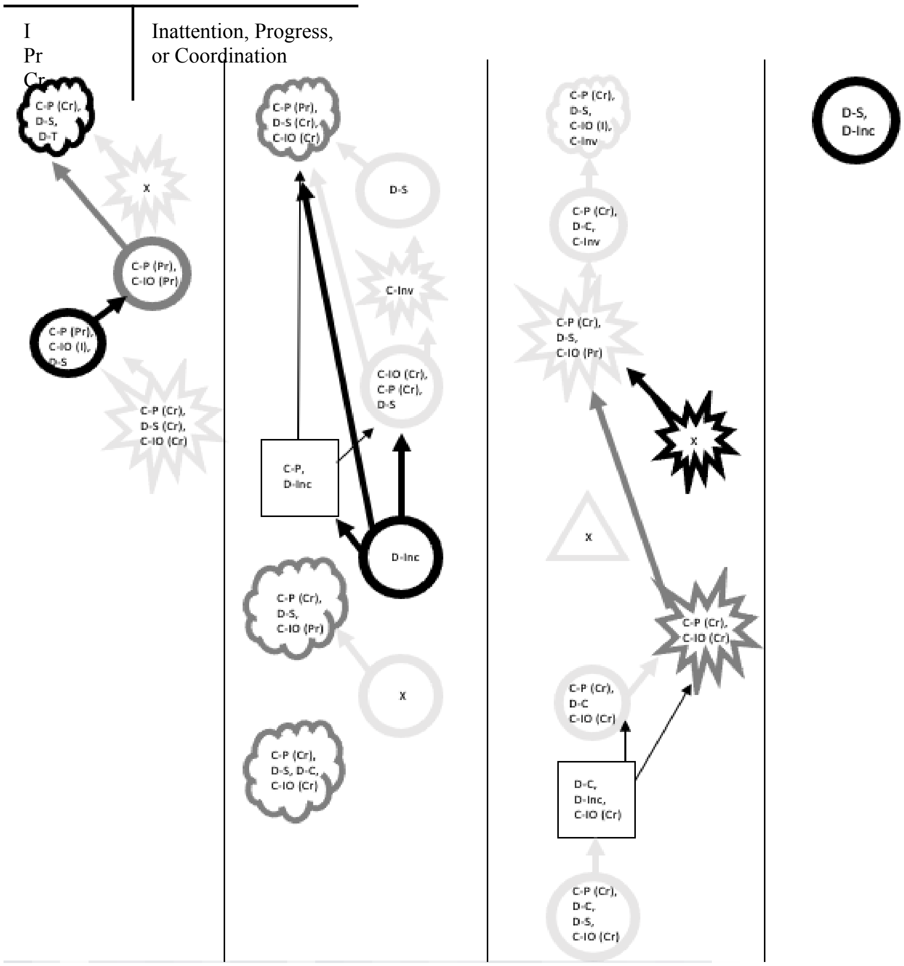
Legend

Macro-Argumentative

Cloud	Argument
Star	Counterargument
Circle	Integration
Square	Instructor
Triangle	Non-Argumentative Move

Epistemic

C-P C-IO C-INV	Composition as Product, Input/Output, or Inverses
D-INC D-S D-RoC D-C D-T	Derivative as Increasing, Slope of Tangent, Rate of Change, Computation, or Tangent Line



Discussion

After analyzing and discussing two sets of discussion posts, we arrived at three major revisions to Weinberger and Fischer's (2006) Argumentative Knowledge Construction framework:

- Inclusion of Toulmin's Scheme for micro-argumentative codes;
- Removal of the original social dimension; and
- Switch from deficit to asset-based view of epistemic dimension.

Toulmin's Scheme is included in the micro-argumentative codes as it provides a way to track the types of data students use in their arguments and the ways students justified their arguments. The original social dimension was removed as we found the macro- and micro-arguments sufficient for describing social moves students made when constructing knowledge. Moreover, the mathematical nature of the discussions tended toward non-social arguments. The switch from a deficit-based view of understanding as a connection between the problem and conceptual space to ways students understand concepts allows us to apply current mathematics education research to evaluate how students' understanding evolves over time. An overview of our current Argumentative Knowledge Construction framework specific to mathematics is presented in Table 2.

Table 2: Current Argumentative Knowledge Construction framework specific to mathematics.

Argumentation	
Macro-Arguments • Argument • Counterargument • Integration • Non-Argumentative Moves **Micro-Arguments** • Claim • Data (Graphical, Symbolic, Tabular) • Modal Qualifier • Warrant, Backing, Rebuttal • Non-Argumentative Statements	**Epistemic** • Prompt-Specific Understandings • Progress Towards Coordination of Relevant Understandings • Broader Ways of Thinking Mathematically (When Applicable) **Participation** • Quantity • Heterogeneity

Conclusion and Next Steps

As is seen by looking at research published in undergraduate mathematics education, as well as the huge addition of online asynchronous mathematics teaching in the last 20 years, we believe that understanding how asynchronous learning may happen is a significant contribution to RUME. Our goal was to adapt Weinberger and Fischer's Argumentative Knowledge Construction Framework to asynchronous mathematics discussions and identify ways to systematically analyze student posts between dimensions. While more work is needed to refine the AKC framework specifically to mathematics, we feel that the use of macro-argumentative codes as a basis for visualizing student posts as a web of interactions will play an important role in analyzing knowledge development as it progresses through a discussion post.

We will continue to analyze further asynchronous discussions in Calculus using the modified framework. One particular type of response we would like to incorporate into the framework is non-content, non-social knowledge construction. For example, the instructor explaining how to use Desmos would not constitute mathematical knowledge in the epistemic dimension, nor would it constitute a social or non-social argument that would be captured in the argumentation dimension. Additionally, we anticipate expanding our work to other mathematics courses, and possibly sciences courses as well, which would require either generalizing what is contained in the epistemic dimension or creating subdimensions that work in unison.

This work is part of a larger design project to engage students and faculty in online mathematical learning. We will use our results and altered framework to support revisions and development of STEM courses in asynchronous environment. We also believe the new framework will be applicable to face-to-face student discussions in mathematics. Understanding how student communication is mathematics learning can be seen as an important area that can change mathematics instruction both online and in-person.

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Modeling South Korean Citizens' Quantification of COVID-19 Vaccination and Infection Risks Using Benchmark Values

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During the COVID-19 pandemic, citizens have had to make new decisions about vaccination and social contact based on their personal risk estimates and the risk estimates of experts and governments. We analyzed survey data with 50 participants and interview data with three participants to model how South Korean citizens quantify risks associated with COVID-19 vaccination and infection and the severity and level of concern they associated with these percentages. We found participants' quantifications for COVID-19 risks and their meanings for their estimated risks are substantially different than experts. Participants used benchmarks such as 10% risk and 50% risk to estimate and reason about their COVID-19 risks.

Keywords: Probability, Understanding Percent Risk

Since the start of the COVID-19 pandemic, mathematics educators have been interested in contributing to the collective effort to reduce risk from COVID-19 using prior mathematics education research (Krause et al., 2021). Citizens' perceptions of risk play an important part in their choices, and their choices impact personal and societal risk (Raiffa, 1982). Doctors and epidemiologists often communicate risks with percentages, and mathematics educators are well positioned to study how citizens' make sense of this mathematical information. We know the concept of percentages (and the related concept of fractions) is difficult for many people, including those with undergraduate degrees and students taking advanced mathematics such as calculus (Byerley, 2019; Mullis et al., 1991; Wright, 2006). Specifically, the RUME community has an interest in preparing undergraduates to make sense of real-world mathematics as well as an interest in informal adult education about mathematics (SIGMAA on RUME, 2022). To contribute to this mission, we studied South Korean citizens' mathematical understanding of risks associated with COVID-19 vaccination and infection.

We focused on risk estimates related to both vaccination and infection because vaccine hesitancy is a top ten global health issue (World Health Organization, 2019; 2020). There is a statistically significant association between COVID-19 vaccination status and US citizens' perception of the relative risk of vaccination and infection (Yoon et al., 2022). The study found that US citizens who thought COVID-19 infection was less risky or equally risky than vaccination were less likely to be vaccinated than those who thought COVID-19 infection was more risky than vaccination. Even though the risks due to COVID-19 infection are orders of magnitude higher than the risks due to COVID-19 vaccination (CDC COVID-19 Response Team & Food and Drug Administration, 2021), only 48% of a random sample of US citizens thought COVID-19 infection was more risky than vaccination (Yoon et al., 2022).

In contrast to the previous study by Yoon and colleagues (2022), we used South Korean participants in this study. It is important to study real-world mathematical thinking in a variety of cultural contexts because much can be learned by investigating similar questions in contrasting cultures (Stigler & Hiebert, 2009). The pandemic response, vaccination campaign, and mathematics education in South Korea are notably different from that in the United States. By early August 2022, more than 87% of the population in South Korea had been fully vaccinated and more than 65% had received booster shots (Korea Disease Control and Prevention Agency, 2022). Many South Korean citizens complain about social pressures to get a COVID-19 vaccine and believe the vaccination mandate is not reasonable despite the high vaccination rates (Han, 2022).

In order to better understand the South Korean public's vaccine hesitancy through a mathematics education lens, it is necessary to understand how citizens perceive the risks associated with COVID-19 infection and vaccination which are often expressed in percentages. Thus, this study investigates two research question: (1) How do South Korean citizens quantify their risk associated with COVID-19 infection and vaccination? and (2) What meanings do participants hold for their percentage estimates?

Literature Review and Theoretical Framework

Risk Perception

Risk has been defined as "a probability of an event combined with the magnitude of the losses and gain that it will entail" (Douglas, 2013) or a "probability-based forecast of the future" (Bernstein, 1996). Experts define risk based on relative frequency of occurrence, and while laypeople can assess frequency if asked, they tend to include other factors in their risk judgements (Covello et al., 1986, 1987; Slovic, 1987). These differences in the way the public and experts perceive risk can lead to miscommunication in which laypeople understand risk differently than experts or government agencies (Botterill & Mazur, 2004; Covello et al., 1986). For example, in the August, 2021 Gallup Poll, US citizens' mean estimate of the risk of dying from COVID-19 if infected and unvaccinated was higher than 50% (Rothwell & Witters, 2021), which differs dramatically from estimates by epidemiologists. Because, on average, citizens are giving such different estimates of risk than epidemiologists, it follows that their meaning for these percentages is likely different than that of epidemiologists. For example, citizens who estimated the mean risk of dying from COVID-19 if infected is 50% might disagree with the mathematically similar statement "half of the people who had COVID-19 died."

Risks in Frequency Formats and Percentage Formats

Prior research has shown that risks described in frequency formats (e.g., 10 of 100) are perceived as more serious than those in percentage formats (e.g., 10%) (Gigerenzer, 2011; Hoffrage et al., 2000; Slovic et al., 2000). Slovic et al. (2000) found that when assessing whether a hospitalized patient with mental disorder would harm others, people perceived a risk described with the frequency format, "20 out of 100 patients", as more serious than the same risk described in the percentage format, "20% of patients". Another study suggests expressing a small risk using a frequency format can frighten people because it is psychologically different from the same risk expressed as a percentage format (Purchase & Slovic, 1999). Another important aspect of understanding COVID-19 risk is numeracy which is defined as an individual's number sense and ability to understand and use numbers with concept of proportion, fractions, percentages, and probability (Dieckmann et al., 2009; Peters, 2012; Peters et al., 2006). Highly numerate people are more likely to be able to interpret the same risk presented in either frequency or percentage as equivalent (Peters, 2012; Peters et al., 2006). Exploring citizens' use of frequency format and percentage format may provide insight into how citizens understand probabilities when estimating risks associated with COVID-19 infection and vaccination.

Using Benchmark Values to Estimate Probability

Benchmarks are initial values that function as a point of reference to compare, assess, and adjust estimates of probabilities. We use the term *benchmark value* in this work to describe percentages that citizens' use frequently and easily assign meaning to. Citizens often use benchmark values as a point of reference, and they adjust estimates up or down from these values. Tversky (1974) suggests that when estimating the probability of an uncertain event, people tend to use a starting value and adjust it to produce the final estimates. Then, different initial values result in different estimates; if the starting point is too high or too low, the subsequent adjusting process yields a final answer that is close to the starting information, which may lead to overestimation or underestimation (Epley & Gilovich, 2006).

For example, a prior study found that many US citizens used the benchmark values, 0%, 50%, and 100% for their estimates of risks associated with COVID-19 infection and vaccination (Yoon et al., 2022). Konold (1989) suggests that people who use the 'outcome approach' in their probabilistic reasoning focus on outcomes of single trials of events by interpreting 0% as meaning 'no', 50% as meaning 'I don't know', and 100% as meaning 'yes'. Yoon and colleagues (2022) used the outcome approach construct to hypothesize the citizens' meanings for those benchmark values. Our theoretical focus on citizens' meanings is inspired by Thompson's work on mathematical meaning (Thompson, 2015). We remained alert that two different people could have different meanings for the same percentage and used the outcome approach as a way to make sense of the different meanings participants might have.

Methods

We collected interview and survey data. Both asked South Korean citizens to quantify their risks associated with COVID-19 infection and vaccination and to discuss how they reason about the risks. First, we conducted a survey with a random sample of 50 South Korean adults on June 13, 2022 using a Korean survey company, Survey Top (https://www.surveytop.co.kr/)[1]. In our sample, there were 23 male and 27 female participants. The mean of participants' age was 41.34 years (*SD* = 14.77). As the level of highest education, 22 citizens reported a high school diploma and/or some college credits, and 28 citizens reported having a bachelor's degree or higher. Out of the 46 participants who received at least one dose of COVID-19 vaccines, 13 citizens were fully vaccinated for COVID-19 (two-doses of Pfizer of Moderna or a single dose of J&J vaccine), 31 citizens were fully vaccinated for COVID-19 with one booster shot, and two citizens had received two booster shots. Four citizens responded that they had never received a COVID-19 vaccine. Survey questions were developed in prior research (Yoon et al., 2022) and included: 1) If you were infected with COVID-19, what do you think your percent risk of hospitalization from COVID-19 if infected. Why? 2) If you were infected with COVID-19, how concerned would you be about getting hospitalized? 3) If you receive a COVID-19 vaccine in the future, what do you think your percent risks of a serious adverse reaction would be? Why? Other questions are also asked such as demographic information, their reasons for vaccination, and general probabilistic questions.

We also recruited three South Korean citizens for interviews. All participants were male, in their thirties, and held an undergraduate degree in non-mathematical fields. Both Chang-seob and Byung-soo (pseudonyms) were fully-vaccinated with a booster shot. Byung-soo was vaccine-hesitant; he received COVID-19 vaccines because of social pressure. Sun-kyu (pseudonym) had

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1 The survey company provides information about the demographics of the panel and how to select a random sample from the panel (https://www.surveytop.co.kr/online-survey-panel).

not received any vaccines. Interviews were conducted through Zoom in Korean by the first author between June 19th, 2022 and July 4th, 2022. Each interview lasted approximately an hour and a half. During the interviews, the first author asked participants to quantify various risks associated with COVID-19 infection and vaccination, their opinions about COVID-19 vaccination, and to interpret meanings of particular percentages in various contexts. The first and third authors, who are South Korean, translated the US survey and interview protocol into Korean. We coded the data using a priori and emergent codes and conducted a thematic analysis (Braun & Clarke, 2006). While conducting our study and analyzing data, we were alert to the possibility that South Korean citizens would use benchmark values to reason about risk and that their meanings for a particular percentage might differ from experts' meanings.

Results

We found that participants' risk perceptions and reasonings were complex, but they commonly used benchmark values as a starting point for estimating their COVID-19 infection and vaccination risks. In both the survey and interviews, we observed that many participants used certain values, 0%, 1%, 10%, and 50%, as benchmark values. In order to answer the research question, we focus on how South Korean citizens quantify risks using benchmark percentages and make sense of the severity of those percentages.

Citizens' COVID-19 Infection and Vaccination Risk Estimates on Survey

In the survey data, we found noticeable clusters at 10% and 50% in citizens' estimates. In response to questions about risks related to both infection and vaccination, 10% was the most frequent estimate. When estimating their personal COVID-19 hospitalization risk if infected, 24% of citizens (12 of 50) estimated 10%. Further, 16% of citizens (8 of 50) estimated their risks of serious adverse reactions to COVID-19 vaccination as 10%. The question about adverse reactions clarified "An example of a serious adverse reaction is an allergic reaction requiring treatment in a hospital. Do not include your risk of common side effects such as fatigue." Another frequent estimate was 50%. For their COVID-19 hospitalization risk if infected and COVID-19 vaccination risk, 18% of citizens (9 of 50) and 12% of citizen (6 of 50) estimated 50%, respectively (Figure 1).

We also wanted to understand citizens' meanings for estimates of 0% because there is a small cluster at 0% (6 of 50) in the estimates of COVID-19 hospitalization risk if infected, and 0% is a value that appeared in our study of US citizens (Yoon et al., 2022).

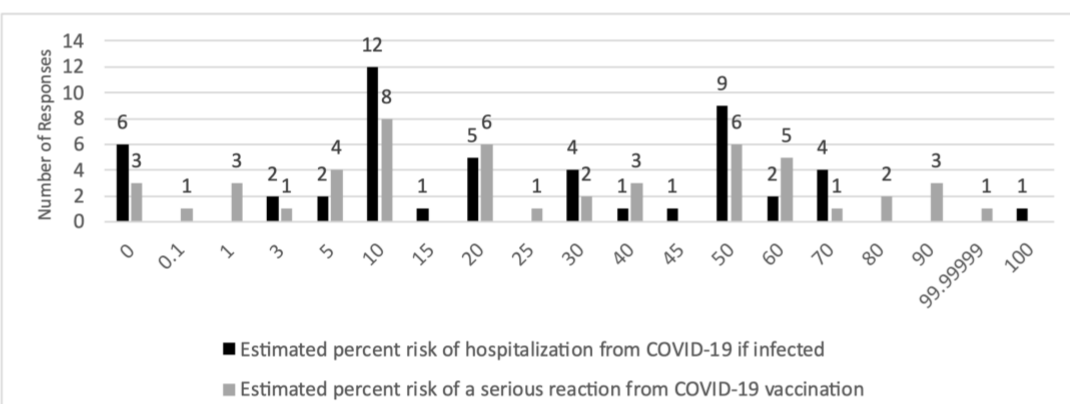

Figure 1. Citizens' estimated percent risk associated with COVID-19 infection and vaccination.

Use of Benchmarks in COVID-19 Infection and Vaccination Risk Estimates

Because 0%, 10%, and 50% were common estimates in the survey and interview data, we decided to focus on interview participants' meanings for these values and how they used them as benchmarks for estimation. Additionally, we discuss 1% as a benchmark value which emerged in our interview data, and we relate interview data with short fill in the blank responses on the survey.

50% as a Benchmark. During the interview, Byung-soo estimated his COVID-19 hospitalization risk if infected and unvaccinated as 50%. When asked how he quantified the 50%, Byung-soo said "I really don't know. I don't know how to estimate that risk as a probability. I have no reason for it". His response suggests his meaning of 50% is "I don't know" which may be explained by the outcome approach (Konold, 1989). Similarly, in the survey data, four of nine citizens who estimated their risk of COVID-19 hospitalization if infected as 50% and three of six citizens who estimated their risks of serious adverse reactions to COVID-19 vaccination as 50%, gave the reason for their estimate of 50% as "because I don't know". Thus, many citizens used 50% as a benchmark if they were unsure of their risk.

0% as a Benchmark. Chang-seob initially estimated his risk of dying from COVID-19 if infected and unvaccinated (in the case of the Omicron variant) as 0% since he thought that it would never happen to him. When asked to estimate his risk of dying from COVID-19 if infected and unvaccinated, he said: "I think my COVID-19 death risk from Omicron is 0% because I think I will not die if I were treated". Chang-seob used 0% as a benchmark to estimate his COVID-19 death risk if infected and unvaccinated given various treatments. Sun-kyu, who also thought that he would not die from COVID-19 if infected and unvaccinated, said "my risk would be convergent to 0%".

Similarly, in the survey, all six citizens who estimated their risk of hospitalization from COVID-19 if infected as 0% explained that they are unlikely to be hospitalized from COVID-19 if infected. An outcome approach explains that a benchmark of 0% means "No, it will not occur" (Konold, 1989). Therefore, 0% is associated with an event that is impossible or unlikely to occur.

10% as a Benchmark. Sun-kyu used a benchmark of 10% for estimating his risks associated with COVID-19 infection and vaccination. Sun-kyu estimated his risk of COVID-19 hospitalization if infected and unvaccinated and his risk of a serious adverse reaction from COVID-19 vaccination as more than 10%. On the other hand, he gave estimates less than 10% for his risk of COVID-19 hospitalization if infected and vaccinated (Table 1).

Table 1. Sun-kyu's estimates of risks associated with COVID-19 infection and vaccination.

Risk of		
COVID-19 Hospitalization, if infected and unvaccinated	COVID-19 Hospitalization, if infected and vaccinated	a serious adverse reaction to COVID-19 vaccine
10~20%	5~10%	30~40%

He transformed 10% into 1 out of 10. In this way, he interpreted 10% risk as "I don't think that 10% risk associated with COVID-19 infection is high. But, one or two out of 10 people is not too low to ignore. So, I would be concerned about it a little". In addition, he estimated his risk of serious adverse reactions from COVID-19 vaccination as 30~40%: "With a risk of 30%

related to vaccine side effects, I could be the one who had severe vaccine reactions, so I would much worry about it". Sun-kyu considered 30-40% as risky because it is more than one or two out of ten people, and so he used 10% as a benchmark and adjusted from a low risk of 10% to a higher risk of 30-40%.

In the survey, we found that all 12 citizens who estimated their COVID-19 hospitalization risk if infected as 10% tended to think that 10% is not very risky on a multiple choice follow up question. Of the 12 citizens, three responded that they were not concerned, five responded that they were slightly concerned, and four responded they were somewhat concerned about getting hospitalized if they were infected with COVID-19. Thus, participants from both the survey and interviews may use a benchmark as 10% to estimate a risk of an event that seem not very risky.

1% as a Benchmark. During the interviews, we observed 1% was used as a benchmark to estimate risk and participants adjusted the 1% upward or downward to reach their final estimates. We also found that participants thought 1% was a small risk and that risks with less than a 1% chance were unlikely to occur. Two interview participants thought COVID-19 risks less than or equal to 1% are not worth worrying about. For example, Chang-seob used 1% as a benchmark for all estimates of his risk associated with COVID-19 infection and vaccination (Table 2).

Table 2. Chang-seob's estimates of risks associated with COVID-19 infection and vaccination.

COVID-19 Risk of			
Hospitalization, if infected (unvaccinated)	Hospitalization, if infected (vaccinated)	Dying, if infected (vaccinated)	A serious adverse reaction to vaccine
1%	0.1%	0.001%	0.001~0.0001%

When the first author asked Change-seob to estimate his risk of hospitalization and dying from COVID-19 if infected and unvaccinated, he said:

> I estimate 1%, at most, that I would be hospitalized from COVID-19 if infected and unvaccinated. I think my risk of dying from COVID-19 if infected and unvaccinated would be from 0.01% to 0.1%. Assuming I am treated, I multiplied 0.01 to the 1% risk of hospitalization from COVID-19 and multiplied 0.1 to the 1% if not treated. Similarly, when vaccinated, I multiplied 0.1 to the 1% and got 0.1% risk of hospitalization from COVID-19 if infected. Also, I multiplied 0.1 to 0.01% risk of dying from COVID-19 if infected and unvaccinated and got 0.001% for my risk of dying from COVID-19 if infected and vaccinated.

Chang-seob started from 1% as his estimated risk of COVID-19 hospitalization risk if infected and unvaccinated. Then, he adjusted the 1% to 0.1% and 0.01% to estimate his risks associated with COVID-19 infection if infected and vaccinated. He thought every other risk in Table 2 was less likely than hospitalization if infected and unvaccinated. He adjusted his 1% estimate downward by multiplying 1% by 0.1, 0.01 and 0.001 to estimate other risks. Thus, Chang-seob used a benchmark of 1% for estimates of his risk associated with COVID-19 infection and vaccination. Additionally, he thought a 1% chance of hospitalization is not risky to him:

> I'm just splitting the risks less than 1%, so I feel like 0.1%, 0.01%, and 0.001% are all the same to me which are extremely low and 0.0001% is almost 0%. If a risk is less than 1%,

I think the probability is low no matter what the value is, because it means less than 1 out of 100 people.

Sun-kyu made a similar judgement about a 1% risk associated with COVID-19 infection and vaccination: “But if the chance has more zeros in the percentage such as 0.000…%, I would feel safe and comfortable with the risk”. He interpreted risks of the form 0.000…% as “1 out of 1,000, 1 out of 10,000, and so on”. Both Chang-seob and Sun-kyu used 1% or 1 out of 100 people and noted that if risks are less than 1%, it is meaningless to differentiate between them.

Discussion and Conclusion

From surveying and interviewing South Korean citizens about their COVID-19 risks, we found they used benchmark values of 0%, 1%, 10%, and 50%, and participants conveyed that 10% and 1% risk are not risky. Interview participants used and interpreted the benchmark values of 1% and 10% for their COVID-19 risk as 1 out of 100 people and 1 out of 10 people, respectively, for who would be at risk. This may have led participants to perceive risks less than 1% as small and negligible because they count the number of people out of 100 people, and 1 person is the smallest countable number from those 100 people. Compared to a sample size of 100, they perceived the occurrence of 1 out of 10 people as something that cannot be ignored but is slightly risky. Thus, reasoning with a frequency format using the number of people out of 10 or 100 may explain the use of benchmark 1% and 10% and why those risks were not perceived as risky.

In contrast, risks under 1% were more likely to be measured in a percentage format by a number of zeros in a decimal. When asked to transform 0.1% and 0.01% into a frequency format, the interview participants were able to correctly get 1 of 1,000 and 1 of 10,000. Nevertheless, they preferred to quantify risks less than 1% in a percentage format. Thus, the participants’ use of denominators (100 and 10) may explain why they thought 1% was not risky, and the smallest risk value they would pay attention to because 0.1% means less than 1 person dies out of 100.

However, if the infection fatality rate of COVID-19 was 1%, and all 50 million South Korean citizens were infected, 500,000 people would die. Although the South Korean citizens in both our interviews and survey viewed risks near 1% as not too serious, it is clear that these risks are serious on a population scale. South Korean citizens’ estimates of risk in our survey are problematic and worthy of attention in the international mathematics education community. Before vaccines were available, the infection fatality rate of COVID-19 was estimated to be about 0.68% (Meyerowitz-Katz & Merone, 2020). Although this is less than 1%, it was still significant enough to cause significant loss of life, overwhelm medical systems, and lower life expectancies. A 10% or even 1% infection hospitalization rate would be catastrophic from an epidemiological perspective.

Measuring small risks is hard. Small risks are represented in both frequencies and percentages. When thinking about the meaning of a percent in the COVID-19 context, it is important to think about that percent of a population of people. Moreover, it is important to understand the equivalent relationship in fractions and percents for risk perception, but many students as well as adults struggle with it (Sharp, 2002). These difficulties in understanding probabilistic information may cause problems in risk communication (Covello et al.,1986). Imagine if an epidemiologist tells us that a disease has a 5% infection fatality rate thinking that this will convey that the disease is extremely serious but what is actually conveyed to people is that they do not need to worry because it is less than their benchmark of 10%. As college instructors, it is important for us to ensure the graduates of our institutions are prepared to make sense of risks and develop more productive meanings for percent risks.

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Students' Perceptions of Active Learning Classroom Community

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Classroom community plays a pivotal role in student success, especially when students have the opportunity to interact with their peers and instructors. This study explores how students perceive classroom community and its impact on their learning. We present findings from an essay reflection assignment where students were asked to share their experiences with learning mathematics. We found that students felt more comfortable to ask questions and were able to develop interpersonal skills in conjunction with mathematical skills because of the classroom community. Conclusions drawn from this research can provide insights for instructors who are interested in fostering strong classroom communities in engaging learning environments.

Keywords: community, active learning, group work, student-instructor relationships

Being part of a classroom community has been described as having feelings of belonging and obligation to one another so that each members' educational needs will be met through commitment to shared goals (Rovai, 2002). Research has shown that these two facets of a classroom community, sense of belonging and social interdependence, are crucial to students' well-being and academic success (Awang et al., 2014; Johnson & Johnson, 2008). Classroom community develops through interactions between students, instructors, and content (Cohen et al., 2003; Lampert, 2001). These interactions often depend on the pedagogical choices of instructors and how students are positioned (e.g. active learners vs. passive recipients). In recent years, research has shown that active learning instructional strategies increase student engagement and success in undergraduate mathematics (Freeman et al., 2014). While several studies have focused on academic student success (Freeman et al., 2014; Laursen et al., 2014; Theobald et al., 2019) and differences in student outcomes in active learning classrooms based on gender (Johnson et al., 2020; Reinholz et al., 2022), few studies have looked at students' perceptions of the classroom community in an active learning setting and how they perceive community to impact their learning.

The purpose of our study is to investigate students' perceptions of an active learning classroom community and how they interacted with their peers, their instructors, and the mathematical content. This proposal focuses on the following research question: How do students report that the community of an active learning class supports them in their mathematical learning? Our work provides insight for other institutions, departments, and instructors who are interested in incorporating more active learning into their undergraduate mathematics courses and highlights areas where more research is needed.

Context

Our study takes place in the context of a mathematics department that has recently undergone several major changes to its first-year mathematics program. Prior to Fall 2021, all first-year mathematics courses at this institution were taught using the emporium model (Twigg, 2011; Webel et al., 2017). Under this instructional model, students would attend a one hour lecture each week and then were required to spend a minimum of three hours in a computer lab working independently on online homework problems. Students were able to get individualized help from undergraduate tutors who worked in the computer lab, but there was little community developed amongst students and their peers. When COVID-19 emerged in Spring 2020, issues associated with the emporium model were exacerbated. Lectures and tutoring hours were held over Zoom making it easy (and normalized) for students to turn off their video cameras and avoid interactions with their peers and instructors altogether.

In Spring 2021, a team of faculty members were hired to improve first-year mathematics courses at this institution. Course redesign efforts began in Summer 2021, and major changes were implemented in Fall 2021. Instead of using the emporium model, courses were scheduled to meet for 75 minutes twice a week, and sections were capped at 40 students. Each section had a designated instructor of record along with two classroom learning assistants. Courses were tightly coordinated with a team of faculty members working together to develop curricular materials, assess student learning, and make revisions to courses throughout the semester. The instructional team's overall goal was to improve student learning in these first-year mathematics courses by creating an inclusive classroom environment where students felt safe to make mistakes and learn from them while working together with their peers. Thus, developing a strong sense of classroom community was central to the course redesign efforts.

Background Literature and Theoretical Framing

Classroom community is difficult to define, and many definitions exist in research literature (Summers & Svinicki, 2007). Baturay (2011) defines classroom community in terms of two components: "(a) social community or the feelings of connectedness among community members and (b) learning community or their common expectations of learning and goals" (p. 564). In our study, we viewed classroom community through the lens of the instructional triangle; i.e. considering how students' experiences in the classroom are influenced by interactions between students, instructors, and mathematical content, and the environment in which these interactions occur (Lampert, 2001; Cohen et al., 2003). Community is developed (or not) through these interactions, and classroom communities are formed when there is a "culture of learning, where students come to see themselves as contributors to their own learning and that of the community" (Bielaczyc & Collins, 2009, p. 10).

Our study is situated within the context of an active learning classroom environment. Several definitions of active learning exist in the literature (Williams et al., 2022). For this study, we define active learning broadly as "classroom practices that engage students in activities such as reading, writing, discussion, or problem solving, that promote higher-order thinking" (CBMS, 2016, p. 1). Another common definition of active learning draws on the four pillars of Inquiry-Based Mathematics Education: students engaging deeply with mathematics, students working collaboratively, instructors inquiring into student thinking, and instructors fostering equity in their teaching (Laursen & Rasmussen, 2019). In this study, we investigate how students perceive the classroom interactions described by these pillars (between students, instructors, and mathematical content) and how these interactions relate to the classroom community.

Finally, our work builds on the SEMINAL Project, which focuses on incorporating more active learning into first-year undergraduate mathematics programs. As part of this project, the second author and colleagues analyzed survey data from over 2500 students taking a range of first-year mathematics courses at several institutions across the United States. In their open-ended survey responses, students discussed five major topics: student-to-student interactions, relationships with instructors, course format, assessment, and students' affective experiences (Uhing et al., 2021). While this data was gathered in a different context, similar themes arose in our findings. In this proposal, we discuss and build on student-to-student interactions and relationships with instructors and relate these themes to classroom community.

Methods

Data for this study come from a Mathography assignment that was assigned to students enrolled in a College Algebra course at a large, Midwestern, metropolitan university during Fall 2021. The assignment was based on Drake (2006) and included prompts such as "Describe what you learned in College Algebra. These can be mathematical topics but also non-math things, like anything you learned about yourself or your ability to do math or what math is." Out of 459 students enrolled in the course at the beginning of the semester, 228 completed the assignment, which was collected at the end of the semester. Students were informed that class assignments would be gathered for research and course evaluation purposes.

Mathographies were analyzed using Dedoose (2022). To establish a preliminary set of codes, deductive coding was conducted with an initial set of codes from the SEMINAL Project (Uhing et al., 2021). In order to refine our analysis, a set of subcodes was created for most initial codes. At least two researchers tagged each response to prompts in the Mathographies with these initial and secondary codes. Researchers then met to discuss and reconcile disagreements until agreement was reached. Excerpts were then sorted by initial codes and the code "Community and Culture" was selected for a second round of coding. Inductive coding was used to create sets of themes for each subcode, which illustrate how students experienced the classroom community.

Findings

Secondary analysis of the "Community and Culture" code yielded four subcodes. We describe themes that emerged under two of them, Student/Student Interactions and Student/Instructor Interactions, in this proposal.

Student/Student Interactions

Group work was a large focus of the College Algebra active learning curriculum. Students spent the majority of class time working together to solve problems posed by the instructor or complete classroom activities. As a result, Student/Student Interactions was the most common secondary code from Community and Culture (n=234 excerpts out of a total of 538 excerpts). Table 1 provides an overview of the most common themes under the Student/Student Interactions code.

Table 1. Major Themes for Student/Student Interactions

Theme	Summary	Student Excerpt
Working in a team	Learning mathematics through positive experiences with peers	"I liked having people around me in my group that can help me understand concepts and I can do the same for others."
Meeting new people	Building social networks and friendships with classmates	"I got a lot out of this class lik em ynew [*sic*] friends that I happened to get partnered up with at the beginning of the class."
Building collaboration skills	Developing collaboration and other interpersonal skills	"I learned how to better manage my time, how to work in a group with efficiency, and how to learn alongside my peers."

Disliking group work	Preferring step-by-step instruction (i.e. lecture) or individual work over group activities	"Something that didn't work so well for me in this class was all the independent work time we had with our groups to work on worksheets/packets."

Working in a team. The notion of working with others was a common, positive response from students as almost 37% of the Student/Student Interactions excerpts referred to the concept of teamwork. In many cases, students felt that working with their peers allowed them to grasp mathematical concepts in a way that they had not been able to in prior math courses. There was a realization amongst students that the ability to collaborate with one another gave them additional resources, other than just the instructor, when they struggled with a mathematical concept. One student's response illustrates this idea: "the use of groups allows students to help each other so more people learn than if the teacher was answering every individual student's question. This also allows students to create some connection to the people in the course."

Several students also mentioned feeling comfortable working in groups and asking their classmates questions. One student remarked, "I think that discussing the problems in a judgement free environment helped me advance in this class a lot." Another student explained how this course felt different from past experiences in which they had felt rushed or judged:

> We were giving each other pointers, and just getting along well and we weren't rushing anyone, or judging one another if one was confused. It felt good on how well we were responding to each other. In the past there would always be someone trying to rush you, or make you feel less if you don't understand, or catch on right away. Even teachers would do that. It made me realize people are respectful and patient when it comes to trying new things.

As illustrated by these excerpts, students felt comfortable working in teams in a "judgement free environment", and as a result were able to more deeply engage with the content and "advance" their mathematical knowledge.

Meeting new people. Since College Algebra is a first-year mathematics course, many of the students who take this course are early in their academic careers. Moreover, during the Fall 2021 semester, when data were collected, many students had just experienced several months of remote or online classes due to COVID-19, emphasizing the significance of this theme. About 18% of the Student/Student Interactions excerpts described how students made new friends or developed close relationships with classmates while taking College Algebra. One student mentioned how they were new to the area and making friends and forming connections was critical in their success in College Algebra:

> I feel that I got friends out of the class and a more in-depth version of a high school math class, except it was college. Friends are important to me because I am new to [this city], and I felt welcomed by everyone in that class. It made me feel like I belonged to a big math family and community.

This excerpt highlights how a strong classroom community can help students build social networks. Even though this student was new to the area, they were able to make friends with other students in College Algebra because of the "welcoming" environment, thus helping them feel a sense of belonging to a "big math family and community".

Building collaboration skills. In addition to developing mathematical skills, some students felt that they were able to develop their collaboration skills as a result of the emphasis on group work in College Algebra. Approximately 10% of excerpts under Student/Student Interactions referenced building these collaboration and networking skills. One student described their experience:

> While I learned many new mathematical skills in this class, which I will no doubt be carrying into my future classes, I think that the most important lessons that I learned in this class did not pertain to math. While in [College Algebra], I learned a lot about myself and about networking. I learned that I could put aside my pride and independence when I need help. I can ask questions and not be judged – something that I had feared before. I also learned that I am capable of networking. By this, I mean that I was able to form relationships with my colleagues and professor that could help me in the future.

Although this student acknowledged that their new mathematical skills will be useful in future classes, they emphasized that the "most important" part of College Algebra was building their collaboration ("networking") skills.

Disliking group work. Although the interactions with their peers resulted in many positive experiences for students, some students felt that this style of learning did not work as well for them. Around 8% of the excerpts indicated some kind of negative experience in relation to Student/Student Interactions. One student conveyed: "the style of learning that I learn best with is lecture-type style where I have an instructor explaining the concept step-by-step and less of me and my table partners working together to figure it out ourselves." In this case, the student explicitly stated that they preferred learning in a passive, lecture-style learning environment compared to the active learning environment in which the class was designed, which is consistent with other research (e.g. Deslauriers et al., 2019).

Not all excerpts tagged under this theme, however, were critical of the active learning environment. Some students were able to provide specific interactions with other students which created a negative experience for them. One student recounted such an experience: "I really liked working in groups however some of them were hard to get along with [...] I didn't feel comfortable asking my group for help in certain cases because another member was very aggressive." Another interaction a student discussed related less to the fact that they did not like working with their peers, but more that they found it easier to compare themselves to other students which resulted in a feeling of inadequacy:

> I remember when we were to work with our small groups over a set of problems that involved identifying parts of a quadratic function via a graph. While all three of them could understand how to find [the] domain, how to locate the vertex, and so much more, I couldn't. I felt like if I asked for help, that I'd be labeled as something rather than intelligent. I was scared to even share my answers.

Although most students who submitted the Mathography assignment enjoyed learning in a more active setting, these excerpts show that some students had uncomfortable experiences in this classroom setting due to poor group dynamics or personal insecurities.

Student/Instructor Interactions

After Student/Student Interactions, Student/Instructor Interactions was the next most common subcode under the Community and Culture (n=186 out of 538 excerpts). Table 2 describes common themes associated with Student/Instructor Interactions and example student excerpts. Overall, comments under this subcode were positive in nature; only six out of 186 excerpts were somewhat negative. The majority of these negative comments referred to students wanting more direct guidance from instructors (e.g., "I also feel like the guessing games waiting for someone to magically conjure an answer out of thin air gets annoying, I would prefer to be guided and shown the way rather than having to find the way myself.").

Table 2. Major Themes for Student/Instructor Interactions

Theme	Summary	Student Excerpt
Creating a supportive, engaging classroom environment	Providing in-class general instructional support to students	"My professor was very engaging and always emphasized participation and asking questions."
Providing opportunities for support outside of the classroom	Interacting with students outside of class time (including tutoring, review sessions, and office hours)	"I also felt better about this when I went into tutoring the next week and my teacher told me I did really well."

Creating a supportive, engaging, classroom environment. Several students expressed how their interactions with instructors in class affected their learning. Some students appreciated how they were able to quickly get support from instructors as compared to other experiences where an instructor would lecture for the duration of the class. One student compared this course with others that they had taken saying, "this was my favorite I was able to talk to the professor or his Aids [*sic*] for help." This student went on to say that "having the constant support was better than just having a professor stand in the front and just do problems all lecture and instead walk around and engage students." Thus, because of the instructional methods and classroom community, students felt like they could get help immediately from instructors in class when they needed it.

As mentioned in the Student/Student Interactions section, students occasionally felt reluctant to ask instructors for help or to ask them a question. Some students wrote that they appreciated when an instructor would approach them and ask how they were doing. As one student commented:

> Both the teacher and the teacher assistants, [TA Name] in particular, were very helpful and were always going around asking if people needed help and were most often able to provide an answer. Communication between everyone was very good and helped me feel more comfortable in the class when asking questions to the teacher or TA.

Since the community of the classroom allowed for instructors to check on individual students and groups, students who were apprehensive about initiating a conversation with their instructors were still able to get the help they needed. Moreover, this type of classroom environment also allowed instructors to build trust with their students and eventually some students felt that they could ask their instructors questions when they did not understand a topic. Another student reflected on this growth:

> I also liked these assignments because I was able to ask my professor questions one-on-one and get the clearer answers that I needed. At the beginning of the semester, I was a very shy and independent student. I think that these peak experiences helped me to "come out of my shell" and become more participative in class. I think that these experiences helped me to realize that it's okay to ask questions.

Overall, students appreciated the in-class support provided by their instructors. These in-class student/instructor interactions allowed students to not only get the help that they needed when dealing with challenging mathematical concepts, but also learn how to ask questions and speak up when they needed help.

Providing opportunities for support outside of the classroom. In addition to their in-class experiences, students also wrote about getting help outside of the classroom. Although getting help with classwork outside of class can be a daunting task for students, one student recounted when they went to their first review session:

> A high point for me in [College Algebra] was when I went to the review session for the class. I felt a bit uneasy and nervous about going there at first because I thought it would be awkward. When I gained the courage to go there, I saw the professors and students from other class periods. The environment was friendly. The professor sat down with me to help go step by step through the [learning outcomes] I had problems with. They answered all my questions, made me feel relieved about the coming [exam], and lifted the stress of not completing this class.

The community aspect of the course and students' relationships with their instructors allowed them to have the courage to ask for help when they needed it and better advocate for themselves outside of class. One student commented:

> What did I get out of this class? Honestly, I feel like this class taught me to approach my professors regarding any questions or concerns I may have. Honestly, that's what they're there for, and really to take the initiative to attend office hours because they are beneficial in the success of this course.

These excerpts illustrate that students felt comfortable enough to approach instructors and ask them for help outside of class because of the open classroom community that was cultivated.

Discussion

The majority of excerpts about Student/Student Interactions were positive, showing that students were able to form a strong classroom community in an active learning setting. For the most part, students felt comfortable working together and asking each other questions even though there were some students who expressed disliking group work and feeling inadequate compared to their classmates. In addition, the excerpts relating to Student/Instructor Interactions show that students appreciated being able to develop close relationships with their instructors in an active learning setting. Due to the supportive classroom community that instructors helped to cultivate, students felt comfortable enough asking questions in class and seeking help from their instructors outside of class.

These findings provide insight for others who are interested in developing supportive classroom communities in active learning settings. To facilitate positive student interactions, instructors should be intentional about structuring group activities and helping students create group norms that promote the sharing of ideas in a "judgement free environment". In addition, instructors should strive to develop individual relationships with their students to help students feel more comfortable asking for help. Although this paper sheds light on interactions between students and instructors in active learning settings, further research is needed to better understand how these types of supportive classroom communities are cultivated.

Positive interactions between students and their classmates and instructors are crucial to creating a strong classroom community in which students are able to excel in their mathematical learning. First-year mathematics courses play a pivotal role in students' overall college success, so it is important that students feel like they belong in these classes. As demonstrated by our findings, a strong sense of community can help students feel both comfortable and confident in their interactions with other students and their instructors, thus supporting students in their learning and helping them to succeed in these courses.

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How do STEM Undergraduates Structurally Conceive Change During Modeling?

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Using data from teaching experiments and theories from quantitative reasoning, we built second-order accounts of students' mathematics on how they conceived change via operating on existing quantities. We report three different ways STEM undergraduates structurally conceived change as they constructed mathematical models for real-world scenarios.

Keywords: Quantitative Reasoning, Mathematical Modeling, Change

A student's conceptualization of change plays a central role in their mathematical models they construct for real-world situations. Researchers have looked into how students' conceptualizations of change inform the mathematical decisions students make while modeling dynamic situations. For example, researchers have investigated students' conception of rate of change, limit, and accumulation through considering students' images of change in dynamic situations (e.g., Carlson et al., 2002; Thompson, 1994a). In addition, researchers have also studied how students' conceptions of change influence how students conceive variation and make sense of dynamic situations (Castillo-Garsow et al., 2013). However, these studies report on accounts of student thinking by focusing on students' variational and covariational reasoning as the student reasons about how values of a quantity are *changing*. We still need an account of how students conceive *change,* as a measurable attribute of a varying quantity, through establishing relations among existing quantities. Thus, we ask: how do STEM undergraduates conceive change while constructing mathematical models for real-world scenarios?

Theoretical Underpinnings

Our research lies within the cognitive perspective of mathematical modeling (Kaiser, 2017). In this perspective, mathematical modeling comprises the cognitive processes involved in constructing a mathematical model for real-world scenarios. We define a mathematical model to be a representation of the relations among conceived quantities.

Quantities are conceptual entities that exist in the mind of an individual. Thompson (1994b) defined *quantity* as a mental construct of a measurable attribute. It consists of three interdependent entities: an object, a measurable attribute, and a quantification. *Quantification* involves conceiving a measurable attribute of an object and a unit of measure and forming a proportional relationship between the attribute's measure and the unit of measure (Thompson, 2011). Quantification takes place in the mind of an individual and, as observers, we can only use an individual's externalized actions to infer whether and how she has quantified an attribute. Czocher & Hardison (2021) presented eight observational criteria that can be taken as indication that a student has conceived measurement process for a measurable attribute of an object.

A relation among measurable attributes is established through operating on quantities. Thompson (1994b) defines *quantitative operation* as the "mental operation by which one conceives a new quantity in relation to one or more already-conceived quantities" (p.10). As a result of a quantitative operation a *quantitative relationship* is created: the quantities operated upon along with the quantitative operation are in relation to the result of operating (Thompson, 1994b). In other words, a quantitative relationship is the "conception of three quantities, two of which determine the third by a quantitative operation (Thompson, 1990, p. 12)." Examples of quantitative operations include *combining two quantities additively*, *comparing two quantities* *additively, combining two quantities multiplicatively*, *comparing two quantities multiplicatively*, *instantiating a rate*, *generalize a ratio*, and *composing two rates or ratios* (Thompson, 1994b). Thompson defined (1990) defined *quantitative structure* as a network of quantitative relationships. For example, the total number of feral cats that visited a backyard during one weekend is a quantity that may be constructed by *additively combining* the number of cats that visited the backyard on Saturday and the number of feral cats that visited the backyard on Sunday. At the same time, the total number of feral cats that visited the backyard on Saturday during the time period 9am to 5pm is a quantity that maybe constructed by *instantiating a rate* of 10 cat-bird interactions per hour for 8 hours.

Borrowing ideas from the aforementioned constructs we define *structural conception of change* as conceiving change as a measurable attribute of an object through forming a relation among constituent quantities (i.e., operating on quantities). We operationalize change as a measurable attribute of a varying quantity, that can be quantified via a quantitative relationship. The research question addressed in this report is: In what ways do STEM undergraduates structurally conceive change while constructing mathematical models for real-world scenarios?

Methods

We present data from a pair of 10-hour individual teaching experiments (Steffe & Thompson, 2000) conducted with undergraduate STEM majors at a large university. The overall goal of the teaching experiment was to investigate how students conceive real-world situations through quantities and relations among quantities. Our participants, Szeth and Pai were both enrolled in differential equations at the time of the interviews. Throughout the teaching experiment, Szeth and Pai worked on 10 and 9 tasks, respectively, that were based on real-world scenarios. In this report we present data from *The Disease Transmission Task*, *The Pruning Task*, and *The Population Dynamics Task*. We focus on these tasks because they exemplify how Pai and Szeth operated on quantities that they had previously constructed, to structurally conceive change.

The Disease Transmission Task: Suppose a disease is spread by contact between sick and well members of the community. If members of the community move about freely among each other, develop a mathematical model that informs us about the dynamics of how the disease would spread through the population.

The Pruning Task: Imagine you have a hedge in your garden of some size, S, and you want it to increase its size even more. You hire a gardener for some advice on growing this particular plant. She advises you that the overall rate of growth will depend both on the extent of pruning and on the regrowth rate, which is particular to the plant species and environmental conditions. Both rates can be measured as a percentage of the size of the plant. The pruning rate can be adjusted to result in a target overall growth rate. Can you derive a model for the rate of change of the size of the plant?

The Population Dynamics Task: Suppose in a laboratory setting, we are looking at large populations of breeding stock in which individuals give birth to new offspring but also die after some time. Suppose that on average each member of the population gives birth to the same number of offspring, α, each season. The constant α is called per–capita birth rate. We also define β as the probability that an individual will die by natural causes before the next breeding season and call it the per-capita death rate. Construct a mathematical model that describes the dynamics of this population.

The interviews were retrospectively analyzed to construct second-order accounts (Steffe & Thompson, 2000) of students' reasonings via inferences made from students' observable activities such as verbal descriptions, language, written work, discourse, and gestures. The retrospective analysis consisted of multiple passes of through the data to arrive at examples that illustrate the different ways Szeth and Pai structurally conceived change. First, we watched the videos in MAXQDA in chronological order and paraphrased each interview by chunks. Next, we created accounts of students' mathematics and the reasons they attributed to their mathematics. Next, we went over the accounts we created and the videos at the same time and refined our accounts by adding details using theories from quantitative and covariational reasoning. We credited a student to have instantiated a quantity if we were able to infer from his reasonings that he had conceived an object, attribute, and a measurement process for the attribute. As evidence of student to have conceived a measurement process, we checked if at least one of the quantifications criteria was met (see Czocher & Hardison, 2021). We used segments of transcripts, where the students engaged in quantitative reasoning along with inscriptions and gestures as evidence for our claims. Next, from these accounts, we selected instances where the students engaged in structurally conceiving change. Next, while watching the videos and going over our accounts of students' structural conception of change, we further refined our accounts by adding details of how Pai and Szeth operated on constituent quantities to form a relation in order to conceive change. Next, we went through our accounts of Pai's and Szeth's structural conception of change and observed any patterns, in terms of operation on constituent quantities, that were consistent throughout the sessions. We made note of them by summarizing the pattern. We watched the videos in their entirety again and made sure all of the different ways of conceiving change were recorded. Finally, we refined our second-order accounts by seeking clarification on utterance and gestures to support our claims on Pai's and Szeth's structural conception of change. We present some of these second-order accounts below.

Findings

Pai structurally conceived change as accrual and as the additive comparison of quantities. In contrast, Szeth structurally conceived change as an instantiation of rate. I illustrate these structural conceptions using Pai's work from *The Disease Transmission Task* and *The Pruning Task* and Szeth's work from *The Population Dynamics Task*.

Change as the Accrual

Conceiving change as the accrual entails an image of accruing the amounts at different points in time, by additively combining the amount present before and the amount that was newly added. In *The Disease Transmission Task*, Pai constructed the expressions in Figure 1 to represent the change in healthy people and change in infected people.

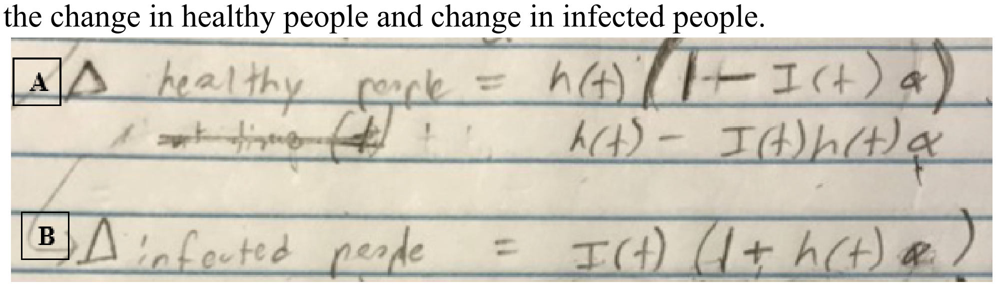

Figure 1. Pai's expression for the change in infected people and change in healthy people

After Pai constructed the above expressions, we asked him by how much the infected people would change. The goal of this question was to document how Pai was distinguishing between "$\Delta\, infected\; people$" and $I(t){\cdot}h(t){\cdot}\alpha$, because these two quantities carried the same meaning to us. Interestingly, Pai said that the "increase [in infected people]" was $I(t){\cdot}h(t){\cdot}\alpha$. Even though we used the term change when we asked him by how much the infected people changed, Pai replied with the "increase" in infected people. We interpret that Pai associated two different meanings to the attributes change and increase of the infected people, while he continued to associate change in infected people with expression B in Figure 1. This interpretation was further evidenced in the excerpt below.

Pai: Well, yeah. Change in the infected people is the base infected people at time t, plus those again, infected at time t. And this is just an increase. $h(t)\cdot I(t)\cdot\alpha$. Is going to be less than... You're going to be adding this to...to the base population.

Interviewer: Okay. I see. So yeah, you are seeing the delta infected people as the change in that sense, the total number of infected people that would be at the end of an infectious period?

Pai: Right.

Interviewer: Whereas the other one, how much... How many are getting infected that you have to add to the base.

Pai: Yeah, it's the increase, but just the increase of people infected in this time period.

Interviewer: Okay.

Pai: Maybe I could write that better saying this is the people infected in time period t [pointing at the expression $h(t)\cdot I(t)\cdot\alpha$]. That was a better statement of it. Like people infected... Well, people infected... In current period t. Whereas $I(t)$ is just how much we had already. Initially, before interactions with other healthy people. If that makes sense.

From the above excerpt, we interpret that Pai quantified change in infected people and increase in infected people differently. While Pai quantified the change in infected people as the "base population" plus how many more got infected, he quantified the increase in infected population as just the how many more people got infected, which to him was $h(t)\cdot I(t)\cdot\alpha$.

Pai continued to reason as to why expression B in Figure 1 represented the change in infected people as "cause that's a change in two - over two time periods because you're adding one. Cause you're adding another time period." From his reasoning, we interpret that Pai imagined to be representing the change in infected people as an accrual of the infected population at different points in time. For Pai, $I(t) + h(t)\cdot I(t)\cdot\alpha$ represented the change in infected population as an accrual of the amounts of infected people across a progression of time. On the other hand, for Pai, $h(t)\cdot I(t)\cdot\alpha$ represented increase in infected people during the moment of interest. Therefore, Pai structurally conceived change in infected people as the *additive combination* of the amount of infected people present at the beginning of the moment of interest and amount of people that got infected at the moment of interest.

Change as the Additive Comparison of Quantities

Structurally conceiving change as the additive comparison of quantities entails constructing change in the amount during a time period $[t_1, t_2]$ as the additive comparison of the amounts at t_1 and t_2, respectively. In *The Pruning Task*, Pai constructed the following expressions to represent the size of the plant at the end of the first and second pruning periods, respectively.

$$S_1 = S_0 - p(S_0) + r(S_0)$$
$$S_2 = S_1 - p(S_1) + r(S_1)$$

In the above expressions, Pai defined S_0 as the size of the plant at the beginning of the first pruning period, S_1 as the size of the plant at the end of the first pruning period, S_2 as the size of the plant at the end of the second pruning period, r as regrowth rate of the plant, and p as the extent of pruning. After constructing the above expressions, Pai constructed the expression below to be the change in size of the plant during the second pruning by *additively comparing* the quantities S_1 and S_2.

$$\Delta S = \ - p\left(S_1\right) + r\left(S_1\right)$$

Next, Pai constructed a general expression for the change in size of the plant during any pruning period of 1 unit, to be the following expression.

$$\Delta S = \ - p\left(S_t\right) + r\left(S_t\right) \quad (1)$$

In the above expressions, Pai structurally conceived the change in size of the plant during a particular period by *additively comparing* the sizes of the plant at the beginning and end of the particular pruning period. Pai's structural conception of change, in this manner was also prevalent in the remaining of his work in this task. We illustrate this episode below.

Pai started working on deriving an expression for the rate at which the size of the plant was changing with respect to time. As Pai was not sure how to proceed, the interviewer directed him to derive an expression for the change of plant's size during a pruning period of length Δt. The intent of this question was to guide Pai towards thinking of the average rate of change of the size of the plant and how it can be manipulated to derive the instantaneous rate of change. Pai constructed the expression below to represent the change in size of the plant during two successive pruning periods.

$$\Delta S =- p\cdot\left(S_{t+1}\right) + r\cdot\left(S_{t+1}\right) - \left(- p\cdot\left(S_t\right) + r\cdot\left(S_t\right)\right) \quad (2)$$

Although expression 2, for Pai, represented the change in size of the plant during two successive pruning periods, normatively it represents how much more the plant changed in size than the previous pruning period.

The interviewer emphasized to Pai that expression 1 is the change in size of the plant during any one pruning period and asked what the change in size of the plant would be during two such pruning periods, given that the change is the same for each pruning period. The intent of the question was to give Pai the opportunity to structurally conceive overall change during two successive pruning periods as the additive combination of the changes in size of the plant, during each pruning period. However, Pai was still referring expression 2 to as the change in size of the plant during two pruning periods. Pai gave the following reasoning:

Pai: Because it's a change of the size of the plant over two periods. This is the change of... It's the second pruning period, $t + 1$ [*pointing at* $- p\left(S_{t+1}\right) + r\left(S_{t+1}\right)$], minus the initial pruning period change [*pointing at* $- p\left(S_t\right) + r\left(S_t\right)$].

When the interviewer pointed out that expression 2, to her, represented a "change minus change," Pai recognized expression 2 did not achieve his goal.

In an attempt to reduce Pai's cognitive load and to aid in unpacking his conception of change, the interviewer modified the scenario to be such that the change in size of the plant during a pruning period would be a constant β instead of ΔS. The interviewer asked Pai what the change in size of the plant would be over two time periods, given that β is the change during each pruning period. Pai said that it would be "The beta of the second pruning period minus beta of the first pruning period". This again serves as evidence that Pai's structurally conceived of change as the *additive comparison* of quantities.

The interviewer pointed out to the fact the quantity he constructed – "the beta of the second pruning period minus beta of the first pruning period" – was change in change. She asked what the overall change would be from the beginning of the first pruning period to the end of the second pruning period. Pai's responded with "You take the end, the change... Well, take the size at the end of period two minus the size of initial before period one. That's overall change." Again, Pai was only measuring the attribute change in size of the plant through comparing the sizes of the plant at different times.

To circumvent Pai's thinking of change as "beta of the second period minus beta of the first period" the interviewer shifted his focus to the quantities S_1, S_2 . The interviewer asked Pai that given $\beta = S_2 - S_1 = S_3 - S_2$, and so on, what the total change in size of the plant be from the beginning of the first pruning period to the end of the third period. To that, Pai answered $S_3 - S_1$ (Figure 2, point A). The fact that Pai was still considering the difference between S_1 and S_2 (instead of $\beta + \beta$) to find the change in size of the plant confirms our conjecture that his structural conception of change predominantly entails the *additive comparison* of the amounts at different points in time. When the interviewer asked him what $S_3 - S_1$ was in terms of β, Pai solved for the system of linear equations to get 2β. He attained 2β through comparing the quantities S_3 and S_1 and not by adding the $\beta's$ during each compounding period (See Figure 2). Only after arriving at 2β, Pai realized that it is "two changes in size β".

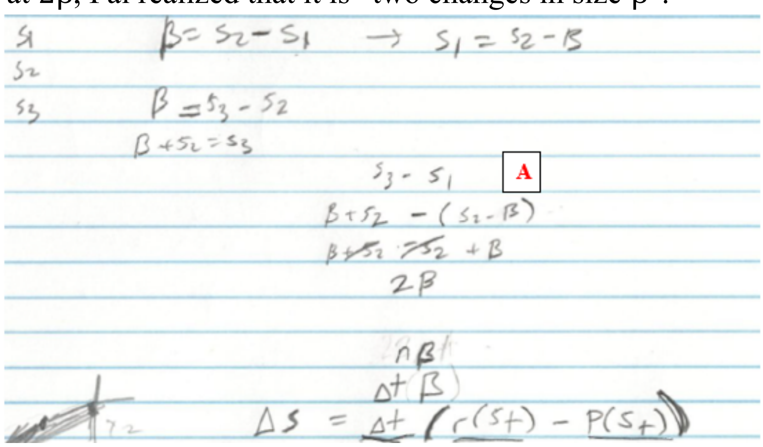

Figure 2. Pai additively compared the sizes of the plant at different times to evaluate the change in size of the plant

Change as the Instantiation of Rate

Structurally conceiving overall change as instantiation of change entails constructing overall change through *multiplicatively combining* the change during an interval of one unit of time and the length of the time period. In *The Population Dynamics Task*, Szeth constructed the expression in Figure 4 to represent the "size of the population at time t."

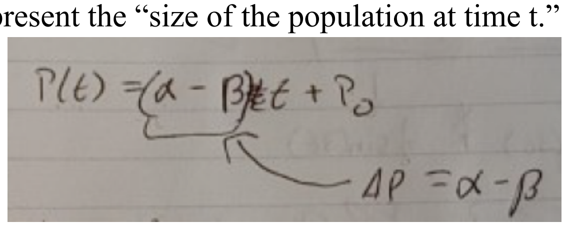

Figure 4. Szeth's expression for the size of the population at time t

In the expression in figure 4, Szeth defined $P(t)$ as the "size of the population at time t," P_0 as the population "that is already there" and α and β were given in the task as the per-capita birth rate of the population and the per-capita death rate of the population of the population, respectively. In the above expression in Figure 4, Szeth nominalized $(\alpha - \beta)\cdot t$ as "how much it [*the population*] is growing by," and $\Delta P = \alpha - \beta$ as the change in the population during one season. Szeth explained that he multiplied $\alpha - \beta$ by t because he wanted a "method within the equation for it to iterate over time [because $\alpha - \beta$] is based on the one season." When Szeth referred to $\alpha - \beta$ as "based on one season" and denoted $\Delta P = \alpha - \beta$, Szeth conceived the change in the size of the population during an interval of one unit of time as $\Delta P = \alpha - \beta$. Then to construct overall change in size of the population during a time period, Szeth *multiplicatively combined* the net change in the population during an interval of one unit of time and the length of the time duration. For Szeth, the length of the time duration was from $t_i = 0$ to $t_f = t$.

Discussion

We presented three examples of differing ways that Pai and Szeth structurally conceived change in terms of operating on constituent quantities. Structurally conceiving change as the accrual entails constructing an image of accruing the amounts at different points in time, by additively combining the amount present before and the amount that was newly generated. Structurally conceiving change as the additive comparison of quantities entails constructing change in the amount during a time period $[t_1, t_2]$ as the additive comparison of the amounts at t_1 and t_2. Finally, structurally conceiving overall change as instantiation of change entails constructing overall change through multiplicatively combining the change during an interval of one unit of time and the length of the time period.

In *The Disease Transmission Task*, Pai structurally conceived change in as an accrual through *additively combining* the amount of infected people at the beginning of the moment of interest and the amount of people who got infected at the moment of interest to construct the change in the amount of infected people. However, in *The Pruning Task*, despite the interviewer's efforts in having Pai to construct change in the size of the plant during two pruning period through additively combining the change in the size of plant during each pruning period, Pai consistently constructed change as an *additive comparison* of quantities. We conjecture that Pai had a strong preference for structurally conceiving change as the additive comparison of quantities, that he dropped the situational attributes of the quantities operated on and the resultant quantities were measuring. In contrast, Szeth structurally conceived change as an instantiation of rate where he unitized the change in size of the population and then multiplied the unit change by the length of the time period.

Knowing the different ways students structurally conceive change provides insights about the mathematical decisions students make during their mathematical modeling activities. This knowledge has implications for the teaching of mathematical modeling. For example, for a student who structurally conceives change as an instantiation of rate, like Szeth, the construction of instantaneous rate of change – typically the desired output while modeling dynamic situations – can be scaffolded by first constructing the change during a time period to the average rate of change during a time period, and finally to the instantaneous rate of change by considering infinitesimally small time periods. For someone who conceives change as an accrual, like Pai, focusing on the amount increased/decreased may yield favorable modeling outcomes. As quantities are building blocks of a mathematical model (Larson, 2013), understanding how students operate on existing quantities to construct a new quantity informs researchers and educators *where* and *how* to intervene in the students' mathematical modeling process.

Acknowledgments

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A Case of a Computational Setting Facilitating Empirical Re-Conceptualization

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The use of computer programming is ubiquitous for research mathematicians, and two common practices are facilitation of experimentation and testing of conjectures. These practices align with empirical re-conceptualization, which is the process where empirical data is used to identify patterns, form related conjectures, and then re-interpret the conjectures from a structural perspective. However, literature on empirical re-conceptualization has only examined student work done by hand. We present case study data of one student, Allen, using the programming language Python to facilitate empirical re-conceptualization as he found the closed-form solution for binomial coefficient C(n,k)*. We discuss why the computational environment was productive for empirical re-conceptualization and provide avenues for future research.*

Keywords: Computation, Re-Conceptualization, Combinatorics, Generalization

Introduction, Motivation, and Relevant Literature

Mathematics as a discipline has adopted computation to progress research that has historically been performed by-hand, to provide additional insights into existing theory, and to pursue new solution methods for difficult problems. As computation—which, for our purposes, refers to the use of a computer to carry out, or otherwise assist in, a solution to a mathematical task—becomes increasingly ubiquitous in the discipline, there has been a push by mathematics education researchers for investigations into the interplay between computational environments, mathematics, and students. This has included investigating how computational environments can be used to facilitate conceptual development in novel and complementary ways (e.g., Sand et al., 2022), how to assess student use of computation (e.g., Buteau & Muller, 2017), how computer programming informs student affect towards mathematics (e.g., Purdy & Lockwood, 2020), and how further research might benefit from theoretical perspectives that accommodate the increasing relevancy and diffusion of computing into mathematics (e.g., diSessa, 2018). Computing is now an authentic mathematical practice (Lockwood et al., 2019), and as mathematics education research seeks ways to foster student development of authentic mathematical practices across subject areas, we feel there is an opportunity to do so for computing. Specifically, Lockwood et al. (2019) identified ways in which mathematicians use computing in their work. In this paper, we build on two possible ways students can use a computational environment that align with those identified in Lockwood et al., (2019): to facilitate experimentation, and to test conjectures.

Experimenting and conjecturing align with the concept of empirical re-conceptualization (ER) (Ellis et al., 2022), which is "the process of identifying a pattern or regularity, forming an associated generalization, and then re-interpreting those findings from a structural perspective" (p. 2). As yet, the literature on ER has only examined student work performed by hand. This work includes either the researchers providing a data set for students to study and conjecture about, or the students creating the data set themselves. We feel that a computational environment can facilitate ER by providing the means to quickly produce large data sets in ways that are intractable by hand, and to test the veracity of conjectures over these data sets. Moreover, this provides an opportunity to investigate how a computational environment can be used in problems students are capable of solving while still having them engage in practices that are authentic to the discipline.

In this paper, we seek to demonstrate an instance of a computational environment facilitating ER for a student, Allen (a pseudonym), as he finds a closed-form expression for the binomial coefficient, *C(n,k),* by writing computer code in Python. We will show that Allen used the computational environment to create data sets based on the parameters *n* and *k*, and he used the data sets to iterate between creation and validation of conjectures. In doing so, we hope to contribute to the literature on computation in mathematics education, and to the literature on conjecturing and generalization, by documenting one way that a computational setting can be used to open the door for students to engage in mathematical generalization. We also contribute to the literature on combinatorics education by discussing how Allen's counting process for *C(n,k)* involved enumerating and tallying the entire set of outcomes, and discussing how he had difficulty connecting his counting process to the closed-form expression. This paper addresses the research question "In what ways might a computational setting facilitate empirical re-conceptualization (ER) as students find closed-form expressions for binomial coefficients?"

Theoretical Perspective on Empirical Re-Conceptualization

Curricular development and mathematical reasoning hierarchies expect students to move from empirical toward deductive reasoning, and they broadly frame empirical reasoning as less sophisticated than deductive reasoning, or as a stumbling block to overcome. This is justifiable, as empirically-found patterns can be faulty, lack explanatory power (Bills & Rowland, 1999), and students may overly rely on a small number of cases to check the validity of mathematical statements (Knuth et al., 2009). Yet, students can be adept at empirical patterning (Küchemann, 2010), and they demonstrate dynamic interplay between patterning empirically and arguing deductively, just as mathematicians do (de Villiers, 2010). Because students have difficulty shifting from empirical to deductive reasoning (e.g., Stylianides & Stylianides, 2009), there is motivation to search for ways of leveraging the productive aspects of empirical reasoning in order to enhance, or otherwise inform, deductive reasoning. As an example, empirical reasoning can give insight into a problem's structure in a way that supports deductive reasoning. Further, students flexibly reason between multiple representations when reasoning empirically (Amit & Neria, 2008), which can be beneficial for deductive reasoning. Ellis et al. (2022) propose the concept of ER as one way that empirical reasoning can be leveraged to enhance deductive reasoning.

Empirical re-conceptualization is "the process of identifying a pattern or regularity, forming an associated generalization, and then re-interpreting those findings from a structural perspective" (Ellis et al., 2022, p. 2). A structural perspective refers to the last four categories of structural reasoning as described by Harel and Soto (2017), whose five major categories are (a) pattern generalization, (b) reduction of an unfamiliar structure into a familiar one, (c) recognizing and operating with structure in thought, (d) epistemological justification, and (e) reasoning in terms of general structures. The first category further elaborates the difference between result pattern generalization (RPG), which is a way of generalizing that attends only to results, and process pattern generalization (PPG), which is a way of generalizing that attends to a process (Harel, 2001). Generalizations formed through empirical reasoning are examples of RPG because only the results (i.e., the empirical data) are used to create the generalization. A structural perspective refers to the other four categories of structural reasoning. ER can thus be evidenced by a shift from RPG to PPG, which entails a shift from pattern generalization to any of the other four categories. In this study, we use this notion of ER to frame our student Allen's work, and we highlight the ways in which the computer supported Allen's ER.

Methods

Participants and Data Collection

The data presented here are from a teaching experiment (TE) (Steffe & Thompson, 2000) that consisted of 3-4.5 hours of contact time each with four students in 60- to 90-minute interview sessions over the course of several weeks during a single academic term. This TE is part of a broader study that aims to investigate how a computational environment can be leveraged to enrich student understanding of mathematics. We focus on a single student, Allen (pseudonym), who was an Electrical and Computer Engineering major recruited from an introductory computer science course at a large university in the western United States. We chose Allen based on his responses to a recruitment questionnaire, which indicated that he had never taken a discrete mathematics course, and he exhibited a strong computer science background. We interviewed Allen in person over three 90-minute sessions, and we focus on the last session in this paper. We chose these data because they exemplify using computational techniques to engage in ER through experimentation and conjecture validation in a sophisticated way. Indeed, Allen's mastery of fundamental computer science ideas indicates that most students would not produce similar results, but we nevertheless believe our findings have useful theoretical implications about using computer programming to engage in ER. For example, our findings might speak to how future studies could create tasks and scaffolding intended to elicit generalizing activity, such as ER, in a computational setting.

During the TEs, the students sat at a desk and worked individually on a desktop computer in the programming environment CoCalc. CoCalc is an online-supported application that allows users to write in Jupyter notebooks, which include cells where Python code can be written and executed. Jupyter notebooks allowed the students to write separate pieces of code for each task, or multiple pieces of code for a single task, and to quickly reference code they had previously written. We also gave them paper handouts of the problem statements and provided a brief introductory guide to Python syntax. To capture the interviews, we videotaped and audiotaped the interviews, took screen recordings of the desktop computer, and scanned the students' hand-written work.

Tasks

Allen worked on a number of counting problems that involved finding the solution by hand, and writing a computer program that enumerates and tallied all possible outcomes. Throughout the TE, he worked on problems involving Cartesian products, arrangement with repetition, arrangement without repetition, and one problem involving selection without repetition. Allen used the multiplication principle to solve these problems by hand (with the exception of the selection without repetition problem), and his computer programs used nested *for* loops to implement an odometer strategy, similar to those reported by De Chenne and Lockwood (2022).

We focus on his work on the Book Problem: "Suppose you have eight books and you want to take three of them with you on vacation. How many ways are there to do this?" The solution to this problem is $C(8,3) = 56$, the number of ways to *select* (without arrangement), three of eight books. We chose these data because Allen initially solved the problem incorrectly, and after finding the correct solution he decided to search for patterns between his initial incorrect answer and the correct answer. The pattern he found made him curious about how the pattern would change with different parameter values, and Allen engaged in ER ultimately to find a closed-form expression for the binomial coefficient, *C(n,k)*. We elaborate on Allen's solution, and his ER, in the Results section.

Data Analysis

We chose this episode because it demonstrates an instance of a computational setting facilitating ER. The first author identified key moments where Allen used his computer programs to reason empirically about the relationship between *C(n,k)* and *P(n,k)*. She then created an enhanced transcript of the episode by including pictures of Allen's work and screenshots of the computer programs he wrote. This analytic process allowed us to make a narrative of Allen's reasoning about *C(n,k)* as it relates to *P(n,k)*, and capture instances of conjecturing, validating conjectures, and re-conceptualizing the empirically-found patterns. Finally, both authors reviewed the enhanced transcript and identified ways in which Allen's work with the computer demonstrated instances of ER.

Results

We first provide a chronological narrative of Allen's work finding a closed-form expression for the binomial coefficient, *C(n,k)*. Then, we discuss three ways in which the computational environment facilitated the ER: (1) isolating generalizable relationships, (2) generating data, and (3) validating conjectures.

Allen began his work on the Book Problem (stated above). The correct solution to this problem is *C(8,3)*, the number of ways to *select* three of eight books, but Allen's initial solution was *P(8,3) = 8*7*6 = 336*, the number of ways to *arrange* three of eight books. The first author asked him how he would verify if his solution was correct, and Allen verbally described a computer program that would enumerate the outcomes. The computer program he described enumerated all *arrangements* of the books, which was consistent with the numerical solution he found, but was incorrect because it overcounted the outcomes. The first author asked him to write down the first ten outcomes his computer program would produce. When he wrote down these outcomes, Allen noticed that the computer program he described would produce both *1,2,3* and *1,3,2*, which represent the same outcome. Based on his realization that *1,2,3* and *1,3,2* represent the same outcome, we infer that Allen understood that different arrangements of the same three books do not constitute different outcomes in this problem.

After realizing his error, Allen was unable to find a closed-form expression that correctly counted the outcomes, but he was able to write a computer program that would correctly enumerate all the outcomes. To do so, he modified his enumeration strategy for arrangements so that a length three number sequence was enumerated only if the numbers in the sequence were in increasing order, as seen in Figure 1. This modification ensures a unique order for any length three number sequence, and so each combination is only counted once. Allen justified this modification by stating "Say I pick book number 5 as my second book. I don't need to pick book 1, 2, 3, 4, or 5 [as the third book] in that case, so I'd only have three books to choose from." After executing his code, Allen found that the correct answer was one sixth of his original, incorrect answer; that is, Allen found that *P(8,3) = 6*C(8,3)*.

After finding that his initial solution was six times greater than the correct solution, Allen stated

> *Allen*: That's interesting, *8*7* is *56* … Completely the third value [the value *6* in *8*7*6*] didn't even really matter it looks like … I just want to see if bringing the total number of books down by one, does that make it equal *7*6*? Is there a relationship there?

This is the first instance where Allen reasoned empirically about a generalizable pattern. He noticed that his initial solution was $8*7*6$ and the correct solution was $8*7$, and he conjectured that $C(7,3)=7*6$. We infer that Allen's conjectured pattern was that the correct answer could be found by first finding his original answer, and then removing the last term in the product. Hence, because $P(7,3) = 7*6*5$, Allen conjectured that $C(7,3) = 7*6$ because the term 5 would be removed. This is an instance of RPG because Allen found the numerical pattern based solely on the results, and he did not provide structural reasoning that justified the pattern, or connect the pattern to his computer program. Allen modified his computer program to find $C(7,3)$, seen in Figure 1, and he found that $C(7,3) = 7*5$. After observing that $P(7,3) = 6*C(7,3)$, and the ratio was not 5 as he anticipated, Allen conjectured that $P(n,3) = 6*C(n,3)$ for every value of n. This conjecture is another example of RPG because he justified the pattern based only on the observed results. To validate this conjecture, Allen wrote a function that would return the ratio for an input value of n (which is called 'book' in his code), and he used the function to check that the ratio was 6 for all n values up to 21. After validating this conjecture for those n values, Allen concluded that the ratio was 6 for all n values.

```
count = 0

for i in range(1,9):
    for j in range(i+1,9):
        for k in range(j+1,9):
            count = count + 1
            print(str(i)+" "+str(j)+" "+str(k))
print count
```

```
count = 0

for i in range(1,8):
    for j in range(i+1,8):
        for k in range(j+1,8):
            count = count + 1
            print(str(i)+" "+str(j)+" "+str(k))
print count
```

```
count = 0

for i in range(1,7):
    for j in range(i+1,7):
        for k in range(j+1,7):
            count = count + 1
            print(str(i)+" "+str(j)+" "+str(k))
print count
```

```
def fun(book):
    count = 0
    for i in range(1,book+1):
        for j in range(i+1,book+1):
            for k in range(j+1,book+1):
                count = count + 1
                #print(str(i)+" "+str(j)+" "+str(k))
    return (book*(book-1)*(book-2))/count
```

Figure 1: Allen's computational solutions for C(8,3), C(7,3), C(6,3), *and* C(n,3).

Allen then conjectured how the ratio of $P(n,k)$ to $C(n,k)$ would change for other values of k. To generate a conjecture, Allen observed that $6=2*3$, and he connected the 3 in the product to the parameter value 3 in $C(n,3)$. He conjectured that $P(n,4) = 12*C(n,4)$ because $12 = 4*3$. This is another instance of empirical patterning, as Allen did not provide, or search for, structural justification for why the parameter value 3 was connected to the ratio. To validate his conjecture, Allen modified the function he previously wrote for $C(n,3)$ to calculate $C(n,4)$. This modification did not appear to be difficult for Allen, and he made the appropriate changes without prompting by the researchers. Using this function, Allen found that $P(n,4) = 24*C(n,4)$, which was not the ratio he anticipated. He then characterized the change in ratio from 6 to 24 as a multiplication by 2^2, not as multiplication by 4, which we take as further evidence that Allen was empirically patterning based on the 2 in the original product $6 = 2*3$. Allen then conjectured that $P(n,5) = 40*C(n,5)$, where the $40 = 2^3*5$. This is another example of RPG, where Allen generalized the results (factors of 2) without structural justification. However, after further modification of his code to calculate $C(n,5)$, Allen found that the correct ratio was 120.

At this point, Allen had found that $P(n,3) = 6*C(n,3)$, $P(n,4) = 24*C(n,4)$, and $P(n,5) = 120*C(n,5)$, and he observed that $24 = 6*4$, and $120 = 24*5$. Based on these results, he then conjectured that "my next guess would be 120 times six, which would be 720," which is to say that he conjectured $P(n,6) = 720*C(n,6)$. This is correct, and in describing the pattern he states "What I noticed is each time you go up, you multiply by the next number. So 6 times 4 equals 24, which multiplied by 5 equals 120, which multiplied by 6 equals 720." To formalize his conjecture regarding the ratio between $P(n,k)$ and $C(n,k)$ for any parameter values n and k, Allen wrote the expression in Figure 2, which he characterized as his original solution, $P(n,k)$, divided by $k!$. Although he had found the correct closed-form expression for the binomial coefficient, he had done so solely through empirical patterning. The first author asked Allen why dividing by $k!$ might yield the correct solution. Allen recognized that $k!$ represents the number of ways to arrange k objects, which he used as justification for why dividing by $k!$ yields the correct answer. We view this justification as a shift from RPG to PPG because Allen moved from using empirical results to create patterns, to justifying those patterns structurally by arguing how the mathematical expression connects to the outcomes he was counting. Hence, this episode demonstrates ER because it presents an instance of finding a pattern (a constant ratio between permutations and combinations), forming an associated generalization (the closed-form expression for the binomial coefficient), and re-interpreting the generalization from a structural perspective (justifying the ratio based on the outcomes being counted).

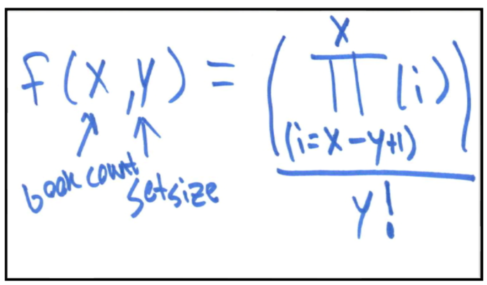

Figure 2: Allen's closed-form expression for the binomial coefficient

Having presented Allen's work, we now discuss three ways in which the computational environment facilitated the ER: (1) isolating generalizable relationships, (2) generating data, and (3) validating conjectures. Together these highlight specific ways in which the computational environment served to support this practice of ER.

Isolating Generalizable Relationships

Allen's generalizing activity in this TE centered on the relationship between $P(n,k)$ and $C(n,k)$, which is the relationship between a mathematical expression he found by hand, and a numerical solution he found computationally. Because $C(n,k)$ is a 2-variable function, isolating this relationship required understanding the relationship as both variables changed. In Allen's code, the variable k was hardcoded as the number of *for* loops, while the variable n was an assigned integer. Allen could easily change the value for n, which he used to his advantage by cycling through values of n in a *for* loop and observing the relationship between $P(n,k)$ and $C(n,k)$ for a fixed value k. However, changing the value for k required writing a new function, which can explain why Allen only examined the relationship for $k = 3, 4, 5, 6$. Having the two variable values manifested differently in the computer programs may have been beneficial, as it forced Allen to only change one variable value at a time. Creating conjectures that generalize the effect of changing one value while keeping the other constant may have been easier than if he had the ability to change both variable values simultaneously.

Generating Data Sets

Allen wrote computer programs to generate sets of outcomes that are intractable to write by hand. To do so, he was required to engage with the sets themselves, as generating them required creating an algorithm that enumerated every outcome. His conjectures evolved as he generated each data set, which may indicate that writing the computer programs is more beneficial to ER than if he were given the data sets by the researchers. In addition, Allen was unsure how to calculate $C(n,k)$ without writing a computer program. Hence, we posit that this episode would not have occurred if not for the computational environment.

Validating Conjectures

Much of Allen's work iteratively moved between conjecturing and validating conjectures. The computational environment seemed to positively impact how he formed conjectures by motivating him to form conjectures that he could validate computationally. Every time Allen formed a conjecture, he immediately wrote a computer program to validate that conjecture. What Allen decided to conjecture about may have been informed by the elements of his computer program that seemed immediately adaptable. For example, Allen could easily change the value of n, and his initial conjectures formed around how n impacted the numerical solution. We posit that Allen's mastery of basic computer programming allowed flexibility in how he could adapt his code to accommodate changes in variable values. This flexibility manifested in his conjecturing by allowing some flexibility in the conjectures he was able to validate.

Discussion

In this paper, we have attempted to demonstrate a case in which a student leveraged a computational environment (and his computational facility) to engage with a combinatorial task, particularly using the computer meaningfully to engage in ER. We see much potential for ER to be a useful construct in considering students' mathematical activity (particularly relating to the practices of generalizing and conjecturing). In light of this, we think it is important to explore different aspects of this construct. Further, given the growing importance and ubiquity of computing in mathematics, and mathematics education, we see value in particularly exploring ways in which ER may be reinforced via the computer (and, reflexively, how a computational environment can specifically support practices such as ER). In this way, our findings contribute to a growing body of literature that demonstrates ways in which computational activity supports students' mathematical thinking and activity. Moreover, our findings also contribute to literature that demonstrates students using a computational environment in sophisticated ways that reflect the practices of research mathematicians.

We were fortunate in this study to uncover a case of a student leveraging the computer in a powerful way to support ER. Further research could more explicitly target not only instances of students using the computer meaningfully, but could investigate ways to support and engender productive use of the computer to support students' engagement in practices such as ER.

Acknowledgments

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Resources for Participation: Examining Multilingual Students' Perspectives on What it Means to Participate in Active Learning

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More college mathematics classrooms are adopting active learning practices like collaborative learning. Participation in these practices typically requires students to engage in verbal communication and interpersonal interactions, which are mediated by language. Despite this, the experiences of multilingual students have often been overlooked in the literature on active learning. Through analyzing interviews with 26 multilingual students, this study explores what participation in active learning meant to these students. It also explores the resource that students perceived to be necessary in order to participate in active learning, for example, being comfortable speaking English. Finally, this study demonstrates how classroom discourses and broader social discourses about language shaped students' perceptions about participation.

Keywords: Multilingual learners, active learning, groupwork, classroom norms and discourses

Introduction

Active learning has become a well-established teaching pedagogy for improving student learning and promoting equity in undergraduate mathematics courses. Although this approach to teaching includes a broad range of strategies, students in active learning classrooms are often expected to work on mathematics collaboratively with peers. For instance, groupwork is a common way that instructor choose to infuse active learning into their courses (e.g., Bennett, 2022). Collaboration is also a core pillar of inquiry-based learning (Laursen & Rasmussen, 2019). As such, groupwork and collaboration are often integral parts of how students experience active learning in undergraduate mathematics courses.

When students work in groups, they are typically expected to participate in talk-intensive activities, like discussing their thinking and arguing for why their solutions are correct (see also Laursen & Rasmussen, 2019; Voigt et al., 2022). These activities also require interpersonal communication, as students learn through social interactions with others. Therefore, in active learning classrooms, participation is often mediated by language and communication.

Active learning has traditionally been linked with promoting equity in the undergraduate mathematics classroom. Analyzing data from over 100 courses, Laursen et al., (2014) found that inquiry-based learning (IBL) helped remove the gender performance gap that was observed in lecture-based courses. However, a growing body of research, questions whether students from underrepresented groups experience active learning in the same way and equitably benefit from it (Ernest et al., 2019; Henning et al., 2019; Johnson et al., 2020; Reinholz et al., 2020). For instance, Ernest and colleagues (2019) demonstrated that women's verbal contributions during groupwork were less likely to be revoiced in whole class discussions.

Given the mediational role of language in active learning classrooms, students' linguistic identities may also shape how they experience active learning. At the post-secondary level, Hwang et al. (2022) documents two contrasting cases of students (Jia and Falon) who felt excluded during groupwork based on their home language identity. At the K-12 level, much has been learned about multilingual students' experiences with groupwork. Numerous studies have found that multilingual students tended to participate more in small group settings than in whole class discussions (Brenner, 1998; Civil, 2008; Gorgorió & Planas, 2001; Planas & Civil, 2013;

Planas & Setati, 2009). In addition, these findings demonstrated how normative discourse about language can shape power and participations structures in the classroom. Planas and Civil (2013) show how students in Barcelona felt the need to use the language of instruction, Catalan, in whole group discussions. This perception contributed to students feeling less comfortable sharing ideas during class.

As globalization increases, undergraduate mathematics classrooms are becoming more linguistically diverse. More research is needed to better understand how multilingual students' experience classroom pedagogies like active learning. To this end, the goal of this study is to explore multilingual students' perceptions about what it means to participate in active learning and how these perceptions are influenced by social and local discourses.

Theoretical Perspectives

A Sociopolitical Perspective on Participation

A situated perspective (Lave & Wenger, 1991) emphasizes the connection between learning and participation. That is, learning can be conceptualized as changes in students' classroom participation (Sfard 2007) where, by working together, individuals access resource for participation and become better able to engage in activities (Wood, 2013). Building on this perspective, a sociopolitical approach (Gutiérrez, 2013) recognizes that power and identity shape how participation structures unfold in active learning classrooms. A sociopolitical perspective argues that learners are not always inherently positioned in the same way in relation to the community that they are trying to participate in (Walkerdine, 1994). For example, students learning English may be positioned as less intelligent and have less access to participate in mathematical discussions (Flores et al., 2015). Thus, participation is subject to the norms and discourses[1] established by a social practices. These discourses constitute what is considered meaningful participation (Lerman, 2000). Discourses also shape what resources for participation are consider legitimate and lead to full participation (Barton & Tan, 2009). For instance, being able to formally articulate mathematical thinking is often framed as an important resource for participation in mathematics classrooms (Moschkovich, 2002).

A Sociopolitical Perspective on Language

Like mathematics, learning a language also requires participating in a new language community (Barwell, 2005). Similarly, dominant social discourses about language constitute what it means to be a legitimate language-user. For example, although English has become a lingua-franca and is spoken in many parts of the world, discourses about nativeness give preferential status to native speakers from western countries (Ferri & Magne, 2021). Subtirelu (2015) claims that these discourses operate to position groups of people as Others (i.e., different from the dominant group). "Nonnative" varieties of language can become socially stigmatized to discredit and exclude the speaker (Birney et al., 2020). Furthermore, native speakers may feel less obligated to "ensure successful communication" with nonnative speakers (Subtirelu, 2015, p. 36) and less willing to listen to and meaningful engage with their verbal contributions (Rios, 2022). A sociopolitical perspective on language recognizes that language and language status are inherently connected to who is afforded power in the classroom (Takeuchi, 2016) and whose linguistic resources are legitimized (Planas & Civil, 2013).

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[1] Gee (2015) defines discourse as: "distinctive ways of being, doing, and saying. They are ways of using words, doing deeds, valuing, thinking, believing, and feeling, as well as ways of using objects, tools, and technologies that allow us to enact or recognize socially meaningful identities" (p. 245).

Research Questions

The research questions that this study addresses are: (1) How did multilingual students conceptualize participation in active learning and what resources did they feel were needed in order to participate? (2) What discourses (both at the classroom level and more broadly) were relevant in shaping students' perceptions about participation?

Methods

This paper draws on data from semi-structured interviews with 26 multilingual undergraduate students whose home languages were not English. The study took place at a large Hispanic Serving Institution. Eighteen of the students interviewed were international students (countries of origin included Bangladesh, China, India, Iran, Peru, Saudi Arabia, South Korea, Trinidad and Tobago, Uzbekistan, and Vietnam). The remaining students were either immigrants (from China or Mexico), refugees from Afghanistan, or transitional students who grew up near the border with Mexico and lived and attended school across both countries (Hamman et al., 2010). Participants were selected based on: (1) their language identities, and (2) their enrollment in a pre-calculus or calculus course that used active learning. Based on students' descriptions during interviews, their instructors typically implemented active learning by having students work on math problems in groups of three or four during class. Therefore, in this study, active learning is operationalized to be instances during class where students work on mathematics collaboratively via groupwork.

All interviews were recorded and fully transcribed. Pseudonyms were assigned to each participant. Data analysis focused on exploring students' perception about participation in active learning and the classroom and broader social discourses that shaped these perceptions. To do this, a data-drive approach to thematic analysis was used (Braun & Clark, 2006). In the first stage of analysis, transcripts were carefully read over numerous times. The data was then coded using an open coding system to identify instances where students talked about participation. Codes were categorized based on whether they reflected how participation was defined or a resource for participation. Descriptors of the codes were developed. The final step of the analysis focused on comparing, contrasting, and connections codes and identifying themes across the data. Although interviews were transcribed verbatim, the quotes presented in this paper have been modified slightly for clarity. No modification impacted the meaning or feel of any of the sentences in the quotes.

Results

Most students in this study described participating less frequently in active learning classrooms compared to their English dominant peers. Figure 1 illustrates how the students conceptualized participation in active learning: that participation meant speaking. Given this framing, students also shared a range of different resources that they perceived to be necessary in order to be active participants in the classroom. These resources were essentially personal attributes, like being comfortable speaking English, being extroverted, being brave, being good at mathematics, being aggressive, and being resilient (the font size in Figure 1 corresponds to how frequently each resource was mentioned during interviews). The second stage of analysis focused on unpacking how local classroom discourses and broader social discourses influenced these perceptions about participation.

In this section, I document what participation meant to students and how this was influenced by local classroom discourses. Next, I discuss two of the resources for participation mentioned by students, being comfortable speaking English and being aggressive, and demonstrate how these resources were influenced by broader social discourses about language. The first resource was chosen because it was the most frequently discussed and the second one was chosen because it represented an interesting case.

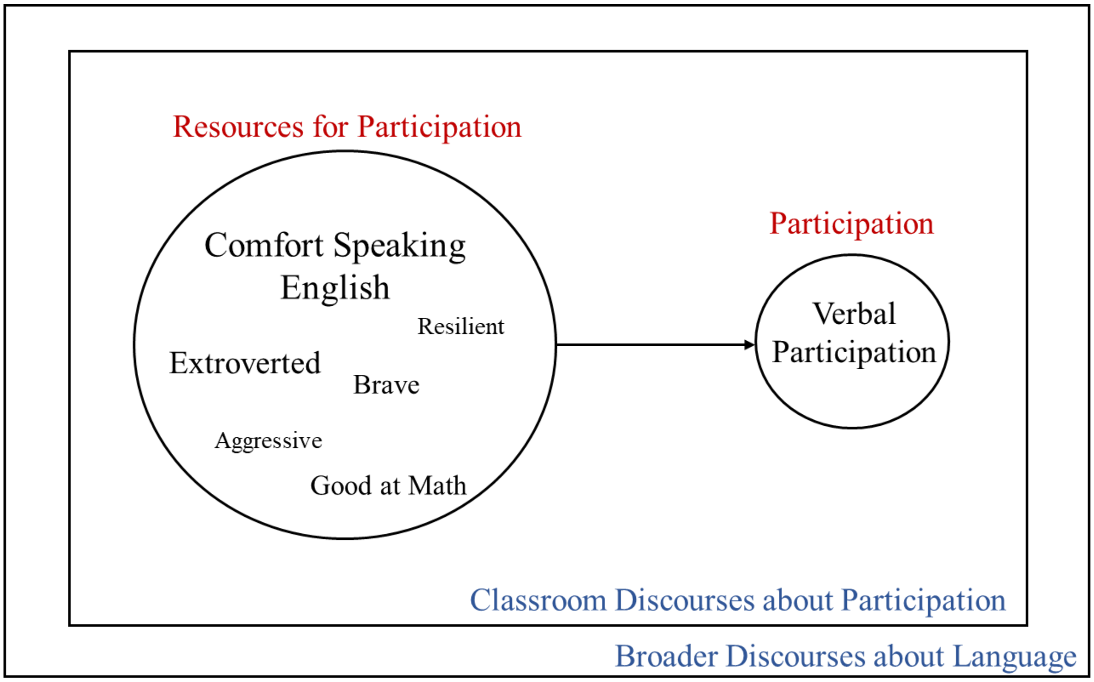

Figure 1. How multilingual students conceptualized participation and necessary resources for participation.

What Participation Meant: *Speaking*

During interviews, students frequently discussed classroom participation in terms of speaking. Whether it was in analyzing their own participation, or the participation of others, students generally associated participation with actions that centered verbal contributions (e.g., contributing ideas, responding to peer ideas, asking questions, etc.). Students shared that participation meant: “you basically speaking in front of everyone” (Dan, Trinidad and Tobago) and is “all about the group work and like I have to speak up in the class” (Sun-Hi, Vietnam).

Students also assessed their own participation in the course based on how often they spoke. For instance, although Dan was invested in the mathematics he was learning, he perceived himself as having poor participation in his calculus course because he rarely spoke during class: “When I'm not engaging much I feel like shoot. Especially when we have a question we don't understand and the other TA comes and help us with the question, my groupmate will be the one asking the question, not me”. Despite being involved in groupwork, Dan was disappointed by his level of participation because he was not the one verbalizing the group’s question. This exemplifies how participation was almost always associated with verbal communication.

Classroom Discourses about Participation. Findings also demonstrated how classroom norms and discourses about participation communicated to students that participation meant speaking. For example, students frequently discussed the teaching moves that their instructors used to elicit more verbal participation during class. Several instructors used cold-calling[2] as a way to ensure that all students were contributing (verbally) to class discussions. Similarly, one students’ instructor did not allow groups to leave until a group member, randomly chosen by the instructor, explained the group’s solutions to the class. These teaching strategies, geared toward

[2] Cold-calling refers to randomly choosing students to answer a question or discuss their solution.

enhancing participation, all aimed at fostering verbal participation. Therefore, these strategies also reinforce to students the norm that participation in the classroom is a verbal activity.

A Resource for Participation: *Being Comfortable Speaking English*

Students felt strongly that English was the only language that could be used in the mathematics classroom. Additionally, every student in the study described the importance of being comfortable speaking English in order to fully participate (verbally) in active learning. Makhmud (from Uzbekistan) shared that the “students who are more comfortable speaking English” are the ones that are “more engaged in the math classes” and are “more active”. Prisha (from India) shared how being comfortable speaking English allowed her to be an active participant in her group:

> If it's a monolingual classroom like we have here in [the US], it pretty much forces people to get out of their comfort zone if they're not comfortable talking in English. If, say I'm not very comfortable explaining my thought process in English, it would hinder my way of communicating and collaborating and it would hinder how I put myself out there, how I talk to my teachers, or how pretty much I express myself in the classroom /…/ I wouldn't be comfortable asking questions. I wouldn't be comfortable putting myself in uncomfortable situations, because just speaking in English alone is uncomfortable for me.

Groupwork in math can be a vulnerable, risk-taking activity where students are expected to produce the “right answer” and asses the correctness of their peer’s ideas. Tamir (from Afghanistan) described how these activities can feel “uncomfortable”. Engaging in them when you are already uncomfortable speaking the language “is pretty a lot to ask from somebody”. This demonstrates the importance students placed on needing to be comfortable speaking English in order to begin (verbally) participating in active learning.

In addition, students felt that if they were not comfortable speaking English in the classroom, and as a result participated less, the responsibility was on them to become more comfortable. For example, Leo felt more comfortable interacting in Spanish and shared that it was “intimidating to try to participate with others” in English. When asked what instructors could do to make speaking in groups feel more comfortable, he responded “It’s not something that they can do. It’s just something that we have to get over. It’s something we have to work on”.

Broader Discourses about Language. In contrast, study findings demonstrated how social discourses about language can impact students’ comfort speaking in the classroom. When Luis was asked if language mattered in his mathematics classroom, he responded “It’s mainly about how comfortable you feel speaking English in your class when you’re not like 100% good at the language”. Although language is a tool for communication, this quote evidences the expectation students can feel to be “100% good” at English. Numerous students described “staying silent” during class because they were afraid of making language mistakes or speaking with an accent. Many students also reported experiencing negative feelings (e.g., feeling nervous, anxious, ashamed, scared, or embarrassed) about their own language-use because they perceived it to be not “good enough”:

> - I don't feel so good when I talk. I don't know it's because of I'm an international student and my English is not perfect. – Nader, international student from Iran
> - Sometimes I really feel like too bad [because] sometimes in classes, I can't always speak as fluent English. – Abril, international student from Peru
> - A lot of times I was embarrassed to use my English because I didn't think it was good enough. – Habib, refugee student from Iraq

In addition, peers and instructors were often less willing to engage with the communicative contributions of multilingual students. For example, Anaís (from Mexico) described how others "avoid" talking to people with an "English accent" and Prisha shared how others are "not very willing to put in the same amount of effort as they would be if somebody was more fluent in English". Makhmud felt that "if you're not good at English, it's boring for others who are listening to you". This highlights that being comfortable speaking English also reflects the sociopolitical contexts of the classroom and the discourses about language that often stigmatize the linguistic resources of non-native speakers.

A Resource for Participation: *Being Aggressive*

As an international student from South Korea, Sun-Hi's prior math education experiences had been with traditional lecturing where students "hardly speak up in the class". For Sun-Hi, active learning required students to "speak", engage in "more participation in class and more interaction with instructor", things she was "not used to". To Sun-Hi, these practices felt "more aggressive" than traditional lectures. In fact, she used the word aggressive to describe active learning more than ten times throughout the interview.

Sun-Hi expressed that in order to participate in the active learning classroom, students also needed to be aggressive. This involved "being able to engage the instructor or disagree with opinions or discussing about the questions". Although Sun-Hi did not explicitly articulate why these forms of participation felt aggressive, she did share that aggressive students were more likely to be noticed by the instructor. It may be the case that when students participated, they seemed to be competing for the conversational floor and the instructor's attention, which felt aggressive to Sun-Hi. Sun-Hi expressed wanting to become more aggressive so that she could also "get noticed from the instructor". She shared: "talking with the instructor helps me a lot, so I think being aggressive has more benefits than not aggressive".

Broader Discourses about Language. Sun-Hi shared that being aggressive was hard for her. As a non-native speaker of English, she felt "not comfortable in English" and "not that confident to speak in class". She also explained that: "When I speak up in class, I'm afraid, what if I'm like, I look like a silly one". Sun-Hi worried that the way she communicated in English might be interpreted by others as "silly". This again demonstrates the sociopolitical nature of language and its impact on students' comfort and confidence speaking English in the classroom.

Discussion

By exploring the relationship between language and participation, the current study reveals another complexity in striving for equity using active learning. Study findings suggest that classroom discourses had an impact on how active participation in the classroom was characterized. By overemphasizing verbal participation (in English), instructors created a narrow window of what active participation could look like (i.e., someone who speaks regularly during class), which had the tendency to exclude language diverse students. Other forms of participation, like writing, listening, gesturing, participating in embodied activities that utilize materials (Nemirovsky & Ferrara, 2009), and communicating using home languages should also be recognized as valid ways that one can be an engaged member of the classroom (see Figure 2).

Furthermore, conceptualizing participation as speaking established a narrowly defined set of available resources for participation. This is particularly true in the context of multilingual students, given the stigma they often navigated in relation to their language identity. For example, discourses about language seemed to suggest to students that only English should be used for communication in the mathematics classroom. This became a missed opportunity for students to leverage their home language in their mathematical sensemaking.

Multilingual students also overwhelmingly perceived being comfortable speaking English in the classroom as a necessary resource for active participation and felt it was something they needed to continue developing. Fitri and Aeni (2021) state that the talk-intensive nature of active learning can help multilingual students practice speaking and feel more confident doing so in the classroom. While this may be true, the responsibility should not be exclusively placed on students for becoming more comfortable speaking. As the data in this study revealed, the sociopolitical realities of the classroom cannot be ignored. The ability to feel comfortable speaking English in the classroom encompasses more than just language skills, as students also often navigate accent bias, lack of access to peers who are willing to communicate with them, and so forth. Therefore, instructors must be aware of the complexities involved with developing comfort speaking in the classroom and work to establish norms that push back on these discriminatory discourses.

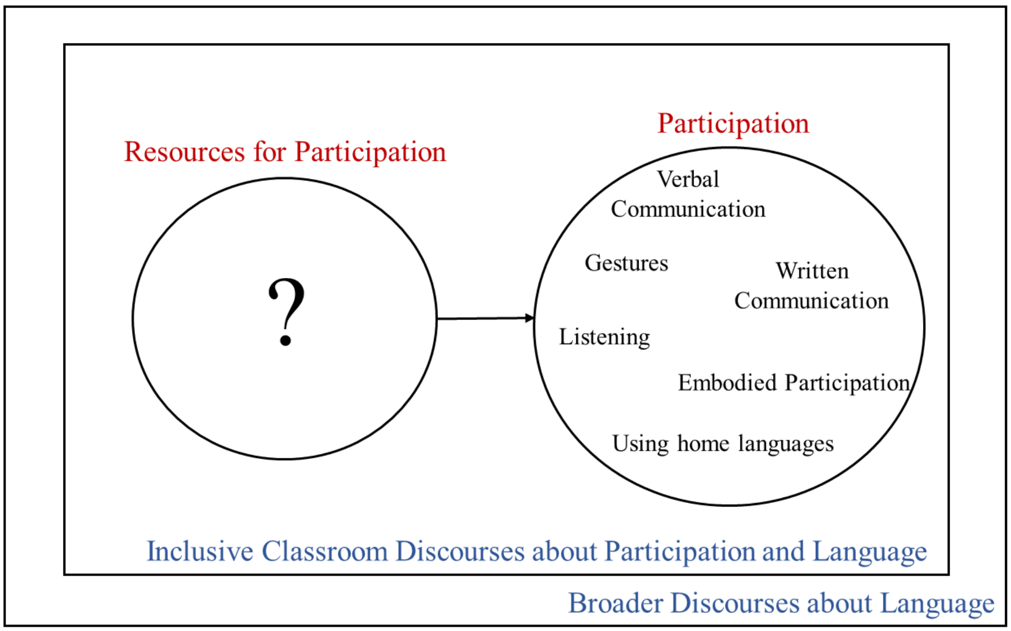

Figure 2. A reimagined and more inclusive model for conceptualizing classroom participation.

The set of resources for participation that students described were often personal attributes they did not possess, but felt they needed to develop (e.g., become more comfortable speaking English in the classroom, become more extroverted, etc.). If instructors work to create more inclusive classroom norms (i.e., norms that validate all students' linguistic resources and expand what counts as legitimate classroom participation), then the set of available resources for participation will likely become more inclusive. Ideally, these resources will embody things that students bring into the classroom (e.g., speaking in their home language, being able to visually express or gesture their understanding). Exploring this can be an area for future research efforts.

Finally, Sun-Hi and other international students described numerous cultural differences they experienced navigating active learning in their mathematics classrooms. Although all students might struggle to adjust to active learning pedagogies, international students might find certain norms and practices very different from previous classroom experiences. Making explicit some of the norms of participation and being mindful of other cultural practices can help students adapt to the new classroom environment.

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Engineering Instructional Practices to Better Support Access and Participatory Equity in Authentic Proof Activity

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The Orchestrating Discussion Around Project (ODAP) was conceived of as an investigation into how instructors might support a student-centered classroom with aims of engaging students in authentic mathematical proof activity. The design-based research focused on engineering tasks and specific high leverage teaching practices to promote student engagement with three focal proofs in Abstract Algebra. In this report, we share a series of project learnings related to supporting students in accessing formal proof activity and promoting more equal participation amongst students. We focus specifically on how complex task launch, structuring group work, and working with student ideas in whole class discussion may be shaped to support more students in authentic mathematical proof activity.

Keywords: design-based research, teaching practices, participatory equity, proof

In the United States, there has been a substantial push for undergraduate mathematics classes to become more student-centered. To this end, several researchers have developed curriculum materials that provide resources for instructors to promote active engagement in classes. In the context of upper division proof-based courses, these innovations often exist within the inquiry paradigm (inquiry-oriented to inquiry-based) where students engage actively in constructing and presenting proofs, or in reinventing abstract definitions and theorems. In general, the field is in consensus that more active classes support deeper learning (Freeman et al., 2014). Others have suggested that active classes or in some cases inquiry in particular can lead to more equitable learning environments (Laursen et al., 2014; Tang et al., 2017; Theobald et al., 2020). Yet, others have argued that instruction geared towards inquiry can also open space for more inequitable interactions as students become more actively involved (Brown, 2018; Ernest et al., 2019). Recent large-scale results reflecting a performance disparity between men and women on a group theory concept assessment (Johnson et al., 2019) in inquiry classes has raised questions about how such instruction may alleviate or exacerbate inequity in classroom environments. That is, when classrooms are opened to be more student-centered, we now have students and instructors engaging with each other in ways that would not be found in a traditional lecture class. Thus, the classroom space can become a place where societal inequities can be reproduced and only certain students fully participate. There are several cases (Haider & Andrews-Larson, 2021; Rasmussen & Kwon, 2007) that illustrate the potential for inquiry-oriented teaching to have a very positive impact on student learning. However, as noted in their recent literature review (Melhuish, Fukawa-Connelly, et al., 2022), we know little about the details of instructional practice that may account for when inquiry approaches are successful in supporting students. As noted by Adiredja and Andrews-Larson (2017), “more work is needed to document if, when, and how these instructional approaches are equitable for all students” (p. 459).

In this paper, we share some lessons we learned during our design-based research project that was designed around participatory learning goals in proof classes: engaging students in authentic activities related to comprehending, validating, and constructing proofs. That is, we explored the question, in what ways might HLTPs be incorporated in the proof setting to support our participatory goals? We made the decision to forefront instructional practices, a set we identify as high-leverage teaching practices, as objects of design. These practices stem from the K-12 literature and have been documented to play a key role in structuring student-centered classes and have the potential to increase access and inclusion. Through cycles of design and engineering we found that our practices supported students in engaging in the type of mathematical activity we valued, yet, if we focused on the who and how of engagement, participation was often unbalanced across students. As we transitioned from the lab setting with small groups of students to the classroom setting, we found a need to disentangle our authentic activity goals into two underlying participation heuristics:

- (Access) Providing access to opportunities to participate in authentic mathematical proof activity.
- (Engagement) Promoting participatory equity in authentic mathematical proof activity engagement.

We share a few key insights from our design cycles. We found that just having high-quality tasks and centering student reasoning was insufficient to support all students in engagement, and thus made several modifications to promote more equitable participation. For the scope of this work, we focus specifically on launching tasks, structuring group work, and selecting student ideas for discussion. We note that our focus on participation is a narrow view on equity, and caution that this exploration reflects just one dimension related to promoting equitable learning environments. While the question is beyond addressing fully in a single project like this, our report thus intends to help address the question, "What instructional practices support students' engagement in authentic proof activity and promote participatory equity in such activity?

Background and Perspective

For the scope of this project, we view learning as participating in activity that resembles a broader community of practice (e.g., Sfard, 1996). That is, the proof-based mathematics classroom is a setting where students can be apprenticed into the work of research mathematicians. To adhere to these goals, we focus on learning in terms of student engagement in what we deem *Authentic Mathematical Proof Activity*. These activities include not just constructing formal proofs, but comprehending and validating them. A full treatment of valued activities can be found in Melhuish, Vroom, et al. (2022).

We assume that that the classroom is a complex system and that instructional practices, those we consider high-leverage teaching practice, can serve to mitigate complexity in student-centered classrooms and promote space where students can engage in authentic activity. We integrate several scholars' (Ball et al., 2009; Woods & Wilhelm) definitions of high-leverage teaching practices to define these practices as those that:

1. can be implemented routinely,
2. use, shape, or otherwise integrate students' mathematical thinking,
3. have potential to increase equity, access, and/or engagement, and
4. are supported by research connecting the practices to students' learning.

Ultimately, we aim to support participatory equity, defined as "fair distribution of both participation opportunities and participation itself as students engage in the learning process" (Shah & Lewis, 2019, p. 423).

For the scope of this paper, we focus on three such practices: complex task launch, structuring group work, and selecting and working with records of student thinking. These three practices can serve to structure a lesson and have been well studied and developed in the K-12 setting. See Table 1 for descriptions and key citations.

Table 1. Definitions and key citations for focal high leverage teaching practices.

High Leverage Teaching Practice	**Descriptions (found in Melhuish, Dawkins, et al., 2022)**
Complex Task Launch (Jackson et al., 2012; Jackson et al., 2013; Khisty & Chval, 2002; Wood & Wilhelm, 2020)	*The teacher engages students in making sense of tasks via attending to relevant mathematical language, ideas, relationships, task contextual features, and development of a common understanding of the task goals and context. This practice occurs prior to or in parallel with problem-solving, but does not scaffold or directly instruct on solutions to the task*
Structuring Group Work (Cohen, 1994; Esmonde 2009; TeachingWorks, 2018; Webb, 2009)	*The teacher structures and manages partner and group work in order to engage all students meaningfully in mathematical activity. This can include scripts, clear mathematical activity expectations, and/or roles that provide guidance for how students are to interact with each other and the mathematics.*
Selecting and Working with Records of Student Thinking (Durkin et al., 2017; Stein et al., 2008; TeachingWorks, 2018; Wilburne et al., 2018).	*The teacher orchestrates mathematical discussion where (1) students publicly present ideas and (2) students are prompted to meaningfully engage with the ideas through analyzing, critiquing, and/or comparing across student ideas.*

Methods

This report summarizes findings from the ODAP design-based research project. This project consisted of six cycles of design and refinement (two in a lab setting, and four in classroom settings) all occurring at a Hispanic-serving, doctoral-granting research institution in the United States. The focus of instruction was engaging advanced undergraduate mathematics students in authentic proof activity related to three focal theorems in Abstract Algebra: Structural Property Theorem, Lagrange's Theorem, and First Isomorphism Theorem. At a prior RUME conference, we shared some initial work designing and implementing for HLTP from the lab setting where we provided data showing that the K-12 practices could be adapted robustly to this setting (Melhuish et al., 2020). In this paper, we focus primarily on later cycles of development and key aspects of design that emerged during the rather complex transition (see Lamberg & Middleton, 2009) from working with small groups of students in a lab to full classroom implementations.

We focused on key hypotheses about how HLTPs could support students in participation in authentic activity. This analysis occurred in several layers. All implementations were video recorded, transcribed, as well as observed by several members of the research team. Observers took field notes on ways that students did or did not engage in hypothesized activity with attention to issues of access and imbalances in participation. Between lessons, we engaged in debriefing and coming to a consensus related to lesson goals, HLTPs, and student activity. We attended specifically to ways our hypothesized relationships between HLTPs, and student participation were or were not realized. After the completion of each cycle, the team reflected more holistically on prior cycles, revisited important points in the data, and in some cases conducted extended analysis using specific frameworks to analyze elements of our study more systematically. For instance, we analyzed student activity (Melhuish, Vroom, et al., 2022), instructional moves (Melhuish et al., 2020) and distribution of authority in small groups (Hicks et al., 2021). After these debriefs and analyses, we made evidence-based modifications to improve the enactments of our HLTPs and developed new hypotheses of characteristics of HLTP and how they support student activity in proof settings. At the completion of the project, we have engaged in retrospective analyses to better describe elements of the HLTPs leveraging our data corpus. In this report, we share examples that span our project and evidence of what played out in proof contexts and particular instances that led to modifications.

Results

In this section, we focus on several adaptations and refinements we made to the three focal HLTPs for reasons related to either the proof context or participation inequities that emerged during the classroom implementations (see Table 2).

Table 2. Questions related to each focal HLTP and student participation.

High Leverage Teaching Practice	Considerations	Key Features
Complex Task Launch	What information do students need to make sense to engage productively in a given task?	• Attention to quantification and referent objects • Discussions of informal ideas that parallel formal tools
Structuring Group Work	What task features are needed to support students in shared engagement amongst group members?	• Eliciting expertise beyond proof construction • Distributing roles and expertise
Selecting and Working with Records of Student Thinking	How can public records represent a multitude of student reasoning? What tools are needed for students to work from each other's records productively?	• Expanding records beyond proof construction • Introducing partial linking ideas

Complex Task Launch

In the proof-based setting, complex task launch involved supporting students in sense making about a focal theorem in ways that may anticipate important conceptual insights or proof structures. Ultimately, complex task launch serves the purpose of supporting access to the major ideas and tools needed to engage in a focal task. Our implementations shared many features of the K-12 versions; however, we found two needs specific to the proof context. First, in addition to questions about meaning, examples, and relationships, we also found a need to ask specific questions about *quantifiers* and *referents.* For example, to engage with the Structural Property Theorem that a group isomorphism preserves commutativity, complex task launch involved unpacking the different definitions involved including isomorphism and abelian. During these conversations, students often provided imprecise explanations, such as articulating "G is abelian" as "*ab*=*ba*." Both quantification (recognizing that a statement applies for all elements in in a group) and referent objects (recognizing that the isomorphism function must be introduced) are essential to the formal proof. Without these tools, many students hit an impasse about how to prove the statement or provided a proof that did not cover all elements in the codomain.

We also found a need to explicitly consider *parallelism* between explorations of theorems and how these link to proofs. For example, in order to prove Lagrange's Theorem, students were positioned to recognize this is a theorem about multiplication, and they were easily able to retrieve a relevant definition of divisibility ($|G| = k|H|$ for some k). However, this formal definition did not evoke the parallel multiplicative structure (G being composed of k disjoint subsets the same size as H) they needed to leverage to create a proof. In our first cycle, the students attempted to create a proof by cases depending on the order of the group, suggesting, "What if we did a couple of cases, like where the order of the group was even, or it was odd?" They introduced ideas of using different remainders to make the argument, soon recognizing the number of cases became unwieldy. We suggest that the formal statement evoked prior knowledge related to number theoretic proofs, rather than drawing on the idea of repeated addition or totaling equal sized sets. We modified this element of task launch in later cycles to incorporate explicit conversation about the *conceptual meaning* of multiplication that better paralleled the proof structure of Lagrange's (e.g., "[L]et's think back to elementary school when we write these things, and we're gonna make a similar type of visual to go with this that's kind of connected to what we mean by multiplication."). In all cycles with these discussions, students built on notions of "repeated addition" or "totalling up" for their proof approaches.

Structuring Group Work

Structuring talk in group work became a major focus of design during the switch between lab setting and whole class setting. In our lab settings, we relied on mechanisms such as think-pair-share and peer review to structure group work. When we switched to whole-class implementations, we found these structures insufficient to disrupt status differentials between students. For example, students were asked to share proofs of the Structural Properties Task written at home and answer "What is one thing that makes sense? What is one thing you have a question about?" in regard to your partner's proof approach. The conversation often devolved into the student with a more complete proof teaching the student with a less complete proof. For example, in one exchange, we noted one of the partner stating they "don't know anything" and the other partner walking them through what was "missing" from their proof. In this case, the introduction of a "function from G to H that is one-to-one and onto."

This led us to modify our focal proof activity. Instead of students peer-reviewing proofs they had written, they were each tasked with making sense of one of two prototypical student proofs provided by the instructor and becoming experts on the proof they read. The student conversations about these proofs were much more balanced both in the quantity of participation and in students' mutual engagement (as opposed to exchanges resembling one-on-one tutoring). This change draws attention to a major theme in our work: *expanding the evoked expertise beyond proof construction.* If the only way to show competence is through constructing formal proofs, we are necessarily amplifying the status of certain students who have become more adept at engaging in this complex and challenging practice.

In other tasks that had less support for equitable small-group discussion, we identified even more foundational issues. For example, when students were tasked with making sense of parts of the First Isomorphism Theorem, some groups of students talked very little and did not appear to engage in meaningful comprehension. In later cycles, we changed the structuring of this task such that each member of the group was responsible for leading a discussion on a specific question, such as identifying the difference between the isomorphism and homomorphism involved in the proof the First Isomorphism Theorem. This structuring served to not only support students' access into the proof, but also *individual and shared accountability* across students.

Selecting and Working with Records of Student Thinking

Finally, when we consider whole class discussion and comparison of student ideas, we found it important to attend carefully to whose ideas made it into the public space. As we anticipated certain approaches to select and sequence, we found the instructor was limited in whose proofs make it to the focus of discussion. In first classroom implementation of the Structural Property Task, the only students with largely complete (though possibly not fully correct) proofs were the more vocal, white men. This informed two modifications: expansion of expertise so that a presenting student would not have to have proven a statement (as discussed in the prior section) and developing *interdependent ways of creating public records*. For example, when students created examples to illustrate the First Isomorphism Theorem, the records were subdivided such that each student had a different role: writing domain and codomain elements, creating the homomorphism map, identifying the kernel, creating the isomorphism map (see Figure 1). In this way, discussions around this student work focused on a truly shared product.

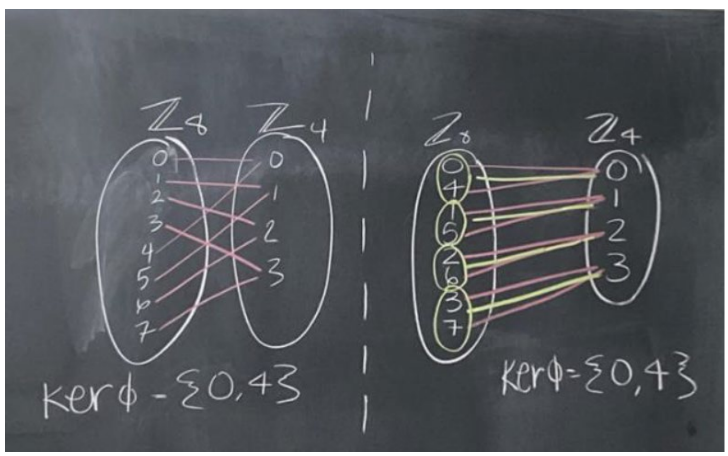

Figure 1. Cycle 5 small group student board work illustrating the FIT

The other major component of comparing and using records of student reasoning was to position students to generalize and see important features of their ideas that may connect to more formal representations. This modification to our task implementation required a large amount of attention and reflection, because, as we compared across implementations, we found that certain discussions could easily devolve into an initiate-respond-evaluate conversation where the instructor was doing majority of the reasoning (and students were left searching for the target insight). To support students in further reasoning and meaningful contribution, we found it useful to introduce *partial information* that linked student ideas and the formal proofs. For example, we provided a partially written isomorphism map for students to complete "β (____) = ____" (First Isomorphism Theorem), a partial function diagram (Structural Property Task), or a series of formal lemma statements that students were asked to pair with their own informal conjectures (Lagrange's Theorem). By providing partial information, the instructor navigated between students' ideas and formal mathematics while providing students explicit tools to increase access and engagement.

Discussion

Working in the formal proof setting, we encountered two substantial hurdles: (1) navigating between formal and informal ideas (Raman, 2003;Weber & Alcock, 2004; Zazkis & Mills, 2017; Zazkis et al, 2016); (2) student status differentials occurring based on students' speed and ease working within the formal proof setting (cf., Weber & Melhuish, 2022). Regarding the first hurdle, we found for complex task launch to provide students access to necessary tools, the conversations had to draw attention to structural ideas (both formal and informal) and help connect imprecise language to quantification or to identifying referent objects involved. Simply unpacking definitions or creating examples was not sufficient to support later proof aims. We also found that orchestrating student discussions to promote comparison and generalization from their proofs and examples required careful attention to ways to connect between formal and informal representations systems (our access heuristic of authentic mathematical proving activity). Often the instructor would introduce an object to focus discussion and promote students in engaging in translating between systems and noticing important ideas.

The second hurdle required us to return to the literature on promoting more equitable group work. As noted by many scholars, active engagement in proof contexts can favor students who are more comfortable with academic language or more quickly engage with the norms of proof (Brown, 2018; Weber & Melhuish, 2022). This can foster divisions of status between students and large inequities in who participates. To better navigate this issue, we incorporated more intentional features related to group-worthy tasks and Complex Instruction (Cohen & Lotan, 1996). By subdividing responsibilities – for example, leading an assigned portion of discussion or taking on a specific role in the construction of an example – and promoting distributed expertise – for example, placing each student in charge of a particular proof approach – we were able to engage more students in authentic activity during group work (our engagement heuristic of authentic mathematical proving activity). This also freed us from constraints we previously experienced in selecting student ideas, as we were no longer tethered to finding certain, anticipated proof approaches or using an example that primarily legitimized only one students' contribution.

We suggest other design researchers consider the dual participation heuristics and engineering of HLTP implementations as they move from small-group lab settings to whole-class instruction. Issues of access and participatory equity become amplified in the latter, where an instructor can no longer interact with each individual student throughout a lesson. HLTPs provide a means to manage some of this transition, but as we found in our project, also require intentional analysis and modification related to our heuristics and the specifics of theorems and proofs involved to meet our participation goals.

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Shifting Students' Feelings and Mindsets about Learning Mathematics through Standards-Based Grading

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Standards-based grading is an alternative assessment method that is gaining traction in undergraduate mathematics courses. Reported benefits for students include reduced math and test anxiety, increased achievement rates, and deeper understanding of concepts. However, evidence is mixed, and researchers have called for qualitative studies to explore students' feelings and experiences with standards-based grading. In this study, we examined 228 College Algebra students' responses to an end-of-semester reflection assignment. We discovered that some students expressed a shift in their feelings and mindset towards mathematics, and that these shifts were connected to the standards-based grading assessment structure. We present these findings, which describe how students viewed their learning and how shifts in their feelings and mindsets related to specific features of the course. We also discuss the internal and external motivations for these shifts and pose questions for future research related to students' experiences with non-traditional assessment structures.

Keywords: Standards-based grading, feelings, mindset, assessment

Traditional grading and assessment methods are the norm in undergraduate mathematics courses, particularly at the introductory level (Harsy et al., 2021). However, in recent years, there has been a growing interest in alternative approaches to assessment. These alternative approaches share several characteristics, a primary one being that grades should reflect students' final understanding of a learning objective after they have had multiple opportunities to demonstrate their learning. One such alternative assessment method is standards-based grading, where grades are intended to reflect students' proficiency in clearly defined learning objectives (Lewis, 2021). Points and letter grades are de-emphasized in favor of thorough feedback given to students after an assessment, with students having multiple attempts to demonstrate their understanding on learning objectives throughout the course.

The analysis presented in this proposal builds on a larger research study that examined how students perceived their mathematical experiences in a newly redesigned, introductory-level, mathematics course. Three themes emerged from preliminary data analysis: community, assessment, and mindset, with students discussing both positive and negative feelings throughout their reflections (Uhing et al., 2022). Through this analysis, we discovered that some students expressed a shift in their feelings and mindset towards mathematics, which prompted us to further examine this phenomenon. Students who discussed this shift primarily did so in the context of the assessment structure, which involved standards-based grading. Thus, the purpose of our work is to better understand shifts in students' expressed feelings and mindset with respect to an alternative grading structure.

Background Literature

Standards-Based Grading

While changing the undergraduate mathematics norm of summative exams to implement a new assessment method can be challenging (Elrod & Strayer, 2018), mathematics education researchers argue that it is a worthwhile endeavor for increased student learning (Prasad, 2020) and decreased test anxiety (Harsy et al., 2021). In a study with 132 undergraduate students in the U.K., students rated less traditional assessments, such as presentations and projects, as some of their least preferred methods of assessment (Iannone & Simpson, 2015). Instead, they preferred more traditional assessments, such as closed book exams and weekly homework assignments. In addition, students preferred assessments that were better discriminators of mathematical proficiency and that seemed the fairest. Notably, this study did not include a standards-based model, where closed-book exams are accompanied with feedback and can be taken repeatedly.

Students in Zimmerman's (2020) large-scale standards-based grading College Algebra course expressed that this assessment structure held them accountable for their learning, gave them more feedback, and led to a deeper understanding. Zimmerman concluded that a standards-based grading approach "removes the extrinsic motivation factor that often fosters unproductive learning behaviors" (p. 1052); however, their precise meaning of "extrinsic motivation factor" is unclear. In our study, we sought to explore different types of motivating factors and how they influence students' shifts in feelings and mindsets about learning mathematics.

Emotions and Expressed Feelings toward Assessment

Wass and colleagues (2020) documented 14 distinct negative emotions expressed by students who completed undergraduate course assessments, such as "satisfaction", "fear", and "worry". The two most common positive emotions were "happiness" (7%) and "pride" (5%) and the most common negative ones were "annoyance" (17%) and "frustration" (10%). They found that students' positive or negative emotions toward assessments corresponded to their performance and argued that understanding students' emotions better could help educators design more effective and supportive assessment structures.

In our study, we use the term "expressed feelings" rather than "emotions" as "emotions are thought of as very primitive, extremely fast, unconscious mechanisms" while feelings are the "conscious and cognitive perceptions" of those emotions (Hansen, 2005, pp.1426-1427). Indeed, much of psychology research is in agreement that there are 27 distinct human feelings, which can be divided into two disjoint groups: positive feelings and negative feelings (Cowen & Keltner, 2017). This dichotomy of feelings allows for examination of a change in feelings valence (e.g., Kleine et al., 2005).

Traditional forms of assessment have been linked to increased feelings of stress and anxiety for students (Lewis, 2020; Prasad, 2020; Wass et al., 2020). A reported benefit of standards-based grading formats is that they are lower-stakes, and thus can reduce anxiety for students (Harsy et al., 2021), although the evidence across disciplines is varied (Lewis, 2020). In mathematics, Lewis (2020) received contradicting responses from students on a survey related to their mathematics text anxiety in Linear Algebra and Differential Equations courses taught with a standards-based grading approach. Specifically, analysis of pre/post survey items showed that test anxiety significantly increased over the semester; however, students self-reported a decrease in text anxiety at the end of the semester and an appreciation for the standards-based grading format in their qualitative responses to open-ended questions. Lewis (2020) called for future research to examine this apparent contradiction using qualitative study methods.

Mindsets about Learning Mathematics

The notion of reflecting on, revising, and reattempting work builds on a growth mindset perspective of learning (Dweck, 2010). A student with a growth mindset believes that through their efforts they can nurture and develop a better understanding of a mathematical concept;

whereas, a student with a fixed mindset believes that mathematical ability is innate and unchangeable. As students enter mathematics classrooms with (sometimes deficit) views of their mathematical abilities, it is up to the practices of growth-mindset instructors to foster "academic growth by encouraging and recognizing students for their growth in learning" (Harsy et al., 2021, p. 1073). Standards-based grading emphasizes that learning mathematics is part of a process; deep understanding of concepts may not happen immediately. In this way, a growth mindset is a potential outcome of an experience with standards-based grading (Dweck, 2007).

In a study with 104 students in mastery-based teaching Calculus and Linear Algebra courses, Harsy and colleagues (2021) found that there was no statistically significant change throughout the semester based on their responses to growth mindset items on a Likert-type survey. Additionally, regarding a growth mindset toward mathematics, they found no difference between students who experienced mastery-based testing and those who did not. However, they did not qualitatively explore the 89 students (85.57%) who disagreed or strongly disagreed with the fixed-mindset statement "Your math intelligence is something about you that you can't change very much." This exploration is a place for further research.

Research Questions

Using the existing literature to frame our study, the research questions we sought to address were:

1. How did students' experiences with a standards-based grading structure in a College Algebra class lead to shifts in their expressed feelings and mindsets about learning mathematics?
2. Which aspects of students' experiences in this course were most prevalent in their expressed feelings and mindset shifts about learning mathematics?

Methods

The data for this study come from a Mathography assignment that was collected at the end of the Fall 2021 semester from students enrolled in College Algebra at a large, metropolitan university. Out of 459 students enrolled at the beginning of the semester in 14 sections of College Algebra, 228 students submitted the assignment. During this semester, instructors were involved in a large-scale course redesign effort that involved developing new curricular materials around active learning and using standards-based grading for the assessment structure. Assessments were called Learning Outcome Assessments (LOAs) and were created based on clearly articulated learning outcomes (i.e., objectives). Students were allowed multiple attempts over the semester to complete each learning outcome. Sections of the course were also coordinated, using the same instructional materials and assessments.

The Mathography data included student responses to prompts that were adapted from Drake's (2006) mathematics story interview, particularly the "peak experience" and "nadir experience" that students experienced during their time in College Algebra. We coded all 228 Mathography assignments in Dedoose individually and met pairwise to discuss and reconcile disagreements until we achieved 100% interrater agreement. Our predetermined codes in Dedoose (2022) included: emotions (subcodes of positive and negative), community and culture, assessment (subcode learning outcome assessments), and personal growth/mindset, which were determined after preliminary analysis of the data (Uhing et al., 2022).

For the analysis of shifts in expressed feelings, we considered excerpts tagged with "assessment" AND "positive emotion" AND "negative emotion" codes. This produced 61 excerpts. We analyzed these excerpts to determine if a negative to positive shift in feelings occurred, which reduced our sample to 48 excerpts. In our initial tagging of excerpts for positive and negative emotions, we were careful to only include excerpts where students voiced an explicit emotion or feeling. For example, we did not include excerpts where students used verbs such as "struggling", as this action could be connected to both positive or negative feelings (e.g., enjoying a challenge or feeling frustrated by a problem). To compare, we also looked at excerpts tagged with "positive emotion" AND "negative emotion" AND not with "assessment". This yielded a total of 16 excerpts, only 9 of which exhibited a shift. We chose not to include these excerpts in our analysis as we wanted to focus on the context of assessment.

For the analysis of shifts in growth mindset, we considered excerpts tagged with "assessment" AND "personal growth/mindset". This produced 150 excerpts, 27 of which we had already included from the expressed feelings set. We analyzed the remaining 123 excerpts to determine if a shift toward a growth mindset occurred, which reduced our sample to 56 excerpts. Most excerpts included explicit shifts from a fixed mindset (e.g., "I didn't think I could learn this.") to a growth mindset (e.g., "It made me realize that if I work hard enough for something then I can accomplish it."). However, some showed more implicit growth in mindset by discussing a "turning point" moment where they "kicked it into gear" or "started to turn things around". Overall, excerpts varied in length but were generally 2-5 sentences in response to a single prompt.

We emphasize that the final excerpts in our analyzed dataset are focused on a *shift* in feelings or mindset, not simply a single state. Several excerpts were tagged with positive feelings as students expressed their enjoyment, satisfaction, and confidence as a result of the assessment structure in this course. However, here we focus on excerpts where students first expressed a negative feeling or mindset, such as concern or nervousness, and subsequently expressed a positive feeling or mindset as a result of some shift during their experience in this course. Then, we used open, data-driven coding (Saldaña, 2016) to analyze the excerpts for the type of shift that occurred.

Findings

In our analysis, we found five primary causes that students attributed to a shift in their feelings or mindset (see Table 1). Three of the causes (the assessment structure, grading scale change, and LOA performance) related directly to the implementation of standards-based grading in the course, while the other two causes (external resources and overall course experience) were more general in nature.

Table 1. Causes of Shifts in Students' Feelings or Mindset

Cause of the Shift	Description
Assessment Structure	Overall structure of the assessment and grading system
Grading Scale Change	Instructors changed the grading scale nine weeks into the semester from needing to complete all 24 learning outcomes (LOs) twice, to having 5 "core" LOs and 19 "supplemental" LOs
LOA Performance	How students did on a particular learning outcome assessment (LOA) (i.e., a specific grade or outcome)

External Resources	Other people (e.g., instructors, TAs, classmates) or other external resources
Overall Course Experience	Reflecting back on their experience in the course as a whole (e.g., talking about the hard work that paid off or things they learned about themselves)

Table 2 shows the frequency across student excerpts for each cause with respect to shifts in feelings and mindset. Excerpts were only coded under one cause for a particular shift (e.g., a student excerpt that related to a shift in mindset was only attributed to a single primary cause, such as assessment structure). We expand on these findings below and provide illustrative student examples.

Table 2. Number of Student Excerpts Relating to Different Causes of Shifts in Feelings and Mindset

Shift	Assessment Structure	Grading Scale Change	LOA Performance	External Resources	Overall Course Experience	Total Number of Excerpts
Feelings	7 (14.6%)	7 (14.6%)	23 (47.9%)	9 (18.8%)	2 (4.1%)	48
Mindset	9 (16.1%)	0 (0%)	21 (37.5%)	10 (17.8%)	16 (28.6%)	56

As shown in Table 2, LOA Performance was the most common cause of a shift in feelings or mindset for students. Several students discussed how getting feedback on an LOA cause them to experience this shift. For example, one student remarked:

> A peak experience that happened to me in [College Algebra], was after the second to last LOA of the semester. I hadn't gotten any LO's the previous assessments and was very concerned about my grade, especially since I had a core LO that I still needed 2 points in. I put in about an hour and a half of studying leading up to the day of the assessment and, and came out of class feeling accomplished. Every question I answered, I ended up getting the points for, giving me confidence that I will end up succeeding in class.

This excerpt shows how performance on an LOA caused the student to feel "concerned" about their grade, which led to them studying more and doing better on the subsequent assessment, causing the student to then feel "accomplished" and confident.

In terms of mindset, several students mentioned how their performance on LOAs caused them to "push themselves" and "work hard" to achieve better results. One student commented:

> A peak experience in my math story took place this year. I looked at my LOA scores and realized that I needed to pick it up or I would have problems passing the class. So, when the next LOA came up, I studied more than I had for any other test, and it paid off and now I am on pace to pass the class if I keep up my study habits for the next couple of tests. This experience says that I as a student am someone who learns well from their mistakes and is a hard worker who wants to achieve his goals.

Instead of giving up after earning a low score on an LOA, this student viewed their mistakes as an opportunity to learn, thus providing motivation to study harder for the next assessment and demonstrating a growth mindset.

Several students mentioned a major change to the grading scale that occurred nine weeks into the semester in their reflections. About halfway through the semester, after reviewing assessment data and having multiple discussions about the assessment structure, instructors of the class decided to reduce the number of learning outcomes that students needed to demonstrate skill on twice. Several students discussed this key event and attributed it to feeling less stressed about the course overall. One student remarked,

> At the start of the year, we were having to complete quite a few LOs and we had to do them twice, which was stressful. Once the grading got changed to only having to do the supplemental LOs twice, it helped me gain confidence.

Another student expressed a similar sentiment:

> A low point in Math 1220 this semester would be right before the grading requirements were changed. I hadn't scored very well on the LOA's at that point and I wasn't sure if I'd be able to meet all the requirements. Overall, I was very discouraged and didn't think I would be able to pass. I have definitely grown since then in my math skills and math confidence.

Both of these students described shifts in their feelings from being stressed and discouraged to feeling more confident in their mathematical abilities. It is also interesting to note that while this grading scale change was expressed as a cause of shifts in feelings (7 out of 48 excerpts), it was not discussed with respect to shifts in students' mindsets (0 out of 56).

On the other hand, shifts in mindset seemed to occur over the duration of the semester, leading to the code Overall Course Experience. For instance, one student discussed their view on "being good at math" and how they had persevered throughout the semester:

> While many people would describe being good at math as simply receiving good grades, that is not necessarily the case. For me, being good at math means persevering when you are not completely sure how to complete something. Being able to make a mistake while learning and growing from it is extremely valuable, not just in math but in life as well. Initially, my goal was to receive a C in the course [...]. However, closer to the end of the semester, my goal is to receive an A in the course. [...] I will be much happier knowing I gave this class my full effort and was rewarded for it.

For this student, the overall experience of being in a course that used standards-based grading helped shift their mindset from wanting to earn a passing grade to wanting to give their "full effort" and feeling "rewarded" for doing their best.

Discussion

The most prevalent cause of a shift in both expressed feelings and mindset was students' LOA performance, meaning how well they did on a specific LOA. This finding surprised us as we were expecting that LOA performance might be more closely related to a shift in feelings (as feelings are more fleeting and transitory) as compared to mindset (which is sustained over time). However, our findings suggest that students' mindsets can also be shifted by their performance on assessments. Moreover, it is clear that even in a standards-based grading environment, students still cared about their grades on assessments as LOA performance was the most commonly referenced cause of both shifts in feelings and mindset.

Zimmerman claimed that a standards-based grading approach "removes the extrinsic motivation factor that often fosters unproductive learning behaviors" (p. 1052). In other words, when students demonstrated a shift in expressed feelings regarding assessment or evidence of a growth mindset, did it appear to be motivated by extrinsic factors? Although the assessment LOA structure was a standards-based grading approach that allowed for multiple attempts, students were still concerned about their immediate results on the individual assessments. This suggests that many students are still motivated by grades and, oftentimes, by their performance on the first attempt of an LOA.

In addition to seeing a shift in mindset, we saw strong evidence for the presence of a growth mindset in several students, although it was sometimes unclear whether or not they previously held more of a fixed mindset toward learning mathematics. This finding supports other research which claims that the structure of standards-based grading courses provides students with multiple chances to succeed and thus helps students foster growth mindsets about learning mathematics (e.g., Harsy et al., 2021). Our findings further suggest that a standards-based grading structure may encourage students to metacognitively consider their internal and external sources of motivation and to persist toward their learning goals, rather than becoming discouraged after a nadir event (e.g., failing the first exam).

Finally, one finding that we are still investigating is the difference between shifting to positive feelings/mindset versus shifting to non-negative feelings/mindset. For instance, we observed a marked difference between those excerpts that expressed a shift from negative to (explicitly) positive emotions (e.g., from worry to confidence) versus those that went from negative feelings to "decreased negative" or neutral feelings (e.g., from stress to a sense of relief/calmness, such as, "I felt relieved" or "I felt better"). In particular, "relief" appeared to be qualitatively different from the other feelings.

This emergent finding may be connected to Lewis's (2020) contradictory research with math test anxiety, which suggests that the feeling of relief plays a role in how students express their feelings towards mathematics assessments. Namely, does the timing of the survey (or Mathography assignment, in our case) matter? If students respond to questions at the end of the semester, are they more likely to express relief and positive feelings, or, is the stress of final grades weighing more heavily on their minds? Kleine and colleagues (2005) stated that future research should examine the direct relationship between achievement-related emotions and academic performance. We are continuing to explore this research finding.

Implications and Limitations

As other researchers and practitioners of standards-based grading have claimed, it is necessary to carefully consider how the course is designed (Collins et al., 2019), what the learning objectives are (Zimmerman, 2020), how the grading system is communicated with students (Kelly, 2020), and what strategies should be used for implementing these alternative grading approaches. In our study, we found that considering these components was not sufficient for *all* students to shift away from a grades/performance mindset to learning mathematics. We were, however, encouraged by the number of students (56 of 228 students, 25%) who demonstrated a shift toward a growth mindset for learning mathematics, especially those who clearly expressed that they initially held a fixed mindset about their math abilities. However, we emphasize that standards-based grading is not a cure-all for mathematics instruction. Implementing this assessment structure should be done thoughtfully and not prescriptively. Moreover, we should continue to explore how students' feelings are entangled with grading and assessment and how we can work (within the confines of our educational system) to encourage a growth-oriented mindset toward learning mathematics.

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Mathematicians' Conceptions of Active Learning and Equitable and Inclusive Teaching: A Set Theory Analysis

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Although research on active learning suggests strong connections to equitable and inclusive teaching, few studies have explored the relationship between the two concepts. For this proposal, we examined how 13 participants in an equity workshop described the relationship between active learning and equitable and inclusive teaching. We used set theory as an analytic tool to categorize their descriptions. We found that all participants saw active learning and equitable teaching as related. The two most commonly held views were that active learning and equitable teaching are related yet independent sets, or that equitable teaching is a subset of active learning. We discuss implications of these findings and pose questions for further exploration.

Keywords: Active learning, equitable and inclusive teaching, set theory

Active learning, equity, and inclusion are often-referenced terms in undergraduate mathematics education. Despite their popularity, these terms are nuanced and often difficult to define (e.g., Roos, 2019). One well-known framework for conceptualizing active learning (AL) is Laursen and Rasmussen's (2019) Four Pillars of Inquiry-Based Mathematics Education, which includes ideas about both AL and equitable and inclusive teaching (EIT). In past work (Williams et al., 2022), we used this framework to explore how various stakeholders in mathematics departments across multiple institutions conceptualized AL. We found that we were unable to categorize definitions of AL based solely on stakeholder role, and overall, stakeholders mentioned aspects of teaching and learning that related to the four pillars of inquiry-based mathematics education (students, content, teachers, and equity) in their definitions of AL. While stakeholders most frequently mentioned the actions of students in their definitions, they rarely acknowledged how equity played a role in their conceptualization of AL. This led us to wonder how members of a mathematics department perceive the relationship between AL and EIT.

For this proposal, we examined how mathematics faculty members enrolled in a year-long workshop focused on equity described the relationship between AL and EIT. We used set theory as a tool for carrying out our analysis. Specifically, if AL and EIT are viewed as two sets, then how do they intersect (or not)? Thus, our guiding research question was: How do mathematics faculty members engaged in work on active learning and equitable and inclusive teaching view the relationship between the two concepts? We provide a brief theoretical framing and background before describing how we applied set theory to understand how participants described the relationship between AL and EIT.

Theoretical Framing

Equitable and Inclusive Teaching

One definition of equity in mathematics education includes three criteria: 1) "Being unable to predict students' mathematics achievement and participation based solely upon characteristics such as race, class, ethnicity, gender, beliefs, and proficiency in the dominant language" (Gutiérrez, 2007, p. 41); 2) "Being unable to predict students' ability to analyze, reason about, and especially critique knowledge and events in the world as a result of mathematical practice, based solely upon characteristics such as race, class, ethnicity, gender, beliefs, and proficiency in the dominant language" (p. 45); and 3) "An erasure of inequities between people, mathematics, and the globe" (p. 48).

Accomplishing these three criteria requires teachers to attend to the dimensions of equitable mathematics, which lie along dominant and critical axes (Gutiérrez, 2007). The dimensions of the dominant axis are access and achievement, while the critical axis consists of identity and power. Studies on learning mathematics have frequently focused on the dominant axis by suggesting that interactive classrooms provide opportunities for students to access rich learning experiences, and by showing that student learning increases in classrooms that use collaborative, student-centered practices (Gutiérrez, 2013; Smith et al., 2021). However, creating equitable and inclusive mathematics classrooms involves thinking about how identity and power influence classroom interactions. In other words, EIT involves paying attention to how personal relationships and societal norms impact students' interactions and experiences (Gutiérrez, 2007). Power dynamics and students' identities affect students' learning experiences, and thus should be considered in mathematics classrooms that are interactive (Adiredja & Andrews-Larson, 2017).

Active Learning

There are several definitions of active learning in research literature (e.g., Williams et al., 2022; Prince, 2004; Stains et al., 2018). In their well-known metastudy, Freeman et al. (2014) provided a definition of AL that explained the difference between AL and passive lecture:

> Active learning engages students in the process of learning through activities and/or discussion in class, as opposed to passively listening to an expert. It emphasizes higher-order thinking and often involves group work (pp. 8413-8414).

Another broad definition was provided by the Conference Board of Mathematical Sciences (2016), which referred to AL as "classroom practices that engage students in activities, such as reading, writing, discussion, or problem solving, that promote-higher-order thinking" (p. 1). While these definitions provide us with some notion of what AL is, there is not a single, commonly agreed upon definition in undergraduate mathematics education. Therefore, recent studies have begun to examine the nuances and complexities of the term AL, and how it is operationalized in undergraduate mathematics classrooms (Williams et al., 2022).

Research showing that AL can lead to improved student outcomes has contributed to a consensus in the field that AL is an equitable practice (Freeman et al., 2014; Laursen, 2019); however, student experiences with AL vary (Johnson et al., 2020; Reinholz et al., 2022). There is evidence from the context of upper-division mathematics courses that when directly compared to lecture-style instruction, AL may result in greater performance disparities between women and men (Johnson, et al., 2020). Moreover, Rios (2022) found that students in introductory-level courses whose preferred math language was not English felt a decreased sense of belonging in English-dominant math classrooms as AL practices increased. These findings, coupled with evidence that AL increases achievement outcomes for students from underrepresented groups in STEM (Theobald et al., 2020), suggest that the relationship between AL and equity is complex and not fully understood.

Although research on AL suggests strong connections to EIT, such as Laursen and Rasmussen's (2019) Four Pillars framework, few studies have explored the relationship between the two concepts. Moreover, these studies tend to focus on student outcomes, rather than how the instructor's view of these concepts may influence their role in the classroom. Thus, more work is needed to understand how members of mathematics departments view the relationship between AL and EIT.

Methods

Data Collection

Data were collected from a total of 13 participants. These participants were undergraduate mathematics faculty members at a variety of institutions across the US who were participating in a year-long equity workshop as part of a larger study, which focused on AL in first-year mathematics courses. Interviews were conducted over the span of the equity workshop in September 2020, January 2021, and May 2021. Data presented in this proposal come from the final interview after the conclusion of the workshop in May 2021. As part of this interview, participants were asked to provide their definition of EIT, and discuss whether and how this definition might have changed over the previous academic year. Participants were also asked to respond to the following prompt: "Some people equate active learning with equitable and inclusive teaching. How do you see these as related, and in what ways are they different?" Participants' responses to this prompt form the data set for this proposal.

Data Analysis

To analyze the data, we used set theory to list the possible relationships between AL and EIT. If we view AL and EIT as two distinct sets (see Figure 1), we can investigate the ways in which these two sets can interact. In this context, there are five possible relationships which correspond to five diagrammatic representations. Table 1 shows the five possible relationships between AL and EIT and provides descriptions from mathematical set theory along with interpretations of how we applied this logic structure to the data in our study.

An implicit assumption that we made in our analysis is that AL and EIT exist in the same domain and that it makes sense for these sets to be intersecting. We also assumed that AL and EIT are clearly defined terms and we can thus determine what is contained in each set. We discuss these assumptions further in our Findings and Discussion sections.

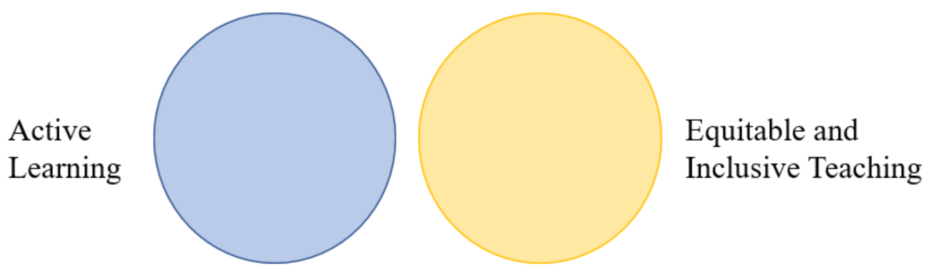


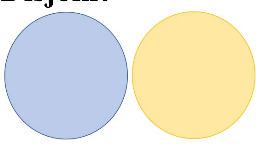
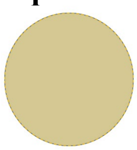
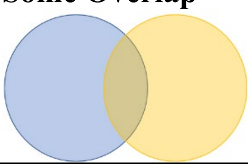
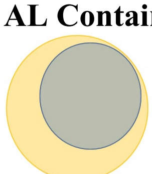
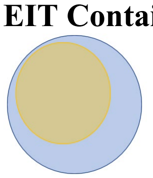
Figure 1: AL and EIT as Two Distinct Sets

After interviews were transcribed, the excerpts where participants were asked about the relationship between AL and EIT were identified and organized into a matrix for coding. Each row represented a participant and their specific response. The columns represented the different possible relationships between AL and EIT (as seen in Table 1). Our goal was to look for evidence of how the participant viewed the relationship between AL and EIT. We then decided on what relationship(s) from Table 1 the participant discussed (either saying that AL and EIT were related or were not related in the following ways) and copied text from their response to support our decision. The decision categories we used were:

- YES: if there was <u>strong</u> evidence in the participant's response that aligned with that relationship
- MAYBE: if there was <u>possible</u> evidence in the participant's response that aligned with that relationship
- NO: if there was <u>no</u> evidence in the participant's response that aligned with that relationship OR there was evidence <u>against</u> that particular relationship (directly or indirectly) in the participant's response

Every participant's response was coded independently by at least two of the authors, who then met to reconcile and discuss any differences (Creswell & Creswell, 2018).

Table 1: Set Theory Relationships between AL and EIT

Relationship between AL and EIT	Set Theory Description	Interpretation
Disjoint 	• The intersection of AL and EIT is empty ($AL \cap EIT = 0$)	Whenever AL occurs, EIT does not occur. Whenever EIT occurs, AL does not occur.
Equivalent 	• AL and EIT are equivalent sets ($AL = EIT$) • AL exists if and only if EIT exists.	Whenever AL occurs, EIT occurs. Whenever EIT occurs, AL occurs.
Some Overlap 	• The intersection of AL and EIT is not empty ($AL \cap EIT \neq 0$) AND • There exists EIT that is not AL, and there exists AL that is not EIT.	AL and EIT can occur together, and they can also occur independently.
AL Contained in EIT 	• AL implies EIT ($AL \rightarrow EIT$) • There exists EIT that is not AL. • EIT is a necessary condition for AL	Whenever AL occurs, EIT also occurs. However, EIT can occur without AL.
EIT Contained in AL 	• EIT implies AL ($EIT \rightarrow AL$) • There exists AL that is not EIT. • AL is a necessary condition for EIT	Whenever EIT occurs, AL also occurs. However, AL can occur without EIT.

Findings

Our overall findings are summarized in Table 2. We provide examples and descriptions of participants' responses in each of the set theory relationships in the following paragraphs.

Disjoint and Equivalent

All participants (13 of 13) provided strong evidence that AL and EIT were not equivalent. Similarly, almost all participants (12 of 13) provided clear evidence that they saw some relationship between AL and EIT (i.e., that these two concepts were not disjoint). For example, Lacy stated, “they’re related, but they’re not equal.” This provided evidence that Lacy saw that there was some relationship between AL and EIT, but that they were also not the same.

Emma was the only participant who potentially saw AL and EIT as disjoint concepts. In her response, she discussed the difference between AL and EIT, describing AL as a “paradigm for learning” and EIT as what the teacher is doing to create an equitable environment in the classroom:

> The definition of active learning is that students need to construct knowledge for themselves and that we can provide them with information and scaffolding and opportunities to construct it for themselves [...] this is different from equitable teaching, right? Equitable teaching is sort of making a classroom where students of any background can learn and that's different from sort of this philosophical understanding of how learning happens.

From this response, it was unclear whether Emma saw AL and EIT as disjoint (possibly in two separate universes without the same domain). One possible interpretation is that Emma saw AL from more of a student perspective (i.e., how students learn and construct knowledge in the classroom), whereas EIT is more focused on the teacher (i.e., what the teacher is doing to create an equitable environment). This led us to consider the possibility that perhaps AL and EIT could be viewed as existing in different universes (learning paradigms vs. teaching strategies) with no potential for overlap because they did not have the same domain. Therefore, we decided to code Emma’s response as a “MAYBE” under disjoint.

Table 2: Evidence of Relationship between AL and EIT

Participant	**Disjoint**	**Equivalent**	**Some Overlap**	**AL Contained in EIT**	**EIT Contained in AL**
Aadaya	NO	NO	NO	NO	YES
Bill	NO	NO	NO	NO	YES
Cassandra	NO	NO	NO	NO	YES
Crystal	NO	NO	YES	NO	NO
Collin	NO	NO	MAYBE	YES	NO
Camila	NO	NO	MAYBE	NO	MAYBE
Mark	NO	NO	YES	MAYBE	NO
Kathleen	NO	NO	YES	NO	NO
Lacy	NO	NO	MAYBE	NO	MAYBE
Emma	MAYBE	NO	MAYBE	NO	NO
Robert	NO	NO	MAYBE	NO	NO
Shea	NO	NO	NO	NO	YES
Thomas	NO	NO	MAYBE	NO	YES

Some Overlap

Most participants (9 of 13) viewed AL and EIT as related to some extent. For some, the evidence for this relationship was straightforward; Robert said, “I think they intersect, but yet they’re not equal.” For other participants (Crystal and Kathleen), their expressions supported this relationship because they claimed AL and EIT are related and provided evidence against both containment relationships. For example, Kathleen said, “I was kind of surprised by that prompt, because they definitely have an overlap, but I don't see them as being equal to each other” (evidence for some overlap); she added, “while I think that active learning benefits all students, and also helps with teaching equitably, I think that teaching equitably is also like, it includes other things as well" (evidence against EIT contained in AL), and "even though [group work] is an active learning strategy, it's not necessarily equitable unless we think about equitable practices" (evidence against AL contained in EIT). Thus, some overlap best fit Kathleen's view of the relationship between AL and EIT.

Six participants (coded as "MAYBE" for Some Overlap) gave responses indicating that they saw AL and EIT as related, but it was not clear which relationship aligned best with their explanation. One participant, Camila, explicitly used set theory in her response and said, "I haven't thought, 'which is a subset of the other,' but it could be that maybe equitable and inclusive learning is a subset of active learning. I'm not quite sure yet." Her acknowledgment of subsets shows she was considering the concepts to at least have some overlap, but then expressed doubt of a final relationship.

AL Contained in EIT

Overall, participants' responses were least aligned with the relationship that AL is contained in EIT. Only one participant, Collin, expressed strong evidence for AL existing within the set of EIT, describing AL as a "specific pedagogical strategy" within the "more ingrained [...] act of learning". He shared that

> "active learning is a specific pedagogical strategy, whereas equitable and inclusive learning is something less explicit and more ingrained in how you're doing the act of learning. [...] active learning sort of, in my mind, inherently has this idea that students are capable of learning independent of their background, so, in a sense, you are creating space for equity by allowing students to discover and have dialogue with the material."

Collin viewed AL as "inherently" equitable, implying that AL exists within the domain of EIT.

EIT Contained in AL

Seven participants' responses aligned with EIT as a subset of AL, with five providing strong evidence and two providing possible evidence for this relationship. Aadaya's response encapsulated this relationship as follows: "I think that they are related but yeah I think one would kind of be incorporated into the other." She elaborated further, stating, "So those are things you could do that maybe make students feel more included in the class when they're doing their active learning activities." In other words, in a classroom in which AL is being implemented, there are additional strategies that can be used to help students feel more included. This suggests a view that EIT is a special case of AL.

Bill described a similar viewpoint with the response, "equitable and inclusive learning … might be more purposeful." He went on to say, "there's things that you can do to be very purposeful as opposed to my current understanding of active learning as just good pedagogy, not necessarily, you know, driving in awareness of inequities." This points to a view that AL alone is not necessarily equitable or inclusive unless the instructor is "purposeful" about using equitable and inclusive teaching practices.

Discussion

Although our set theory framework allowed for the possibility of five clearly defined relationships, an overall glance at Table 2 shows that most of our participants' responses fell into two categories. The strongest evidence aligned with the relationships involving some overlap of the two concepts or containment of EIT in AL. We originally conjectured that EIT might be a larger, umbrella term that contained AL. In other words, AL as a set of teaching practices might be situated within equitable and inclusive teaching as a broader ideology (e.g., Roos, 2019). However, participants' responses primarily pointed toward the reverse containment relationship, where EIT exists within AL. This finding is consistent with studies that showed that not all AL classrooms and practices are equitable for all students (e.g., Johnson et al., 2020; Rios, 2022). We continue to explore how this finding aligns with the Four Pillars framework (Laursen & Rasmussen, 2019), where equity as a pillar could be interpreted as one element contained within the larger set of AL, or that equity is a necessary condition of AL and thus AL is contained in EIT. Furthermore, the equity workshop was situated within a larger study on AL (SEMINAL Project; see Smith et al., 2021), where instructors were first introduced to AL in mathematics instruction before connecting this concept to EIT. Thus, the primary focus on AL might have prompted participants to think about EIT within an AL context.

We also emphasize that participants shared their perceptions of the relationship between AL and EIT *at the end* of a year-long workshop on equity and equitable teaching practices. In other words, it seemed reasonable to presume that participants would have similar notions of EIT, AL, and the relationship between them. However, our data showed that participants expressed varied notions of the domains in which these terms exist (e.g., learning theories, pedagogies, specific teaching practices) and how they function in a mathematics classroom, much like Roos's (2019) findings about the role of inclusion. Our findings illustrated that even after participating in a workshop focused on equity, instructors and other mathematics faculty can still hold diverse views of the relationship between AL and EIT. This has major implications for instructors who implement these pedagogical concepts in their classes and for future professional development centered on AL and EIT. Understanding this relationship and being aware of the potential differences in stakeholders' views can inform how mathematics departments frame their collective efforts toward implementing AL and EIT, which is imperative to systemic and sustained change (Williams et al., 2022; Laursen, 2019).

In some ways, our work generated more questions than answers. For example, does having one containment view of the relationship (compared to the other) change how educators implement these practices in mathematics classrooms? Based on someone's view of the relationship, are some ideas privileged over others? Is there one "right" way of viewing the relationship between EIT and AL? Is it possible to hold multiple views of this relationship simultaneously (as responses from some of our participants suggest)? While we cannot definitively say which relationship between AL and EIT is "true" or even if one exists, our work raises questions about the implications of different views of this relationship, particularly for researchers and educators who are interested in instructional change efforts.

Finally, we acknowledge that our analysis brings to light a tension between straightforward mathematical logic and messy teaching constructs. Mathematics often involves creating well-defined categories in order to organize objects. For this analysis, we used set theory as an analytic tool to classify complex, fluid concepts within an exploratory qualitative study. As mathematicians, the structure of set theory is compelling and satisfying; however, as education researchers, we grappled with how to fit participants' responses into precisely one set relationship. Further exploration is needed to better understand the relationship between AL and EIT and the implications it has for improving undergraduate mathematics education.

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Teaching Experience as a Frame for Contributions of Academic Mathematics Coursework to Teacher Learning

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We analyzed end of course reflections of graduate students in mathematics courses for practicing teachers. Participants were asked questions about mathematics content, habits of mind, and whether being a concurrent mathematics teacher while taking the graduate course impacted the value that they saw in either area. Participants were also asked to reflect on whether they believed the course would have been valuable when they were an undergraduate student. We first coded these reflections with Hoffman and Even's (2018, 2021) framework including essence, doing, and worth of mathematics. As a result of our analysis, we suggest extending their framework to mathematics and mathematics teaching, and to include content. Discussion includes suggestions for future research to understand how teaching mathematics reframes learning mathematics, and the importance of framing to understanding the uptake of mathematics courses for both prospective and practicing teachers.

Keywords: Content Courses for Teachers, Mathematical Knowledge for Teaching, Masters Degree Programs

Introduction & Background

We open by recalling a conversation that one of the authors often encounters when teaching a content course for secondary teachers, although this proposal will address content courses for practicing elementary, middle, and secondary teachers. The conversation begins when the topic of abstract algebra is brought up, a course where prospective teachers first learn modular arithmetic. A prospective teacher says, "I hated abstract algebra. I learned in that class that I don't like math." Yet, when practicing teachers in a graduate summer course, from the same district, learn modular arithmetic once more, they will say, "I really liked this class! I want to find a way to teach mods to my kids."

The two courses, the undergraduate abstract algebra course, and the graduate summer course, are taught by faculty in the same department. They address similar topics. Across different years, different faculty members, and different cohorts of teachers, these contrasting conversations repeat, as if in a time loop. Why does the conversation change, as teachers in this district cross the threshold from future teachers to current teachers?

While there is evidence that some teachers benefit greatly from taking graduate level coursework (University of Waterloo, n.d.), and there is evidence that content-focused professional development does impact teacher productivity across elementary and secondary (Harris & Sass, 2007), there is also evidence that it is pedagogical coursework and development that may ultimately make a larger impact on student outcomes (Baumert et al., 2010; Ottmar et al., 2015). Moreover, in some cases, the additional coursework may address content that overlaps with undergraduate coursework. There is a problem: the efficacy of undergraduate and graduate course taking, in terms of impact on teachers' teaching or in perceived usefulness of the course, is debatable and idiosyncratic.

The puzzle for us is this: mathematics teachers need more mathematics than they have learned in elementary, middle, and high school, in order to teach at that level. Yet taking mathematics as an undergrad--a natural place for candidate teachers to prepare mathematically to teach mathematics--seems to be met with adversity. There is a seeming paradox in how experience with teaching may mediate the uptake and relevance of mathematics courses.

Hence, we are interested in why teachers may find graduate and undergraduate courses useful or not. Our overarching research question is: How does teaching experience mediate teachers' uptake of mathematics? We address: *(RQ1) How do practicing teachers describe the contribution of graduate mathematics coursework on their learning? (RQ2) How would the same practicing teachers assess the relative value of this same mathematics coursework, had they taken the course as an undergraduate, prior to any teaching experience?*

In this study, we use *teachers* to refer to PK-12 classroom teachers, *students* to refer to PK-12 students, and we only use *instructor* to refer to tertiary level instructors.

Related Literature

In the existing empirical literature on the contribution of academic mathematics experiences to teacher learning, many practicing and prospective secondary teachers have been documented to find their mathematics experiences disconnected from teaching (e.g., Goulding, Hatch, & Rodd, 2003; Zazkis & Leikin, 2010) to be difficult in unexpected and sometimes unpleasant ways compared to secondary mathematics experiences (see Corriveau & Bednarz, 2017 for a review). At the elementary level, some prospective elementary teachers have found courses to be emotionally stressful and detrimental to their mathematical confidence, and to be disconnected from their teaching (e.g., Hart, Oesterle, & Swars, 2013). The literature also suggests that instructors of mathematics courses for elementary and secondary teachers design courses to combat disconnects and to support students' learning (e.g., Lai, 2019; Li & Superfine, 2018).

This literature has gaps. First, the literature does not distinguish between viewpoints of teachers before and after teaching. As our experience suggests, there may be a change of viewpoint that occurs after teachers enter the classroom. Second, the literature tends to report results from either multiple institutions, so it is hard to disentangle results from context, or from small case studies, meaning that it is hard to generalize. Our study addresses these gaps.

In designing our study, we observed that multiple guiding documents, in the US and beyond, emphasize the value of mathematical *content* and mathematical *habits of mind* (e.g., Ministry of Education, 2007; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). We also used the results of Hoffman, Even, and Mytlis on the contribution of academic mathematics to teacher learning (Hoffman & Even, 2018, 2021; Mytlis & Even, 2021). Their findings suggest that teachers and mathematicians view contributions in three categories: (1) the *essence* of mathematics, that is the nature of mathematics, or "what is mathematics"; (2) *doing* mathematics, or "how is mathematics done"; and (3) the *worth* of mathematics, for its own sake or for its applications. We interpret the *doing* of mathematics to be synonymous with *habits of mind*. We used content and habits of mind as a proxy for what may be explicitly taught in a mathematics course, and we use the results of Hoffman, Even, and Myrtis as a hypothesis for the learning that teachers perceive from these courses.

Conceptual Framework

Our past experiences influence our current perception. From our expectation of success at a task, to how much we value doing a task, to reaction time, our past experience can mediate how we respond to a current moment (e.g., Eccles & Wigfield, 2020; Pearson & Brascamp, 2008). In the case of teachers, prospective and practicing mathematics teachers differ in one significant respect: only one of them has past experience of classroom mathematics teaching. We hypothesize that this difference in past experience mediates how they experience mathematics learning, so that prospective teachers and practicing teachers may view the same mathematics content differently. Our study is an initial step to test this hypothesis by posing to practicing teachers who were enrolled in mathematics courses a counterfactual: would they have found this course just as valuable had they taken the course as an undergraduate prior to teaching?

We investigate potential differences in terms of framing (Goffman, 1974): how a person defines a situation. For instance, if one teacher defines a situation of learning as a "hoop to jump through", and a second teacher defines the same situation as "a place to learn how to do mathematics better", they may end up having different experiences of the same situation.

Participant Context

At Indigo University, the STEM Education Center, in collaboration with the Department of Mathematics, offers graduate mathematics courses taken by K-12 mathematics and science teachers (all institutional names are pseudonyms). Although many teachers enrolling in these courses are local, there are also teachers who enroll from out of state, as well as from countries outside the US. These graduate courses have been offered for almost two decades, and typically feature high end-of-course evaluations from the enrolled teachers.

We anticipated this site would be a fruitful context for our research questions for several reasons. The variety of backgrounds of teachers made it more likely that these teachers would have come from different undergraduate institutions. In this way, their collective viewpoints about experiences with undergraduate mathematics would reflect more than one particular institution or pedagogy. Moreover, we were interested in the case of a success story with mathematics. Frankly, we thought it would be more interesting and productive to document ways that experiences with graduate mathematics coursework are useful, rather than the opposite.

Data & Method

Participants included 46 practicing teachers who were enrolled in 9 mathematics courses in the summer masters program at Indigo University. Participants consented to sharing an end-of-course reflection assignment, which formed the data source for analysis. In total, the 46 teachers submitted 61 separate responses, as some teachers were enrolled in more than one course. Among these were responses from courses intended for elementary teachers and for middle and secondary level teachers. Instructors who taught the participants agreed to include three common prompts for reflection. One prompt posed to teachers the counterfactual situation that they had taken the course prior to classroom teaching:

> Suppose you had taken this course as an undergraduate, before having any significant teaching experience. Would you have found the experiences with habits of mind less or more valuable than now? Would you have found the content less or more valuable than now? For what reasons?

This was Prompt 3 of the reflection assignment. We report only on analysis of responses to this prompt. Prompts 1 and 2 asked about how their experiences with habits of mind and course content might change the way they taught an activity or topic in the future, and defined habits of mind as "approaches to creating mathematical ideas or working on math problems". Example habits of mind and course content specific to the particular course were included in Prompts 1 and 2 (e.g., "looking at a problem from multiple perspectives" and "using precise definitions" were example habits of mind; and "Pascal's triangle" and "linear regressions" were example mathematical content).

Analysis for RQ1: Characterizing the contribution of masters coursework in mathematics to teacher learning

We used a combination of inductive and deductive coding (Miles, Huberman, & Saldana, 2018). We began with deductive coding of responses according to Hoffman and Even's (2018) categories of the contribution of academic mathematics to teacher learning: Essence, Doing, and Worth of Mathematics. However, we noticed statements describing contributions of the masters' course to teachers' learning that did not fit neatly into Hoffman and Even's descriptions, and wanted to differentiate between contributions for the purpose of mathematics from those for teaching. We inductively expanded the categories as shown in Figure 1 (next page). In tabulating the coding, we elected to use the unit of analysis of teachers rather than responses. We noticed that some teachers who had enrolled in different courses used the same phrasing across responses to different courses, and that this phrasing was essential to our coding.

Analysis for RQ2: Comparing relative value of coursework before and after teaching

We initially categorized responses as stating that the graduate course was *more valuable before teaching*, *more valuable after teaching*, *valuable either way*, or *none* (meaning, the response did not explicitly make a comparison of value before or after teaching). Then, as we noticed teachers stating that some aspects were more valuable before teaching and others more valuable after teaching, we inductively coded for the aspect of relative value (e.g., experience with habits of mind, content, technology).

Finally, we excerpted *frame* statements from responses, that is statements that explained something to the effect of, "Teaching experience made a difference in how I learned in the course because …". We surveyed these frame statements for themes in how teaching experiences framed the contributions of the mathematics courses to teacher learning. In tabulating this coding, we elected to use the unit of analysis of responses rather than teachers, because the comparison of contributions was course-dependent, and some teachers stated different comparisons for different courses.

Figure 1. Categories of contribution to teacher learning for RQ1

Contribution		Description: Statement that the course experiences …
Essence	Essence of Math	Changed a teacher's conception of "what is mathematics", or will help them convey to students a conception of "what is mathematics"
	Essence of Math Teaching	Changed a teacher's conception of the nature of teaching, including the possibilities of what teaching can look like
Doing	Doing of Math	Contributed to a teacher's mathematical habits of mind, or that the course will help teachers cultivate students' mathematical habits of mind, including struggling or persevering through mathematics
	Doing of Math Teaching	Statement that the course exposed teachers to teaching practices or activities they see as valuable or beneficial
Worth	Worth of Math	Statement that the course improved a teacher's sense of the value of mathematics, or helped a teacher convey to students the value of mathematics, either for its own sake, or because of its applications to science, technology, etc.. or that the course
	Worth for Math Teaching	Statement that the course improved a teacher's sense of the value of mathematical knowledge for the purpose of teaching
Content	Content Knowledge	Statement that a teacher developed specific content knowledge, or that the course will help teachers support students' acquisition of specific content knowledge, including connections among topics and better explanations (Shulman, 1986)
	Pedagogical Content Knowledge	Statement that a teacher developed specific pedagogical content knowledge, including detecting the potential of activities to facilitate learning; knowledge of student conceptions; or knowledge of or about multiple representations and explanations (Baumert et al., 2010)

Results

Characterizing the contribution of masters coursework in mathematics to teacher learning

Across our participants, the most frequently cited positive contribution of coursework was to the categories of doing mathematics (16 participants; 34.8%) and content knowledge (12 participants; 26.1%). Our prompt specifically mentioned content and habits of mind, and hence it is unsurprising that these categories are mentioned frequently. Table 1 summarizes the results.

Our main differences from Hoffman and Even are (1) introducing Content as a category and (2) extending contributions to both mathematics and mathematics teaching. This delineation allowed us to record that one possible benefit of these courses is the teaching itself: teachers mentioned the benefits of exposure to “great habits of teaching” (Doing of Math Teaching), or “realiz[ing] that how I am teaching now is not fulfilling what my expectations are for my students” (Essence of Math Teaching). Among our participants, 12 cited content knowledge as a contribution, for example, discussing the value of learning about concepts of calculus, or properties of number lines. Teachers explained the value in terms of being able to see more connections or providing better explanations in the future. All but 12 teachers cited a contribution of the graduate course in the responses analyzed.

Table 1. Contributions of coursework in mathematics, with unit of analysis being teachers (n = 46)

Contribution	Math/Teaching	# Participants mentioning contribution (%)
Essence	of Math | Math Teaching	0 (0%) | 2 (4.3%)
Doing	of Math | Math Teaching	16 (34.8%) | 7 (14.2%)
Worth	of Math | Math Teaching	0 (0%) | 2 (4.3%)
Content	CK | PCK	12 (26.1%) | 5 (10.9%)

Analysis for RQ2: Comparing relative value of coursework before and after teaching

We found that teachers parsed the relative value in terms of habits of mind, content, the course as a whole, and technology. We coded *valuable either way* when teachers wrote about benefits in both directions. We coded more *valuable AFTER teaching* only when teachers wrote about benefits in one direction (resp. *BEFORE)*. See Table 2. We only coded *course as a whole* when the teacher did cite a specific aspect, but unambiguously stated relative value.

Table 2. Aspects of contribution and their relative value, with unit of analysis being responses (n = 61)

	Course as a whole	Habits of Mind	Content	Technology
More valuable AFTER teaching	4 (6.6%)	24 (39.3%)	18 (29.5%)	1 (1.6%)
More valuable BEFORE teaching	3 (4.9%)	9 (14.8%)	11 (18%)	0 (0%)
Valuable either way	1 (1.6%)	6 (9.8%)	6 (9.8%)	0 (0%)

We found in total 27 framing statements. Analysis of framing statements suggests that overwhelmingly, teachers found an aspect more valuable after teaching because they could envision how the aspect could be applied to their teaching – and that they could not do so before. As one participant stated, “"Without teaching experience, I would not be thinking about those things as we go along." For brevity, we discuss only habits of mind here. Participants mentioned that habits of mind were useful now, and not previously, primarily because they now saw the value of habits of mind to students ("... now that I have had experiences working with students, I understand the importance of being able to solve using multiple strategies") or because they could better empathize with students ("When I actually had the experience of teaching in a classroom, I was surprised how many students struggled with the content. Having to struggle through habits of mind problems helped me to see what my students go through daily. I think if I would have taken this class earlier, it would not have resonated with me as much.").

Discussion & Conclusion

We undertook this study because of a local phenomenon that teachers seemed to have differing evaluations of the value of similar mathematics coursework, depending on whether they took the material after or before teaching experience. We posited that teaching experience mediated the uptake of course experiences in some way. Consequently, we examined how practicing teachers describe the contribution of graduate mathematics coursework to their learning, and how they would assess the relative value of this same coursework, had they taken the course as an undergraduate, prior to any teaching experience. We posed to teachers in an end-of-course reflection assignment the counterfactual situation of taking the same course as an undergraduate.

We found that most teachers cited some contribution of the graduate course to their learning, including the emphasis of the course on habits of mind, or the course's content. We also found that the plurality of responses indicated that teachers felt this course was more valuable after teaching than prior to teaching. To investigate why this may be the case, we examined framing statements. In these statements, a common theme was that without teaching experience, teachers did not realize the worth of habits of mind, or of content, for cultivating these aspects in students. Another common theme was that struggling in graduate coursework gave teachers empathy for their students, which would ultimately help the teachers be more understanding of students in the future.

We view our main contributions to be twofold. First, we extend Hoffman and Even's (2018, 2021) theoretical framework, and elaborate the possibilities for how academic mathematics coursework can contribute to teacher learning. Not only may they learn about mathematics, but they may also rethink their own teaching, or find teaching practices and activities modeled by the instructor they wish to bring into their classroom (cf. Wasserman & McGuffey, 2021). Second, we introduce the notion of framing as an explanation for why teachers may find courses useful or not, and how utility may shift over time. Prior work on teachers' conceptions of academic mathematics experiences have not differentiated between viewpoints before and after teaching experience. But as our results suggest, undergraduates and practicing teachers may frame their mathematics learning very differently. We suggest that further work investigating these frames can contribute to improving courses for both populations, as well as potential differences to work into course design with intention.

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The Potential Impact of Opportunities to Apply Mathematics to Teaching on Prospective Secondary Teachers' Competence

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In the past two decades, there has been a trend in materials for mathematics courses for prospective secondary teachers: more opportunities for teachers to "apply mathematics to teaching". That is, materials increasingly highlight how mathematical knowledge learned in the course can be useful in secondary teaching, and provide opportunities for teachers to harness this knowledge in simulations of teaching. There is little known about the effects of this curricular reform on teachers' competence. In this report, we use data from the Mathematics of Doing, Understanding, Learning, and Educating for Secondary Schools MODULE(S^2) project to examine the potential impact of using such curricular materials. The data include over 300 prospective secondary teachers' responses to 3 sets of Likert pre-/post-term surveys addressing: mathematical knowledge for teaching; expectancy for enacting selected core teaching practices; and valuing of enacting these practices. We found mean increases across the survey results. We conclude with directions for future research on the impact of this curricular reform.

Keywords: Content Courses for Secondary Teachers, Mathematicians, Mathematical Knowledge for Teaching, Expectancy, Value

Introduction & Background

Many secondary teachers do not find their tertiary mathematics experiences usable in secondary mathematics teaching (e.g., Goulding, Hatch, & Rodd, 2003; Winsløw & Grønbæk, 2014; Zazkis & Leikin, 2010). In response to this problem, multiple scholars, including within the RUME community, have advocated for mathematics courses for prospective secondary mathematics teachers to provide explicit opportunities to connect university mathematics with secondary mathematics teaching (e.g., Álvarez et al., 2020; Lai, 2019; Wasserman et al., 2017).

In these scholars' view, the disconnection problem in the US may be a consequence of historical attempts to emphasize tertiary-secondary connections through *mathematics* (e.g., The Panel on Teacher Training, 1971) at the expense of attention to *mathematics teaching practice.* To rectify this situation, secondary mathematics teacher educators have in the past decade or so developed materials that articulate links to teaching practice (e.g., Álvarez et al., 2020a, 2020b; Bremigan et al., 2011; Buchbinder & McCrone, 2020; Hauk et al., 2017; Heid et al., 2015; Lischka et al., 2020; Sultan & Artzt, 2011; Wasserman et al., 2017). An instructional device common to each of these materials is *applications of mathematics to teaching* – where teachers respond to a description of a secondary teaching situation by leveraging material learned in a tertiary course (cf. Bass, 2005; Stylianides & Stylianides, 2010).

Since the advent of this curricular reform, there have been relatively few studies describing the potential impact of the use of materials featuring applications of mathematics to teaching. As far as we know, there are two exceptions (Buchbinder & McCrone, 2020; Wasserman and McGuffey, 2021) which though promising, also featured fifteen and six teachers only. Hence advancement of research and practice in secondary mathematics education faces a problematic gap: we do not know much about the impact of a growing curricular trend in materials for mathematics courses for secondary teachers.

The purpose of this report is to take a first step in addressing this gap, by examining changes in teachers' competence. As explained below, we view competence as including mathematical knowledge for teaching, expectation of success of enacting core teaching practices, and valuing of core teaching practices. The research questions guiding this report are:

Over the course of a semester-long experience featuring applications of mathematics to teaching that were coordinated with mathematical content learned,

1. *To what extent did teachers' mathematical knowledge for teaching change?*
2. *To what extent did teachers' expectation of success for carrying out selected core teaching practices change?*
3. *To what extent did teachers' valuing of these selected core teaching practices change?*

We note that while statistics and mathematics are distinct disciplines, in secondary education, "mathematics" is often seen as including statistics, a practice that we will follow in this paper. Unless otherwise indicated, we use "teacher" to mean a prospective secondary mathematics teacher, "student" to mean secondary student, and "instructor" to mean a university instructor.

Conceptual Perspective

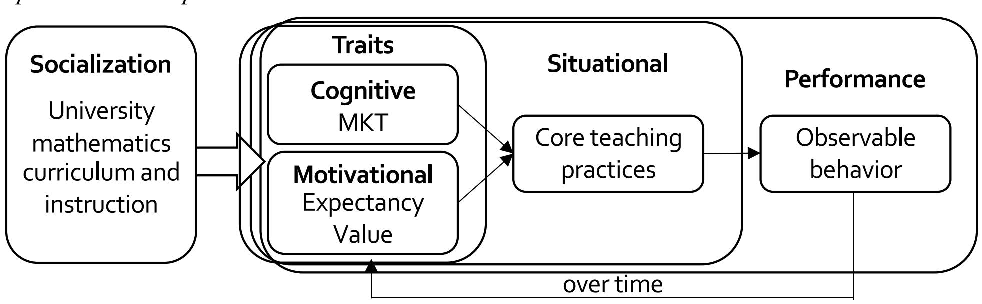
We follow Blömeke et al.'s (2015) notion of competence for teaching to include cognitive, motivational, and situational resources. Cognitive and motivational resources are also known as teacher traits; we see these as malleable. Cognitive resources include mathematical knowledge for teaching (MKT; Ball et al. 2008; Thompson & Thompson, 1996), the knowledge entailed in carrying out recurrent work of teaching. Within motivation, we focus on expectancy and value. Expectancy is the expectation of success at enacting a task in a particular situation (Wigfield & Eccles, 2000). Value is the subjective value of enacting the task, and it encompasses utility, enjoyment, and personal fulfillment (see Eccles & Wigfield, 2020, for a review).

Situational resources include teachers' understandings and conceptions of core teaching practices. These practices are those that benefit students in equitable ways; are learnable by teachers; and have the potential to promote teaching improvement over time (Grossman et al., 2009). Example core teaching practices include generating questions and discussion that promote students' reasoning, and learning about student understanding using their explanations.

We follow Eccles and Wigfield's (2020) framing that one's choices and performance feed iteratively into one's expectancy and value, and vice versa. If one experiences successful performance in mathematics and its applications to teaching, one is more likely to try again in the future. Conversely, if one perceives failure, one might desire to avoid core teaching practices.

From this perspective, an important role of teacher education is socialization: providing opportunities for teachers to learn and succeed, so as to encourage them to choose to engage in core practices in the future. Figure 1 summarizes our conceptual perspective.

Figure 1. Perspective on Competence



Data & Method

Study Context

Our report examines change in teachers' competence over the course of using MODULE(S^2) project materials in four different mathematical areas: algebra, geometry, mathematical modeling, and statistics. Materials for each area were intended to be used across one semester, and each featured 6 extended prompts. These extended opportunities specified a secondary student-level task, a goal for the scenario involving engaging students in a mathematical practice, and in most cases text or images of sample work drawn whenever possible from secondary student work obtained from secondary classroom teachers consulting for the project. Teachers were to respond to half of the prompts by writing a narrative describing how they would use the given responses to set up a whole class discussion, and the other half by videotaping themselves responding to sample student contributions as if they were personally responding to a student in their own class. For all responses, teachers were asked to make specific reference to the students' thinking, elicit student conceptions, and pose questions to advance mathematical understanding. All materials came in instructor-facing and teacher-facing versions, with the instructor-facing providing guidance for enacting core teaching practices in their own instruction.

Participants

Participants were 368 teachers enrolled in tertiary mathematics courses using MODULE(S^2) materials with 65 instructors at 54 different institutions across the United States and Canada. These participants consented to participation and completed various of the instrument forms detailed below. We defined "completion" as completing the majority of questions on that form.

Courses were taught at institutions ranging from large public research universities to small private colleges to regional public universities, and from those that served predominantly white populations to those that served predominantly minoritized populations.

Phases of Study and Instruments

Research occurred in two phases, before and after the first year of the covid-19 pandemic. Phase One spanned the project's Years 1 and 2, with data collected in two content areas per year. Phase Two spanned the project's Year 4, with data collected in all four content areas.

MKT measures. At the beginning and end of the term using MODULE(S^2) materials, we measured teachers' content knowledge for teaching. These include items from the Exponential, Quadratics, and Linear assessment (Howell et al., 2016); Geometry Assessments for Secondary Teaching (GAST; Mohr-Schroeder et al., 2017); Anhalt and Cortez's (2016) instrument on preparation for teaching modeling; and sample items based on the Levels of Conceptual Understanding of Statistics repository (LOCUS, n.d.), and an assessment item developed by Randall Groth (personal communication). To analyze change in teachers' content knowledge for teaching, we used Cohen's d to quantify effect size for knowledge change in each area. As of writing, we scored only assessments from Phase One data collection, so we only report those results.

Expectancy and value measures. At the beginning and end of the term, we measured each teacher's expectancy and value for enacting selected core teaching practices (SCTPs) in the area emphasized by their course. In Phase One, we used the same form in pre- and post-term administrations. After analyzing these results, and in view of conversations with piloting faculty, we posited that the large pre- ratings potentially obscured changes in expectancy and value for participants. In particular, faculty reported that teachers enrolled in their class remarked on how much they learned that they did not know could be learned. In Phase Two, to take this perception into account, we modified the post-term form to include a retrospective pre-rating for each item on the post-term measures. Moreover, in Phase Two, conversations with piloters and among the research team suggested adding an SCTP. See Table 1 for SCTPs included in each phase. See Figures 2 and 3 for a sample Expectancy item with a retrospective pre- rating; the post-term Value forms featured similar items. Underlined portions indicate parts specific to a content area.

We analyzed categorical shifts from pre- to post-test on the expectancy and value instruments across SCTPs and content areas using descriptive statistics of differences paired by teacher. We are interested in *the effect size* of any changes, rather than simply asking *whether* a difference exists. We use Cohen's d to quantify effect size for expectancy and value for enacting each SCTP; it measures the practical significance of the observed difference by accounting for variability in the difference. Common benchmarks for interpreting Cohen's d are 0.2 for a small but non-negligible effect, 0.5 for a medium effect, 0.8 for a large effect, and 1.3 for a very large effect (Sullivan & Feinn, 2012).

Table 1. Selected core teaching practices emphasized in materials used by study participants

Code	Definition	Phase One	Phase Two
SCTP1	Ask questions so that middle and/or high school students make conjectures.	✓	✓
SCTP2	Ask questions and lead discussions that help middle and/or high school students come up with mathematical explanations.	✓	✓
SCTP3	Ask questions that help middle and/or high school students make connections between different representations of the same idea		✓
SCTP4	Ask questions so that middle and/or high school students understand how to build on their thinking and what to revise.	✓	✓
SCTP5	Analyze middle and/or high school students' responses to understand their reasoning	✓	✓

Figure 2. Sample items used in Phase One pre- and post-term and Phase Two pre-term only

Expectancy item and set up, with content area and key concept emphasized	Value item and set up, with content area emphasized (no key concepts cited)
Suppose you are teaching middle or high school algebra students how to think about functions in terms of how changes in the value of one variable may impact the value of the other variable. How well does this statement describe how you feel?	How much do you personally agree with the following ideas about teaching algebra in middle or high school?
I would be comfortable regularly asking questions so that middle or high school students make conjectures.	*I think it is important to regularly ask questions so that middle or high school students make conjectures.*
Not at all Very much 0 1 2 3 4 5	Not at all Very much 0 1 2 3 4 5

Figure 3. Sample item in Phase Two Expectancy Measure, post-term only

Expectancy item and set up, with content area and key concept underlined		
Suppose you are teaching middle or high school algebra students how to think about functions in terms of how changes in the value of one variable may impact the value of the other variable.		
Looking back, how well did these statements describe you at the beginning of the course, AND now at the end of the course?		
	BEGINNING of course	NOW, at END of course
I would be comfortable regularly asking questions so that middle or high school students make conjectures.	Not at all Very much 0 1 2 3 4 5	Not at all Very much 0 1 2 3 4 5

Results

Changes in mathematical knowledge for teaching (MKT)

Table 2 indicates that all areas exhibited a mean increase in teachers' MKT. All effect sizes are above the threshold for large practical significance (Cohen's $d > 0.8$). All post scores are less than 50% of the maximum score, and pre- and post- means across all areas are roughly comparable. As context for interpreting this finding, we turn to validation studies of measures for content knowledge for teaching geometry, where there has been significant work at the secondary level with reported raw scores. Mohr-Schroeder et al. (2017) validated GAST forms to measure mathematical knowledge for teaching geometry with a sample of predominantly practicing teachers. In contrast, our participants were strictly prospective teachers. On full GAST forms, the mean score was 20 points out of 30 possible. Mohr-Schroeder et al. did not compare prospective and practicing teachers' performance in their sample. However, in reports of their results with a different instrument (MKT-G), Milewski et al. (2019) reported that most prospective teachers' scores were comparable to those in the lower half of practicing teachers' scores. In view of these results, it is not surprising that the teachers in our sample scored, overall, in a lower range than those in Mohr-Schroeder et al.'s sample, and apparently in a similar range over all areas. Finally, we note that the indicated sample sizes include only those teachers who completed both pre and post forms. Because not all piloters administered the post form, participant numbers are lower in some areas.

Table 2. Paired difference results for MKT, reported in terms of percentage of maximum score on each area assessment

Area	Pre-Mean	Post-Mean	Mean Δ	SD_Δ	Cohen's *d*
Algebra (*n*=9)	20.8%	39.2%	18.3%	13.5%	1.36
Geometry (*n*=63)	25.6%	35.9%	10.3%	10.1%	1.02
Mathematical Modeling (*n*=20)	31.3%	44.4%	13.1%	17.9%	0.73
Statistics (*n*=40)	26.6%	42.2%	15.6%	16.4%	0.95

Changes in expectancy and value for enacting selected core teaching practices (SCTPs)

Tables 3-8 summarize our findings. For brevity, the table does not include all 32 + 40 combinations of expectancy and value (Phase One: 4 SCTPs × 4 content areas ×{expectancy, value}; Phase Two: 5 SCTPs × 4 content areas ×{expectancy, value}), but we discuss their characteristics now. In Phase One, all *expectancy* effect sizes among SCTPs are non-negligible (Cohen's $d > 0.2$). Eight of the 16 indicate at least medium practical significance (Cohen's $d > 0.5$), and 1 indicates large practical significance (Cohen's $d > 0.8$). Among effect sizes for *value*, 9 indicate non-negligible practical significance, with most of them being for SCPT1 and SCTP2.

Retrospective pre-ratings were overall lower than actual pre-ratings, confirming our hypothesis that teachers, when looking back, now believe they perceived less expectancy and value at the beginning of term, as compared to their perceptions at the actual beginning of term. The lowest mean expectancy *retrospective* pre-rating for any SCTP in any content area was 2.58, compared to 3.20 for *actual* pre-ratings. The lowest mean value *retrospective* pre-rating for any SCTP in any content area was 3.46, compared to 4.28 for *actual* pre-ratings.

All effect sizes for *actual difference in pre/post expectancy* are non-negligible, 16 indicate medium practical significance, and 8 indicate large practical significance. Effect sizes for actual pre/post increases in expectancy in Phase Two are slightly larger than those in Phase One; this may be due to improvements to the materials, or simply to differences in students and instructors. All effect sizes for *retrospective difference in pre/post expectancy* are non-negligible and in fact are above the threshold for large practical significance and 15 indicate very large practical significance (Cohen's $d > 1.3$). Among effect sizes for *actual difference in pre/post value*, 14 indicate small practical significance, and 1 indicates medium practical significance. All effect sizes for *retrospective difference in pre/post value* are above the threshold for medium practical significance, and 11 indicate large practical significance.

Table 3. Phase One paired difference results for pre/post expectancy, with min and max Cohen's d across SCTPs

Area	Pre-Mean	Post-Mean	Mean Δ	min Cohen's *d*	max Cohen's *d*
Algebra (*n*=54)	3.840	4.293	0.453	0.288 (SCTP5)	0.498 (SCTP1)
Geometry (*n*=50)	3.583	4.040	0.457	0.292 (SCTP1)	0.446 (SCTP2)
Mathematical Modeling (*n*=26)	3.421	4.320	0.899	0.675 (SCTP2)	0.839 (SCTP1)
Statistics (*n*=44)	3.317	4.119	0.802	0.502 (SCTP4)	0.613 (SCTP2)

Table 4. Phase One paired difference results for pre/post value, with min and max Cohen's d across SCTPs

Area	Pre-Mean	Post-Mean	Mean Δ	min Cohen's *d*	max Cohen's *d*
Algebra (*n*=54)	4.384	4.565	0.181	0.075 (SCTP5)	0.336 (SCTP2)
Geometry (*n*=50)	4.455	4.510	0.055	0.088 (SCTP5)	0.206 (SCTP1)
Mathematical Modeling (*n*=26)	4.289	4.548	0.259	0.196 (SCTP5)	0.340 (SCTP2)
Statistics (*n*=44)	4.489	4.705	0.216	0.094 (SCTP4)	0.450 (SCTP2)

Table 5. Phase Two paired difference results for pre/post expectancy, with min and max Cohen's d across SCTPs

Area	Pre-Mean	Post-Mean	Mean Δ	min Cohen's *d*	max Cohen's *d*
Algebra (*n*=69)	3.50	4.23	0.74	0.76 (SCTP1)	0.91 (SCTP5)
Geometry (*n*=27)	3.44	4.37	0.92	0.67 (SCTP3)	0.83 (SCTP4)
Mathematical Modeling (*n*=26)	3.35	4.36	1.02	0.71 (SCTP2)	0.99 (SCTP1,3)
Statistics (*n*=68)	3.45	4.06	0.62	0.42 (SCTP2, 5)	0.5 (SCTP1)

Table 6. Phase Two paired difference results for pre/post value, with min and max Cohen's d across SCTPs

Area	Pre-Mean	Post-Mean	Mean Δ	min Cohen's *d*	max Cohen's *d*
Algebra (*n*=70)	4.60	4.75	0.15	0.12 (SCTP2)	0.28 (SCTP1)
Geometry (*n*=27)	4.60	4.82	0.22	0.22 (SCTP3)	0.54 (SCTP2)
Mathematical Modeling (*n*=26)	4.43	4.69	0.26	0.20 (SCTP3)	0.46 (SCTP1)
Statistics (*n*=67)	4.47	4.65	0.18	0.07 (SCTP3)	0.40 (SCTP1)

Table 7. Phase Two paired difference results for retrospective pre/post expectancy, with min and max Cohen's d across SCTPs

Area	Pre-Mean	Post-Mean	Mean Δ	min Cohen's *d*	max Cohen's *d*
Algebra (*n*=54)	2.72	4.23	1.52	1.61 (SCTP5)	1.68 (SCTP1,4)
Geometry (*n*=50)	2.75	4.37	1.58	1.51 (SCTP1)	1.56 (SCTP3)
Mathematical Modeling (*n*=26)	2.79	4.36	1.56	1.76 (SCTP3)	1.91 (SCTP4)
Statistics (*n*=44)	2.95	4.06	1.11	1.20 (SCTP3)	1.29 (SCTP1)

Table 8. Phase Two paired difference results for retrospective pre/post value, with min and max Cohen's d across SCTPs

Area	Pre-Mean	Post-Mean	Mean Δ	min Cohen's *d*	max Cohen's *d*
Algebra (*n*=54)	4.384	4.75	0.96	0.79 (SCTP5)	0.99 (SCTP3)
Geometry (*n*=50)	4.455	4.82	0.81	0.56 (SCTP3)	0.88 (SCTP4)
Mathematical Modeling (*n*=26)	4.289	4.69	1.11	0.83 (SCTP5)	1.11 (SCTP4)

Statistics (*n*=44)	4.489	4.65	0.62	0.67 (SCTP3)	0.82 (SCTP2)

Discussion & Conclusion

We sought to examine changes in teachers' competence before and after a semester-long experience featuring applications of mathematics to teaching. Our results address a gap in the literature concerning the impact of curricular reform on teachers' competence. We analyzed data about teachers' cognitive and motivational resources, specifically MKT, and expectation of success and value for enacting core teaching practices when teaching a particular content area. Overall, we found that teachers' MKT, expectation of success, and value increased after this semester-long experience.

A significant strength of our results is that they come from teachers across a variety of institutions, with different instructors, across four distinct content areas. Our results on motivation are limited in that, like most studies in this area, we rely on self report. Teachers may have felt pressure to respond more positively than the reality. Nonetheless, the overall increase in all variables, across all areas, indicates that it is plausible that expectation of success and value did improve, even if it is to a lesser extent than reported. Hence, we theorize that teachers benefit when using materials that coordinated mathematical learning opportunities with applications of mathematics to teaching, and these benefits come in the form of increased competence for teaching.

Any conclusions that can be drawn from these results are limited in two crucial ways, which also point to directions of future research. First, correlation is not causation. Although we designed the materials to enhance teachers' competence, the data here do not say what factors exactly supported the mean improvement over time we found across our data. We cannot say whether the benefits derive from the materials' design, the confluence of taking the content course alongside a pedagogically-focused methods course, a time effect that when teachers take similar pre/post forms they will generally improve, or another factor. Beyond technical studies of how teachers' knowledge appears to change or not simply by having been exposed to an assessment of MKT, we also suggest that examining potential mechanisms for change is also important. Future studies, for instance, could investigate whether certain instructional practices are associated with changes in teachers' competence; whether teachers' knowledge changes as a result of propositional knowledge or change in mathematical practice; or how opportunities to learn mathematical content and applications of mathematics to teaching interplay.

Second, we have not addressed directly the problem that many teachers themselves have been documented to find content courses not useful. We and other scholars believe that teachers would find applications of mathematics to teaching useful, but we cannot conclude this based on the analysis reported here. We are currently in the process of analyzing the data from MODULE(S^2) project to address this gap.

Finally, we observed that teachers' retrospective pre- ratings of their motivation were generally lower than their actual ratings from the beginning of term. This finding suggests practical and methodological questions. The practical question is: What changes their perspective? The methodological question is: in what areas of competence would self-reports change retrospectively, and what are the affordances and limitations of retrospective ratings?

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What Instructional Factors do Prospective Secondary Teachers Attribute to their Learning?

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Although it is well known that motivational and cognitive resources influence secondary teachers' instructional quality, less is known about the tertiary instructional factors that influence secondary teachers' development of these resources. To address this gap, we report on factors that prospective secondary teachers attribute to their learning. We draw on survey responses of 70 prospective secondary teachers enrolled in mathematics courses for teachers using Mathematics of Doing, Understanding, Learning, and Educating for Secondary Schools (MODULE(S^2)) materials in one of four content areas. We triangulate response themes with data from 300 prospective secondary teachers on their perceptions of instructional practices used in a mathematics course for teachers using the same suite of materials. Then, we compare these themes with literature documenting implementation of mathematics curricula in these courses. We argue that coordinating mathematics content, applications of mathematics to teaching practices, and tertiary instructional practices are key to success of these mathematics courses.

Keywords: Content Courses for Secondary Teachers, Mathematical Knowledge for Teaching, Instructional Practices

Although prospective teachers take many tertiary mathematics courses (e.g., Hill, 2011; Tatto & Bankov, 2018), many find their experiences in these courses irrelevant to secondary teaching (e.g., Goulding, Hatch, & Rodd, 2003; Zazkis & Leikin, 2010). One response to this problem is to incorporate applications of mathematics to teaching into content courses for secondary teachers (e.g., Álvarez et al., 2020a, 2020b; Bremigan et al., 2011; Buchbinder & McCrone, 2020; Hauk et al., 2017; Heid et al., 2015; Lischka et al., 2020; Sultan & Artzt, 2011; Wasserman et al., 2017). Emerging empirical scholarship on the impacts of such applications points to the promise of this solution (e.g., Buchbinder & McCrone, 2020; Wasserman & McGuffey, 2021). However, as with any curricular innovation, it is not just the curriculum materials that matter, but also how the curriculum materials are deployed (e.g., Cohen, 1990; Stein et al., 2007).

Our purpose in this report is to identify instructional factors in successful tertiary-level mathematics courses for prospective secondary teachers. By success, we mean that the prospective teachers develop their competence for teaching; and by competence, we include teachers' cognitive and motivational resources for teaching. We focus especially on the resources of mathematical knowledge for teaching, value commitments, and expectation of success in mathematics and teaching. We address the research questions: *(RQ1) What instructional factors do prospective secondary teachers attribute to their learning in a mathematics course for teachers? (RQ2) To what extent do tertiary instructors' instructional practices associate with prospective secondary teachers' increase in their expectation of future success?*

In the remainder of the report, unless otherwise noted, we use "teacher" to refer to prospective secondary mathematics teacher, "instructor" to refer to the instructor of a tertiary-level mathematics course, and "student" to refer to secondary students.

Background & Conceptual Perspective

Design and impact of mathematics courses for secondary teachers

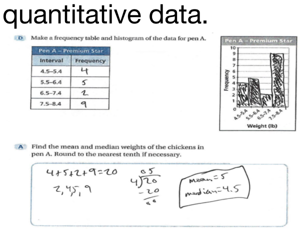
A teacher's instruction benefits from robust mathematical practice and knowledge of secondary content from an advanced perspective (Baumert et al., 2010; Sword et al., 2018), and course design has historically reflected this understanding (e.g., CUPM, 1961; CBMS, 2001; Murray & Star, 2013; Tucker et al., 2015).

However, results into the last decade also suggest that design principle for mathematics courses around *only* mathematics falls short: many teachers saw their mathematics courses as disconnected from their future teaching (e.g., Goulding et al., 2003; Zazkis & Leikin, 2010). Even when teachers enrich their understanding of the secondary mathematical concepts that they will teach, they may still exit tertiary mathematics believing that they may have become better mathematicians, but not better mathematics teachers (Wasserman & Ham, 2013).

In response to the inefficacy of mathematics courses for teachers, a number of scholars have advocated for the inclusion of *applications of mathematics to teaching* in these courses (Álvarez et al., 2020; Artzt et al., 2011; Bremigan et al., 2011; Heid et al., 2015; Lai, 2019; Lischka et al., 2020; Wasserman et al., 2017). By applications of mathematics to teaching, we mean opportunities for teachers to respond to secondary teaching scenarios using content addressed in the course. Two examples of such applications are shown in Figure 1. Bass (2005) and Stylianides and Stylianides (2010) argued that drawing on mathematical knowledge to address problems of teaching is enacting a form of applied mathematics. Their arguments are consistent with principles of a practice-based theory of professional education (e.g., Ball & Cohen, 1999).

Figure 1. Example snapshots of applications of mathematics to teaching

Two parts of a student's work on a task are shown below. Identify what the student has done well when analyzing the data, and areas to work towards understanding about analyzing univariate quantitative data.



As students are working independently on rotations of a flag around a point, you observe two students with the following work completed.

Record a video of yourself providing a response to both Student 1 and Student 2. Build on their thinking to help the student finish their thought, prompt the student to investigate an error, or help the student move forward in their thinking.

Pilot studies examining courses where such applications are used suggest that, in contrast to previous studies, teachers are able to articulate how course content connects to future teaching (e.g., Fukawa-Connelly et al., 2020; Wasserman & Galarza, 2018). Further, the 15 teachers in Buchbinder and McCrone's (2020) study went on to incorporate task designs from their tertiary mathematics course into a lesson for secondary students while student teaching. Teachers in Wasserman and McGuffey's (2021) study of 6 teachers attributed teaching moves to their experiences in a real analysis course featuring these applications. Attributions were predominantly of two kinds: experiences with applications, and instructional practices that were modeled by their real analysis instructor.

Instructional practices in middle and secondary grades and at the undergraduate level

Multiple studies with middle and secondary students suggest the beneficial impact of core teaching practices (e.g., Grossman et al., 2009) that maintain and elevate cognitive demand, and elicit and build on student thinking (e.g., Gates Foundation, 2012; Voss et al., 2011). Yet Banilower et al.'s (2013) national survey suggests that many secondary teachers may not teach in these ways. They found that only 55% of secondary teachers focus on developing students' mathematical practices, including mathematical justification. Banilower et al. do not suggest reasons for their findings. However, based on other literature, we do know that teachers' practices may depend on their self-efficacy (Zee & Koomen, 2016), values (e.g., Schoenfeld, 2010), and pedagogical content knowledge (Baumert et al., 2010).

Perhaps unsurprisingly, the principles of effective instruction for middle and secondary grades carry through to the undergraduate level. Undergraduate instructors are increasingly aware of benefits of eliciting undergraduates' reasoning and supporting undergraduates' collaboration (e.g., Laursen & Rasmussen, 2019). These teaching practices benefit student outcomes overall (e.g., Bressoud & Rasmussen, 2015).

Summary of perspective based on the above literature review

Altogether, we take the view that mathematics courses for teachers are a productive place to incorporate the kinds of teaching practices that have been shown to be effective at the middle and secondary level, where many candidates of secondary mathematics teacher education programs will end up teaching. These teaching practices are consistent with trends at the undergraduate level and benefit undergraduate students. Further, there is evidence that teachers may transfer instructional practices experienced at the tertiary to the settings of their future secondary teaching (Buchbinder & McCrone, 2020; Wasserman & McGuffey, 2021). Hence one way to move the needle on secondary teachers' practice may include tertiary instructors' modeling of effective teaching practices.

Data & Method

Study Context

Our report examines teachers' perceptions of their experiences in a mathematics course for teachers, during a term when the course used MODULE(S^2) project materials in one of four different mathematical areas: algebra, geometry, mathematical modeling, and statistics. Materials for each area were intended to be used across one semester, and each area featured 6 extended applications of mathematics to teaching, termed "Simulations of Practice". The examples from Figure 1 are excerpts of these Simulations of Practice. These activities asked teachers to build on sample student thinking, and to generate questions that elicited student reasoning. Simulations were designed to apply the mathematics learned through the materials. All materials came in instructor-facing and teacher-facing versions, with the instructor-facing providing guidance for building on teacher thinking, and generating questions that elicited teachers' reasoning. Elsewhere we have analyzed teachers' pre- and post-term expectation of success and value of carrying out core teaching practices, and found mean increases in these variables across all content areas (see Lai et al. in these proceedings). Instructors received support from the project team in the form of summer workshops and meetings with materials developers throughout the academic year.

Participants

Data were drawn from responses of 368 teachers enrolled in tertiary mathematics courses using MODULE(S^2) materials with 65 instructors at 54 different institutions across the United States and Canada. These participants consented to participation and completed various of the instrument forms detailed below. We defined "completion" as completing the majority of questions on that form. Courses were taught at institutions ranging from regional public universities to large public research universities to small private colleges, and from those that served predominantly white populations to those that served predominantly minoritized populations.

Instruments, Analysis, and Phases of Data Collection

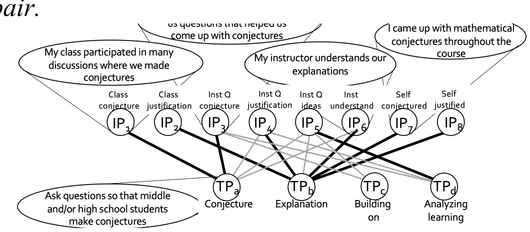
RQ1: Teacher Perceptions of Learning (RQ1). We distributed a survey at the end of the term where we asked teachers to identify factors that influenced their learning. These open-ended questions asked teachers, "What did you learn about doing [content area] as a result of this course? What was most helpful about this course for learning to do [content area]?" and "What did you learn about teaching [content area] as a result of this course? What was most helpful about this course for learning to teach [content area]?"

We examined factors influencing change in expectations of success or value in mathematics or mathematics teaching. We coded responses for *expectation of success:* confidence or facility in aspects of doing mathematics, learning mathematics, or teaching mathematics (Eccles & Wigfield, 2020); *value*: importance, benefit, worth, or enjoyment ascribed to aspects of doing mathematics, learning mathematics, or teaching mathematics (Eccles & Wigfield, 2020); and *course attribution*: attributing change in expectation of success or value to instruction, where instruction includes course activities, norms, or interactions (e.g., Cohen et al., 2003).

RQ2: Pre/Post Expectation of Success in Teaching Practices (TPs). We measured each teacher's expectation of success for enacting selected teaching practices (TPs) in the area emphasized by their course. Our items for this construct use phrasing from Eccles et al.'s (1993) study. All items used a Likert scale from 0 (not at all) to 5 (very much) and read: "Suppose you are teaching middle or high school [content area] students [about key concept]. How well does this statement describe how you feel? 'I would be comfortable [TP].'" Key concepts aligned to curriculum materials (e.g., covariational reasoning was an algebra key concept). We analyzed categorical shifts from pre- to post-test on the expectancy and value instruments across TPs and content areas using descriptive statistics of differences paired by teacher.

RQ2: Perception of Instructional Practices (IPs). We surveyed teachers' perception of the extent to which they experienced instructional practices (IPs) similar to TPs. Item phrasing mirrored that of TPs, phrased from the perspective of learning ("I did ...", "My class did ...") and teaching ("My instructor ..."). We identified clusters of TPs that corresponded to or supported each other: for instance, an instructor or teacher's capacity to build on learner thinking supports the capacity to ask questions that elicit conjectures. We used the Pearson correlation coefficient *r* to measure correlations between teachers' expectation of success in a TP and IPs whose modeling corresponded to or supported the TP. For brevity, we do not list all TPs and IPs in full, but only a selection of them. These are shown in Figure 2.

Figure 2. Correlations explored between TPs and IPs. Bold line indicates a corresponding TP-IP pair. Gray line indicates supporting TP-IP pair.



Phases of Data Collection. Research occurred in two phases, before and after the first two years of the covid-19 pandemic. Phase One spanned the first two years of the project, with data collected in two content areas per year. Phase Two spanned the fourth year of the project, with data collected in all four content areas. RQ1 analysis has only been completed with Phase Two data at time of writing. RQ2 analysis is reported with Phase One and Phase Two data.

Results

Instructional factors prospective secondary teachers attribute to their learning

Table 1 summarizes the percentage of teachers who made at least one statement regarding expectation of success, value, or course attribution to their learning, across their entire response to the Teacher Perception of Learning survey. This table also shows the number of total statements of expectancy, value, or course attribution.

Table 1. Coding results for responses to Teacher Perceptions of Learning.

	% participants mentioning expectation of success	% participants mentioning value	% participants mentioning attributions	# Expectation of success statement (Pos + Neg)*	# Value statements (Pos + Neg)*	# Attribution statements (Pos + Neg)*
Algebra (*n* = 28)	82.1%	60.7%	67.9%	37+0	25+2	27+2
Geometry (*n* = 6)	100.0%	83.3%	50.0%	12+0	7+0	5+0
Math Modeling (*n* = 23)	78.3%	95.7%	73.9%	30+0	48+2	25+5
Statistics (*n* = 13)	92.3%	76.9%	61.5%	21+1	14+2	9+3

*Pos = positive statement, neg = negative statement

Across all areas, teachers overall described increased facility in content knowledge and working with students. In algebra and geometry, multiple teachers cited increased knowledge of "why things work", describing "deeper" levels of understanding (e.g., "Being challenged to dig deeper into these ideas will be helpful in my future career"). In statistics and mathematical modeling, multiple teachers described little previous knowledge of these topics, and feeling more confident about teaching the topic as a result of the course.

When looking across instructional factors that teachers reported as influential, the most common elements across all areas are applications of mathematics to teaching, or simulations of practice (22 mentions; e.g., "The videos we had to create where we looked at a student's answer… get them to think where they might come up with the answer on their own without me giving them the answer I found very beneficial and helpful!") and discussion with other teachers (35 mentions; e.g., "Having conversations with peers and being given time to absorb and reflect on ideas was really helpful.") All the above factors were mentioned across all areas. Teachers also cited curricular structure and content (e.g., "Assignments that led us to figuring out an idea before giving the definition of that idea really helped make the definition impactful"), opportunities to develop conceptual understanding (e.g., "The thing that was most helpful was discussing why things work and seeing how it all connected"). For brevity, we only list themes with at least 5 mentions found in attribution statements. See Table 2.

Negative statements were comparatively rare, and differed by area. For instance, in Algebra, two teachers stated that the depth of content was inappropriate for secondary teaching. In Mathematical Modeling, all five negative attribution statements were from one instructor's course and all described repeated experience with modeling as redundant. In Statistics, two stated that they still felt uncomfortable with statistical concepts, and one stated they had insufficient opportunity to apply statistics to teaching. In statistics and modeling, three described "unnecessary" inclusion of social justice issues.

Correlating changes in teachers' expectancy and instructional perception

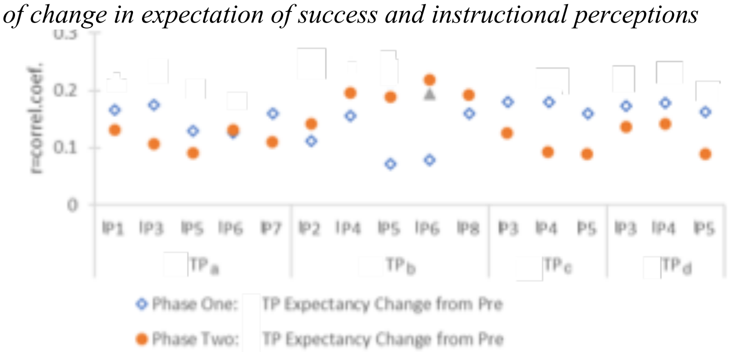
Figure 3 indicates positive correlations between pairs of variables examined. Although all correlations are relatively small ($0.07 < r < 0.24$), that they are all positive indicates an overall pattern that larger increases in expectation of success can be predicted by teachers' more positive experience of their instructional environment, in particular, their perception of how much they experienced core teaching practices during the content course.

Table 2. Attribution categories for instructional factors

Attribution category	# statements (Pos + Neg)*	Content areas of mentions◊	Example
Discussions with other teachers (including instructor role in facilitation)	35+0	A, G, M, S	"In this class, the teacher was probably the most helpful. She did a great job pushing us to talk and discuss each problem. Then looking back, you can see the results of those discussions. Being able to do that myself will be a massive help."
Simulations of practice	22+0	A, G, M, S	"The videos we had to create where we looked at a student's answer… get them to think where they might come up with the answer on their own without me giving them the answer I found very beneficial and helpful!"
Course structure and content	7+4	A, G, M	"When I heard the phrase "in the future your students will ask you.", I never really thought about it, but after witnessing it first hand and with the exact same topics from class... I was shook and thankful that I have this class to teach me fundamental techniques and strategies to help with my future class. Thank you!!!!!!!!!!! :)" "Although at some point it felt like I was doing the same assignment but with a different skin. What I mean by that is I felt like the assignment was the same for each new topic. The only difference was that we were given a new topic. I felt like there was nothing new to really learn after the first few weeks of the class."
Doing math modeling	11+0	M	"I think viewing and practicing modeling problems ourselves made it easier to see what modeling is and does."
Written materials	3+2	A, G, S	"The way that the book, and the class as a whole, took us step by step through each new content element was incredibly helpful." "The book was not helpful"

* Pos = positive, Neg = negative
◊ A = Algebra, G = Geometry, M = Mathematical Modeling, S = Statistics

Figure 3. Correlations of change in expectation of success and instructional perceptions



Discussion & Conclusion

We set out to examine the instructional factors that teachers attribute to their learning, and to what extent instructors' practices associate with teachers' increase in their expectation of future success. We found that discussions with other teachers and applications of mathematics to teaching were mentioned most as attributions to increased competence. In some ways, these results are not surprising. The findings bear out the working hypotheses of a practice-based theory of professional education: when teacher preparation is explicitly and intentionally linked to the practice of teaching, it is more likely to be effective.

Our findings are important because teachers attributed usefulness to specific features of curriculum and instruction. We strengthen results from existing smaller studies (e.g., Buchbinder & McCrone, 2020; Wasserman & McGuffey, 2021) by establishing the impact of instructional practices and applications of mathematics to teaching in four different content areas, with over 50 different instructors in over 50 different institutions.

Generalizations from this study are limited by the fact that piloting instructors volunteered to participate and had support, so they may have been more equipped to enact the curriculum as intended. The survey responses are based on teachers' self-report, and they may have felt compelled to respond more positively than they felt, or to not write as many negative comments. Nonetheless, given the sheer number of mentions of applications of mathematics to teaching, and discussion with other teachers, along with the overall small but positive associations between instructors' practices and teachers' expectation of future success in related practices, we conclude that applications of mathematics to teaching are a key innovation for secondary teachers to see the usefulness of the course content. We also conclude that instructors' practices do shape the impact of the course on teachers.

The practices identified by teachers in our study as well as in Wasserman and McGuffey's study, such as facilitating productive whole class discussions, are consistent with principles of inquiry-based mathematics education (e.g., Laursen & Rasmussen, 2019). Yet we emphasize that our conclusion is not that inquiry-based mathematics education is needed (though we believe this), but that practices of such instruction are part of a bigger picture. This picture includes both how the course taught and what is taught. Instructional practices are how a course is taught; what is taught includes the opportunities to learn from the curriculum, including activities such as the applications of mathematics to teaching. Our curriculum features intentional coordination of content with applications of mathematics to teaching. We propose that this coordination is an essential feature of curriculum for secondary teachers, if the course is to be perceived as useful, and to support the development of teachers' cognitive and motivational resources for teaching. Moreover, we posit that the instructional practices should be consistent with the images of teaching practice in applications of mathematics to teaching.

Looking forward, we see the need for further studies to document the impact of instructors' practices on teachers' development, for instance conducting case studies of courses with higher and lower mean gains in teachers' competence. With the hope that this curricular reform takes even greater hold, we also suggest studies into how instructors enact materials with applications of mathematics to teaching.

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Beliefs and Rationales For Instructor Questioning Choices During Class Discussion of the First Isomorphism Theorem

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Previous studies have established teacher questions as a significant interaction component in proof-based courses whether lecture-based or student-centered. However, little research has been conducted to investigate behind the types of questions asked in class. In this report, we share an analysis of three instructors' lessons on the First Isomorphism Theorem. We characterized instructor question types and then conducted interviews with each instructor to investigate how their pedagogical beliefs relate to the types of questions they ask in class. We found that the instructors held similar pedagogical beliefs about student learning, but diverged in their views on the purpose of the course. Both beliefs on learning and their beliefs on purposes of the course linked to the nature of the questions asked in class.

Keywords: Instructor beliefs, instructor questioning, proof presentation

Instructor questioning is one of the primary ways students are invited to participate in mathematical proof courses (Artemeva & Fox, 2011, Melhuish et al., 2022; Paoletti et al., 2018). We conjecture that the types of questions instructors asked are shaped by their beliefs about teaching, learning, and proof-based courses. While we know broad information about instructors' pedagogical decisions, investigations into the how and why of instructional practices in proof-based courses are still needed.

In this paper, we set out to address the research question: how do abstract algebra instructors' pedagogical beliefs relate to the types of questions they ask in class? In proof-based courses, research on teaching practices can be sorted into two categories: lecture style teaching and student-centered teaching (Melhuish et al. 2020). Analysis of teaching actions, such as types of questions asked, is largely limited to lecture settings where instructors frequently engage in "chalk talk" (Artemeva & Fox, 2011) writing proofs on the board, adding verbal commentary, and asking questions. In contrast, in student-centered instruction studies, most focus has been on student activity and tasks with the role of the instructor backgrounded. There has also been some work has focused on the mathematical and noticing activity of instructors and how these shape in-the-moment instruction (e.g., Johnson and Larsen, 2012; Johnson 2013). With these different foci, several questions are open including the nature of questioning in more active classes, and rationales for how and when instructors ask questions across both settings. This study makes a comparison between instructors who teach a lecture style classroom versus those who teach a student-centered classroom. Specifically, we plan to investigate the way that instructor pedagogical beliefs shape the questions asked in class. For most classes, questions are one of the most common ways for students to be invited to participate in the class (c.f., Melhuish et al., 2022; Paoletti et al., 2018).

Relevant Literature

Researchers have investigated instructor actions in lecture and rationales for their teaching styles. Paoletti et al. (2018) found that instructors typically ask factual and next step type questions (see Table 1 for definitions question types) during lectures in proof-based classes. Fukawa-Connelly et al. (2016) surveyed mathematicians and found that lecturers chose the style because of concerns about time and covering material. Woods and Weber (2020) reported similar findings. They also discussed other goals and orientations of lecturing instructors and how these can affect the instructor's in-class behaviors such as lecture allowing for mathematical precision and formalism. That is, the literature largely suggests that lecturers are intentional and have robust rationales for the way they teach (Melhuish et al., 2022).

Instructors also diverge on views of advanced courses and what the overall emphasis should be (e.g., Alcock, 2010). For example, Johnson et al. (2013) conducted interviews with mathematicians implementing an inquiry-oriented (IO) abstract algebra curriculum and found that some instructors held different goals for the abstract algebra course (with or without the IO curriculum they were implementing). She found that one of her instructors emphasized mathematical practice and activities, while the other instructors had a heavier focus on the abstract algebra content. Each of these instructors had beliefs about what students should gain and learn from the course and then implemented and focused the course on these beliefs, even when using the IO curriculum.

A recent case study was conducted by Rupnow (2021) where instructors were interviewed on their beliefs about mathematics instruction and their abstract algebra classes were observed to determine how these beliefs were enacted. In terms of questioning, Rupnow observed that the IO instructor asked more questions than the lecturer. Rupnow reported on two instructors, one using an IO curriculum where students were more actively working on mathematics in class while the other instructor more traditionally lectured. The instructors' beliefs impacted the structures of the class and these structures changed over time due to additional beliefs such as those related to coverage.

Conceptual Framing

The framing underlying our study draws on Leatham's (2006) sensible system framework. In this framework, teachers are viewed as sensible actors who can and do hold many beliefs. These beliefs work together in a sensible system, with some holding more power than others and some beliefs leading to other beliefs. A teacher's belief may seem contradictory to an observer's point of view, but from the teacher's point of view all their beliefs are consistent and work together. Some relevant beliefs may reflect orientations towards teaching, learning, and course goals.

We then make the assumption that beliefs play a mediating role in the nature of questions asked in classrooms. To parse question types, we adopt Paoletti et al.'s (2018) analytic framework (Table 1) of types of questions that were asked by instructors in proof-based classrooms.

Table 1: Paoletti et al.'s (2018) Question Type Framework

Question Type	Definition
Fact	Questions that elicit a closed form mathematical response. These questions do not call for a course of action from the students.
Next Step	Questions that elicit recommendations from students on what to do next within a proof or example. The purpose of these questions is to create logical progressions.
Proof Framework	Questions that elicit higher-order levels of proof structure. Such questions targeted proof structure such as type of proof or how to generically prove a construct such as one-to-one.
Warrant	Questions that elicit justification responses to a statement or claim.
Evaluation	Questions that elicit the truth-value of a statement. Typically answered with a yes or a no.
Convention	Questions that elicit mathematical convention of proof writing or notation.
Other	Questions that do not fit into any of the other question types.

Methods

Data for this study was collected for a larger NSF-funded design-based research project[1]. Data were collected in three instructors' abstract algebra classes at a large research institution in the United States on days they covered three key theorems in the class. This paper focuses on the lessons addressing the First Isomorphism Theorem (FIT). Instructors A and B used an inquiry-oriented approach to help students in comprehending the FIT. Instructor A was involved in the creation of the materials, and both Instructors A and B used these materials in their classes. This lesson was implemented across two 80-minute (160 minutes total) class days, with each instructor taking a total of approximately 61 and 65 minutes, respectively, of whole class instruction. Instructor C covered the FIT via an interactive lecture in one 80-minute class day with their lesson taking 54 minutes. All instructors have had prior experience teaching abstract algebra and had strong student evaluations. Instructor A has a Mathematics Education Ph.D., and their primary area of research is teaching and learning in advanced proof classes. Instructor B holds a Mathematics Ph.D. with a focus in geometric group theory but has transitioned to mathematics education research. Instructor C holds a Mathematics Ph.D., and his primary area of research is graph theory and combinatorics. Focal lessons from each instructor were video recorded and transcribed. At the end of the academic year, semi-structured stimulated recall interviews (Schepens et al., 2007) were conducted with each instructor. For each participant, a member of the research team went through the transcript data of classroom observations and identified all questions that met the following two criteria: (a) the question had to have been posed to the whole class and (b) the question had to have adequate wait time of at least 3 seconds. The first criterion was implemented so we could analyze the ways in which instructors ask questions posed to the class and how these questions may reflect their beliefs. The second criterion was established to rule out questions where the instructor posed the question and immediately answered it. Questions that were asked towards individual students or groups of students during small group discussions or independent thinking were not considered for this analysis.

[1] DUE# 1836559

Once all questions asked by the instructor had been identified, the research team independently went through each of the participants' questions to identify a representative set of questions. That is questions of a similar type (such as the multiple instances where instructors would ask students for the definition of a concept). The research team then met to determine which questions would be part of the stimulated recall interviews. Instructor A was interviewed first by the whole research team and the data was initially used as a pilot for interview protocol question creation and interview practice. Two members of the research team interviewed Instructor B and two other members of the research team interviewed Instructor C.

In addition to having instructors provide rationales for their questions, they were also prompted to answer questions about how students learn, the purpose of the course, and the meaning of mathematics. The interviews were then analyzed by two research team members independently to open code for salient instructor beliefs. The questions asked during the lesson were coded using the above analytic framework by two members of the research team. Any discrepancies in coding were resolved through discussion. After completion of this initial analysis, the interview transcripts and question rationales were considered in light of the types of questions asked, instructor beliefs, and the rationales provided by the instructor to develop initial explanatory mechanisms for the types of questions asked.

Results

To contextualize our results, we provide the frequency of types of questions in Table 2. Note that Instructor C's question profile is reminiscent of most lecturers from Paoletti et al.'s (2018) study. Instructor C asked mainly fact questions (63%[2]) with next step as the second most frequent (19%). Instructor A asked "other" questions most frequently (43%) and fact questions (40%) as the second most frequently asked question type. While Instructor B asked fact and other types questions equally frequently with both having the highest frequency in comparison to the remaining question types (34%). We also note that unlike in Rupnow (2021), the number of questions asked in the lecture class and the active classes were not substantially different. Rather, we see that the type of questions asked in the inquiry proof comprehension lessons were qualitatively different than those asked in the lecture class on the same theorem.

Table 2: Frequency of question types

Instructor	Facts	Next Step	Proof Framework	Warrant	Evaluation	Convention	Other	Total
Instructor A	23	0	3	5	2	0	25	58
Instructor B	23	3	4	5	5	4	23	67
Instructor C	39	12	6	1	2	2	0	62
Total	87	15	13	11	9	6	45	186

[2] All displayed percentages are rounded percentages of total questions asked by that instructor

In order to make sense of these qualitative differences, we turn to the instructors' stated beliefs. All three instructors shared similar ideas about how students learn focusing on students being actively engaged in the learning process. Instructor A explained that students need to be "actively engaging in things by making connections to prior knowledge" elaborating that this is hands-on and not just mimicry. Instructor A elaborated that students "need to have opportunities to really think through ideas." Instructor B explained that experience and problem solving are the most important things for students to learn, stating "you learn through problem solving." Adding that to accomplish this, Instructor B tries "to set-up a learning experience so there are opportunities to be active where being active will lead to at least some degree of success; some degree of intuition building." Instructor C stated that, "You learn by doing." For Instructor C, they viewed this idea of doing by having students engage in proof production, whether in-class by answering questions or by reproducing proofs on their own that they may have already seen. Each of these instructors viewed similar notions about learning in having students be active in their courses and attended to providing opportunities for students to engage in the course.

While the instructors provided similar ideas about learning, they diverged on their primary foci in describing the aims of the course. All instructors had a shared focus on advanced mathematical content and thinking more broadly with Instructor A discussing apprenticeship into research mathematics, Instructor B stating that 90% of the course is "what it does for you as a mathematics learner" and Instructor C explaining the course as an "introduction to more advanced mathematics." The instructors also all valued abstract algebra content-specific aims, such as better understanding underlying mathematical systems and structures students have seen. However, the respective weights and interpretations of each component differed. Throughout the interview, Instructor A tended to focus on understanding the concepts specific to abstract algebra, Instructor B focused on the problem-solving involved in proving and sense-making, and Instructor C focused on developing knowledge to succeed in constructing formal proofs.

We believe the similarities and differences in beliefs, as expressed in the responses to how students learn and the aims of the course play a large role in the questions asked in class. All three instructors believe students should be active in their learning, and this manifests in different ways through the questions. Further, we hypothesize that the belief in the purpose of the course also interacts with the questions asked by further pushing their narratives in what the students should be focused on when discussing or contributing to the content.

Throughout the interview with Instructor A, we saw many references to the importance of having students wrestle with complex ideas within class and a focus on abstract algebra specific content. For example, when asked about the purpose of turn-and-talks (a frequent occurrence in Instructor A's class), Instructor A stated that they want to get "more students actively involved in the conversation. And give them a chance to tackle something that is kind of complicated." For example, Instructor A gave students question prompts such as, "What in the world is a quotient group?... And I don't care if it's a formal definition or an informal explanation, but what is a quotient group for us?" (Other type question). This question had students actively engaging in their learning but also focuses them on the abstract algebra content of the course. Instructor A provided further rationale that they choose to ask questions about content-related issues that might be "stumbling blocks" for students. For example, Instructor A prompted another turn-and-talk sequence with "What is the image of G? What do we mean by that? What is that thing representing?" (Other type question). This prompt focused students on attending to the difference between phi as a function and phi(G) being the image of a function. They chose the turn-and-talk mechanism to specifically give more students opportunity to talk and actively develop their conceptual understanding. This reflects the stated value of having all students draw on their knowledge and participate in making sense of complex ideas.

Unlike Instructor A, Instructor B was using curriculum materials that were developed externally. His value on problem-solving led him to note that without this task he may have "handed off actually more of it to the students" in terms of constructing the proof rather than just comprehending it. Instructor B's focus on process and problem-solving came through via questions even though his lesson contained many of the same features as Instructor A's. For example, Instructor B prompted his students to construct an isomorphism map that takes a coset to its image (first requesting what to call the input (convention question) then where it might map to (next step question)). He noted, "I kind of wanted students to experience coming up with that." Instructor B further added that this also gave students more agency, stating "I feel like if I can hand a little bit of that agency of coming up with the map off the students, then then they own a little bit more of that now." A student then suggested mapping to the representative element ($\beta(aK) = a$). Instructor B then asked, "So what might be problematic about saying this image is going to be just lowercase a here?", an *other* type of question. Instructor B is encouraging students to engage in analyzing their mistakes and rectifying, a key part of problem-solving. Another example is during the student presentations of each portion of the FIT proof. Instructor B gave students the prompts "focus on helping the class understand why this stuff is happening. Why did they do this?" Instructor B focused on problem-solving and processes, allowing students to be the owners of their learning and frequently putting them into situations in which they can productively struggle with the concepts and the proving process.

For Instructor C, many of his questions asked for next steps and facts during the proving process. These questions were asked to engage students in the production of the proof. He explained that "students will tune out if they don't feel ownership in what you're doing right, so that's certainly… if I am going to do something, I want them to be interacting with me." Instructor C further elaborated that in an instance of a complex theorem, such as the FIT, his role is to break the proof into "bite sized pieces" for students to develop understanding of the idea in each of the pieces. For example, Instructor C asked the proof framework question "What else do we have to do?" This question had students attend to the different pieces of the FIT proof, specifically the different pieces of showing a mapping is an isomorphism. Instructor C also modeled the types of actions students would want to take in their own proving endeavors by asking next steps questions ("How can I use that ψ is onto?") and for definitions ("So what does ψ from A to B being a homomorphism mean?") to contribute to the proof construction. Instructor C noted that one of the major goals of the course is for students to use definitions to structure proofs and that students struggle with this idea. Instructor C's way of creating active learning is to engage students by asking small, manageable questions for the students to handle while he supports them in using the formal proof system, definitions, and attending to appropriate precision.

Discussion

We selected a common lesson focus across three instructors to better understand the nature of questions asked in an advanced undergraduate class and ultimately the rationales for selection of questions. Two of the instructors implemented an inquiry-oriented lesson where students engaged in comprehending the proof of the First Isomorphism Theorem. The third instructor presented the theorem and constructed its proof through an interactive lecture. Our first contribution is illustrating that the three instructors had many commonalities in terms of how they believed students learn. All instructors discussed students being actively involved and doing mathematics. Furthermore, throughout the interviews, all three instructors talked about students taking ownership over the mathematics in the class. However, we can note that the type of activity asked of the student varied. Instructor B had much of the activity divided between group work and whole class instruction with a focus on what students could construct or problem-solve. Instructor A used a combination of group work, but also a turn-and-talk strategy where students engaged with partners briefly before responding to whole class discussion questions about points of difficulty related to the abstract algebra content. Instructor C used interactive discussion focusing on targeted questions that allowed students to contribute to the construction of the proof while also modeling ways to structure a proof based on using formal definition.

As noted in Melhuish et al.'s (2022) literature synthesis, the choices that instructors make in advanced classes are well grounded and rational in view of pedagogical goals. Instructor C provides additional insight into the results of Paoletti et al.'s (2018) paper on lecture instruction in proof-based courses. The questions Instructor C asked were in service of students being both actively engaged in the proof construction process, while also being constrained enough such that the instructor can model nuances in this new way of thinking: using definitions to construct proofs.

In contrast, Instructor A and B would have been considered anomalies in terms of their question profiles; however, quite similar to each other. However, where we saw substantial differences was in their questions in the "other" bin. Instructor A frequently used their knowledge of where students might struggle with abstract algebra ideas and notation to provide students time to recall prior knowledge and use their own words to make sense of ideas. Instructor B commonly focused on elements of problem-solving and letting students construct components where possible. These results contribute to the larger narrative about instruction in inquiry settings and surface somewhat distinct questioning profiles.

The overarching beliefs about course purpose provide some insight into these differences with Instructor A focusing heavily on abstract algebra content, Instructor B on the problem-solving process, and Instructor C on formal proofs and their connection to definitions. If we turn to Alcock's (2010) identification of four modes of thinking in proof classes, we could suggest that Instructor C tended towards syntactic thinking (generating proofs of statements by formal structure), Instructor A towards semantic thinking (developing understanding of a statement via attention to concepts), and Instructor B towards creative and critical thinking (examining objects to support proving and checking correctness.) Of course, any such classification would oversimplify the instruction and instructor rationales as elements of each thinking can be identified across instructors. Overall, we suggest that these orientations may play a more substantial role in questioning types in an advanced class than more general beliefs about learning and mathematics which tended to be rather uniform (at least from the surface) for our three instructors.

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New Graduate Teaching Associates and Student-Centered Instructional Methods

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The importance of student-centered approaches took on new importance when the COVID-19 pandemic caused a tectonic shift in college instruction in 2020. In Fall of 2021, I reached out to all 13 of the new graduate teaching associates (GTAs: graduate students who were instructor-of-record for undergraduate courses) in the department. Eight of the 13 completed three rounds of survey and two rounds of interview. The goal was to find out what they knew about student-centered instruction and how that changed over time. GTAs spoke of the importance of supporting students in evaluating knowledge claims, learning how to learn, how to collaborate, and how to seek help. Their greatest opportunities for professional growth at the end of the year were with teaching that supported students to: become capable of self-assessment, to be resilient, and in building skills in what to do when they (students) do not know what to do.

Keywords: graduate teaching assistant, graduate teaching associate, GTA, novice instructor, student-centered

Many departments rely on graduate students to teach introductory courses and course-adjacent lab/discussion sessions. Such teaching experiences provide financial assistance to graduate students and reduce the workload of the faculty (Muzaka, 2009). There has been evidence that the work of graduate teaching assistants and associates (GTAs) leads to increased retention of undergraduates (Zehnder, 2016). Also, undergraduates learn more and have higher pass rates in classes that de-center instructors (e.g., where class time is spent on activities, collaborative group work; Freeman et al., 2014; Laursen, et al., 2014). Over the past 30 years more effective support for GTAs to learn about teaching has emerged (Deshler et al, 2015). An additional factor in the development of teaching skills since 2020 has been the COVID–19 pandemic and the formidable challenges of synchronous online instruction (Lopez, 2021).

Activities and discussions about teaching, course-specific instructor interaction, and, more generally, working with graduate student teaching-peers increase GTA confidence and the desire to teach in the future (Ellis, 2015; Stes et al., 2010). Graduate students have little prior experience with instruction that is student-interactive (Oleson & Hora, 2014). Yet, due to changes in K-12 education, today's graduate students are likely to be teaching undergraduates who do have experiences with student-interactive mathematics instruction (Yonezawa, 2015). In analysis and reporting results, I used Goodyear and Dudley's (2015) theoretical perspective that describes types of engagement entailed by student-centered approaches.

This study, at a large master's-granting institution, explored what knowledge about student-centered instruction GTAs built in their early experiences of teaching. The courses taught by GTAs as instructor-of-record were pre-calculus, business calculus, and a statistics co-requisite support course. Each of the courses had established curricula (e.g., materials for activities and assessment) and regular course coordination meetings. GTAs in this study were offered several forms of support for learning about teaching. There was a week of pre-semester workshop sessions, university-sponsored professional learning opportunities, and a 3-hour per week semester-long course in the mathematics department. This course, Mathematics Graduate Teaching Workshop, focused on developing and refining teaching skills for student-centered instruction. Half of the GTAs in this study also participated in one or more university-sponsored "Teaching Squares" – where four instructors agree on a topic (e.g., "making group work work") and commit to meeting for at least ten hours across the semester as they rely on each other to address the shared topic in their own teaching. The research question driving the study was: *How does GTA knowledge about student-centered instruction develop over time when GTAs are in online professional development for online teaching?*

Methods

Participants

Many GTAs at the study site had little to no experience with active student-centered instruction as either learners or teachers. The new GTAs in the study were also dealing with online struggles. The GTAs were new to synchronous technology for teaching, particularly for generating and maintaining online student interaction. To recruit participants for this research, I sent an email to all of the 13 new GTAs in the mathematics program in Fall 2021. Of these, 9 consented to participate and 8 of the GTAs completed all three surveys and both interviews. The pseudonyms of participants encode some basic information. One syllable indicates a person with no teaching or tutoring experience or professional learning about teaching. A two-syllable name indicates a person with some experience as a tutor or teacher of groups of people but no formal preparation for it. A three-syllable name is used for people with at least some experience in both professional learning for teaching and with tutoring or teaching groups of people.

Instrumentation

The data gathering tools in this study were three surveys and two interviews. There was a survey at the beginning of the Fall 2021 semester with a follow-up interview mid-semester. There then was another survey at the beginning of the Spring 2022 semester with a follow-up interview mid semester. Finally, there was one last survey at the end of Spring 2022. The main survey item for this was repeated across all three surveys: *In a few sentences, please explain what "student-centered instruction" means to you.* Surveys contained other questions. Survey 2 had a set of questions that asked GTAs to consider their teaching from their students' point of view. Survey 3 asked for GTAs' reflections on student-centered teaching and what they had learned during their professional activities across the year and a set of questions about personal experiences as learners with student-centered instruction. I conducted two follow-up interviews with each participant using the Zoom web conferencing tool. Each GTA consented to recording of each interview. Interviews were completed by mid-Fall (#1) and end of Spring (#2).

Data Analysis

There were three questions directly about student-centered instruction in survey 1, two in survey 2, and one in survey 3. I used three coding passes through that data to write a summary of each GTA's progress over time, as reported in the Cases section of the results. Interviews were transcribed into a spreadsheet, one row per utterance (i.e., a complete thought, a new thought or topic was a new utterance). I assigned a code to each row. The initial code ranged anywhere from one word to a sentence and was intended to capture and summarize the essence of the utterance. After this initial coding of each interview, I went through each interview again. The goal of the second coding was to consider each of the other interviews when generating new categories that grouped together initial codes that had something in common. In some cases, the commonality was the topic, like "first week expectations." In other cases a new category code connected across initial codes. The third pass through the interviews involved noticing themes across the existing codes. I used pivot tables to organize and review utterances and categories (from the second pass). From the interviews there were a total 537 utterances across 19 different themes. The next section includes descriptions of the central, most discussed, themes.

Results

First, I share the main themes that emerged from the interviews. Then I connect the themes and survey findings to directly address the research question through cases that illuminate GTAs' thought processes while they were trying to implement student-centered instruction as a first-time instructor. In what follows, "a few" means two to three, "several" means three to four, "many" means at least five, "most" means six or more, and "all" means all who responded.

Knowing and Doing Student-Centered Instruction

The challenges of implementing student-centeredness. In interview 1, a few of the interviewees had a similar statement to Patricia who said "I'd like to know where the line crosses from student-centered teaching to like babysitting and hand-holding." Many reported struggling with a different teaching style than they were used to themselves. Robert found student-centered instruction challenging because it required more of him than he had expected:

> *Robert:* To be student-centered they (the instructor) need to put a lot of effort into the way they lecture or design the class because you need to have enough activities that need to be engaging enough, it can't be too difficult, but it needs to drive the point (of the lesson).

Patricia also suggested the department's courses had a higher pass rate than first-year courses at other universities because student-centered instruction made the course too easy for students. The following semester, in interview 2, a few GTAs discussed dealing with student push-back. Queenie said, "I felt like every time I tried to implement anything student-centered or group worthy my students would kind of riot a little bit." Also, most of the GTAs reported having a tough time balancing students taking the lead and the demands of keeping "on track." For example Oliver reported, "I might be intervening too early in group work and telling them the answers before they can figure it out, but …we only have so much time."

Learning without a referent experience. In interview 1, some GTA comments agreed with Queenie's, "Encouraging students to do math in a student-centered way . . . I think it's very difficult for me. I never even experienced it; I don't know how I can implement it." Many of the interviewees weren't sure what they would like to learn, with Oliver saying, "There are things that I don't need to know and I can point to those easily, but the thing is I don't know what I need to know." In interview 2, a few of the GTAs felt it was hard to implement some of the lessons from the workshop, for example:

> *Tom:* I can't say that I always saw the point of what we were talking about [in the workshop] ...We were being taught about a bunch of things without really being told how to actually go and use that knowledge to improve our own teaching.

At the same time, a few female GTAs encountered new experiences (e.g., with male students) that they did not know how to deal with, as an example:

> *Patricia:* I find myself constantly trying to make sure that I'm keeping up a very professional demeanor and the right level of respect and professionalism. I haven't heard any of my male peers speak about that . . . I think it would have been helpful to have a discussion about how to handle something like that.

Student participation, engagement, and communicating with students online. There were many comments about black screens and frustration at trying to get students engaged and not being sure how to change that, for example:

> *Robert:* I was expecting college students to want to get a good grade and be sort of motivated . . . So, on the first day of class, everything [cameras and microphone] was turned off. No people are really talking. Had to overcome that.

Some GTAs talked about reaching out to students who were falling behind. The Graduate Teaching Workshop helped Patricia figure out how, it helped her in "knowing how and when to reach out to students who are not staying caught up on work or showing up to class all the time." In interview 2, GTA comments focused on disappointingly low attendance and disengagement by students in Zoom class meetings. GTAs noted "fluctuations" and "students seeming lost." For all the GTAs, arriving in a silent breakout room and eliciting any response was a recurring challenge in their first semester of teaching. Several GTAs echoed Queenie who said "online it was like pulling teeth to get them to share their screen." Sandra had challenges trying to communicate with students who were not doing the work saying:

> *Sandra:* Students who wouldn't show up to class and put in effort, they still wanted to pass, but they weren't doing any work. So having to reach out to them and figure out what was going on was challenging.

This sentiment was shared by most and they all struggled with connecting with students online. All of the GTAs who later taught in person the second semester noted an improvement in communication and student engagement.

Less frequent themes that emerged from interviews. Though not universal, multiple GTAs reported on: learning to be a teacher and what made them feel more like a teacher, giving and receiving peer support, joys and challenges of planning, time management, working with the university's Learning Management System (Moodle), grades and grading, and balancing the role of teacher with other roles (e.g., graduate student).

Cases: How GTA Knowledge about Student-Centered Instruction Changed Over Time

This section draws on survey and interview data to offer five cases that answer the research question. These cases characterize five ways GTA knowledge on student-centered instruction changed over time. The primary value of the surveys was the snapshots of GTA knowledge at different points in time. In this section, I use the seven touchstones from Goodyear and Dudley (2015) to organize the comparison of the open-ended survey responses across time and link that information to the interview themes. This framework uses seven ways teachers support engaging students in: 1. how to learn; 2. how to collaborate; 3. how to evaluate knowledge claims; 4. how to seek help; 5. how to become (formative) assessment capable; 6. how to be resilient; 7. what to do when they do not know what to do.

For each GTA, I report on the evidence from the survey items and the previously discussed themes to characterize the change in that person's knowledge about student-centered instruction and indicate theme by #1 through #7. I have grouped the cases where there were large similarities (e.g., the growth for Nicholas was similar to that for Mateo and to save space only Mateo's case is included). At the end of the section, Table 1 summarizes the results.

Case A: Mateo (& Nicholas). Mateo's first definition of student-centered instruction was that "it is more about what the student is doing rather than the instructor." He also mentioned how student-centered instruction "is about conceptual change as opposed to information transmission" and that it "related content to what the student already knows" with a focus on the "development of learning skills." His definition showed evidence of his knowledge including supporting students in learning how to learn (#1). In survey 2, Mateo said that student centered instruction "means giving up most of the control of what happens in the classroom to the students," which suggests evidence of attention to supporting students in decisions about how to learn (#1). Mateo noted that he felt that he was effective at providing feedback to students, with a focus on his job of assessing students rather than supporting students to assess themselves. His third definition stated that "running more carefully-designed activities in groups" was important to student-centered instruction. This suggests knowledge about supporting his students to collaborate (#2). In interview 1 he realized the importance of wording questions to get answers that aren't just a yes or no which would promote critical thinking. This could be considered evidence of him supporting his students to evaluate knowledge claims (#3).

Case B: Oliver. Oliver's first definition stated that a "student-centered approach to instruction focuses on how students learn rather than the material." He also mentioned a need to "give space for students" and to allow for "a hands-on approach to learning" to accomplish this. This showed evidence of knowledge for supporting students in learning how to learn (#1). His definition of student-centered instruction in survey 3 (he did not respond to the prompt on survey 2) included that "student-centered teaching puts more weight on the will of those who lack knowledge [the students]." Meaning that "students create the focus of the course, rather than the teacher." This definition also suggests evidence of knowledge related to students learning how to learn (#1). In interview 1 Oliver noted the value of student-centered instruction saying that "in order to understand something you should be working with your hands." This suggests further evidence related to students learning how to learn (#1). In interview 2 he said he was maybe "intervening too early in group work" This awareness about giving more time for struggle, supporting students to persist in problem solving provides some evidence of his knowledge in supporting his students to evaluate knowledge claims (#3).

Case C: Patricia (& Robert). In the first survey, Patricia said that she thought student-centered instruction was "focused on student interests and needs" and that it included "getting students involved in the direction" of the course. This could be evidence of knowledge of supporting students in how to learn (#1). In survey 2, Patricia's definition of student-centered instruction included "creating a classroom environment in which students have an equal voice with their professor" and "actively doing work rather than passively listening to a lecture." This gave evidence of her growing knowledge about supporting students in how to learn (#1). She said she was effective at "giving helpful feedback" and encouraging "students to ask myself and each other questions." This was evidence of her knowledge expanding to include supporting students in how to seek help (#4) and how to collaborate (#2). In survey 3 Patricia said "the focus is on getting the student actively participating in their own learning" (i.e., supporting students in how to learn, #1). Patricia also said she used group activities instead of lecturing so that "students can explore concepts and learn from each other" (collaborate, #2). In interview 1, Patricia said the orientation workshop helped her be more supportive of students who needed disability resources (knowledge growth related to supporting students to seek help, #4). In interview 2, she said the workshop course helped her "to ask better questions that would lead to more discussion, rather than just like a yes or no answer." This encouragement of critical thinking provides evidence for supporting students in evaluating knowledge claims (#3).

Case D: Queenie (& Sandra). Queenie's definition of student-centered instruction in the first survey mentioned "allowing students the space to form their own understanding and ask questions". This shows evidence of knowledge related to supporting students in learning how to learn (#1) and how to seek help (#4). She also said that student-centered instruction "requires group work and students discussing what they are learning" while giving "the tools to explain back what they are figuring out." These show evidence of knowing to support students in how to collaborate (#2) and how to become formative assessment capable (#5). By survey 2 Queenie was less sure about a definition for student-centered instruction, saying "all teaching has to be student-centered" since teaching students is the focus. She also said that "teaching becomes motivational." In survey 3 her definition of student-centered instruction was mostly the same as her first definition saying that it included “giving space” and for students to “come to me with their own ideas.” In both interviews, Queenie mentioned implementing group worthy activities – evidence of her knowledge including support for students to learn to collaborate (#2).

Case E: Tom. Tom’s definition in survey 1 stated that student-centered instruction “describes an approach to teaching where each student’s needs and background are considered.” This could be evidence of knowing to support students in a variety of ways in learning how to learn (#1). He also mentioned that it “involves a high level of student feedback and interaction.” This statement provides some evidence of knowledge about supporting students in how to collaborate (#2) and how to become formative assessment capable (#5). In the second survey Tom stated that student-centered instruction puts “students in the driver seat” and makes students “active members in the classroom and in their own learning.” This is more evidence related to how to learn (#1). In interview 1 Tom said, “I’ve tried to be more careful in how I handle student questions... thinking of understanding why something happened, not just what happened.” This showed evidence of him supporting his students in evaluating knowledge claims (#3). In interview 2 Tom said he would like to learn to create activities that will help students discover knowledge on their own. This was evidence of supporting students in learning how to learn (#1).

Summary. Table 1 summarizes, grouped by case, what GTAs described in defining student-centered learning. Notice there was evidence for all GTAs of supporting students in learning how to learn (#1) and, with one exception, learning how to collaborate (#2). Most of the GTAs had evidence of supporting their students in evaluating knowledge claims (#3). For several of the GTAs there was evidence of supporting their students in how to seek help (#4). There was sparse evidence of GTA awareness or use of supporting their students in how to become assessment capable (#5), how to be resilient (#6), and what to do when not sure what to do (#7).

Table 1: Summary of GTA reporting on aspects of student-centered instruction.

Case	Name	1. learn	2. collaborate	3. evaluate knowledge claims	4. seek help	5. become assessment capable	6. be resilient	7. what to do when unknown
A	**Mateo**	X1, X2	X3	Y1				
	Nicholas	X2	Y1, X3	Y2				
B	**Oliver**	X1, Y1, X3		Y2				
C	**Patricia**	X1, X2, X3	X2	Y2	Y1, X2			
	Robert	X1, X2	Y2	Y1	X2, Y2			
D	**Queenie**	X1, X3	X1, Y1, Y2, X3		Y1, X2			
	Sandra	X1, X2, Y2, X3	X1, X2, X3		X1, X2, X3			
E	**Tom**	X1, X2	X1	Y1		X1		

Note: X1=survey 1, X2=survey 2, X3=survey 3; Y1=interview 1 and Y2=interview 2.

Discussion

The question driving the study was: How does GTA knowledge about student-centered instruction develop over time when GTAs are in online professional development for online teaching? The short answer: Slowly. The surveys indicated that the majority of the new GTAs at the master’s granting study site in the 2021-2022 academic school year built substantive knowledge about student-centered instruction (as indicated in Table 1). At the same time, consideration of each GTA experience shows differences in the nature and timing of those changes. In Survey 1, most GTAs' ideas seemed to be based on the literal meaning of the phrase "student-centered." By Survey 2, GTAs had a formal definition and talked about what the goals were for the *students* in student-centered instruction but said little about what the *instructor* did. As time progressed, more GTAs included the teacher, using examples from their own instruction in Survey 3 and some expanding on ideas in Interview 2 with additional classroom examples.

Not only GTA knowledge changed. The GTAs perceptions of themselves and students changed in many ways. GTAs were not just learning about student-centered instruction for teaching, many were experiencing it as a student for the first time in their own graduate courses. Most of the GTAs had not experienced student-centered instruction as a learner before teaching. Moreover, no one reported they had been a college student in the class they were teaching. Finally, for most, experience of online learning was only under the conditions of the pandemic.

In the surveys and interviews all of the GTAs mentioned supporting students to learn how to learn as a goal, but did not offer specifics about what instructional moves or approaches might achieve that goal in the examples they provided. GTA focus was on "getting" students engaged – a teacher-centered view about having students look at and do the things the teacher and/or curriculum wanted/expected. That is, for #1 in Table 1, all of the GTAs appeared to include learning how to learn as an important component in student-centered instruction, but they were still working on how to accomplish it.

Throughout the surveys and interviews all but one GTA mentioned the use of group work as an important part of student-centered instruction. They all seemed to accept the idea that working with others could be an effective form of learning in addition to or instead of lecturing only. The one GTA, Oliver, who did not mention it was using a curriculum that had activities that required group work, but he did not directly mention using group work in the surveys and interviews. Thus, for #2 in Table 1, the GTAs in this master's granting department were unlike the GTAs in previous studies among doctoral students (Beisiegel, 2012, 2017, 2019).

Directions for future work include:

- Following GTAs who start in a master's program and continue on to a doctoral program; capturing what ways their PD in their master's program may have influenced their knowledge about teaching and how that compares to their peers who went directly into doctoral programs.
- Purposefully examine the future graduate experience and teaching development of today's undergraduates, who are experiencing student-centered instruction (unlike the GTAs in this study).
- What are the benefits and pitfalls of extended PD for master's program GTAs who are instructor-of-record in terms of frameworks in previous work (e.g., Beiseigel, 2019)?

Limitations and Delimitations

The main findings are from a study of a particular GTAs in the specific context of teaching online synchronous courses at the study site and are not intended to be generalizable. Rather, the goal is transferability: giving sufficient detail to support the reader in interpreting the results for a situation similar to that presented here. This small study did not gather or analyze information about societal structures and systemic influences (and consequences) of student and instructor perceptions. Nonetheless, a useful example can anchor future research and GTA development.

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Exploring How Computation Can Foster Mathematical Creativity in Linear Algebra Modules

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The mathematics education community has simultaneously experienced a call for mathematical creativity and the implementation of computing within the classroom. What has not been explored are the ways in which these two calls are complementary. This study investigates how computation, enacted through coding, can bring about mathematical creativity within a series of linear algebra computational notebooks designed using the Understanding by Design framework. Specifically, through the analysis of a group of students, this paper highlights how a computational prediction and reflection cycle enables exploration and student originality while also facilitating connections to multiple representations.

Keywords: Computation, Linear Algebra, Mathematical Creativity, Coding

Mathematics – a field of beauty, creativity, and innovation, yet what is echoed by students is not how mathematics allows students to explore, but rather the ways in which it constrains and does not necessitate creativity (Silver, 1997). Students state mathematics is "a dead subject [with] hundreds of methods and procedures to memorize that [students] will never use, and hundreds of answers to questions that they have never asked" (Boaler, 2022, p. 35). This decline in creativity within school mathematics has coincided with the evolution of computation. Computation is now a scientific pillar (Skuse, 2019) as well as a pedagogical tool within the mathematics classroom (Castle, 2022; Lockwood, 2022). Further, many students associate coding with freedom to create and express themselves (Isomöttönen et al., 2020) which is in sharp contrast with students' views of mathematical creativity. Therefore, a natural question that arises is what if computation could offer mathematics education more, specifically with relation to mathematical creativity. This study is part of a larger research project that aims to address the ways in which computation enacted through coding can aid mathematical understanding and promote mathematical creativity, while preparing students for careers in the upcoming computational world. Within this paper, I seek to answer the following research question: *What are the affordances and constraints that computation, as enacted in coding through a creative environment, provides for students' learning of linear algebra concepts and procedures?*

Theoretical Perspective

Cultural Historical Activity Theory (CHAT) is based on the of the work of Vygotsky, Leontiev, and Engeström (1987). It provides a framework for studying collective activity, specifically emphasizing the role of mediated activity, a key consideration when considering computing. Within CHAT, subjects are the people involved in the activity, whereas the object is the aim of the activity. Subjects' actions are mediated by artifacts or tools. Further, it is imperative that individuals are understood within the context of their actions, and that the social elements must be brought in. Therefore, within this study individuals cannot be divorced from their group and context. There are different levels at which you can analyze the relationship between subject and object: activities, actions, and operations (Leontiev, 1974). Activity is composed of actions, and actions are composed of operations (Batiibwe, 2019). Operations are automatized or routinized behaviors and many times do not require conscious effort (Jonassen & Rohrer-Murphy, 1999). The conditions determine the operations. At the next level up, the goal an individual holds results in actions, but the goal is also affected by the conditions. This is the functional level that uses planning and problem-solving to complete the activities. Finally, it is motive that generates activity, which is composed of actions, and the motive determines the goal. This hierarchical structure is used in order to operationalize activity theory within observations, as there are multiple units of analysis that can be used to build up the overall activity. As the action level is what is of interest within this study, applying this framework necessitates establishing what actions are manifestations of mathematical creativity, thereby providing a way of documenting how the relationship between the students and the aim (mathematical creativity) is mediated by the tool (Jupyter notebooks).

Mathematical Creativity

In order to engage with the notion of mathematical creativity it is important to define what I mean, as there is a plethora of definitions. Based on literature (Haylock, 1997; Savic, 2016; Savic et al., 2017; Silver, 1997; Sriraman et al., 2013) I define mathematical creativity as:

> A process that can be fostered, is context-dependent, and manifests in fluency, flexibility, and originality with respect to the mathematics which results in the connection and abstraction of ideas that bring a new perspective that is of use to the community and is many times the result of prolonged work and reflection.

It is important to note that the focus is on creativity as a process, rather than a trait. This is an important distinction as it means there is the possibility to design opportunities to foster creativity and it is not a stagnant attribute of students. Further, I draw from mathematical creativity rubrics (Blyman et al., 2020; Henriksen et al., 2015; Savic et al., 2017) in order to establish an operationalization of mathematical creativity - specifically six dimensions of creativity: *originality* (ability to try novel or unusual approaches towards a problem), *flexibility* (ability to use multiple methods for solving the same problem), *fluency* (ability to apply the same mathematical idea, concept, or procedure, to a variety of problems and situations), *visualization* (development and use of illustrations, either physical or mental, to clarify or present concepts), *elaboration* (ability to establish meaningful connections between concepts), and *risk* (willing to take action where result is unknown, or a novel approach, to advance problem solving process).

Methodology

This paper reports on data collected from a study investigating the role of computation in developing mathematical creativity. This study consisted of 8 different students recruited from an introduction to computational modeling course (Silvia et al., 2019) who met recurringly with a small group (2-3 members) for a total of 6 weeks. Each week students engaged with a Jupyter Notebook designed to introduce new linear algebra concepts while fostering opportunities for mathematical creativity and understanding for a total of two hours. Afterwards students would complete a reflection about the module. Additionally, I conducted semi-structured pre- and post-study interviews with all participants as well as pre- and post-surveys. For the purposes of this paper, I will focus on a sole group composed of three students: Ivy, Alex, and Kylie, all of whom were non-mathematics STEM majors. This was done for space considerations, allowing an in-depth exploration into their experience across the study and for a more nuanced understanding.

Module Design

An underlying portion of this study centers around the design of the computational modules, accomplished by implementing Understanding by Design framework (Wiggins & McTighe, 2005). This process employs backwards design where the learning goals are first established, then potential evidence for meeting the goals, and finally the activity design itself using the prior work. A textbook, syllabi, and curriculum analysis yielded potential topics and desirable understandings. As this was not an official course, nor a prerequisite, I had latitude in selecting the broad understandings that the modules would center on, specifically for students to develop geometric understandings of linear algebra ideas and properties, as well as how to represent and model systems using their understanding of these concepts. Learning goals were established for each module and the evidence for meeting these goals were written out. Finally, pairing the tasks with opportunities to foster mathematical creativity as well as computational pedagogies such as use-modify-create cycles (Lee et al., 2009) resulted in the final construction of these modules.

Data Analysis

For the results shared within this paper, I transcribed and coded all interviews, observations, and reflections along the mathematical creativity dimensions and whether they were mediated through the notebook. The unit of analysis for coding was individual actions, as determined from the CHAT framing. I then wrote analytical memos detailing the initial themes of mathematical creativity followed by grouping codes along the dimensions of creativity and then along whether the action was mediated, resulting in sets of common elements. These groupings gave rise to two main themes within this set of students' experiences: (1) computation enabled experimentation via prediction and reflection cycles and (2) prediction and reflection facilitated the connection of multiple representations.

Results

Computation Enabling Experimentation Within Mathematics via Prediction and Reflection

During the fifth module, students were introduced to Markov chains. They wrote a function that would take in a state vector, a stochastic matrix, and the number of observations, and return the final state vector. They used the function to predict the population levels of a related suburb and city after 1, 2, and 5 years. The following exchange occurred after writing their functions:

Kylie: I'm just like running them all. Uh, it basically just keeps going in the same direction.

Alex: I think eventually it would, I don't know if we did, like, a thousand that would probably make our computers freeze, but um, would eventually it balance out somewhere.

Ivy: Maybe we can try like 10 and see what happens.

Alex: Maybe balance out, balance out isn't the right word. But eventually it would like 50-50, the directions would've reversed. Yeah.

Ivy: Let's try this. Yeah. It reversed at 10. So, the city is less than the suburb.

Alex: I mean like eventually they would get like eventually a year would go by and the city would increase while the other one would decrease.

Within this example, a key point is that students were never explicitly prompted to explore beyond the 5-year mark. Rather, they were designing a function in order to be able to easily model the Markov chains, and then testing that function to check predictions for a model. After completing the task at hand, the group demonstrated both originality and risk by asking a question that was of interest with a coupled prediction. Note that they had not specifically encountered the concept of steady state yet. After this prediction they tested their hypothesis, using their function, and then reflected on the output itself. Within this example, the computational environment enacted through coding enabled a cycle of prediction and reflection, which ultimately later led to a concept of steady state vectors. The coding enabled function creation, which allowed for an efficient way to engage in experimentation. The students were able to create a prediction and then check their result.

The fourth module focused on linear transformations and matrices as linear transformations. Prior to the discussion, the students had just finished up with figuring out how to take a matrix of points that defined a cat outline and double the cat image in size. They had experimented with this and graphed the resultant set of points. The next prompt within the notebook was *"It will now be your turn to explore! See what happens when you try different matrices. Use the cell below to document your thoughts or whatever you find easiest! Try and state what you think will happen before you run the code"*. This was followed by multiple cells designated for students to try different matrices. This portion was specifically designed for students to experiment.

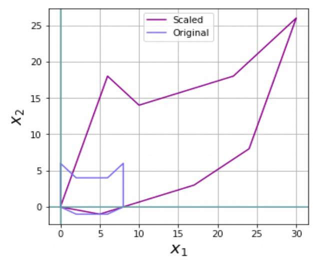

Figure 1. The output from Ivy's Jupyter Notebook when using a matrix transform along a set of points.

Ivy: I wonder if you like had like 3, 1, what would happen like anything that's not a zero.
Alex: Or you wanna do 3, 1, 3, 1. *(Referring to a 2x2 matrix)*
Kylie: Do you think it will work?
Ivy: Um, we can say, we don't think it will work.
Alex: What do you mean? What do you think?
Kylie: I guess it will just change. Like it'll disproportionately change the width and length.
Alex: Uh, I think the X values will be three times X plus Y and the Y values will be, uh, Y plus three X. So I think it, the X and Y will turn out to be the same, if that makes sense…
Ivy: So you think that it's gonna, they're gonna be multiplied by three, but then the value, like a value's gonna be added to them as well…
Kylie: Okay. And then what do we think that's gonna do the graph. It's gonna make it larger and move it?
Alex: I don't think it'll look like a cat anymore.
Kylie: Yeah. It's gonna move him around. So it's gonna like break up the lines. That is correct. It looks really weird. Oh! *[Output graphic shown in Figure 1.]*
Ivy: It kind of looks like it just stretched linearly. Like, kind of like on a line.
Kylie: It's it looks like if you took a side of it and just pulled yeah. You like took one of the cat ears and pulled it.
Ivy: So it's still cat.
Alex: I think it, it kinda looks like you took, like, we took the lines and multiplied them each by like some number, but it looks consistent, you know? You know what I'm saying?
Ivy: Kinda like, it just looks linear to me. I don't know why.
Alex: So for the next one, what do you guys wanna do?
Kylie: What about if we did it like negative numbers?
Alex: Oh, it'll flip it over. Like the X, Y or something?

This process continued on and the students each completed 6-8 different trials, deriving their own novel transformations, and plotting the visualization. From here, students were then asked to define the matrices that would correspond to the points undergoing: reflection across each axis, lines, and through the origin, as well as contractions and expansions, and shears. During each of these phases students then took what they had learned through experimentation to abstract, make meaningful connections, and create general representations of the types of transformations and the corresponding matrix. If they were usure, then they would experiment, using the insight they gained during the first part. Then they used code to validate their approaches and show the transformation on a set of points.

During this activity, there was specific reference to 'exploring' within the prompt. However, it was previously shown that students engaged in experimentation via prediction and reflection even when not prompted. This new excerpt highlights the potential ways in which the prediction and reflection cycle can take place. Specifically, the students were able to predict the ways that a 2x2 transformation matrix would affect an image and how the corresponding vertices would shift. The experimentation highlighted their initial choice in what matrix to start with and followed by different modifications. This allowed for their thoughts to guide the exploration. Further, during this experimentation, students were not simply 'pushing buttons'. Rather, they actively engaged in the sense making and making predictions based on their current understanding of the materials. This exploration piece allows for an inquiry-oriented approach to transformations, and the computational environment is something that enabled this to be fostered, especially the visualization. This was drastically different than previous mathematics experiences for most students. When reflecting on the experience as a whole, Alex stated:

> "I know it was a math, you know, teaching us math, but it didn't feel like math the same way as like solving integral does … it was more so recognizing like what was changing … it felt much more like recognizing patterns... Yeah. Just, just figuring that out. It didn't feel so much like, step by, I move this over here and then simplify this, it felt more like, playing a game, trying to, trying to figure out the way to get to the end goal."

This quote highlights the fact that the experimentation portion felt different to Alex than previous math classes. Although he does not explicitly name experimentation, he discusses this concept of changing pieces and looking for patterns to get to the end goal, which mirrors the notion of prediction and reflection. Within the previous examples from the observations, the group engaged in making changes in order to figure out the corresponding effect, thereby developing a deeper understanding of the system. This experimentation fostered via computation countered Alex's prior notions of mathematics, and specifically gave a framing of exploration rather than following a series of given steps, thereby promoting opportunities for creativity.

Prediction and Reflection Facilitating Connections to Multiple Representations

The second area highlighted during this pilot study was the ways in which the computational environment facilitated students making connections to multiple representations. To start, consider the following excerpt from the group's sixth module. It shows the interplay between group members and the ways in which the development and running of code supports multiple representations. Within this activity, participants were provided a function that takes in three vectors and displays the corresponding parallelepiped. Kylie was sharing her screen with the group at this time, and began by plotting the identity matrix, which represents a unit cube. The group ensured the determinant was 1 since the volume of the cube was 1. They read through the plotting function to ensure they knew what was occurring and how to use the function. This is where the following excerpt picks up. The exchange has actions added within brackets in italics; also, some of the points of general conversation were removed for the sake of space.

Alex: So, let's use the columns of A as our vectors. Okay. So, um, I guess that kind of makes sense. Cause the determinant is like doing the, it's kinda like doing the cross products a couple times. Right. And cross products, like the area of a parallelogram. So anyway, um, that's our parallelepiped… Cool. What would cause the determinant of A, to be zero? Uh, if it were, how do you get a volume of zero?

Ivy: Would it be like a 2D shape?

Alex: Oh yeah! Yeah that makes sense.

Ivy: But how would we express that?

Alex: We could try different matrices. So v_1, v_2, and v_3 are probably all gonna be in the same plane is my guess. I guess. Yeah but how do we –

Ivy: Should we set one of them to all zeros? No –

Kylie: We could try it! I guess we don't need to plot it right now. We're just looking at the determinant…. Uh okay. So we are trying. Yeah. Here is our original -

Ivy: I'm just gonna change the first value to be zero for all of them. See if that does. Yeah. That is how you can do it… Cause technically you're saying v_1 has like zero value…

[Kylie then sets one of the columns of A to be 0 and uses linAlg.det() to check determinant]

Alex: Okay so that one is like on the same plane? …

[Kylie then runs the plotting code with the new matrix that has a determinant of 0]

Kylie: … How does – no. Cause it's all on the same plane, because one of them is zero. What must be true for one of them to not be zero?

Alex: They must be on the same plane

Ivy: So if we have three vectors that are all on the same plane or that they're all, oh wait, like 2d plane…

Alex: Right. Well there's no volume if they're all on the same.

Kylie: Plane. Yeah okay. No, no, okay, got it. Got it. We're good.

Ivy: Which means that they must be linearly dependent!

Alex: … Oh yeah. So we can say they're linearly dependent - if the determinant is zero.

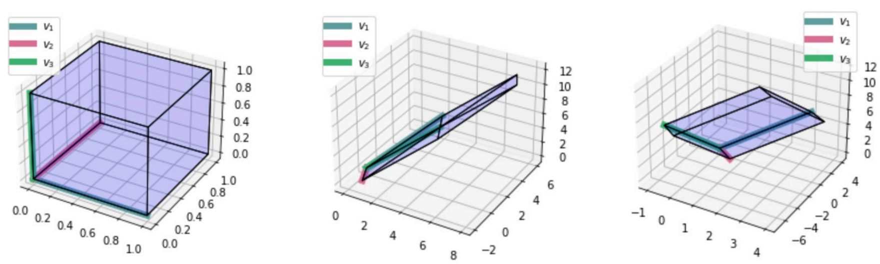

Figure 2. Visualizations Kylie created during the exploration of determinants, trying: (a) unit basis vectors, (b) linearly independent vectors, (c) linearly dependent vectors

The group then goes on to check another set of known linearly independent vectors to verify the determinant is not zero. The three visualizations produced through this experimentation and reflection are shown in Figure 2. The activity does engage the previously discussed prediction and reflection cycle, but also leverages multiple representations as well. This in turn enabled students to discover the relation between the determinant being zero and the linear dependence of vectors, proving opportunities for flexibility and visualization. During the excerpt, the computational representations were the code itself and the actual output of the code, referring both to the visualization of the vectors in 3D as well as the numerical value of the determinant. That is, they were representations that existed via the computer alone. The code itself had students decompose the matrix into a set of vectors in order to calculate the volume to determine the determinant while simultaneously using the full definition of a matrix to calculate the determinant. It was up to the students to determine the test matrices, how the matrix was then used to determine the parallelepiped, and then calculate and visualize the determinant. Besides the scaffolded plotting code, students are enacting all of these concepts, necessitating a switching between resources. Further, when they discussed the notion of determinant, they draw on the algebraic notion alongside the graphical interpretation, leading to the discovery of the impact of linear dependence on the determinant.

Computation enabling prediction and reflection, as previously discussed, means that students are maintaining a constant balance between computational representations. Namely, the code itself and the output, such as Ivy coordinating the input and execution of code with the output value of the determinant. Therefore, since students are already engaging in this process of bridging representations (lines of code and output) then there is a natural way of connecting the tradition of multiple representations, namely students balancing both the graphical, numerical, and algebraic representation. All of these representations were brought into the excerpt, and this facilitated not only flexibility but also the creation of personal visualizations and elaborations.

Discussion and Conclusion

This work highlights the potential that computation has for engaging students in mathematical creativity, specifically when enacted through coding. Within the focal group of students, this was accomplished by computation naturally supporting experimentation through cycles of coding predictions and reflections. Specifically, students adapted their code to follow their own ideas and fluidly check how this altered the result. This cycle of prediction and reflection is not new to linear algebra (Wawro, 2009) nor computation (Lockwood, 2022). However, what this study contributes is specifically how computation enacted through coding can be leveraged to bring about this cycle to promote mathematical creativity. Students naturally explored utilizing the code and engaged in this cycle with and without prompting. These cycles contained multiple dimensions of creativity, especially along originality and risk, which is consistent with research on students' epistemic agency within computational notebooks (Odden et al., 2021). Further, these cycles of prediction and reflection facilitate connections to multiple representations. During the determinant cycle, the group continually bridged and connected their code with the output produced. The modification of the code led to the determination of multiple cases of linear (in)dependence where they then pivoted to make connections between the concepts they had already encountered and the code output. This enabled for a natural bridging between the graphical, algebraic, and numerical understandings of a determinant. A key point within the literature is how students are able to develop great insight and creativity when developing geometric interpretations of key linear algebra concepts (Larson et al., 2008). This is one of the particular strengths of these specific modules but more broadly within computation enacted through coding. Many programming languages have supports built in for visualization which allows students to plot different objects that were previously unattainable by hand. Further, the act of coding coupled with students' control over the representation enable not only originality but also constant maintenance of multiple representations. Within this paper evidence was presented for how computation enacted via coding can lead to opportunities for fostering mathematical creativity through experimentation within computational prediction and reflection cycles, promoting connections to multiple representations. As the relation between computation and mathematical creativity is relatively new, future work should entail focusing on computation as a pedagogical tool for mathematics education in addition to exploring the potentially unique insights that computation offers students.

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What Do Students Do To Develop Understanding of Definitions in Proof-Based Courses?

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Advanced, proof-based mathematics courses typically include learning goals of developing understanding of the relevant content of the class. The content always includes definitions of concept in that they are foundational to theorems and proofs. This study explores what three students believe it means to understand a definition, the actions they take to develop that understanding, and the rationales they use to explain why their actions are productive. The students held divergent beliefs about what it means to understand a definition. While all cited lecture attendance and homework as critical activities in their learning, they diverged on the use of outside resources and the rationales for why they might or might not be useful.

Keywords: proof-based mathematics, student understanding, definitions, learning goals

Mathematical definitions serve an essential purpose in proof-based courses, which require students to attend to these definitions in a distinct way: as stipulated rather than descriptive (Edwards & Ward, 2004). Definitions are often provided during lectures; however, we know little about what students understand from these settings. In one of the few fine-grained studies of what students gain from specific lectures, Lew et al. (2016) described that students may not apprehend the content that professors are attempting to convey. Melhuish et al.'s (2022) recent literature review points to a lack of empirical studies focused on student learning or what students might do outside of class to develop understanding.

As understanding and using definitions constitute a major content goal of proof-based courses, we explore two primary research questions and two that provide context and explanation for the findings of the first two:

- What activities do students engage in to try to develop understanding of a definition?
- Why do they engage in those activities?

To provide context, we also explore:

- What do students believe a definition is in advanced mathematics and why?
- What do students believe it means to understand a definition and why?

Literature Review

In formal mathematics, definitions stipulate the nature of a mathematical concept, and any object that meets the definitional criteria is an example, and any that does not is a non-example. Such definitions are referred to as *stipulative* (Edwards & Ward, 2004). Alcock (2010) suggested that mathematicians desire students to understand the stipulative nature of mathematical definitions. At the same time, extant research suggests that students do not consistently hold similar beliefs about definitions (c.f., Edwards & Ward, 2004; Alcock & Simpson, 2004; Krupnik, et al, 2018), and may operate with the definitions as if *descriptive* (that is they are extracted and created from use) rather than *stipulated.* Some studies have pointed to complexities in students' thinking of definitions attending to issues of mathematical criteria and communicative goals (Zazlavsky & Shir, 2005) or ways students may reinvent stipulative mathematical definitions (Zandieh & Rasmussen, 2010).

In terms of lecture courses, mathematicians have argued that much of the work of learning, such as learning definitions, occurs outside of class (Wu, 1999; Weber & Fukawa-Connelly, in press.). This may involve homework activities (Rupnow, et al, 2020) and other types of at home activities. Yet, we know little about what types of activities students intend to and do engage in at home to meet learning goals. If we focus on definitions, we can find some papers on what mathematicians do to learn new definitions and content (e.g., Parameswaran, 2010; Wilkerson-Jerde & Wilensky, 2011). In the context of definitions, Parameswaran (2010) interviewed and surveyed a set of mathematicians finding they studied examples, used definitions to prove theorems, explored equivalent definitions, and resolved cognitive conflicts while trying to understand a new definition. Others have pointed to how mathematicians' use intuition and diagrams (c.f., Johansen & Misfeldt, 2020; Sinclair& Gol Tabaghi, 2010) in connection with formal definitions. That is, there may be reason to hypothesize that understanding a definition in a formal proof context, may also involve drawing on not just definition, but also concept image (as defined in Tall & Vinner, 1981).

Theoretical Perspective

The practical rationality framework, initially designed to analyze teacher decision making at the K-12 level (Herbst and Chazan, 2003), posits that decision-making is regulated by rationality. More specifically, the knowledge, beliefs, and goals of the teacher mediate the obligations and norms from stakeholders (Chazan, et. al. 2016; Webel 2013) and ultimate decisions. Since its development, it has been extended to investigate the students' decision making. Webel (2013) argued for attention to "both social and individual factors in describing how particular student actions come to take place in mathematics classes" (p. 25). In our study we hold that the beliefs students held and the actions that they took were rational considering the obligations placed on them and their experiences within the mathematics discipline.

Methods

The results discussed here are a part of a pilot study aimed at exploring students' beliefs about definitions. Data collection took place at a large research university in the southern United States. Our first round of data collection was an online questionnaire in which students were asked about what activities they engaged with in between classes that helped them better understand definitions and why they think these activities helped them. This survey was given to students in two undergraduate math classes (modern algebra, real analysis) a few days after their professor had introduced a new definition in class (direct products, continuity). Two students in modern algebra and one student in real analysis completed both the questionnaire and the interview.

Data Collection Methods

The goal of the survey was to document in-the-moment activities that students claimed to engage in to develop understanding of new definitions. During the follow-up interview, students were asked questions about their background, their beliefs about definitions – including what it means to understand a definition – and questions about how they come to understand definitions. For questions relating to how they developed understanding, we encouraged the students to think back to their most recent activity to understand a new definition which often focused on a homework assignment. They were also given the opportunity to respond to other contexts such as preparing for exams or their general practice with new definitions.

Data Analysis Methods

To analyze our data, we began with an open coding scheme in which each interview was first transcribed and then coded for beliefs and activities. All four authors coded Kurt's transcript for both beliefs and actions. For Kris' and Dave's transcripts, each transcript was coded for beliefs by two members of the research team and for actions by three members of the team. We then met to reconcile the coding and any discrepancies were worked out through discussion. Using these codes, we built profiles of each student that addressed the following five areas: what is a mathematical definition, what is the purpose of a definition, what does it mean to understand a definition, what actions are taken to understand a definition, why the student holds each of these beliefs. Note that these are from the student's perspective. The results of this process are addressed in the sections below.

Data and Results

Table 1. Definition Profiles of Participants

Participant	What is a definition?	What is the purpose of definitions?	What it means to understand a definition
Kurt	Rigid criteria categorizing a set of objects ("black and white, either in or out")	The purpose of a definition is to create and prove claims about objects that fit the definition.	• Determine whether an object meets the definition • Developing Intuition
Dave	Axiomatic "Foundation" of mathematics that cannot be changed	The purpose of definitions is to construct and prove mathematical propositions.	• Knowing the definition • Being able to apply the definition in proofs
Kris	Fixed and cannot be argued as incorrect	The purpose of a definition in mathematics is to contribute to the development of propositions and proofs	• Understanding its purpose • Knowing when to use in proofs

Before providing an overview of the at-home activities, we first share a brief overview of the three students' beliefs (Table 1). All the students held stipulated views of mathematical definitions with all three making explicit comments about the definitions being rigid. We also note that all shared a common view that definitions were in service of proofs and propositions. However, the three students reported somewhat distinct views on understanding definitions.

Additionally, we note that none of the students claimed to take any specific activities to learn a definition the day it was presented in class. As we detail below, they all describe some activities that they take to learn definitions, and have well-developed rationales, none are taken immediately after the presentation.

Kurt

Kurt explicitly described four activities that he engages in to try to develop understanding of a definition and we inferred one additional activity, repetition. The first three that he named are attending class, looking at his notes, and doing the homework. He also claimed to look at online videos when those were insufficient or when he thought animations might be helpful.

Kurt explained that he went to lecture diligently, took notes, and asked questions if he was not understanding. In particular, he explained for continuity,

> I've mostly been relying on lectures for that too, like the first day that it was introduced to us. I was like I don't really get it, so I just I kept like, you know, going to class and like OK like let's try this again and my professor was really good about when we were like getting in deep with continuity being like OK class.

He noted that he asked questions like, “When something isn't continuous, why does it break? Why doesn't it fit that definition? You know? Why does it break? So, I was like, can you show us like a picture of like the epsilon bubble and the delta bubble like show us like how its discontinuous function like breaks that so to speak.”

Outside of class he then sought out YouTube animations to further develop his understanding with visuals in the cases he found lecture insufficient elaborating that, “OK, here's that curve, and here's what happens when we do this and this and this, that could be really handy. That's something that's kind of you can't replicate on a chalkboard super easily, you know.” He appreciated another perspective but noted sometimes what the professor said was simpler.

Kurt also explained homework as his most valuable tool to develop his understanding of definitions, explaining that he needs to “play with [the definition] himself” and understand how to apply them in proof. This was a prevalent theme throughout the interview.

Reflection. Kurt emphasized several activities that aligned with his view on understanding a definition, including attention to examples (categorizing in or out), and developing intuition from animations (often found online.) He did not see memorizing a definition as part of understanding nor did that influence his activity and uses homework as the primary way to achieve understanding. Although Kurt did not explicitly say using definitions to prove as part of understanding, this was stated as a purpose of definitions and the primary activity for Kurt, likely reflecting the nature of homework prompts.

Dave

When we consider the actions that Dave takes to understand a definition, we can summarize his activity by saying that he believes that he tends to do better on the coursework when he simply does the homework and try to understand each question conceptually. This approach is believed to lead to remembering definitions and proof strategies. If he does get stuck there is a preference of talking with the professor, but, if necessary, he will use the internet. We noticed that Dave has a nuanced stance on the actions he takes. For instance, he will make statements such as “maybe it's just me. But memorizing does not help. I think understanding the big picture helps a lot more than any amount of memorization.” We do not take this statement from Dave to mean that he is against knowing the definition. Instead, we take this to mean that Dave holds the belief that definitions are learned progressively and through doing mathematics as opposed to the use of flash cards or other memorization tools. Also in his internet use, there is a strong preference towards seeking help from the professor before turning to the internet as the internet often lacks the ability to give insight and creates confusion with different notation usage.

In further consideration of Dave’s actions, he seems to take actions that he believes will support him in better developing his concept image of the topic at hand. During the interview Dave provided an instance in which talking to the professor was of greater benefit than the internet. In this dialog he stated

> And then [Dr. X], I went to his office hours once and I was like I don't understand what this looks like. Help me. So, then he pulls out his Rubik's Cube and he's like see how the face doesn't go anywhere. This is the stabilizer. Like I love stuff like that. I think that's way more helpful to me in the grand scheme of things than like just knowing that

definition, then trying to apply it, you know. That I can think about it better if I asked directly for help from the professor versus random Internet strangers online.

In this case, directly talking with the professor allowed Dave to develop a richer concept image of the stabilizer than turning to "random internet strangers."

The rationality for each of these actions was traced back to Dave's intro to proof course. In the following excerpt, Dave describes his experience transition into proof-based mathematics courses

> I can trace it back to intro to advanced math. Umm? There was just kind of like this. I guess wall, right? Because when, when you're like doing computation, computation, computation for years. You hit calculus 3 computation, differential equations computations. And then suddenly, you're just thrown in this theorem land where like everything's completely different, and there's no computation. It's just you need to know how to use definition to prove big idea. It's just a completely different beast, I guess. Like in terms of like math. And I didn't see it coming, so I had to shape up.

In this quote, Dave describes a form of distinction between the computational mathematics that he had been doing and the theoretical mathematics that he was beginning. This distinction between theoretical and application-based mathematics forced the development of new activities for learning.

Reflection. Like Kurt, Dave's activity was tied to his homework. Dave's description of understanding definition aligned with his activity where he explicitly teases apart the ideas of "knowing how" and "knowing what", both of which are a part of understanding. There is more value placed on the "knowing how" category, that is applying definitions to proofs, which we can see in the quote above dealing with the stabilizer.

Kris

Like his peers, Kris uses both lecture and homework as a primary means to engage in definitions. He also attends office hours and seeks out information from the internet. Kris claimed that his two primary activities to develop understanding were to attend lecture and do the homework, "But in my opinion, I think it's, that's the only two ways to help me to understand the definition is you just do the homework or see what Dr. X, like in know the instructor, what they expand for the definition."

Kris also acknowledged the importance of extended time noting that it "takes time to understand the definition and what's the purpose…" and a single lecture is too short, and he may need to revisit ideas on the weekend. He also explains that during lecture it helped when the professor paused and gave "a little bit of time to think about the definition" and how important that time was for him to understand.

While Kris said that those are the only two ways to develop understanding, he specifically describes other activities that he uses to develop understanding including office hours and using the internet as a resource. He would opt for the internet if his instructor was unavailable to meet searching out," Where is this definition come from? And the work, the purpose of this definition for sometimes it helped me to understand more, but some not all the time." He also would seek out different points-of-view on the definitions such as reading online arguments between different people. He also sometimes asked his peers for what the "purpose" of a definition is.

Kris's last observation from his coursework is that while he tries to be able to state theorems in his own words, he does not do this for definitions. He explains that "if I change the words in my, I have a different story than a definition." He elaborated how this led to him developing incorrect quantification in prior classes. That prior experience led to his decision to "force [himself] to memorize" the definition exactly which can also help when they need to be stated on exams.

Reflection. Like other cases, homework and lecture played a substantial role in Kris's activity were using definitions to prove statements was a key activity and addressed the "purpose" of a definition. However, we can see one substantial way that Kris's case differed from the prior. He placed outsized value on memorizing a definition exactly as stated. This may reflect his beliefs on definitions as the "inarguable" parts of mathematics. He also did not go to the internet for animations or intuition, but rather history, which again may reflect how he sees the roles of definitions as foundational.

Table 2. Actions taken by Participants

Participant	Actions taken to understand a definition	Beliefs and Rationale for Actions Taken
Kurt	• Attending class • Reviewing notes • Doing homework • Using online resources	• He needs to "play with it [the definition] himself" and understand how to apply them in proof. • Justifies actions using prior experiences in math courses and advice from friends.
Dave	• Doing homework • Talking with professor	• Memorization is not useful to developing the ability to use the definition in writing proofs. • Justified actions based on the goals of understanding/ desired learning outcomes set for himself and prior experience in math courses
Kris	• Attending class • Doing homework • Talking with professor, online resources	• Kris wants to know the historical uses and evolution of a definition, and this can be found via lecture and online

Discussion

The purpose of this study was to address two primary questions:

- What activities do students engage in to try to develop understanding of a definition?
- Why do they engage in those activities?

In addressing these two questions, we also examined two additional questions:

- What do students believe a definition is in advanced mathematics and why?
- What do students believe it means to understand a definition and why?

Regarding these first two questions, through both our survey and interview, we found that there are a range of activities that students engage in when trying to understand definitions and some consistencies. All the students suggested that doing homework and attending lectures were their primary means of developing understanding. Kurt also relied on additional resources, including those found on the internet, as part of their coming to understand. In contrast, Dave preferred not using outside resources like the internet. However, he does acknowledge that he has looked at them in the past. To account for differences in what students saw as useful, we return to the notion of the rationality of students and consider the goals that the students had. We do hold that each of the students utilized resources that were rational from their perspective as they considered their own learning goals. Webel (2013) had stated the importance of understanding the students' goals when trying to examine the motives of their behavior.

The students had relatively similar backgrounds in terms of the mathematics coursework that they had taken. The students always engaged in the activities they believed would best help them achieve their learning goals but had developed very different sets of learning behaviors and rationales to explain them. Kurt, for example, believed that he should develop understanding of the definition through repeated exposure via lecture and practice with homework tasks. In contrast so as to be able to reliably classify sort candidates into the class of example and non-example. In contrast, Kris memorized definitions for exams, but sought to understand the historical reasons that they were created and what theorems they lead to by using lectures, office hours, and resources found on the internet. Dave wanted to know when to use a given definition (akin to Weber's (2001) *strategic knowledge*) and so did the homework, go to office hours, and sometimes use the internet. Dave was leery of the internet due to changes in 'notation' that might confuse him. That is, different goals for understanding appear to support different collections of behaviors.

Through documenting both the activities students engage in when coming to understand a definition and the rationale behind why students engage in activities, we make a minor contribute to the literature on how students engage with definitions as there was little existing literature. We note two limitations that also suggest next directions for study. First, due to our limited sample of three students, we have no reason to believe that they are representative of students in advanced mathematics courses, and so one way to explore that would be to replicate the study with a larger sample. As a second limitation, we found that students provided incongruous responses during the survey and their initial claims during the interviews. We found that students recalled different behavior when prompted with their own survey responses than when asked to describe their behavior without prompting. At the same time, because the survey did not attempt to capture the logic for their behavior, more research is needed in order to explore the in-the moment decision-making processes that students use to seek out outside resources and evaluate them. Finally, in addition to replication, we noticed that preferences and sense of self-efficacy may influence their behavior. Thus, it would be of interest to examine the relationship between students' actions and their self-efficacy.

While we note that this is a small group of students, all three noted that they did not recall their professors describing any practices or behaviors that would support coming to understand definitions outside of class to come to understand a definition. The research on classroom instruction (c.f., Pinto, 2019; Fukawa-Connelly, 2012) suggests that mathematicians engage in modelling behaviors that could support coming to understand the concept in ways that go beyond memorizing the words of the definition. There also is reason to believe that mathematicians want to promote conceptual understanding (Nardi, 2007). Thus, these students do not believe that they have given any instruction about how to develop this understanding or what this understanding might entail, which could explain the divergence in their beliefs about what it means to understand a definition. As a result, we suggest that if mathematics professors have specific actions that they want students to take, they need to more explicitly teach them. At the same time, the literature does not currently allow us to link teaching activities and students' learning very well (c.f., Melhuish, et al, 2022). As a result, explorations of explicit instruction about coming to understand, how that relates to behaviors students actually carry out, and the types of understandings that result would be a valuable addition to the research literature.

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Differences in Students' Beliefs and Knowledge Regarding Mathematical Proof: Comparing Novice and Experienced Provers

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Learning to interpret proofs is an important milepost in the maturity and development of students of higher mathematics. A key learning objective in proof-based courses is to discern whether a given proof is a valid justification of its underlying claim. In this study, we presented students with conditional statements and associated proofs and asked them to determine whether the proofs proved the statements and to explain their reasoning. Prior studies have found that inexperienced provers often accept the proof of a statement's converse and reject proofs by contraposition, which are both erroneous determinations. Our study contributes to the literature by corroborating these findings and suggesting a connection between students' reading comprehension and proof validation behaviors and their beliefs about mathematical proof and mathematical knowledge base.

Keywords: logic and proof, belief about mathematical proof, mathematical knowledge base

Learning to interpret proofs is an important milepost in students' mathematical development and maturity. Such development is especially crucial since proof is a structure unique to the field of mathematics (Balacheff, 2008; Fawcett, 1938). Though much variability exists in how transition-to-proof courses are delivered in the U.S., over 80% of them attend to principles of formal logic (David & Zazkis, 2020). Presumably among the many facets of students' development with regard to proof is their ability to correctly discern whether a proof justifies a given theorem. While undergraduate students' comprehension and behaviors have been a focus of research in the reading and validation of proofs (e.g., Dawkins & Zazkis, 2021; Selden & Selden, 2003), this study aims to provide insights of how students' beliefs and knowledge about proofs might be associated with their reading and validation thereof. Building on the existing research (e.g., Dawkins & Roh, 2022), this study explores the accuracy with which students validate theorem-proof pairs, the reasons they offer for their decisions, and the similarities and differences exhibited by students with different degrees of proof experience. By studying similarities and differences between these groups, we address the following research question: What differences exist between novice and experienced provers in how they read proofs and characterize the relationship between proofs and theorems?

Theoretical Perspective

In this study, we employ the lens of radical constructivism (Glasersfeld, 1988). Under this view, knowledge does not objectively reflect reality, rather it is stored in the mind of an individual learner who has organized their activity and experience idiosyncratically into schemes. As such, we designed our investigations to understand our participants' schemes regarding proofs of theorems in order to build models for their thinking.

As a way of organizing and interpreting our findings about students' schemes regarding proof, we introduce the constructs of *beliefs about mathematical proof* and *mathematical knowledge base*, both of which may differ from student to student. The former refers to general notions that students hold regarding the practice of proving or the properties that a proof should have. For example, a student might believe that a proof needs to make explicit the structure of their proof e.g., direct proof, contrapositive, while another would accept a proof which only implies the structure. The latter refers to content-specific knowledge that students accept without justification. For example, a student might conceive that the sum of two continuous functions is continuous and accept a proof that used this argument as valid. Another student might reject a proof which doesn't justify this claim. In either case, such knowledge is only relevant in proofs that pertain to functions and their analysis.

Though ideas in one's mathematical knowledge base do not require justification, students were asked on multiple occasions to explain why certain ideas were true. To more fully describe their understanding, we rely on *warrants* (Toulmin, 1958), the reason a prover gives for why their evidence is germane to their argument. In particular, we use the *warrant-types* described by Inglis et al. (2007) to make sense of our participants' knowledge bases.

Research Methodology

As part of a larger study, we conducted clinical interviews (Clement, 2000) with undergraduate students with various levels of proof experience at a large public university in the United States from spring 2020 to spring 2022. We recruited eight students who had already taken at least two proof-oriented courses by spring 2020. We labeled these participants experienced provers. To compare and contrast these provers' conceptions about proof, in the springs of 2021 and 2022, we recruited four students who had not yet taken any proof-oriented mathematics courses at the university level, labeling them novice provers. The second author of this paper served as the interviewer of all participants while the remaining authors served as witnesses. In the discussion of results, we label participants with E or N (indicating their experienced or novice prover classification), a number from 1-8, and a pseudonym.

Each clinical interview lasted between 60 and 120 minutes. Some interviews in the spring of 2020 were conducted in person in a space other than their regular classroom while the rest of the interviews were conducted remotely. To facilitate retrospective analysis, we video- and audio-recorded all interviews. Participants completed all annotations on tablet computers, allowing us to collect digital copies of their work.

Interview Tasks

The tasks for the clinical interviews consisted of a series of theorem-proof pairs. After showing a theorem and at least one proof associated with it, we asked a student participant to think aloud while reading and interpreting them. When the student had indicated that they had sufficiently reviewed the theorem and proof, we asked whether the proof proves the theorem. If they determined that the proof did not prove the theorem, we asked if there were other statements that it proved. As the student responded to our questions, the interviewer asked follow-up questions in tandem to understand the student's reasoning for their decision.

While we asked all student participants the same questions for each theorem-proof pair, these pairs were not the same across the three all data collection periods. In spring 2020, we offered our experienced provers five different theorems (Theorems 1, 2, 4, 6, and 9 in Figure 1), each of which was accompanied by two or three different proofs. In spring 2021, we provided one theorem (labeled Theorem in Figure 1) to two of our novice provers and four associated proofs. In spring 2022, we presented the remaining novice provers four different theorems (Theorems $\alpha, \beta, \gamma,$ and δ in Figure 1), each with a single proof of its converse or contrapositive.

For all of the theorems in Figure 1 below, we provided students additional information such as relevant definitions or supporting theorems which may be needed for their reading of the proofs. We informed students that the proofs we provided were mathematically valid, but proofs associated with a theorem may not necessarily prove the theorem.

Theorems presented to experienced provers in Spring 2020:

Theorem 1: If x is a multiple of 6, then x is a multiple of 3.
(Associated proofs are direct, disproof of converse, and contraposition)

Theorem 2: If x is a multiple of 2 and a multiple of 7, then x is a multiple of 14.
(Associated proofs are direct and proof of converse)

Theorem 4: If $ABCD$ is a rhombus, then the diagonal AC forms two congruent isosceles triangles.
(Associated proofs are direct and disproof of converse)

Theorem 6: For any line segment AB, if a point X is on the perpendicular bisector of AB, then $AX = BX$.
(Associated proofs prove the converse and prove directly)

Theorem 9: If f and g are continuous on $[a, b]$, $f(a) = g(b)$, and $f(b) = g(a)$, then there is a c in $[a, b]$ such that $f(c) = g(c)$.
(Associated proofs are direct, disproof of converse, and contraposition)

Theorem presented to novice provers in Spring 2021:

Theorem: For any integer x, if x is not a multiple of 3, then $x^2 - 1$ is a multiple of 3.
(Associated proofs are direct, inverse, converse, and contrapositive)

Theorems presented to novice provers in Spring 2022:

Theorem α: Given a line segment AB, for all points X, if X is on the perpendicular bisector of AB, then $AX = BX$.
(Associated proof proves the converse)

Theorem β: For any triangle XYZ, if no two angles are congruent, then the triangle is scalene.
(Associated proof proves the contrapositive)

Theorem γ: For any integer x, if x is a multiple of 4 and a multiple of 21, then x is a multiple of 84.
(Associated proof proves the converse)

Theorem δ: For any integer x, if x is not a multiple of 3, then it cannot be written as the sum of three consecutive integers.
(Associated proof proves the contrapositive)

Figure 1. Theorems and types of proofs associated with them

Data Collection and Analysis

To facilitate our analysis, we transcribed each interview and created detailed field notes to describe how students processed each proof. We analyzed data in hopes of building a theory grounded in the available data (Strauss & Corbin, 1998). We first coded each line of each transcript by describing student behavior e.g., reviewing given definitions, drawing diagrams, deciding on the validity of a proof. We further coded the transcripts to attend to students' reasoning underlying their responses to questions, which revealed five different phenomena which we present in more detail shortly. These phenomena gave rise to two ways of categorizing students' conceptions – belief about mathematical proof and mathematical knowledge base.

Results

The goal for our research was to characterize the differences between how novice and experienced provers understood proofs, theorems, and the relationship between them. We begin by discussing their commonalities in order to provide a reference for their differences. We found that students' comprehension and validation of proofs are associated with their beliefs about mathematics proof and mathematical knowledge base. Various sub-categories of each construct emerged from our analysis. In particular, we found two different sub-categories of students' beliefs and three sub-categories of their knowledge. Once we identified all students with these categories, we compared and contrasted novice and experienced provers. Our findings are summarized below (See Table 1).

Table 1. Summary of conceptions about mathematical proof

Conception	Phenomenon	Novice	Experienced
Beliefs about Mathematical Proof	Valid proofs require logically sequenced arguments.	✓	✓
	Valid proofs require correct overall structure.	×	✓
Mathematical Knowledge Base	Arguments rely on empirical evidence.	✓	×
	Arguments rely on definitions.	×	✓
	Arguments rely on logically sound principles.	×	✓

Summary of Conceptions of Proof of Novice and Experienced Provers

Both groups of provers exhibited a belief that valid proofs require correctly sequenced arguments, yet only experienced provers believed that proofs must also follow the correct structure i.e., assumptions and conclusions are correctly identified. Regarding mathematical knowledge base, novice provers primarily argued using empirical evidence while experienced provers preferred arguments based on definitions. Lastly, experienced provers alone showed consistent sensitivity to logically sound principles.

Beliefs about Mathematical Proof

This category pertains to what students generally believe a prover should do when formulating a proof or what properties a proof ought to include. Our findings refer specifically to the characteristics that students believe contribute to the validity of a proof, or lack thereof.

Valid proofs require logically sequenced arguments. A logically sequenced argument is such that each line in a proof is both justified by the ones that precede it and justifies the ones that follow. Put alternately, students attended to the body of the proof without necessarily attending to the assumptions and conclusions. In each of the following excerpts, one each from a novice and experienced prover, participants discussed why this coherent flow is necessary in a valid proof.

Interviewer: "Can you explain why [this proof doesn't prove the theorem]?"

Priya (E4): "Because they're not justifying their steps. When they don't justify their steps, the steps they've omitted don't indicate that they understand what's going on. It just seems like they're fudging because they know where they need to go."

Interviewer: "Can you explain why the proof proves the theorem?"

Carl (N2): "If x wasn't on the perpendicular bisector…Therefore, the lines AX and BX would not be equal. Because it is on the perpendicular bisector, triangles AMX and BMX would be equal. By the SAS theorem, the triangles would be equal."

Priya referred to the desired conclusion as "where they need to go," thereby acknowledging a proof framework, but rejected the arguments used to arrive there. Carl accepted his proof based solely on the links between arguments and did not attend to the hypothesis and conclusion.

Valid proofs require correct overall structure. If the category above pertains to the body of a proof, this category pertains to its head and tail, the assumptions and conclusions, respectively. The belief that a proof's validity depends on its hypotheses and conclusions, and the flow from the former to the latter, was prevalent only among experienced provers.

Interviewer: "Can you explain why [the proof doesn't prove the theorem]?"

Nate (E6): "In proof 1.2, your given assumption is actually what we're trying to prove…This statement is what we're needing at the end of our proof…So they're starting at the opposite end of the proof…This is saying that if x is a multiple of 3, then it's not a multiple of 6, which is not what we're actually trying to prove."

Interviewer: "What is your top criterion for this to be a valid proof?"

Heather (E2): "It starts by assuming that the if condition is true."

Nate and Heather each attended to how the proof began and ended. Nate rejected the proof because it assumed the wrong premise. Heather cited the correct assumption as her top validation condition. In each case, the student attended to the ends of the proof rather than the body thereof.

Mathematical Knowledge Base

A student's knowledge base refers to the information that they have at their disposal which helps them read, interpret, and formulate proofs. Most relevant to this study is the set of content-specific tools that students have which allows them to analyze and compare arguments.

Arguments rely on empirical evidence. Under this approach, students cited particular examples to substantiate their claims. On several occasions, novice provers used one or more particular examples directly before declaring that a theorem was indeed valid.

Interviewer: "Can you explain why [proof 1 proves for any integer x, if x is not a multiple of 3, then $x^2 - 1$ is a multiple of 3]?"

Joaquin (N4): "I didn't realize that the proof would approach the problem like this…It says let x be an integer that is not a multiple of 3…We could pick 8, 7 even…For me personally, I experimented with some numbers. For example, I let k equal 1."

Joaquin's acceptance of the theorem stemmed from his ability to satisfy it with several spontaneously chosen examples. Though he convinced himself of the validity of the proof inductively, we do not necessarily claim that he would have accepted a proof by example. Nonetheless, his empirical reasoning was fairly common among novice provers.

Arguments rely on definitions. Rather than using particular examples, provers in this category reasoned arbitrarily i.e., using examples which represent all examples. Put alternately, experienced provers reasoned from definitions and properties rather than from examples.

Interviewer: "Can you explain in your own words what this theorem states?"

Priya (E4): "Given any integer x, if x satisfies the property of being a multiple of 6, meaning there is some number that multiplied by 6 gives you x,…There is another number that when multiplied by 3 gives you x."

Whereas Joaquin reasoned via empirical evidence, Priya reasoned arbitrarily and directly from the definitions. Joaquin's and Priya's preferred modes of reasoning were common among other novice and experienced provers, respectively.

Arguments rely on logically sound principles. Provers in this category were adept at employing logic, most notably for this study contrapositive equivalence and converse independence (CE/CI). Novice provers did not consistently exhibit understanding of CE/CI. Nevertheless, the manners in which experienced provers justified these principles varied greatly.

Interviewer: "Can you explain why [this proves if x is not a multiple of 3, then $x^2 - 1$ is a multiple of 3]?" (proof proves the converse)

Violet (N3): "They're showing that x is not a multiple of 3 by saying that x equals k times 3…But it can't be since in the theorem it says that x is not a multiple of 3…I think it [proves the theorem] because they're showing in their work that x is a multiple of 3, because they're assuming that it's a multiple of 3."

Note that Violet attended to the arguments in the body of the proof but exhibited no sensitivity to the overall structure of the proof.

Experienced provers, on the whole, reliably recognized CE/CI. Significant differences however existed in the way they justify these ideas. For example, some participants took CE/CI as given, but did not provide a justification.

Interviewer: "Why do you think that since this disproves the converse that it does not prove the theorem?"

Priya (E4): "Because the converse is not logically equivalent to the original."

I: "What do you mean that they are not equivalent?"

Priya (E4): "That's a good question. Like how do I know that two things aren't logically equivalent? I guess at this point, that's just an inherent fact to me."

Provers who reasoned about CE/CI in this fashion perhaps viewed CE/CI as a belief rather than knowledge since it is neither requires nor is accompanied by warrant.

Other experienced provers were able to warrant CE/CI with concrete examples which were specific to a particular context. These contexts were not always overtly mathematical in nature, as shown by the excerpt below.

Interviewer: "You said this proves if Q, then P, right? My question is why a proof of if Q then P is not a proof of if P then Q."

Mark (E7): "Let's say I say that if an animal is a blue jay, then it is a bird…This is the example I always think of when I have to think of if-then statements."

Though not explicitly stated by the student, it can be reasonably presumed that since the statement he gave had a false converse, its purpose is to illustrate general converse independence through a particular example. Note that while empirical evidence was primarily used by experienced provers, this was not exclusively the case.

Finally, our participants also justified their knowledge of CE/CI through abstract warrants which were not beholden to any particular contexts. Such justifications most often took the form of truth tables, subset relationships, and logical manipulations (see Figure 2).

Mark (E7): "It works from logic that for an implication to be true, either the hypothesis is false or the conclusion is true. Since they have the same truth table, we know that the statements are going to be equivalent." (see figure 2, left image)

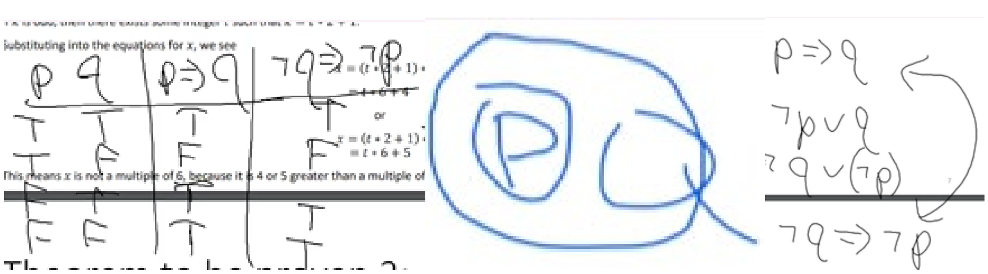

Figure 2: Mark's truth table and Euler diagram as well as Nate's syntactic argument

Interviewer: Can you explain why the proof demonstrates [if Q, then not P]?"

Mark (E7): "So it could be the case that x is a multiple of 3 and not a multiple of 6. It could be the case the Q and we don't know anything about P. So, what the theorem says is that if I am anything in P, then I will also be in Q. But what this shows is that if I'm in Q, then I might not be in P." (see Figure 2, middle image)

Interviewer: "How can you tell that the proof of the contrapositive also proves the theorem?"
Nate (E6): "I think the easiest way would be through logic. The statement is P implies Q. That's the same is not P or Q. Then, if we do double negation, we get not Q implies not P. So, these two are exactly the same." (see Figure 2, right image)

Discussion and Conclusion

The goals of this study were to characterize the ways in which undergraduate students interpret proof-texts, their relationships to underlying theorems, and to describe the differences between novice and experienced provers. Though our tasks were designed to gauge reading comprehension through student behaviors, we learned much about their conceptions of proof, suggesting that the phenomena are related.

With regard to beliefs about mathematical proof, both groups of provers asserted that a logical linking of ideas should be present in a proof. Our research is thereby consistent with prior literature (e.g., Ko & Knuth, 2013; Selden & Selden, 2003; Dawkins & Zazkis, 2021). Similarly, we found that students with more mathematical development were more likely to attend to the assumptions that are made at the outset of the proof and the overall structure, which is also consistent with prior studies (e.g., Heinze & Reiss, 2003; Weber, 2008). Indeed, experienced provers validated proofs correctly more often than novice ones, though experienced provers make occasional errors. Our findings in this regard support the work of Inglis and Alcock (2012).

The results of our study also highlight the different ways in which our participants justify the ideas of contrapositive equivalence and converse independence. Our findings are consistent with prior literature. In their discussion of modeling arguments, Inglis et al. (2007) discuss the warrants used by graduate students of number theory. Though their participants were more mathematically developed than ours, parallels exist between our findings and theirs. Inglis et al. (2007) do not discuss participants who offer no warrant, but our provers who readily asserted the ideas of CE/CI but could give no reason for their validity exhibit what Krupnik et al. (2018) call *psychological knowledge*, a belief that an idea is true which the knower cannot justify. Our participants who warranted CE/CI with a single example e.g., the blue jay, parallel what Inglis et al. (2007) call the inductive warrant-type, wherein a prover evaluates a conjecture using one or more specific examples. Inglis et al. (2007) also describe a structural-intuitive warrant-type, wherein a prover uses a mental or visual structure to support a conjecture. This is consistent with our experienced provers who used set-theoretic notions and Euler diagrams to justify CE/CI. Inglis et al. (2007) describe the deductive warrant-type as reasoning solely from axioms. Since our provers who used truth tables and logical manipulations were relying on the base relationships between propositions in an implication, their reasoning was consistent with this warrant-type.

Our findings suggest that students with formal training in proof validate proofs with greater reliability but greater attention may be paid to the justification for CE/CI. Ongoing studies are testing instructional interventions using set-theoretic activities to effect deeper conceptual understanding of proof structures (Dawkins et al., in preparation).

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“I want this to be the last time. I don't want to teach myself anymore”: Relearning Factorization from Middle school through College

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We report results of a longitudinal case study of one Intermediate Algebra student Sierra’s experiences relearning factorization from 8th grade through Precalculus. This data comes from a larger project investigating six developmental mathematics students’ perceptions of the experience of relearning individual topics and reflections on relearning across two semesters of college algebra courses. Such a student-centered perspective is critical to developmental mathematics educators who are currently grappling with high rates of failure and attrition in such courses nation-wide. In examining Sierra’s learning experience as one of relearning, this report sheds light on how students may use their previous algebra learning experiences to make predictions about the kind of understanding they will come to have about particular topics, and to motivate particular behaviors and affective states during instruction. Furthermore, we illustrate how such topic-level experiences may be used to shape learning experiences in subsequent mathematics courses involving algebra.

Keywords: developmental mathematics, relearning, factoring

This report centers on student perceptions of the experience of relearning, i.e., the experience of learning about content one has already tried to study in a previous course (Amman & Mejía-Ramos, 2022). In particular, we focus on traditional developmental math courses: semester-long courses whose content focus mirrors that of middle and high school pre-algebra to algebra courses. Such courses persist in making up a significant portion of developmental math offerings (Kosiewicz, Ngo & Fong, 2016) and lie at the heart of scrutiny in the developmental math space. We argue that understanding students’ expectations and perceptions surrounding the relearning process can help explain their motivations, behaviors, and affective dispositions while learning in courses with previously seen content (Amman & Mejía-Ramos, 2021). While relearning is not exclusive to developmental mathematics courses, it is a uniquely central feature of these courses (due mainly to the prevalence of Algebra in United States middle/high school curricula). This work contributes to a small but growing body of literature that seeks to open up the “black box” of teaching and learning in developmental mathematics courses (Grubb, 2001; Sitomer et al., 2012; Mesa, Wladis & Watkins, 2014) to better understand their historically high failure rates (Chen, 2016) and reputation as deterrents to college completion (Bailey, Jeong, & Cho, 2010).

In a previous report (Amman & Mejía-Ramos, 2021) we discussed students’ perceptions of the experience relearning an Intermediate Algebra topic that they had seen in a previous algebra course. Although the topics studied were presented within the first third of the semester, students described having complex relearning experiences. Contrary to current depictions of developmental mathematics students, which collapse their learning needs across individual topics to describe them as needing either a “refresher” of course material or an intensive review of “basic skills” (e.g. Grubb & Gabriner, 2013; Cox & Dougherty; 2019), these students’ experiences shifted according to both the sub-topic under discussion and whether the interview was predictive or retrospective. The data presented in the current report come from a larger study motivated by the question of how such relearning experiences might evolve over the course of a semester, as students are exposed to topics of varying complexity, and as their familiarity with the notion of relearning grows. Specifically, we report findings related to one student's relearning experiences throughout two semesters of algebra-based, college-level coursework. In doing so, we seek to contribute to educational theory concerning the variety of possible impacts of relearning on one's understanding of content seen before, as well as the behavioral and affective associations with relearned content that distinguish it from learning for the first time.

Literature Review

Typical profiles of developmental mathematics students describe them as having a lack of sufficient algebra coursework in high school (e.g. Kinney, 2001; Grubb & Gabriner, 2013; Boylan, 2011). However, current research suggests that many developmental mathematics students completed at least Algebra II in high school, and that a substantial portion were enrolled in mathematics courses for all four years in high school (Benken et al., 2015; Ngo, 2020; Moran, 2008). For instance, in their study of 306 first-year students placed into Intermediate Algebra at California State University, Benken et al., (2015) found that nearly 60% of these students had taken courses beyond Algebra II in high school. Additionally, 66% of participants had taken mathematics all 4 years of high school, and 23% had spent 3 or 4 of those years just attempting to pass coursework at or before Algebra II. Likewise, Ngo (2020) investigated the percentage of college students who take "redundant" math courses, or courses whose content is either at the same or lower level than their highest completed math course in high school (or lower than what they would be predicted to pass given their results on a 12th grade math assessment). He found that while roughly 20% of all college students take redundant math courses, that percentage increases to roughly 40% when looking at developmental math students specifically. This suggests that roughly 40% of students in these developmental mathematics courses had completed coursework above the level of at least Algebra I in high school. Thus, counter to the belief that a sizeable number of developmental math students enroll in such courses because they were never exposed to algebra concepts in high school, these studies suggest that a sizable number of students have taken algebra prior to developmental mathematics enrollment. While these studies do not make explanatory claims to help us understand student outcomes, they provide valuable insights into the widespread nature of relearning in developmental math classes. In attempting to explain phenomenon such as high failure rates and attrition, it would seem critical to better understand student relearning experiences in this context.

In contrast to the lack of research on student perceptions of their experience with developmental mathematics content, there is a growing number of studies that focus on describing how developmental mathematics instructors view students in terms of their readiness for college mathematics and the detriments to their success mathematically. These studies suggest that instructors perceive many developmental math students as lacking in basic skills that are both meant to be formed in high school courses and critical for mathematical success (Er, 2018; Reichwein, Schneider & Onwuegbuzie, 2014; Grubb & Gabriner, 2013). For instance, in their interviews with instructors from over 20 community colleges in California, Grubb and Gabriner (2013) found that instructors perceived the majority of developmental math students as being placed into such a course due to a need for an in-depth review of basic skills, whereas a minority of "brush up" students are misplaced due to a time lapse since their previous mathematics course and a need for only a quick review of the material in order to be college ready. However, instructors surveyed by Benken et al., (2015) thought it was problematic to approach the course as a "review" rather than an "opportunity to learn new content", assuming such an approach stems from an overconfidence in students' skills as opposed to a true need for a quick review. Furthermore, there appears to be inconsistencies in which skills specifically instructors are referring to. Whereas some studies of instructor perceptions relate a lack of basic skills toa likelihood to engage in unproductive learning behaviors such as self-handicapping strategies (Mesa, 2012), others note poor study skills (Reichwein, Schneider & Onwuegbuzie, 2014), while still others note basic mathematical skills related to arithmetic and algebraic procedures (Grubb & Gabriner, 2013).

An interesting question is how instructors' perceptions relate to those of their students. For instance, Cox and Dougherty's (2019) interviews with 25 community college pre-algebra students indicated that while many students perceived the course to be useful in "refreshing" their memory, others found 'reviewing' to be a waste of time given their prior experience with the content. Consistent with findings reported by Mesa (2012), 18 students indicated that they "wanted to 'really understand'" the reasoning behind their use of procedures in the course in way that they had not accomplished previously (p. 256). However, it was relatively uncommon for students to indicate that they felt this had been accomplished by the end of the course. Indeed, a critical finding of Mesa (2012) was the vast discrepancy she found between instructor perceptions of students and students' perceptions of themselves. Namely, she found that students indicated preferring mastery over performance goals, a lack of engagement in self-handicapping behaviors, a disposition towards hard work, and an expectation that instructors would challenge them. As illustrated above, instructors perceived students in a near complete opposite manner. The results of these studies raise interesting questions about how student perceptions of their own learning needs, behaviors, and affective dispositions relate to those of their instructors. These results, as well as the muddling of "refresher" students and "basic skills" students led us to conduct the current investigation into student perceptions of their experience learning about content seen before.

Theoretical Framework

Consistent with Amman and Mejía-Ramos (2021; 2022), we conceptualize the experience of relearning as broader in scope compared to previous work on memory in cognitive psychology (Ebbinghaus, 1885; Bahrick, 1979; Rawson, Dunlosky & Janes, 2020) and mathematics teacher education (Zazkis, 2011). We define relearning in mathematics as the experience of learning about content (T1) one has already tried to study in a previous mathematics course (T2). Importantly, the target content must remain stable across the two learning experiences. That is, the mathematical content in the second learning experience cannot be simply reviewed or relevant for the purpose of teaching something else. In such scenarios, we typically consider students to be using their *prior knowledge* in service of learning about new content. The definition of relearning used here is broader in the sense that it does not require specific learning outcomes at T1 or T2 (e.g., work on memory in cognitive psychology traditionally requires content to be fully memorized and recalled in order for relearning to occur). Similarly, this definition does not characterize specific relearning experiences as successful or unsuccessful. Instead, it is deliberately general to allow researchers to more broadly describe how students' existing understanding of content can be (or be perceived to be) impacted by relearning, without a predetermined framing in mind.

In developmental mathematics courses, relearning occurs at multiple levels. The results presented here consider two such levels. The first is what we refer to as "topic-level" relearning in which the focus is on a student's experience relearning a single topic (e.g., the content of a single chapter or unit) in an algebra course. Topic-level relearning was the sole focus of our earlier work (Amman & Mejía-Ramos, 2021), in which we asked questions about what it was like for students to relearn content from one of two specific textbook chapters in an Intermediate

Algebra course. In the results of that study, we provided preliminary descriptions of student perceptions of the impact of relearning on their understanding of a particular topic, such as *confirming my understanding, jogging my memory,* and *reconstructing my memory with guidance*. These perceived impacts of relearning were used to guide the current investigation while still allowing for such descriptions to be expanded or refined (see Amman, 2022 for details). The second level pertains to "course-level" relearning. Course-level relearning experiences focus on holistic perceptions of what learning is like overall in a course, and is the perspective more commonly taken by studies that ask about student perceptions of their course experience, without focusing on any particular piece of content (e.g., students got a "refresher" of the subject of algebra in Cox & Dougherty, 2019). However, as demonstrated by the current study, course-level relearning perceptions are not necessarily simple aggregates of topic-level experiences formed at the end of a course, but may also already be formed by students early on in the semester and may help explain subsequent patterns in topic-level experiences. In the current study, we present findings that describe the relationship between one student's topic-level and course-level perceptions of relearning as she continued to encounter content involving the sub-topic of factorization throughout a developmental math course.

Methods

We conducted six longitudinal case studies each spanning two semesters between Spring 2021-Spring 2022 enrolled in either Elementary or Intermediate Algebra at a four-year university. Elementary and Intermediate Algebra are non-credit-bearing courses whose syllabi reflect those of a typical high school algebra course. Sierra, the pseudonym given to the participant whose data is the focus of this report, began the study as a freshman Intermediate Algebra student in Fall 2021 followed by Precalculus in Spring 2022. All participants were recruited at the beginning of the semester via email for a series of paid interviews focused on their experiences with relearning in their current math course and in the following semester. Interested students were given a recruitment survey in which they were shown topics that would be taught in their current course and asked to describe how well they remembered and how confident they were in their understanding of them. Participants were chosen to reflect an even distribution of starting course, memory, confidence in understanding of each topic, and semesters of experience with college algebra courses. Sierra, like the other participants, indicated that she knew she had learned *about* each course topic before, even if she could not always remember exactly what she had learned, and she had previously completed Algebra II, and Precalculus in high school. This indicated that much of her upcoming learning experience was likely to be characterized as relearning. Data collection consisted of six one-hour interviews per student, surveys and, when possible, responses to questions on course exams. For the sake of space, we describe only interviews 1-4, those focused on topic and course-level relearning, for this report.

Interviews 1-4 occurred in the first semester of study enrollment. In order to understand how previous learning experiences influenced students' relearning experiences, each interview contained sections for students to both predict and reflect on aspects of upcoming and recent relearning experiences, respectively. Specifically, students were asked to predict how their memory and current understanding of a topic would influence how they approached the task of relearning that topic in class (behaviorally and affectively) and what the anticipated impact of that relearning experience would be on their understanding. In terms of reflections, students utilized their own exam responses and their predictions from previous interviews to reflect on their relearning experience. At the end of every interview, we asked students to reflect on their course-level relearning experience by describing, overall, how they thought the experience of relearning had been different from learning new content in mathematics for the first time. Responses to this item were carried over to subsequent interviews for further reflection in order to understand students' evolving perception of relearning throughout the semester.

Formal analysis proceeded in multiple rounds. The first round focused on building descriptions of individual cases utilizing the main variables outlined in the theoretical framework. They are: perceptions of similarity to and confidence in understanding from the T1 learning experience, predictions and reflections of the impact of relearning on understanding, behaviors associated with relearning, and affective dispositions associated with relearning. The second stage involved cross-case comparisons resulting in preliminary themes that either connected variables within a type of relearning or helped explain linkages between variables across different types of relearning. This process involved engaging in explanation-building utilizing all data sources for an individual participant, then moving to the next participant in order to locate commonalities across experiences and produce evidence of rival explanations if that participant experienced a similar outcome despite different combinations of the study variables (Yin, 2009). The result of this stage of analysis was a set of preliminary themes that were refined over a period of several further rounds of analysis for consistency.

Results

How Relearning at the Topic-level Influences Behavior and Affect

Sierra came into her first interview with a vivid memory of a frustrating history with learning (and relearning) how to factor polynomial expressions. She recalled missing the first day of math class in 8th grade when factorization techniques were introduced. To quickly catch up, Sierra went online to "whatever was available" to teach herself what she had missed. While she performed well in Algebra I, her technique for learning about factorization did not allow the content to "stick"; she forgot everything when she reencountered the topic in her high school Algebra II class. Once again, Sierra felt she needed to catch up to the rest of the class and turned to outside sources to re-teach herself the factorization techniques that she was expected to already have learned. She then promptly forgot these techniques and relearned them again in her high school Precalculus course. Sierra was looking forward to using her time in Intermediate Algebra to end this cycle. In her first interview, she stated, "I want this to be the last time. I don't want to teach myself anymore." Sierra consistently received good grades in high school and was a member of an honors program at the University—she was confident that her main concern was not whether she could reteach herself factorization and perform well on assignments, but on her inability to recall any of what she had taught (and retaught) herself concerning this subtopic in the past. Specifically, she identified her behavior of inconsistently turning to various sources outside of the class each year as resulting in her struggle to remember what she had learned the following year. She predicted that by closely adhering to her professor's approach to factoring, she would be able to reorganize her knowledge in a way that would make it easier to remember in the future (Amman, 2022, termed this particular relearning outcome *Reorganization*).

Unfortunately, Sierra's anticipation of this change in behavior and subsequent change in understanding was not realized. Sierra found that she *remained stuck in her previous understanding* (Amman, 2022). In her second interview, Sierra reviewed her responses to items on the first exam involving factoring, almost all of which she received full credit for. Despite her high performance, Sierra lamented that she found herself stuck in her old relearning approach in which, "I just had to learn it [myself] again, like every time when it comes to factoring." Just as before, she found herself turning to outside sources to reteach herself, and once again did not feel she had addressed the root cause of her memory issues. Indeed, when encountering future problems involving factorization in class, she remarked in later interviews, “[my memory] wasn’t really helpful...I kind of knew I was on the right track because I remembered seeing it, but it didn't help me in terms of solving the actual problem.” As a result, Sierra found that her understanding of factoring after relearning was, “the same. I just get there faster [every time I learn it].” Sierra explained that she was able to pick up on factoring techniques more quickly than in previous algebra courses, but this was not accompanied by a feeling of improved understanding that could have come from the implementation of a more conceptual organizing principle of content that she had hoped for. When asked what had made her feel like she needed to use her old relearning approach again, she stated, “I think it’s more to like a mental thing. Like once I see the word factoring, I'm like, ‘oh I know I'm not going to do well on this unless I teach myself again.’” Here, we see that Sierra’s previous experiences with factoring triggered a prediction (“I’m not going to do well”) concerning the outcome of her topic-level relearning experience before her instructor had even begun the first lesson on factoring. This prediction was then used to motivate behaviors (“unless I teach myself again”) that impacted how she decided to approach studying this topic. Sierra’s previous experience learning this material also had an impact on her affective response to relearning it. She mentioned feeling discouraged from the moment the class had begun the factoring sub-unit and struggled with these negative feelings associated to her previous experiences throughout the rest of the classes in which factoring was mentioned. For instance, while reviewing for her first exam, she questioned, “why am I still doing this in college? That's what I was asking myself a lot while I was studying…I feel like every time I revisit this topic it's a little bit discouraging. Because it's not sticking and I don't know why.” In this way, Sierra’s ability to recognize previously learned material and make predictions surrounding the outcome of relearning had significant consequences for her (and other participants) in terms of her behaviors and affective states during course instruction.

How Topic-Level Relearning may Shape Perceptions of Course-Level Relearning

Sierra’s experience relearning factorization also came to impact her course-level relearning perceptions. In contrast to the idea expressed by other participants that previous experience must make relearning easier than learning something new, Sierra came to view her previous algebra learning experiences (in middle and high school) as a potential hindrance in the course. Discussions around this perception began in her third interview, after polynomials (and the factoring sub-topic) had been covered and she had experienced the shift from predicting that she would be able to reorganize her knowledge of factorization to subsequently reflecting she had remained stuck in her previous understanding of the topic. At that point in the semester, Sierra described her overall outlook on relearning in the course as, “I feel like as opposed to other subjects, I feel like math is all about confidence…If I don't have that confidence and I'm confident that I don't remember it, past [Sierra] didn't know how to do it, or couldn't figure out how to do it, then it’s gonna be a doozy trying to figure out how to solve it.” She explained in her fourth interview that her confidence throughout the semester came to be tied to her understanding of factoring, given that subsequent course topics such as combining rational expressions and solving quadratic equations involved factoring as a subtopic. In interview 4 she described changes in her confidence throughout the semester by saying “I was like ‘um what are doing?’ [every time] factoring came back in,” indicating that her confidence would dip down repeatedly every time factoring was reintroduced in a new topic. In essence, Sierra’s experience relearning about factorization seemed to have influenced her perspective of what to expect for the course as a whole: she saw an inability to use what she had previously learned as a warning sign that her relearning experience was about to be difficult and did not allow for the possibility of an improved experience relative to her first time learning these topics. In summary, Sierra's topic-level experience came to inform her overall outlook on relearning in the course, transforming her from a student who in Interview 1 was "looking forward to" intellectually grappling with questions like "Why am I getting this answer? Why do I have to do this step?" to a student who came to view the presence of such questions as hits to her confidence and indicators that her relearning experience would be more difficult than it should.

Sierra enrolled in Precalculus the following semester, another course which she had previously taken in high school. Although she was unable to achieve the impact of relearning on her understanding of factorization that she had predicted, she mentioned hoping that the combination of the short break between Fall and Spring semesters and the increased speed with which she arrived at her understanding of factoring would improve her results next semester. Unfortunately, this was not the case. In her fifth interview she noted that she had done poorly on her first exam, and that, "the ones that I struggled on, it was factoring, of course. The whole entire test was factoring." In this way, Sierra's topic-level experience could be understood as part of a self-reinforcing cycle that comes to inform her experiences with relearning not only within her algebra course, but also in future courses involving said algebraic topics.

Conclusions

By presenting Sierra's case, we intend to contribute to theoretical accounts of the experience of relearning as instantiated in developmental algebra courses. Studies in reforms of developmental math courses have begun to suggest that the most successful reforms are multifaceted in terms of the modifications they make to course experiences (e.g. Hodara, Jaggars & Karp, 2012; Yamada & Bryk, 2016). That is, successful reforms address placement into courses, instruction while students are enrolled in such courses, and alignment with students' overall goals for college after the course is over. Similarly, the results of this study suggest that fully understanding student experiences with relearning in developmental mathematics requires a multifaceted approach. Whereas topic-level perceptions help us to understand the decisions Sierra was making and the impact of the course on her understanding of factoring on a local level, her course-level perceptions allow us to understand the motivations behind patterns in her experiences that were more than the sum of her topic-level perceptions in terms of their impacts beyond one semester. We would argue that in our efforts to better evaluate developmental mathematics courses and understand exactly how they impacting students, we need to be aware of the limitations of taking the perspective of students at any one "level", and ensure that the claims we want to make about student experiences are aligned with this. For instance, co-requisite algebra courses are designed to improve alignment with students' overall goals for college and modify the algebraic content presented in class accordingly (e.g. Logue, Douglas & Watanabe-Rose, 2019). However, there are very few studies that seek to investigate the hypothesis that student *understanding* of algebra in these courses will be improved as a result of an ability to see more directly the purpose of relearning particular algebra concepts related to students' majors. This suggests we would do well to focus on student perceptions of relearning at both the topic- and course-level in these courses: how students view the relearning of specific topics as understanding of these topics is needed in the context of the college-level course, and how they view relearning of algebra as a whole in the special context of a co-requisite course. We suggest that future work seeking to qualify the impact of such courses on student understanding takes into account the particular components of relearning focused on here in order to be able to best explain student approaches to and reflections on course material.

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Explanations and Justifications Regarding Converse Independence

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This paper presents six categories of undergraduate student explanations and justifications regarding the question of whether a converse proof proves a conditional theorem. Two categories of explanation led students to judge that converse proofs cannot so prove, which is the normative interpretation. These judgments depended upon students spontaneously seeking uniform rules of proving across various theorems or assigning a direction to the theorems and proof. The other four categories of explanation led students to affirm that converse proofs prove. We emphasize the rationality of these non-normative explanations to suggest the need for further work to understand how we can help students understand the normative rules of logic.

Keywords: logic, proof, converse independence

Introduction

Proof-based mathematics education research has long attended to how students think about the relationship between mathematical proof and the truth of mathematical claims. Within mathematical practice, deductive proof is the dominant standard for declaring a theorem true. Similarly, we would like students learning proof-based mathematics to adopt normative standards for what kinds of proofs justify a given mathematical claim. In this paper, we specifically consider theorems that are universally quantified conditionals, meaning they can be expressed in the form "For all… , if P, then Q." Since many theorems in undergraduate mathematics are of this form, introduction to proof texts introduce standard proof techniques related to such statements: direct proof, converse proof, contrapositive proof, and proof by contradiction. Of these four types of proof, all prove the given theorem except for the second. A converse proof assumes Q is true and concludes why P must be true as a consequence. We call the principle that the converse proof never proves the given theorem *Converse Independence* (hereafter, "CI"). In common mathematical parlance, the theorem states that P implies Q and the converse proof proves that Q implies P, which are taken to be related, but independent claims.

Previous studies (e.g., Hoyles & Kuchemann, 2002; Yu, et al., 2004) have noted that students often strongly associate a conditional and its converse statement. It has also been suggested that making a distinction between a statement and its converse may be even more difficult when the statement and its converse are both true (Imamoglu & Togrol, 2015), though no explanation was provided for how students should understand CI in this case. CI is meaningfully different for theorems where the statement and converse are both true. Durand-Guerrier (2003) and Dawkins (2019) both discussed how students interpret mathematical statements to refer to the objects that make them true. This referential way of reasoning emphasizes a challenge: how might students distinguish converses that are not distinguished by their truth-values? Our study investigates the question: how do students' reason about the relationship between a converse proof and a conditional theorem, especially in cases where the theorem is true biconditionally?

We answer this research question by characterizing novice undergraduate students' explanations and justifications regarding the relationship between a converse proof and a conditional theorem. These findings help elaborate Hoyles & Kuchemann's (2002) findings of how students might interpret the relationship between converse statements. In other words, our study offers qualitative insights into the previously documented ways students closely associate converse claims. We shall emphasize the coherence and rationality of students' non-normative explanations, which suggest a need for greater attention to this topic in introductory logic.

Literature Review

There are three primary task types that have been used to study how students relate a conditional statement and its converse: inference tasks, statement tasks, and proof tasks. First, some studies consider the inferences that students make or endorse based on a conditional claim. The statement "If P, then Q" is taken to justify inferring Q given knowledge that P is true (often called *modus ponens*). However, according to mathematical logic the conditional statement above does not justify inferring P given knowledge of Q (*affirmation of the consequent*), though students often make this inference or endorse it as appropriate (Evans & Over, 2004; Inglis & Simpson, 2008; Alcock, et al., 2014). Attridge and Inglis (2013) compared inference task performance between UK secondary students studying mathematics and their peers studying English literature (and not mathematics). The year of mathematics study reduced the frequency of *affirmation of the consequent* responses more than the year of English study did, suggesting that general mathematics instruction influences this aspect of CI.

A second type of research task relevant to CI invites students to make judgments about conditional statements and their converse statements. For instance, some scholars (e.g., Hoyles and Kuchemann, 2002; Yu, et al., 2004) have shown that many middle school students believe that conditional statements and their converses say the same thing. Both studies explained this finding in light of the way that students interpreted the truth/falsehood of the statements. Many students affirmed and denied statements using affirming examples (which were provided in the task statement) or counterexamples (which the task did not provide). Accordingly, many students judged the false conditional claim to be true based on affirming examples, so the truth-value did not distinguish the statements.

Introduction to Proof textbooks generally justify CI either using 1) the truth-table or 2) example statements. The former argument (converses have different truth tables) assumes that students are using the truth-table definition to interpret conditional claims, though this has been long shown a poor model of how people interpret such statements (e.g., Evans & Over, 2004; Schroyens, 2010; Inglis, 2006). The second textbook argument is worth considering in more detail. Hammack's (2013) Introduction to Proof textbook provides an instance of this type of argument. It presents one example of a true conditional claim with false converse ("a is a multiple of 6 $\Longrightarrow$ a is divisible by 2") and explains, "Therefore the meanings of $P \Longrightarrow Q$ and $Q \Longrightarrow P$ are in general quite different… a conditional statement and its converse express entirely different things" (p. 44). This justification relies on an implicit meta-theorem about the nature of logic, which we shall call the *Fundamental Warrant of Logic (FWL)*. It states an argument of a given form is only valid if all examples of this argument with true premises have true conclusions. Thus, a single example of some form of argument that yields false results invalidates all other arguments of that same form. In this case, the argument in question would be "If (P⟹Q), then (Q⟹P)." The FWL asserts that the relationship between a statement and its converse must be the same for all conditional statements. The extent to which students agree with this justification or find it compelling is an open question that our study will begin to address.

A third type of task used in research invites students to consider the relationship between a conditional theorem and a converse proof. This is a novel type of task used in the sequence of experiments in which our data originated (see Dawkins & Cook, 2017; Dawkins & Roh, 2022), which we shall describe in the methods section.

Theoretical Framing

Our study draws upon the tradition of Piagetian constructivism both in theoretical orientation (e.g., von Glasersfeld, 1995) and methods (Steffe & Thompson, 2000). Piaget was fond of using logic to model children's reasoning (e.g., Inhelder & Piaget, 1958, 1964). However, the nature of his claims have been widely (mis)understood as claiming that adolescent reasoning comes to conform to formal logic in some strong sense (thought is the mirror of logic). This would violate Piaget's principle of the constructive independence of knowledge. We understand that Piaget used logic as a convenient organizing tool to describe patterns of reasoning (logic was the mirror of thought) that for the student were embedded in more complex systems of meaning (Piaget & Garcia, 2011). Only upon conscious reflection on the form of statements and reasoning can someone abstract conscious understandings of logic (see Beth & Piaget, 1966). A helpful distinction here is between what students construct *in activity* and what they have *reflexively abstracted* to some higher level of re-presentation (von Glasersfeld, 1995). In this study, we do not assume that students have stable re-presentations of logical structure or the abstract relationships between a statement and its converse. Instead, we seek to describe their often tenuous and provisional ways of reasoning about CI that may shift based upon the context of the statements and the task (what Thompson, et al., 2013, called *in the moment meanings*). As such, we present categories of explanations and justifications that students give in particular moments, which are neither mutually exclusive nor necessarily stable throughout their participation in our study. In other words, we seek to understand the subjective rationality of how students begin to reason about such logical relationships.

Methods

The data from this study comes from a larger series of constructivist teaching experiments (Steffe & Thompson, 2000) with undergraduate students from three large public universities in the United States. A total of six teaching experiments were conducted with pairs of students over the course of five years, resulting in a total of 12 participants. None of the students had previously taken a proof course. A key goal of the experiment was to support them in constructing more normative understandings.

The goal of this paper is to characterize students' explanations and justifications regarding whether a converse proof proves a conditional theorem. These explanations and justifications arose in response to What does it prove? tasks. These tasks involve presenting students with a theorem and an associated proof, and students are asked to decide, "Does the proof prove the theorem? Why or why not?" as well as "If the proof does not prove the theorem, what statement does it prove or disprove?" This invites students to attend to the structural relationship between the theorem and the proof and to begin comparing such relationships across theorem/proof pairs. In this paper, we focused on analyzing data from three What does it prove? tasks involving converse proofs (or disproofs). Table 1 presents the three theorems and the structure of the associated proof.

Theorem 1	**Theorem 2**	**Theorem 2'**
For every integer x, if x is a multiple of 6, then x is a multiple of 3.	For any integer x, if x is a multiple of 2 and a multiple of 7, then x is a multiple of 14.	For any integer x, if x is a multiple of 4 and a multiple of 6, then x is a multiple of 24.
Proof 1.2 Let x be an integer that is a multiple of 3. Then x could be 15, which is not a multiple of 6. Thus, it is not necessarily the case that x is a multiple of 6.	**Proof 2.2** Let x be an integer that is a multiple of 14. … Thus, x is a multiple of 2 and a multiple of 7.	**Proof 2.2'** Let x be an integer that is a multiple of 24. … Thus, x is a multiple of 4 and a multiple of 6.

Table 1. The three converse What does it prove? tasks studied in this report.

Though we engaged in the experiments overall to promote set-based reasoning (Dawkins, 2017), these tasks were implemented without instructor guidance toward the normative way of reasoning regarding CI. Since our goal was to categorize students' initial responses regarding CI, the analysis in this paper approached the data as task-based interviews (focusing only on the initial implementation of the three tasks above). Consistent with teaching experiment methodology, the researchers continuously generated and tested hypotheses about students' ways of reasoning. We thus had developed models of how each student reasoned about CI, and were struck by the coherence of students' explanations and justifications, especially those that were non-normative (i.e., argued that the converse proof did prove the theorem). Using comparative coding methods (Glaser, et al., 1968), we identified six categories for students' explanations and justifications for whether a converse proof proved the original theorem (henceforth we shall say "does prove" or "does not prove" always referring to the theorem that is converse to the proof). As noted above, these ways of reasoning were not always stable, nor were they mutually exclusive. However, some of the non-normative categories of explanation persisted with certain participants, suggesting they may be important for future work.

Results

In this section we describe the six categories of explanations and justifications regarding CI. The first two categories led students to the normative interpretation that converse proofs do not prove while the latter four led them to affirm that the converse does prove. Overall, four of the 12 students initially decided Proof 2.2 did not prove Theorem 2.

Category 1: Proof Rules Should be Universal

Students who were observed using this category of reasoning believed that proof rules should be universal, that is, a proof rule cannot be accepted unless it works in all cases (similar to Hammack's, 2013, explanation that implicitly invokes the FWL). Students using this category acknowledged that a statement and its converse can have different truth values (Theorem 1) and used this to conclude that there are cases where one can prove the converse of an implication while the implication itself is false (like Theorem 2'). Therefore, they claimed that a converse proof cannot be accepted as proof of an implication because it would only lead to the correct conclusion when the implication and its converse are both true (Theorem 2). One student, Theo (all names are pseudonyms), provided a justification for why Proof 2.2 did not prove Theorem 2 that fit into this category. He explained (imagining the truth-sets of the hypothesis and conclusion as circular regions):

> In this case, the circle of the "if" exists inside the "then," but it encompasses the whole "then." They're the same set. So, in 2.2, when we switch them around, the "if" and the "then" are still the same sets, but, if we had an example where the "if" is a subspace of the space, and then you switch it around, it's not necessarily true. In this case, it is, but, in general, if you switch them, it might not work.

Theo recognized that the truth sets were the same in this case, and thus each was contained in the other. He used this to explain why the implication and its converse are both true for Theorem 2. It is important that Theo treated all conditional theorems and proofs as instances of the same relationship, and judged that the relationship should remain invariant across the different theorems. Therefore, he claims that the proof of a converse cannot be accepted as proof of an implication because it would not always lead to the correct conclusion.

Category 2: Proofs Should Match the Theorem Direction

Students who used this category of reasoning understood that conditional statements and proofs both had an inherent direction. Accordingly, they judged that a proof should match the theorem's direction. This is distinct from the first category because it was not based on comparing proof relationships across different theorems. One student, Moria, gave an explanation that serves as an example of this category:

> I don't feel like the converse is inherently proving the theorem and if you do that across it's like the converse is proving the theorem… I think it has to do with the trickle down what you're starting with necessarily, if I start with 14, it's going to be a multiple of 14, that's just, it's not going to prove anything… Using those two [Proof 2.2 proving Theorem 2] is saying, "if you have a multiple of 14, it will be a multiple of 14" is what that kind of 2.2 to the theorem 2 is saying.

Moria claimed that combining Proof 2.2 and Theorem 2 would be tautological because the hypothesis of the proof is the same as the conclusion of the theorem. Since she viewed theorems and implications as paths from one claim to another, she argued that the proof began at the ending point of the theorem, which is "not going to prove anything." This led her to claim that Proof 2.2 did not prove Theorem 2 (affirming CI).

Category 3: Equivalent Properties Allows for Substitution

Students who were observed using this category of justification claimed that the properties in the theorem's hypothesis and the properties in the theorem's conclusion were equivalent. They then used this to argue that the direction of the proof did not matter because the properties being related were the same. One student, April, justified that proof 2.2 proved theorem 2 by writing the equation "2*7*k=14*k" and claiming that this shows that the properties "multiple of 2 and multiple of 7" and "multiple of 14" are equivalent. Another student, Jean-Luc, also provided this type of explanation. His partner had argued for CI, but he responded:

> I guess we're having trouble because multiple 14 breaks down into seven and two and then seven and two break down right into 14. So, I feel like it's saying the same thing, but I totally understand what you're saying. It's reversed.

While Jean-Luc acknowledged that the direction of the proof was the reverse of the direction of the theorem, his belief that "multiple of 14" and "multiple of 2 and multiple of 7" are "saying the same thing" led him to see no conflict between the proof and the theorem. Since he viewed "multiple of 14" and "multiple of 2 and multiple of 7" as being the same property, he treated them as though they were synonymous. This sense of identity led him to deny CI since the order did not distinguish the proof from the theorem.

Category 4: Set Equality has No Direction

Students who provided this category of explanation acknowledged that the set of objects that satisfied the hypothesis was equivalent to the set of objects that satisfied the conclusion. They then used this to argue that since the hypothesis and conclusion were referring to the same set of objects, they were interchangeable. This category is very similar to the previous; they differ with regard to whether the student is attending to sets of objects or properties. Mathematically speaking, the sets of objects are truly equal while the properties differ in definition. For example, the set of integers that are multiples of both 2 and 7 is the same as the set of integers that are multiples of 14. Students giving such justification thus did not find grounds to distinguish a converse proof from the conditional theorem. One student, Phil, reasoned in this manner:

> I was saying I feel like it's true because all multiples of 2 and 7 are all multiples of 14. So, if it's a multiple of 2 and a multiple of 7, then it's going to be a multiple of 14 because multiples of 2 and 7 are 14, 28, and 42, and then to whatever degree you want to go to.

Phil claimed that Proof 2.2 still proves Theorem 2 because both the hypothesis and the conclusion are referring to the same set of objects. Therefore, switching them around in a statement does not change the meaning of the statement.

Category 5: The Theorem, the Proof, and my Knowledge All Agree

Students who used this category of reasoning focused on what they knew to be true when deciding whether a converse proof proved. If there were no contradictions between the theorem, the proof, and what they knew to be true, then they affirmed that the proof proved. In biconditional situations, this led them to affirm the converse proof. One student, Carl, used this category of reasoning to justify why Proof 2.2 proved Theorem 2:

> I said that it proves that if x is a multiple of 14, then x is a multiple of 2 and 7. Which is kind of like the opposite, but not really, of Theorem 2. But because the statement is true it doesn't really matter.

Carl argued that because Theorem 2 and its converse are both true, the order of the proof "doesn't really matter." In other words, Carl did not distinguish the truth of the claim from the efficacy of the proof in justifying the claim, since both the theorem and the proof described accurate facts about the same objects and properties. This reflects the common phenomenon that people judge arguments very differently depending upon whether they believe the conclusion of the argument (Inglis & Simpson, 2007; Evans & Feeny, 2004).

Category 6: Error-free Proofs Prove

Students who used this category of justification did not attend to the direction of the proof at all. Instead, students only focused on dissecting the line-by-line content of the proof for errors. Two participants, Joaquin and Violet, had a long discussion about whether the inferences made within the proof were valid. They sought to determine whether they believed each line in the proof followed from the previous line. Since they concluded that this was the case, they also concluded that proof 2.2 did in fact prove theorem 2. Interestingly, they frequently stated that the theorem was "true" and the proof was "true," seemingly assigning the same epistemic roles to the two kinds of text. Whereas the previous type of explanation notes the reversed order and dismisses its importance, this kind of explanation does not entail attention to an explicit link between the proof and the theorem beyond discussing the same topics.

Discussion

We present six categories of student explanation and justification regarding the relationship between a converse proof and a conditional theorem. There is some overlap with Hoyles and Kuchemann's (2002) description of four types of reasoning about CI: converses are the same in general, converses are distinct in general, converses are the same with reference to data, and converses are distinct with reference to data. Our first category matches the view that converses are distinct in general. It depended upon students perceiving all conditional theorems and proofs as instances of the same relationship (in accordance with the FWL). Our second category is similar, though it depended upon students according some significance to the direction of a theorem and proof. Our latter four categories all correspond to the view that converses are the same with reference to data. We did not observe the view that converses are the same in general (possibly since we presented Theorem 1) or that converses are distinct with reference to data for Theorem 2 (since this theorem is true biconditionally).

Students rejecting CI (for Theorem 2) variously attended to properties, sets of objects, or to their judgments about what is true. In each case, they did not see a reason to distinguish converse proofs from the given theorem, even when their partners provided explanations from Categories 1 and 2. Indeed, we are struck by the coherence of these non-normative explanations. Once students perceive some kind of identity (of property or set), it is hard to understand why they would perceive it as having some direction. Furthermore, explanations in Categories 5 and 6 raise more fundamental issues of how students learn to distinguish what a proof accomplishes from what they believe to be the case. It is worth noting that many students providing explanations in Categories 3-5 recognized the reversed order and also recognized that the shared structure among the proofs that follow converse order. Nevertheless, they affirmed that converse proofs can prove the theorem and that this relationship differed by context. We interpret this as a rejection of the FWL as applied to this type of proof, which poses a major challenge for the teaching of logic and proof that we hope will be addressed in future research in this area.

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What Do the Calculus I Students Say About the Impact of IBL on Their Math Anxiety

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Math anxiety negatively affects student learning and academic performance. Students with high math-anxiety exhibit physical, mental, and emotional symptoms. These symptoms often have a short-term and long-term impact on students' mathematics learning and their performance both inside and outside of school. This study investigated the effects of inquiry-based learning (IBL) on Calculus I students' math anxiety, compared to lecture-based instruction. The short version of the Mathematics Anxiety Rating Scale (MARS-S) was used as a pre- and post-test to identify the students whose anxiety from pre- to post-test—greatly increased, greatly decreased, and did not change much. The selected participants were interviewed one-on-one to understand their perceptions and experiences of learning Calculus I. The results showed that some activities, such as the opportunity to work in groups, optional and ungraded homework, and the instructor's welcoming, caring, and amicable nature decreased IBL students' anxiety. On the other hand, the instructor's readiness to explain the material in class when students asked him to do so and his care for student success decreased lecture-based students' anxiety. However, the tests and exams and anticipating the instructor's call for a response increased anxiety among both groups of students.

Keywords: math anxiety inquiry-based learning, collaboration

Math anxiety affects student learning and academic performance. Highly math-anxious students exhibit physical, mental, and emotional symptoms. Physical symptoms include nausea, sweaty palms, and increased cardiovascular activity (Ashcraft, 2002; Chang & Beilock, 2016). Mental symptoms include an inability to concentrate and mind blanking (Plaisance, 2009; Ruffins, 2007). Emotional symptoms include extreme nervousness and apprehension (Mattarella-Micke et al., 2011). These symptoms often have a short-term and long-term impact on students' mathematics learning and their performance both inside and outside of school. In a short term, students may begin to dislike mathematics and take fewer mathematics courses, and in the long term, they tend to avoid mathematics and mathematics-related courses (Godbey, 1997; Hembree, 1990).

Due to the substantial impact of math anxiety on mathematics learning and mathematics performance, it is essential to diagnose the causes of math anxiety and to determine some potential interventions to reduce such anxiety. Therefore, this study investigated the effects of inquiry-based learning (IBL) pedagogy on Calculus I students' math anxiety, with lecture-based instruction as a comparative group.

Literature Review

Math anxiety has been a part of the human experience for centuries. The verse, "Multiplication is vexation ... and practice drives me mad" goes back at least to the 16th century (Dowker et al., 2016). In 1957, Dreger and Aiken introduced the concept "number anxiety," and math anxiety received increasing attention thereafter. Richardson and Suinn (1972) conducted the first formal study of math anxiety, who characterize math anxiety as "feelings of tension and anxiety that interface with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (p. 551). Since then, studies on math anxiety have been substantially investigated.

Traditional lecturing, which is a predominant mode of instruction in college mathematics courses across the United States (Stains et. al., 2018) and is ineffective in helping students learn mathematics (Boaler, 2008), could be one of the possible reasons for evoking math anxiety among students. The lecture-based does not offer substantial opportunities for students to share each other's ideas and experiences with their teachers and peers. On the other hand, IBL, which is an active learning pedagogy, provides extensive opportunities for students where they can work in pairs or groups to make conjectures, gather information for problem-solving, and present their work to groups and to the whole class (Kogan & Laursen, 2014). Through a comparative study, Laursen et al. (2014) reported that students in IBL math-track courses achieved greater learning gains than their non-IBL peers in cognitive, affective, and collaborative areas. Similarly, Laursen et al. (2011) found that the IBL students were involved more in interacting with each other, with the instructor, and they were more involved in setting the course pace and direction. It is also reported that IBL enhances students' conceptual understanding (Jensen, 2006), communication skills, confidence, and self-efficacy (Laursen et al., 2011). Considering the benefits of IBL as a ground, this study sought to examine the relative changes in the scores of Calculus I students' math anxiety, using a short version of the Mathematics Anxiety Rating Scale (MARS-S).

Method

Research Context and Participants

The students, who were enrolled in Calculus I course and were taught using either IBL or lectures during Spring 2021at a university located in the Midwestern United States were the sample for this study. Students, who were engaged via IBL pedagogy were IBL group and those who were lectured were lecture-based group. In this study, about 65% (n = 15) of the students off the 23 from the two IBL sections and about 41% (n = 20) students off the 49 from one of the lecture-based sections responded to both the pre- and post-MARS survey. Table 1 shows the distribution of IBL and lecture-based participants by their gender and academic standing.

Table 1. Distribution of IBL and lecture-based students by their major and academic standing.

Characteristics	IBL Group		Lecture-Based Group	
	Frequency	Percentage	Frequency	Percentage
Gender				
Male/Man	5	33.3%	8	40.0%
Female/Woman	10	66.7%	11	55.0%
Non-Binary	0	0.0%	1	5.0%
Academic Standing				
Freshman	11	73.3%	15	75.0%
Sophomore	2	13.3%	1	5.0%
Junior	2	13.3%	4	20.0%
Senior	0	0.0%	0	0.0%

Regarding the instructors, the IBL instructor had 15 years of experience in teaching Calculus I at the university and high school level via active learning, including an IBL. The lecture-based instructor had 2 and a half years of experience in teaching Calculus I at the university level and 14 years of experience in teaching undergraduate-level mathematics courses via a lecture-based approach. Both the instructors taught remotely using audio-visual conferencing platform; the IBL instructor taught using Microsoft Teams, whereas the lecture-based instructor taught using Zoom. Throughout the semester, both IBL and lecture-based classes had class meetings every Monday, Wednesday, and Friday.

The IBL instructor engaged students collaboratively in sequentially organized pre-tasks and tasks in and out of the class. Students were supposed to practice the pre-tasks before the class for a better understanding of the material during the next day's class meeting. The instructor usually began the class by welcoming each student and briefly describing the tasks and activities for that day. Then, the students were sent to Teams breakout rooms, where they shared each other's ideas, asked questions, made conjectures, and solved problems while they were working with their small group members. The instructor visited each group at least once, or as needed and prompted students if they had any questions or concerns. In the end, the students were returned to the main room, where the instructor facilitated whole class discussion. On the other hand, the lecture-based instructor began the class by asking students whether they had any questions or concerns from the previous class. If they had, then, the instructor solved the examples or explained the concepts as needed. After that, the instructor usually began the lecture by solving preselected examples using the Notability app from his iPad. Occasionally, the instructor paused during the lecture and asked some questions to the whole class. Students were never sent to breakout rooms and never provided opportunities for group discussions.

Data Collection and Analysis

Qualtrics online survey was used to collect the pre- and post-MARS data from both IBL and lecture-based groups after receiving an institutional review board (IRB) approval. The MARS-S survey is the 30-item anxiety measuring instrument that was developed by Suinn and Winston (2003). The pre-MARS survey, in conjunction with a demographic questionnaire, was administered during the second week and the post-MARS was administered during the eleventh week of the class. Based on the change in anxiety scores from pre- to post-test, 9 students from the IBL group (3-greatly increased, 3-greatly decreased, and 3 did not change much) and 3 students from the lecture-based group (1 student from each of the categories) were invited for a semistructured interviews. Additionally, eight classes of each of the two sections of IBL group and eight class of one lecture-based group were observed remotely. The data thus collected were transcribed using NVivo, a qualitative data analysis software, and generated several codes. Thematic analysis was conducted to identify various emergent themes. The results were then organized based on the themes generated.

Results

In this section, I present the major findings from the analysis of the data obtained from observations and interviews. While doing this, I compare the findings from both the IBL and lecture-based groups.

The in-class group work and lecturing reduced IBL students' anxiety; out-of-class group chats reduced lecture-based students' anxiety. IBL students were partly lectured and partly engaged in collaborative work in each class; lecture-based students were fully lectured, but some students managed to chat via GroupMe. Because of the collaborative learning opportunities, IBL students became closer to each other and were not scared of asking questions, responding to their groupmates' questions, proposing their own solution strategies, and listening to their peer's various perspectives. Such types of activities made them feel relaxed and also reduced their anxiety. On the other hand, lecture-based students initiated out-of-class collaboration via GroupMe. They maintained it throughout the semester, where they felt comfortable asking questions and responding to each other. Cooperation and mutual support received by these students reduced their anxieties and frustrations with learning Calculus I.

Table 2 and 3 below show sample of the emergent themes, related codes, and student quotations, respectively, for the IBL and lecture-based groups.

Table 2. Emergent Themes, Related Codes, and Sample Student Quotations for IBL Group

Theme	Related codes	Sample quotation
Both lecturing and collaborative learning	• The instructor delivered minilectures • Students were engaged in small group works	He typically starts off with a small lecture … and then, we'll typically go into a group setting into our breakout rooms.
Students received optional and ungraded homework	• Assigned optional homework • Homework was not graded • Students were less stressed	I know they're just used for practice and prep for the exams and for class. So, I don't feel as stressed about them.
Instructor questioning	• Frequently asked short-answer-type questions • Asked to respond in chat	He would give each individual student a certain problem to do … and randomly call on people for it.
Engaging in behaviors that decreased anxiety	• Working in group • Asking questions of the instructor outside of class • Talking to groupmates	I think doing the task in a group reduced it for sure, and ... watching the videos in class, like that does not stress me out or anything.
Engaging in behaviors that increased anxiety	• Thinking of being called on by the instructor • Responding to the instructor questions in front of the colleagues	I would definitely say I don't enjoy going to class because I'm scared, he's going to call on me and ask me a question I don't know the answer to.

Instructors' readiness to meet and help students at any time decreased IBL students' anxiety; the long wait for the instructor's email replies increased lecture-based students' anxiety. The IBL instructor's readiness to meet with students at any time and discuss their questions, concerns, or problems in and out of the class decreased IBL students' anxiety. Students could ask questions to the instructor at any time—during the class, at the end of the class, right after the class, or by scheduling a virtual meeting—and discuss their problems. They could approach the instructor at "eleven o'clock" at night or "7:00 am that morning;" he was always ready to meet and talk with them. Students could also send their questions or concerns via email, to which he replied promptly. This kind of flexibility of the instructor made students calm down and reduced their anxiety. Alternatively, although the lecture-based instructor encouraged students to reach out to him with their questions, students had to wait for long hours for his responses. Students were disappointed and frustrated for not receiving a response to their emails for several days. Liz, for example, mentioned, "We have an assignment that's due on Friday, and I need help. And I can't afford to wait three days for you [the instructor] to email me back because this piece of information is necessary for me to do 60 percent of the assignment."

Table 3. Emergent Themes, Related Codes, and Sample Student Quotations for Lecture-Based Group

Theme	Related codes	Sample quotation
Students received too many online homework problems	• Too many online homework problems • The number of online homework problems gradually decreased	He gave us homework assignments that had eighty-seven problems. ... But that was really the only thing that made me feel anxious about the assignments.
Initiating and maintaining out-of-class collaboration	• Worked together through a GroupMe chat application • Liked helping each other	We all try to help each other out so I would say it's very collaborative.
Instructor questioning	• Frequently asked short-answer-type questions • Asked to respond in chat	He would give each individual student a certain problem to do … and randomly call on people for it.
The instructional environment that decreased students' anxiety	• Instructor's willingness to review the materials • Recitation classes • Recorded lecture videos	His [instructor] willingness to kind of like go over things makes me feel like very comfortable.
The instructional environment that increased students' anxiety	• Tests and exams • online homework • Online learning • Proctored-track exams	I still get nervous every time when quizzes [are] announced or an exam is announced because like I said, this material has been hard.

Optional tasks and ungraded assignments decreased IBL students' anxiety, and the overwhelming number of online homework problems, especially at the beginning of the semester, increased lecture-based students' anxiety. In addition to the midterms, the IBL instructor engaged students in both optional and mandatory works, which were rarely graded. His pretasks were optional and ungraded but recommended for students to complete before class. However, the tasks, mock exams, and group competitions were mandatory and were required to complete during class but were rarely graded. Students did not feel stressed working with the optional and ungraded tasks and activities because they did not impact their grades; rather, these activities reduced IBL students' anxiety by developing confidence and problem-solving ability. Lecture-based students, on the other hand, experienced stress, frustrations, and anxieties when they confronted as many as 87 online problems as homework assignments during the first week. Students were anxious, thinking that they might continue receiving problems in the same manner. However, their anxiety reduced as the online homework problems decreased as the semester progressed.

Both IBL and lecture-based students felt anxious about being called on during the lectures and whole-class discussions. Students in both groups experienced extreme anxiety in the class, more specifically during the whole-class discussions, thinking that the instructor might randomly call on them for a response at any time. Recall that IBL student Camila stated, "the rest of the time, I am comfortable until he directly calls on me." These students were worried about their colleagues' judgments when they could not respond to the instructor's questions correctly and instantly. Likewise, lecture-based students were worried about being called on during the lecture for similar reasons. Unlike the IBL students, lecture-based students were usually busy "trying to write everything down" that the instructor wrote, and they experienced nervousness when they did not respond quickly to the instructor's abrupt questions. One of the lecture-based students, Liz, stated "Occasionally he would … call on someone, and if they wouldn't know, he'd be like, you should know this. I would be, like, woo, I should know this by now, which would freak me out a little bit."

Tests and quizzes, online learning, and proctored-track exams increased both IBL and lecture-based students' anxiety. Both IBL and lecture-based students were anxious about taking exams, remote learning, and proctored-track exams. Although midterms and final exams were cumulative for IBL students, they were more anxious about taking the final exam than the midterms. In contrast, the lecture-based students were anxious about taking the tests and the final exams because they weighed the largest proportion of the overall grades. Also, lecture-based students were anxious to see a few questions on the exams because they would lose huge points if they did even a single problem incorrectly.

Online learning was another aspect that induced anxiety among both groups of students. IBL and lecture-based students occasionally missed a part of or the entire class due to the poor Wi-Fi connection, which added an extra pressure on students. Moreover, proctored-track exams caused anxiety among both groups of students. They were anxious about being watched by someone remotely and hearing a loud noise when someone asked clarifying questions to the instructor,

Instructors' questions increased both IBL and lecture-based students' anxiety. Instructor questions—either during lectures or whole-class discussions—increased IBL and lecture-based students' anxiety. Three out of nine IBL students said that they feared the instructor's questions, specifically when they were unsure of the answers and were still processing the information. Maria said she is terrified of instructor questions thinking of giving an incorrect answer in front of her colleagues, who were not from her small collaborative group. Although lecture-based students experienced similar types of anxieties in responding to instructor questions, they were also anxious by the instructor's verbal pressures, such as "There's been a couple of times when he [the instructor] said, you should know this, and I did not know it, and that made me very anxious," Recall what Liz from the lecture-based class said before.

Discussion

Findings from the analysis of the interviews and observations show that the factors, such as the instructor's amicable and cordial nature, readiness to help students at all times, in-class group work, and optional and ungraded homework assignments decreased IBL students' anxiety. However, some other factors, such as thinking of being called on for a response and asking questions in front of their peers, increased these students' anxiety. On the other hand, the classroom activities, such as the overwhelming number of online homework problems that were due every week, thinking of being called on for a response in class, responding to the instructor's questions in front of their colleagues, instructor's questions, and fast-paced teaching increased lecture-based students' anxiety. Still, other factors, such as the instructor's readiness to explain the materials in class reduced the anxiety of lecture-based students.

Based on the findings from the present study, I recommend that although some factors of the instructional environment, student behaviors, and instructor behaviors have been found to be increasing anxiety among both groups of students, there are many other factors that have lessened their stress, frustration, and anxiety. It is suggested to the instructors of Calculus I and other similar mathematics courses to implement IBL in their classes or at least transition toward the student-centered approaches to instruction because of the benefits of such instructional practices on students' ability to critical thinking, reasoning, and problem-solving. It is also suggested to avoid the instructional activities that have been found to increase students' anxiety.

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Using Professional Learning in Practice: The Case of Novice College Instructors

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We report on findings from a mixed methods study on preservice elementary teacher learning outcomes as a result of being enrolled in courses where their instructors were participating in professional development designed for those new to teaching content courses for future teachers. The study contributes to the current sparse literature within the RUME community on student performance as a result of professional learning opportunities for mathematics faculty. Results indicate improved undergraduate learning outcomes among preservice teachers when instructors engaged in a professional short-course that both fostered growth of instructors' own Mathematical Knowledge for Teaching (MKT) for teaching undergraduates and provided instructors with opportunities to learn about and support prospective teachers' development of MKT (for teaching children).

Keywords: professional development, mathematical knowledge for teaching, teaching for robust understanding

There have been many calls in the last decade to devote time and effort to the professional development (PD) of faculty in mathematics departments who teach the courses for future K-8 teachers (Flahive & Kasman, 2013; Masingila et al., 2012; Masingila & Olanoff, 2022). Indeed, Greenberg and Walsh (2008) noted that mathematics faculty regularly seek professional support in the work of teaching prospective teachers. To the best of the authors' knowledge, there have been two documented efforts in the literature to meet the needs of this call. Castro Superfine and colleagues (2013, 2014) have discussed the design model for their NSF funded *Mathematical Knowledge for Teaching Teachers* (MKTT) project, while we (Jackson et al., 2020) elaborated on the design features of our NSF funded *Professional Resources and Inquiry into Mathematics Education* (PRIMED) *for K-8 Teacher Education* project. In the former, the use of a research-based curriculum by experienced teacher educators was associated with large learning gains by undergraduates (Castro Superfine et al., 2013). However, the work did not include explicit attention to investigating prospective teacher learning outcomes as a result of faculty participation in professional development or include instructors who were new to teaching future teachers. This report aims to provide at least a partial snapshot of the relationships among *novice* instructor PD and undergraduate learning. The study is part of a larger (PRIMED) project. Here we share results of examining measures of particular target outcomes – related to mathematical knowledge for teaching – among participating faculty and their undergraduate students.

Theoretical Foundations

Mathematical Knowledge for Teaching and Its Assessment

The types of knowledge required of mathematics teachers have emerged from several decades of research and development that have grown from seeds sown by Shulman (1986). In 2008, Ball and colleagues described *mathematical knowledge for teaching* (MKT). As initially conceptualized, MKT consists of *subject matter knowledge* and *pedagogical content knowledge.* Subject matter knowledge included *common content knowledge*, *specialized content knowledge*, and *horizon content knowledge*. In particular, common content knowledge is used in everyday activities by teachers and others such as mathematicians, engineers, or homemakers. By comparison, specialized content knowledge is the knowledge that is specific to the task of teaching. Within pedagogical content knowledge, there are *knowledge of curriculum*, *knowledge of content and students*, and *knowledge of content and teaching*. Knowledge of curriculum includes awareness of the content and connections across standards and texts . Knowledge of content and students blends knowledge for doing mathematics with knowledge for unpacking it in ways relevant to how students think about, know, or learn (Hill, et al., 2008). Knowledge of content and teaching is a knowledge of teaching moves for making unpacked meanings accessible to students.

The Learning Mathematics for Teaching (LMT) instrument developed by Hill, Schilling, and Ball (2004) is a validated and reliable tool for assessing MKT development. The instrument assesses common and specialized content knowledge in different domains of K-8 mathematics including number and operations; in addition, some items assess knowledge of content and students. Though the instrument was initially validated for the purposes of measuring MKT among in-service teachers, in recent years it has been used to measure MKT development in courses for future teachers (Flake, 2014; Namakshi et al., 2022). A sample released item from the LMT instrument assessing specialized content knowledge is shown in Figure 1.

3. Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35 x 25 125 +75 875	35 x25 175 +700 875	35 x 25 25 150 100 +600 875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

Figure 1 Sample Item from LMT Instrument (Hill et al. 2004)

Teaching for Robust Understanding

Teaching for Robust Understanding (TRU, Figure 2) is a research-based framework designed to attend to particular aspects of instruction (Schoenfeld, 2016). The framework was developed for research and has proven useful in practice. The TRU framework is used to explore the attributes of equitable and robust learning environments that support each student in becoming a knowledgeable, flexible, and resourceful disciplinary thinker. The TRU framework is a powerful tool because it provides a working definition of *effective mathematics instruction* that includes attention to equity along with language for describing characteristics of classroom activity. Given the centrality of educational equity in the PRIMED project, the TRU framework served a dual purpose: as a resource to guide participants in examining their own and peers' responsiveness to prospective teachers in curricular and instructional choices, and as a resource for the developers in the design, development, implementation, and evaluation of the PRIMED short-course itself.

The Mathematics	Cognitve Demand	Equitable Access to Content	Agency, Ownership, and Identity	Formative Assessment
The extent to which the mathematics discussed is focused and coherent, and to which connections between procedures, concepts and contexts (where appropriate) are addressed and explained. Students should have opportunities to learn important mathematical content and practices, and to develop productive mathematical habits of mind.	*The extent to which students grapple with and make sense of central mathematical ideas and their use. Students learn best when they are challenged in ways that provide room and support for growth, with tasks that are not low level but not impossible. The level of cognitive demand should be conducive to what has been called productive struggle.*	*The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core content being addressed by the class. Classrooms in which a small number of students get most of the "air time" are not equitable, no matter how rich the content: all students need to be involved in meaningful ways.*	*The extent to which students are provided opportunities to "walk the walk and talk the talk" – to contribute to conversations, to build on others' ideas and have others build on theirs – in ways that contribute to their development of agency (the willingness to engage), their ownership over the content, and the development of positive identities as thinkers and learners.*	*The extent to which classroom activities elicit student thinking and subsequent instruction responds to those ideas, building on productive beginnings and addressing emerging misunderstandings. Powerful instruction "meets students where they are" and gives them opportunities to deepen their understandings.*

Figure 2: Teaching for Robust Understanding (TRU) Framework, Schoenfeld (2016)

Effective Design and Implementation of PD for Instructors of Future Teachers

Castro Superfine and Li (2014) discussed features of high quality professional development that they considered effective for teacher educators:

(1) Participants should be engaged in learning that is grounded in the content of teaching and learning. In particular, it must pay special attention to the MKT that is used in the work of teaching undergraduates who are preservice K-8 teachers *in addition to* bringing awareness to the instructional challenges of teaching MKT (for K-8 teaching).

(2) The PD must create the possibility of cognitive disequilibrium. That is, instructors are challenged to think about what they believe future elementary teachers need to know about MKT and why. They must also reflect on how future teachers learn about MKT.

(3) Participants must be provided with opportunities for collaboration in order to form a professional community of practice. Thus, effective PD allows for the mutual sharing of ideas and lessons learned from the work of teaching future teachers.

(4) Learning opportunities and tasks within the PD must be embedded in or directly related to the particular specialized work of teaching future teachers. PD tasks are situated in the actual work that instructors do every day in their own local institutional contexts.

Whereas the four points above address the design and development aspects of effective PD, Guskey (2000) highlighted the critical stages involved in evaluation of PD (Figure 3). In Level 1, evaluation focuses on participants' experience of the PD. Participants are questioned in surveys, interviews, and/or focus groups about how useful they feel the experience was and of what use the PD was to them professionally. In Level 2, the focus is on measuring the knowledge and skills that instructors gained as participants in the PD – specific learning goals are outlined before the PD together with a plan for assessing those goals and assessments typically include things such as pencil-and-paper instruments, reflections, journals, discussion boards, portfolios. At Level 3, evaluation turns attention to the local contexts within which participants operate professionally; common questions at this level attend to how change was (or was not) supported at the departmental/college/university levels. In Level 4, evaluation examines if and how the knowledge and skills from the PD are actually applied in the instructors' professional practice – evidence occurs through such things as direct observations, documented instances in reflective journals or portfolios, and interviews with participants and/or their supervisors. At Level 5, evaluation considers if and how participation in the PD impacted student learning outcomes – indicators of student achievement may include cognitive aspects such as performance in things like course grades, assessment scores, or scores from standardized tests. However, affective and psychomotor outcomes are also commonly measured in lieu of/in addition to cognitive skills by examining such things as student attitudes and beliefs, dispositions, and classroom behaviors.

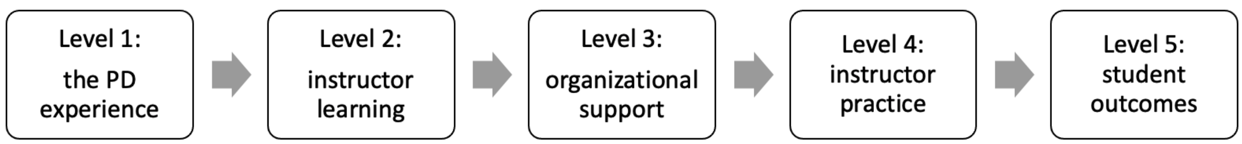


Figure 3: Five Critical Levels of PD Evaluation (Guskey, 2000)

The emphasis in the current study was to dig into the connections among **Levels 2, 4 and 5**. In particular, the aim in this report is to illustrate how the PRIMED design – which had the four principles documented by Castro Superfine and Li (2014) and incorporated use of the TRU framework – had downstream effects on Levels 2, 4 and 5. In this vein, the research question this study addressed was: *What is the nature of relationships among participation in PRIMED activities, college instructors' mathematical knowledge for teaching, and that of their students?*

Methods

For the PRIMED project, each participating college instructor agreed to complete various research tasks and agreed to include assignments in their courses through which their students completed assessments of the mathematical knowledge needed for teaching grades K-8 (the LMT). Instructors (n=15) and undergraduates (n=476) providing data for the study were from 12 different institutions of higher education (2 public community colleges, 7 master's granting [6 public, 1 private], 3 doctorate-granting [2 public, 1 private]).

The PRIMED design and learning goals are briefly described here (see Jackson et al., 2020 for a detailed description). The PRIMED professional learning experience was through the Canvas online course management system and was a hybrid: two of the five modules (Module 2 and Module 5) included both asynchronous and live online activity-based work by instructor teams from different colleges and universities; the other three modules were sets of self-paced asynchronous web-based activities completed by instructors individually with asynchronous discussion board interaction and the option of online meetings of the instructor teams. The short-course included assessments of instructor learning in each module. Participant teams rarely met in real time during the asynchronous modules and tended to check-in with each other by email or through the online discussions in Canvas.

Performance on the LMT was used as one of the outcome measures for Guskey's Level 2 (of instructors) and for Level 5 (of undergraduates). Participants who fully engaged in the short course completed three lesson experiments in which they enacted PD principles as part of their instruction using task-based active learning techniques. In addition to surveys, the project team collected artifacts from the course such as online discussion board posts about lesson experiment efforts by instructors, in-module assessments, and intake/exit interviews with each individual participant.

Results

For Guskey's Level 4 effects we examined details related to the second of the three lesson experiments. For Levels 2 and 5 we looked at gains on the LMT among the undergraduate future teachers (students) and level of PRIMED participation of instructors.

Evidence of Undergraduate Learning from Lesson Experiment Reporting

In the second lesson experiment, which occurred about halfway through the short-course, instructor participants were asked to search the practice literature in teams (of 2 or 3) for a task that had already been developed and implemented in a course for future K-8 teachers. After locating the task, each team agreed on two of the TRU dimensions to pay particular attention to while implementing the task in their own classes. The participants were asked to observe each other and debrief about the observation as each partner implemented the task. Each team was given the prompts below to document their experiences about the entire process:

Before the Lesson Prompts:

1. Which TRU framework dimensions have you selected? What do you hope to learn about yourself as an instructor by paying attention to those dimensions?
2. What is the task you have chosen? Say a little about why you chose it.
3. In terms of the selected TRU dimensions, what do you anticipate will be your greatest challenge with the lesson? How do you plan to address that challenge?

After the Lesson Prompts:

4. How was thinking about the two TRU dimensions challenging? How was it useful/beneficial?
5. How might paying attention to the TRU framework inform future task selection? Implementation?
6. [Bonus question!] Did any of the students get stuck at some point? What did you do or say to help them advance? Would you do the same thing in the future? Why or why not?

Notice that the prompts were written to have the participants actively reflect on their practice, implementing the recommendations set by Castro Superfine and Li(2014). The lesson experiment itself was structured within the work that they did as instructors, another aspect that followed the recommendations.

For the sake of space, we report here on the responses of one participant Sheila (a pseudonym) to demonstrate aspects of her own professional learning as well as that of her future teacher students. Sheila chose a task titled "Maren's Garden" in which the undergraduates in her class had to determine how many gardens could be planted with 4 bags of soil if each garden required ¾ of a bag of soil. Sheila also asked students how much soil would be left. In the Maren's Garden activity, students were asked to solve the problem in two different ways, one of which had to include pictures/diagrams. Sheila indicated that she was aware that future teachers might find it challenging to make sense of what the fractional part of the answer represented in terms of the context of the problem (i.e., that the 1/3 in the answer 5 1/3 referred to a unit of a garden rather than a bag of soil). Sheila indicated in the first prompt that she had chosen to focus on Cognitive Demand and Agency because she wanted the future teachers to make connections with their previous procedural knowledge of fraction division. For prompt 5, Sheila answered:

> Paying attention to the TRU framework will help me to choose tasks that give students opportunities to make connections between concepts or skills that they already know. For example, in the task I selected (which required students to compute 4÷3/4 in more than one way), we were able to compare/contrast solution methods and even provide a reason why multiplying by the reciprocal made physical sense in the problem!

In answering the prompt 6, Sheila noted that:

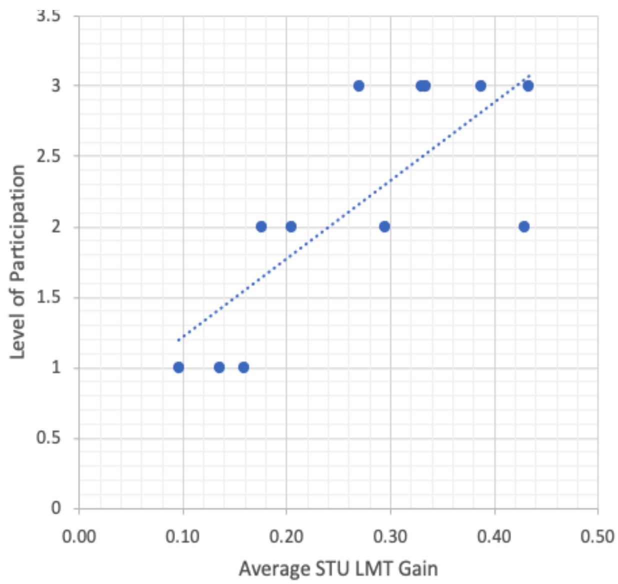
> Yes, students got stuck, and it was a great opportunity for learning! They got stuck when comparing their answers from two methods (one of which had to be visual). In the visual method, it was found that 4 bags of soil divided among gardens that each required 3/4 of a bag of soil meant that 5 full gardens could be made with 1/4 bag of soil remaining. Using the calculation, however, 4÷3/4=4×4/3=16/3=5 1/3. Students were puzzled why the calculation showed a left over part of 1/3 compared to the picture, which showed ¼.
>
> I asked them to think about what the fractional part represented in each case and then gave them some time to think about this. In the end, some of the groups realized that the calculation was counting gardens and so the 1/3 represented part of a garden, not a bag of soil. I would do this activity again and implement it in the same way.

For Sheila, an important part of her job as a teacher educator was to make sure that her students could develop their specialized content knowledge, to make connections between representations (i.e., the symbolic and diagramatic in this case). Sheila found it important to let her undergraduate students explore for themselves in order to become agentic, owners of their own learning. That this was happening was evidenced in her report of the future teachers exploring the distinction between the units involved in the problem.

Association Between Undergraduate Learning and Instructor Participation in PRIMED

The overall mean gain for undergraduates on the LMT was 0.29 standard deviations (sd), indicating they increased knowledge of target ideas. Other research on prospective teacher learning has indicated that one semester of instruction in a carefully designed course by an experienced teacher educator can lead to gains of up to 1sd (Castro Superfine et al., 2013). However, typical instruction when college instructors have professional development to strengthen their teaching has seen gains among their prospective teacher students of around 0.25 sd (Laursen et al., 2015). Here, the instructors were novices with teaching future teachers and had a comparable downstream effect (i.e., the average of 0.29sd across all instructors).

The research team also examined the relationship between an instructor's level of participation and their undergraduate students' average gain on the LMT instrument. Level 1 was assigned to the lowest amount of participation (less than one-third of the short-course activities engaged in/completed). For these instructors the average gain was about 0.13sd. Level 2 indicates moderate participation (up to two-thirds of activities completed; for example, parts of some modules not accessed or missing parts of one or two lesson experiment planning, implementation, or reflection activities). Learning gains among students in the classes of Level 2 participants averaged about 0.28sd. Level 3 indicates high participation in most activities and at least two completed lesson experiments. For the students of Level 3 instructors the average LMT gain was about 0.35sd. Instructors reported on their perceived participation in the exit interview: based upon evidence presented within the course, the research team agreed with all of the participants except for one who ranked herself at Level 2. Compared with the other participants, her engagement was Level 3. As the plot of the regression line in Figure 4 shows, there appears to be a robust correlation between rising average LMT gains and increasing levels of participation (r^2~0.6).



Level of Instructor Participation	Average of Student LMT Gain (sd)
1	0.10
1	0.14
1	0.16
2	0.18
2	0.20
2	0.29
2	0.43
3	0.27
3	0.33
3	0.33
3	0.39
3	0.43

Figure 4. Association and table of values for level of PRIMED participation and student LMT gain.

Conclusions and Future Research

We see from the lesson experiment and LMT data that there was a positive relationship between PRIMED participation and the knowledge for teaching gained by their students. While limited in scope due to the size of the study, we see that instructors implemented their intended lesson experiment curriculum with careful attention to the TRU framework and, as illustrated in the case of Sheila, their future teachers gained specialized content knowledge. The more instructors participated, the more opportunities they had to put into play a Castro Superfine and colleagues' (2013, 2014) action knowledge – an analog of knowledge of content and teaching for teacher educators (see Jackson, et al., 2018).

While certainly not evidence for a causal effect (Yoon et al., 2007), the study reported here contributes to the literature on ways to connect PD impact with undergraduate learning impact. A real challenge for the RUME community is to design and conduct large enough studies (or multiple repetitions of moderate sized studies like ours) to explore robust causal effects. Methods for addressing this challenge are part of ongoing work in educational research at all levels (e.g., Kim et al., 2018; Rosenbaum, 2005).

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Student Reasoning About the Least-Squares Problem in Inquiry-Oriented Linear Algebra

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The method of Least Square Approximation is an important topic in some linear algebra classes. Despite this, little is known about how students come to understand it, particularly in a Realistic Mathematics Education setting. Here, we report on how students used literal symbols and equations when solving a least squares problem in a travel scenario, as well as their reflections on the least squares equation in an open-ended written question. We found students used unknowns and parameters in a variety of ways. We highlight how their use of dot product equations can be helpful towards supporting their understanding of the least squares equation.

Keywords: linear algebra, least squares, dot product, student reasoning, inquiry

Linear algebra courses frequently include the topic of least squares approximation. A central focus of the least squares problem is when the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ has no solution. When this occurs but some kind of solution is needed, one tries to "find an $\boldsymbol{x}$ that makes $A\boldsymbol{x}$ as close as possible to $\boldsymbol{b}$" (Lay et al., 2016, p. 362). This is consistent with a subspace orientation to the least squares problem (i.e., finding the best approximation in a subspace to a vector not in a subspace), which we leverage in our design research, as compared to a statistical interpretation. The set of least squares solutions to $A\boldsymbol{x} = \boldsymbol{b}$ are the vector(s) $\widehat{\boldsymbol{x}}$ that are solution(s) to $A^T A\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$. How can we, as instructors, help students come to understand why that equation is relevant to solving the least squares problem? Or, how can we, as curriculum designers, engage students in the guided reinvention of that equation? In this paper we pursue the research question: How do students use literal symbols and equation types as they solve an experientially real least squares problem, and what reactions do they share about the least squares equation $A^T A\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$?

The work presented in this paper is from a project (DUE-1915156/1914841/1914793) that aims to create research-based curricular materials for the guided reinvention of core concepts within an inquiry-oriented linear algebra class. This work is guided by making sense of student thinking and the complexity of mathematical ideas, which informs the refinement of the curricular materials. In pursuit of our research question, we examine written data from 13 students and analyze the variety of ways they use literal symbols and equations when solving a least squares problem in an experientially real (Gravemeijer, 1999) task setting. We also analyze the students' reactions to the least squares equation $A^T A\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$. This analysis will inform a future conceptual analysis of the mathematics as well as refinements to the task sequence.

Theoretical Framing and Literature Review

Our approach to design research is informed by what we refer to as the Design Research Spiral (Wawro et al., 2022), which is based on the design research cycle (e.g., Cobb et al., 2003), is composed of five phases, with revisions occurring between each phase based on ongoing analyses of student thinking and reflections on the mathematics. Within the first phase, the Design phase, our work is based in Freudenthal's philosophy of mathematics as a human activity (1973) and the design principles that emerged from his work in Realistic Mathematics Education

(RME) (Gravemeijer, 2020; Gravemeijer & Terwel, 2000; Treffers, 1987). RME design principles include didactical phenomenology (the means for creating the task setting of the phenomena to be organized), emergent models (a process through which students can progress from a less formal understanding of the phenomena to a more mathematized organization of the phenomena), and guided reinvention (a mechanism by which students can reinvent mathematical ideas guided by the task structure and their interactions with the instructor and their peers).

There is little literature on the teaching or learning of least squares approximation within the context of linear algebra. Turgut (2013) designed a lesson to teach least squares as a line of best fit using Mathematica. His lesson invited students to use commands in Mathematica to plot dots, form matrices from points, transpose a matrix, and take an inverse of a matrix in finding the best possible solution. A topic we view as related to the teaching and learning of least squares is dot product. Donevska-Todorova (2015) identified three definitions of the dot product within three modes of description (arithmetic, geometric, abstract-axiomatic) and created an applet to promote students' geometric understanding of the dot product. Cooley et al. (2014) developed a module to teach dot product, focusing on the cosine definition. Their task included comparing frequency vectors and determining if an author wrote two different texts. Dray and Manogue (2006) found projection to be essential in understanding dot product. They claimed the geometric approach of the dot product benefits students in many applications of physics and engineering.

As we examined students' work on the question shown in Figure 2, we were struck by the variety of ways literal symbols and equations were used as students completed the problem. In undergraduate mathematics, literal symbols are used in many ways. It is important to understand students' interpretation of literal symbols and how they are used when solving problems. For our analysis, we draw from Philipp (1992) and Drijvers (2003) for unpacking the nuance in literal symbol use. Drawing on works such as Keiran (1988), Küchemann (1978), and Usiskin (1988), Philipp uses literal symbol to describe the mathematical use of a letter; he provides seven literal symbol uses: labels, constants, unknowns, generalized numbers, varying quantities, parameters, and abstract symbols; most relevant to our work is unknown, varying quantity, and parameter. Philipp states that an "unknown involves the use of a literal symbol when the goal is to solve an equation" (p.558), such as in the role of x in $8x + 4 = 28$. Philipp's use of varying quantity is consistent with Knuth et al.'s (2005) definition of variable as "a literal symbol that represents, at once, a range of numbers" (p. 70), such as x and y in $y = 3x + 5$. Finally, Philipp describes parameters as generalized constants, such as m and b in the linear equation $y = mx + b$.

Drijvers (2003) focused on design research related to the concept of parameter, which he sees as "an 'extra variable' in a formula or function that makes it represent a class of formulas, a family of functions and a sheaf of graphs" (p. 60). Drijvers delineates four roles that a parameter can assume: placeholder, changing quantity, generalizer, and unknown. First, a *parameter as a placeholder* plays the role of a constant value that does not change; whether known or unknown, its value is fixed, and filling in different known values relate to different situations rather than variations of the same situation. Second, a *parameter as a changing quantity* represents a numerical value that takes on a dynamic character of systematic variation. It runs smoothly through a reference set, affecting the complete, global situation set rather than a single situation (e.g., p and q in $y = (x - p)^2 + q$). Third, a *parameter as a generalizer* does not stand for a specific number but rather for an exemplary number or set of numbers. It facilitates seeing the general in the particular, formulating solutions at a general level, and solving concrete cases at once by means of a parametric general solution (e.g., t in a parametric solution $x = t(1,2,3)$). Fourth, the *parameter as unknown* facilitates "selecting particular cases from the general representation on the basis of an extra condition or criterion. In such situations, the parameter acquires the role of unknown-to-be-found" (p. 69) (e.g., solve for t for a specific x in the above). Finally, these roles are not fixed and can change in the solution process.

The Task Setting

Our task sequence leverages the subspace-oriented version of the least squares problem (the best approximation to a vector not in a subspace is its orthogonal projection onto the subspace). We designed an experientially real task setting called *Delivering Mail to Gauss*, which is based on the Magic Carpet Ride sequence (Wawro, et al., 2012) in the Inquiry-Oriented Linear Algebra (IOLA) curriculum (Wawro et al., 2013). Even though it is a fantasy setting, we have found that students can immediately engage with the idea of different transportation modes, each traveling forward and backward in a single vector direction. The first Delivering Mail task asks students to use three specific travel vectors (the same vectors in Task 3 of the Magic Carpet Ride sequence) to travel to Gauss in $\mathbb{R}^3$ so that they can deliver his mail. This differs from the Magic Carpet ride task in that Gauss is now in a location outside of the span of the travel vectors. Using previous knowledge, students determine Gauss cannot be reached and the travel vectors span a plane in R^3. Students are then told their cousin has a drone they can use to deliver Gauss's mail, on the condition that they get as close as they can to Gauss using the travel vectors before they use the drone. They then determine where to travel to, how to get there with the travel vectors, along what vector the drone would travel, and what distance the drone's trip would be (Figure 1a).

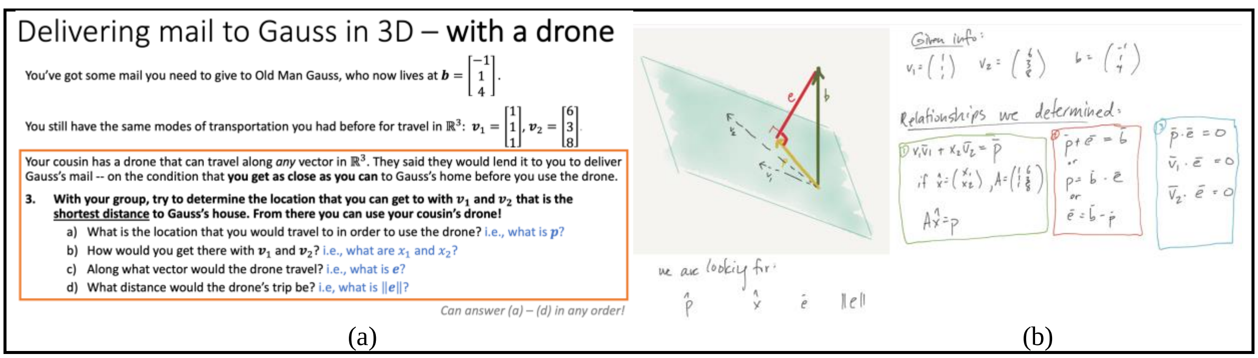

Figure 1. The setup of Task 3 and the class's shared development of known relationships to help solve the task.

There are many aspects of the problem to symbolize. Gauss's location is denoted as $\boldsymbol{b}$. The three travel vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$ are in the first task statement, but once students realize the three span a plane, they work only with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. The sequence is designed to foster students' exploration of what location on the plane would be closest to Gauss, and students consistently suggest the location that creates a path *orthogonal* from the plane to Gauss (Lee et al., 2022). With an instructor's suggestions for which literal symbols to choose, the class uses vector $\boldsymbol{p}$ to denote where on the plane they should travel to and use the drone, **e** as the drone's path to Gauss, and $||\boldsymbol{e}||$ as the distance of the drone's trip. The class symbolizes the relationship between these as $\boldsymbol{p} + \boldsymbol{e} = \boldsymbol{b}$. To denote how to get to $\boldsymbol{p}$ using the travel vectors, the scalars x_1 and x_2 are used to mean how much and in what direction to travel on $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ so that $x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2 = \boldsymbol{p}$, which can be written as a matrix equation $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$ where A is the matrix with columns $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and $\widehat{\boldsymbol{x}}$ is the vector with components x_1 and x_2. Finally, the instructor leads the class in a derivation that the dot product of two orthogonal vectors is zero, which results in the class denoting $\boldsymbol{v}_1 \cdot \boldsymbol{e} = 0$, $\boldsymbol{v}_2 \cdot \boldsymbol{e} = 0$, and $\boldsymbol{p} \cdot \boldsymbol{e} = 0$. All of these relationships are summarized in Figure 1b, which shows one instructor's written work that recorded the relationships during class.

Students then use these known relationships to solve for $\boldsymbol{p}$, $\widehat{\boldsymbol{x}}$, $\boldsymbol{e}$, and $||\boldsymbol{e}||$. Students make progress in a variety of ways. If students combine $\boldsymbol{v}_1 \cdot \boldsymbol{e} = 0$ and $\boldsymbol{v}_2 \cdot \boldsymbol{e} = 0$ into a system of equations and write it as an augmented matrix, the instructor could notate the coefficient matrix as A^T and leverage the student work towards the matrix equation $A^T\boldsymbol{e} = \boldsymbol{0}$. The final task prompts students to "Combine three of our main equations, $A\widehat{\boldsymbol{x}} = \boldsymbol{p}$, $\boldsymbol{p} + \boldsymbol{e} = \boldsymbol{b}$, and $A^T\boldsymbol{e} = \boldsymbol{0}$, to come up with one *general equation* (symbols only, no specific numbers) that would help us determine $\widehat{\boldsymbol{x}}$ for any A and $\boldsymbol{b}$. The only unknown in your *general equation* should be $\widehat{\boldsymbol{x}}$ (i.e., no $\boldsymbol{p}$ or $\boldsymbol{e}$)." Student work on this task allowed for the guided reinvention of the least squares equation $A^TA\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$ (abbreviated LSE), where $\widehat{\boldsymbol{x}}$ is the least squares solution to $A\boldsymbol{x} = \boldsymbol{b}$.

Methods

The data for this paper come from in an in-person introductory linear algebra class at a large, public, research university in the Mid-Atlantic US. The course had 27 students, of which 13 both gave consent and completed the assignments analyzed in this paper. In the university system, 8 of these students chose he/him/his pronouns (pseudonyms begin with "M"), 4 chose she/her/hers pronouns (pseudonyms begin with "W"), and 1 did not choose pronouns (pseudonym P1). Most were second-year students by credit hours and were general engineering majors. The prerequisite was a B or higher in Calculus I or a passing grade in Calculus II. The data analyzed in this paper come from student responses to two written reflections. After most class sessions, students were asked to complete a reflection by the end of the day and submit their work via an online learning management system. Students were asked to spend 5-10 minutes on a reflection, for which full credit was awarded based on effort rather than correctness. Reflection #1 (Figure 2) was given the day that the class completed their solutions to the Delivering Mail to Gauss tasks and the reinvention of $A^TA\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$ (the LSE). The purpose of this reflection was to learn more about how students were making sense of the various aspects of the least squares problem and what solution approaches they would use. Reflection #2 (Figure 3) was given the following day to learn more about how students were making sense of the least squares equation.

Create your own example of two travel vectors in $\mathbb{R}^3$ and a location for Gauss in $\mathbb{R}^3$that you cannot reach with your travel vectors. Then solve for at least two of the following: the vector closest to Gauss that you can reach, how you would get there with the travel vectors, and the distance from that location to Gauss. Show your work and/or explain your thinking

Figure 2. Reflection prompt #1.

We started Least Squares Approximation with the "delivering mail to Gauss" scenario, eventually deriving the equation $A^TA\widehat{\boldsymbol{x}} = A^T\boldsymbol{b}$ as a way to directly solve for $\widehat{\boldsymbol{x}}$. We are really curious about your reaction to the equation. In 2-3 sentences, please share with us how you are making sense of it, your thoughts, or any questions you may have.

Figure 3. Reflection prompt #2.

To analyze the data, we began by creating thick descriptions for student responses to Reflection Prompt #1. In doing so, we were struck by the variety of ways in which most students leveraged the known relationships from Figure 1 to reach a solution for the travel scenario that they created, rather than using the LSE formalized in class that day. In order to capture the nuance of students' solution processes, we decided to focus on the diversity in their use of literal symbols and equation types. We studied the related literature and found Philipp's (1992) characterization of *literal symbols* and Drijver's (2003) characterization of *parameter* to be particularly appropriate; thus, we coded the data within their frameworks (summarized in the Theory section). We also studied the literature related to students' use of various types of equations in linear algebra, and we found Zandieh and Andrews-Larson (2019) to be most helpful. Their work, which is grounded in their prior research on three interpretations of $A\boldsymbol{x} = \boldsymbol{b}$ (Larson & Zandieh, 2013), characterizes students' symbolizing while solving linear systems. We coded the data in a way compatible with their approach, analyzing the various *equations types* (e.g., vector equation, matrix equation) students brought to bear in their solution process. For Reflection #2, we engaged in open coding to make sense of the variety of student responses. The first two authors independently coded all the data, conferred with each other to resolve any differences, discussed the data with the author team, and further refined as needed.

Results

Overall, in Reflection Prompt #1, we found that students use literal symbols as unknowns in three ways: as vectors, vector components, and scalars. We found that students used literal symbols as parameters in three ways: as placeholder, generalizer, and unknown. Students also leveraged six equation types in their problem solving: matrix equation, vector equation, system of linear equations, augmented matrix, dot product equation, and quadratic equation. Because of space, we focus on a vignette from one student. We chose M7 as a paradigmatic example because of the broad range of literal symbols and equation types that he used. A limitation of our data is that it is written data only; we cannot know how the students were thinking about the various symbols they wrote. Instead, we focus on how the literal symbols seemed to function in use; for this reason, our analysis makes claims such as "$\boldsymbol{e}$ and $\boldsymbol{p}$ are unknowns" rather than "the student reasoned about $\boldsymbol{e}$ and $\boldsymbol{p}$ as unknowns." To help the reader follow the analysis, we use italics for literal symbol use and underlining for equation type within the vignette. We do not label the equations that are of the type *unknown* = *determined value,* which communicate when a student completes a solution process for the unknown vector, component, parameter, or scalar.

M7's Work on Reflection Prompt #1

At the top of his page (see Figure 4), M7 wrote the relationships $\boldsymbol{e} \cdot \boldsymbol{p} = 0$, $\boldsymbol{e} \cdot \boldsymbol{v}_1 = 0$, and $\boldsymbol{e} \cdot \boldsymbol{v}_2 = 0$, which had been established in class (Figure 1). These three equations are <u>dot product equations</u>; within them are four literal symbols: $\boldsymbol{e}$ and $\boldsymbol{p}$ are *unknown vectors*, and vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are each a *parameter-as-placeholder*. M7 shifts to a <u>system of equations</u> $e_1 + 2e_2 + 3e_3 = 0$, $e_1 + e_2 + e_3 = 0$ created from the latter two dot product equations. This introduces three new literal symbols– e_1, e_2, and e_3 –which are each *unknown components.* We note that e_1, e_2, and e_3 are the components of the $\boldsymbol{e}$; to eventually solve for unknown vector $\boldsymbol{e}$, M7 decomposed it into unknown components. M7 transitions to an <u>augmented matrix equation</u>, using it four times as he carries out row reduction. M7 expresses the solution resulting from the row reduction as a <u>system of linear equations</u> $e_1 = e_3$ and $e_2 = -2e_3$, which again use e_1, e_2, and e_3; in this instance, however, the three literal symbols are now used as *varying quantities* because the system expresses how they are related and change together. It seems that M7 next compacted this information into $\boldsymbol{e} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$. M7 did not explicitly write the "$\boldsymbol{e} =$", but he substituted the vector in for $\boldsymbol{e}$ twice in his subsequent work. Thus, at this point in his work, M7 used the literal symbol t as a *parameter-as-generalizer* to represent all possible solutions for $\boldsymbol{e}$.

M7 then writes the <u>vector equation</u> $\boldsymbol{b} - \boldsymbol{e} = \boldsymbol{p}$. Here, $\boldsymbol{b}$ is a *parameter-as-placeholder* and $\boldsymbol{p}$ is still an *unknown vector.* However, we see a shift in $\boldsymbol{e}$ from vector unknown to *parameter-as-placeholder*; this is evidenced by the subsequent <u>vector equation</u> in which M7 substitutes in component-wise versions of both $\boldsymbol{b}$ and $\boldsymbol{e}$, using the parameterized version of $\boldsymbol{e}$, where again t functions as a *parameter-as-generalizer*. M7 then simplifies that vector equation into $\boldsymbol{p}$ in terms of t. Next, M7 next brings in the very first known relationship he had written, $\boldsymbol{e} \cdot \boldsymbol{p} = 0$, but now that dot product equation is written with component-wise expressions in terms of t for both $\boldsymbol{e}$ and $\boldsymbol{p}$. Thus, we see a shift in the role of t to that of *parameter-as-unknown*. This use of t continues in M7's simplification of the dot product equation into a quadratic equation in t; two additional quadratic equations are written as M7 simplifies in order to solve for t. This leads to M7's solution $t = 0, \frac{1}{6}$, where t as a literal symbol is a *determined value* (i.e., the values of t that make the equation true, determined through a solution process). Choosing $\frac{1}{6}$ as an *assigned value* for t (Alaee et al., 2002), M7 gets exact solutions for $\boldsymbol{e}$ and $\boldsymbol{p}$ via substitution. M7 boxes the $\boldsymbol{p}$ vector and writes "closest vector we can reach." M7 completes his work by using a vector equation to write $\boldsymbol{p}$ as a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, with the literal symbols s and t as *unknown scalars*. Although M7 again uses t as a literal symbol, we see no evidence that the two uses of t were connected in any way. M7 transitions to a system of linear equations in s and t and solves for the unknown scalars $s = -\frac{1}{2}$ and $t = 4/3$; M7 concludes by explaining how to use the travel vectors, presumably to reach $\boldsymbol{p}$, which is consistent with his algebraic solution.

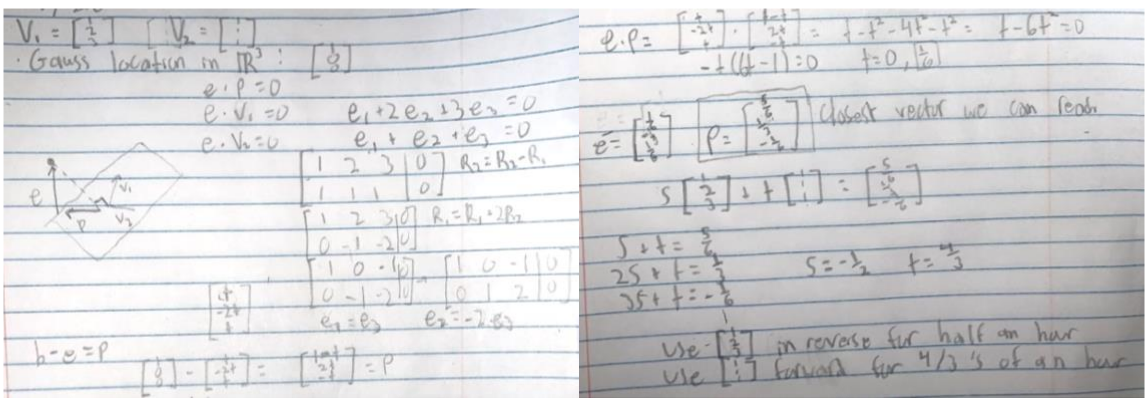

Figure 4. Student M7's written solution for Reflection 1.

We emphasize that the first augmented matrix M7 wrote corresponds to the matrix equation $A^T\boldsymbol{e} = \boldsymbol{0}$; he did not use the literal symbol A^T to notate his work, and we have no evidence he recognized the coefficient matrix as A^T. We point this out as an implicit use of A^T that grows out of the student's own problem-solving, which is important in terms of RME-inspired curriculum design and the class's use and understanding of the least squares equation $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$.

Student work on Reflection Prompt #2

When asked about their sense-making, thoughts, or remaining questions they may have about $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$ (Figure 3) after the following class, students shared a range of reactions about both individual parts of the equation and it as a whole.

Reactions to individual parts of the equation were typically related to the interpretation of its components. Among these, the most common topic invoked by students was that of A^T, both its properties and its function in the LSE. For example, W4 shared, "I'm wondering how $A\boldsymbol{x} = \boldsymbol{b}$ and $A^T A\hat{\boldsymbol{x}} = A^T\boldsymbol{b}$ don't have the same solutions since it's just multiplication A^T to both sides." We note that W4 wrote $\boldsymbol{x}$ rather than $\hat{\boldsymbol{x}}$ in her LSE, helping us understand that the nuance between $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ may not be straightforward for students. W7 wrote, "It may be because I'm still a little confused as to how transposing a matrix effects [sic] the original image, but I still don't understand how it adds possible solutions to the equation." We interpret the first part as W7 trying to make sense of what A^T means as a linear transformation, which we see as a valuable curiosity. We interpret the second part as W7 grappling with how the two equations are related and what it means to be a solution, which seems related to W4's response. We do have evidence that some students understood the utility of the transpose matrix; for example, M6 wrote "$A^T\boldsymbol{e}$ does the job of dot product each column of A by $\boldsymbol{e}$, that's why $A^T\boldsymbol{e}$ would yield the zero vector."

Other students commented on the efficiency of the LSE or contextualized their understanding on how to use the LSE within the class's work together. For example, M2 wrote "It condensed a mess of variables and diagrams into a single expression," and M8 said "The formula is quite simple to use" (M8). It appears that some students' comfort in using the equation related to their understanding of its derivation. For example, W5 wrote: "I would not know what to do with [the equation] if I didn't understand the derivation." It is unclear what W5 means by "what to do," possibly meaning use the equation to solve a least square problems or knowing what each literal symbols means in the context of least square problems. W6 wrote: "The equation makes sense to me based on how we derived it based on what we knew. However, I don't really understand how it all works together/why it all works." It is unclear what the distinction is for W6 between the LSE making sense and understanding why it works.

Discussion

The method of Least Squares is an important topic in linear algebra, although it is not always discussed in a first course possibly because of the background needed to understand all aspects involved in the SLE solution method. The first task in the least squares sequence, *Getting Mail to Gauss*, is straightforward enough to introduce in the first or second week of an introductory linear algebra course (such as after the second task of the *Magic Carpet Ride* IOLA sequence, Wawro et al., 2012). As explored in this paper and in our previous work (Lee et al., 2021), however, the solution process may bring to bear equation types and solution strategies learned across the entire introductory course. M7's work above illustrates the range of equation types and literal symbol use that this student has knowledge of and can flexibly move between in reconstructing a successful solution method.

One of the key features of the LSE is the presence of the matrix A^T. The student work in this study (not all of which we could share in limited space) suggests some connections the students were making between the LSE and A^T. While some students used A^T immediately in Reflection #1, such as M8, others such as M7 derived it through their solution process without labeling with the literal symbol A^T, such as M3, M7, and P1. The use of the array of numbers that experts think of as A^T was not problematic for students such as M7 when using these as coefficients in a system of equations or within the related augmented matrix equation. Students in this class were familiar with converting between systems of equations (or augmented matrices) and matrix equations of the form $A\boldsymbol{x} = \boldsymbol{b}$. So, converting equations such as M7's initial work into the expression $A^T\boldsymbol{e} = \boldsymbol{0}$ was not itself problematic for students; however, thinking of this array as a matrix that A, $A\hat{\boldsymbol{x}}$, or $\boldsymbol{b}$ can be multiplied by seemed to be something students wondered about. The questions some students such as W7 and W6 wondered about on Reflection #2 seemed to be about the meaning or role of A^T when multiplied by the other expressions in the LSE. Given that students in an IOLA classroom tend to be familiar with reasoning about a matrix times a vector as a transformation of that vector (Andrews-Larson et al., 2017), these students may have wondered what transformation A^T imparts on input vectors.

Our future work involves further analyzing the nuances in student solution strategies and their conceptual understanding of the LSE, using this to make adjustments to the task sequence, and developing a conceptual analysis for the mathematics in least squares approximation.

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Two Vignettes on Students' Symbolizing Activity for Set Relationships

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Mathematicians often use set-builder notation and set diagrams to define and show relationships between sets in proof-related courses. This paper describes various meanings that students might attribute to these representations. Our data consist of students' initial attempts to create and interpret these representations during the first day of a paired teaching experiment. Our analysis revealed that neither student imputed or attributed our desired theoretical meanings to their diagrams or notation. We summarize our findings in two vignettes, one describing students' attributed meanings to instructor-provided set-builder notation and the other describing students' imputed meanings to their personally-created set diagrams to relate pairs of sets.

Keywords: Symbolization, representations, student thinking, sets, set relationships

Introduction and Literature Review

Mathematics is a unique science in which objects of analysis are inaccessible to the five senses and can only be visualized indirectly using various representations (Duval, 2006). Theories of representation abound in the mathematics education literature (e.g., Duval, 1999, 2006; Godino & Font, 2010; Goldin, 2008; Radford, 2013; Vergnaud, 1998). Previous researchers have called for increased visual representations (e.g., Arcavi, 2003) and identified privileged forms of representation for specific mathematical topics (e.g., González-Martín et al., 2011). Diverse representations provide opportunities to construct shared meanings, which invite further investigation into how students invest representations (standard or not) with meaning.

This paper investigates undergraduate students' symbolizing activity about sets and set relationships. We define *symbolizing activity* as a process of mental activities that entails students' creation or interpretation of a perceptible artifact (writing, drawing, gesture, verbalization) to organize, synthesize, or communicate their thinking. We refer to *symbolization* as the status of completing the symbolizing activity and perceptible artifacts as *symbols*. Our definition differs from Tillema's (2010) communication-focused symbolizing activity by including individuals' creation of personal representations to reflect on their thinking.

As part of our investigation into the role of set-based reasoning in students' comprehension of conditional statements (Dawkins, 2017; Dawkins et al., 2021), we created an instructional sequence for students to investigate sets using set-builder notation and diagrams. We present two vignettes detailing the various meanings students imputed to these representations during their initial exposure to these ideas. We provide the following research question to contextualize the vignettes: *What differences in thinking did students exhibit as they (a) made sense of set-builder notation and (b) created set diagrams to describe relationships between sets?*

Theoretical Perspective

One problem with studying representations in isolation from student thinking is that students can impute various ideas to the same symbol. For instance, Gray and Tall (1994) stated that mathematicians utilize algebraic notations (e.g., the numeral 6) fluidly to refer to either a process (e.g., putting together two and four pennies) or a stabilized concept (e.g., a pile of 6 pennies). Thompson (2002) showed differences between individual student meanings for a point on a graph, even after students agreed upon a group definition. Alternately, Eckman and Roh (2022) showed that students might generate one notation to reason about a process they conceive and a different convention to describe a concept they abstract from reflecting on the process.

We propose two constructs to describe the representations we investigate in this paper. First, Eckman and Roh (2022) used the term *personal expression* to describe students' imputation of meaning to a self-generated algebraic expression. In this paper, we expand the definition of *personal expression* to cover all forms of students' mathematical representation. There are two components to personal expressions: a meaning and a perceptible artifact to which the student imputes their meaning. We use the term *meaning* in the constructivist sense (Thompson, 2013; Thompson et al., 2014) that individuals construct and maintain cognitive structures through their experience. A *perceptible artifact* includes any action or product a student produces to convey their meanings (writing, drawing, gesture, verbalization), which another individual might observe with his five senses. Our definition of personal expression is related to de Saussere's (2011) notion of *signifier* and *signified*, which also informed Glasersfeld's (1995) definition of symbol.

When a student creates an expression to organize or synthesize her thinking, she creates a *personal expression*. We consider all non-student generated expressions that require the student to anticipate the expression creator's intended meaning to be *communicative expressions*. There are three components to a communicative expression: (a) the creator's intended meaning, (b) the interpreter's evoked meaning, and (c) the perceptible artifact the creator uses to convey their intended meaning. For instance, Jill might create the personal expression $S = \{x \in \mathbb{Z} | x \text{ is prime}\}$ to denote her image of the set of prime numbers. If Jill presented her personal expression to Jack, Jack would perceive $S = \{x \in \mathbb{Z} | x \text{ is prime}\}$ as a communicative expression to which he would need to assign meaning. However, Jack's evoked meaning from Jill's personal expression may not reflect Jill's intended meaning. In summary, whether a perceptible artifact is a personal or communicative expression is in the eye of the beholder. The expression $S = \{x \in \mathbb{Z} | x \text{ is prime}\}$ is personal to Jill because she created it and communicative to Jack because he must interpret it.

Methodology

The data we present in this paper come from an ongoing project to develop models of students' abstraction of logic for conditional statements (Dawkins, 2017; Dawkins et al., 2021). During the study, we conducted six paired constructivist teaching experiments (Steffe & Thompson, 2000), consisting of 8-12 sessions lasting 60-90 minutes each. We focus on the first day of the Spring 2022 teaching experiment. Our students, who chose the names Sarah and Carl, were enrolled in Calculus 3 at a large public university in the United States. The second author served as the teacher-researcher, with all other authors serving primarily as witnesses.

Students' work was collected via video recording, a shared whiteboard application, and photographs of physical board work. We analyzed the data using the principles of open coding (Strauss & Corbin, 1998). As our initial codes emerged, we realized that some findings aligned with previously proposed constructs (Dawkins et al., in preparation; Sellers et al., 2021). During axial coding, we combined our unique codes with these constructs to describe meanings students exhibited during their symbolizing activity. We synthesized our findings into two vignettes. The first vignette describes meanings that students might attribute to *communicative expressions* of set-builder notation. The second vignette describes how students might construct set diagrams as *personal expressions* to express their image of the relationship between two sets.

Results

For each vignette, we first present a theoretical model of a beneficial meaning for representing sets or set relationships. We then offer two alternative meanings from our data that students exhibited in their symbolizing activity.

Vignette 1: Students' Meanings for Set-builder Notation (Communicative Expressions)

We initially presented Sarah and Carl with pairs of sets (defined using set-builder notation) and asked them to posit relationships between the sets (see Figure 1). Our examples constituted *communicative expressions* because students interpreted our notation. In this vignette, we report on students' imputed meanings to set-builder notation while comparing set pairs α, β and α, γ.

Given the set $\mathbb{T}$ of all triangles, answer these questions about each pair of sets:

1) Is there anything in both sets?
2) Does one set contain all the members of another?
3) Can you say anything more about the relationship between the sets?
4) If you use an oval region to represent one set, how would you portray the other in relation?

$\alpha = \{\Delta ABC \in \mathbb{T}: \Delta ABC \text{ is isosceles}\}$	$\beta = \{\Delta XYZ \in \mathbb{T}: \Delta XYZ \text{ is equilateral}\}$
$\alpha = \{\Delta ABC \in \mathbb{T}: \Delta ABC \text{ is isosceles}\}$	$\gamma = \{\Delta RST \in \mathbb{T}: \angle R \cong \angle S\}$
$\alpha = \{\Delta ABC \in \mathbb{T}: \Delta ABC \text{ is isosceles}\}$	$\eta = \{\Delta JKL \in \mathbb{T}: \Delta JKL \text{ is not isosceles}\}$

Figure 1. An excerpt from Task 1. The prompt is shortened for brevity, and not all pairs of sets are shown.

A beneficial way to interpret communicative expressions for set-builder notation. We first present a theoretical meaning that did not emerge in our data but we considered beneficial for students to compare two sets appropriately (see Figure 2).

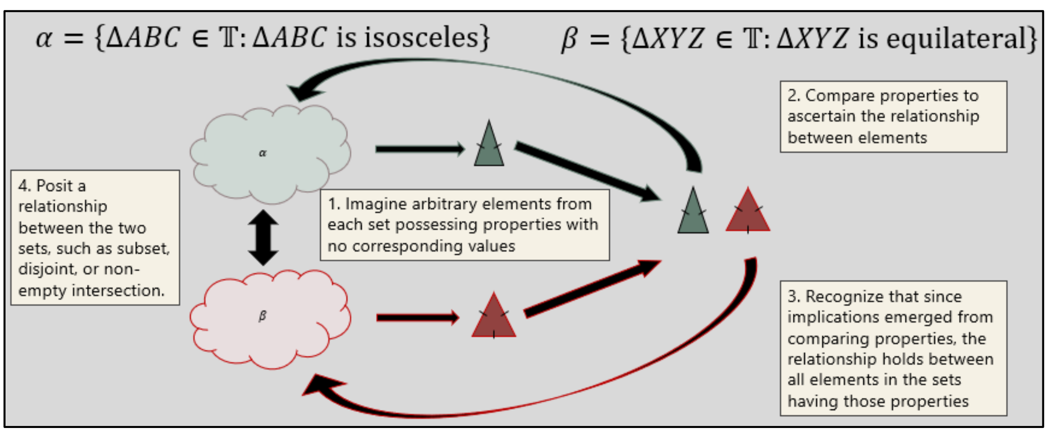

Figure 2. A productive meaning and resulting method for comparing sets α and β.

A student determining the relationship between sets α and β might first imagine arbitrary elements, ΔABC and ΔXYZ, from each set (Figure 2, step 1). These arbitrary elements contain no specific measurements for characteristics such as angle measure. The student would then compare the properties of the two elements to determine their relationship (Figure 2, step 2). For instance, the student might imagine that since the isosceles triangle has at least two equal sides and the equilateral triangle has exactly three equal sides, the equilateral triangle can be considered isosceles. The student would then infer that since every element in set β has exactly three equal sides, all equilateral triangles can be considered isosceles (Figure 2, step 3). Finally, the student would conclude that if all equilateral triangles are isosceles, then set β must be a subset of set α (Figure 2, step 4). Conventionally, we call this meaning for ΔABC or ΔXYZ an *arbitrary particular*. However, our theoretically propitious meaning was distinct from the meanings exhibited by Carl and Sarah. In the following data-driven examples, we show two meanings these students attributed to set-builder notation while comparing pairs of sets.

Meaning 1a: Particular $\boldsymbol{\Delta ABC}$. When Sarah read the teacher-researcher's communicative expressions $\alpha = \{\Delta ABC \in \mathbb{T}: \Delta ABC$ is isosceles$\}$ and $\beta = \{\Delta XYZ \in \mathbb{T}: \Delta XYZ$ is equilateral$\}$, she imagined specific triangles with corresponding values and labels unique to each triangle.

Sarah: I have a question. (Interviewer 1: Ok.) The left-hand side set (α) is congruent to the right-hand side set (β) but they're two different triangles…or are they the same triangle? Like, they are different sets of triangles, right?

(omitted dialogue)

Interviewer 2: Sarah, just to clarify, was your question in part about set α has the letters ABC and set β has the letters XYZ?

Sarah: Yes.

Interviewer 2: And so, because there are different letters, you weren't sure if the triangles were the same triangles?

Sarah: Yeah, I just got a little bit confused on that.

In this example, Sarah questioned interviewer 1 (second author) whether two triangles from sets α and β (which she considered congruent) could be regarded as the same triangle when comparing sets. Eventually, interviewer 2 (first author) asked whether Sarah's confusion emanated from denoting elements of set α with ΔABC and elements of set β with ΔXYZ, which she confirmed. In other words, Sarah comprehended that a triangle ΔABC from set α could be congruent to a triangle ΔXYZ in set β but was unsure whether ΔABC could exist within set β because the vertices of ΔABC in the communicative expression for set α were not labeled with the letters for set β. We call Sarah's evoked meaning for the expression ΔABC a *particular* triangle. We compare Sarah's meaning with our theoretical meaning in the vignette 1 summary.

Meaning 1b: Spontaneous particular $\boldsymbol{\Delta ABC}$. When Carl compared the the communicative expressions ΔABC and ΔXYZ, he imagined various possible pairings between elements in set α and set β and the relationships that might occur for each comparison.

Interviewer 1: What do you think, Carl (about the relationship between sets α and β)?

Carl: Yeah. Um, I thought that it was. I don't think of like, the sets. I thought it more like, it could, like have a good chance of being 100% the same triangle. But also, there's also a good chance that it's close, similar, but not quite. Like 70 or so percent chance.

Interviewer 1: You're talking about one specific triangle?

Carl: Yeah, like comparing ABC to XYZ.

Carl's explanation shows that he was considering two distinct situations: (1) comparing a triangle with exactly two equal sides from set α with a triangle from set β and (2) comparing a triangle with exactly three equal sides from set α with a triangle from set β. Carl's probabilistic language also indicates that he imagined how often the triangles he selected spontaneously were likely to be in both sets. We thus say that Carl's evoked meaning for the expression ΔABC was of a *spontaneous particular* and not an *arbitrary particular* triangle. We further discuss how Carl's *spontaneous particular* meaning emerged in his set diagram personal expressions in vignette 2.

Carl's *spontaneous particular* meaning ΔABC and ΔXYZ is analogous to what Sellers et al. (2021) called an *MQ4* meaning for a quantified variable. Students exhibit an *MQ4* meaning when they spontaneously select elements within a domain of universal discourse, make inferences without exhausting all elements, and may or may not repeat this process to make (potentially different) inferences about other elements.

Summary of vignette 1. In this vignette, we have shown three meanings that a student might have for ΔABC in the communicative expression $\alpha = \{\Delta ABC \in \mathbb{T}: \Delta ABC\ is$ isosceles$\}$. A student exhibiting the *arbitrary particular* meaning (see Figure 2) imagines triangles defined by the properties described in the set-builder notation and would be capable of making general comparisons between sets. Sarah's meaning of *particular* triangles allowed her to imagine elements of sets α, β and γ (Figure 2, step 1) and make rudimentary comparisons between these elements (Figure 2, step 2). However, her image of triangles with fixed values for various characteristics precluded her from discerning the appropriate relationship between the entire sets of objects (Figure 2, steps 3, 4). Carl's meaning of *spontaneous particular* triangles allowed him to imagine random pairings of elements from sets α and β (Figure 2, step 1) and compare them (Figure 2, step 2). In effect, Carl constructed relationships of likelihood, not relationships of necessity as are privileged in mathematical logic, which are essential for proving.

Vignette 2: Students' Meanings for Set diagrams (Personal Expressions)

We also invited Sarah and Carl to construct set diagrams to represent the relationships they envisioned between pairs of sets (see Figure 1, question 4). We consider the diagrams that Carl and Sarah generated (even if they exhibited the conventions of Euler diagrams) to constitute their *personal expressions* for organizing their thinking about various pairs of sets.

Regions that partition: A beneficial way to construct a set diagram. We first present a beneficial meaning that did not emerge in our data which a student might leverage to construct a set diagram to compare sets α and β (see Figure 3). First, the student would imagine the universe of discourse, $\mathbb{T}$, the set of all triangles. The student might then draw a box to metaphorically gather all triangles into an enclosed entity (Figure 3, part 1). Second, the student would consider set α, the set of isosceles triangles. The student might represent α by drawing an oval region inside the box to simultaneously gather all isosceles triangles and partition them from other triangles (Figure 3, part 2). The student would recognize that the region outside the oval denoting α constitutes the complement set to α (α^c). Third, the student would utilize the *arbitrary particular* meaning to determine that all elements of β exist within set α (Figure 3, part 3).

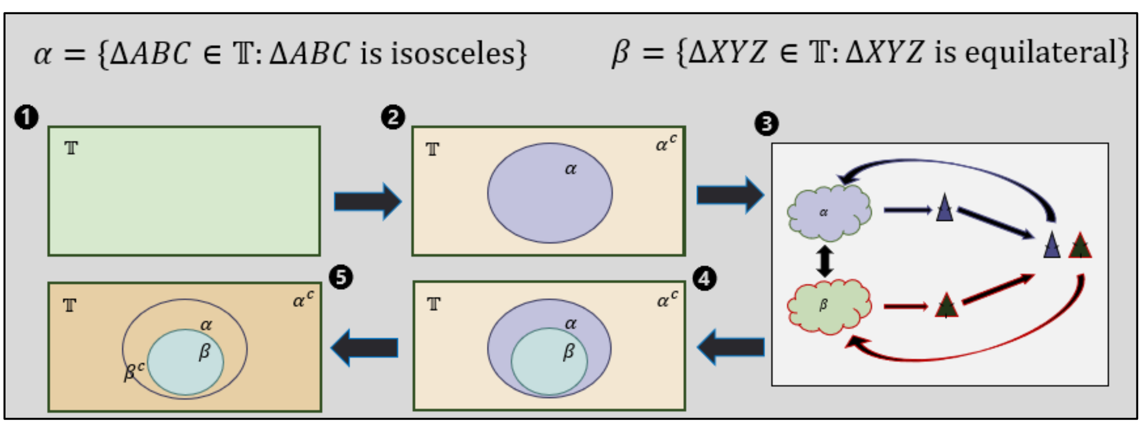

Figure 3. A productive method to produce a set diagram comparing sets α and β.

The student might then draw an oval region to represent set β within his previously created region for α (Figure 3, part 4). The student would realize that his actions (a) gather equilateral triangles from the universal set $\mathbb{T}$ into the oval region β and (b) denote the regions outside the oval for set β as the set of all non-equilateral triangles (β^c; Figure 3, part 5). We have previously used *regions that partition* to describe students' set diagrams constructed through this propitious meaning (Dawkins et al., in preparation). In the following data-driven examples, we report two other meanings that Sarah and Carl attributed to their diagrams comparing set pairs α,η and α, γ.

Meaning 2a: Regions that gather. When Sarah drew her set diagram *personal expression* to relate sets α and η (which are disjoint sets), she drew one oval region to represent α and a second, non-overlapping oval region to represent η (see Figure 4). Sarah then drew isosceles and equilateral triangles to represent the elements she imagined in set α and a scalene triangle to represent the elements she imagined in set η (see Figure 4). When the interviewer asked Sarah what she imagined the region outside α and η to represent, Sarah responded that this area was irrelevant but she could imagine the region as empty if she chose. Sarah's comment indicates that she created her personal expression to represent solely her images of α, η, and their relationship.

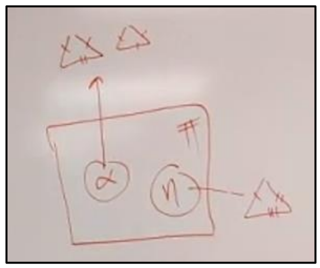

Figure 4. Sarah's set diagram comparing sets α and η (regions that gather).

We have previously defined students' diagrams to which they gave meaning solely to the inside of drawn regions as *regions that gather* (Dawkins et al., in preparation). A student who creates a diagram to express *regions that gather* typically ignores areas outside their drawn regions (i.e., no partitioning). These students use set diagrams to highlight sets of interest, not construe relationships between all the elements within the universal domain.

Meaning 2b: Regions that distinguish. While comparing sets α and γ, Carl initially concluded the two sets were equal and drew a single ovular region (see Figure 5). Carl then claimed that he could further clarify his diagram by drawing a second oval region inside the first. Carl explained that the new oval region denoted instances where he was comparing triangles from α and γ with exactly three congruent sides. He then stated that the region outside of the interior oval (but inside the exterior oval) represented instances where he was comparing triangles with exactly two congruent sides. In other words, Carl introduced a local partition to distinguish two classes of elements he perceived within his gathered set containing elements of sets α and γ. We use the term *regions that distinguish* to describe Carl's separation-of-elements-into-cases meanings he imputed within a locally gathered region of his set diagram.

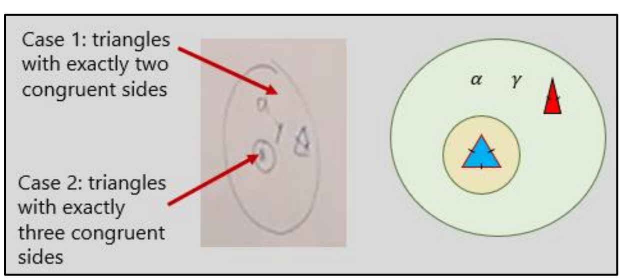

Figure 5. Carl's set diagram for comparing sets α and γ and a digital reproduction (regions that distinguish).

Summary of Vignette 2. In this vignette, we have shown three meanings that students might attribute to the regions they draw in a set diagram *personal expression*. Sarah's meaning allowed her to imagine gathering elements of a similar type, and she drew *regions that gather* to denote her grouping action. Sarah's meanings allowed her to create oval regions to represent the sets α and η and successfully determined that the sets were disjoint (Figure 3, steps 2-4). However, Sarah indicated that the regions outside the ovals were irrelevant to the task. Later in the interview, Sarah claimed that exterior regions contained non-examples of the sets but gave no indication that she considered these elements as the complement of the sets portrayed by her ovular regions (Figure 3, steps 2, 5). We also note that some students creating *regions that gather* may not include a box representing the universe of discourse (see Dawkins et al., 2021; Figure 3, step 1). Carl's meaning allowed him to imagine sorting comparisons between classes of elements, and he drew *regions that distinguish* to denote his sorting of these cases. Carl provided meaning to all areas within his outermost region (Figure 2, steps 2-4). However, Carl created his set diagrams not to partition the universe into sets possessing or not possessing properties (Figure 3, steps 2, 5) but to sort comparisons of set elements. Finally, a student imagining gathering elements into one set while simultaneously creating a complement set draws *regions that partition* to denote this partitioning action.

Discussion and Conclusion

Our research question for this paper was related to students' differences in meaning when creating *personal expressions* and interpreting *communicative expressions* for sets and set relationships. In vignette 1 we described three meanings, one theoretical and two emerging from our data, that students might possess for communicative expressions of set-builder notation presented to them by an instructor. In vignette 2 we described three meanings, one theoretical and two emerging from our data, that students might attribute to personal expressions they create to diagrammatically represent set relationships. Our results show that students can (a) invest only one portion of a conventional meaning to an expression (e.g., regions that gather, particular) or (b) aatribute meanings that allow local comparisons of element classes but fail to support claims about set relationships (e.g., spontaneous particular, regions that distinguish).

Our findings further work done by previous mathematics educators. For instance, we provided an expanded definition of Eckman and Roh's (2022) *personal expression* and proposed *communicative expressions* to describe the role of symbols in mathematical communication. We also added *regions that distinguish* to Dawkins et al.'s (in preparation) description of the meanings students attribute to set diagrams. Finally, we utilized Sellers et al.'s (2021) *MQ4 meaning* to inform our descriptions of *spontaneous particular* and *regions that distinguish,* extending their MQ framework beyond the context of interpreting quantified variables.

Our vignettes also have relevance for instructors. Vignette 1 highlights that students may attribute very different meanings to a communicative expression of set-builder notation than their instructor intended when creating the symbol as a personal expression. Vignette 2 reveals that students' imputed meanings to local regions of their personal expression set diagrams may vary across students and differ from convention. Therefore, we recommend that instructors regularly facilitate classroom discussions about personal and communicative expressions and the potential meanings the expression creators and interpreters attribute to these expressions.

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What is a "Math Person?": Students' Interpretation of Identity Terminology

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We investigated seven undergraduate students' definitions of two terms used in mathematics education research: "identity" and "math person." Students' interpretations of these terms were compared to an existing identity framework (Cass et al., 2011; Cribbs et al., 2015). Participants' definitions for "identity" varied widely, emphasizing different aspects of personal, social, and mathematics identities. In contrast, students were fairly uniform in describing a "math person" as an individual who enjoys and/or is proficient in mathematics, which is consistent with known factors related to mathematics identity. This second finding bears relevance given the use of self-identification as a "math person" in measurements of students' mathematics identity. These results guide our development of survey items meant to assess aspects of student affect in the mathematics classroom, with the ultimate goal of providing instructors actionable insights into their classroom climate.

Keywords: Math Person, Identity, Mathematics Identity

Research has shown that the development of a strong mathematics identity is critical to students' persistence in STEM and success in their math courses (Meece et al., 1990; Boaler & Greeno, 2000; Gonzalez et al., 2020). However, *identity* and *mathematics identity* have been conceptualized and measured in different ways by different scholars (e.g., Sfard & Prusak, 2005; Cass et al., 2011). As part of a project aimed at measuring various aspects of student affect, we reviewed instruments developed to measure identity and became curious about students' interpretation of these constructs – and of the survey questions intended to measure these constructs. With an eye toward understanding the student perspective, as well as to reducing measurement error in our own work, we ask:

(1) How do students conceptualize "(mathematics) identity" and "math person?"
(2) How do students' perspectives relate to usage of these terms in prominent mathematics identity frameworks?

To begin answering these research questions, we interviewed seven undergraduate students enrolled in university mathematics courses and considered their responses in relation to the mathematics identity framework used extensively by Cribbs and colleagues (Cribbs et al., 2015; Cass et al., 2011; Godwin et al., 2016; Dou et al., 2019).

Mathematics Identity & Relevant Literature

The framework we use for mathematics identity (Cass et al., 2011; Cribbs et al., 2015) has its origins in studies of science identity (Carlone & Johnson, 2007), has been refined over time (Hazari et al., 2010), and has been positively linked to students' persistence in STEM majors and career choices (Cribbs et al., 2015; Cass et al., 2011; Verdín & Godwin, 2015; Godwin et al., 2016). In this framework, *mathematics identity* refers to how one views themselves in relation to mathematics; this is distinct from, but not entirely independent of, *personal identity* (one's individual characteristics) and *social identity* (one's characteristics as a member of a group or community). In empirical studies, self-identification as a "math person" (or "science person" etc.) has repeatedly been used as a measure of mathematics (science, etc.) identity (e.g., Cribbs et al., 2015; Gonzalez et al., 2020; Shanahan, 2008; Hazari et al., 2010; Carlone & Johnson, 2007). Three constructs have been shown to contribute to a positive mathematics identity, as measured by seeing oneself as a "math person": competence/performance, interest, and recognition (Cribbs et al., 2015). *Competence/performance* refers to a student's beliefs about their ability to understand and perform in mathematics, *interest* as a student's desire to learn mathematics, and *recognition* as how students think that others perceive them and their mathematical abilities. In this study, we explored seven students' interpretations of "identity" and "math person," and compared these to these interpretations, respectively, to the identity factors (mathematics, personal, social) and mathematics identity factors (competence/performance, interest, recognition) described above.

Studies have demonstrated, with varied methods and populations, that mathematics identity can be linked to several student outcomes. Here, we highlight a few of these findings related to students' persistence and success in STEM classes. In a large study of undergraduate calculus students, Cass et al. (2011) found that three of the factors related to mathematics identity (interest, performance, recognition) were significant predictors of students choosing careers in engineering. In another study, Boaler and Greeno (2000) conducted interviews with high school students and found that those planning to leave the discipline often lacked interest in the procedural nature of the mathematics being taught. This link between mathematics identity and retention in STEM courses has been examined in studies alongside other outcomes, such as success in math courses. For instance, Meece et al. (1990) employed surveys with middle schoolers in suburban communities and found that students' ratings of the importance of math predicted their future enrollment in mathematics courses. They also determined that students' performance expectations in their math courses strongly predicted their subsequent grades. More recent studies have further explored this relationship between students' mathematics identity and performance. For example, Gonzalez et al. (2020) demonstrated a positive correlation between mathematics identity and mathematics GPA for Black secondary students in a large-scale survey across the U.S. From these findings and others, we consider fostering a positive mathematics identity in students to be an important component of their persistence and success in STEM. This study presents a first look at understanding terms associated with mathematics identity, so that we may more accurately and sufficiently assess it in students.

Methods

In Spring 2022, we conducted IRB-approved interviews with undergraduate students at a large public research university in the southwestern United States. These interviews are part of a larger project intending to add questions to the mathematics department's end-of-course evaluations for assessing the current atmosphere in mathematics classrooms at the university. A preliminary version of such a survey included items aimed at measuring students' (1) sense of community, (2) sense of belonging, (3) sense of inclusion in mathematics classrooms, and (4) self-perceptions of ability and identity. As part of the design process, we developed an interview protocol to better understand how students interpret terms and respond to questions.

One purpose of the interviews was to gain insight into the design of the selected questions to inform the revision and refinement of the survey items. Accordingly, the interview was structured to investigate how participants interpreted the terms and questions and to understand how these interpretations informed their responses. This study employs clinical interview methodology using open-ended, think-aloud interview questions (Clement, 2000). The interview protocol consisted of two major parts. First, students were asked to define each of six terms: “sense of community,” “sense of belonging,” “sense of inclusion,” “math person,” “identity,” “classroom climate.” Second, participants responded aloud to free-response and Likert-style items from the survey, while explaining the reasoning behind their responses and their interpretations of the questions as written.

Invitations to participate were distributed via email from the instructor of record to students in all levels of undergraduate mathematics classes. Seven students participated in the interviews and were each compensated with a $15 e-gift card. These students had a wide range of experience with mathematics at the university, with enrollments from college algebra to upper-division proof-based courses. Interviews lasted between 30-60 minutes and were conducted remotely, using web-based video conferencing software.

Data analysis was conducted on de-identified transcripts of the seven clinical interviews. The transcripts were automatically generated by the video conferencing software and then verified and edited for accuracy by members of the research team. Participant responses for the first set of interview items (defining terms) were analyzed using conventional content analysis (Hsieh & Shannon, 2005). This process involved identifying themes and patterns in the students’ responses *without* using preconceived codes. Though our analysis involves comparing students’ responses to an existing mathematics identity framework (Cass et al., 2011; Cribbs et al., 2015), we decided not to use the existing framework for creating initial codes from the interview transcripts. We wanted to ground our developed codes in the key thoughts and concepts from the students’ responses, without preconceived notions of what “identity” or “math person” could mean. Our codes were then refined through several iterations with constant comparison. Final primary codes capture the essence of student responses, with secondary codes and modifiers used as needed for additional specification.

Although this coding process was implemented for all parts of the transcripts, we restrict our attention to students’ definitions of “identity” and “math person” in this study. We leverage our codes to understand how participants define these terms and compare their definitions to the factors that influence mathematics identity formation (Cass et al., 2011; Cribbs et al., 2015). In particular, we examine how these students’ definitions for “identity” relate to the personal, social, and mathematics components of identity and how their definitions for “math person” relate to the performance/competence, interest, and recognition factors of mathematics identity.

Results

Identity

We first present themes which emerged from students’ responses to the item, “What does ‘identity’ mean to you in the context of the mathematics classroom?” Note that, due to time constraints while interviewing, only five of the seven students responded to this question. Representative excerpts from each student are shown in Table 1, with key sections bolded; the right-hand column displays the codes assigned to each excerpt.

There was little to no overlap in themes between the students’ definitions of “identity.” Two students, Student1 and Student2, described “identity” in a general sense, which was decontextualized from the mathematics classroom. Student1 described “identity” as how someone views themselves (and thinks others view them), with particular focus on one's personality and interests. On the other hand, Student2 specifically called for going beyond “just who you are,” and instead defined “identity” as including one’s morals and values. Both students interpreted identity as an "inner" part of oneself but differed in whether they focused on a person's actions or values.

The remaining three students thought of "identity" specifically in the context of their mathematics classroom but were influenced by different components of the classroom. Student3 conceptualized "identity" as what one contributes to the mathematical and social environment of the classroom. They focused on the interactions that one has with their peers and TAs, as well as the effects these interactions have on establishing a hierarchy within the class (teacher > TAs > students). Student4's definition of "identity" was less about interpersonal interactions and more about one's math ability. The impact of the teacher is still significant in this definition, but only insofar as the teacher is able to foster the student's mathematics identity. Student5 also highlighted the similarities between "math person" and "identity," focusing on a person's priorities and interests related to math as central to their identity in the math classroom. However, unlike the previous two students, this definition omitted interactions with people in the classroom and instead emphasized a student's personal motivations.

Table 1. Excerpts of student responses to "What does 'identity' mean to you in the context of the mathematics classroom?" with coding. Emphasis added.

Excerpt	Coding
Student1: It's, it's who you are and what makes you different…If I can play guitar, which I'm trying to do, but I suck at it. **If I suck at playing the guitar, that adds to my identity**, because there are not as many people in the world who suck at playing the guitar. And so you keep adding that. Well, I also like video games. Well, people who suck at guitar and who play video games is even smaller in between. Well, I also like creative writing, and **it keeps getting smaller and smaller until you're your own unique person**.	How you view yourself; How others view you; Skills; Likes and Interests
Student2: Because I think of identity more of like an inner, like a personal level rather than just who are you, what are your credentials. What are, like, **what are your values, what are your morals, what do you support**? How do you, like, do you respect people?	Personality (vs. achievements); Values/morals; Respecting others;
Student3: So, I guess I would identify identity in, like, the math class as, like, like, your own personal identity. Like, **what you do, how you contribute to the classroom at all, if you do**. So, like, if you are someone that's like, "Hey, I'm really good at math. I can help you. Like, here's my phone number, like, I can help tutor you." I feel like that's, like, your sense of identity. Um, I feel like in the math class, because there is very little peer interaction, there isn't really a lot of identity."	Making Contributions (to class); Interactions (student-student)
Student4: I think identity in a math class has quite a bit to do with the teacher. Because in my, in my earlier days, like in middle school and high school, when I was doing math, **I didn't have a very good**	Teacher-dependent; Academic identity;

teacher. So, I was, my main instinct was just to think, "well, my teacher isn't good. I'm not understanding, even though they tried to explain. So, **I'm not very good at math…**So, yeah, it all boils down to, firstly, **if you're a math person**, which we addressed already. But also, your teacher and how you're, how you develop in the math class over the years.**"**	"Math person"; Developing skills
Student5: Um, I don't think [personality traits and interests are as] significant as…like it kind of, **it kind of goes hand in hand with like my definition of a math person**. Like, are you more single minded like, into like, you know, your major and your career, or are you, you know, like balancing kind of a lot of like interests and priorities? Because I know some people who will spend like hours studying for a class that I might spend a couple…you know, like, it just, it just, you know, **some people are just like really into getting everything and getting the theory and I do, I do think that like contributes to identity**.	Personality (not as important); Priorities (single-minded vs. broad interests, studying); Inconsistent definition (self); Likes and interests (concepts, theory)

These responses can be considered in light of an overall framing of identity in mathematics education literature (Cass et al., 2011; Cribbs et al., 2015). Student1 and Student2 did not contextualize their responses in terms of a mathematics classroom; instead, their responses most closely align with descriptions of personal identity (Cass et al., 2011). That is, they centered on interests and values related to the individual person. On the other hand, Student3, Student4, and Student5 contextualized their definitions within the mathematics classroom. While they also referred to aspects of personal identity, these students emphasized the social and mathematics identities that they formed in the classroom. They focused on their ability to do mathematics in the classroom, as well as the various interactions in the classroom that influence their development as a "math person." Within these responses, different components of mathematics identity were mentioned by different students: For example, Student4's response incorporates elements of competence/performance and interest, while Student5's response mostly focused on interest alone.

Math Person

In this section, we present and analyze participant responses to the item, "What does 'math person' mean to you?" Recall that "math person" is a term used to evoke a person with a positive mathematics identity (Cribbs et al., 2015). We identified more commonalities in these responses than in the identity question, with multiple students' entries receiving the same codes. Table 2 presents operationalized definitions of codes assigned to more than one student, as well as how many students (out of seven) were assigned each code. Since these codes were produced through conventional content analysis of the interview data (Hsieh & Shannon, 2005), each code may not be directly associated with the mathematics identity factors from Cribbs et al.'s (2015) framework.

Table 2. Codes for student responses to "What does 'math person' mean to you?"

Code: Description	*N*
Skills: Abilities related to successfully performing a task or action	6
Likes/Interests: Topics or activities that elicit feelings of interest or like, with or without enjoyment	5
Disinterested: Topics or activities that do not explicitly elicit feelings of interest	4
Enjoyment: Pleasure or satisfaction derived from an action or interest	4
Field of study: Reference to field(s) of study other than math	4
"Non-math person": Participant mentions "non-math" people	3
Counter-example: Participant uses a counter-example to explain a concept	3
Math is …: Subjective description of mathematics or its qualities	3
Priorities: Activities or topics that are preferentially attended to	3
Developing skills: Action of improving or expanding skills, intentionally or otherwise	2
Intelligence: Quality of being intelligent/smart, potentially in relation to others	2
Interviewee academic details: Participant gives general academic information about themselves (e.g. major or classes), with or without prompting	2
Logical: Characterized as being able to use formal/informal logical reasoning	2

A majority of the definitions of "math person" included a discussion of one's capacity to perform mathematical tasks, which we coded as *Skills*. These were sometimes explicit statements, e.g., "being good at math" as a prerequisite for being considered a "math person." Other times, students referenced skills that "math people" possessed, such as "calculating on the spot" or being able to "rearrange numbers in their head." One student who did not explicitly state the significance of skills still thought of "math people" as people who "think logically." In some capacity, all participants conceived of a "math person" as someone who thinks with and about mathematics.

Enjoyment of mathematics or mathematical thinking was also an important characteristic of "math people" for five of the seven students. This was observed in explicit statements suggesting that *Enjoyment* is essential, such as "one of the big denominators of if you're a math person is if you enjoy it." It was also implied by statements of *Likes and interests* in reference to logical or quantitative thinking, such as finding the "puzzle aspects of mathematics fun." However, one student did directly note that enjoyment was not essential, stating that "you don't have to like math to be a 'math person.'" This student's reasoning was that, after a certain point, performing mathematics becomes intuitive and does not necessarily require enjoyment (much like breathing). This is not so much a statement of lack of enjoyment as it is a statement suggesting that, after a certain level of mathematical skill attainment, enjoyment may not be a prominent factor in being considered a "math person."

Our findings with respect to "math person" were mostly consistent with a mathematics identity framework (Cass et al., 2011; Cribbs et al., 2015). The competence/performance factor can be seen in the participants' consistent focus on the *Skills* necessary to be a "math person." To them, a "math person" is someone who can perform calculations quickly or has some natural competence when it comes to mathematics. The interest factor is also clearly apparent in participants' mentions of *Enjoyment* and *Likes and interests*. However, recognition was only seen in one of the seven students' descriptions of a "math person." Student3 described a "math person" as "your person to go to" if you have a question, although they also noted that recognition by peers was not a definitive requirement for being a "math person." Overall, the seven students we interviewed did not incorporate external perceptions in their definitions.

Discussion and Next Steps

Participants' definitions of "identity" in the mathematics classroom were highly varied. Some students described only aspects of personal identity, while others referred to all three identity components (personal, social, and mathematical) in their responses. Additional research would be needed to understand this variation in response. In contrast, students were mostly consistent in describing a "math person" as someone that enjoys mathematical processes and has the skills necessary to be proficient at math. These components are closely related to the interest and competence/performance components of a mathematics identity framework (Cass et al., 2011; Cribbs et al., 2015). While students did not explicitly incorporate the recognition component into their definitions, this could be because they were doing the recognizing themselves by defining characteristics of a "math person." That is, the phrasing of the interview question could have made the recognition component of mathematics identity less salient in student responses.

We take these results as evidence that we might want to explicitly define "identity" or avoid the term in surveys altogether, and that using the term "math person" on our survey could be productive. Immediate next steps in the larger project will explore how students interpret terms for other affective constructs linked to student success and persistence in STEM, such as "sense of belonging" (Strayhorn, 2018). We will interview additional students from varied backgrounds from within the university to further enrich our understanding of these constructs. These results will be used to develop a questionnaire that captures meaningful information about the impact of a mathematics course on students' affect, while being brief enough to be appended to end-of-term course evaluations. The current study contributes towards this goal by providing insight into students' understanding of mathematics identity terminology, so that we may create more accurate and actionable items for instructors to assess and foster this form of identity in their students.

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Exploring Relationships Between First-Semester Calculus Students' Self-Regulation Strategies and Their Mathematics Identity and Mathematics Self-Efficacy

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This mixed-methods study examines the self-regulation strategies that first-semester undergraduates use in a first-semester calculus course through quantitative analysis of student survey responses and qualitative analysis of student interviews. The relationship between these self-regulation strategies and both mathematics identity and mathematics self-efficacy was found to be positively correlated. The strategies of metacognitive self-regulation, effort regulation, and time and study environment were found to be the most commonly used strategies by academically successful students based on a large-scale survey given to 188 first-semester calculus students as well as from interviews that occurred throughout the Fall 2021 semester.

Keywords: Calculus, Undergraduate Mathematics, Self-Regulation, Self-Efficacy, Mathematics Identity

The transition from high school mathematics to undergraduate mathematics presents challenges for many students in the United States (Clark, 2005; Gueudet, 2008, Ulriksen et al., 2017). These challenges include difficulty with foundational mathematics content, lack of self-regulation strategies, low degrees of self-efficacy, social challenges of being in a new environment, and institutional challenges of interacting with a new institutional system (Hernandez-Martinez et al., 2011 and Seymour & Hunter, 2019). High failure rates in college calculus as reported by Bressoud et al. (2015), Rasmussen et al. (2019), and Seymour and Hunter (2019) highlight the need for academic intervention for incoming college students. This intervention includes not only attending to the mathematics content understanding needed for success in first-semester college calculus, but also developing effective study strategies and self-regulation strategies employed by successful calculus students. This case study presents findings from a larger study that explores the self-regulation strategies of first-semester calculus students and how those strategies relate to their mathematics self-efficacy and mathematics identity.

To adjust to these challenges, students employ self-regulation strategies as researched by Zimmerman and Pons (1986) in a high school setting and by Wolters (1998) and Johns (2020) at the undergraduate level. In a mathematics setting, self-regulation strategies can be thought of as actions used to overcome some obstacle when working on tasks associated with a mathematics classroom. This can be in relation to specific mathematics problems and strategies used when experiencing some difficulty, or more general strategies or practices regarding tests and studying. In this report findings related to the following research questions are presented and discussed: "In what ways do students adapt self-regulation strategies from high school mathematics to college-level mathematics?" and "How does a student's mathematics identity and/or mathematics self-efficacy relate to their implementation of self-regulation strategies?"

Conceptual Framework

Zimmerman and Pons (1986) and Wolters (1998) identified categories of self-regulation strategy usage in high school students and undergraduate students respectively. The categories developed by these researchers include seeking information, keeping records and monitoring, organizing and transforming information, seeking assistance, self-evaluating, setting goals, providing rewards, changing environmental conditions, and identifying internal motivational factors such as interest, self-efficacy, emotion, attention, and willpower. Zimmerman and Pons argue that as students use the strategies and reflect on how they use the strategies, that their overall strategy usage will improve.

Johns (2020) examined the self-regulation strategies used by students enrolled in a first-semester calculus course, especially as it correlated with academic performance. In Johns' study she examined student responses to the Motivated Strategies for Learning Questionnaire (Pintrich et al., 1991) which gave a quantifiable measure of what strategies students used in a particular course setting. This data was then correlated with overall course grades for her 424 participants, and it was shown that there was a positive correlation between student course grades and the number of self-regulation strategies used.

Researchers find that positive mathematics identity and high degrees of mathematics self-efficacy contribute to students' persistence in a mathematics-based major and higher academic achievement. For example, the ways in which students view themselves as mathematics learners can be a motivating force for whether they will further their mathematics career (Cribbs et al., 2015; Gutiérrez, 2013; Ulriksen et al., 2017). Also, high measures of self-efficacy have been correlated with higher academic achievement, more frequent completion of academic goals, and academic persistence (Bandura, 1997; Hackett & Betz, 1989; Tinto, 2017; Usher & Pajares, 2009). Hernández-Martínez et al. (2011) argue that during times of transition and difficulty there is a restructuring of mathematics identity that can occur for students, such as in the transition from high school to college mathematics.

How self-regulation strategies are used and developed by undergraduates in mathematics courses has not been widely studied while attending to transition to college mathematics and connections between self-regulation strategy usage, mathematics identity, and mathematics self-efficacy. Gueudet (2008) outlined several of the challenges that arise during the transition to tertiary school, especially related to individual, social, and institutional phenomena. For the individual phenomena, Gueudet describes that undergraduates beginning the transition often lack flexibility of switching between different thinking modes, such as practical and theoretical thinking. Sonnert et al. (2020) found that high school preparation was a significant factor on retention and performance for first-year students, but as the distance from high school grows the influences lessens. Though for those first-year students it was preparation in mathematics, and to a lesser extent attitudes about mathematics, that had the largest impact on performance in a calculus class. This is attributed to the highly cumulative nature of mathematics. Self-regulation strategies play an important role in helping students to adapt to and overcome these transitional challenges.

Methodology

This report focuses on two individual student interviews for each of the six first-semester undergraduate students participants enrolled in a first-semester calculus course and participant responses to a large-scale survey. Interviews occurred throughout the Fall 2021 15-week semester. The study occurred at a large, urban research university in the Southwestern United States. All first-semester calculus courses meet two (for 80-minute lecture periods) or three (for 50-minute lecture periods) times a week for lecture. Two additional weekly 50-minute meetings engage students in a collaborative lab activity facilitated by the instructor and graduate teaching assistant (GTA) for one class meeting and traditional recitation activities facilitated by the GTA during the other meeting. Each of the courses have departmentally coordinated exams written by the course instructors, these exams are typically offered the fourth full week of the semester, the ninth full week of the semester, and the final exam during the last week of the semester. Of the five interviews, the two individual interviews occurred within a week of the first midterm and within a week of the final exam. These interviews lasted 20-30 minutes and focused on students' mathematics background, self-efficacy, and identity as well as differences between their experiences in high school and college mathematics. The other three interviews were task-based group interviews, lasting 50 minutes to an hour, occurring a week before each midterm and the final exam.

The initial surveys were sent to thirteen first-semester calculus sections which included approximately 800 students. Of those invited, 188 students completed the initial survey which included background information, the MSLQ (Pintrich et al. 1991), a survey to measure mathematical identity (Kaspersen, 2016), select questions from the Factors Influencing College Success in Mathematics (Sonnert et al., 2020), and a survey to measure the sources of self-efficacy (Usher & Pajares, 2009). Scores for each category were calculated using the scoring rubric provided with the MSLQ. Once self-efficacy, mathematics identity, and self-regulation strategy usage had all been scored, a scatterplot was created. Each score for the MSLQ strategies could range from 0 to 7, where 0 indicated that a participant never used the strategy and higher scores indicated a stronger, more frequent, use of the strategy. From the MSLQ the authors state that a score above 3 indicates an adequate use of the strategy. A weighted average was calculated from the MSLQ scores and related to both the self-efficacy scores and mathematics identity scores on a graph along with the correlation coefficients. For self-efficacy, a low score would indicate that a student is not confident in their ability to complete mathematics tasks, while a high score indicates a strong confidence in that ability. The mathematics identity score relates to how strongly a student perceives themselves in relation to mathematics. A low score would indicate that the student does not see themselves as being particularly mathematically minded, while a high identity would indicate that students see themselves as being highly mathematically minded.

To obtain an interview pool of first-time first-semester freshmen with differing measures of self-efficacy, mathematics identity, and self-regulation strategy usage on the initial survey, 38 of the students were invited to participate in a sequence of five interviews. Of these, seven accepted and began the interview sequence, with one student dropping out after the third interview. Table 1 provides information of the interview participants, including their pseudonym, major, and the highest level of mathematics course they took in high school.

Table 1: Interview Participant Information

Participant	Major	Highest Mathematics Course Taken in Secondary school
Aria	Civil Engineering	AP Calculus AB
Cyndy	Biology	Precalculus
Frank	Computer Engineering	AP Calculus AB
Jane	Architectural Engineering	Calculus
Jay	Computer Science	Trigonometry
Sunny	Computer Science	AP Calculus BC

A case study methodology was chosen to explore individual experiences of undergraduates' transitions to college mathematics and their uses of self-regulation strategies. A mix of qualitative data from the interviews and quantitative data from the initial surveys helped formulate our depiction of the students' transition. Content analysis (Stemler, 2000) methods were used to analyze interviews. For each interview an audiovisual recording was made and transcribed verbatim. Participants were given pseudonyms. To code the data, the first author used codes derived from synthesized categorizations by Zimmerman and Pons (1986) and Wolters (1998) and developed codes related to statements related to mathematics identity, self-efficacy, and differences between high school and college mathematics experiences. To ensure inter-rater reliability, the second author coded a subset of the interviews for comparison. Most codes were consistent across coding instances, and any differences were resolved.

Findings

Through the analysis of the initial survey and interviews, findings related to the relationship of self-regulation strategies to students' mathematics identity and mathematics self-efficacy were determined. First, the results of the initial survey and the quantitative data collected will be presented. Secondly, the results of the individual interviews will be presented as they inform the research questions.

From the 188 initial survey participant responses, a positive correlation between self-regulation strategy usage and mathematics self-efficacy and identity can be shown. In the following two figures (Figure 1 and Figure 2) we see the general trend of the data where self-regulation strategy usage, mathematics identity, and mathematics self-efficacy scores were determined as described in the previous section.

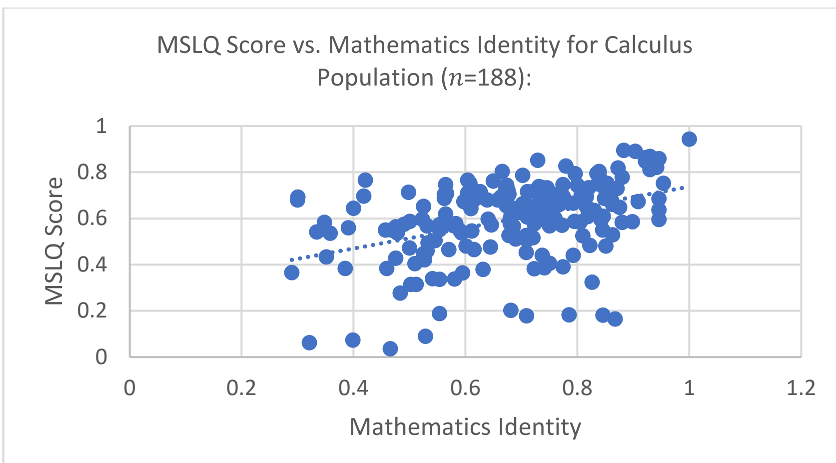

Figure 1. Population of Calculus participants' (n = 188) score obtained from Motivated Strategies for Learning Questionnaire versus Mathematics Identity score. Pearson's $r = 0.4101$

Figure 1 shows that as mathematics identity scores increase student use of self-regulation strategies tends to increase also. There is a moderately positive correlation between participants use of self-regulation strategies and their mathematics identity scores.

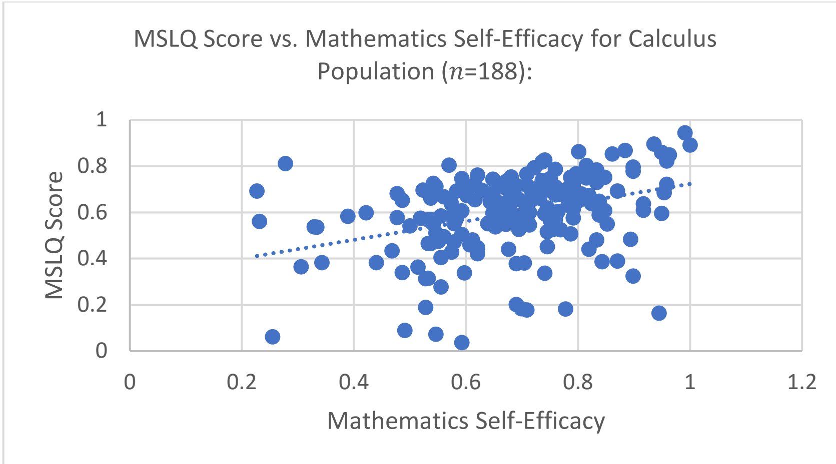

Figure 2: Population of Calculus participants' ($n = 188$) score obtained from Motivated Strategies for Learning Questionnaire versus Mathematics Self-Efficacy score. Pearson's $r = 0.3589$

As in Figure 1, Figure 2 shows a moderately positive correlation between the use of self-regulation strategies as determined by the MSLQ score and participants mathematics self-efficacy. Aside from the correlation data, data about individual participants' self-regulation strategies was found using the MSLQ. Comparing the scores from the MSLQ with participants' self-reported expected course grades it was found that three major strategies emerged as being frequently used by successful undergraduates in a first-semester calculus course. These three strategies were metacognitive self-regulation, effort regulation, and time and study environment. These strategies were also present with the academically successful interview participants as will be described later in this paper.

Along with the initial questionnaire data, data about participants' self-regulation strategy usage as it relates to mathematics identity and self-efficacy was examined through the individual interviews that occurred at the beginning and end of the semester. Within the interviews participants spoke about their general study habits in high school and how they studied for each midterm in their college calculus course. They also reported on their confidence as a mathematics student and how they saw themselves as mathematics learners. Based on their responses to the initial survey, interview participants had self-efficacy and identity scores that ranged from medium to high, but no participants with low scores agreed to be a part of the interviews. The examination of those participants with higher scores of mathematics self-efficacy and mathematics identity was used to identify what strategies students tended to employ. This analysis provides a preliminary framework of self-regulation strategies that could benefit students in their transition to undergraduate mathematics courses. From the initial questionnaire, while the three strategies of metacognitive self-regulation, effort regulation, and time and study environment were present in the interview participant responses, examining the successful interview participants' strategies also revealed they had a high frequency of organization and elaboration of course content.

From both interviews key elements emerged regarding how participants adjusted their study habits for exams from high school mathematics to college mathematics can be shown. Moving to a college setting, participants discussed how they expected to study on the calculus midterms, before having taken any mathematics midterms, and then during the final interview they discussed how they studied for the first and second midterms. The strategy most used in a high school course by these six students involved self-evaluating understanding by using test reviews or memorization tools. The next most used strategy was to seek peer assistance. Several of the participants reported that they did not expect to change their study habits significantly, but for those that did, the participants reported that they would seek external resources to further their understanding of the course content. Participants reported using more self-regulation strategies than they at first anticipated. While seeking external resources was one of the most reported additions to studying for the college mathematics midterms, most of the participants also reported adjusting their routine of studying, indicating that they spent more time studying for college exams than in high school. A couple participants, Jane and Jay, also reported adding other strategies including setting performance goals such as striving to earn a B on a particular midterm because that is what is needed to pass the course. Participants reported that this increase in strategy usage was heavily influenced by the first midterm grade and perceived difficulty of future exams; the first midterm either being lower-than-expected in Cyndy, Jane, and Jay's cases, or in the cases of Aria, Frank, and Sunny having a high midterm and resolving to keep the grade high which required additional studying.

Along with changing their strategy usage to adjust to the difficulty of the course exams, participants also discussed whether their perceptions of their mathematics identity and mathematics self-efficacy had changed. Based on the initial survey, Jane and Sunny had the highest scores of self-efficacy and mathematics identity, with all the other participants having a score indicating a medium degree of self-efficacy and mathematics identity. Cyndy and Jay reported that their confidence in their ability to work mathematics tasks was lower than when starting the course, although Jay did add that by becoming less confident it made her realize that she needed to spend more time and effort studying and preparing for exams, especially through self-evaluation of her understanding. Aria, Frank, Jane, and Sunny each reported raising their confidence levels throughout the semester, citing that their understanding of the material and the requirement of needing to know the material at a deeper level than high school improved their confidence. Jane did add that although her grade lowered, like Jay, this change in performance helped her reflect on how to better her study habits and require her to increase the effort she put into her mathematics course. Only two participants reported any change to their mathematics identity during their first semester of college. Jay reported that she did consider herself good at mathematics during the first interview, but during the final interview no longer held that belief of herself. Sunny on the other hand increased her perception of herself as a mathematics student. In general, data from the interviews and initial surveys show that these interview participants had higher rates of self-regulation strategy usage as their mathematics identity and mathematics self-efficacy increased, although in some cases like Jay, it was through the reflection that her self-efficacy had decreased that motivated her to better her study habits. Additionally, while the overall use of strategies tended to increase, in the case of Jane it was reported that she did not use certain beneficial strategies, such as seeking assistance from peers or instructors, because of her high degree of mathematics identity which led her to believe that she needed to be able to solve the problems independently to be good at mathematics.

Discussion and Conclusion

Participants in the study adapted their self-regulation strategies to the college setting by changing their study habits in response to exam performance and the increased difficulty of the course. This adjustment tended to involve some reflection about their study habits, an increase to time spent studying, or improved effort while studying which could take the form of attending to test reviews more frequently or seeking assistance from peers or instructors. Participants reported expecting a need to increase or change their high school study habits, but not to the degree that might be necessary. This finding aligns with Collier and Morgan's (2008) report that indicated undergraduate students did not expect to spend as much time per week studying in general undergraduate courses as what was recommended by the faculty that taught those courses. Almost all participants added or changed some aspect of their self-regulation strategy of setting a routine by increasing the time spent studying to adjust to the difficulty of the course, and specifically the difficulty of early exams. Participants were inclined to continue using strategies that worked for them in high school, but after recognizing the higher cognitive demand of their college course and the difficulty of their exams, participants adapted their study habits.

To inform the relationship between self-regulation strategy usage and mathematics identity and self-efficacy, we see a positive correlation between these factors as measured by the MSLQ, the mathematics identity survey by Kaspersen (2016), and the self-efficacy survey by Usher and Pajares (2009). Within the interviews we also saw that participants' discussions of their mathematics identity and self-efficacy affected their self-regulation strategies. As some participants, when their self-efficacy or identity was challenged by the difficulty of the course, were motivated to increase their use of self-regulation strategies to better their academic standing in the course. This positive correlation between self-regulation strategies versus mathematics identity and mathematics self-efficacy supports Hackett and Bets' (1989) call for promoting positive development of students' mathematics identity and self-efficacy at a young age. This also relates to Tinto's (2017) recommendation that development of self-efficacy can be accomplished through careful monitoring of first-year college students to provide social and academic support when facing difficulties.

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University Students' Evaluations of Quantified Statements in Mathematics: A Survey Study

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In this paper, we describe the results of administering a survey we created to investigate students' meanings for a universally quantified variable. Respondents with various levels of proof-course experience reviewed statement interpretations showcasing different meanings for quantified variables, indicating their preferred examples, which examples they believed to be viable, and a truth value for the statement. We developed the various interpretations from meanings described in our previous qualitative work on student quantification (e.g., Sellers et al., 2021). Our results suggest that students with a range of proof-course experience may benefit from reviewing diverse interpretations of a quantified statement. Still, students with less proof experience tended to accept more interpretations as viable or chose inappropriate truth values. In particular, we identify two non-normative interpretations most students believed were viable. We recommend proof-course instructors explicitly address these meanings in their classroom to foster student understanding of quantified statements.

Keywords: Quantification, proof education, scaling qualitative research, survey design

Introduction

Research has shown that students' interpretations of quantified mathematical statements play a crucial role in their understanding of advanced mathematical concepts in real analysis (Roh, 2010; Roh & Lee, 2017) and their proof writing and comprehension (Dawkins et al., 2021; Selden & Selden, 1995). Our previous work (e.g., Sellers, 2020; Sellers et al., 2018, 2021) has described five distinct ways students might interpret quantified variables. We have also shown that students' meanings for quantified variables can change from moment to moment, with the highest instability in meaning shown by calculus and transition-to-proof students (Sellers et al., 2017).

In this paper, we report the results of administering a survey we designed to (a) investigate the prevalence of these quantified variable meanings among collegiate students and (b) provide insight into instructional interventions that might promote students' appropriate quantification. In the survey, respondents with various levels of proof experience identified: (a) meanings they believed produced viable interpretations of a quantified statement, (b) which interpretations they preferred, and (c) a truth value for the statement. Our research questions emerged through reflection on what we might learn about students' quantification through a survey. Specifically, we investigated:

1. How does exposure to various interpretations of the quantified variable in a mathematical statement impact students' selection of the truth value for the statement? Do students' selections vary by proof-course experience?
2. Which meanings for quantified variables do students consider to be viable? Do students' selections vary by proof-course experience?
3. What meanings for quantified variables do students who select an inappropriate truth value prefer or believe to be viable after viewing the various interpretations?

Theoretical Framework

Sellers et al. (2018, 2021) propose the *Meanings for Quantified Variables* (MQ) framework to explain possible meanings students might have in a given moment for a quantified variable (see Table 1). We adopt the constructivist definition of *meaning*, grounded in the notion that individuals construct meanings in their minds through experience (Thompson et al., 2014).

Table 1: Categories of Meanings for Quantified Variables

Meaning	Mental Actions
MQ1	Checking if the predicate $P(x)$ holds for at least one value of x in X
MQ2	Checking if the predicate $P(x)$ holds for exactly one value of x in X
MQ3	Checking if the predicate $P(x)$ holds for all values of x in X
MQ4	Checking if the predicate $P(x)$ holds for spontaneously chosen values of x in X
NQ	No quantification (or absence of meaning for a quantified variable)

We describe how a student with each meaning might interpret a universally quantified statement, which we refer to as Statement 1 (S1): *For each real number x, let* $f(x) = x^4$*. Consider the following statement about the function* f*: For all real numbers x,* $\sqrt[4]{f(x)} = x$. The predicate $P(x)$ for S1 is $\sqrt[4]{f(x)} = x$, the domain of discourse X is the set of all real numbers $\mathbb{R}$, and the truth value of the statement is false. A student who uses MQ1 would check whether at least one real number satisfies the predicate $\sqrt[4]{f(x)} = x$. A student who uses MQ2 would check whether precisely one real number satisfies $\sqrt[4]{f(x)} = x$. A student who uses MQ3 would check whether all real numbers satisfy $\sqrt[4]{f(x)} = x$. A student who uses MQ4 would spontaneously choose real numbers and individually test whether each value satisfies $\sqrt[4]{f(x)} = x$. A student who exhibits no meaning for a quantified variable (i.e., NQ) might adopt an algebraic approach to investigating the predicate $\sqrt[4]{f(x)} = x$ rather than address how the quantified variable affects his interpretation of S1. A student normatively interpreting the quantified variable x in S1 would likely conclude that S1 is false by using MQ3 to identify a negative real number as a counterexample. However, other students might determine that S1 is false using MQ2, NQ, or MQ4, arriving at a normative truth value using an inappropriate quantification.

Survey Design and Methodology

Our survey instrument contained one universally and one existentially quantified statement. In this paper, we focus on S1, the universal statement. For S1, we included seven interpretations for x based on the MQ framework (see Table 2): one interpretation for each of the five meanings, and a second interpretation for both MQ1 and MQ3. In the first MQ1 and MQ3 examples, we included only values of x that satisfy the predicate $\sqrt[4]{f(x)} = x$ (which we called "E" for "examples"). In the second MQ1 and MQ3 interpretations, we included only values of x for which the predicate fails (which we called "CE" for "counterexamples").

For both quantified statements in the survey, the respondents answered questions in four distinct phases (see Figure 1). In Phase I, the students viewed and selected a truth value for the statement (options included *true*, *false*, *cannot be determined*, and *other*). The students then read a definition of "viable:" *capable of working, functioning, or developing adequately* (Merriam-Webster, n.d.). We included this definition to minimize the differences in how students might understand the term "viable" in our survey prompts.

Table 2: *Text of the Hypothetical Interpretations for S1 on the Survey*

Meaning	**Text of hypothetical interpretation for S1 on survey**
NQ	I would just plug in the function formula to the equation in the statement. The statement means that x is a real number and $\sqrt[4]{f(x)} = \sqrt[4]{x^4} = x$.
MQ1-E	I would expect to see there is at least one real number for x so that $\sqrt[4]{f(x)} = x$. If $x = 2$, then $\sqrt[4]{f(2)} = \sqrt[4]{16} = 2$. So, I have at least one real number for x so that $\sqrt[4]{f(x)} = x$.
MQ2	I would expect to see there is only one real number for x so that $\sqrt[4]{f(x)} = x$. But I found more than one real number for x so that $\sqrt[4]{f(x)} = x$. If $x = 2$, then $\sqrt[4]{f(2)} = \sqrt[4]{16} = 2$. If $x = 3$, then $\sqrt[4]{f(3)} = \sqrt[4]{81} = 3$.
MQ3-E	I would expect to see no matter what number I select for x, if x is a real number, then $\sqrt[4]{f(x)} = x$. If $x = 2$, then $\sqrt[4]{f(2)} = \sqrt[4]{16} = 2$. If $x = 3$, then $\sqrt[4]{f(3)} = \sqrt[4]{81} = 3$. I imagine that if I were to have unlimited time, I could keep selecting a different real number for x each time and see that I would have $\sqrt[4]{f(x)} = x$.
MQ1-CE	I would expect to see that there is at least one real number for x so that $\sqrt[4]{f(x)} = x$. If $x = -2$, then $\sqrt[4]{f(-2)} \neq -2$ because $\sqrt[4]{f(-2)} = \sqrt[4]{16} = 2$. Well, I will keep trying to check other real numbers to see whether I have at least one number for x so that if x is a real number, then $\sqrt[4]{f(x)} = x$.
MQ3-CE (Normative)	If I choose a real number -2 for x, $\sqrt[4]{f(x)} \neq x$ because $\sqrt[4]{f(-2)} = \sqrt[4]{16} = 2$. So there is at least one real number x such that $\sqrt[4]{f(x)} \neq x$.
MQ4-E/CE	It could be $\sqrt[4]{f(x)} = x$ or it could be $\sqrt[4]{f(x)} \neq x$. For instance, if $x = 2$, $\sqrt[4]{f(x)} = x$ because $\sqrt[4]{2^4} = 2$. On the other hand, if $x = -2$, $\sqrt[4]{f(x)} \neq x$ because $\sqrt[4]{(-2)^4} = 2$.

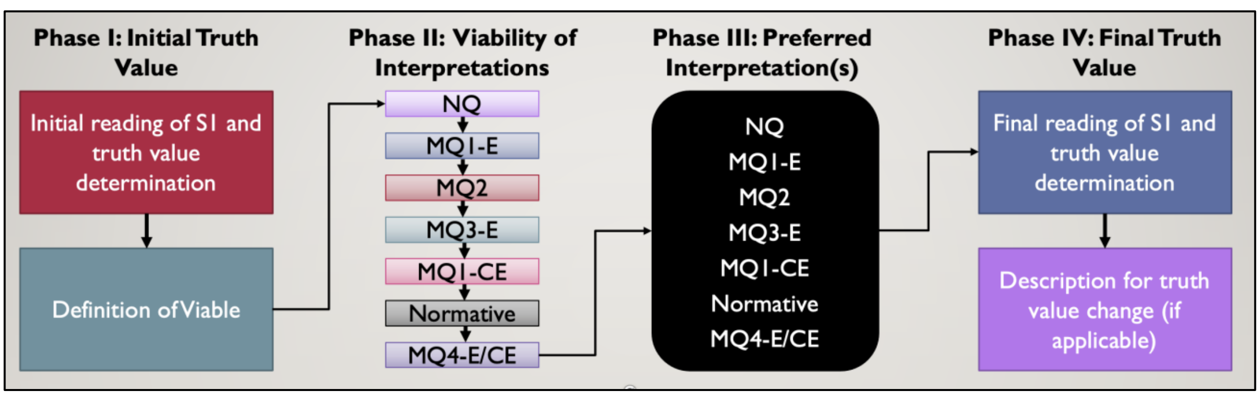

Figure 1: Visual Depiction of the Four Phases of the Survey for Statement 1

In Phase II, students read individual interpretations of the quantified statement, indicating whether they believed each example to be viable. In Phase III, students who indicated at least one interpretation was viable (a) reviewed their previous viability choices and (b) selected one or more preferred interpretations to reason about the truth value of S1. In contrast, students who claimed that no examples were viable described an alternative interpretation they considered viable. In the final phase, Phase IV, students again read the quantified statement and assigned a truth value. Students whose final truth value for S1 differed from their original truth value were prompted to explain why their choice changed.

We collected responses from 108 students enrolled in calculus courses and a proof-based geometry course at two universities in the United States. After comparing students' overall survey completion time, item completion times, and truth-value-change justifications, we deleted eight responses that appeared to be unreliable. The final sample of 100 students included 37 respondents who self-reported no proof-based course experience, 39 who reported taking one proof course, and 24 who reported participating in more than one proof course.

Results

In this section, we describe the results of our preliminary descriptive analysis of the data in relation to our three research questions regarding (a) the impact of the interpretations on students' truth value selections, (b) students' beliefs regarding the viability of the various hypothetical examples, and (c) which interpretations students who persisted in claiming S1 is true preferred or believed to be viable. We devote a subsection to each research question.

RQ1: Impact of Hypothetical Interpretations on Students' Truth Value Selections

We found a significant difference in the distribution of students' chosen truth values before and after reviewing the hypothetical interpretations, which we show in Table 3. Specifically, 69% of the 100 students initially indicated S1 is true, and 70% of the overall sample concluded that S1 is false after viewing the interpretations. This change in students' perception of S1's truth value seems to have emerged from their reflection on our survey items. For example, two students who changed their truth value from "true" to "false" explained that "I viewed instances where this statement couldn't be true" and "I did not initially think…negative numbers would not be viable solutions, as presented by multiple students."

Table 3: Initial and Final Truth Value Selections for S1 (Overall Sample of 100 Students)

Initial Chosen Truth Value	**The statement is true**	**69%**
	The statement is false	24%
	We cannot determine if the statement is true or false	6%
	Other (Please explain)	1%
Final Chosen Truth Value	The statement is true	26%
	The statement is false	**70%**
	We cannot determine if the statement is true or false	3%
	Other (Please explain)	1%

**Highlighted rows indicate the most selected truth value among respondents*

When we separated students' responses by proof-course experience, we discovered that students with little or no proof experience initially selected S1 is true in much greater numbers than those with more than one proof-based course (see Table 4). We also noted a large gain in the percentage of students who indicated that S1 is false after reviewing our interpretations (~50 percentage point increase for each group). Still, due to the near-universal initial response that S1 is true by students with no proof experience, we found that over 40% of calculus respondents still maintained that S1 is true after reviewing all seven interpretations.

Table 4: Initial and Final Truth Value Selections for S1 (100 Students Sorted by Proof-course Experience)

		0 Proof Courses ($n = 37$)	1 Proof Course ($n = 39$)	More than 1 Proof Course ($n = 24$)
Initial Chosen Truth Value	**True**	**34 (92%)**	**26 (67%)**	9 (38%)
	False	3 (8%)	10 (26%)	**11 (46%)**
Final Chosen Truth Value	**True**	15 (41%)	6 (15%)	5 (21%)
	False	**21 (57%)**	**30 (77%)**	**19 (79%)**

**Highlighted cells indicate the most commonly selected truth value among respondents. Students who selected "Cannot be Determined" or "Other" for their initial or final truth value are not included in this table.*

RQ2: Students' Determination of Viability for Individual Interpretations

We chose the order in which the hypothetical interpretations were presented to students strategically, considering that some students may not initially consider negative real numbers for values of x. We first presented an algebraic example, followed by arguments utilizing positive real numbers for x, and concluded with interpretations using negative real number values for x.

The following table, Table 5, shows the percentage of students in each proof-experience category that indicated a particular interpretation is viable. We found that at least 50% of students with no proof experience accepted 6 of the 7 interpretations as viable. Although students with proof experience tended to select fewer interpretations as viable, we noticed that a majority of students with proof experience still believed NQ, MQ3-CE (the normative interpretation), and MQ4 examples were viable. In contrast, the percentage of students who accepted the MQ1-E, MQ2, and MQ3-E interpretations decreased for each proof experience category. Our data imply that for our respondents, proof course experiences may have impacted their viability beliefs about some interpretations (i.e., MQ1-E, MQ2, MQ3-E) but not fully convinced them that other interpretations are unviable (i.e., NQ, MQ4).

Table 5: Viability Selections for S1 (100 Students Sorted by Proof-based Course Experience)

Proof Experience	**NQ**	**MQ1-E**	**MQ2**	**MQ3-E**	**MQ1-CE**	**MQ3-CE (Norm)**	**MQ4-E/CE**
0 courses (n = 37)	**27 (73%)**	**23 (62%)**	**20 (54%)**	**30 (81%)**	13 (35%)	**24 (65%)**	**20 (54%)**
1 course (n = 39)	**22 (56%)**	12 (31%)	10 (26%)	**24 (62%)**	16 (41%)	**29 (74%)**	**26 (67%)**
More than 1 course (n = 24)	**13 (54%)**	7 (29%)	4 (17%)	9 (38%)	9 (38%)	**20 (83%)**	**14 (58%)**

**Interpretations highlighted indicate more than 50% of respondents selected "viable"*

RQ3: The Preferred and Viable Interpretations of Students who Selected S1 is True

A vast majority of students in our sample concluded that S1 is false after viewing various hypothetical interpretations we provided in our survey. Still, a non-trivial minority of students

(26%) continued to insist that S1 is true (see Tables 3, 4). In this section, we examine the preferred and viable interpretations indicated by these students.

The following table, Table 6, contains two rows. The first row shows the percentage of the 26 students who insisted S1 is true that selected a particular meaning among their preferred interpretations. The second row displays the percentage of the 26 students who selected "true" that indicated an interpretation was viable. From these two rows, it appears that students who insisted that S1 is true preferred examples that only utilized positive real number values for x (i.e., MQ1-E, MQ3-E).[1] We also found that many students who indicated that S1 is true rejected interpretations that used negative values of x. For instance, no student who claimed S1's final truth value is true preferred the normative counterexample-based interpretation. Additionally, only 23% of these students claimed the normative interpretation is viable (whereas 91% of the students who claimed S1 is false stated the normative example is viable).

Table 6: Preferred and Viable Interpretations of the 26 Students whose Final Truth Value for S1 was "True"

	NQ	**MQ1-E**	**MQ2**	**MQ3-E**	**MQ1-CE**	**MQ3-CE (Norm)**	**MQ4-E/CE**
Students preferring interpretation ($n = 26$)	**12 (46%)**	**6 (23%)**	2 (8%)	**8 (31%)**	1 (4%)		3 (12%)
Students indicating interpretation is viable ($n = 26$)	**18 (69%)**	**14 (54%)**	11 (42%)	**20 (77%)**	7 (27%)	6 (23%)	9 (35%)

**Interpretations highlighted indicate the 2-3 highest (or lowest) percentages in each row*

Discussion and Conclusion

Our study aimed to gain insight into meanings for quantified variables that students preferred or believed to be viable for interpreting statements. Our survey constitutes an attempt to study these meanings, which emerged from our previous qualitative work (e.g., Sellers et al., 2017), on a larger scale. Our survey varied from our clinical interviews (Clement, 2000) in two distinct ways. First, in the survey, we gave students various statement interpretations and asked them to indicate which examples they believed were viable, while in the interviews, we modeled students' naturalistic interpretations of statements. Second, in the survey, we asked students to denote one or more provided arguments they preferred to use for interpreting the statement, while in the interviews, we assumed students' expressed ideas were their preferred interpretation.

Our first research question was related to how students' reading of various interpretations might impact their truth value choice for S1. Our results show that students selected an appropriate truth value for S1 at much higher rates after viewing the interpretations, although 41% of calculus respondents still maintained that S1 is true after reviewing our examples. This finding indicates that for many calculus students, pondering various examples of statement interpretations may not be sufficient to develop normative meanings for quantified statements.

Our second research question was related to the meanings students believed to produce viable interpretations of S1. Our results show that students with no proof experience might accept a broad range of non-normative meanings for the variable x. Additionally, our findings indicate that many students (regardless of proof experience) believe that NQ and MQ4 are viable

[1] We do not include the NQ example because 61% of students who selected S1 is false believed it to be viable.

quantifications. However, students' use of NQ or MQ4 is problematic because they might utilize these non-normative meanings to provide the appropriate truth value of S1. We thus encourage calculus and proof-based course instructors to facilitate student discussion activities to compare NQ, MQ4, and normative interpretations of quantified statements in their curricula.

Our third research question was related to students who maintained S1 is true after reviewing various interpretations of the statement. Our results show that these students preferred algebraic examples or those that only utilize positive real number values of x. These students' preference for NQ was unsurprising (to us), since we have previously reported that calculus students often exhibit NQ while interpreting statements (Sellers et al., 2017). Our finding that many students who maintained that S1 is true also believed that MQ1-E and MQ3-E are viable is a novel contribution to the quantification literature. We conjecture that these students might have relied more on examples than quantification to interpret S1. In this case, we would consider these students' rejection of MQ3-CE and MQ1-CE to be a natural consequence of their focus on interpreting S1 through examples. Specifically, these students may have believed that values of x that did not satisfy the predicate $f(x) = \sqrt[4]{x}$ were irrelevant to the interpretation process because they constituted "non-examples" of S1.

In summary, our survey results both verify and extend our previous qualitative findings. For instance, Sellers et al. (2021) described the existence of the five meanings in their MQ framework. In this paper, we extend these findings by showing (a) which meanings students prefer or believe are viable to interpret a universally quantified statement and (b) patterns in the meanings of students who believe a false universally quantified statement is true. We also provide insight into how teachers might implement Sellers et al.'s (2021) suggestion that instructors construct activities to address students' non-normative meanings for quantified variables. Our findings highlight one possible activity: asking students to compare NQ, MQ4, and normative interpretations for a universally quantified variable in a mathematical statement. In future studies, we plan to investigate further the prevalence of non-normative quantification among students and potential instructional interventions to facilitate students' construction of mathematical logic.

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What is so difficult about teaching precalculus for the first time? Graduate students' perspectives on learning to teach.

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This study seeks to explore and understand the challenges of teaching precalculus for the first time from the perspective of a novice Mathematics Graduate Student Instructor (MGSI). This qualitative multiple-case study followed three MGSIs through a semester of teaching to understand their needs and inform efforts to support instructors. Findings indicated novice MGSIs encountered many challenges including covering course content, estimating class time, facilitating activities, maintaining student engagement, and adjusting to variation in students' preparation. However, balancing dual responsibilities of student and instructor, working with students' negative responses to engaging teaching methods, and interpreting students' performance on assessments emerged as the most difficult challenges to overcome.

Keywords: Graduate Student Instructors, Challenges, Professional Development, Precalculus

Many graduate students aspire to careers in academia (Golbeck et al., 2016) and desire to be effective instructors. Research suggests graduate teaching assistants are concerned about their teaching effectiveness, communicating with students (Feezel & Meyers, 1997), dealing with student conflict (Park, 2004), and desire support for working with undergraduate students (Boyle & Boice, 1998). This study focuses on mathematics graduate student instructors (MGSIs), defined as a graduate student who serves as full instructor of record for an undergraduate mathematics course as part of their assistantship. Thus, they present course material, assess student learning, and assign final course grades. MGSIs are commonly assigned to teach introductory mathematics courses, despite being students themselves, having limited access to professional development (PD), and limited teaching experiences (Deshler et al., 2015; Ellis, 2014; Speer et al., 2005). They may face challenges including balancing teaching with their own coursework or research, awareness of low status of teaching at a research university, difficulties getting and interpreting feedback, working with undergraduates who bring negative mathematical experiences to their classrooms, and anxiety in dealing with these challenges (Hauk et al., 2009). Further, their previous experiences and competency with mathematics likely vary greatly from the undergraduate students they teach (Deshler et al., 2015). Additionally, MGSIs may need to develop pedagogical content knowledge such as understanding common student difficulties, different ways students approach tasks, and illuminating examples to share with students (Speer, 2008).

Although many universities have developed PD opportunities and supports for MGSIs, they vary greatly in depth and breadth (Belnap & Allred, 2009). Formal PD courses can improve STEM graduate students' college teaching self-efficacy, particularly for teaching methods, course planning, and assessment of student learning (Connolly et al., 2018). Efforts to develop and improve such programs benefit from a rich understanding of MGSIs needs and teaching experiences and focusing on novice or less experienced instructors is important as universities continually use new MGSIs to provide instruction to undergraduates.

To deepen the understanding of MGSIs' needs, a larger semester-long study explored how MGSIs understand their planning and teaching for student learning and their efforts to incorporate active learning methods into an undergraduate course by examining reasons they gave for making pedagogical decisions and identifying their perceived challenges[1]. The larger study sought to address the research question: During their first semester teaching a new undergraduate mathematics course, how do MGSIs plan (design and reflect) and implement their plans, with a focus on their goals for student learning? To address this broad question, four sub-questions were created. Due to space limitations, this paper focuses on findings to only the final sub-question: *What challenges do MGSIs describe as they reflect on their planning and classroom instruction?*

Theoretical Framework

Jackson (1968) described three phases of teaching: preactive, interactive, and postactive. The preactive stage could be thought of as planning or what teachers do to prepare for class. This could include writing lesson plans, making decisions about content, selecting mathematical tasks, identifying instructional goals, or preparing materials. The interactive phase consists of the time spent in the classroom with students. Thus, it describes the time when the lesson plan is executed. The postactive stage of teaching follows the interactive stage and includes the teacher's reflective or evaluative thinking and their consideration of future lesson modifications.

However, Clark and Peterson (1986) note that many researchers do not differentiate preactive and postactive thinking when studying teacher planning. They argue that teaching is a cyclical process and that a teachers' reflections may influence their plans for subsequent classes. Thus, they conceptualized teacher planning to include both preactive and postactive thinking. Similarly, the Mathematical Association of America defined design practices as "the plans and choices instructors make before they teach and what they do after they teach to modify and revise for the future" (Abell et al., 2018, p. 89) reiterating the notion that these phases of teaching may be combined. Yet Clark and Peterson do contend that the "kind of thinking that teachers do during interactive teaching does appear to be qualitatively different from the kind of thinking they do when they are not interacting with students" (1986, p. 258) and maintain the distinction between planning and the interactive phase of teaching.

Methods

Methodological Approach

In order to deeply explore MGSIs perspectives on planning and teaching, a qualitative multiple-case study methodology was selected. Case study is defined as "an empirical method that investigates a contemporary phenomenon (the "case") in depth and within its real-world context" (Yin, 2018, p.15). The research design provided ample reflection time and space for MGSIs to describe their goals, decisions, and challenges. The site, timeframe, course, and participants defined the bounds of this case (Yin, 2018). Each case is comprised of an MGSI serving for the first-time as a full instructor of record for precalculus, an undergraduate service course in the mathematics department, as part of their teaching assistantship during fall 2019 at a large research university in southeastern United States.

Participants and Setting

All MGSIs teaching precalculus in fall 2019 were invited to participate in this study and three male MGSIs in their early twenties volunteered. All three were in their second year of graduate school, cared about their teaching, and were considering academia as a potential career track. None had experience teaching precalculus and all had served as a calculus teaching

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assistant during their first year of graduate school. At the end of the study, each MGSI chose a pseudonym. Chen was an international graduate student and the other two, Willie and Patrick, were domestic graduate students. These MGSIs were actively receiving university support for learning to teach as they were provided with a peer mentor, were enrolled in a one-credit, year-long pedagogy course, and had access to lesson plans that included problem sets and teaching suggestions.

Precalculus is the prerequisite course to the calculus sequence which most STEM majors are required to complete. It is a four-credit hour course covering topics in algebra and trigonometry and is considered a service course, meaning the majority of students enrolled in the course are not mathematics majors. The course typically meets Monday through Thursday and enrolls about 38 students in each section. At this university, precalculus is a common teaching assignment for second-year graduate students.

Data Collection

To answer the research question and provide a rich understanding of MGSIs challenges and teaching experiences, data was collected throughout the fall 2019 semester and included (1) interviews, (2) lesson observations, (3) weekly journals, (4) course documents, and (5) a focus group. MGSIs were interviewed at the beginning of the semester to understand their prior teaching experiences and beliefs about teaching and student learning. They were asked to select a total of three lessons, roughly one each month, where they planned to seek student engagement. These lessons were observed and video-recorded, lesson plans were collected, and followed by a semi-structured interview (50–100 minutes) that explored the MGSIs' goals, planning and lesson design, and perceived challenges. Other forms of data included weekly journal entries (responding to 3–6 prompts), participant observations from the pedagogy course for MGSIs, their course assignments, and interviews with their peer mentors. At the end of the semester, MGSIs participated in a focus group seeking to gain insight into their overall experience as first-time instructors of record. The focus of this study was on understanding the MGSIs experience of learning to teach undergraduate precalculus.

Data Analysis

The purpose of this study was to understand MGSIs' needs by examining their thinking and teaching experiences in order to inform efforts to support their development. Thus, dramaturgical coding was chosen as the coding scheme. According to Saldaña (2016), it is useful for exploring underlying psychological constructs and "attunes the researcher to the qualities, perspectives, and drives of the participant" (p. 146). It includes six types of codes: objectives, conflicts, tactics, attitudes, emotions, and subtexts. Objectives typically referred to MGSIs goals for student learning. Conflicts, or obstacles, referred to aspects of teaching MGSIs identified as challenging or difficult and are the focus of this paper. Tactics provided insight into MGSIs' lesson design and classroom practices. The attitude codes included the MGSIs orientations about teaching and learning. Analysis of a pilot study suggested the emotion and subtext codes were not useful for answering the research questions, thus they were omitted from later analysis. A fifth category called influences was added to capture references to external influences on MGSIs planning.

Seven sources of data were coded for each MGSI: their semester-long journal, three lesson interviews, the background interview, and two written assignments from a pedagogy course. All codes were created in a hierarchical manner; each code was placed under the categories of objectives, tactics, conflicts, attitudes, or influences (level 1). To maintain MGSIs perspectives in the analysis, coding included in vivo codes (Saldaña, 2016). This paper focuses on the analysis of the conflict codes which included codes such as “run out of time” and balancing workload. Code mapping (Saldaña, 2016) included reviewing quotes from categories at levels 2 and 3 to confirm that quotes described the same concept, discussing unclear passages with a colleague, and listing, comparing, and organizing codes into categories by rearranging or combining similar codes, was utilized at four points in the analysis: after coding the first piece of data (Chen’s journal) and again after completing coding the data from each MGSI. The codes were then condensed into themes, by grouping level 2 codes into larger categories. The analysis sought to identify areas MGSIs described as problematic from their perspective. As MGSIs expressed many common difficulties, findings are reported only through cross-case analysis.

Throughout data collection and analysis, a research log was created containing memos to document the research process, reflect on lesson observations and interviews, generate future questions, discuss concerns, and contemplate next steps in the research process (Saldaña, 2016). All interviews were transcribed using software and were cleaned by the author. All coding was done in Dedoose and coding proceeded one MGSI at a time so that each instructor could be studied individually before making comparisons across cases. Tentative findings were shared with and reviewed by MGSIs for trustworthiness.

Findings

MGSIs limited teaching experience likely contributed to some of their challenges, particularly as they were trying out new teaching methods and thus struggling to facilitate activities or estimate time students would need to complete problems or tasks. Further, they were teaching the course for the first time and still grappling with determining the depth and length of time to spend on mathematical concepts. The difficulties with managing class time and course pacing could easily be attributed to their stage of beginning to learn to teach and are not elaborated upon in this paper due to space limitations. Further, MGSIs noted times in the semester, such as before breaks or near the end of the semester, when students seemed generally less engaged. MGSIs were able to manage most of these challenges with support from mentors and peers. Thus, this paper focuses on other more enduring challenges highlighted in the table.

Table 1: Summary of Challenges and Aspects of the Challenges

Phases of Teaching	Challenges
Preactive: Planning	(1) **Limited time for planning**; (2) Course content coverage
Interactive: Implementing Lesson Plans	(1) Estimating time needed for tasks or activities; (2) **Students’ responses to teaching methods** (3) Facilitating lesson logistics; (4) Maintaining semester-long student engagement
Postactive: Reflecting on Teaching Experiences	(1) **Interpreting students’ performance on quizzes or tests**; (2) Interpreting students’ course preparation

Preactive Challenges: Preparing to Teach

MGSIs are simultaneously both a student and an instructor. As a result, Willie expressed in his journal that he was “essentially working two full time jobs.” Learning to balance these responsibilities and mange workloads was a challenge for all MGSIs in this study. Throughout the semester, MGSIs found weeks where the time needed for their own coursework conflicted with their responsibilities as an instructor. Both Patrick and Chen identified this as their greatest teaching challenge in the final journal entry. Chen reflected,

> Time management is probably the most difficult in the past semester. This semester is the first time that I taught MATH 115 [precalculus]. We meet four times a week, so, I have to do a lot of lecture preparation (which I have been trying to align with homework, quiz, and exam). Even though we have the lesson plans in online material and worksheets, I still usually go over the book, look for extra material online, and compare my notes to other peoples' notes. These planning worked fine when I am not busy, however, toward the end of the semester, I find it difficult to balance planning the lesson with focusing on my own study.

Patrick identified the same challenge and stated, "there were weeks this semester where it felt as though I had to choose between doing well in my role as a MATH 115 instructor and doing well in my courses. This is a no-win situation." In the focus group, MGSIs collectively agreed they spent most of the day teaching, holding office hours, or preparing for teaching in some way and then would go home and complete their own graduate coursework in the evening or over the weekend. These MGSIs chose to spend a large amount their time on preparing to teach and wanted to be effective instructors. Willie explained,

> I feel like if you want to have a good class, it takes a lot of preparation. It's pretty much writing all those assignments and grading. You know, if you really want to be a good teacher, you're going to spend more time.

MGSIs reported that planning active learning activities tended to take more time to prepare than traditional lecture-type lessons. Experienced faculty often cite lack of time to redesign courses as a reason for not incorporating active learning (Johnson et al., 2018; Plush & Kehrwald, 2014). Thus, it may be that examples of these types of activities are more valuable to share with new instructors. To a lesser extent, MGSIs occasionally mentioned concerns about feeling behind schedule, pacing material, or selecting problems of the right difficulty level.

Interactive Challenges: Implementing Lesson Plans

A conflict MGSIs experienced with implementing lesson plans came from some students' responses to MGSIs efforts to create student engagement. MGSIs experienced conflict when their students either provided feedback indicating they disliked an activity or appeared frustrated by or not engaged in an activity. These challenges often related to lessons when MGSIs planned activities intended to engage students. MGSIs tended to reference a few students or a group of students rather than an entire class when reflecting on this difficulty. For example, Chen reflected in an interview on feedback he received from a few of his students on both a mid-course evaluation and directly following a discovery type lesson he designed to help students make meaning of asymptotes. He stated, "they don't like when I do this kind of stuff, have them struggle and conclude an idea from the lesson. A few people don't like that." Similarly, Patrick reflected on student feedback related to a jigsaw activity in his journal. He wrote, "many comments suggested that they prefer a standard lecture because the way that I explain topics is clearer and more organized" than their peers' explanations.

Beyond preferring lecture-style instruction, MGSIs occasionally encountered students who opposed non-lecture teaching methods. Willie discussed at length difficulties he encountered with one group who struggled to complete a jigsaw activity about quadratic functions. Willie shared in the interview,

> I just remember it was three red faces right in front of me. I was like, Oh no, that's not good! I tried to help. [It] just wasn't sticking. They weren't putting in effort. They just want to come into class, take notes and leave is what it felt like. They were unhappy.

Willie confirmed the students were unhappy with the *teaching method*, explaining, “I had one of them come up to me and said, can we not do this again? So yeah, she was not happy with the method.” Willie tried to provide hints to move the group along, but their progress lagged well behind other groups and their attitudes did not improve.

Although these MGSIs participated in some PD related to facilitating student-to-student interactions, MGSIs perceived success with group work varied greatly in this study. Patrick and Chen repeatedly expressed difficulties getting students to engage in group work, indicating that students tended to work alone despite their efforts to encourage collaboration. Patrick stated in his journal, “I am having trouble getting my students to interact with each other.” And elaborated in the first lesson interview,

> Maybe I can find some way to make my lessons so that they are helping each other a little more or at the very least so that it's not just silent in my classroom. Sometimes it is. A lot of times it is… I'll say you can work on these problems. You can work with people around you or people next to you. They just work by themselves mostly. I don't know if it's a matter of they just want to try it [alone] or what.

Despite the limited success, both continued to try various approaches to group work. Chen shared, “I'm not gonna give up doing group work so, but I have to like find a better way to do it.” Both suggested time of day could have impacted their students’ engagement. Many students appreciate aspects of group work yet report dissatisfaction if they prefer to work at a different pace than their group, are uncomfortable, shy or introverted, or need more clarity on a topic (Uhing et al., 2021). MGSIs perceived these types of teaching challenges as difficult to overcome, and during the focus group they expressed a consensus understanding that it is not possible to please all students.

Postactive Challenges: Reflecting on Teaching

MGSIs discussed challenges related to understanding their precalculus students, particularly their performance on quizzes or tests, but also around students’ prior preparation, and how students appeared to prepare for their current course. In this study, quizzes and tests refer to written assessments of students’ knowledge of mathematical content which MGSIs collected and graded. MGSIs were troubled that some students at times struggled to demonstrate mathematical knowledge, understand concepts, recall facts or procedures on quizzes or tests, or repeated errors that MGSIs attempted to address in their teaching despite MGSIs best efforts to include student-centered teaching methods.

MGSIs’ feelings, and perhaps efficacy for teaching, seemed to be negatively impacted by student performance on graded assessments and at times they pondered to what extent they were accountable for student learning. In some ways, this presented an emotional challenge for MGSIs as they expressed feeling responsible for student performance on quizzes and tests. Patrick wrote in his journal: “When quiz scores are lower than normal, it's hard not to feel like it's my fault somehow.” Chen stated in the third lesson interview, “I feel like as an instructor I should be somewhat responsible for how they learn.” Willie also expressed feelings of responsibility for student learning during the third lesson interview when he asked, “I mean 70 is like the minimum that I want it. So, when I get below it, I'm like, did I do something wrong?” MGSIs contemplated their responsibility for student learning. Chen expressed this in his journal when reflecting on things he could do to help students remember concepts and concluded “I don’t know if it’s the students or my fault that they don’t learn, I don’t know.” In the final journal entry Willie expressed,

> The most difficult thing for me was watching kids not do well and knowing I've tried everything I could and offered everything I could to help them. It made me feel bad as a teacher and bad for the student. However, I have to learn that I cannot help everyone!

Processing students' performance on tests or quizzes and communicating with students about low scores can be difficult for instructors. MGSIs responded to students low scores in a variety of ways including curving scores, offering extra credit, dropping quizzes, and allowing students to correct exams for points. The experience of seeing students struggle with problems on a quiz or test that had been, from the instructor's perspective, clearly covered during class was a source of conflict for these MGSIs whose goals included helping students develop mathematical thinking and procedural skills so that they could succeed in both current and future courses (Monastra & Yee, 2021). This tension may cause distress, disappointment, or discouragement for MGSIs and may have been especially salient in this study as MGSIs spent a lot of time reflecting on their goals for student learning. Further, MGSIs put a great amount of effort into their teaching and then did not always receive the benefit of large amounts of student success.

Discussion

This study's findings suggest that MGSIs need opportunities to reflect on and discuss their teaching, support and training for working with undergraduate students, and planning time. Cognitive coaching techniques (Costa & Garmston, 2018) may be helpful for promoting reflection. Graduate student PD can address a wide range of topics, yet often focuses on generic teaching skills (Park, 2004) or "administrative responsibilities and mechanical aspects of teaching" (Speer, 2008, p. 313). These topics are important and necessary, yet they may not be sufficient for supporting MGSIs. Further, their teaching does not always lead to the desired results. MGSIs may experience tension between their goals for student learning and the actual learning they see demonstrated on tests or quizzes. Acknowledging and reflecting on these tensions by pondering questions about instructors' responsibility for motivating their students may be an important area for PD. Allowing MGSIs to reflect on who should be accountable for student learning may lead to fruitful discussions.

Further, MGSIs may encounter some negative responses or pushback from their students. Student attitudes have been identified by faculty as a reason to not utilize reform materials or teaching methods (Deslauriers et al., 2019; Henderson & Dancy, 2007) and student stress can impact instructors' problem selection on assessments (Yerushalmi et al., 2010). Further, undergraduate STEM students perceive they learn more and prefer well-delivered lectures when compared to the increased cognitive effort required in active learning settings (Deslauriers et al., 2019). Discussing productive ways to respond to students' attitudes towards teaching methods, tips for creating learning environments that promote student engagement with tasks, and discussions or suggestions related to setting class norms, features of high-quality tasks for group discussion, various models of group work, and question posing techniques would appear to be beneficial for graduate student instructors. Some of these topics were addressed in these graduate students' PD, which emphasizes the point that support cannot be limited to a one-time training. Rather novice instructors need *sustained* discussion and models of teaching methods as well as opportunities to share success and struggles with implementing various teaching methods in their classrooms. Finally, PD providers and mathematics departments must be aware of the time strain some graduate students may be experiencing as time is identified as a barrier to instructors adopting reform teaching practices (Henderson & Dancy, 2007; Johnson et al., 2018). Future research into the impact of graded assessments on MGSIs self-efficacy or teacher identity and how MGSIs understand student motivation would be helpful.

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Instructional interventions and teacher moves to support student learning of logical principles in mathematical contexts

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This study explores how instructional interventions and teacher moves might support students' learning of logic in mathematical contexts. We conducted an exploratory teaching experiment with a pair of undergraduate students to leverage set-based reasoning for proofs of conditional statements. The students initially displayed a lack of knowledge of contrapositive equivalence and converse independence in validating if a given proof-text proves a given theorem. However, they came to conceive of these logical principles as the teaching experiment progressed. We will discuss how our instructional interventions played a critical role in facilitating students' joint reflection and modification of their reasoning about contrapositive equivalence and converse independence in reading proofs.

Keywords: logic and proof, instructional interventions and teacher moves, contrapositive equivalence, converse independence

The purpose of this study is to explore how students might learn logical principles and how instructional interventions and teacher moves might support students' learning of logic. We focus on two logical principles: contrapositive equivalence and converse independence. By contrapositive equivalence, we refer to a logical principle that a conditional statement has the same truth value as its contrapositive. By converse independence, we mean a logical principle that a conditional statement does not necessarily have the same truth value as its converse. These two logical principles are foundational for mathematical justification: the former provides a logical account that proof of a conditional statement is also a proof of its contrapositive, and the latter provides a logical account that proof of a conditional is not a proof of its converse.

It is critical for students in proof-oriented mathematics courses to know and use contrapositive equivalence and converse independence for their proof activities. However, empirical studies have reported students' challenges with using these logical principles: In Stylianides et al.'s (2004) study, many mathematics undergraduates did not use the contrapositive equivalence as a valid inference. Dawkins et al. (2021) reported a similar phenomenon in which undergraduate students with no proof experience in college conceived that the proof of the contrapositive would not provide a proof of the original conditional statement. Dawkins et al. (2021) also documented calculus students conceiving a proof of a conditional statement as proof of its converse when both the conditional and its converse are true.

While issues with student learning of these logical principles have been studied widely in proof research, these empirical studies have not focused much on how instructional interventions and teacher moves might provide support for students to learn these logical principles (Melhuish et al., 2022; Stylianides & Stylianides, 2017). In this paper, we document a case of two undergraduate students, Carl and Sarah, as a possible account for students coming to understand and might use these two logical principles. We also examined how the instructional interventions and teacher moves we designed and implemented might have played a role in students developing set-based reasoning for these logical principles. This study addresses the following research questions: (1) How might students make progress in learning to use contrapositive equivalence and converse independence by engaging in set-based reasoning? and (2) how, when, and what types of instructional interventions and teacher moves could encourage or facilitate their set-based reasoning for learning these logical principles?

Theoretical Framework

We employ Piaget's genetic epistemology as our theoretical lens for this study. From this perspective, we assume that individual students idiosyncratically organize their experiences within mental schemes (Glasersfeld, 1995; Piaget, 1971; Piaget & Inhelder, 1969). The schemes organized by an individual student's unique experience would provide space of implications for her reasoning (Thompson et al., 2014). On the other hand, individual students' ways of reasoning are not accessible to observers; we, as researchers, propose viable models of their ways of reasoning through their behaviors and utterances. We suggest and use a triad of relationships a student might need to construe when validating a proof-text paired with a conditional statement to be proven (**Error! Reference source not found.**):

- What the student knows: this may include mathematical propositions, logical principles, or any relationship that the student conceives from the given conditional
- What the student construes from the given conditional to be proven: a student might posit a relationship that the given conditional construes (to them).
- What the student believes the given proof-text attempts to prove: A student might posit a relationship that the given proof-text construes (to them).

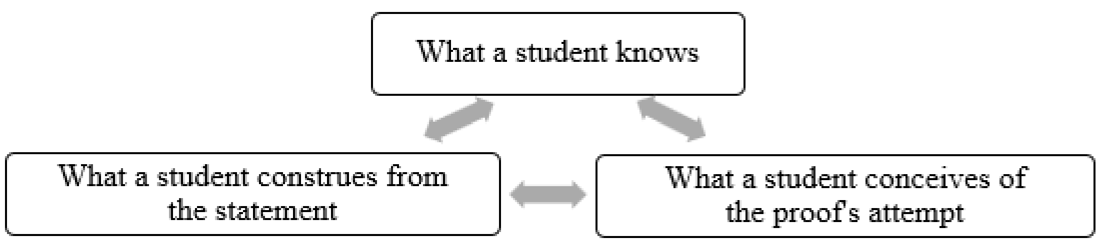

Figure 1. Three relationships a student may construe while reading a proof-text paired with a statement

The three relationships a student construes would likely support her reasoning when examining if the proof-text proves the statement to be proven. However, students may not find compatibility among the relationships they have construed. In such cases, instructional interventions and teacher moves could play a critical role in supporting students in connecting these relationships (e.g., Ellis et al., 2019; Mata-Pereira & da Ponte, 2017). Using this triad as an analytic tool, this paper provides empirical evidence that instructional interventions and teacher moves could leverage students' learning of contrapositive equivalence and converse independence.

Research Methodology

As part of a more extensive study aiming to develop constructivist models of students' abstraction of logic for proof of conditional statements, we have conducted exploratory teaching experiments (Steffe & Thompson, 2000) since 2018. The exploratory aspect of the experiments allowed us to repeat six iterations of the design-implementation-analysis cycle over five years to refine the instructional tasks and our models of students' ways of reasoning about logical principles.

This paper documents our findings from our sixth iteration of the teaching experiments conducted at a large public university in the United States. Two undergraduate students, Carl (engineering major) and Sarah (mathematics major), were recruited from calculus 3 at the beginning of the semester in the Spring of 2022. We selected Carl and Sarah out of 14 students who completed the screening survey (Roh & Lee, 2018) as they met our selection criteria. Their survey responses indicated they would have sufficient mathematical knowledge to comprehend conditional statements and proof-texts in our designed tasks yet need to learn logical principles to validate mathematical proofs. Both students also reported they learned proofs in geometry in high school but had not yet taken any proof-oriented courses in college.

The exploratory teaching experiment was organized once a week for 12 weeks for 75-minute interviews. For the first (intake) and last (exit) interviews, we conducted clinical interviews (Clement, 2000) by meeting each student individually to assess their use of logic to validate proofs of conditional statements. We conducted exploratory teaching interviews for the rest of the ten interviews (Sellers, 2020). We met with both students and implemented the tasks we designed to help students leverage set-based reasoning. The tasks for the teaching interviews consisted of 4 tasks: (1) set theory tasks (defining sets by shared properties); (2) truth conditions tasks (evaluating truth values of conditional statements); (3) what-does-it-prove tasks (reading and comparing proof-texts paired with a conditional statement), and (4) abstraction task (comparing various proofs across different mathematical contents to abstract general proof frames for conditional statements). With these tasks, we tried to create a student-centered learning environment to encourage students' reflection and modification of their reasoning.

Results

At the intake interviews, Carl and Sarah exhibited the opposite of the normative mathematical logic regarding contrapositive equivalence and converse independence. Specifically, they both responded that a proof of the contrapositive of a given conditional does not prove the statement. In contrast, a proof of the converse of a given conditional proves the statement when both the original conditional and its converse are true. Their responses at the intake interviews indicate the absence of these logical principles in the students' reasoning, or at least the intake interview tasks did not evoke the students to use these logical principles. However, their reasoning about logic shifted during the teaching interviews in which we implemented the what-does-it-prove (WDIP) tasks (Days 5-8). Carl and Sarah first made sense of the contrapositive equivalence and later began to make sense of the converse independence. We describe how contrapositive equivalence became these students' knowledge base for proof by contrapositive, yet created resistance to develop converse independence. We also document how our instructional interventions designed to leverage set-based reasoning and teacher moves facilitated students' reflection on mathematical proof and their eventual recognition of converse independence.

Teacher Moves Leveraging Student Progress in the Contrapositive Equivalence

On Day 5, the first day for the WDIP tasks, the interviewer presented Theorem 1 ("For every integer x, if x is a multiple of 6, then x is a multiple of 3") with three associated proof-texts, Proof 1.1 (direct proof), Proof 1.2 (disproof of converse), and Proof 1.3 (proof of the contrapositive). Carl and Sarah immediately attended to the first and last lines of Proof 1.1 and said that Proof 1.1 would prove Theorem 1. Afterward, these students frequently examined if the first line and last line of other proof-texts matched the if-part (the premise) and the then-part (the conclusion) of the theorem to be proven, respectively. These students' attention to the first and last lines of the proof-texts enabled them to find the compatibility among what relationship they know about the premise and conclusion of the given theorem, what relationship the given theorem describes, and direct proof attempts to prove the given theorem.

On the other hand, these students' tendency to check the matches between the first line of a proof-text and the premise of the theorem statement may have hindered them from discerning why proof of contrapositive indeed proves the given theorem, even though proof of contrapositive does not start with the if-part of the given theorem statement.

Sarah: I said it [Proof 1.3] proves something different. I guess it [Proof 1.3] proves the complements of the original one that if x is not a multiple of 3, then x is not a multiple of 6, which [is] base[d] off of how we were seeing the complement, remember? That could possibly be true [inaudible].

Carl: Yeah. I said the same thing […] It's, yes, if we're allowed to say that if A is a subset of B, and B's complement is a subset of A's complement.

In the dialogue above, Sarah and Carl used set languages, such as subsets and complement sets. Carl used letters A and B to name the truth sets for the premise and conclusion of Theorem 1 and used these letters to interpret what Theorem 1 says and what Proof 1.3 proves in terms of subset relationships. Sarah then claimed that what Proof 1.3 attempts to prove is the contrapositive of Theorem 1 and could also be true. Carl agreed to interpret Theorem 1 and Proof 1.3 in terms of subset relationships if they were allowed to say the set relationships $A \subseteq B$ and $B^c \subseteq A^c$. However, the students' use of set language itself did not indicate their use of contrapositive equivalence. They responded that both Theorem 1 and its contrapositive are true in this case but drew two different diagrams to represent what Theorem 1 construed to them and what they believed Proof 1.3 attempts to prove. Sarah drew a diagram to represent what Theorem 1 meant to her: the truth set P of the premise of Theorem 1 is a subset of the truth set Q of the conclusion of Theorem 1. She also drew another diagram for Proof 1.3, in which Q^c is contained in P^c (**Error! Reference source not found.** left).

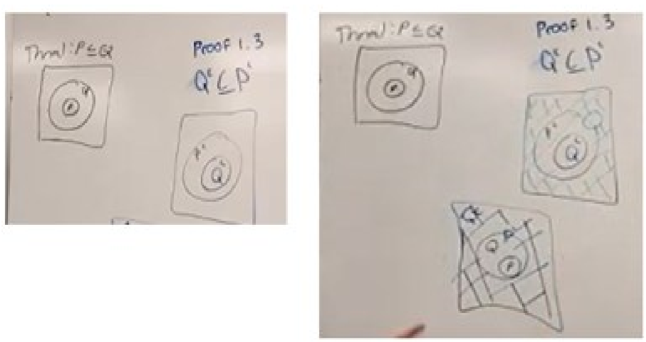

Figure 2. Sarah's diagrams for Theorem 1 & Proof 1.3: (left) initial; (right) after the teacher intervention

At this point, the interviewer intervened with Carl and Sarah by inviting them to use only one diagram for both Theorem 1 and Proof 1.3. This teacher intervention enabled Sarah to use her diagram for Theorem 1 to shade the region corresponding to the complements of P and Q. (i.e., If $P \subseteq Q$, then $Q^c \subseteq P^c$). By Instructor's request, Sarah was also able to use her diagram for Proof 1.3 to describe Theorem 1 (i.e., If $Q^c \subseteq P^c$, then $P \subseteq Q$). Sarah's revised diagrams (**Error! Reference source not found.**, right) are indicative of her progress in the conceptualization of contrapositive equivalence (P $\subseteq Q$ iff $Q^c \subseteq P^c$). After looking at Sarah' revised diagrams, Carl agreed with Sarah and illustrated his diagram, which was similar to

Sarah's diagrams. But he also added another case in which if two sets A and B are equal, then A's complement and B's complement are also equal (If $P = Q$, then $Q^c = P^c$). Figure 3 illustrates our model of Sarah's reasoning in which she began to use contrapositive equivalence as her new knowledge to comprehend a proof of contrapositive as a proof of the original conditional. From that point, the contrapositive equivalence became robust knowledge for Carl and Sarah for the rest of the teaching experiment. Here, we see the teacher intervention supported these students to use what they came to know (contrapositive equivalence) to connect

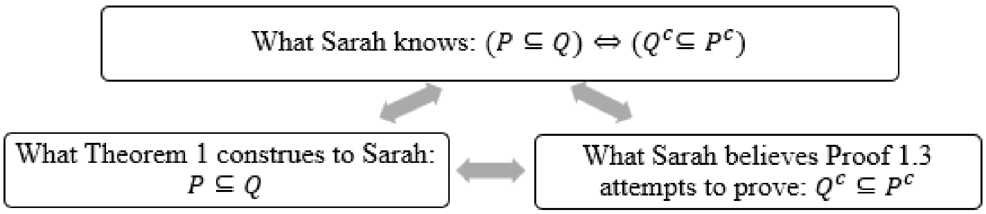

Figure 3. A model of Sarah's way of reasoning with contrapositive equivalence

what relationship Theorem 1 construed to them ($P \subseteq Q$) with what they believed Proof 1.3 attempts to prove ($Q^c \subseteq P^c$).

Teacher Moves Leveraging Student Progress in Converse Independence

On Day 6, the interviewer presented Theorem 2 (for any integer x, if x is a multiple of 2 and 7, then x is a multiple of 14) with two associated proof-texts: Proof 2.1 (proof of converse) and Proof 2.2 (direct proof). Carl and Sarah responded that Proof 2.1 proves Theorem 2 despite the reversed order because they believed Theorem 2 and its converse are both true. To be more specific, Carl explained that he knew the set of all multiples of 2 and 7 (P) is the same set as the set of all multiples of 14 (Q), i.e., $P = Q$, and to him, Theorem 2 interprets the subset relationship $P \subseteq Q$. Carl also believed Proof 2.1 proves the reversed subset relationship $Q \subseteq P$. While Theorem 2 and Proof 2.1 form different subset relationships, he could infer Theorem 2 from Proof 2.1 by substituting P to Q and Q to P in the subset relationship $Q \subseteq P$. He said that since he already knows $P = Q$, by using his knowledge, he could infer Theorem 2 from what Proof 2.1 proves.

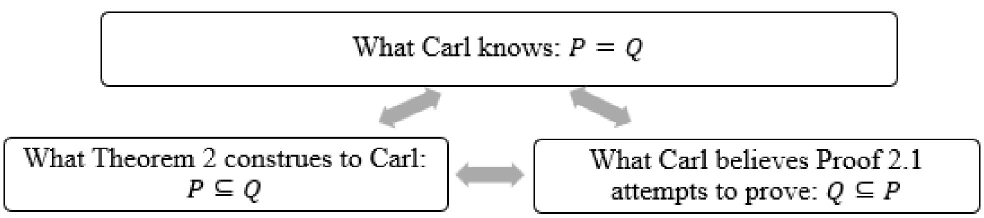

Figure 4. A model of Carl's reasoning about why Proof 2.1 proves Theorem 2

On Day 7, the interviewer presented Theorem 4 (Given any quadrilateral ABCD, if ABCD is a kite and parallelogram, then ABCD is a rhombus) with Theorem 4.1 (proof of converse) and Theorem 4.2 (direct proof). Carl continued claiming, and Sarah agreed, that although Proof 4.1 proves the converse of Theorem 4, it also proves Theorem 4 because both Theorem 4 and its converse are true. We infer that these students meant the two subset relationships $P \subseteq Q$ and $Q \subseteq P$ are indistinguishable to them when $P = Q$. In our model, they conceived that Proof 4.1 proves the converse of Theorem 4. But since they already knew the premise and the conclusion of Theorem 4 represents the same truth sets ($P = Q$), they went further to infer that Proof 4.1 proves Theorem 4 as well.

On Day 7, Carl claimed, and Sarah agreed, that they should be allowed to use what they know without justification. He compared proof of converse with proof of contrapositive. Although Proof 1.3 does not justify contrapositive equivalence, Carl accepted that Proof 1.3 (proof of contrapositive) proves Theorem 1 because he knew contrapositive equivalence. Carl then claimed with an analogy that we should also accept Proofs 2.1 and 4.1 (proofs of converse) even though these proofs do not provide justification for $P = Q$, since he already knew these sets were equal. Furthermore, Carl claimed, and Sarah agreed, that proof does not necessarily justify explicitly what they already know. This is similar to how one may apply a known theorem without reproving it. The distinction between the ways mathematicians cite prior knowledge and how Carl wanted to cite prior knowledge is quite subtle, and we see Carl's reasoning as subjectively rational. The more central question was: how would Carl and Sarah justify $P = Q$ instead of saying they already know it?

On Day 8, the interviewer presented Proof 4.4 () as an instructional intervention, which resembled our model of Carl's reasoning on Day 7. As a version of proof of converse, Proof 4.4 explicitly stated the equal set relationship that "we already know" without justification. Both students responded that Proof 4.4 would prove Theorem 4 directly because it explicitly stated $P = Q$ in the proof-text and thus provided warrants ($P = Q$) to support Proof 4.1 ($Q \subseteq P$) infers Theorem 4 ($P \subseteq Q$).

Proof 4.4: Let P and Q be subsets of $\mathbb{Q}u$ defined as follows:
$P = \{ABCD \in \mathbb{Q}u: ABCD\ is\ a\ kite\ and\ paralleogram\}$;
$Q = \{ABCD \in \mathbb{Q}u: ABCD\ is\ a\ rhombus\}$.
From Proof 4.1, we proved $Q \subseteq P$.
And we already know $P = Q$.
Then $P \subseteq Q$.
Thus, given any quadrilateral $\blacksquare ABCD$, if $\blacksquare ABCD$ is a kite and is a parallelogram, then $\blacksquare ABCD$ is a rhombus.

Figure 5. Proof 4.4: A model of Carl's reasoning in accepting Proof 4.1 (proof of converse) as a proof of Theorem 4

While the interviewer acknowledged that the students already knew $P = Q$, she invited them to state what it would mean for two sets P and Q to be equal. Students' responses to this instructional intervention uncovered the absence of meaning for equal sets in these students' reasoning: They were not sure how to say two sets are equal. Carl merely suggested that two sets are equal ($P = Q$) when their complements are equal ($P^c = Q^c$) (see **Error! Reference source not found.**, left). Sarah then suggested combining Theorem 4 and its converse (see **Error! Reference source not found.**, right) as a meaning for equal sets.

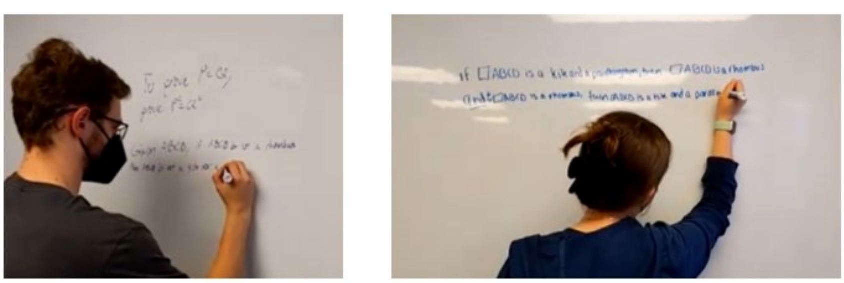

Figure 6. Students wrote their meaning of equal sets: Left: Carl; Right: Sarah

After the interviewer invited Carl and Sarah to state their meaning of equal sets, Sarah shifted her interpretation, revealing a sense of circularity. In particular, her interpretation of equal sets by this conjunction of a conditional and its converse helped her to realize the absence of justification for $P = Q$ in Proof 4.4. When Carl asked if "we are given $P = Q$ from Proof 4.4 or not," Sarah responded to Carl that "No, Proof 4.4 isn't even really proving $P = Q$. It's just saying it is." When Carl asked again if that ($P = Q$) is "given or inferred," Sarah again responded to Carl that "it [Proof 4.4] never says [justifies] $P = Q$." From this student dialogue, we could see Sarah's reasoning had evolved regarding the converse independence. While engaging in the sequence of activities and responding to the interviewer's prompts, she concluded that we should use only what we had already proved without justification again. Since they already justified contrapositive equivalence by diagrams (Figure 2), Sarah believed that they could say Proof 1.3 (proof of contrapositive) proves Theorem 1 even though Proof 1.3 does not justify the contrapositive equivalence. However, she contended that "Proofs 2.1 and 4.1 (proofs of converse) do not necessarily prove their original theorems because the theorems ask "if P, then Q" ($P \subseteq Q$) but the proofs instead prove "if Q, then P" ($Q \subseteq P$). To prove the theorem to be proven, the proofs would have to prove $P = Q$ so we can assume $P \subseteq Q$ *and* $Q \subseteq P$." While Carl did not accept Sarah's claim on Day 8, he exhibited his acceptance of converse independence at the exit interview. Specifically, Carl determined, "Proof γ (proof of converse) proves $B \subseteq A$ but doesn't really prove Theorem γ ($A \subseteq B$) [because] Proof γ lies on the reader inferring $A = B$. Basically, if proof γ added an extra line, proving $A = B$, then it would be equal to it. Then it would prove the theorem. [but] I don't think they do it."

Discussion

In this paper, we documented how Carl and Sarah generalized contrapositive equivalence and converse independence across proofs of conditional statements as they were engaging with the WDIP tasks. The interviewer's prompting to use only one diagram to interpret two subset relationships helped the students make the line of inference between Proof 1.3 (proof of contrapositive) and Theorem 1 explicit, such that they affirmed it by their "prior knowledge" regarding the contrapositive equivalence. Later, Carl used this idea to justify why proof of converse proves the original conditional. For him, proof of contrapositive and proof of converse both relied on his prior knowledge, which he called "prove indirectly." Indeed, in one case, his prior knowledge was logical knowledge about the generalizable contrapositive relationship; in the other, it was local mathematical knowledge that the situation described by Theorem 4 related two equal sets of quadrilaterals. By introducing Proof 4.4, the interviewer's move of reflecting the form of Carl's justification back to the students allowed them to move forward in critiquing the justification. However, to see the conflict, which is what mathematicians usually call "circularity," Sarah needed to interpret set equality as the conjunction of two subset claims. This made it clearer how asserting $P = Q$ without proof was tantamount to asserting $P \subseteq Q$ without proof. Understanding the logic of the relationships between proofs and theorems is quite challenging. Still, we see how these instructional moves supported Carl and Sarah in apprehending the structure of their own arguments to evaluate them more precisely. We are pleased with how our interventions allowed these students to wrestle deeply with these matters of justification, though more work is needed to explore this arena of learning.

Acknowledgment

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Students’ Inferences Using a Conditional Statement in Probability: The Relationship between Independence and Covariance of Random Variables

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The purpose of this study was to investigate how students in a probability course interpret a theorem and how they apply the theorem when asked to determine whether two random variables are independent given the value of the variables’ covariance. We considered how responses of students with prior training in logic compare to other students, as the theorem is a conditional statement. We used a quasi-experimental design to investigate the impact of two instructional treatments, using isomorphic colloquial statements and using Euler diagrams. Participants largely gave conventional truth values for variants of the conditional statement prior to instruction, but many of them used the converse and/or did not use the contrapositive when given covariance values and asked to determine whether two variables were independent. Our results do not consistently show a significant difference in students’ choices based on their prior logic training or either treatment.

Keywords: Probability, Conditional Statements, Logic, Covariance, Independence

Students are frequently presented with conditional statements in undergraduate mathematics courses that do not place a large emphasis on proof; consider as examples the Intermediate Value Theorem and the Divergence Test for Series in Calculus and the Existence and Uniqueness Theorems in Differential Equations. Challenges arise as students with no formal training in logic contend with deciding when and how to apply these tests and theorems while instructors sometimes make assumptions that logical inference is more intuitive than it is. While studying students’ use of formal mathematical logic has long been of interest for the training of students majoring in mathematics, it has taken more time for researchers to begin exploring how students’ interpretations of conditional statements in courses like calculus might impact their success (Case, 2015; Durst & Kaschner, 2020; Sellers et. al, 2021).

In this study we consider how Probability students interpret and use the following theorem about random variables: “If X and Y are independent random variables, then $\mathrm{Cov}(X, Y) = 0$” (Casella & Berger, 2002, p. 171). The statement’s converse, “If $\mathrm{Cov}(X, Y) = 0$, then X and Y are independent” is logically false. (Covariance is a measure of a linear relationship, and variables may be clearly related without being linearly related.) In a probability course, students may be asked to use their knowledge about the independence or covariance of two random variables to make inferences about the other, and more importantly, to recognize in which situations they can do so.

Our study differs from the majority of prior work on students’ understanding of conditional statements in two ways. First, our goal is not to help students redevelop conventional rules of logic to then use more broadly. Rather, we are providing students with the truth values of a statement and its three variants (subsequently referred to as the contrapositive, converse, and inverse) and we want to know which analogies and connections might be useful in helping students use this particular conditional statement to make accurate inferences about independence. Secondly, because of the nature of these problems, students have the option to entirely bypass the use of logical inference. While calculations may be cumbersome, covariance can be calculated directly and independence determined using the definition of independence. That is, students may choose to avoid using the logic of conditional statements altogether to answer the problems we give them. We therefore pose the following questions:

1. *How do students use covariance values to determine whether two random variables are independent?*
2. *Are their choices influenced by prior formal logic training?*
3. *Are their choices influenced by connections to isomorphic colloquial examples and/or the use of Euler diagrams?*

Review of Literature and Theoretical Perspective

An important and consistent finding in prior studies of students' interpretations of conditional statements is that students very often interpret conditional statements as biconditionals (Hoyles & Küchemann, 2002; O'Brien et. al, 1971; Wason, 1968). This presents as students assuming inverse and converse statements are true even though they are not. Possibly more significant is the finding that even the same student often does not make the same logical inferences consistently; instead, their inferences depend on the context, particularly when it is a familiar one. (Dawkins & Cook, 2017; Durand-Guerrier, 2003).

The results above must be interpreted in light of which "reasoning domain" the researchers used (Sylianides et. al, 2004). Logic can be taught using meaningful everyday (colloquial) statements, meaningless everyday statements, mathematical statements, or formal syntax (Hub & Dawkins, 2018). Claims have been made that students have less difficulty interpreting verbal and mathematical statements than abstract or symbolic ones (Case, 2015). Moreover, it has been hypothesized that everyday statements may be used to help students develop logical principles. Epp (2003) argues "it is helpful to introduce each principle with examples of sentences whose "natural" interpretation agrees with the one used in standard logic" (p. 895). Stylianides et. al (2004) found evidence to support this. Specifically, students who initially identified a contrapositive statement as false changed their minds after seeing an isomorphic statement in the verbal domain.

However, formal logic conventions do not align with the way conditional statements are used in everyday language. Epp (2003) provides the following example. A parent may say, "You can go to the movie if you finish your homework" when what she intends is "You can go to the movie if and only if you finish your homework," but the latter rarely gets stated. "If you don't finish your homework, you cannot go" is implied. So while they only state a conditional, the intention is biconditional. It is not surprising that students would interpret conditional statements in mathematics similarly. We interpret the theorem relating independence and covariance as a "generalized conditional statement" (Durand-Guerrier, 2003). There is an implicit universal quantifier within the antecedent, "X and Y are independent random variables," so mathematicians interpret the conditional as "$For\ all$ pairs of random variables X and Y, if X and Y are independent, $\mathrm{Cov}(X, Y) = 0$." Accordingly, mathematicians assign a positive truth value to the statement and its contrapositive and negative truth values to its converse and inverse. Students often, and reasonably so, read conditional statements as "open statements" which have no quantifier and no truth value (Durand-Guerrier, 2003).

Some authors have posited that explicit training in logic and/or exposure to more mathematics should improve students' ability to make correct logical inferences. Durst and Kaschner (2020) for example, show that students often label statements such as "If the limit of a sequence is 0, the corresponding series converges" as "Sometimes True, Sometimes False." They argue that explicit training in logic would assist students in adopting the conventions of formal logic. However, there has been no clear relationship established between current performance and either prior exposure to either mathematics or logic instruction (Attridge et. al, 2016; Cheng et. al, 1986; Inglis & Simpson, 2008).

Our position is that students, in identifying some statements as "sometimes true," have viable reasons for doing so. These students are not conforming to conventional rules of logic but may, in fact, be being more specific. Additionally, while adhering to these conventions may be essential for success in mathematics, we question whether this should be a goal for engineering students. Engineers are trained to become comfortable with uncertainty and this training leads to a reasonable and necessary discomfort with absolutes (Cardella & Atman, 2005; Gainsburg, 2006). Recognizing this, we narrow our focus to identifying analogies that students may use to convince themselves of when independence and covariance may be used to determine the other.

Recently Dawkins and colleagues have argued for developing in students a "subset meaning for conditional truth" (Dawkins & Cook, 2017; Hub & Dawkins, 2017). Their work suggests that given a conditional statement, students initially struggle to determine whether the set of elements that satisfy the *if* are contained in the set of elements that satisfy the *then*, or vice versa. Hub and Dawkins make the case that using Euler diagrams, which provide an illustration of this subset relationship, may facilitate the development of the subset meaning which will then help them move towards consistently making normative logical inferences. For example, a student who places set P within set Q can recognize in the visual representation that elements outside Q are also outside P.

Methods

Data Collection

The data was collected during in-class instruction and testing in the spring of 2022 in a semester-long Applied Probability course for engineering and computer science students at a large public institution in the United States. Prerequisites for the course include multivariable calculus and an introduction to programming. All students from the four sections who were present during this lecture were invited to participate and 72 students provided consent. Of these 72 students, 53 had previously taken or received credit for Discrete Mathematics, which provides an introduction to logical inference and proof techniques.

The second and third authors each taught two sections of the course on the same day when they covered the relationship between covariance and independence of random variables. Each instructor taught one "control" and one "treatment" section, though the treatments differed. After the initial presentation of the theorem in each section, the instructors asked each student to determine whether each of the following statements below (subsequently referred to as the inverse, converse, and contrapositive) were true or false and collected their responses prior to any discussion.

a. If X and Y are not independent, then $Cov(X,Y) \neq 0$.

b. If $Cov(X,Y) = 0$, then X and Y are independent.

c. If $Cov(X,Y) \neq 0$, then X and Y are not independent.

The instructors then explained briefly why statement (c) was true but statements (a) and (b) were false. Specifically, they shared an example of two random variables that were dependent but had no linear relationship (thus having a covariance of 0) and briefly reiterated that covariance is specifically a measure of the *linear* relationship between the variables.

At this point in the lesson, instruction in the control and treatment sections diverged. In the Control sections, which closely resembled how instructors of this course have presented this lesson in past semesters, the instructors completed 2 examples where they first determined whether two random variables were independent and then found the covariance of those variables. In the first example, the variables were independent and therefore the statement of the theorem could be applied directly to decide that the covariance must be 0. In the second, the variables were determined to be dependent, so the covariance had to then be calculated.

Instructor 1 presented the Treatment 1 students with the colloquial statement: "If it is raining, there are clouds" and asked them to determine whether the other variants of this statement, such as "If it is not raining, there are no clouds" were true. This was completed as a whole class discussion and subsequently the instructor made a direct comparison between the truth values of the rain/clouds statements and variants of the theorem.

Instructor 2 presented an Euler diagram illustrating the universal set of all pairs of random variables. (Note that in our conditional statement, the elements of sets P and Q are *pairs* of random variables, which is a potential complication for students in the process of abstraction.) In the diagram, an oval representing the pairs of independent random variables was completely contained within an oval representing the pairs of random variables with a covariance of zero. The class then discussed how a situation could be in the outer oval without being in the inner oval, but it was impossible to be in the inner oval without also being in the outer oval.

At the end of class for all four sections, students individually provided open-ended responses to two problems in which they first computed covariance of two random variables and then were asked to determine whether those variables were independent and justify their answers. In the first example, the covariance was 0 so independence had to be determined using the definition of independence. In the second example, the covariance was not 0 so dependence was guaranteed. We will subsequently refer to these examples as the "In Class Converse" problem and the "In Class Contrapositive" problem, respectively.

We also collected data on a midterm exam and the final exam. The midterm contained a similar converse problem and the final exam a similar contrapositive problem. The midterm and final exam were given 12 days and 6 weeks after the in-class instruction, respectively and there was no significant intervention between assessments.

Analysis

The first and third authors first open coded responses to the two in-class problems. When coding the midterm and final exam problems, we built on and refined the coding schemes from the in-class problems since they were very similar to the in-class problem 1 and 2, respectively. At this stage we were attending to correctness as well as the type of justification provided. The second author then used the coding scheme to code all 4 problems and we refined the coding scheme to reach agreement on approximately 95% of responses across the 4 problems.

For ease of comparing groups by treatment and by exposure to prior logic instruction, we opted to complete a second round of coding to condense the number of codes for each type of problem. For the contrapositive problems, we considered whether the student used the fact that a covariance other than 0 implies variables are dependent (which we labeled "using the contrapositive"). For the converse problems, we asked whether students used the fact that covariance was 0 to (incorrectly) claim that the variables were independent (which we labeled "using the converse"). This resulted in a binary response for each student for each of the 4 problems that we then used in the comparisons discussed below.

Results

Prequiz Truth Values

The truth values that students assigned after being presented with the theorem but prior to instruction were largely in line with conventional logic. Of the 72 participants, 59 of them (82%) stated the inverse was false and (though not the exact same group of students) 59 also claimed the converse was false. 64 students (89%) selected true for the contrapositive statement. 49 students (68%) gave conventional truth values for all three statements. Prior logic training did not appear to impact responses; 68.4% of students who did not have credit for Discrete Math gave conventional truth values for all three statements while 67.9% of those with credit for Discrete Math did the same.

Types of Responses

For the in-class and midterm converse problems, we focused on whether students used the fact that covariance was 0 to (incorrectly) claim that the variables were independent (which we will subsequently refer to as "assuming the converse"). 38% of students used this argument on the in-class converse problem and 31% used it on the midterm. Only 25% used the definition of independence on the in-class problem to correctly conclude that X and Y were dependent. However, the response of 17% of students in class and 10% of students on the final exam was that independence could not be determined since the covariance was 0. These students made a conventional logical inference, although they failed to recognize that independence can be determined directly through use of the definition of independence. This highlights another challenge of introducing and asking students to use this conditional statement; students may focus on the application of the theorem and neglect viable and necessary alternatives. We saw improvement in using the definition of independence on the midterm; 40 of the students (56%) used the definition of independence on the midterm problem to reach the correct conclusion that X and Y were dependent. The remaining students used incomplete or incorrect arguments, some of which referenced the definition of independence and some which did not.

On the in-class contrapositive problem, 90% of students used the covariance being a value other than 0 to (correctly) claim that X and Y were dependent. In contrast, only 44% of students used this argument on the final exam problem. On the final exam problem, 33% of students "bypassed" the theorem and used the definition of independence correctly. This appears to coincide with the increase in students using the definition of independence correctly on the midterm converse problem. 13% of students provided multiple justifications; specifically, they stated that X and Y must be dependent because the covariance is not 0 and also used the definition (either by stating $P_X(x)P_Y(y) \neq P_{X,Y}(x,y)$ for all x, y or by providing an example.) We grouped these students with those who did not use the contrapositive in our analysis below.

From Covariance to Independence

We might expect students who claimed the converse was true on the prequiz to be more likely to use the covariance of 0 as justification to claim independence on the in-class converse problem. (We highlight the cells in Table 1 below where responses on the prequiz and in-class problem are consistent from our perspective.) However, students who said the converse was false on the prequiz were actually slightly more likely to use covariance being 0 as justification for independence in the example problem than students who had said the converse was true in the prequiz.

Table 1. Converse Prequiz Truth Value versus In Class Problem Approach

Prequiz Response	Used Converse	Did Not Use Converse	TOTAL
Converse: False	22	37	59
Converse: True	5	8	13
TOTAL	27	45	72

A smaller percentage of Treatment 2 students assumed the converse on the In Class Problem than Control 2 students. In contrast, a smaller percentage of Control 1 students assumed the converse than Treatment 1 students. However, these differences were not statistically significant. We built a binary logistic regression model for the response on the In-Class Converse problem for each instructor using Previous GPA, prior logic experience, and treatment as independent variables. For each instructor, p-values for all three variables were each at least 0.1. Trends on the midterm Converse problem closely resembled those on the in-class problem except that on the midterm, treatment for Instructor 1 was a marginally significant predictor ($p = .042$) with Control students less likely to assume the converse.

There was greater consistency between the truth value students selected on the prequiz and the approach to the in-class problem for the contrapositive than for the converse (see Table 2). Specifically, 58 of 72 students claimed the contrapositive was true and used covariance being nonzero to decide X and Y were dependent. However, of the eight students who declared the contrapositive was false on the prequiz, seven of them used covariance being nonzero as their sole justification for the variables being dependent. Furthermore, six students claimed the contrapositive was true but did not use the covariance value in determining independence.

Table 2. Contrapositive Prequiz Truth Value versus In Class Problem Approach

Prequiz Response	Used Contrapositive	Did not Use Contrapositive	TOTAL
Contrapositive: False	7	1	8
Contrapositive: True	58	6	64
TOTAL	65	7	72

Treatment 2 students were more likely than Control 2 students and Control 1 students more likely than Treatment 1 students to use the contrapositive on the In Class Problem. On the final exam problem, Treatment 2 and Control 2 students performed comparably and Control 1 students again used the contrapositive more than Treatment 1 students. These results were not significant when using binary logistic regression and neither GPA nor prior logic training were statistically significant predictors for whether students used the contrapositive.

Discussion and Limitations

Students in this study were more likely to provide conventional truth values for variants of a conditional statement than they were to apply those only when appropriate to determine whether two random variables are independent. Students who provided conventional truth values were not more likely than others to use the contrapositive or avoid using the converse. This may partially explain why prior logic training was not a statistically significant factor impacting problem response. We posit that many students are able to identify contrapositive and converse statements and have learned conventional truth values for the same, but do not hold the same meanings for “true” and “false” as those in the mathematics community. This may help explain why one of the control groups outperformed the treatment group. Each treatment focused on helping students reach agreement with the conventional truth values of the contrapositive and converse statements. Yet, our data suggests that this is not the primary issue. The examples used in instruction with only the control sections accomplished something the treatments did not; they made explicit that the theorem could not be used to solve all problems and that alternative methods to calculating covariance or determining independence are available.

We saw a marked increase in using the definition of independence from in-class to midterm/final exam on both converse and contrapositive problems. It is natural as an instructor to view this shift positively for the converse problems and somewhat negatively for the contrapositive problems. It may be that some students did not have confidence in their ability to apply the theorem correctly during exams and therefore decided to always calculate covariance and determine independence directly. Alternatively, it is possible students simply didn’t think to use the theorem. Students also may have been less likely to use the definition of independence on the in-class problems because of availability bias (Tversky & Kahneman, 1973); they had discussed the theorem more recently than the definition of independence in class, but between class and the midterm and final exam had more time to review. Other explanations are possible of course and this is one question that should be explored in the future with student interviews.

A primary limitation in this study is the single source of data (student written responses) and our inability to perfectly dichotomize students by their understanding of the conditional statement and its implications. For example, a student who chooses to use the definition of independence when the covariance is 0 does not necessarily believe the converse is false. The student may, in fact, answer this way with little to no knowledge of the theorem or logic principles. As another example, some students provided multiple justifications to one of the problems and we do not know why. They could have been simply checking or confirming their work with a second method. Alternatively, students who use the contrapositive correctly may simply be reiterating a rule they’ve been taught but do not necessarily believe it is always true. Future studies may follow up these types of responses with interviews and slightly modified problems so that students can be more carefully classified. We are particularly interested in the 22 students who claimed the converse was false but then used covariance being 0 as justification for X and Y being independent on the in-class converse problem.

Another limitation is using credit for Discrete Math as a proxy for whether the student had received prior instruction in logic. We cannot guarantee that some of the students who did not have credit for Discrete Math had not taken other courses in logic. Moreover, we did not capture performance in Discrete Math but only whether credit had been achieved.

Finally, the instruction was limited to one 50 minute class period, not by choice but because of the restrictions imposed by the course schedule. This once more highlights the common challenge of how to best prepare students who are not math majors for courses that rely on an understanding of logical implication but do not have time allotted for explicit logic instruction.

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Analyzing the Reasoning Types potentially Required by Tasks in an Open-Source Introductory Statistics Textbook.

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This study examined the tasks in the exercise sections of a popular open-source statistics textbook in order to characterize the types of reasoning the tasks are likely to elicit from readers. To achieve this, I examined the narrative sections of the book alongside the solved examples to determine the extent to which the kind of guidance provided was relevant in solving the tasks in the exercise sections of the book. Findings show that although most tasks could be solved by closely following the procedures and methods in the book (imitative tasks) there were other tasks that could elicit more nuanced forms of reasoning (creative mathematical reasoning tasks). These tasks were more prominent on the tail ends of the exercises section than on the beginning parts. Implications of these findings are discussed.

Keywords: Creative mathematical reasoning, Task analysis, Mathematics textbooks, Imitative reasoning.

Mathematics textbooks are among the most important resources for the teaching and learning of mathematics at the college level (Geteregechi & Waswa, 2020). Both instructors and students rely on mathematics textbooks to meet various learning needs such as self-study, practice, among others (Van Steenbrugge et al., 2013). Although it is difficult to exclusively characterize the role of textbooks in students learning of mathematics (Tarr et al. 2008), researchers (e.g., Stylianides, 2009) have reported that textbooks have a significant influence on how students make sense of and learn mathematics concepts.

One of the most important aspects of student learning is the kinds of reasoning that they engage in. Instructors often want to understand students' ways of thinking so they can better support their learning. However, what is often ignored is the sources that may inform students' thinking. In this paper, I argue that textbooks play a significant role in informing students' ways of thinking and thus there is need to examine the opportunities they present. Although there are many studies on the kinds of opportunities that mathematics textbooks provide for their readers (Thompson et al., 2012), not many such studies focus on how the narrative section of the texts influence the kinds of reasoning that students use for future task solving including tasks found in the exercise sections of the texts. I define the narrative section as the parts of a textbook where the readers learn the intended concepts and ideas. For most textbooks, the narrative section comprises of descriptions, illustrations, solved examples, among others. Exercise sections, on the other hand, comprise of the tasks intended to be completed by the readers of the text.

Most studies examining mathematics textbooks focus on calculus and geometry textbooks hence leaving a dearth of research in other important areas such as statistics. Furthermore, not many studies examine tasks from the perspective of the kinds of reasoning they are likely to elicit from readers. The term reasoning is often used without a clear definition of what it is and how to characterize it which means many studies on textbook analysis do not provide ways to characterize the potential of tasks to elicit specific kinds of reasoning. In this study, therefore, I sought to answer the following research questions:

1. What kinds of reasoning can be expected from tasks in a popular introductory open-source statistics textbook?
2. How are these tasks distributed within the exercises section of the book?

Related Literature

Most studies analyzing opportunities for reasoning in mathematics describe general characteristics associated with higher order thinking. Dolev and Even (2015), for example, examined the extent to which tasks in mathematics textbooks required students to provide justifications for claims made. The study compared six textbooks with a focus on geometry and algebra strands. Their findings indicated that geometry tasks tended to have more opportunities for justifying than algebra tasks across all textbooks. Similarly, Otten et al. (2014) examined the reasoning-and-proving opportunities available in secondary geometry textbooks and reported that although there were numerous reasoning and proving opportunities in both the narrative section and the exercises section of the books, these opportunities were not an object of focus in these books. While these studies provide insights into the kinds of tasks in textbooks, they rarely classify tasks themselves based on the reasoning types they are likely to elicit. This is due, in part, to the fact that the studies did not approach the analysis with a specified definition of reasoning. Thus, this study uses Lithner's well defined framework on reasoning to characterize tasks according to the kinds of reasoning they are likely to elicit. There are, however, some studies that have used specified definitions of reasoning in conducting their analysis. For example, Sidenvalli et al. (2015) analyzed the reasoning types required by secondary mathematics textbooks and compared them to the types of reasoning actually used. The study reported that most tasks (80%) could be solved by imitating the procedures given in the book. Studies such as this are rare for college-level textbooks especially ones that focus on statistics concepts.

Frameworks

Lithner (2008) defined the term reasoning as "the line of thought adopted to produce assertions and reach conclusions in task solving" (p. 257). Argumentation is the part of reasoning that one uses to ground the truthfulness or otherwise of assertions. Argumentation could be based on intrinsic properties (relevant and correctly applied mathematical features in a given situation) or non-intrinsic ones. If one is trying to tell which fraction between 7/8 and 8/9 is bigger, for example, an intrinsic property would be the quotient while a non-intrinsic one would be a focus on the magnitudes of the numbers themselves (e.g., 9 is bigger than 8 so 8/9 is bigger). Lithner (2008) categorized mathematical reasoning into two broad categories namely imitative reasoning (IR) and creatively founded mathematical reasoning (CMR). IR is analogous to rote learning and is further categorized as memorized imitative reasoning (MIR) or algorithmic imitative reasoning (AIR). MIR involves complete recall of a solution to a problem and is common especially in tasks requiring proofs of mathematical claims and theorems. AIR, on the other hand, involves following certain solution procedures/algorithms to a certain class of problems without a deeper understanding of the process.

Reasoning is classified as CMR, if it meets the following conditions:

1. Novelty. A new (to the reasoner) reasoning sequence is created or a forgotten one is recreated.
2. Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible.

3. Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning. (Lithner, 2008, p. 266)

CMR is further classified as local or global. Local CMR can be thought of as an improved form of AIR. Local CMR happens when someone makes non-trivial modifications to parts of an already known solution strategy or algorithm in solving a new task. Global CMR happens when a solver has no known solution strategy or algorithm and thus engages in CMR throughout the task. Using these classifications of reasoning, I define tasks accordingly. A task that requires imitative reasoning is called an IR task while a task that requires CMR is labelled a CMR task. CMR tasks are further classified as global or local while IR tasks can be algorithmic (AIR) or memorized (MIR).

In making the above classification of tasks, I recognize the fact that different solvers may experience the same task differently depending on several extraneous factors. A task may be a local CMR task for one person while being a global CMR task for another. Since the focus of this study was to examine the tasks themselves, the study only considered the guidance provided in the narrative section of the book and the extent to which such guidance would likely elicit different kinds of reasoning among readers of the book.

Method

Data and Setting

The text analyzed in this study is the fourth edition of *OpenIntro Statistics* by David Diez, Mine Centikaya-Rundel and Christopher Barr. I chose this book because it is one of the most popular open-source books being used as the main text or as an additional reference "at community colleges to the Ivy League" (OpenIntro, n.d). The book is divided into nine chapters each with several sections. Each section has one exercise set with several exercise tasks for readers to complete. I randomly sampled four tasks from selected sections within chapters five, six, seven, and eight. I chose these chapters because their primary focus was on statistical inference, which according to previous studies, is one of the areas that many students struggle in (Makar & Rubin, 2018). Chapter five had three sections with a total of 26 tasks while chapter six had five sections with a total of 38 tasks. Chapter seven on the other hand had five sections and a total of 46 tasks. For chapter eight, only one section focused on inference (i.e., inference for linear regression) and had six tasks. Previous studies (e.g., Author, 2020) have shown that for many textbooks, more cognitively demanding tasks tend to appear towards the end of the exercises. Since I were not sure whether this is the case for the book that I analyzed, I wanted to ensure our sample had at least one task from all parts of the exercises (i.e., initial, middle and last). To achieve this, I used stratified sampling where I split the questions into the first 25%, middle 50% and upper 25%. I then used simple random sampling to select one task from the first stratum, two from the second, and one from the last. Some selected tasks had several parts (like a, b, c etc.) that were related and built on one another. Whenever such a task was selected in the random sampling, I purposefully chose the last one in the series since it was likely to incorporate all the others in some way. In the end, I had 54 tasks in the sample, representing 47% of all tasks within the target sections of the book.

Within the narrative of the text, I focused mainly on the solved examples, guided practice items, and the solved case studies. I also brushed through the other parts of the narrative section (e.g., introduction and motivation) for descriptions of procedures and methods of solving certain kinds of tasks. This information was useful in characterizing the reasoning forms that the tasks in the exercise section of the book might elicit.

Analytic Procedures

To analyze the data described above, I used a modified version of Lithner's framework as provided in Mac an Bhaird et al. (2017). As described earlier, argumentation is an important component in any attempts to characterize reasoning. However, since the aim in this study was to characterize exercise tasks according to the kinds of reasoning they are likely to elicit from their readers and not the reasoning of specific students, I thought that examining the model solutions alongside the argumentations given in the book would be more meaningful. This is because the solutions in the book provide a model that readers may use with varying degrees of modification when solving the exercise tasks. While I recognize that a reader can use knowledge gained elsewhere to solve problems in the book, I believe that for most readers, the methods described in the book would play a significant role in their task solving process. Indeed, Mac an Bhaird et al.'s (2017) noted that the solved examples provided in math textbooks and lectures play a key role in students' task solving processes. I then solved the sampled tasks from the exercise sections by following the methods given in the book whenever possible while keeping track of the number and nature of any required modifications on the book methods. I defined modifications as any alterations on the book methods without which a successful solution would be impossible. Trivial alterations such as a different problem context and numerical values were not considered modifications. Another expert also worked on the problems separately then we met to discuss. The interrater agreement on the modifications was 94%. I labelled tasks that required no modification of procedure in the book as IR tasks and those that required at least one modification or that did not have a procedure in the book as CMR tasks. I classified CMR tasks as local CMR if they required only one modification and global CMR tasks if they required at least two modifications or if there was no algorithm or procedure for solution in the book.

Results

To concretize the analysis described above, I start by using a task (see Figure 1) drawn from a section called "Confidence Intervals for a Proportion" under the chapter titled "Foundations for Inference".

> **5.14 Coupons driving visits.** A store randomly samples 603 shoppers over the course of a year and finds that 142 of them made their visit because of a coupon they'd received in the mail. Construct a 95% confidence interval for the fraction of all shoppers during the year whose visit was because of a coupon they'd received in the mail.

Figure 1. A task from Inference for Single Proportion

We were able to solve this task by closely following the solution model to one of the guided exercises (see Figure 2) in the book. Although the problem does not provide a percentage value for the success rate (in this case percentage of shoppers who made a visit because of a coupon), this is considered trivial given that most readers of the book would be college or high school students. The rest of the solution requires using different numbers and substituting them in the formulas appropriately. The critical value associated with the new confidence interval can be confusing to some readers but the process of finding it was provided elsewhere in the book and could be applied in this new situation.

GUIDED PRACTICE 5.15

In the Pew Research poll about solar energy, they also inquired about other forms of energy, and 84.8% of the 1000 respondents supported expanding the use of wind turbines.[9]

(a) Is it reasonable to model the proportion of US adults who support expanding wind turbines using a normal distribution?

(b) Create a 99% confidence interval for the level of American support for expanding the use of wind turbines for power generation.

Solution:

[9](a) The survey was a random sample and counts are both ≥ 10 ($1000 \times 0.848 = 848$ and $1000 \times 0.152 = 152$), so independence and the success-failure condition are satisfied, and $\hat{p} = 0.848$ can be modeled using a normal distribution.
(b) Guided Practice 5.15 confirmed that $\hat{p}$ closely follows a normal distribution, so we can use the C.I. formula:

$$\text{point estimate} \pm z^{\star} \times SE$$

In this case, the point estimate is $\hat{p} = 0.848$. For a 99% confidence interval, $z^{\star} = 2.58$. Computing the standard error: $SE_{\hat{p}} = \sqrt{\frac{0.848(1-0.848)}{1000}} = 0.0114$. Finally, we compute the interval as $0.848 \pm 2.58 \times 0.0114 \rightarrow (0.8186, 0.8774)$. It is also important to *always* provide an interpretation for the interval: we are 99% confident the proportion of American adults that support expanding the use of wind turbines in 2018 is between 81.9% and 87.7%.

[10] No, a confidence interval only provides a range of plausible values for a parameter, not future point estimates.

Figure 2. Sample guided practice exercise and solution

A reader solving the task in Figure 1 can provide a similar sequence of argumentation as given in the guided practice and other parts within the text. For this reason, I categorized the task as an AIR task. Next (Figure 3) I consider an example of a CMR task.

5.11 Waiting at an ER, Part I. A hospital administrator hoping to improve wait times decides to estimate the average emergency room waiting time at her hospital. She collects a simple random sample of 64 patients and determines the time (in minutes) between when they checked in to the ER until they were first seen by a doctor. A 95% confidence interval based on this sample is (128 minutes, 147 minutes), which is based on the normal model for the mean. Determine whether the following statements are true or false, and explain your reasoning.

(a) We are 95% confident that the average waiting time of these 64 emergency room patients is between 128 and 147 minutes.

(b) We are 95% confident that the average waiting time of all patients at this hospital's emergency room is between 128 and 147 minutes.

(c) 95% of random samples have a sample mean between 128 and 147 minutes.

(d) A 99% confidence interval would be narrower than the 95% confidence interval since we need to be more sure of our estimate.

(e) The margin of error is 9.5 and the sample mean is 137.5.

(f) In order to decrease the margin of error of a 95% confidence interval to half of what it is now, we would need to double the sample size. (Hint: the margin of error for a mean scales in the same way with sample size as the margin of error for a proportion.)

Figure 3. A sample CMR task

Although there were several solved examples and guided exercises in the narrative section of the text, none of them was presented in a way that can be directly used to answer the task in Figure 3 (part d). The examples provided step by step procedures for creating confidence intervals but did not provide ways of assessing the effect of sample size on the margin of error. As a result of this, I concluded that a solver would have to reuse the procedure for creating confidence intervals in a different way that goes beyond a mere plugging in of numbers. Specifically, a solver will need to think about the procedure in a reverse manner since the confidence interval is already provided. These two modifications meant that the task was further classified as a global CMR. In an analogous manner, I examined all sampled tasks within the chapters focusing on statistical inference. I summarized the results in Table 1.

Table 1. Classification of Task Types by Chapter and Distribution

Chapter	IR Tasks			CMR Tasks					
				Local CMR			Global CMR		
	First 25%	Mid. 50%	Upper 25%	First 25%	Mid. 50%	Upper 25%	First 25%	Mid. 50%	Upper 25%
Foundations of inference	5	3	1	0	1	1	0	0	1
Inference for Categorical data	4	6	2	0	1	1	0	1	1
Inference for numerical data	7	6	2	1	2	1	0	0	1
Linear Regression	2	2	1	0	0	1	0	0	0
	18	17	6	1	4	4	0	1	3

The general trend prevalent in Table 1 is that the text has more IR tasks than CMR tasks. Out of the 54 sampled tasks, 41 (76%) were found to be IR tasks. These tasks tended to appear more at or below the 50th percentile. Indeed, out of the 41 IR tasks, 35 (85%) appeared at or below the 50th percentile. This trend appears to be reversed for CMR tasks, especially for global CMR ones. Three out of 4 global CMR tasks were all found at or above the 75th percentile of tasks. Similarly, for local CMR tasks, 8 out of 9 tasks were at or above the 50th percentile.

Discussion

This study sought to explore the kinds of exercise tasks in a popular open-source statistics textbook based on the kinds of reasoning the tasks are likely to require for successful solution. As noted earlier, most studies on textbook analysis are based on calculus and geometry strands and do not use well specified definitions of reasoning. Thus, this study addresses this gap by using a well-specified framework for mathematical reasoning to characterize mathematical tasks based on the kinds of reasoning they are likely to elicit from readers. The study targeted one of the most challenging parts of statistics (inference) within a popular open-source textbook used in several colleges for introductory statistics courses.

In general, the study found that the processes described and/or implemented within the narrative sections of the book could be followed with little to no modifications for successful solution of most tasks within the exercise sections. This means that opportunities for engaging in deeper thinking were limited at least in the exercise sections. This finding should not be taken to mean that there are no opportunities for deeper thinking in the textbook. In fact, the narrative section did have examples that provide opportunities for deeper thinking but only of the reader attempts them before looking at the solution provided. Nevertheless, there were several tasks that provided opportunities for deeper thinking in the sense that solvers would have to engage in creative reasoning for successful solution. These tasks require major modifications to the procedures described in the book and a reader would need to think deeply about the concepts for successful solution. This finding is similar to those reported by Sidenvalli et al. (2015) which indicated that mathematics textbooks tended to have more imitative reasoning tasks than creative reasoning tasks.

In regard to the distribution of the tasks within the exercises section, I found a general trend in which imitative reasoning tasks appeared at or below the 25th percentile while creative reasoning tasks appeared at the tail end (above the 75th percentile). A possible explanation for this trend could be that the authors intended readers to practice with basic concepts and factual tasks before they could engage with more cognitively engaging ones.

One possible implication of this study is for instructors using the text as a resource in their classrooms. When assigning exercises, it would be helpful to have students complete at least one task from the initial, middle and the last parts of the exercise sections. Depending on the goal of assigning the exercises, it might also be helpful for instructors to modify the tasks in some way before assigning them to the students.

One limitation of this study is the lack of student voices in characterizing the reasoning likely to be elicited by the tasks. Nevertheless, the fact that the study examines the tasks in relation to the narrative section still provides useful information because the arguments provided in the narrative section are likely to be repeated by students solving the tasks.

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Navigating AB705 and a Pandemic: An Investigation of Students' Experiences With Placement and Advising

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Recent state policies have pushed for more diversity in the STEM pipeline. One proposed solution is the California Assembly Bill (AB705) (California Community College, 2018), which requires students to take and pass gateway math courses within their first year at community colleges. In theory this policy seems like a solution, however pragmatically there is a lot to be learned about its implementation and consequences for students. In this paper, we address the research question: What are students experiences with placement and advising at a local two-year college. We conducted a descriptive qualitative case study (Yin, 2009) that draws upon s student outcome data and focus groups. We focus on 5 STEM majors' experiences. Findings suggest most students passed their gateway math courses within a year, however students described negative placement and advising experiences: receiving deficit-oriented suggestions from counselors and limited counselor availability. We discuss implications in the paper.

Keywords: Gateway Mathematics Courses, Placement, Advising, Guided Self Placement (GSP)

Introduction

California Assembly Bill 705 (AB705) was created to increase the diversity of students pursuing STEM (California Community College, 2018). AB705 requires that students enroll and complete transfer-level mathematics courses, such as college algebra, trigonometry, or pre-calculus by the end of their first year. In order to meet state mandates, institutions were forced to adjust structures and procedures, while still navigating various levels of students' preparation. Despite AB705 being well intentioned, one potential consequence is that the increase in students taking STEM courses sooner, has not led to a decline in student failure rate in these courses. In fact, the failure rate in these courses is the same or in some cases higher. This creates a paradox of sorts, as administrators see the increase in "throughput" as a positive, while instructors see the unchanging or higher failure rates as a negative result. This paradox raises concerns about what issues are underlying the implementation of the state mandate. In this paper we describe students' experiences with placement and advising at a local two-year college. Specifically, we consider the role of counselors in placement and advising students. It is important to note that the study took place during the pandemic. We highlight the pandemic because it is an additional layer of complexity that potentially impacts student placement and advising. As a result, this paper has two purposes: understanding students' experiences with placement and advising and highlighting the challenges of ensuring that students are appropriately placed in gateway mathematics courses. The research question that guides this paper is How do students feel about their placement and advising compared to how they did in the course they were placed into?

Literature Review

Several researchers have investigated gateway mathematics courses at two-year colleges. For example, one thread of inquiry researchers have tried to understand is student placement and the subsequent role that advising plays in the placement process. By gateway mathematics courses, we mean courses that are credit-bearing and transferable to four-year institutions (e.g., College Algebra, Precalculus). With respect to student placement and advising, there are several key findings worth highlighting. First, Kosiewicz and Ngo (2019) using self determination theory, found students who have autonomy to select their courses are successful in enrolling and passing those courses. This finding is important as it highlights the value and importance of student agency. In contrast, other research using Behavioral Decision Theory (BDT) has found that students often do not make good decisions on which course to take (Payne, Bettman, & Johnson, 1992). The tension between findings that show students successfully enrolling and passing their course, alongside studies that show students do not make good decisions raises the question: Under what circumstances is it possible for students to exercise their autonomy and agency in course selection so that they can be successful?

There is literature that points to the idea of directed self placement (DSP) meaning that students have some autonomy, but also have guidance in the placement process. Tovar (2015) highlighted how crucial it is for counselors and students to develop a good relationship with one another in order to allow counselors to have a positive impact on supporting students to take and pass courses throughout their college career. Based upon this literature it seems it is possible that students can be placed in the correct courses and pass them. By correct courses we mean courses that students are prepared to take and likely to pass. It also seems that key individuals such as counselors play a pivotal role. However, one area that needs further study is understanding how students navigate having more agency and autonomy in selecting courses as a result of state mandate. Additionally, we need more understanding of the role that counselors in particular play in this process to ensure that students are placed in the appropriate course. Our study attends to these questions by focusing on students' experiences with placement and advising during the implementation of AB705.

Theoretical Framework

Self Determination Theory

Self determination theory (SDT) is a framework at the macro level utilized to comprehend personal intrinsic motivation and how social circumstances can influence individual intrinsic tendencies to not only grow, but also develop (Ryan and Deci, 2005). Moreover, social determination theorists have found that when a student is in an educational environment that captures three psychological needs, autonomy, competence and relatedness, student academic achievement is enriched (Kosiewicz and Ngo, 2019). Autonomy refers to the students using their own interest to decide how to take action.

In this study, we will focus on students' autonomy in order to understand how students decide to place themselves into their courses. From a SDT perspective, we expect that students' individual autonomy will make them feel like they were placed into the correct course. On the other hand, BDT suggests that students underestimate their abilities because of factors such as personal experience, demographics, and psychological features (Kosiewicz and Ngo, 2019). As a result, students tend to make suboptimal decisions. Our study intends to explore and highlight student experiences when it comes to placement and advising using both SDT and BDT. The implementation of AB705 removes the high stakes of a placement test in order to properly place students into their respective courses. Furthermore, this removal gives students more autonomy as additional measures such as high school GPA, high school courses and high school grades are taken into consideration when placing students into courses. Since students are aware of these additional measures that are being considered, they realize their current skills and abilities and use their autonomy to ultimately decide which math course they want to take.

Social Capital Theory and "Cooling Off"

Social capital theory is a framework that considers all the resources such as social networks and interpersonal relationships created and maintained by individuals when support is being sought (Bourdieu, 1986). Additionally, Tovar (2015) found that counselor and student communication impact not only how students interact with faculty members, but also their acclimation to social and academic environments in college. This highlights the importance of fruitful interactions and relationships between students and counselors. One may deduce that a lack of developing and nourishing this relationship between students and counselors may be detrimental to a student's overall experience navigating college. Using social capital theory as a framework, we hypothesize that counselors are supportive and helpful in advising students, and predict that as a result of this support, students will have positive experiences with counselors. It is important to note that although counselors are meant to act as agents of the institution who are supposed to provide students with the help, guidance, and support they need to be successful in their academic endeavors, Bahr (2008) emphasizes that counselors are subject to "cooling off" students. Bahr (2008) defines the "cooling off" process as the progressive detachment of a student from his or her personal academic goals. Furthermore, Bahr (2008) explains that this is executed by counselors using factors from students such as level of preparation, skills and abilities in order to lead them down a lesser path of achievement rather than the path students intend to take. In our study, we intend to explore the role counselors have on students and how student experiences with advising affect placement.

Collectively we leverage SDT to understand the ways students exercise their autonomy with respective placement and advising. BDT is used to capture if students are not appropriately placed and/or do not pass their course to understand what mechanism might contribute. Lastly, we use "cooling off" as a lens to describe how counselors place and advise students.

Methods

Context

Focus groups and interviews were conducted at Southern Hispanic Serving Institution (SHSI). When collecting the data, courses taken by students and taught by instructors at this institution were college algebra and precalculus with trigonometry. This HSI has diverse students with 90% racial-ethnic minorities and the Hispanic population making up 70% of the population. Furthermore, the most common demographic for instructors at this institution was female professors, followed by male professors and female assistant professors.

Data

The data used for this study comprised student interviews and student outcomes. Student interviews were done in the format of focus groups. There was a total of 19 students interviewed across gateway math courses. Student interviews focus on students' experiences navigating placement and advising with a focus on counselors. We define counselors as fine non-department university staff who work with students on course selection. The last data used in this paper is student outcome data. Student outcome data contained the courses students took for Fall and Spring and whether the student passed a course. This data included only gateway mathematics courses. It is important to note that we define *passing* as a student who does not need to take the course again. For example, if a student earned a C in College Algebra, then they would not need to retake College Algebra.

Data Analysis

Analysis of the qualitative data was done using MAXQDA software. Interview transcripts were coded using open coding with a focus on advising and placement. In the first round of coding, the first author coded the student transcripts. After coding the first and second author met to discuss first level codes. That is, segments of the data that focused on placement and advising. In the second round of coding the first author developed subcodes to further unpack the first level codes. The first author then met with the second author to confirm the subcodes. Lastly, the first author looked across the coded transcripts for similarities and differences between and among students' transcripts.

Findings

In this section we provide data on students' course trajectory to get a sense of the courses students took and the letter grade they received, which indicates whether they passed. Next, we address the research question: *How do students feel about their placement and advising compared to how they did in the course they were placed into?*

After the passing of AB705, the SHSI created an assessment called Guided Self Placement (GSP) that considers additional measures such as high school transcripts, last math course taken, etc., in order to help students be successful in their respective placed courses. In some cases, if a students' assessment determines that they are going to require additional help to be successful in their math course, the student needs to enroll into a support course for the math course they were placed into. Regarding placement, most of the nineteen students interviewed in focus groups believed they were placed into the correct math course. Additionally, students who felt they were not due to either the difficult transition to online classrooms during a pandemic and/or lack of proper foundation to be successful in their placed course. Despite students feeling that they were placed into the correct course, multiple students described having negative experiences while trying to get placed into the proper course. There were two categories of negative experiences students had with placement, interactions with counselors and counselors not having adequate time. Below I provide student examples to illustrate students' negative experiences with placement.

Counselor Suggestions Through Deficit Orientation

Kali is a student who enrolled into Precalculus with Trigonometry after taking the SHSI's Guided Self Placement (GSP). When consulting a counselor about her math placement, the counselor questioned Kali's ability to be successful in the course by saying "oh a lot of people take Applied Calculus I first because they think it's easier, are you sure you want to take Precalculus with Trigonometry?" This comment shows how counselors partake in the "cooling off" process described by Bahr (2008). Kali stated that the counselor's comment, "...kind of rubbed me the wrong way." Kali's reaction to the comment highlights how although it may seem like counselors have the students' best interest at heart, they may not properly convey that. As a result, a weak relationship is developed between Kali and this counselor, which limits the social capital Kali has access to. Having limited access to social capital can be detrimental to student experience, but regardless of what she was told, Kali felt like she was placed into the correct course. This shows that through her own agency and autonomy, Kali decided Precalculus with Trigonometry was the correct fit for her level of preparation. Lastly, she was able to prove the counselor wrong by receiving a passing grade.

After taking Intermediate Algebra I and II, Saul is a student who decided to reach out to his math instructor to receive guidance on which class to take next. He ended up enrolling in

Precalculus with Trigonometry. When Saul decided to consult with a counselor to get advice, the counselors completely disregarded what Saul said and did not give him advice to help him take the proper steps toward achieving his academic goals. Instead, the counselor advised Saul to change his major, which leaves Saul in a tough position since he knows what major he wants to pursue and decided to attend this SHSI in order to save money. This interaction highlights how counselors accomplish “cooling off” by not listening to students and suggesting them to stray away from their academic goals. Furthermore, it highlights the importance of listening to a students’ wants and needs, and having the counselors receive the proper professional development to ensure those needs are met. Even having a negative experience with advising, Saul used his own agency and autonomy to ultimately enroll in and pass the Precalculus with Trigonometry.

While only two experiences are highlighted in this section, these experiences are powerful because they highlight the counselors' deficit perspective on student trajectories in STEM (e.g., Change of Major). Since only two students were able to personally interact with counselors, this shows the limited availability of counselors. We will say more about limited counselor availability in the next section and how it affects student placement and advising.

Limited Counselor Availability

Nicole is a student who was placed into Analytic Geometry and Calculus II. The path to getting the correct placement was not easy. Nicole had to talk to her high school teacher for guidance because her discrete math class did not transfer credit to the SHSI, but the material covered in the discrete math course should place her in a higher level of math. After speaking to a counselor about her misplacement, Nicole was placed into Analytic Geometry and Calculus II. This particular experience highlights the lack of coordination between high schools and local community colleges. Furthermore, it shows that students rely on former math instructors for guidance on which math course to take when they transition into college, which shows Nicole utilized her autonomy to ultimately decide what course she wanted to be in. Nicole states in her interview that getting counseling is a bit hard. As a result, she relies on the SHSI’s student plan that was made with a counselor in order to help her choose the classes she needs to achieve her academic goals. Although it was difficult to receive advising, through her own agency and autonomy, Nicole enrolled into and received a passing grade in her placed course.

Evie is a student who enrolled into College Algebra with support after filling out the SHSI’s GSP. When the GSP assesses that a student is going to need support to be successful in the course they place into, the student is required to enroll into the support course. Evie took initiative and decided to reach out to counselors to get advice on which courses to take. She states, “Well, I went to the counselor and um, I just, I did-they didn't have time to do a counseling meeting um at the time. So, it was just some lady in the front. She just could see me for five minutes.” Evie not being able to access helpful resources such as counselors shows that Evie’s social capital is limited. Additionally, it shows a need for properly trained counselors for students to be able to receive the proper resources to be successful. Overall, despite not being able to access counselors, Evie utilized her autonomy to ultimately enroll into College Algebra with support and received a passing grade.

Domi is a student who enrolled into Precalculus after using the SHSI’s GSP. When attempting to reach out to counselors about which math course to enroll into, Domi did not receive a response. As a result, Domi attempted to enroll into Calculus, but due to not meeting the prerequisites was unable to. After this experience, Domi reached out to a friend who told him to fill out the GSP. The path Domi had to take in order to be properly placed into his math course is common for students. Additionally, not being able to access counselors during a time of need shows that there are more students needing help than counselors can attend to. Although Domi was unable to seek support from institutional agents (e.g., counselors, faculty/staff), he utilized his autonomy to reach out to others in order to ultimately make a well-informed decision on which course he should enroll in. Additionally, Domi's experience highlights the power of GSP as it helps students place themselves in their math courses. Overall, although Domi had to reach out to resources other than counselors, he still felt like he was placed into the correct course and received a passing grade.

Discussion

The purpose of this study was to investigate students' experiences with placement and advising at a two-year college in the aftermath of passing AB705. In our study, we noticed that fifteen out of the nineteen students passed the gateway mathematics courses they were placed into within a one year time frame. Furthermore, we found that most students felt like they were placed into the correct mathematics course. Although this was the case, students described having negative experiences when interacting with counselors. These experiences were grouped into two categories, Counselor Suggestions Through Deficit Orientation and Limited Counselor Availability. These findings highlight points made in the literature about counselors "cooling off" students, and not developing positive relationships with students, which can limit the access to resources within a students' social capital. An important resource students mentioned being helpful was being a part of the First Year Experience (FYE), which gives students direct access to counselors who can dedicate themselves fully to them. On the other hand, students who did not have a resource such as FYE had a more difficult time accessing counselors. As a result, although not focused on this paper, there seems to be a lack of counselors needed to ensure that students are able to obtain and access the resources they need, which brings attention to limitation of resources available in a students' social capital. Other work might also attend to the ways in which counselors are trained, as several interactions between counselors and students highlighted the deficit lens. It is important to note that this study was conducted during the pandemic, which may have contributed to the experiences students described. However, the concerns raised are very significant while learning to navigate a pandemic.

Limitations

While our study focused on student experiences with placement and advising, our study did not allow us to interview many students. In addition, student focus groups were conducted over Zoom, which may have contributed to why students may have resisted being interviewed as they were just beginning to use the platform. Collecting data over Zoom may not give diverse perspectives. Moreover, we also recognize that when this data was collected students were learning how to navigate the world in a pandemic, which could heavily influence their responses and experiences at the SHSI. As a result, we must acknowledge that the self-reported data we gathered may be subject to bias. These limitations highlight the importance of further research of student experiences with placement and advising during a pandemic.

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Participatory Equity in One Undergraduate Linear Algebra Class

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Inquiry and active learning instructional methods have largely been regarded as equitable and beneficial for students. However, researchers have highlighted math classrooms as racialized and gendered spaces that can negatively impact marginalized students' experiences in such spaces. In this study, I examine the development of one argument, and whose ideas are solicited and leveraged, in an inquiry-oriented linear algebra course with an eye toward participatory equity. I found that gender related most to the inequity of participation in argumentation and that only men participated in generalizing activity. This study adds to the growing literature addressing equity in inquiry and active learning math settings.

Keywords: Inquiry-oriented instruction, Participatory equity, Argumentation, Linear Algebra

Researchers have found active learning to be beneficial for students (Esmonde, 2009; Freeman et al., 2014; Laursen et al., 2014; Rasmussen et al., 2020; Yackel & Cobb, 1996) and link active and cooperative learning environments to higher student achievement and success in STEM courses compared to traditional, lecture-based teaching methods (Freeman et al., 2014; Theobald et al., 2020; Zakaria & Daud, 2010). These student-centered approaches generally involve argumentation; however, argumentation can take many forms depending on who contributes and connects between ideas in the arguments. While many of these approaches have been linked to positive outcomes, researchers have also found whole-class and small-group discussions in mathematics classes to be gendered and racialized spaces, which can create negative mathematical experiences for women and students of color (Boaler, 2008; Ernest et al., 2019; Hand et al., 2012; Jackson & Cobb, 2010; Leyva et al., 2020; Reinholz et al., 2022; Wilson et al., 2019). Some students might have their mathematical methods consistently explored more often or more thoroughly, while others might feel as though their ideas are overlooked. The emphasis on certain ways of participating in mathematical activity involve cultural assumptions and expectations about what is normal and acceptable (Boaler, 2002; Nasir & Cobb, 2007).

Studies examining students' success, achievement, or growth in active learning mathematics classrooms often omit the gendered and racialized experiences that transpire in these classrooms. Much of the research detailing the latter tend to background the mathematics. This study fills a gap in the research by analyzing argumentation in an inquiry-oriented linear algebra course while attending to racialized and gendered interactions. My goal is to illuminate ways in which certain interactions in student-centered classrooms, such as who gets called on or whose ideas get taken up, can lead to racialized and gendered experiences in mathematical practices, such as argumentation. One research question guided this analysis: *In the context of an inquiry-oriented linear algebra class, how are arguments being developed and whose ideas are solicited and leveraged during this development?*

Theoretical Background: Equitable Participation and Argumentation

Equitable Participation

Leyva et al. (2020) theorize the ways in which mathematics classrooms are racialized and gendered by locating classroom events at the intersection of societal discourses about ability in STEM and explicit and implicit institutional logics related to access and authority (e.g., use of certain mathematics courses to prevent students from entering certain majors). With this lens, they highlight the ways in which students experience a range of classroom events that provide different opportunities and support to students in math class that vary systematically by race and gender. Reinholz and Shah (2018) refer to equity related to who is allowed to participate in classroom practices, such as discussions, as participatory equity. Ernest et al., 2019 describe participatory equity as concerning “the fair distribution of both participation and opportunities to participate in core aspects of the learning process” (p. 155). Reinholz and Shah (2018) explain that participating in mathematical discussions is part of the learning process, which can present a variety of settings for inequities. They explain that while it might not be intentional, women and students of color tend to be systematically given easier tasks and fewer opportunities to participate in doing mathematics.

Argumentation

I use Toulmin’s (1958/2003) model of argumentation not only to examine the mathematics being discussed by students and the instructor but also to organize the instructor’s prompts for contributions and responses to those contributions. Toulmin originally used this model to examine “argumentation in the traditional sense of one person convincing an audience of the validity of a claim” (Conner et al., 2014, p. 404). Mathematics education researchers (e.g., Andrews-Larson et al., 2019; Conner et al., 2014; Rasmussen & Stephan, 2014) have shifted the way this model is used, to examine the development of ideas through argumentation. An argument is a series of statements involving a combination of claims, data, warrants, rebuttals, and backings as defined below (Conner et al., 2014; Toulmin, 2003).

- *Claim*: statement to be validated
- *Data*: evidence validating the claim
- *Warrant*: statement connecting data to the claim
- *Rebuttal*: statements asserting when the warrant would not be valid
- *Backing*: “support that give warrants authority” (usually unstated) (Andrews-Larson et al., 2019, p. 4)

Data or warrants can become a *Data/Claims* or *Warrant/Claims*, respectively. For instance, if a claim that once needed validation becomes data to validate another claim, it is a Data/Claim.

Methods

Study Context, Participants, and Data Sources

This study focuses on a single instructional unit (systems of linear equations) in one undergraduate linear algebra class at a public institution in the Southeastern United States. According to institutional data, this university is categorized as a primarily Black institution. The class was taught using a hybrid format, where some students were in-person and some online via Teams. Students could decide each day how they wanted to join the class (in-person or online). The class used OneNote, a platform where the instructor and students could see the tasks, insert text or images, write notes, and see what each group did every class (making sharing group work easier in the hybrid format). The instructor for this course, Dr. Pi (all names are pseudonyms), used inquiry-oriented instruction to teach most, if not all, of the topics. All gender and race data were self-reported by students via survey. The participants were students enrolled in the course who consented to participate, which included four women, eight men, and no students self-identifying in any other gender categories; six students self-identified as Black, two as two or more races, one as Asian, one as Hispanic or Latinx, one as White, and one preferred not to respond. The data sources were video and audio recordings of whole class discussions (across 12 days), records of the chat in Teams, and field notes.

Data Sources and Analysis

Equitable participation analysis. To examine the equitable participation in the arguments described below, I coded students' contributions using Reinholz and Shah's (2018) EQUIP observation codes. The unit of analysis was a *sequence* of talk. A sequence begins when a student initially speaks and ends when another student speaks, thus starting a new sequence of talk. Because of the reliance on another student speaking, sequences vary in length depending on how long the instructor and student are speaking. If two students are going back and forth in conversation, then each time a student speaks starts a new line of code. For this study, I used Author's (2019) modified EQUIP codes. I generally focused on Dr. Pi's Solicitation Method and Evaluation. I counted the number of sequences involved in argumentation and the number of sequences involving contributions from women and from men. I used these numbers to calculate equity ratios (Reinholz & Shah, 2018), which compare the percentage of sequences involving women or men to their percentage in the population. An equity ratio of one means the ratio of a group's contributions in discussions (in this case, argumentation) was proportional to their representation in the class. An equity ratio less than one means a group is underrepresented in argumentation, where an equity ratio greater than one means an overrepresentation.

Argumentation analysis. I identified 12 different arguments taking place throughout the systems unit, which was eight class days long. For a discussion to be identified as an argument, students had to make vocal mathematical contributions. The arguments were large in grain size, where I allow for subarguments in the development of arguments (Conner et al., 2014). I identified the arguments by topic using my knowledge of the tasks and what happened in class. I identified all mathematical whole class discussions in the video data and selected which ones I wanted to further analyze based on observed trends, which will be described later in this section. After transcribing, I did the argumentation analysis first, which involved identifying the different arguments. I then examined the overall flow of ideas, later adding the evidence provided by students and Dr. Pi. I then labelled the different contributions as components in the arguments. In the 12 arguments I identified, I noticed some important trends that were relevant to the research question as they directly related to equitable or inequitable participation in argument development. The trend highlighted in this paper is that no women contributed to any arguments after the second half of the fourth day. This happened to be when the class started moving toward general activity (Gravemeijer, 1999), in which they generalized knowledge they had developed about solution sets so far in the unit. Instead, women only participated in situational activity in which students worked with systems in the context of a meal plan.

Findings

EQUIP Findings

Equity ratios were calculated based on a population of 8 people (due to consent), 2 women and 6 men. There were 98 sequences of talk throughout the unit, 5 (5%) from women and 93 (95%) from men. The 5 sequences involving women were all made within the first 4 (of 8) days of the systems unit, when the task sequence intended for mostly situational activity. Women had 5% of the total sequences but comprised 25% of the total population, as compared to men having 95% of the sequences while comprising 75% of the population (shown in Table 1). This produces an equity ratio of .2 for women and 1.267 for men. Thus, men were participating in argumentation far more than their representation in class, where women were involved in argumentation much less than their representation. Overall, men spoke more in general and, when women did speak, it was earlier in the unit, pointing to more contributions from women during situational activity than generalizing activity. When I examined the solicitation method by gender, I found that women were called on and not called on two times each (50% of the time for each code) (shown in Table 2). Table 2 shows men spoke most of the time without being called on (66.7%), but also spoke a fair amount being called on (29.4%). Sometimes men were called on was because they had previously spoken without being called on and Dr. Pi later asked them to elaborate or made a connection back to their idea. Women were also only solicited to give contributions answering a "what" type question, where men were solicited to give answer what, how, and/or why questions.

Table 1. Sequences and equity ratios by gender.

Gender	% Of Sequences	% Of Population	Equity Ratio
Women (2)	5%	25%	.2
Men (6)	95%	75%	1.267

Table 2. Solicitation method by gender.

Gender	Called On	Not Called on	Called on Volunteer	Called on Group
Women (2)	2 (50%)	2 (50%)	-----	-----
Men (6)	30 (29.4%)	68 (66.6%)	2 (2%)	2 (2%)

I found that the groups by race that participated in arguments almost proportionally to their representation in the class were the Asian student (equity ratio of 1.2 shown in Table 3) and the Black students (.82). The four Black students consisted of two women and two men. One man, Noel, had about 83% of the sequences within this group. He was both called on and not called on about equally as often. It could be argued that the White student had an equity ratio relatively close to 1 (.72). Alberto, who identified as multiracial, contributed considerably more than his representation in the class. He contributed to arguments more than three times his representation in the class. He was often not called on and had some of the longest sequences of talk. Lastly, the Hispanic/Latinx man, Jorge, was involved in argumentation much less than his representation in the class. He was generally called on and often had short sequences of talk.

Table 3. Sequences and equity ratios by race/ethnicity.

Race/Ethnicity	% Of Sequences	% Of Population	Equity Ratio
Black (4)	41%	50%	.82
Asian (1)	15%	12.5%	1.2
Hispanic/Latinx (1)	3%	12.5%	.24

White (1)	9%	12.5%	.72
2 or More Races (1)	32%	12.5%	2.56

Argumentation Findings: Contributions from only men during generalizing activity

The selected argument (shown in Figure 2) was chosen because it highlights and is representative of the type of generalization that occurred in the class, where students were generalizing knowledge about solution sets of systems by drawing on what they learned previously in the meal plans task. It lasted about 16 minutes and took place on the sixth day of the unit. Students were discussing a task in which they match four different systems to images of what their graphs would look like before solving the system or using technology to graph. In particular, students were discussing which graph out of six choices corresponded with System 1, $\begin{cases} x+y+z=210 \\ 5x+7y+10z=1500 \\ x+y+z=500 \end{cases}$. Dr. Pi started the discussion by looking at and reading out each group's choices for which graph matches:

> "*So y'all say that system one, the third equation will be parallel to equation one, but intersects equation two* [Data/Claim 11]. *So, we believe it's graph E* [Claim 12] … *They said for D, E, F* (shown in Figure 1)*, that they have no common points of intersection* [Data 10] ... *But due to the way systems three and four equations were made, they had connection either to D, E, or F."*

All the groups agreed that System 1 has no solution (Data/Claim 1); thus, System 1 corresponds with D, E, or F (Data/Claim 2). Dr. Pi then explained what the same group said about two of the equations, "*we know that* $x+y+z=210$ *and the money equation,* $[5x+7y+10z=1500]$ *do intersect,*" (Rebuttal 3, Data/Claim 11) with which the class agreed. The group also said that "*the scalars of* $x+y+z=210$ *and* $x+y+z=500$ *are the same and would probably lie on the same plane or be parallel to each other*" (Data/Claim 3). This was revoiced by Dr. Pi and other students, notably Noel who pointed out which planes in the graph were parallel.

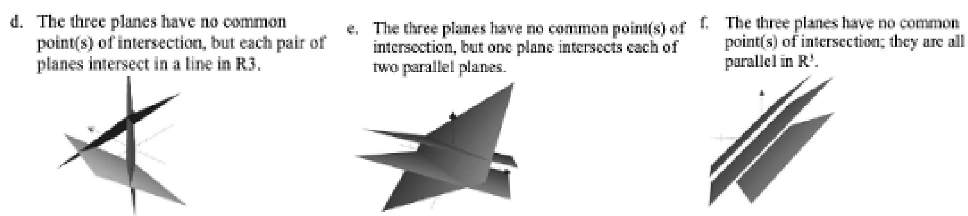

Figure 1. Options D, E, and F as possibilities for the graph of System 1.

This argument was driven by students' ideas. Dr. Pi voiced some of the argument by reading out each group's decision and explanation and drew in students to explain their reasoning or reason with other students' ideas. Dr. Pi did not make many connections during this argument, but rather engaged students in sensemaking about their own and others' decisions and making connections between those. In the end, students articulated what it means for a system to have no solution: there are no intersections between any of the three planes, or there is no common intersection between all three planes. Dr. Pi emphasized the latter as correct, and Aaron concluded E represents System 1.

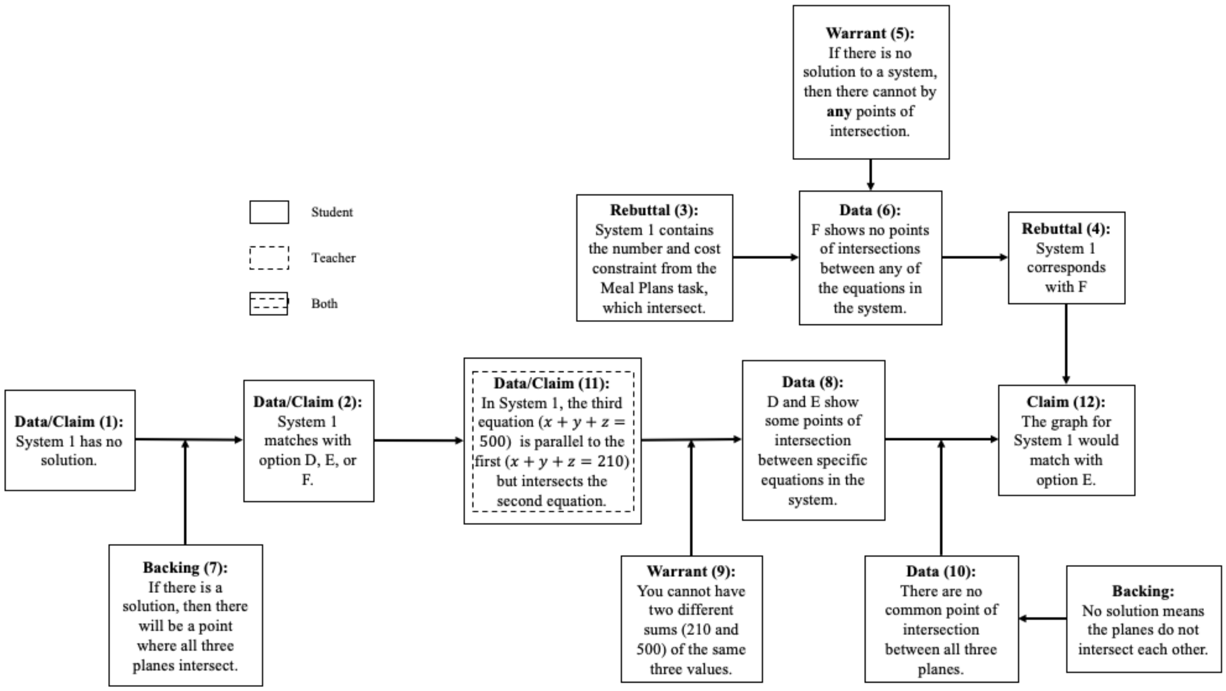

Figure 2. Map of the argument: Contributions from only men during generalizing activity.

Argument with EQUIP. Recall that this argument was selected because it is representative of the generalizing argumentation in which students engaged later in the unit. Also recall that no women verbally contributed to this type of argumentation during this unit, which I noticed during the argumentation analysis but confirmed through the EQUIP analysis. The EQUIP analysis showed that many of the students who did participate in developing this argument spoke without being called on, which could contribute to women not participating because men tend to dominate open discussions (Reinholz et al., 2021; Author, 2019). There was one instance in which Dr. Pi called on a group to see if they wanted to change their answer, which still elicited a response from a man. These findings of more men participating when Not Called On or Called On Group are consistent with findings from Author (2019).

Dr. Pi started the discussion by reading out different groups' ideas, so several argument contributions were provided by students but voiced by Dr. Pi. He started with a group's idea that System 1 has no solution because two of the equations are parallel planes, so System 1 must match with D, E, or F. To further think about this, the class looked at a graph of the system that a group developed to find which two planes were parallel. When discussing the colors of the planes, Alberto said, "*Leave the color identification to the girls.*" Dr. Pi made an uncomfortable noise in an effort to push back a bit on this comment. Alberto added that he said this "*because* [he is] *colorblind.*" This presented a moment when the women in the class could have felt uncomfortable and framed as less capable than the men, or like they do not belong. Alberto was suggesting that men need to make more mathematical contributions while the women can be left to decide colors. There were no women attending in-person in class that day and students joining virtually always had their cameras off, so it was hard to tell women's reactions. This comment created a slight disruption in the class discussion before Dr. Pi continued.

At one point after reading the groups' decisions, Dr. Pi asked if the group from Breakout Room 1 wanted to change their answer. Aaron spoke on behalf of the group and explained that he still thinks System 1 matches F because there should be no intersections when a system has no solution. Alberto interjected, without being called on, that a solution to a system would be somewhere *all three planes* intersect, not just two of them. In the end, Dr. Pi leveraged Alberto's contributions leading Aaron to agree that System 1 matches with E. In this argument, Alberto, Noel, and Aaron had the largest number of talk sequences. Alberto had 33% of the 15 sequences of talk, Aaron had 26.67% of the sequences, and Noel had 20% of the sequences, so they comprised 79.67% of the sequences of talk were contributed by three students in the class. This tended to be the trend throughout the generalizing arguments, where only a few students, always men, contributed most of the arguments. Alberto, who made a comment that suggested women's role in a mathematics classroom is to identify colors on a graph, was ultimately positioned as making the mathematically correct contribution that was publicly accepted by the authority figure, Dr. Pi. Not only were women not part of the development of the argument, but they were also learning mathematics in a context where a fellow student (a man) who made a sexist comment regularly played a central role in the mathematics that was developed.

Discussion

In this paper, I found that the data showed that the most distinct participation inequities in this class related to gender more than race. Women contributed between zero and three times per class, while provided between 4 to 30 sequences. Women's overall representation was substantially less than what would be considered equal, let alone equitable. Women participated in argumentation during situational activity and not formalizing activity. This could be related gender stereotypes in which women are framed as procedural and algorithmic thinkers as opposed to creative and abstract (Leyva, 2017). This could also be related to a larger stereotypical narrative in society (and STEM classrooms) around what women can and cannot do. Furthermore, this could contribute to similar outcomes between women enrolled in inquiry courses and women enrolled in non-inquiry courses (Johnson et al., 2020), as abstraction and generalization are core components of inquiry-oriented instruction. The current study adds to literature highlighting the nuance and complexities of active learning and discussions in mathematics classrooms.

Additionally, Alberto tended to dominate discussions and, because he usually spoke without being called on, there was limited space for other students to contribute. Instructors can sometimes implicitly, and unintentionally, allow students to drive the mathematical discussion, even when the other students provide more mathematically sound arguments (Black, 2004). Alberto also publicly commented during whole class discussion, saying Dr. Pi and Noel should "*leave color identification to the girls*." This was a multi-faceted situation in that Alberto felt comfortable enough in the class to make this comment but there are also implications for how this could have affected women in the class. It was normal for Dr. Pi and students to make jokes during class, but these jokes generally did not reference anyone's gender or race. This comment implied that men's role is to do the mathematics and women can take on a more superficial role (e.g., identify colors). Despite making this comment, Alberto spoke more than most students in the class and had his mathematical reasoning heavily publicly featured in almost every argument. These findings together point to the need for further examinations into the nature of interactions in inquiry and active learning classrooms, even when there appears to be strong argumentation taking place. Exploring these interactions can reveal inequities taking place when using a type of instruction thought to be the most equitable.

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Identifying Tensions in a Small Group Work on an Open-Ended Modelling Task

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We analyze students' cooperative small group work on an open-ended mathematical modelling task. Introduction of new ideas and techniques challenged traditional students' views on mathematics and conventional solution routines. This caused tensions in the student activity system pointing towards primary and secondary contradictions. To identify contradictions, their discursive manifestations were used.

Keywords: mathematical modelling, activity theory, contradictions, discursive manifestations of contradictions

The role of mathematics in life sciences has been continuously growing over past decades. Biology eventually replaced physics as the principal research partner for mathematics, and "after a century's struggle, mathematics has become the language of biology" (Steen, 2005, p. 22). An emergent need for professionals combining the knowledge of biology and mathematics stimulated changes in biology education, "the need for basic mathematical … literacy among biologists has never been greater" (Gross, Brent & Hoy, 2004, p. 85). New demands on the education of future biologists created major challenges for both disciplines calling for the reform of undergraduate biology education which "is burdened by habits from a past where biology was seen as a safe harbour for math-averse science students" (Steen, 2005, p. 14). New educational trends promoted interdisciplinary integration suggesting that "concepts from biology should be integrated within the quantitative courses that life science students take, and quantitative concepts should be emphasized throughout the life science curriculum" (Gross, Brent & Hoy, 2004, p. 86). Higher education institutions were encouraged to create strong interdisciplinary curricula integrating life sciences, mathematics, physics, and information science (Steen, 2005). In applications, mathematical modelling (MM) is often used to describe real-world problems in mathematical terms and analyze the real system gaining a deeper insight allowing to understand the future and take necessary actions. Motivated by the idea that MM can serve as a "didactical vehicle both for developing modelling competency and for enhancing students' conceptual learning of mathematics" (Blomhøj and Kjeldsen 2013, p. 151), in this paper we analyze the work of biology undergraduates on an open-ended MM task.

Design of the Study

An extra-curricular MM project for the first-year undergraduate biology students at a large Scandinavian university was designed to engage students into solution of biologically meaningful tasks. A group of twelve volunteer students (nine female and three male) met for five four-hour sessions where they were introduced to fundamentals of mathematical modelling (both theory and worked out examples) and worked on similar tasks in small groups of three to five students. Only two students had previously taken mathematics courses and ten were enrolled at the same time in a regular first-semester compulsory mathematics course. Our main goal was to demonstrate applications of different mathematical ideas and tools in life sciences and increase students' motivation for learning mathematics. Students' work on MM tasks was as a cooperative learning understood as "students working together in a group small enough that everyone can participate on a collective task that has been clearly assigned … without direct and immediate supervision of the teacher" (Cohen, 1994, p. 3). In the very first session students were introduced to basic ideas of modelling and a model of a modelling cycle illustrated with relevant examples. To get an idea of students' initial preparedness for mathematical modelling tasks, they were asked to work on an open-ended task 'Rabbits on the Road' (Harte, 1988, pp. 211–213).

> Driving across Nevada, you count 97 dead but still easily recognizable jackrabbits on a 200-km stretch of Highway 50. Along the same stretch of highway, 28 vehicles passed you going the opposite way. What is the approximate density of the rabbit population to which the killed ones belonged?

The work of two groups of students was video recorded and transcribed. The data set also includes answers to two self-administered questionnaires on a 5-point Likert scale on students' perception of mathematics and its relevance for biology. For more details about the activity, selection of tasks, and organization of students' work we refer to Rogovchenko (2021).

The research question we address in this paper is: *What contradictions arose during a small group work of biology undergraduates on an open-ended modelling task and how they were manifested*?

Activity Theory and Contradictions

Activity theory (AT) provides a versatile tool for inquiry into various aspects of interactions in a teaching and learning process. The main unit of analysis in AT is the activity system composed of six core elements. A mediational triangle is formed by a subject (an individual or a group of individuals) dealing with an object (a desired outcome) via mediating tools or artefacts (material and conceptual). The object embodies the meaning and purpose of the system, and the analysis of the system is based on point of view of the subject. The activity of the subject is directed towards the object and is transformed into outcomes through the tools. The base of the triangle represents the contextual characteristics of the activity system. It is composed of the community (individuals and groups that share with the subject an interest in the same object), the rules (explicit or implicit) that regulate the actions of the subject towards an object, and the division of labor (horizontal and vertical) stipulating the division of tasks between the community members. The model of the activity system we use in this paper is represented in Figure 1. With the main focus on the student group, the six core elements of the activity system are contextualized as follows. The subject is a student group acting towards the object which is related to desired outcomes (engaging in a MM activity; learning new practices; gaining new skills, solving assigned problems). To achieve the goals, tools (instruments, mediating artefacts) are used including MM tasks; problem-solving strategies; mathematical symbols, concepts, procedures, and results. The object and tools can eventually change impacting in turn the activity. A wider community includes the MM project group, fellow students, family, friends, educational officials, academics influencing the activity system. The system is regulated by the rules -- social-mathematical norms, university regulations, class culture, educational pedagogy, expectations of peers. The division of labor defines the roles and responsibilities of community members along with their expectations of each other's roles.

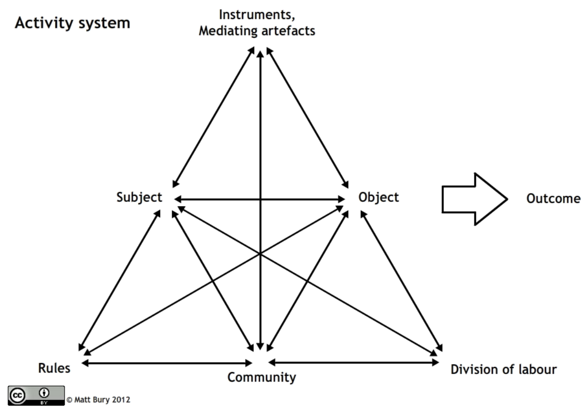

Figure 1. A model of the second-generation activity system. Adapted under CC-by-SA-3.0, Bury (2012).

Contradictions and Their Discursive Manifestations

Each activity system continuously evolves through collective actions; it is not isolated and interacts with other activity systems. In activity systems, "equilibrium is an exception, and tensions, disturbances, and local innovations are the rule and the engine of change" (Cole & Engeström, 1993, p. 8). External influences are appropriated by the system and modified into internal factors which impact the system's development. The third generation of AT identifies contradictions as "historically accumulating structural tensions within and between activity systems" emphasizing "the central role of contradictions within an activity system as sources of change and development" (Engeström, 2001, p. 137). Contradictions should not be perceived as system's deficiencies but rather as opportunities for transformations that change the activities. Implementation of new teaching approaches, ideas and tools disturbs activity systems and creates tensions which manifest contradictions in the system that should be resolved to improve the quality of education. The process of using contradictions to promote learning and change is referred to as expansive learning and should be understood as "construction and resolution of successively evolving contradictions" (Engeström & Sannino, 2010, p.7). Our recent research discusses tensions arising in the modeling project analyzing two activity systems for students and for the project team (Rogovchenko, 2023).

Contradictions are the systemic phenomena which develop in time; they cannot be identified directly. Contradictions should be "approached through their manifestations" (Engeström & Sannino, 2018, p. 49), and we use the term 'tensions' for the latter. Tensions signal internal contradictions in the activity system both within its core elements (primary contradictions) and between these elements (secondary contradictions). In a network of interacting systems tensions may point towards tertiary and quaternary contradictions (Engeström, 1987) which are out of the scope of this paper. Engeström and Sannino (2011) introduced a methodological framework for the identification and analysis of discursive manifestations of contradictions distinguishing four main types described in Table 1. We use this tool to analyze the transcripts of students' small group work on a MM task.

Table 1. Types of discursive manifestations of contradictions (adapted from Engeström & Sannino, 2011, p. 375).

Manifestation	Features	Linguistic cues

Dilemma	Expression or exchange incompatible evaluations	“On the one hand […] on the other hand”, “yes, but”
Conflict	Arguing, criticizing	“no”, “I disagree”, “this is not true”
Critical conflict	Facing contradictory motives in social interaction, feeling violated or guilty	Personal, emotional, moral accounts, narrative structure, vivid metaphors
Double bind	Facing pressing and equally unacceptable alternatives in an activity system	“we”, “us”, “we must”, “we have to”, pressing rhetorical questions, expressions of helplessness

Methodology

Our research focus is mainly on the student group whose traditional views on mathematics and conventional solution routines were challenged in the project when new ideas and techniques of MM were introduced. It was expected that tensions in the activity system would arise, in particular, in relation to student engagement, understanding of a modelling task, comprehension of the mathematical content, and approaches to solution of a modelling task. Since tensions in the system were anticipated, deductive thematic analysis of the data was conducted to identify the presence of contradictions in our activity system. The list of tensions-related themes was generated, and the search for discursive manifestations of contradictions was made by the authors based on the descriptions and linguistic cues in Table 1. Full session transcripts were analyzed for two groups working on the task, the transcripts of a 42-minutes long recordings with 5,076 words and 7,874 words respectively. The word count includes speaker labels (Student 2 or Teacher), time stamps (01:00 or 01:26-01:38), and short comments of transcript writers (“inaudible” or “students work silently”). The initial screening of full transcripts was made, and eight episodes were selected for the analysis in this paper. Separate coding was used for identifying four types of discursive manifestations of contradictions (as in Table 1) and four types of tensions related to (i) student engagement, (ii) understanding of a modelling task, (iii) comprehension of the mathematical content, and (iv) approaches to solution of a modelling task. Tensions that were identified were in turn coded in relation to core elements of the student activity system and associated primary or secondary contradictions. Both authors independently coded each of the eight episodes; the count of discursive manifestations and tensions is shown in Tables 2 and 3.

Table 2. Discursive manifestations of contradictions identified by the authors.

Manifestation	First author	Second author
Dilemma (D)	3	4
Conflict (C)	4	5
Critical conflict (CC)	6	6
Double bind (DB)	7	6

Table 3. Tensions identified by the authors.

Tension	First author	Second author
Engagement (E)	2	2

Understanding of the task (U)	8	9
Comprehension of the mathematical content (MC)	2	2
Approaches to solution (A)	6	7

The disputed issues were reviewed again, the authors discussed the codes applied in the episodes and negotiated common meanings coming to the agreement.

Analysis

Eight episodes were selected from the transcripts to illustrate tensions identified through discursive manifestations of contradictions. For each episode we provide the codes used in the analysis of students' work.

Episode 1

Student 1: Eh, and then we have to assume how many cars are running over a rabbit then.
Student 3: Oh, God…
Student 2: Not everyone runs over a rabbit!
Student 1: No, it is not so! And that is not even a half either, it is maybe five percent. (C, A)
Student 2: Right.
Student 1: Five percent is one out of 20.
Student 3: But one out of 20 sounds like quite a lot of rabbits then. (D, A)
Student 1: Yes.
Student 3: So, one out of 20 is not so wrong. Take five percent then.
Student 1: Five percent.
Student 3: We just have to put… we just have to put something like this. (DB, A)
Student 1: But no! We have 97 dead [rabbits]. But if we assumed that they died in the last 24 hours, then we divide it by the number of cars. Then we find that percentage. (DB, A).

Episode 2

Student 5: Can't we just divide the number of individuals in the area and find out how many are there in each square meter?
Student 1: But we don't know what the area is, we only know the stretch. We don't know how wide it is. (DB, U)

Episode 3

Student 3: No, but let's just assume, as she says, that we have 20 miles that way and 20 miles that way, then we have 20 miles on the stretch.
Student 1: Yes.
Student 3: But then there is still a very large area, for only 20,000 rabbits. (D, U)
Student 1: Yes, I think, I don't know if this is what they mean by density.]So, density.
Student 3:]No.
Student 1: There is no point in calculating density per hypothetical square kilometer. (C, A)
Student 2: You never know how big it is as well…
Student 1: So, I do not know if that's what they mean by density, or if it is just a part of the population they're looking for. It is like 'What is the approximate density of the rabbit population?'

Student 3: To obtain density, we have to have the whole area; we can't use just a stretch. You must have something like a square. (DB, A)

Episode 4

Student 4: Yes, you have to say how many remain alive, that is something.
Student 2: Yeah. Not all [rabbits] who jump over there die.
Student 2: No, and there were only 28 cars, and you don't know for how long the bodies have been laying there. (C, U)
Student 4: No, that's exactly it. ‘Easy recognizable,’ then it's like that, then it's like they have just been ran over, so you can see what they are. (C, U)

Episode 5

Student 4: No, you find kilometers per rabbit. Ninety-seven, what did you get?
Student 3: Hm?
Student 4: Ninety-seven divided by two hundred
Student 3: That's half a rabbit per kilometer
Student 4: Yes, approximately one half
Student 2: So, half a rabbit dies per kilometer
Student 4: Yes, per kilometer
Student 3: Every two kilometers there is a dead rabbit.
Student 4: Yeah, but that’s hell is not right, it's not logical at all! (CC, U)

Episode 6

Student 3: Since he] drives two hundred kilometers an hour
Student 2: What if we just round it [velocity] up to a hundred?
Student 4: Sure.
Student 3: It makes it much easier to round up to a hundred.
Student 4: Yes, but we have, we have this bizarre thing, so it will be the same anyway. (CC, U)
Student 2: Hmmm, this was a frustrating task. You think it's a math problem, but then you know nothing. So, you can't solve it. (CC, U)
Student 4: Yes, we have to, we have to [solve it] somehow. Find our own numbers. (DB, U).

Episode 7

Student 1: Won't it be too many, because it gets a bit more complicated when you start pulling so many different factors into
Student 4: Yeah, but it is only getting … The answer will just be better and better (D, U)
Student 1: That's about it.
Student 4: Then there will be fewer factors that will screw us up. (CC, U)
Student 2: Just assume something about all the factors in the end!
Student 1: I wonder if it is possible to solve this task here. (CC, U)
Student 4: Sure, it is.

Episode 8

Student 4: I'm really just looking forward to getting an answer, because this was really annoying! (CC, U)

Student 2: Hmm, but I think I will be just as annoyed then, because there is no correct answer. But it will be fun to hear what can be assumed. (C, U)

Conclusions

Activity theory acknowledges that contradictions can potentially transform the activity system, but this may not necessarily happen. Contradictions are present in each activity system; they may originate from multiple perspectives, cultural and historical traditions, different interests of members of an activity system. Experiencing problems, conflicts, disturbances that originate from the contradictions in the activity system, individuals or groups of individuals attempt to change the system to alleviate tensions. Nevertheless, resolution of contradictions by individual actions alone is not possible, significant changes in the activity system require cooperative actions leading to the development of a historically new form of activity. Ultimately, "an expansive transformation is accomplished when the object and motive of the activity are reconceptualized to embrace a radically wider horizon of possibilities than in the previous mode of the activity" (Engeström, 2001, p. 137).

In our MM project, primary contradictions were observed within the subject (engaged learners vs passive participants), tools (new methods of population dynamics vs student's traditional mathematics toolkit), rules (student-oriented learning in MM sessions vs teacher-centered pedagogy in regular classes), object (scientifically based understanding of phenomena vs intuitive interpretation of the reality), division of labor (lack of scaffolding vs increased volume of the independent student work). Secondary contradictions were manifested between the rules and object (students' engagement with extra-curriculum modelling tasks vs the need to perform well in the exam; instructor's wish to infuse teaching innovations in students' learning vs curriculum and time constraints), between the tools and division of labor (new thinking required by MM tasks vs learners' expectations how the learning of mathematics should be organized), and between the community and object (creating numerate mathematicians and students' conventional perceptions of mathematics and their previous bad experience with it).

Although the work on an open-ended modeling task was challenging for students, they mentioned the work with assumptions as the best thing in the project: "It made me think in a different way than usual and it was exciting. Making assumptions was new to me." "The assumptions. To understand what assumption is important and which is not." We observed that students interest in mathematics was gradually increasing, and they were opening up for new ways of learning the subject. Students' answers prompted that the use of MM in teaching biology undergraduates stimulates their interest in mathematics and its interdisciplinary applications: "It was interesting to try to solve problems without knowing all about them." "It was social and challenging; it was also very interesting to experience new educational methods." Students' positive feedback on the project indicates possibilities for the expansive learning in the activity system which can transform students' learning of mathematics by enriching the standard curriculum with biologically meaningful mathematical tasks.

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A Student's Application of Function Composition when Solving Unfamiliar Problems: The Case of Zander

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Function composition is highly useful for making sense of many fundamental ideas in calculus and related areas of mathematics. However, of the large body of research investigating students' understandings of function, very little has been centrally focused on function composition. In an effort to contribute to the existing findings pertaining to function composition, this interview-based study involved an exploration of a first-semester calculus student's use of function composition when solving unfamiliar problems. The student's responses revealed potential evidence of his understanding of function composition, function as an action or process, and covariational reasoning.

Keywords: Function Composition, Function, Covariational Reasoning

Introduction and Literature Review

Functions serve a fundamental role for modeling dependence relationships between variables in every area of mathematics and physical science. Thus, it is not surprising that the idea of function is heavily emphasized in mathematics curricula, and has been heavily studied in mathematics education research (early examples include Breidenbach et al., 1992; Carlson, 1998; Vinner & Dreyfus, 1989). Function composition has also been mentioned as an area of importance, but often briefly or in passing. To date, few studies in mathematics education related to the idea of function have had function composition as their central focus (a few exceptions include Bowling, 2014; Engelke et al., 2005; and Kimani, 2008). However, applications of function composition can be found in many important ideas studied in calculus, including the limit definition of derivative, related rates problems (Engelke, 2007), and the chain rule (Clark et al., 1997; Cotrill, 1999; Engelke, 2007). Thus, function composition is an area worth studying further, particularly in the context of students' experiences prior to and during calculus.

Many of the existing studies pertaining to function composition have been based on the primary theoretical frameworks used to explain students' understandings of function more generally. Among the most common are Breidenbach et al.'s (1992) characterization of students' perspective of function in terms of specific actions or procedures to compute a result, or an "action" view of function, versus a general process that associates input with output values, or a "process" view of function (empirical studies using this framework have included Ayers et al., 1988; Bowling, 2014; and Engelke et al., 2005). Other frameworks used in function composition studies have included Carlson et al.'s (2002) framework for covariational reasoning (Bowling, 2014), a framework to study the impact of representational context (e.g., symbolic, tabular, graphical, word problem) on a student's conception of functions in a problem (Kimani, 2008), and Sfard's (1992) framework for function in terms of operations or in-progress activities, referred to as an "operational" view of function, or in terms of a single object that could be operated upon, referred to as a "structural" view of function (Kimani, 2008).

Several findings from existing studies pertaining to function composition have been focused on students' reasoning relative to the above frameworks. For example, studies employing the action and process framework have found that a process view of function can play an important role in student success with unfamiliar function composition problems, which the students would not be able to rely on a memorized procedure to solve (Ayers et al., 1988; Bowling, 2014; Engelke et al., 2005). Across multiple studies, another common finding is that students are often more comfortable solving function composition problems with functions defined symbolically than problems with graphical, tabular, or applied (word problem) contexts (Engelke et al., 2005; Kimani, 2008), which could be linked to the symbolic substitution approach to function composition commonly presented in high school curricula (e.g., Bowling, 2014). These studies have provided insight into how students' understandings of function might inform their approaches to solving function composition problems.

In addition to the above findings based on existing frameworks, a few studies have resulted in findings related to students' reasoning specific to function composition. One such finding was that, when solving related rates problems involving variables which the students found difficult to relate directly, students found a "middle variable" to be helpful for identifying familiar functions that could be composed to relate the primary variables of interest (Engelke, 2007). This finding suggests a possible extension of existing function frameworks to account for reasoning involved in function composition. In an effort to continue extending existing frameworks, this study was intended to address the question: In what ways does a calculus student apply their understanding of function composition, variable, and function to solve unfamiliar problems?

Theoretical Perspective

The analysis for this study involved a characterization of the types of functions and relationships between values a student might construct. For this work, two main frameworks for explaining students' understanding of function were leveraged: action and process view of function (Breidenbach et al., 1992) and covariational reasoning (Carlson et al., 2002; Castillo-Garsow et al., 2013). Each of these frameworks are described in more detail below.

Action and Process View of Function

Someone using an action view of function perceives a function as a specific set of computations or other describable, repeatable activities to get a particular result from given information. With an action view, a function is the method by which a result can be related to the given information (often both the result and given information are numerical values). For instance, someone with an action view of function might associate $f(x) = 3x + 2$ with the series of computations: 'multiply x by 3 then add 2'. On the other hand, someone using a process view of function perceives a function as an association between values which could be specified in different ways and is not tied to a specific rule. With a process view of function, someone might associate $f(x) = 3x + 2$ with an assignment of every real number (each of which would be a value of x) to a number that is 2 units greater than the number 3 times its value. With a process view, someone might imagine the rule of assignment according to the equation above, a graph, a computer program, a verbal description, an applied context, or a variety of other ways.

Covariational Reasoning

A commonly adopted perspective of covariational reasoning for modeling students' understandings of function is that of Carlson et al. (2002). Re-framed to account for the idea of function as a dependence relationship, Carlson et al. (2002) defined covariational reasoning as imagining a situation with multiple variables that change together such that the value of one variable depends on the value of the other. Underlying the idea of variable according to this perspective is an idea of quantity as conceiving of a feature of an object which one can imagine measuring (e.g., Thompson, 2011). The situation in which one might imagine a quantity could be a tangible, real-world situation, or it could be more abstract. For example, someone could quantify their distance from a wall in a number of feet, or they could quantify the x-coordinate of a point in the Cartesian plane as the horizontal distance of that point from the origin, measured in some specified unit used to scale the graph.

Carlson et al. (2002) characterized students' covariational reasoning into five stages of mental activity (MA1 – MA5). A description of the first three (used in this study), framed in terms of constructing a dependence relationship between variables is as follows: (a) MA1: identifying two varying quantities and noting that the value of one depends on the value of the other, (b) MA2: a qualitative description of the direction of change in one variable with respect to the value of the other; and (c) MA3: a description of an amount of change in one variable with respect to the value of the other. Castillo-Garsow et al. (2013) proposed two additional forms of covariational reasoning: chunky continuous covariation, through which someone imagines the variables' values changing in completed chunks or intervals, and smooth continuous covariation, through which someone imagines the variables' values changing gradually, smoothly, or through each moment in time. In addition to the forms of covariation above, in some situations, someone might conceive of a fixed quantity, or, in an algebraic context, associate a symbol with a fixed value (which they may or may not imagine as a quantity).

Methods

Data Collection and Task Design

The researcher (author of this paper) conducted a semi-structured interview with a first-semester calculus student, Zander, attending a large, public university in the Southwestern United States. Zander's calculus course involved a research-based curriculum, which could have impacted his understanding of function, covariation, and function composition differently from traditional calculus curricula. While solving the interview tasks, Zander was asked to think aloud. As he described his thinking and solution approaches, the researcher sometimes asked follow-up questions to further inform hypotheses about his thinking. Scanned images of any written work from the interview were also collected for further evidence of Zander's thinking.

The interview tasks were designed to be unfamiliar to most students, so that a solution would probably not be immediately available from experiences with similar problems. The researcher anticipated that a student's reasoning about the task context and solution attempts would give rise to some form of function composition, while also providing opportunities to generate hypotheses about their understanding of function and types of covariational reasoning.

Data Analysis

The author intended to leverage existing frameworks (as described in the theoretical perspective) and generate new explanatory models for students' applications of function composition in solving the tasks. In alignment with this goal, the author analyzed the interview using a process similar to that described by Simon (2019). Analysis consisted of three primary phases: (a) generating hypotheses about a student's thinking at specific moments in the interview; (b) generating hypotheses about a student's thinking over the course of the interview; and (c) modeling the student's understanding of function in the context of the interview.

Results

To illustrate the multiple types of function composition Zander appeared to use during the interview, data will be reported from two tasks in contrasting representational contexts.

The Equation Task

Let x be a variable that takes on real numbers, and let f be a function defined by $f(x) = x^2$. Suppose that y is a variable related to x through the relationship $y = 5x$. If y increases from -20 to 20, what will happen to the value of $f(x)$?

Figure 1. The Equation Task Prompt

The equation task was designed with the hypothesis that most students will imagine the equation $y = 5x$ as a way to directly relate a value of x to a value of y and the equation $f(x) = x^2$ as a way to directly relate a value of x to a value of $f(x)$, though the nature of the relationships students construct could vary widely. The prompt provided numerical information with which to associate y and requested that the student state a conclusion about $f(x)$. No equation or other function definition was provided to relate a value of $f(x)$ with a value of y, posing a potential challenge for many students in finding a relationship between y and $f(x)$.

Zander's Response to the Equation Task

Zander first responded to the equation task by stating an equation of the form $x =$ [expression containing y], but shortly afterward, indicated that the equation ultimately would not matter and gave a more general description of how he imagined the value of $f(x)$ might change with the value of y.

Zander: so x is equal to y divided by five I think, and—but y is from minus twenty (inaudible) but it does not really matter at the end because, uh, x squared, because it's, uh, x squared, so I think the value of y, I mean the value of f x will increase, when you go from minus twenty to twenty for y.

Zander's initial response suggested some details about his goal for solving the problem he appeared to perceive in the task and the way in which he might describe two variables he imagined, y and $f(x)$, changing simultaneously. However, there was almost no evidence as to why Zander had initially stated that "x is equal to y divided by five". Upon probing for further evidence, Zander's response revealed that writing an expression for x in terms of y was most likely part of a strategy that would allow him to compute—or imagine computing—a specific value of $f(x)$ from a specific value of y.

Zander: So, like, the…I think (inaudible) a function we're supposed to plug in the value of x, and then we're supposed to get another answer which is like our value of y or it could be something else too. So, in this we're like given two things. First of all, it's a function which is f x equals to x squared, so if we have a value of x, we just plug it in for, I'm talking about f x equals to x squared, and then we'll get our y, right? So if we plug in one, it'd be one. Plug in two, it'd be four. So, but (inaudible) to figure out what our y will be, uh, I mean our x will be, we are given another equation, which is that y equal to five x. So, I think from this equation, y is equal to five x, we're supposed to figure out what our x will be, and then we can plug in that x into f x equals to x squared and then we can figure out uh, another, like the, another function (inaudible) y for that one.

From Zander's statement at the beginning of the above excerpt, he appeared to be imagining a situation in which he would consider a specific value of x, perform a calculation, and obtain a specific value of y resulting from that calculation. It is not completely clear from Zander's overall response how he imagined the relationship between x and y to yield the equation $x = y/5$; however, given what appears to be a computational use for the equation with specific values of x and y, it is plausible that Zander imagined manipulating the equation using a "solving" or "switching" procedure. More evidence would be needed to confirm this. Zander also mentioned an overarching strategy for figuring out a value of $f(x)$ from a value of y that involved two separate calculations: a calculation that would give him a value of $f(x)$ from a value of x, and a calculation that would give him a value of x from a value of y.

The Cylinder Task

A chemist pours a liquid into two empty, differently-sized cylinders. The smaller cylinder has a radius of 2 cm, and the larger cylinder has a radius of 3 cm.

(a) If the chemist pours the same volume of liquid in both cylinders, and the height of liquid in the smaller cylinder is 11 cm, what is the height of liquid in the larger cylinder?

(b) If the chemist pours the same volume of liquid in both cylinders, as the height of liquid in the smaller cylinder increases from 0 to 11 cm, how does the height of liquid in the larger cylinder change?

Figure 2. The Cylinder Task Prompt

The cylinder task is loosely based on a problem from the Precalculus: Pathways to Calculus workbook (Carlson et al., 2020). This task was designed in anticipation that if a student wanted to relate the height of liquid in the smaller cylinder and the larger cylinder numerically, they would need some intermediate value (which they could imagine as fixed or varying) they could individually relate to both heights. Then, they could use the intermediate value to compose the relationships, forming a relationship between the height of liquid in each cylinder. One anticipated intermediate value would be the volume of water in each cylinder, since the task context specifies that the volume would be fixed, and it might occur to many students to leverage the formula for the volume in a cylinder with respect to radius of the base and height: $V = \pi r^2 h$ (this formula was not provided in the task context, but was given to a student upon request).

Zander's Response to the Cylinder Task

Zander appeared to engage in two primary forms of function composition in his response to the cylinder task, both of which involved a relationship between the volume of liquid in each cylinder and the height of liquid in each cylinder. The first composition involved computing a single value for the volume of liquid in the smaller cylinder, substituting the result into the formula for volume in a cylinder with respect to radius of the base and height of liquid in the cylinder, and solving for the height of liquid in the larger cylinder. Zander's computation of a single value for the volume in each cylinder and solving for the corresponding height in the larger cylinder enabled him to achieve the goal of getting a numerical answer. The type of function composition he appeared to use in this instance could be regarded as grounded in an action view of function (Breidenbach et al., 1992) and a fixed-value understanding of the volume and height of liquid in each cylinder. However, his description of the relationship between the height in each cylinder shortly before this series of calculations suggests that he imagined the height and volume as variables with which he could make a general, qualitative comparison. In fact, his first inclination for finding the height of liquid in the larger cylinder was to use the relationship between the radii of the cylinders to form a mathematical analogy with which to compare the heights.

Zander: Um so first of all, I'd say that the height of the liquid in the larger cylinder is going to be less than what it is for the smaller one, because the radius is less, which means that for the first one, it is like kind of small, so it will fill up more. And the second one is a larger cylinder, it has a radius of three, so basically its base is bigger, so it could fill in the same volume but in like less height, that is what I'm trying to say here.

Following his description, Zander tried to use a proportion to express the comparison mathematically and ultimately calculate a value for the height in the larger cylinder, but he was not confident that the equation he wrote would result in the correct calculation.

Zander: I'd say, when our radius was two, our height was eleven, and then that'd be, is equals to when our radius is three, what is going to be our height. I'm not really sure if this is like in a ratio or not, but it should be because after all it's like a cylinder, it's just different size. The volume is going to be the same, the amount of liquid poured is the same so, this should be like in a ratio with each other.

When asked why he decided to set up a proportion, Zander's response suggested he was trying to guess what equation he could set up to calculate the desired value: the height in the larger cylinder. He noted that there was little information given in the problem overall that he could use to achieve this goal. Thus, the interviewer asked him what other information would be useful for him to know, and after only a short pause, he stated that it would be useful to know the formula for the volume in the cylinder.

Interviewer: Um, what information would be useful for you to have, that would allow you to figure out the height of water in the larger cylinder?

Zander: (inaudible) Radius. Uh, I think for this, we need to know the formula for the volume of the cylinder. If we do know the formula of the volume, I don't remember it that well, maybe it is—I don't know, I don't remember.

Once the interviewer gave him the formula for the volume in the cylinder, he made a statement that suggested he imagined the formula as a way to express a dependence relationship between the enclosed volume, height, and radius of a cylinder.

Zander: Yeah, so our pi is always constant, so if you're making comparison, it does not really matter. We are only dependent on two things for the volume. The first one is our radius, and the height.

Following his use of the volume formula to calculate the height of liquid in the larger cylinder, Zander rejected the proportion he initially had set up, concluding that it would not be possible to have a greater height of liquid in the larger cylinder than in the smaller cylinder (he had calculated a greater height from his initial proportion, whereas the volume formula produced a smaller value for the height in the larger cylinder).

The second form of function composition Zander used emerged when prompted to describe how the height of liquid would vary in the larger cylinder. Zander described a comparison of two graphs he imagined: one for the height of liquid in the smaller cylinder with respect to volume, and the other for the height of liquid in the larger cylinder with respect to volume.

Zander: Our x is going to be the volume and our y is going to be the height…so if the volume is increasing for the smaller one, our height is also going to, height is going to increase in both of these, so we do know that the graph is going to be like linear in this, it'd be increasing, but at a different rate for the smaller one, it would increase at like a really high rate so our slope is going to be really high; however for the larger one it's not.

Following his description of the graphs and a discussion about how he conceptualized the "rates" he was comparing, Zander described what first appeared to be evidence of a smooth continuous conception (Castillo-Garsow et al., 2013) of the covariation of each height with respect to volume. After further discussion, the researcher concluded that Zander was most likely employing a chunky continuous conception of covariation, since he stated that the slope of the graph should be the same over any size of interval for the volume.

Zander: So like we can, in this equation, we can find out our, like, height at every single moment as volume changes, so as of here, our volume was one thirty eight point one six, and we figured out that our height would be four point eight, so we can increase that volume to one forty and then we can figure out our height. This is what we can do. Then we could like have a slope at this point, and I hope that this slope is the same for the, every single interval in there, otherwise it won't be linear.

Conclusion

In response to both tasks, Zander frequently alternated between computing specific values and using algebraic manipulation procedures, and making general descriptive comparisons between what he seemed to imagine as variables. Zander appeared to imagine the functions as a means of relating the direction of change in two variables (Carlson et al., 2002). Thus, it appears that Zander leveraged a process view of function (Breidenbach et al., 1992). However, whether he used an action or process view of function in particular parts of his solution depended on his goal at that point in his thinking process. When his goal was to calculate a specific value or define an algebraic equation, he described a series of computations between individual values, suggesting he employed an action view of function in these situations.

In addition, Zander appeared to engage in several types of function composition. One form of function composition involved a series of computations and substitutions to compute one specific value given another (i.e., computing the height of liquid in the larger cylinder from a known height of liquid in the smaller cylinder). A second form of function composition involved algebraic manipulations to link together two equations, followed by a two-step sequence of computations to relate several numerical values, one-at-a-time, input to the first equation to corresponding numerical values output by the second equation (i.e., using the equation $x = y/5$ and subsequently the equation $f(x) = x^2$ to compute values of $f(x)$ from values of y). Finally, a third form of function composition involved a sequence of covariational relationships through which he appeared to imagine a chunky continuous, or possibly even smooth continuous variable (Castillo-Garsow et al., 2013) in relation to another by comparing his visualization of two graphs (i.e. comparing volume-height graphs he constructed in response to the cylinder problem).

Zander provides an example of how one student can use several different forms of function composition, covariation, and function to solve unfamiliar problems. His case suggests that the type of function composition that might be deemed applicable in a problem situation depends on the experiences with covariation and function someone associates with a problem's representational context and the goal someone has for a solution at particular points in the solution process.

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Magnitude Bars and Covariational Reasoning

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In this report, I examine the recommendations made by researchers to use magnitude bars (or dynagraphs) with students to support their reasoning about quantities. More specifically, I look at implications of an undergraduate pre-service secondary mathematics teacher's reasoning when using tasks with magnitude bars designed to support students' meanings for geometric formulas in dynamic contexts. I illustrate the ways in which a student reasoned, ways that both resonated with researchers' recommendations but also introduced unanticipated difficulties that resulted from the introduction of the magnitude bars. Specifically, I show two different ways of comparing magnitudes a student illustrated in the Painter Problem-using amounts of change reasoning and length comparison reasoning. The results of this study contribute to the literature on using magnitude bars to support students' reasoning by illustrating the different ways in which a student can reason with magnitudes.

Keywords: Pre-Service Teachers, Covariational Reasoning, Dynamic Geometry, Representations

In RUME 2017, Stevens, Paoletti, Moore, Liang, and Hardison (2017) provided principles for designing tasks that promote covariational reasoning—in which students reason about two quantities changing in tandem (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). Among their considerations to use quantities that are not monotonic, to use shapes strategically, and to use different representational systems, the authors suggest to "use varying segment magnitudes to represent a quantity's magnitude in a situation" (p. 933). In this report, I first (i) review how and why these researchers and others have argued for using dynamic magnitudes to support students' reasoning (ii) describe the methods of a study with pre-service secondary mathematics teachers (hereafter, PSTs) that considered the aforementioned design principles from Stevens et al. (2017), (iii) describe the results of the study by illustrating one PST's reasoning, and (iv) discuss how analysis of the larger study included insights into the affordances and limitations of using varying segment magnitudes that do not have any numerical values or units labeled (hereafter, magnitude bars). Specifically, the results section includes how magnitude bars were able to support students' covariational reasoning but also provided additional sources of confusion about what quantities were being represented. In sum, this report is of a study that aims to answer the question, "What role does the use of magnitude bars play in students' reasoning about relationships between quantities in dynamic geometric contexts?".

The Cases for Varying Magnitudes

This section is a summary of researchers' justifications for using dynamic magnitudes to support students' mathematical reasoning. It includes recommendations from researchers who consider the role of technology in reasoning and are interested in students' cognition.

Varying bars as a way for students to construct productive meanings for graphs

As mentioned in the introduction, Stevens et al. (2017) proposed the following task design principle for designing tasks that promote covariational reasoning—using varying segment magnitudes to represent a quantity's magnitude in a situation. Specifically, the authors stated that using the magnitude bars instead of prompting for a graph immediately offers researchers "insights into students' reasoning while minimizing the influence of the ways of thinking they have developed for graphs (e.g., iconic translations, issues of function/dependency, ways of thinking based on figurative thought)" (p. 933). They also described how the magnitude bars served to help scaffold students into constructing graphs by placing the magnitude bars orthogonally and constructing a multiplicative object (i.e., a point on the graph in this case) and then an emergent trace of all the instantiated pairs (see also a similar use of dynagraphs from Antonini, Baccaglini-Frank, & Lisarelli (2020). This emphasis on using different representations to support productive meanings for graphs relates to the task design principle of Marton's Variation Theory (Kullberg et al., 2017; Marton, 2015) that Johnson (2022) described, though she recommended specifically using different forms of the same type of graph to support students' meanings for rate (building of literature such as Johnson (2012)). Liang & Moore (2021) extend the idea of partitioning magnitudes and its role in students' covariational reasoning. They noted how students' partitioning activities can be rooted in figurative thought by illustrating cases in which students struggled to relate partitions constructed in situations with partitions constructed in graphs intended to represent relationships between quantities from those situations. Thus, collectively, these authors have concluded that magnitude bars can be used to support productive meanings for graphical representations via covariational reasoning but also note the importance of attending to how partitions of magnitudes relate across representation systems.

Varying bars as having semiotic potential for function

Hollebrands, McCulloch, & Okumus (2021) considered the theory of semiotic mediation to discuss the design idea that a tool (such as the dragging tool in dynamic geometry tasks) has semiotic potential that a teacher can exploit to "guide students to produce *mathematical signs* (definition, proof, mathematical conclusion, generalization, etc.). Here semiotic potential of an artifact relates to the "mathematical meanings related to the artifact and its use" (Bartolini Bussi & Mariotti, 2008, p. 754). For example, they described how dynagraphs elicited students' understandings of the notion of function, referencing, the work of Antonini, Baccaglini-Frank, & Lisarelli (2020). *Dynagraphs* are like magnitude bars where students drag the bar representing the independent variable and can observe the effects of the bar representing the dependent variable, but unlike magnitude bars, they are typically placed on a number line with values labeled. In summary, these authors argued that interacting/dragging the bars in the dynagraphs has the semiotic potential for students to find patterns and generalizations related to ideas of function.

Magnitude bars as a way for students to understand measurement

The idea of motivating ideas about measurement through motion stems from Piaget and colleagues (1960). They and other researchers (e.g., Curry, Mitchelmore, & Outhred, 2006; Kamii, 2006) have reported that students start to conceptualize a length or magnitude via motion (e.g., stepping, sweeping, and pointing motions across straight paths). Moreover, Barrett et al. (2012) proposed a hypothetical learning trajectory that intends to build on this meaning for the quantity by introducing the iteration of single units (eventually without gapping, overlapping, or inconsistent unit size.). These researchers, although not explicitly mentioning magnitude bars, describe an image of movement associated with a length quantity to support students in learning about measurement.

Magnitude bars as a way for students to compare quantities

Lastly, the idea of comparing two magnitude bars to each other at different lengths shows up in the literature as well. These researchers have discussed how comparing dynamic magnitudes can support students at various levels. Yeo (2021) described how "dragging mathematical objects plays a critical role to mediate and formulate fraction understandings" (p. 1) And at the undergraduate level, Beckmann & Izsák (2020) described how a variable-parts perspective (in which the unit measurement varies and is then multiplicatively compared to the output measurement) can be considered when representing measurements graphically to support learning about average rate of change, the Fundamental Theorem of Calculus, and Trigonometry. These researchers highlight how the multiplicative comparison of dynamic magnitudes can support the learning of several different mathematics topics from the elementary to the undergraduate level.

Methods

In part of a semester-long teaching experiment (Steffe & Thompson, 2000), three undergraduate students in a preservice secondary mathematics education program at a large public university in the southeastern U.S individually interviewed with a teacher-researcher and observer(s) to reason about formulas via dynamic geometric objects. Each student participated in 12-15 teaching sessions. Due to space constraints, I discuss only one students' work (Lily) on a single dynamic geometric context (called the Painter Problem), one that she worked on over the course of six total sessions, resulting in about six hours of video data. I video-recorded and screen-captured students' work on a tablet and scans were made of student work.

The overall goal of the teaching experiment was to analyze the mental operations needed to support students in constructing productive meanings for formulas (see Stevens (2019) for full details), but this report focuses specifically on the role of magnitude bars in their reasoning by attending to the ways in which Lily reasoned with magnitude bars in the Painter Problem. To do so, I attended to the descriptions Lily provided for her reasoning when discussing magnitude bars, focusing specifically on (i) what quantities (if any) she was referencing in the situation when describing the magnitude bars (ii) what the goals were for Lily when she was reasoning with the magnitude bars and (iii) what mental operations she was describing when trying to achieve those goals. In particular, I look at students' covariational reasoning—the mental actions involved in reasoning about changing quantities. Of note in this proposal are directional covariational reasoning (i.e., as one quantity increases/decreases another quantity increases/ decreases), and amounts of change reasoning (i.e., as one quantity changes by equal amounts, another quantity changes by increasing/decreasing/equal amounts) (see Carlson et al., 2002).

Task Design

As mentioned, the tasks used in the teaching experiment as a whole considered the design principles outline in Stevens et al. (2017), and the Painter Problem involving a dynamic rectangle described here is no exception. The context of a dynamic rectangle varying in one dimension to discuss area in relation to width has been used by several other researchers (e.g., Ellis, 2011; Kobiela, Lehrer, & VandeWater, 2010; Matthews & Ellis, 2018; Panorkou, 2020). In this version, the students are given a dynamic geometric sketch in which they can drag the bottom right corner to the right or left. They can also press the "Paint" button to watch the rectangle sweep out to the right (maintaining the constant height of the paintbrush used to create the painted area) and "Reset" to move the bottom right point back to the beginning location (thus removing the rectangle). The novel addition to this varying rectangle in the Painter Problem is the inclusion of magnitude bars. Doing so meets the task design principle of using varying magnitudes from Stevens et al. (2017). In Part II of the problem (Figure 1), the students were told that the orange bar represented the length that a painter had swept out the roller (i.e., the width of the rectangle) and that they were to decide which, if any, of the purple bars represented the area of the rectangle. The top purple magnitude bar was an appropriate bar to represent the area of the rectangle because it has a linear relationship with the orange bar, whereas the bottom purple bar increases at an increasing rate with respect to the orange bar.[1]The students could move around the dots, which they used to mark specific widths of the rectangle and endpoints of the magnitude bars. Other design principles from Stevens et al. (2017) were also met. For example, the choice to allow the student to move the rectangle to the right or left by dragging the points met the design principle of not providing monotonically increasing quantities. The choice to keep the rectangle shape was strategic in that it provided students with a proportional relationship (other tasks considered different relationships between a length dimension and (surface) area). And lastly, throughout the interviews, students were asked to represent the relationship between quantities using different representations (with a formula being the focus of the teaching experiment), meeting that design principle.

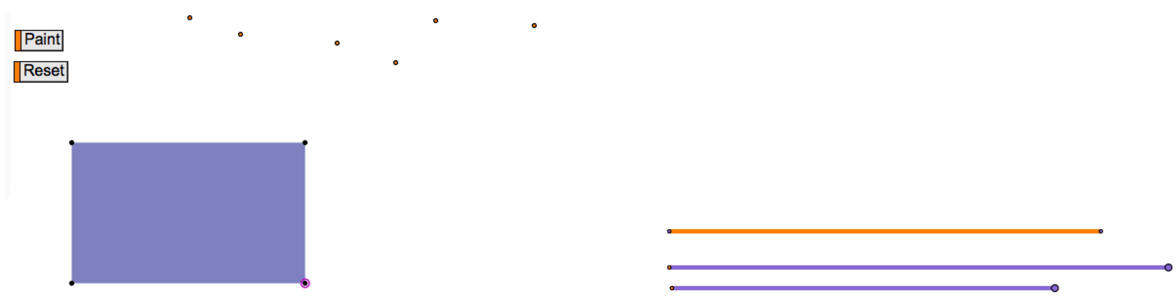


Figure 1. The Painter Problem Part II.

Results

In this section, I describe how Lily reasoned with magnitude bars while considering the relationship between the area and height of a dynamic object, the rectangle in the Painter Problem. Specifically, I attend to her descriptions of (i) the quantities she constructed with the dynamic geometry software (ii) the quantities she constructed with the magnitude bars, and (iii) the quantities she represented in her formulas.

For some context, in the Painter Problem Part I (i.e., prior to the introduction of magnitude bars), Lily had made three conclusions. First, she concluded that the area painted increased as the length that roller swept out increased (a directional covariational relationship as defined by Carlson et. al (2002)). Second, she knew to multiply the length and width to get the area of the rectangle (specifically, her formula was $A=bh$). Lastly, Lily had referenced the idea that "multiplication makes bigger".

Reasoning with the idea that multiplication makes bigger

When Lily started the Painter Problem Part II, she immediately chose Bar #1 (the top purple bar) as appropriate because "the value of the area is bigger than the value of the base, and I think,

[1] In Part III of the problem, the students were allowed to determine a number of partitions that would populate along the length of the rectangle. They could also see measurements of the length and area based on their chose unit sizes. The results of this part of the Painter Problem are not discussed in this section. See Stevens (2019) for details.

it can't-I don't think the area can be smaller than the base." She then discussed three different cases of the magnitude bars (which she called "segments") (Figure 2): when the purple segment (area magnitude) was less than, equal to, or greater than the orange segment (length magnitude). Since there were cases in which Bar #2 was not longer than the orange bar, and since multiplication makes larger means that the purple bar should always be longer than the orange bar to represent the multiplicative relationship between length and area, then Bar #2 could not be appropriate. Bar #1 satisfied her condition because its magnitude was always greater than the orange bar. Thus, Bar #1 was Lily's choice.

(a) *(b)* *(c)*

Figure 2. Lily's three cases for Bar #2 in the Painter Problem Part II: (a) area less than length (b) area equal to length and (c) area greater than length.

Lily felt unsatisfied with her justification for Bar #1, worried she was overgeneralizing the rule that "multiplication makes bigger". Thus, she decided to test various numerical values with her formula $A=bh$ to decide if it was a viable generalization. Her results are in Table 1.

Table 1. Lily's list of numerical calculations using A=bh to compare A (which should correspond to the value of the purple bar) and b (which corresponds to the value of the orange bar).

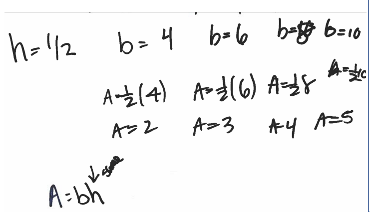
	Painter Problem Part II		
	(Cases) If…	(Conclusions) Then…	Numerical Example
1	$h = 1$	$b = A$	$b=3, h=1 \Rightarrow A=3$
2	(unspecified)	$A < b$	$b=6, h=½ \Rightarrow A=3$ $b=2, h=½ \Rightarrow A=1$
3	$b > h$	$A \leq b$	$b=6, h=½ \Rightarrow A=3$ $b=2, h=½ \Rightarrow A=1$ $b=6, h=1 \Rightarrow A=6$
4	$b < h$	$A \geq b$	$b=.5, h=4 \Rightarrow A=2$ $b=.5, h=1 \Rightarrow A=.5$
5	h is a constant	"If the area starts smaller, it should stay smaller. But if the area starts bigger than the base, then it should stay bigger."	$h=½ \Rightarrow$ "It would always make the area half the base."

As a result of Lily's reasoning with numerical values, she stated that she could no longer rule out Bar #2 as a potential viable candidate based on "my discovery that the area can be smaller than the base, the value of it" (Case 2). But after constructing Case 5, she decided to rule out Bar

#2 again because "[*dragging the corner of the rectangle back and forth*] I don't think [the area bar] should be less than [the base bar] and then greater than [the base bar]." Beyond saying "I don't think" and referencing the idea that "multiplication makes bigger", however, Lily struggled to relate her conclusion to the rectangle in the dynamic situation.

Reasoning with Amounts of Change

In a follow-up session, I asked Lily to consider the case where both purple magnitude bars were always bigger than the orange bar (essentially considering the case where the orange magnitude starts at its location in Figure 1c and extends to its maximum length). After some time, she stated that for equal changes in the base (she specifically stated, "like plus two, plus two, plus two" while dragging the corner of the rectangle in small chunks to the right), the purple magnitude bar should indicate a "constant increase in area." After some effort, and some additional numerical calculations to support her conclusions (see Figure 3), Lily reasoned using amounts of change; namely, that for the rectangle situation, for equal changes in the base, the amount of area increased by a constant amount. When she made that conclusion, she had no difficulty in deciding that Bar #1 was appropriate. That is, she could identify the magnitude bars as "increasing at a constant amount" (for Bar #1) and "increasing by increasing amounts" (for Bar #2). For example, for Bar #2, she described the situation as adding two "square units" for the first change in length and then four "square units" for the second successive equal change in length. Her discussion of the values was consistent across the situation and the magnitude bars; she associated an increasing number of boxes in the situation with a magnitude bar with increasing changes in magnitude (for the same successive change in the orange magnitude).



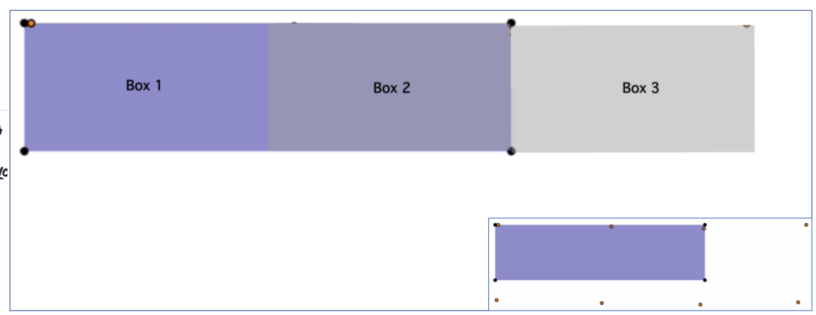


Figure 3. (left) Lily's numerical amounts of change calculations and (right) Lily's amounts of change in the situation after focusing on magnitudes. Bottom right corner shows Lily's markers for amounts of change and main figure shows post hoc edits for the reader to indicate the three boxes corresponding to her amounts of change in area.

Discussion and Implications

Researchers have already identified several reasons to include dynamic magnitude bars when introducing students to topics ranging from graphical representations to functions to fractions to trigonometry to the Fundamental Theorem of Calculus. These recommendations stem from ideas that the bars provide a means for comparison (via segmenting, iterating, counting, making multiplicative comparisons, etc.) and that the dynamic nature of the segments supports means for constructing length as a quantity or for observing patterns and generalizing.

In the results of this study, I reported on how the introduction of the magnitude bars supported Lily in covariational reasoning. Lily did not need the magnitude bars to make the conclusion about a directional covariational relationship between quantities. That is, already in Part I, she claimed that as the length the paint roller has swept out increases, the amount of paint increases. However, given the nature of the initial prompt (i.e., to describe the relationship between the two quantities), there was no intellectual need to compare the quantities beyond a directional covariational relationship. Further inquiry was needed but magnitudes are not the only option. If, for instance, she was asked to create a graph, the graph, whether intentional or not, would have been either straight or curved, giving the researcher a natural place to motivate a discussion about rate of change (e.g., Stevens, & Moore, 2016). Or given a formula, the student could be asked to discuss the form of the equation (y=kx, y=mx+b, $y=x^2$, y=sin(x), etc.) to motivate discussions of rates of change. But both of these methods of prompting reasoning about rates of change rely on perceptual features of the representation (i.e., the straightness of the line, the arrangement of the letters) and steps away from the quantities at hand. In doing so, students can turn to relying on conventions they have learned (e.g., shape thinking (Moore & Thompson, 2015)) to justify responses. Instead, by providing students with magnitude bars that represent different relationships between quantities and asking the students to indicate which one is appropriate, students are intellectually motivated to identify differences between the bars that go beyond the directional covariational relationship. For Lily, that prompted her to consider amounts of change covariational reasoning. That is, she considered equal changes in length on both the sketch of the situation and the orange magnitude bar representing that length, observed the resulting equal changes in area, and then identified which of the two magnitude bars also had equal changes. The larger teaching experiment showed similar gains by other students, with more or fewer struggles revolving around the use of numerical values to support their reasoning (see Stevens, 2019). Thus, this study contributes to the knowledge about magnitude bars by illustrating that they can provide a way for students to reason covariationally about amounts of change that do not rely on their understandings of graphs or equations.

Though Lily demonstrated amounts of change reasoning with the magnitude bars, she also demonstrated that it should not be assumed that students will necessarily reason quantitatively. Namely, Lily also compared each purple bar's magnitude to the orange bar's magnitude at specific points. This reasoning was not productive for her because she started relying on generalizations she recalled about mathematical operations (i.e., multiplication makes bigger) rather than attending to how the magnitudes were representing the growing rectangle. Thus, this study also contributes to the knowledge about magnitude bars by illustrating that students' ways of reasoning about the magnitude bars may not result in conclusions that they can relate to the quantities they are being asked to consider in the situation.

In conclusion, the results of this study show that magnitude bars have the potential to support student's covariational reasoning about formulas without relying on numerical values or understandings of different representation systems. I call for the research community to consider how different ways of reasoning with magnitude bars could be productive when learning different mathematical ideas. For example, although Lily's first way of reasoning about the bars did not support her in making a conclusion about the amounts of change relationship between quantities she was asked about, perhaps her comparison of quantities' values would be productive for different mathematical goals.

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A Linear Algebra Instructor's Ways of Thinking of Moving Between the Three Worlds of Mathematical Thinking Within Concepts of Linear Combination, Span, and Subspaces

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Linear algebra is an important topic for many STEM students and presents unique challenges for teaching and learning. In this study, we analyzed one mathematician's instructional materials and spoken language. Our theoretical framework is based on Tall's (2008) three worlds of mathematical thinking and Harel's (2008) ways of thinking. The goal of the study was to examine the complexities of a mathematician's ways of thinking while moving between the three worlds of mathematical thinking, while orchestrating the important aspects of linear combination, subspaces, and span.

Keywords: subspace, linear combination, span, ways of thinking, three worlds of mathematical thinking

Literature Review

Establishing the current state of linear algebra education research, a survey paper by Stewart, Andrews-Larson, and Zandieh (2019) highlights the students reasoning on the topics of span (e.g., Wawro, Zandieh, Rasmussen, & Andrews-Larson, 2013) and subspaces (e.g., Britton & Henderson, 2009). However, most studies focus on these topics and student comprehension, leaving a gap in the investigation of teacher understanding. Furthermore, the survey paper helps indicate extensive research into linear combinations and span, but less so in subspace.

When discussing Subspace, the idea of "Span" and "Spans" is inevitable. These two words are quite similar but mean different things invoking a type of polysemy, which can create challenges for the learning of subjects when it occurs in mathematics (Kontorovich, 2018). This issue overlaps with the understanding of subspace, something that research has documented as a challenge for student understanding (Britton & Henderson, 2009). Specifically concerning subspaces, one study showed that students prefer to use geometric intuition to understand subspaces (Wawro, Sweeny, & Rabin, 2011). Furthermore, students struggled with understanding how many subspaces are in a given space (Fleischmann & Biehler, 2018). Similar struggles exist in learning the concept of linear combination. When researching undergraduates' ability to determine if something is linearly dependent or not, instead of utilizing the characteristics of linear combination, the students preferred to use matrix operations (Aydin, 2014). Navigating these challenges presents important considerations for an instructor's teaching methods.

Looking to how this information is communicated, a teacher's knowledge base is an important aspect of education. A teacher's knowledge base and its relation to student understanding is a deep facet of education that has been studied (Tallman & Frank, 2020). This knowledge base may be accessed through the educator's ways of thinking (Harel, 2008). In their view Tallman and Frank (2020) find for a given topic, "this study provides support for the notion that mathematical ways of thinking comprise a fundamental aspect of teachers' professional knowledge base" (p. 92). A teacher's knowledge base has overlap with how they present information. In the literature, there has been an increasing interest in examining this topic in how an instructor conveys information to students and conducts procedures in the classroom. For example, Stewart, Troup, and Plaxco (2019) discussed an instructor's reflections on his lectures and movements between different mathematical worlds. The study concludes that the instructor was able to give careful attention to the needs of the students after consulting with his own reflections and knowledge base.

Emerging studies have begun examining how digital resources can be used to help students understand linear algebra topics. Previously, introducing drawings into linear algebra education has been a topic of study. Gueudet-Chartier (2004) argues that "the concreteness that seems to lack in linear algebra could be more efficiently provided by the use of drawings" (p. 500). Research into the learning of span has shown that visualization can be helpful in making students more flexible in their reasoning of the topic (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012; Wawro, Rasmussen, Zandieh, & Larson, 2013b; Cárcamo, Fortuny, & Gómez, 2017). Further, it has been shown that using GeoGebra to assist students in understanding subspace, specifically the zero vector is beneficial (Caglayan, 2019). These studies suggest that linear algebra concepts can be enhanced by visualization tools. Similar studies were conducted on students showing that these students benefited from a more embodied representation (Hannah, Stewart, & Thomas, 2013).

Theoretical Framework

The theoretical framework for this study will utilize Harel's (2008) ways of thinking and Tall's three worlds of mathematical thinking (2010). Harel (2008) introduced the notion of a mental act as actions such as interpreting, conjecturing, justifying, and problem solving, which are not necessarily unique to mathematics. These several types of mental acts are the foundation of ways of thinking defined as "a cognitive characteristic of a mental act" (p. 269). These ways of thinking have the characteristics of being able to abstract, generalize, structure, visualize, and reason logically. Tall (2010) describes the embodied world as "our operation as biological creatures, with gestures that convey meaning, perception of objects that recognize properties and patterns… and other forms of figures and diagrams" (2010, p. 22). In his view, "The world of operational symbolism involves practicing sequences of actions until we can perform them accurately with little conscious effort. It develops beyond the learning of procedures to carry out a given process (such as counting) to the concept created by that process (such as a number)" (2010, p. 22). The formal world "builds from lists of axioms expressed formally through sequences of theorems proved deductively with the intention of building a coherent formal knowledge structure" (2010, p. 22). Using the above frameworks, the goal of this narrative study is to examine the complexities of a mathematician's ways of thinking while moving between the three worlds of mathematical thinking while orchestrating the important aspects of linear combination, subspaces, and span.

In analogy with Tallman and Frank (2020), we consider moving between the embodied, symbolic, and formal worlds as a characteristic of mental acts and, as such, a way of thinking. It is through Tall's three worlds lens of mathematical thinking that we evaluate the teaching moves of the instructor.

Methods

This qualitative narrative study (Creswell, 2013) took place in the Fall of 2021. The participants in this research consisted of a mathematician and co-author. The mathematician, thereafter, referred to as the instructor, specialized in geometry, linear algebra, and number theory. The teaching material of note is part of a first-year linear algebra course from a university located in the southwest region of the USA taught by the instructor. There were 30 students enrolled in this course. In addition, a mathematics educator, and a research undergraduate assistant from another university in the USA were part of the research team. The data consisted of (a) the instructor's notes (which he authored) on linear combinations, span, and subspaces (in this order) that was available to students (Figure 1), (b) his notes on the blackboard (Figure 2a & b), and a GeoGebra interactive example (Figure 2c), (c) two of his recorded video lectures, and (d) students' weekly reflections and homework. The data analysis for this study consisted of transcribing the second lecture, finding pertinent sections in the notes related to the lecture concepts, and coding the different themes. These different components of analysis were done with the instructor as part of the research team. Harel's (2008) ways of thinking and Tall's three worlds of mathematical thinking (2008) were employed as a framework to analyze the data. We identify and analyze the instructor's one specific way of thinking, namely, the disposition of moving between languages as well as Tall's (2008) three worlds of mathematical thinking (embodied, symbolic, formal). The data from students' work is in the process of being analyzed for future research.

Results

The instructor intentionally ordered his lecture with the goal of building on intuition with more embodied understandings before generalizing to reach into new topics. The instructor gave a variety of connections between topics to aid in understanding. In his lecture, he reminds students of the definitions in the notes and refers to them. Descriptive ways of how to understand and apply these concepts are placed on the board. In the first and second lectures, diagrams of two- and three-dimensional space are drawn on the board to emphasize embodiment (Figures 2a & b).

Linear Combination and Span: the language and movements between the three worlds

We noted the instructor used a variety of informal and geometric language to support the concepts and their definitions. In the notes, the instructor used words like "reach", "line", and "parallelogram" to describe linear combination and span (Figure 1). He repeatedly used the informal language of "mixture" to refer to linear combinations. For example, he asked, "Which vectors? The collection of how many mixtures? Some mixtures? … All mixtures. Right, …all mixtures you can make with these not some, not half, not a few, but all mixtures right." Here the instructor relies on language to give meaning to linear combinations as his way of thinking about the topic. An instance of the instructor's disposition to attend to language is when they anticipated and addressed student confusion stemming from the polysemy "span." Figure 2b breaks the topics of possible linear combinations "span the noun" and spanning sets making subspaces "span the verb" into two categories and highlighting the most crucial pieces of the definition. Harel (2008 considers 'anticipating' as a mental act. This certain mental act motivated the teaching act described above. The instructor further clarifies the distinction between an object and the relationship between two objects: "Right, that's very different than the noun right this is just a set (gestures to W = Span(S) this is like a true false kind of thing (gestures to S spans W) it either does or doesn't span." During the lecture (see Figure 2b), the instructor said: "I would call span the mixture set associated to any collection of vectors, okay and then we could talk about uh span the verb you know if if you know if given a set of vectors right if it's mixture set is the subspace in question, we would say it spans the subspace." Following, the core mixture questions (Figure 1) are set up in such a way as to help internalize what is important about the concept and give these characteristics of linear combinations meaningful language while also pushing towards the concept of span. During the first lecture, some examples with matrix representations are written on the board to assist in symbolic understanding (Figure 2a).

During the second lecture, the word "mixtures" is used. This way of thinking about the concept of linear combinations involves changing the language to give the term linear combination additional "meaning". The instructor poses the question, "So let's say what's the noun associated to this, what is the noun given a collection of vectors, what can we make with it? ... Yeah …so we can make mixtures." The instructor utilizes the movement between the mathematical worlds of thinking throughout the lecture. In one such example (Figure 2), the instructor moved from the formal to embodied and acknowledged the possibilities of movements to symbolic: "the span of those guys it equals the line, and you can see that visually and if we wanted to, we could verify that symbolically". For the topic of span, the notes provided a definition of span with emphasis on the word "all" this is done to emphasize any possible linear combination is made up in this set. Meanwhile, the embodiment of the definition invokes the use of visualizing a line that represents all linear combinations of a single vector. This is further extended to the notion of two vectors and a plane. In analyzing the concept of span, in the first lecture, the instructor drew two vectors, proceeded to analyze their linear combinations visually via a parallelogram, and then moved to the symbolical representation (Figure 2a). Next, he showed symbolically how to create an arbitrary vector in R^2 as a linear combination of the original vectors. Performing examples in this order suggest that the instructor comes from a place of visualizing the span first and symbolizes in the matrix form. Visualization continues to be included in the second lecture, with vectors being drawn on the board and displayed in GeoGebra (Figure 2b & c).

Subspace: the language and movements between the three worlds

In the notes (Figure 1), the instructor provided a nonstandard formal definition of subspace relying on the concept of span. His notes continued to discuss a "quick" way to determine if a region is a subspace by checking if it is closed under scaling and addition, but he phrased it in geometric language as "line containment" and "parallelogram containment", respectively. In the first lecture, the notion of subspace, having a set of "mixtures" and then a spanning set, are all written on the board (Figure 2a). With these components, the same language of the notes is used where a plane is drawn on the board in a projected three-dimensional space. The use of line test and parallelogram test are employed with the inclusion of passing through the origin.

During the second lecture, the instructor used GeoGebra as an activity and demonstration to discuss single vectors, zero vectors, and span vs. spans. With the focus changing to subspace, a GeoGebra three-dimensional model is placed on the top of the board, where the instructor manipulated different vectors and created a variety of subspaces (Figure 2c). After the group activity, an example from the class dialogue continued as follows:

"So, let's see, A plus B spans the line well that's a very good point again what is A plus B?... So there's there's also there's all sorts of relationships here so so you could say A plus B spans but A plus B goes by another name D. So, there's all sorts of algebraic relations between these things."

Vectors represent quantities we see in the world around us every day. One of the most fundamental ideas associated to quantities is *mixing quantities*. This core idea will be central to everything going forwards.

The idea of mixing vectors is made mathematically precise through two natural operations: scaling and addition. To reason about them verbally, I will discuss them in the context of cereal, where a vector represents one serving size of a given cereal (as in Example 2.2). *Can you reason about them in other contexts?*

Definition 2.8. If $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, then a **linear combination** of them is a vector of the form $x\mathbf{v} + y\mathbf{w}$, for real numbers $x, y \in \mathbb{R}$.

More generally, a linear combination of vectors $\mathbf{v}_1 \ldots \mathbf{v}_n \in \mathbb{R}^m$ is an vector

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n, \qquad \text{where } x_1, \ldots x_n \in \mathbb{R}.$$

CORE MIXTURE QUESTIONS

If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^m$ is a collection of vectors, then we seek to understand:

1. *Reach.* What are all the possible vectors that can be made by mixing vectors in S?
2. *Redundancy.* Is one of the vectors in S already a mixture of the other vectors in S?
3. *Realization.* Given a vector $\mathbf{w} \in \mathbb{R}^m$, can $\mathbf{w}$ be made by mixing vectors in S, and if so, how?

To begin, let us start with the question of *reach*. In this situation, we have fixed vectors but can select any and all serving sizes for them. The collection of all possible mixtures which can be made with S is made precise by the notions of *span* and *subspace*.

Definition 2.12. If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subset \mathbb{R}^m$, their **span** is the collection of *all* linear combinations of $\mathbf{v}_1 \ldots, \mathbf{v}_n$. Symbolically,

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\} = \{x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n \mid x_1, \ldots, x_n \in \mathbb{R}\}.$$

Definition 2.16. Any subset $W \subset \mathbb{R}^m$ which is the span of a collection of vectors is a **subspace**.

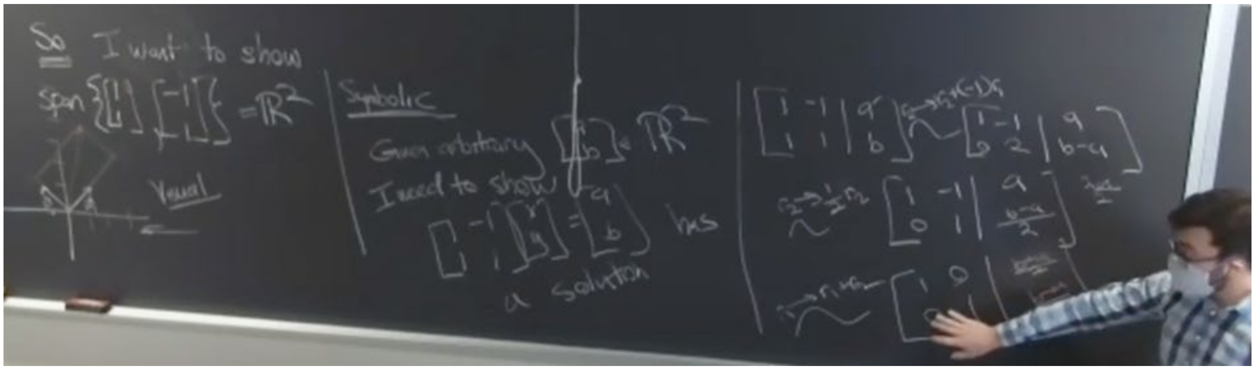
Observe that for our purposes, a span and a subspace are essential the same thing. The reason why we might use the language of one instead of the other is the following: a span means we explicitly know the mixing set S while for a subspace, we know such an S exists, we just do not necessarily know what it is.

The containment properties of span can help us determine *which subsets of* $\mathbb{R}^m$ *are subspaces*.

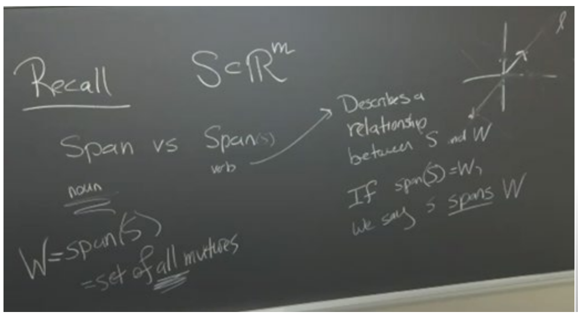
Computational Quick Check 2.17 (Subspaces)**.** If $W \subset \mathbb{R}^m$ is a subset, then W is **not** a subspace if you can find either:

1. **(Line Failure.)** A vector $\mathbf{v} \in W$ for which the line $\operatorname{span}(\mathbf{v})$ is *not* contained in W, or
2. **(Parallelogram Failure.)** Vectors $\mathbf{v}, \mathbf{w} \in W$ for which the mixture parallelogram associated to $\mathbf{v} + \mathbf{w}$ is not contained in W.

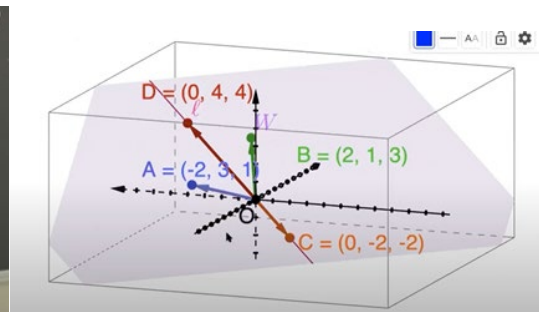
Figure 1. Linear combinations, Span, and Subspaces from the instructor's notes.



(a)



(b)



(c)

*Figure 2****.*** *The instructor is writing on the board during lectures one and two and engaging students via GeoGebra.*

During this episode, the instructor focused on how the zero vector and the zero-subspace fit into 3D space. Here is a sample of subsequent discussions:

Instructor: [I]s a single vector a span? No, right, so we have to go back to our precise definition. So, a single vector is not a span right, but a line here is a span. What other things in somehow depicted here are spans of something?

Student 1: Like a point.

Instructor: Uh which point?

Student 2: Any point on the line.

Instructor: Is Anything on the line, Is any point a span? This is a span? (Gestures at a drawing point of a line on board). This is a subspace? No because right a single point cannot be a subspace unless there's one exception.

Here the instructor is connecting the different ideas about what a single vector can generate while also asking them to recall special cases. This response is supported by the picture of a line placed on the board earlier in the lesson (Figure 2b). "Does this guy span the trivial subspace, zero? Some people say no. No! As they take mixtures of this is bigger, it's bigger it's not equal." Here the instructor used geometric language of relative size to support the idea that the span of a non-zero vector cannot be the trivial subspace.

Discussion and Conclusions

Upon analyzing the data in this study, we observed the instructor relied heavily upon reasoning in many representations, which impacted his choices in instruction. We observed how this informed his selection of demonstrations and his attention to language.

We also found that these ways of thinking impacted the choices that he made in instruction. He consistently moved from embodied to symbolic to formal worlds, and back again. His choice of language moved from informal to geometric to formal, and returned to informal. This was observed in both his prepared lecture and his spontaneous responses to questions. The weaving of formal language with informal and familiar language made the topics more accessible. The lecture began using descriptive geometric words such as “sweep out”, “stretch” and “reach” to articulate how these objects look. The instructor used informal terms during the lecture to familiarize informal terms like “mixtures”, “span the noun”, “span the verb”, when talking about these concepts. In their responses to questions, students also used this informal language, indicating that the familiarization may have been effective. The prepared and distributed notes allowed students to focus on active participation within the class. The lecture weaved through multiple understandings of each concept allowing for engagement with familiar concepts, language, and embodied objects, only to then progress to understanding the mathematical machinery of the subjects before being exposed to their deeper meaning. Here the use of language is crucial again as the lecture begins using descriptive words such as “sweep out”, “stretch” and “reach” to articulate how these objects look.

Furthermore, the use of GeoGebra during the lectures supports this meaning by seeing all the different possible combinations of vectors and having students identify what they created. The instructor gives meaning to span and subspace in similar ways by providing a definition and then constructing a visual representation of the topic. The instructor employed the use of language to illustrate how the zero subspace looks, citing that adding anything besides zero would be “bigger” than the zero subspace. He made a clear distinction between the words span and spans when a discrepancy appeared about what a singular vector can make. Following these results, our future research will include an analysis of students’ weekly surveys and some of their homework.

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Synthetic vs. Analytic Formalism of Matrix Representations in Linear Algebra: What it Means to Diagonalize a Linear Transformation

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The contributed report highlights linear algebra students' understandings of the matrix representations of certain linear transformations defined on with respect to given bases in a DGS-MATLAB assisted pedagogical environment. The research focused on the diversity of strategies that students utilized specifically in the process of diagonalizing linear transformations in a series of in-depth qualitative interviews within a framework of modes of description and thinking in linear algebra (Dreyfus, Hillel & Sierpinska, 1998; Sierpinska, 2000). Data analysis revealed three main categories: (i) synthetic-geometric mode manifested as the visual alignment strategy in which students visualized each image vector as a multiple of the corresponding preimage vector; (ii) analytic arithmetic mode prevailed in the process of expressing the image vectors in terms of the preimage vectors; (ii) analytic structural mode was observed in the process of determining the diagonal form of the linear transformation with reference to the transition matrices and the commutative diagram.

Keywords: matrix representation of linear transformations; transition matrices; diagonalization; eigenbasis; dynamic geometry software; modes of description and thinking in linear algebra

Theoretical Background and Motivation for the Study

The pedagogy of linear algebra has been a focus of research interest for many years (Dogan-Dunlap, 2010; Dorier, 1991, 1995, 1998; Dorier, Sierpinska, 2001; Gueudet-Chartier, 2006; Harel, 1987, 1989, 1990; Parraguez & Oktac, 2010; Pavlopoulou, 1993; Robert & Robinet, 1989; Rogalski, 1994; Sierpinska, 1995, 2000; Sierpinska, Dreyfus & Hillel, 1999; Sinclair & Gol Tabaghi, 2010; Thomas & Stewart, 2011; Zandieh, Wawro & Rasmussen, 2017). Hillel (2000) characterized linear algebra students' understanding of matrices and linear operators to be at the interoperational level of thinking – although they were expected to communicate at the transoperational level of thinking. Hillel (2000) further posited the requirement that at the trans-level linear algebra learners should be able to: (i) determine the matrix representation of a linear transformation relative to a given basis; (ii) determine the coordinate representation of a vector in a vector space relative to a basis; (iii) think about these representations (coordinate representations of vectors and matrix representations of linear transformations) as "objects of inquiry in their own right;" (iv) think about "the general conditions under which a vector or a linear operator can have a particularly desirable representation" (p.206).

Harel and Kaput (1991), and Harel (1985, 1990) formulated the concreteness principle, as a fundamental approach for the teaching and learning of linear algebra, originated from Piaget's (1977) idea of conceptual entities. According to this principle, "for students to abstract a mathematical structure from a given model of that structure the elements of that model must be conceptual entities in the student's eyes; that is to say the student has mental procedures that can take these objects as inputs" (Harel, 2000, p.180). Concreteness principle requires that "students build their understanding of a concept in a context that is concrete to them" (p.182). He recommends MATLAB as a tool that would help students visualize vectors and matrices as concrete mathematical objects, in accordance with the concreteness principle. Aligned with the concreteness principle (Harel, 2000) and in search for a desirable matrix representation for a linear transformation as brought forward in Hillel (2000), this study focuses on students' questioning of the essence of diagonalization of linear transformations in a DGS-MATLAB-assisted learning environment.

Modes of Description and Thinking in Linear Algebra

Representations are central to the teaching and learning of linear algebra. Harel (1989) established, among other things, that geometric representations contribute significantly in linear algebra students' concept image formation (p. 57). The present report is guided by the theoretical viewpoint that identifies three modes of description (language) with the corresponding three modes of thinking as demonstrated by linear algebra students (Dreyfus, Hillel & Sierpinska, 1998; Hillel, 1997, 2000; Sierpinska, 2000): (i) Geometric description (synthetic-geometric mode of thinking): point, line, plane, geometric transformation (translation, reflection, rotation, projection), vector as a directed line segment (arrow with a head and tail); (ii) Arithmetic description (analytic-arithmetic mode of thinking): n-tuples, matrices and operations on matrices including the inverse of a matrix, the adjoint of a matrix, partitioned matrices, systems of linear equations and their solutions; (iii) Algebraic description (analytic-structural mode of thinking): theory of vector spaces, subspaces, inner product spaces, linear transformations.

Context and Method

Qualitative-descriptive interview data were collected over three years in a university in the United States, as part of a research project which was designed for the purpose of enhancing mathematics and mathematics education majors' content knowledge of advanced mathematics, with particular focus on geometry-linear algebra connections using DGS-MATLAB technology. The data for the analysis of matrix representations of linear transformations concept came from the videotapes of interview sessions in a computer lab that included a total of sixteen math majors who successfully completed a one-semester long linear algebra course, who were interviewed by the author individually on separate days. The university linear algebra course math majors took had covered the first eight chapters of Ron Larson's "Elementary Linear Algebra" textbook. Math majors were familiar with the MATLAB and DGS as they had already used them during the preceding interview sessions on linear algebra topics; they were not provided any training or instructional help during the interviews.

Table 1. Interview tasks: Find the matrix A' for $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to the basis β'.

Linear Transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$	Basis β'
$T(x,y,z) = (x+z, -x+3y+z, 2z)$	$\beta' = \{(2,1,0),(1,0,1),(0,1,0)\}$
$T(x,y,z) = (3x+2y+z, 2z, 2y)$	$\beta' = \{(1,-5,5),(-3,1,1),(1,0,0)\}$
$T(x,y,z) = (x+2y-2z, -2x+5y-2z, -6x+6y-3z)$	$\beta' = \{(1,1,3),(1,0,-1),(1,1,0)\}$
$T(x,y,z) = (2y, x-y, z)$	$\beta' = \{(-1,1,0),(2,1,0),(0,0,1)\}$
$T(x,y,z) = (3x+2y-3z, -3x-4y+9z, -x-2y+5z)$	$\beta' = \{(1,-3,-1),(3,0,1),(-2,1,0)\}$

Sample Interview Outline: Why did you decide to begin with the standard matrix A for T? How would you (What does it mean to) obtain the matrix A' for T relative to basis β'? Visually? Analytically? How are matrices A and A' related? What are your thoughts about the types of the matrices A and A' in relation to each other? What made you embrace the visual approach? The analytic approach? Is T diagonalizable? What does it mean for a linear transformation to be diagonalizable.

The qualitative interviews were based on a semi-structured interview model (Kvale, 2007) in the course of which the interviewer followed-up with probes and questions on the interviewees' responses. Math majors were asked to respond to a variety of interview questions with primary focus on the matrix representations of linear transformations in a manner leading to students' questioning of the essence of diagonalization. During the interviews, math majors were asked to think aloud, to clearly indicate their problem solving procedure, and to explain their reasoning in detail. Math majors were granted access to scratch paper along with MATLAB and DGS to facilitate and clarify their explanations during the interviews. All interview tasks were designed in such a way that they could be explored via both analytic and synthetic approaches, in accord with the guiding theoretical framework used in the study. Table 1 outlines the linear transformations along with an interview outline that were used during these interviews. Though it was optional, all math majors were very eager and passionate about using MATLAB and DGS, primarily for checking their work, visualizing analytic approaches, testing conjectures, or providing examples and counter-examples.

Analysis of data, which consists of videotaped qualitative interviews along with MATLAB-DGS work and inscriptions, was carried out using thematic analysis (Boyatzis, 1998). The thematic analysis was primarily used to describe how research participants came up with a diversity of innovative and creative ways for understanding and making sense of matrix representations of linear transformations and their connections to other important concepts of linear algebra. After transcribing all interview videos, using thematic analysis, the author reviewed the interview videos along with the original transcripts line by line to gain access to the modes of description and thinking adopted by students as they explored matrix representations of linear transformations and their relationship to other core concepts of linear algebra, in accordance with the research objective. The last cycle of data analysis process consisted of a holistic review of the corpus of data multiple times, in accordance with constant comparative methodology (Glaser & Strauss, 1967).

Results

Overall, students provided a diversity of synthetic and analytic strategies in their exploration of the diagonal form of linear transformations. This section considers the visual alignment approach (synthetic-geometric mode), the expressing of the image vectors in terms of the preimage vectors approach (analytic-arithmetic mode), and the transition matrix approach (analytic-structural mode), respectively. In each interview task, students were asked to find the matrix representation A' for the given linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ relative to a specifically chosen nonstandard basis β'. Because most of the time all sixteen students began each task by retrieving the standard matrix A for T, the analysis that follows primarily focuses on what happened next.

Synthetic-Geometric Mode: The Visual Alignment Approach

Upon obtaining the standard matrix A for T, six out of sixteen students consistently embraced the synthetic-geometric approach in an organized attempt to arrive at the matrix A' for T relative to β'. Typical student approach was to first introduce A along with the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, respectively on the DGS. Second step was to obtain the *matrix-applied* basis vectors $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$, denoted by $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, respectively on the DGS (Figure 1). Upon noticing the parallel relationships $\mathbf{v}_1 \parallel \mathbf{u}_1, \mathbf{v}_2 \parallel \mathbf{u}_2, \mathbf{v}_3 \parallel \mathbf{u}_3$, visually, students in this category immediately focused on the scale factor $\lambda_1, \lambda_2, \lambda_3$, for each parallel vector pair, which, they mentally straightforwardly determined as $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$, respectively. The final step was to simply write down the diagonal matrix $A' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

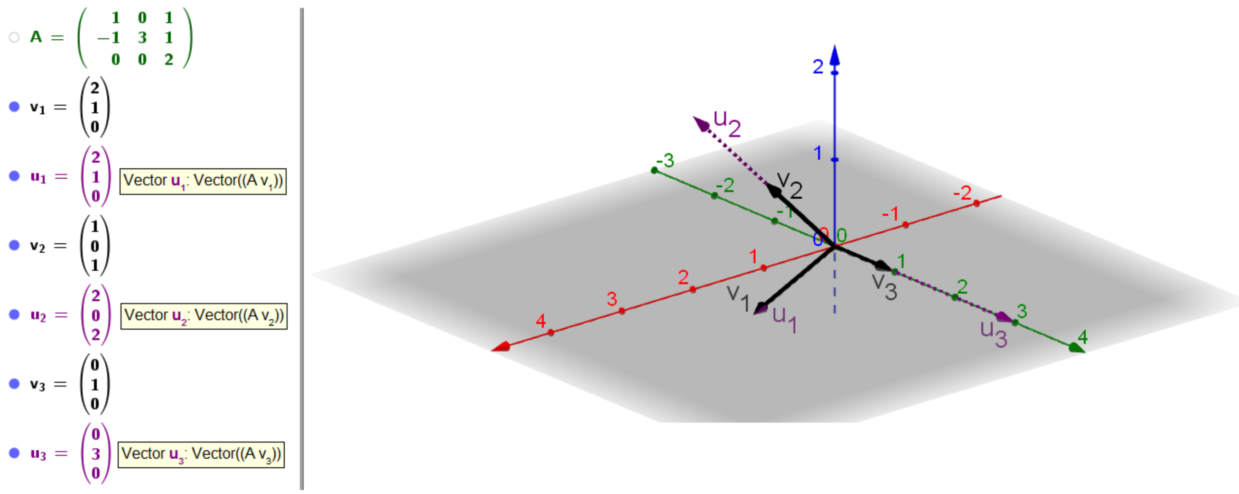

Figure 1 Typical student synthetic-geometric approach for Task 1.

Students' synthetic-geometric description of the parallel vector pair (i.e., the basis vector $\mathbf{v}_i$ and the matrix-applied basis vector $A\mathbf{v}_i$) relationship appeared to have functioned in two steps: (i) when $\mathbf{v}_i$ and $A\mathbf{v}_i$ were aligned in the same octant, the corresponding eigenvalue λ_i was given by the ratio of the length of $A\mathbf{v}_i$ to the length of $\mathbf{v}_i$; (ii) when $\mathbf{v}_i$ and $A\mathbf{v}_i$ were aligned in opposite octants, the corresponding eigenvalue λ_i was given by the negative ratio of the length of $A\mathbf{v}_i$ to the length of $\mathbf{v}_i$. Upon the interviewer's probing whether T is diagonalizable, S1 responded: "*Well in this case A' is a diagonal matrix so I would say yes T is diagonalizable*." S2, who embraced the visual approach like S1, was more specific in her explanations: "*Only because all these* [pointing to the image vectors] *are multiples of these* [pointing to the preimage vectors] *the linear transformation is diagonal*." In the synthetic-geometric approach, students simply focused on each image vector as a multiple of the corresponding preimage vector, without considering the remaining two preimage vectors. This was what made the linear transformation diagonal.

Analytic-Arithmetic Mode: Expressing the Image Vectors in Terms of Preimage Vectors
Six students consistently embraced the analytic-arithmetic mode. Among these six, three students directly applied the linear transformation to each vector of the given nonstandard basis, whereas the other three applied the standard matrix representation of the linear transformation to the coordinate vector representation of each basis vector. This was the slight difference in the thinking of these two groups of students.

Expressing $T\mathbf{v}_1, T\mathbf{v}_2, T\mathbf{v}_3$ in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. In Task 2, for instance, S3 first evaluated the *linear transformation-applied* basis vectors $T\mathbf{v}_1 = T(1,-5,5), T\mathbf{v}_2 = T(-3,1,1), T\mathbf{v}_3 = T(1,0,0)$ and obtained the corresponding image vectors $(-2,10,-10), (-6,2,2), (3,0,0)$. Simply by comparing the preimage-image vector pairs was she then able to retrieve the eigenvalues as

$\lambda_1 = -2, \lambda_2 = 2, \lambda_3 = 3$, respectively via mere inspection. This was enough for her to deduce the desired matrix representation as $A' = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Expressing $A[\mathbf{v}_1]_\beta, A[\mathbf{v}_2]_\beta, A[\mathbf{v}_3]_\beta$ in terms of $[\mathbf{v}_1]_\beta, [\mathbf{v}_2]_\beta, [\mathbf{v}_3]_\beta$. Among the three who embraced this *representation* approach, only one student felt the need to specify the coordinate vector representation notation $[\mathbf{v}_1]_\beta, [\mathbf{v}_2]_\beta, [\mathbf{v}_3]_\beta$ where β denoted the standard basis. The other two students, without being specific, still were able to interpret it as the representation of the vector *object*. S4, who embraced the representation approach for Task 3, explained: "*I'd first apply A to each basis vector*." Upon juxtaposing the three image vectors (written as 3×1 column matrices $\begin{bmatrix} -3 \\ -3 \\ -9 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$), respectively, she first suggested $A' = \begin{bmatrix} -3 & 3 & 3 \\ -3 & 0 & 3 \\ -9 & -3 & 0 \end{bmatrix}$ as the matrix representation of T relative to β'. Upon the interviewer's probing how she knew whether her suggestion was the desired matrix representation, she responded: "*I compare each column to the standard basis vectors… No, that's not right. I am supposed to be comparing them to the nonstandard basis vectors and it looks like each column is a factor.. I mean multiple of each basis vector so I am changing my answer to a diagonal matrix*." This way, S4 obtained $A' = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ as the desired matrix representation.

Analytic-Structural Mode: The Transition Matrix Approach

Among the four consistently adopting the transition matrix approach, two students emphasized the matricial relationship by drawing a commutative diagram. Except for one student, S5, all students began each task by first writing the transition (change of basis) matrix as $P = \beta'$. S6, for instance, began her exploration of Task 4 by setting $P = \beta'$ while explaining: "*Okay I remember this very well.. this is the nice case.. P is the transition matrix from β' to β so I don't have to do anything I remember this very well*." With reference to her commutative diagram, she proposed the matricial equation $P^{-1}AP = A'$ and verified her proposed diagonal form $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ on MATLAB (Figure 2a). Likewise, Figure 2b demonstrates S7's MATLAB procedure based on the transition matrix approach in her exploration of Task 2.

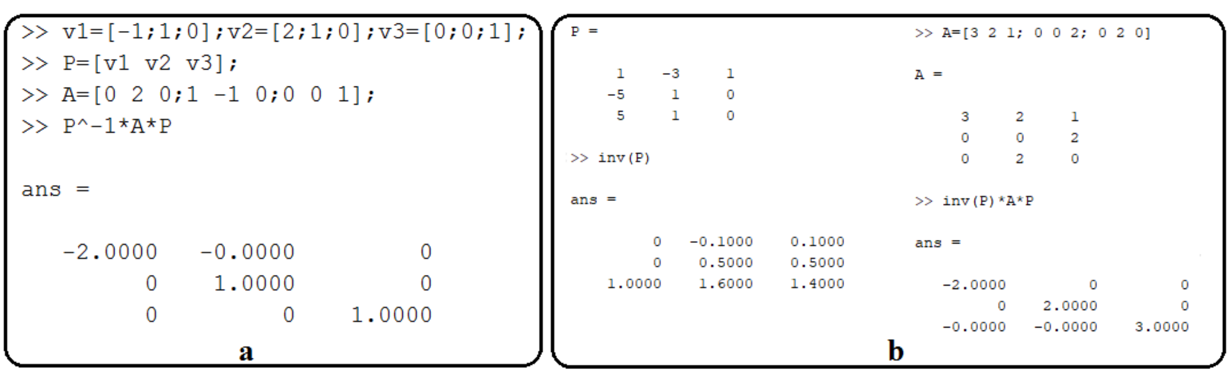

Figure 2 Transition matrix approach.

Difficulties Observed within the Three Modes of Description

This section highlights some of the difficulties students encountered in their exploration of the matrix representations of linear transformations in all three modes of description.

Retrieving the zero eigenvalue from the matrix applied eigenvector. This difficulty occurred in the synthetic-geometric approach in the exploration of Task 5. All six students who embraced the visual approach successfully graphed the given basis vectors along with the matrix-applied basis vectors. Obviously, $\mathbf{v}_2$ and $\mathbf{v}_3$ were both parallel to $A\mathbf{v}_2$ and $A\mathbf{v}_3$, respectively, in the same direction, with corresponding eigenvalues 2, and 2, respectively (Figure 3). It took quite some time for all students to realize that 0 was indeed the eigenvalue corresponding to $\mathbf{v}_1$. S8, for instance, felt the need to switch to the analytic-arithmetic mode to resolve his confusion regarding $\mathbf{v}_1$. In his first attempt, he used the matrix approach by writing $A\begin{bmatrix}1\\-3\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}, A\begin{bmatrix}3\\0\\1\end{bmatrix}=\begin{bmatrix}6\\0\\2\end{bmatrix}, A\begin{bmatrix}-2\\1\\0\end{bmatrix}=\begin{bmatrix}-4\\2\\0\end{bmatrix}$. Inconclusive, he switched to the transformation notation by explicitly writing $\begin{cases}T\mathbf{v}_1 = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3\\ T\mathbf{v}_2 = 0\mathbf{v}_1 + 2\mathbf{v}_2 + 0\mathbf{v}_3\\ T\mathbf{v}_3 = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 2\mathbf{v}_3\end{cases}$. This way of writing enabled him to obtain the diagonal matrix $A' = \begin{bmatrix}0 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}$ while explaining: "*now it looks like a matrix.. a diagonal matrix with eigenvalues on the diagonal*." In theory, all students seemed to accept the possibility of having 0 as an eigenvalue, however, when it came to visualize or algebraically work it out, difficulties arose, as also shown in the present report.

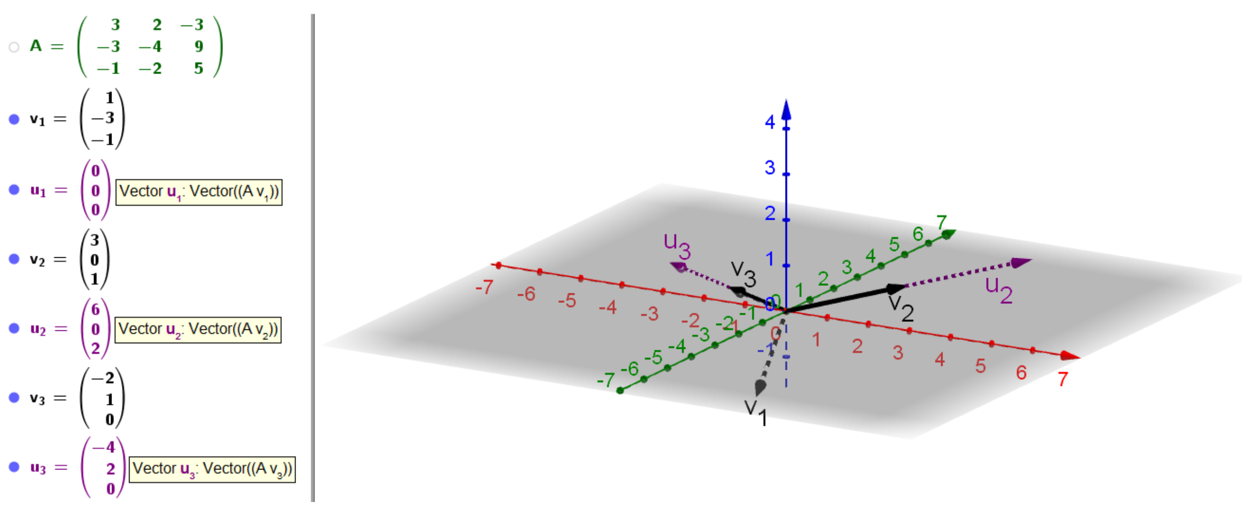

Figure 3 Typical student synthetic-geometric approach for Task 5.

Retrieving the negative eigenvalue from the matrix applied eigenvector. This difficulty occurred in the synthetic-geometric approach in the exploration of Task 3. Upon graphing the basis vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and the *matrix-applied* basis vectors $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$, respectively, S9's first reaction was that that $\mathbf{v}_2$ and $\mathbf{v}_3$ were eigenvectors of A corresponding to the same positive eigenvalue $\lambda = 3$ (Figure 4a). Regarding $\mathbf{v}_1$, S9's first conjecture was that, rather than being a scalar multiple of $\mathbf{v}_1$, $A\mathbf{v}_1$ was perhaps a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$. Upon using some techniques such as rotating the view, hiding the *xy*-plane, changing the scale by zooming in/out, hiding $\mathbf{v}_2$ and $\mathbf{v}_3$, etc., she ultimately came to the conclusion that $A\mathbf{v}_1$ was indeed "*three times* $\mathbf{v}_1$ *but pointing in opposite directions so the eigenvalue must be -3 and not 3* (Figure 4b)."

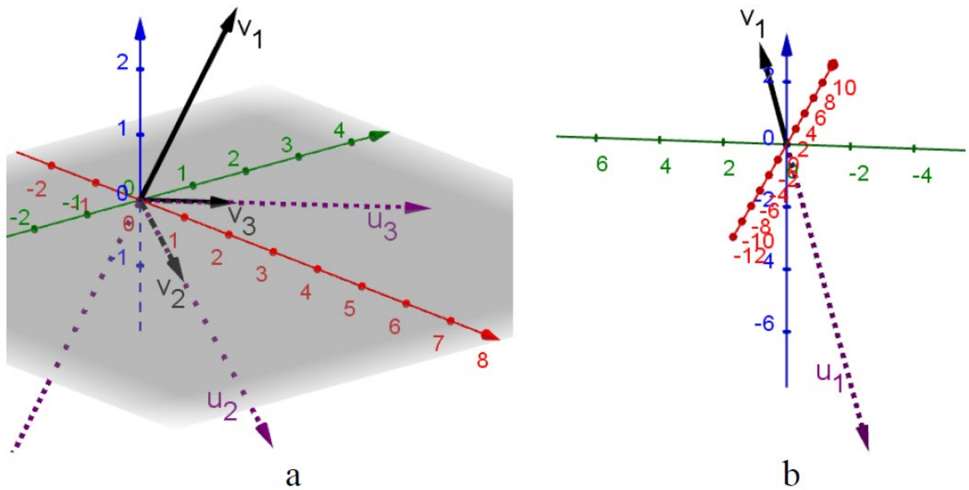

Figure 4 S9's synthetic-geometric approach for Task 3.

Issues within the analytic-arithmetic mode. S10, who embraced the analytic-arithmetic mode in Task 2, used a combination of *representation approach* and *object approach* by explicitly writing $\begin{cases} A\mathbf{v}_1 = \cdots = -2\mathbf{w}_1 + 10\mathbf{w}_2 - 10\mathbf{w}_3 \\ A\mathbf{v}_2 = \cdots = -6\mathbf{w}_1 + 2\mathbf{w}_2 + 2\mathbf{w}_3 \\ A\mathbf{v}_3 = \cdots = -3\mathbf{w}_1 + 0\mathbf{w}_2 + 2\mathbf{w}_3 \end{cases}$ (where $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ denoted standard basis vectors for S10) which enabled him propose an incorrect matrix A'. Only after switching to the analytic-structural mode was he able to obtain the correct diagonal matrix representation as a result of computing $P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (in a manner similar to S6 and S7's approaches described above but without using MATLAB). It can be postulated that S10's willingness to make use of the standard basis vectors within the analytic-mode is perhaps what caused the emergence of an incorrect matrix A'.

The issue was not the fact that she managed to obtain the correct answer at the end; the issue was, rather, that it took S11 quite longtime to solve three sets of systems of equations. S11, who embraced the analytic-arithmetic mode in Task 3, began with the *representation approach* by explicitly writing $\begin{cases} A\mathbf{v}_1 = \cdots = -2\mathbf{w}_1 + 10\mathbf{w}_2 - 10\mathbf{w}_3 \\ A\mathbf{v}_2 = \cdots = -6\mathbf{w}_1 + 2\mathbf{w}_2 + 2\mathbf{w}_3 \\ A\mathbf{v}_3 = \cdots = -3\mathbf{w}_1 + 0\mathbf{w}_2 + 2\mathbf{w}_3 \end{cases}$. However, she failed to see the eigenvalue-eigenvector relationships, which led her to write and solve three sets of systems of linear equations (each with three unknowns and three equations).

Issues within the analytic-structural mode. S5, who embraced the analytic-structural mode in Task 3, began by defining the inverse of the transition (change of basis) matrix as $P^{-1} = \beta'$ (as opposed to the others who defined the transition matrix as $P = \beta'$). This way, she obtained $A' =$

$P^{-1}AP = \begin{bmatrix} 13 & -20 & 0 \\ 8 & -13 & 0 \\ 4 & -8 & 3 \end{bmatrix}$. Upon the interviewer's probing how she knew this was the correct matrix representation for T, she switched to the analytic-arithmetic mode (based on pure *representation approach* unlike S10 who blended *representation approach* and *object approach*) and obtained another incorrect matrix for T due to the involvement of the standard basis vectors like S10. Inconclusive, she felt the need to switch to the synthetic-geometric approach via which she ultimately was able to obtain the correct matrix representation as $A' = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Conclusion and Discussion

This report delved into mathematics students' interpretations of a classical problem in linear algebra, diagonal matrix representations of linear transformations with respect to eigenbases. All three modes of description in linear algebra appeared to have been adopted by the students; in some cases, students felt the strong need to switch between modes or utilize multiple modes of description. All students insistently questioned the meaning of having an eigenbasis such that the matrix representation for T relative to the given eigenbasis is diagonal. Using these multiple modes of description also enabled students acknowledge the importance of adopting a nonstandard basis with respect to which certain linear transformations revealed diagonal matrix representations. Some difficulties arose within each mode of description; notwithstanding, students addressed these difficulties by (i) paying close attention to the diagonalization structure; (ii) delving into the fundamental linear algebraic structures within and between modes description; and (iii) pondering deeply upon the desirableness of the diagonal matrix representation of the linear transformation under consideration.

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The Need for Conditional Reasoning Tasks with Mathematical Content

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This theoretical report argues that conditional inference is central to undergraduate mathematics, and that there is potential for using and adapting conditional inference tasks from cognitive psychology to study ways in which reasoning develops (or does not develop) with mathematical expertise. Specifically, developing mathematical expertise might improve conditional reasoning in mathematics only, or it might improve conditional reasoning in a way that transfers to abstract and/or everyday contexts; alternatively, it might develop conceptual understanding, so that people become better at conceptually valid mathematical inferences but more vulnerable to invalid responses typically seen in tasks with everyday content. This report argues that, to test and distinguish these possible developmental mechanisms, we need new conditional inference tasks with mathematical content. It also describes the first stage of a project designed to develop (and later use) such tasks, initial outcomes of which will be reported at the conference.

Keywords: logic, reasoning, conditional, if-then

Introduction

Conditional reasoning is central to undergraduate mathematics, especially in upper-level courses. Students learning to understand and construct proofs need to work in mathematically valid ways with conditional (if-then) statements. They need, for instance, to make standard interpretations in relation to implicit quantification. Mathematically experienced people tend to infer universal quantification, reading "If 3 divides x, then 6 divides x" as (something like) the proposition "For all $x \in Z$, if 3 divides x, then 6 divides x" and thus interpreting the statement as false (Hub & Dawkins, 2018; Solow, 2005). Less mathematically experienced people might not infer universal quantification, and reasonably claim that the truth value of "If 3 divides x, then 6 divides x" is undetermined because it depends on the value of x (Durand-Guerrier, 2003).

Importantly for this report, students also need to recognise that some inferences from conditional statements are valid and some are not. Four standard inference types – *modus ponens*, *denial of the antecedent*, *affirmation of the consequent*, and *modus tollens* – are illustrated below.

Modus Ponens (MP)
If x is less than 2, then x is less than 5;
x is less than 2;
Therefore, x is less than 5.

Denial of the antecedent (DA)
If x is less than 2, then x is less than 5;
x is not less than 2;
Therefore, x is not less than 5.

Affirmation of the consequent (AC)
If x is less than 2, then x is less than 5;
x is less than 5;
Therefore, x is less than 2.

Modus Tollens (MT)
If x is less than 2, then x is less than 5;
x is not less than 5;
Therefore, x is not less than 2.

With universal quantification assumed, MP and MT inferences are valid under the mathematically normative material interpretation of the conditional, in which a conditional statement is false if and only if its antecedent is true and its consequent is false (Velleman, 2006). DA and AC inferences are not valid, but endorsing them is natural if the conditional "if A then

B" is interpreted as the biconditional "A if and only if B" (Blockowiak, Castelain, Rodriguez-Villagra & Musolino, 2022).

In cognitive psychology, conditional inference has been studied extensively via tasks that systematically vary content across the four inference types (Oaksford & Chater, 2020). Some research uses everyday content with statements like "If Jenny turns on the air conditioner, then she feels cool" (De Neys, Schaeken & d'Ydewalle, 2005). Other research uses abstract content with imaginary letter-number pairs and statements like "If the letter is A then the number is 1" (Evans, Clibbens & Rood, 1995). Participants are typically asked whether or not they endorse inferences of the four types. With variations depending on factors such as believability of the conditional statements (Evans, Handley & Bacon, 2009) and availability of counterexamples (Cummins, 1995; De Neys, Schaeken & d'Ydewalle, 2003), people often endorse inferences that are not normatively valid.

Departures from normatively valid interpretations need not be problematic in everyday life, where we tolerate ambiguity and rely on context to infer intended meanings (Evans & Over, 2004). Indeed, reasoning research is much occupied with developing models that describe human reasoning more accurately than does normatively valid logic (Oaksford & Chater, 2020). However, departures *are* problematic in mathematics, which makes strict distinctions between inferences available from true conditional statements like "If x is less than 2, then x is less than 5", and those available from true biconditional statements like "x is even if and only if $3x$ is even". Failure to recognise invalid inferences can lead to making or accepting invalid mathematical arguments (Inglis & Alcock, 2012; Selden & Selden, 2003), so mathematicians care greatly about valid conditional reasoning.

Wider society cares about valid reasoning too, and mathematical education is valued partly because there is widespread belief that it develops logical reasoning skill in general (Inglis & Attridge, 2016). This is known as the *theory of formal discipline* and it is certainly plausible. Mathematics has clean conceptual boundaries (Vinner, 1991): a number either is or is not less than 2. So logical relationships in mathematics are clear-cut in a way that everyday ones are not. Consequently, studying mathematics – especially at advanced levels – might help people to become sensitised to logical structures and to transfer this understanding to general contexts. Empirical evidence on this process is, however, minimal and mixed, as described below.

Conditional Reasoning and Mathematics

Research indicates that, as the theory of formal discipline predicts, post-compulsory mathematical study does develop conditional reasoning toward normatively valid interpretations. However, it does so unevenly and imperfectly. Research in the UK (Attridge & Inglis, 2013) and in Cyprus (Attridge, Doritou & Inglis, 2015) found that among 16-18 year-old students, those studying mathematics intensively became better able to reject invalid DA and AC inferences in abstract conditional inference tasks. However, they still endorsed just under half of DA inferences and more than half of AC inferences (Attridge & Inglis, 2013). These students also became slightly more likely to reject normatively valid MT inferences (Inglis & Attridge, 2016), a finding in line with work showing that individuals with higher general intelligence scores and higher overall conditional inference scores tend, if anything, to endorse fewer MT inferences than those with lower scores (Inglis & Simpson, 2009; Newstead, Handley, Harley, Wright & Farrelly, 2004). This means that mathematical study seems to shift students away from naïve interpretations of conditionals but not toward a material interpretation; the shift is better understood as toward a *defective* interpretation in which a conditional statement is viewed as irrelevant when its antecedent is false (Attridge & Inglis, 2013).

Indeed, although individuals might tend to endorse conditional inferences in line with particular interpretations, this does not mean that they apply those interpretations consistently (Evans, Handley & Bacon, 2009). Reasons relevant to pedagogical design are evident in detailed qualitative research in undergraduate mathematics education. For instance, it would be a mistake to think that students considering conditionals with mathematical content are necessarily thinking about logic. Their reasoning might instead be built on content-specific and pragmatic factors (Dawkins & Cook, 2017), so that they benefit from experience designed to focus their attention on assigning truth values consistently across contents (Dawkins & Norton, 2022; Dawkins & Roh, 2022; Hub & Dawkins, 2018). Similarly, it would be a mistake to think that students understand the connections within and between conditional statements and the truth tables often used in teaching. Instead, they might treat the components of both statements and tables as separate entities, which prevents them from effectively reasoning about negation and about conditions under which statements would be false (Hawthorne & Rasmussen, 2015)

We see the impact of imperfect conditional reasoning in undergraduate students' evaluations of mathematical arguments: secondary students and undergraduates often fail to distinguish arguments for the converse of a conditional statement from those for the statement itself (Dawkins & Roh, 2022; Hoyles & Küchemann, 2002; Selden & Selden, 2003). Professional mathematicians do not make that mistake: they reliably identify and reject arguments that prove a converse (Inglis & Alcock, 2012). However, their conditional reasoning remains intriguingly imperfect. They do not, for instance, perform as well as might be expected on the Wason selection task (Wason, 1968). This task involves cards with a number on one side and a letter on the other: participants are shown cards labelled D, K, 3 and 7 and asked which should be turned over to determine whether the rule "every card that has a D on one side has a 3 on the other" is violated. Inglis and Simpson (2004) found that mathematicians largely avoided the common error of selecting D and 3, but fewer than half gave the normatively valid answer D and 7. This task involves evaluating the truth of a conditional rather than making inferences from a conditional that is assumed to be true, but the result is in line with the findings on mathematics students: mathematicians neglect the case corresponding to MT (if not-3 then not-D).

Unsurprisingly, given these findings, there is no standard way of teaching conditional inference. Mathematicians explicitly teach logical reasoning in textbooks (e.g., Houston, 2009; Velleman, 2006) and in introduction-to-proof courses (Davis & Zazkis, 2019), but there is debate on how to do this. Some, for instance, believe that teaching should leverage everyday reasoning and previous mathematical knowledge, others that everyday content muddies the waters and that mathematical education should focus on abstract logical structures (Alcock, 2010; Epp, 2003; Hawthorne & Rasmussen, 2015). Books designed to support students at the transition to proof offer short explanations of truth values in relation to conditional statements, perhaps emphasising the difference between a conditional statement and its converse and explaining why we consider a conditional to be true when its antecedent is false (e.g., Houston, 2009; Solow, 2005; Velleman, 2006). But they do not usually explain any research on everyday reasoning or acknowledge that professional mathematicians apparently need not be perfect logicians. If we wish to support students in developing mathematical expertise, there is pragmatic reason to investigate what that development might look like and where it ends up.

Possible Mechanisms and the Theory of Formal Discipline

Of theoretical interest is that although the theory of formal discipline seems to be partially valid, the mechanism by which it operates remains largely unexamined. The mechanism suggested in the Introduction requires that studying mathematics sensitises people to logical structures and that this understanding transfers to other contexts. This appears not to be fully correct: Inglis and Attridge (2016) suggested that mathematics students' lower endorsement of all three of DA, AC and MT inferences might result from their greater experience of error checking, which leads them to be more productively skeptical about all conditional inferences (see also Inglis & Simpson, 2004). But the larger point stands: if mathematical expertise makes people more productively skeptical, then those with more expertise should perform in similar ways across tasks with all types of content. However, mathematical study could promote different reasoning in mathematics without promoting transfer. If so, people with more mathematical expertise should perform more normatively on mathematical tasks but not on tasks with everyday content. In that case, abstract tasks make an interesting and potentially intermediate possibility: the evidence so far has used exactly these tasks, showing that mathematics students become better at rejecting DA and AC inferences (Attridge & Inglis, 2013). Performance with abstract content might represent the best that we can expect in terms of valid reasoning, or performance might be even better with meaningful mathematical content.

Alternatively, mathematical expertise might be less about sensitivity to logical structures or general skepticism, and more about deeper conceptual understanding: greater familiarity with mathematical examples, definitions and theorems might enable people to "understand the terrain" (cf. Goldenberg & Mason, 2008; Michener, 1978) and avoid errors by thinking about mathematical situations as they would about "real" situations. If so, people with more expertise might exhibit patterns of mathematical reasoning that reflect those seen for everyday content. For instance, conditional inference is known to be affected by believability. Individuals are less likely to endorse inferences of all four types when a conditional is less believable (Evans, Handley & Bacon, 2009). Similarly, conditional inference is affected by counterexample availability. For a causal conditional "If p then q", people are less likely to endorse DA and AC inferences if there are many *alternatives* that might cause q in the absence of p; they are less likely to endorse MP and MT if there are many *disablers* that might intervene to prevent p from causing q (De Neys, Schaeken & d'Ydewalle, 2005). People can usually think of several alternatives and disablers for "If the brake was depressed, then the car slowed down", for instance (Cummins, 1995). If greater expertise makes mathematics more "real", then it should be possible to manipulate believability and counterexample availability in mathematical content so as to induce everyday-like response patterns. This might support an account in which mathematicians do not really need formal discipline because expertise is primarily a matter of content knowledge.

These possible mechanisms for the theory of formal discipline – or for its relative unimportance – are not currently distinguishable because research has not examined conditional reasoning with mathematical content. Cognitive psychologists have almost exclusively used tasks with abstract or everyday content, even when studying mathematics students (Morsanyi, McCormack & O'Mahony, 2018) or investigating the theory of formal discipline (Attridge & Inglis, 2013). Research in mathematics education has been small-scale (Hub & Dawkins, 2018) or not systematically related to all four inferences (Hoyles & Küchemann, 2002). To understand how mathematical expertise develops conditional reasoning, and to relate that development to research in cognitive psychology, we need tasks with mathematical content.

From Theory to Empirical Research

The talk for this report will present the theoretical argument given above and describe the development of two new conditional reasoning tasks with mathematical content. (The larger project of which this work forms part will use these alongside tasks with abstract and everyday content to study how conditional reasoning develops with mathematical expertise.) The new tasks will both comprise conditional inference items of the form shown in the Introduction. One (MCB) will have basic mathematical content familiar to all educated adults; inferences will likely be from conditional statements such as "If x is less than 2, then x is less than 5". The other (MCA) will have advanced content familiar only to people with upper-level undergraduate mathematical experience; inferences will likely be from conditional statements such as "If $f: R \rightarrow R$ is differentiable at a then f is continuous at a". Both tasks will be structured using a standard *negations paradigm*, systematically varying negated elements across antecedents and consequents of conditional statements (see Attridge & Inglis, 2013; Evans, Handley, Nielens & Over, 2007).

Specific content for the MCB and MCA will be developed in Fall 2022. Mathematicians experienced in teaching proof-based courses will learn about factors known to affect everyday and abstract conditional inference; they will take these into account when generating potential items. This will not necessarily be straightforward for mathematical content. For believability, there are obvious possible manipulations: switching a true conditional like "If x is less than 2, then x is less than 5" for its false converse "If x is less than 5, then x is less than 2" might interfere with performance in predictable ways; switching it for a conditional with no obvious truth value, like "If x is less than 2, then y is less than 5" might render it more like an abstract task. For counterexample availability, the picture is more complex. It might be straightforward to affect DA and AC inferences: alternatives to "If x is less than 2, then x is less than 500" should be more accessible than alternatives to "If x is less than 2, then x is less than 5", for instance (see Hamami, Mumma & Amalric, 2021, for related results in geometric contexts). But the absolute nature of truth in mathematics makes it difficult to manipulate disablers that might affect MP and MT inferences: nothing could interfere to prevent x being less than 2 from "causing" x to be less than 5, and there is no readily inferable cause in "If x is less than 2, then y is less than 5". It will be interesting to see whether and how mathematicians attempt to address these issues, as well as whether and how they deal with implicit quantification.

With potential items in place, the mathematicians will collaboratively narrow down the list with the aim of constructing MCB and MCA tasks that mimic the systematic variation in tasks with everyday and/or abstract content. This will be accomplished using a Delphi-like process in which the mathematicians review, revise and prioritise one another's suggestions; individual interviews will capture the reasoning behind their suggestions, revisions and prioritisations. Later, the items will be tested with undergraduate participants to establish the extent of their comparability with standard abstract/everyday tasks. The talk for this report will present the outcomes of the Delphi process and the interviews: it will exhibit the items developed and present a qualitative analysis of the mathematicians' input in light of the theory outlined in this report.

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Re-envisioning a Value Framework for Examining Transformational Communities of Practice in Mathematics Education

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Communities of practice (CoPs) provide a structure that allows individuals with a common goal or purpose to come together to engage in collective learning (Wenger-Trayner & Wenger-Trayner, 2015). The COMmunity for Mathematics Inquiry in Teaching (COMMIT) Network was developed to support regional Math CoPs, called COMMITs, composed of undergraduate mathematics faculty interested in using active learning and inquiry teaching approaches in their courses. For the past three years, we have utilized the value framework, presented by Wenger et al. (2011), to better understand the layers of value faculty experience as they engage in COMMITs and the broader network. In doing so, we have identified modifications to the original model from Wenger et al. that may further interrogate and articulate the layers of value that can be used to help advance and sustain CoPs longterm. We present our theoretical adaptation in the following report.

Keywords: Community of Practice, STEM Faculty, Theories of Change, Value Creation Framework

In recent years, mathematics faculty at institutions of higher education have begun to focus on the use of evidence-based teaching practices, such as active learning and teaching with inquiry, to help support undergraduate student success. Despite mounting research that backs the use of such practices (e.g., Chen & Yang, 2019, Cook-Sather et al., 2016; Freeman et al., 2014; Healey et al., 2014, 2016; Werder & Otis, 2010), many math faculty still engage in traditional lecture style instruction, which is often less effective in supporting student success (Jaworski & Gellert, 2011). Attempting to integrate innovative teaching techniques in isolation can make sustained implementation challenging (Banta, 2003), therefore communities of practice (CoPs) offer a pathway to connect like-minded faculty and create a support system for instructional change. CoPs are defined as "groups of people who share a common concern or a passion for something they do and learn how to do it better as they interact regularly" (Wenger-Trayner & Wenger-Trayner, 2015). They help the group to identify common goals, develop and share resources and ideas, and engage in shared learning to foster transformational change.

Our research team is part of a grant funded project focused on supporting regional CoPs of undergraduate mathematics faculty as they attempt to implement active learning and teaching with inquiry strategies into their courses. The project developed the COMmunity for Mathematics Inquiry in Teaching (COMMIT) Network as a way to loosely connect each region to help the regional CoPs (COMMITs) mature and shift toward sustainability. We identified the value creation framework developed by Wenger et al. (2011) as a tool with which to examine the role of the COMMITs and the broader network in supporting faculty members in their efforts to implement instructional innovations in their courses. We have also examined value-add from the COMMIT level through regional leaders, to determine the facets of the COMMITs and the connecting network that best support all participants (including the leaders themselves) in their pursuit of instructional change. Our preliminary findings support the use of the value framework as a mechanism for exploring this work (Gomez Johnson, Jakopovic, von Renesse, 2021; Gomez

Johnson, Jakopovic, Rech et al., 2021). However, through our data analysis we have established several modifications to the original value framework that we hypothesize will more effectively describe the patterns emerging within value-added data at the CoP and network level. In this report, we share our use of the value creation framework, as well as our proposed modifications and the rationale behind the changes.

Teaching with Inquiry and the Role of Communities of Practice

There is an established body of research that shows how active learning and instructor effectiveness can positively impact students' learning, attitudes, and educational experiences (Freemen et al., 2014; De Vlieger et al., 2016; Laursen & et al., 2019). Incorporating active learning teaching strategies in STEM classrooms can impact students' academic success, overall attitude toward content, and their retention rates in STEM courses (Bowen, 2000), as well as lesson the achievement gap for students from historically underrepresented groups in STEM (e.g., people of color, women) (Freeman et al., 2014; Haak et al., 2011; Hrabowski & Henderson 2017; Laursen et al., 2014; Theobald et al., 2020). Active learning classrooms situate students as partners in the learning process with the instructor rather than acting as passive observers (Cook-Sather et al., 2016; Healey et al., 2014, 2016; Werder & Otis, 2010).

A survey by Rassmusssen et al. (2019) found that many mathematics departments in institutions of higher education report believing that active learning is "very important" or "somewhat important," however these beliefs translated to programmatic implementation only 15% of the time. Garnering buy-in from mathematics faculty can be challenging, as active learning practices challenge the historically accepted cultural norms of mathematics teaching. CoPs have the potential to serve as a vehicle for change (Gehrke & Kezar, 2017). Like other areas of social change, this requires disruption to systems that have been historically inequitable and act against and across existing boundaries (Hooks, 1994). For mathematics education, participation in praxis—engagement in reflection, learning, practice, and action—reduces the gap between theory and practice for faculty. In CoP structures, this allows members to defy norms about who belongs in a community or field, what is pursued, how it is organized, and how it works (Drane et al., 2019; Freire, 1968/2000; Hooks, 1994). CoPs can also give a voice to counter-narratives and people who have been historically excluded from stories and spaces (Solórzano & Yosso, 2002). Given the organic nature of CoPs, they transform and evolve at key points in their development or cease to be sustained if they no longer serve a useful purpose to individual members (Stuckey, 2004). For this reason, attention to the value-added aspects of participation at the COMMIT regional and network levels is important to not only examine long-term sustainability impacts, but CoP creation and implementation factors as well.

Theoretical Framework

We established our research design to study the COMMIT Network project as a community-based approach to instructional and institutional change using situated learning theory (Lave & Wenger, 1991; Wenger, 1998) as the theoretical perspective. Rather than focusing solely on the traditional conception of teacher/pupil learning, situated learning theory conceives of learning as something that occurs as individuals engage in CoPs and social networks (Lave & Wenger, 1991). CoPs involve intricate levels of interaction that can make examining the factors leading to the success or failure of these communities a challenging endeavor. The value creation framework positions CoPs within "a dynamic process in which producing and applying knowledge are tightly intertwined and often indistinguishable" (Wenger et al., 2011, p. 21). As Figure 1 illustrates, the original framework includes five cycles, or layers, of value creation—immediate (in the moment resources, information, connections), potential (for the future), applied (piloted implementation), realized (actualized implementation), and transformative (broad dissemination to others) value. One cycle does not necessarily lead to the next and the importance of the various cycles can differ for different stakeholders and at different points in the life of the CoP. Strategic and enabling value layers attend to the programmatic processes, structures, and ways of evolving and sharing value with others that support these five cycles.

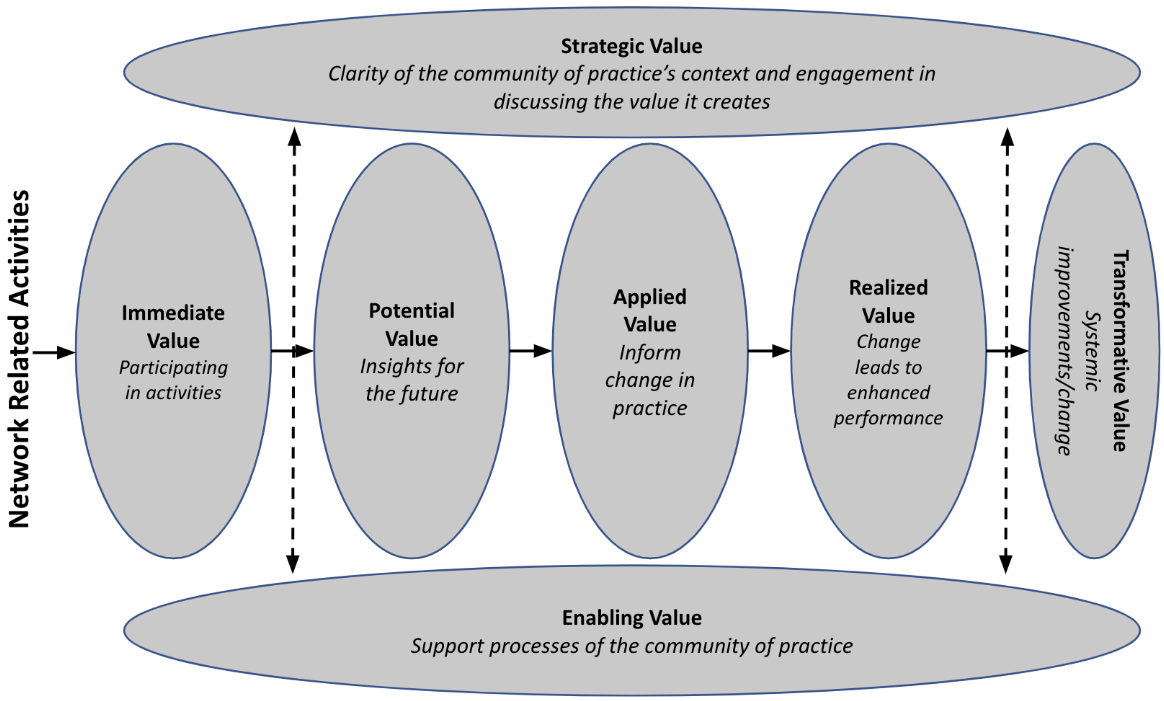

Figure 1. Value creation framework (Wenger et al., 2011).

The value framework in its original form has been utilized in a range of studies around CoPs. For example, Booth and Kellogg (2015) used the value framework specifically to examine value cycles within online CoPs. They identified labels for each type of value as follows: immediate value - productive activities, potential value - knowledge capital, applied value - promising practices, realized value - "return on investment" (i.e. seeing the payoff/benefit of the new knowledge implementation in practice). Clarke et al. (2021) applied the value framework to develop "ground narratives" that allowed them to identify themes within the broader community. In our project, we sought to better understand how the larger network of COMMITs might help to engage stakeholders and expand faculty engagement in the COMMITs. Therefore we adopted the value framework to investigate the experiences of COMMIT participants and leaders and follow the trajectory of the COMMITs themselves, within the context of the larger network. Through our ongoing analyses we gradually began to re-envision the framework. Specifically, we encountered nuances of the value layers emerging as we shifted away from looking at the value of individual participants and toward identifying the value the COMMIT Network afforded regional leaders and their CoPs. In the following sections, we present the findings from our research that support the need for the hypothesized revision to the value framework at the network level value.

Research Methodology

Data analysis is an ongoing, recursive process that involves the examination, interpretation, and reinterpretation of data and findings (Patton, 2002; Richards, 2009). Design based research positions theory development and refinement within the complex, social contexts within which interventions are examined (Brown, 1992; Tabak 2004). In this section, we report on the findings from our research as a way to illustrate both the need for and envisioning of the hypothesized value framework revisions, as this process is intrinsically embedded into the iterative nature of qualitative research design.

Context

Our research on the COMMIT Network is part of a project funded by a National Science Foundation grant (No.1925188). The authors of this report are researchers on the grant leadership team, initially tasked with gathering and analyzing data to better understand how the COMMIT, supported by the network, could recruit, develop, and retain mathematics faculty use of active learning and inquiry-based teaching practices. During the initial two years of the grant, our research focused on identifying the layers of value that individual COMMIT faculty members self-reported after engaging in COMMIT events and activities. At the end of Year 2 of the grant, we gathered additional data from members of the regional COMMIT leadership teams in an effort to expand our examination of value beyond individual participants to the regional COMMIT and COMMIT Network levels.

Participants and Data Collection

Midway through Year 2 of the project, we surveyed individual CoP participants representing a total of five regional COMMITs in the Math COMMIT Network to gather a first round of data. Participants in this initial data set included responses from 156 faculty members who participated in COMMIT workshops and events in their region. The participants included faculty from a broad range of institution types. Coverage of survey responses represented faculty from K-12 schools (4%), two-year colleges (10%), doctoral granting institutions (18%), and predominantly undergraduate institutions (68%). We created an online survey that included a combination of multiple choice and open-ended follow up questions for participants to identify the value, if any, they found through engaging in COMMIT activities and to what extent they planned to use ideas from these events in their teaching practice.

At the end of Year 2, we used social network analysis to identify leaders within each of the regions (Gomez Johnson et al., 2021b). We then gathered a second phase of data, conducting a total of 19 semi-structured interviews with COMMIT leaders representing eight of the regions in Spring 2021. The interviews focused on the experiences of leaders as individuals, as well as representatives of their regional COMMIT. We collected and analyzed the interview data to help us begin to understand the collective value these CoPs identified as a result of engaging in the COMMIT Network. The analysis of this second data set became the area of focus in our re-imagining of the value framework, as we shifted from talking about the value found by individuals within their group to the value the COMMITs found within the Network. The analysis of this second data set led us to uncover more complex, nuanced sub-layers of value that were not completely captured within the original value framework.

Data Analysis

Our initial analysis of the COMMIT participant surveys involved first deductively coding the open-ended survey prompts for instances where faculty identified any immediate, potential, or transformative value they found by engaging in COMMIT activities. In the second round of coding, we used *a priori* codes where we identified responses that aligned with one of the Four Pillars of inquiry based teaching: 1) students engage deeply with coherent and meaningful mathematical tasks, 2) students collaboratively process mathematical ideas, 3) instructors inquire into student thinking, and 4) instructors foster equity in their design and facilitation choices. (Laursen & Rasmussen, 2019). We determined that these four codes did not completely capture the essence of all participant responses, therefore we conducted a third round of descriptive coding (Miles et al., 2014; Saldaña, 2021) where we identified additional emergent codes in the data. These codes included resource sharing, ideas for technology integration, forms of assessment, evolving beliefs about teaching mathematics, and the usefulness of the COMMIT Network. Participants regularly self-reported finding immediate (in the moment), potential (for future use), and transformative (worth sharing with others) value around these five topics (Gomez Johnson, Jakopovic, von Renesse, 2021). Reflecting on this first set of findings, we determined that, for this data set that focused on the experiences of individual participants, the five layers of value that are described by Wenger et al. (2011) appropriately captured the essence of participant responses.

With the semi-structured COMMIT leader interviews, we began in a similar fashion, coding in a first pass deductively for evidence of the five layers of value. As we coded for transformative value, we identified the need for a distinction between enacted transformative value (dissemination and change that has taken place) and the "potential to transform" (future possibilities to promote teaching with inquiry to the broader community). Therefore, we inductively added a sixth code. This came about due to leaders' reflections on the value they identified the COMMIT Network providing to their regional COMMITs (versus identifying the value-add for individual participants). At times, leaders shared the ways in which they envisioned taking steps toward transformative action in their COMMIT, but these were phrased as future aspirations and ongoing goals rather than work currently happening within the region (Jakopovic & Gomez Johnson, 2021).

Reframing the Value Creation Framework

As we reflected on this need for an additional code to more accurately capture the essence of value creation at the CoP level, we realized there is a potential connection between the unit being studied (individual participant or COMMIT leader/representative) and the ways in which "transformative value" manifests itself. With faculty participants, their descriptions of "potential value" encompassed the ways in which their participation in the COMMIT Network and regional COMMITs might influence their future, individual teaching practice. Their responses also included ideas as to how they might "transform" teaching by sharing this information with close colleagues or peers. When we analyzed the COMMIT leader interviews, however, they identified opportunities where the Network provided ideas and supports that have the potential to create transformative value within and across COMMITs. For example, one regional leader explained,

> I think we have a lot to offer. I think especially because we might be ahead, at least in the idea of [our cultural renaissance], There is a huge focus on respecting [our] culture. Even in the aspect of math, that can be shared in the broader network nationwide and reaching

> out to Native American/first nations to collaborate and share ideas. [Our region] has had this ability because of the renaissance of [our] culture happening in the 70s, we are a little further along that we can share, not even inquiry based, but also focus on the place where you are and the people you are teaching.

A leader from a different region shared similar comments with regard to ways their COMMIT had the potential to support other COMMITs, stating, "We have been trying to learn from the others in the network and what they are doing. It is nice to figure out what they are doing. It has been inspiring for us. We hope to contribute like that in the future. I would hope that it's reciprocal." These recurring observations based on our coding and analysis evolved as we made further shifts from studying individuals to COMMIT leaders and have led us to hypothesize a reimagined visualization of the value creation framework, as shown in Figure 2.

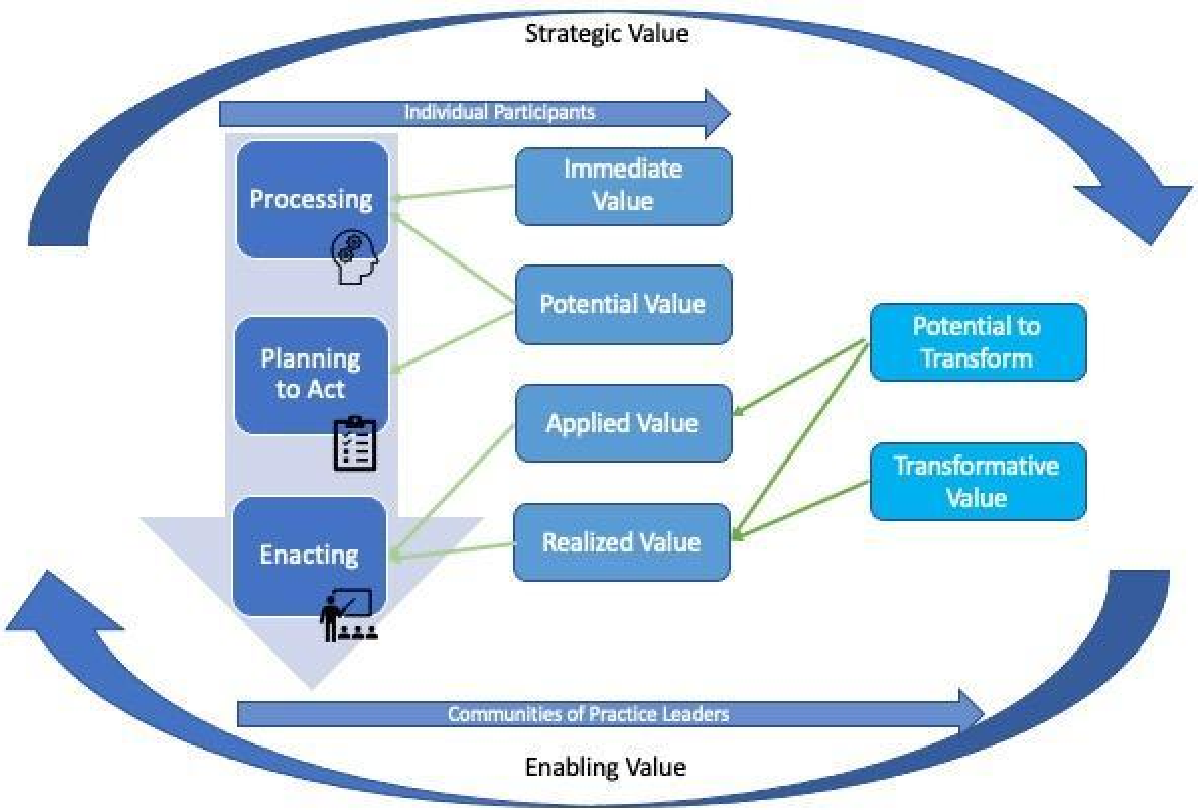

Figure 2. Value creation for community of practice leaders.

We realized that the ways in which value is created through the COMMIT Network for faculty participants is different from that of COMMIT leaders. In the figure above, we attempt to articulate our vision of this hypothesis. The five layers of value creation all lead participants to process and consider new ideas, to make plans and imagine acting on the new information, and eventually to enact these plans in their practice. For individual faculty members, this most often encompasses the first four layers of value: immediate, potential, applied, and realized. They consider new ideas, plan for them, and then implement them in their mathematics courses. They may consider sharing these ideas with others, yet in our findings this typically happened due to the support of their COMMIT and the larger COMMIT Network through established structures like regular meetings, book clubs, and online workshops.

In our consideration of the value COMMIT leaders experienced in their COMMITs and in the Network, however, we overwhelmingly saw more evidence of their plans to share at the regional and network level in addition to the ways they identified their COMMITs as currently acting to share and transform with other COMMITs in the Network.

Implications and Conclusion

To date, we have found the value creation framework first established by Wenger et al. (2011) to be an invaluable tool for studying the potential impacts of STEM CoPs on instructional transformation, particularly at the individual participant level. CoPs are meant to meet the needs of individual participants in a specific area or domain, however, evolution and sustainability are often a challenge once their needs are met (Wenger et al., 2002). Knowing the contextual factors impacting CoPs, we recognize that the ways in which value is defined shifts when moving from studying individual participants to regional COMMITs and broader network levels. Our goal moving forward is to continue refining the model based on new and additional data analysis, to more clearly define how leaders interact with their COMMITs and drive not only transformation of practice but also sustainability of the community as a resource. We believe this work can inform others who are studying COMMITs and sustainability, particularly in subject-specific areas, such as mathematics and more broadly STEM education.

Additional studies in different contexts and subject areas can contribute to our developing understanding of CoP sustainability within the framework of value-add. For example, research is currently underway with other STEM CoPs to further examine if and how this work may be similar or different based on variables such as geographic location, participant demographics, etc. This additional research within this project and other STEM projects is needed to help generalize our findings and proposed model. Our team plans to continue our investigation in this area, with future work focused on considering how strategic and enabling value (see Figure 1) may also need to be reimagined when shifting from the value a regional CoP can provide individuals to the value the CoP can find within a broader network structure.

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Characterizing a Teacher's Ways of Thinking about Teaching the Idea of Sine Function

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Over the past thirty years, researchers have developed constructs to investigate teachers' knowledge base and how it relates to their teaching. These constructs include Mathematical Knowledge for Teaching (MKT), Mathematical Meanings for Teaching (MMT), and Key Pedagogical Understandings (KPUs). In this report, I describe each of these constructs, their uses, and how they are related. I then describe a recently introduced construct, Ways of thinking about Teaching an Idea (WTTI) (Carlson, Bas-Ader, O'Bryan, & Rocha, in press) and illustrate the usefulness of this construct for investigating a teacher's mathematical understanding of sine function and how this understanding is related to the teacher's instructional goals and actions.

Keywords: Mathematical Knowledge for Teaching, Meanings, and Ways of Thinking

Introduction

Mathematics educators have asked the question, "what do teachers need to know to teach students?" for quite some time. In the late 1800s, debates about teachers' knowledge primarily focused on teachers' experience with the subject matter (Shulman, 1986). A century later, emphasis on teachers' subject matter knowledge lost traction as new policies concerning teachers' evaluation and testing were implemented. During the 1960s and 1970s, research programs shifted their focus to teachers' pedagogy with little regard for teachers' content knowledge. Shulman (1986) described the education community's disregard for teachers' content knowledge as the "missing paradigm problem" and responded by proposing that researchers "blend properly the two aspects of teachers' capacities" (p.5) and begin focusing on teachers' pedagogical content knowledge, a type of knowledge that "goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (p.6). Following Shulman (1986), teachers' knowledge in the context of teaching became a focus of mathematics education research. This area of research on teachers' knowledge is now known by many as *Mathematical Knowledge for Teaching (MKT)*(Thompson & Thompson, 1996).

Mathematical Knowledge for Teaching

In the last three decades, researchers have turned their attention to teachers' MKT to improve the quality of mathematics teaching and learning in the United States. While the literature on teachers' MKT is vast, various research groups' foci and ontological stances differ significantly.

A Practice-Based Theory of MKT

Ball and colleagues (e.g., Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2005) sought to answer the question, "what do teachers do in teaching mathematics, and in what ways does what they do demand mathematical reasoning, insight, understanding, and skill?" (Ball, Hill, & Bass, 2005, p. 17). Ball's work on teacher knowledge began in 1990 when she studied 252 pre-service teachers' understanding of division with fractions. In this study, most teacher candidates could correctly complete the procedures of dividing fractions. However, few were able to describe a situation that represented the division of fractions accurately to students (Ball, 1990).

Following Ball's (1990) findings of teachers limited mathematical understandings, Ball and Bass (2003) proposed focusing on the mathematical work of teachers, including how and where teachers use their mathematical knowledge in practice. Ball and Bass' analyses revealed that the mathematical knowledge needed for teaching differs from that of mathematicians and other practitioners of mathematics. Namely, mathematics teaching requires teachers to have knowledge of (1) the work involved in problem-solving, (2) how to "unpack" their mathematical understandings, and (3) how to help students in making connections across mathematical domains (ibid). These empirical analyses lead to Hill, Schilling, and Ball's (2004) proposal that teachers' MKT is multidimensional, and Shulman's (1986) original category of subject matter knowledge be subdivided to include teachers' *common content knowledge (CCK)* and *specialized content knowledge (SCK*).

In the past twenty years, these researchers have explored the multi-dimensional nature of teachers' knowledge when teaching and classified teachers' MKT into five types of specialized knowledge (Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2005). These researchers have also identified positive links between teacher knowledge and student performance (Ball, Thames, & Phelps, 2008; Hill & Ball, 2004; Hill, Ball, & Schilling, 2008; Hill, Schilling, & Ball, 2004, Hill et al., 2008).

A Piagetian Theory of MKT

While Ball and colleagues (e.g., Ball, 1990; Ball & Bass, 2003; Ball, Hill, & Bass, 2003; Hill, Rowan, & Ball, 2005) primarily focused on identifying how teachers must know mathematics to teach it, Silverman and Thompson (2008) focused on the nature of teachers' MKT and how it develops. Silverman and Thompson (2008) made two propositions, (1) teachers' MKT is grounded in personally powerful understandings of mathematical concepts, and (2) teachers' MKT is created through the transformation of those concepts from an understanding having pedagogical potential to an understanding that has pedagogical power.

Mathematical understandings are personally powerful if they carry throughout an instructional sequence, are foundational for learning other ideas, and play into a network of ideas that do significant work in students' reasoning (Thompson, 2008). According to Silverman and Thompson (2008), *Key Developmental Understandings (KDUs)* (Simon, 2006) are an example of mathematical understandings that are personally powerful. A KDU is an individual's understanding of an idea that is critical to their learning of other related ideas (Simon, 2006). We say an individual has developed a KDU for a mathematical idea if they (1) make a conceptual advance or have "a change in [their] ability to think about and/or perceive a particular mathematical relationship" (Simon, 2006, p. 362) and (2) develop an understanding for the idea through their own activity and reflection on their activity.

Silverman and Thompson (2008) propose that "developing MKT involves transforming [personal] KDUs of a particular mathematical concept to an understanding of: (1) how this KDU could empower their students' learning of related ideas; (2) actions a teacher might take to support students' development of it and reasons why those actions might work" (p. 502). More concretely, the development of MKT involves transforming teachers' KDUs into *Key Pedagogical Understandings (KPUs).* A KPU is a mini theory that a teacher has regarding how to help students develop the meanings she intends (Byerley & Thompson, 2017). As such, these researchers developed a framework to describe the process by which teachers' MKT-their KDUs and awareness of their KDUs- develop.

Comparing Research on Teachers' MKT

There are two distinguishing characteristics in research programs focused on teachers' MKT. The first involves the focus of each program. Ball and Colleagues' investigations primarily focused on what teachers do in their teaching and the knowledge required to engage in these teaching actions (Ball & Bass, 2003). In contrast, Silverman and Thompson (2008) focus on the cognitive mechanisms that enable teachers' behaviors. These researchers seek to uncover the nature of teachers' knowledge and how it develops. According to Silverman and Thompson (2008), the second distinguishing feature of MKT research involves their ontological stance that an individual's knowledge is constructed and is idiosyncratic. Ball and colleagues (e.g., Ball 1990, Ball & Bass, 2003, Ball, Hill, & Bass, 2005) describe teachers' knowledge in the sense of a truth about a reality external to the knower. In comparison, Silverman and Thompson describe teacher knowledge in the sense of Piaget and von Glasersfeld (1995)- as schemes and ways of coordinating them that explain how a person might act when teaching. Moreover, Silverman and Thompson describe a teacher's knowledge base in terms of her individual cognitive structures and thought patterns. Silverman and Thompson use the phrase *Mathematical Knowledge for Teaching (MKT)* to describe teachers' schemes or meanings for the ideas they teach and hold at a reflected level.

Mathematical Meanings for Teaching

In 2016, Thompson proposed using the construct *Mathematical Meanings for Teaching* (MMT) rather than MKT to make explicit that he was using the word *knowledge* to describe an individual's schemes or meanings for an idea. An individual's meaning is the space of implications resulting from assimilation to a scheme (Thompson, 2016). Thus, to say an individual has a *meaning* for a word, symbol, expression, or statement means that the individual has assimilated that word, symbol, expression, or statement to a scheme. A scheme is a mental structure that "organize[s] actions, operations, images, or other schemes" (Thompson et al., 2014, p. 11). When defining *Mathematical Meanings for Teaching*, Thompson (2016) extended the construct *mathematical meaning* to account for characterizations of teachers' actions related to teaching. As such, a teacher's mathematical meanings for teaching an idea include (1) the meanings (schemes) the teacher uses while teaching or thinking about teaching, and (2) the teacher's image of the meanings (schemes) they want students to develop for the idea.

The construct *Mathematical Meanings for Teaching* differs from the construct *Mathematical Knowledge for Teaching* in two distinct ways. First, Thompson's use of the word *meaning* connotes something personal (i.e., meanings existing in the mind of the knower) to readers rather than *knowledge*, which seems less personal and disjoint from the knower (Thompson, 2016). Second, and arguably most important, the construct *Mathematical Meanings for Teaching* is used to describe the *status* of an individual's understandings relative to teaching. Although Ball and Thompson defined teacher's MKT differently, both researchers used the construct MKT to describe teacher knowledge as a target for teachers to *achieve*, not a *state* of teacher's knowledge. For instance, Ball, Hill, and Bass (2005) use the example of multiplying two integers to demonstrate that a teacher's sole ability to compute 35 x 25 correctly is insufficient knowledge. These researchers argue instead that teaching involves knowledge of an effective way to represent the meaning of the algorithm to multiply. In this example, Ball, Hill, and Bass (2005) state what teachers must be able to do and know, but they do not make explicit the ways of thinking that enable the teacher to do or know. As a second example, Silverman and Thompson (2008) outline a framework for developing teachers' knowledge that supports conceptual teaching of a particular mathematical topic. This framework frames teachers' MKT as something that must be attained as opposed to a status of teachers' knowledge that may be advanced.

The Affordances of Attending to Teachers' MMT

Attending to teachers' mathematical meanings for teaching an idea has many affordances. Investigations into teachers' meanings enable researchers to focus their attention on the mathematical conceptions teachers have and the implications of these understandings on students' learning. In particular, focusing on teachers' MMT enables researchers to shift their focus from the behaviors that a teacher exhibits while teaching to the cognitive mechanisms, meanings, and ways of thinking that enable these behaviors. This shift in focus to teachers' meanings allows mathematics educators to explain why teachers act as they do and how we can improve their teaching (Thompson, 2013; 2016).

As one example, Thompson (2013) described his observation of a ninth-grade teacher's lesson on the point-slope and point-point formulas. In this example, the teacher can write an equation of a line through a point when provided the slope. However, when provided two points and asked to write an equation of a line through the two points, the teacher is unable to do so. Thompson's (2013) efforts to make sense of and characterize the instructor's meaning for constant rate of change led to him uncovering that the instructor's difficulty resided in her schemes for variation, slope, division, and rate of change. As one example, the instructor expressed a meaning for slope as "rise over run" where rise and run represented "two things changing in chunks" (Thompson, 2013, p. 81). With this way of thinking, the instructor could not determine the y-intercept as her "chunk" did not place her at *x=0*.

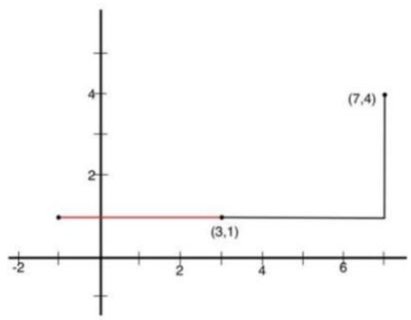

Figure 1. An example provided in Thompson (2013, p. 81)

By investigating instructors' meanings for ideas, we, as mathematics educators, are positioned to enhance our understanding of teachers' practice and develop ways to improve it.

While few mathematics educators have investigated teachers' MMT, the results of the few are alarming. Thompson and colleagues (e.g., Thompson, 2016; Byerly & Thompson, 2017; Yoon & Thompson, 2020; Thompson & Milner, 2018) have repeatedly uncovered the unproductive nature of U.S. mathematics teachers' mathematical meanings for teaching. Yoon and Thompson (2020) have also shown that U.S. teachers' meanings for foundational mathematical ideas are less productive than their South Korean counterparts. The South Korean teachers' conveyance of productive meanings implies that it is reasonable to expect U.S. teachers to be able to construct and convey mathematical meanings in this way (ibid).

One approach for improving US students' mathematical understandings is to support US teachers in advancing their mathematical meanings. Musgrave and Carlson (2017) have demonstrated that teachers' meanings for a foundational mathematical idea can become more productive if these teachers are engaged in interventions designed to support teachers' development of coherent mathematical meanings. "A focus on MMT would also foster the field's conceptualization of bridges among what teachers know (as a system of meanings), how they teach (their orientation to high-quality conversations), what they teach (meanings that an observer can reasonably imagine that students might construct, over time, from teachers' actions), and what students learn (the meanings they construct)" (Thompson, 2013, p. 82).

The Need for a New Construct

Although attention to teachers' mathematical meanings for teaching an idea is needed to support teachers' development of coherent mathematical meanings; researchers who have investigated teachers' MMT have claimed that teachers' development of strong mathematical meanings does not necessarily lead to their conveying these meanings to students (Carlson, O'Bryan, & Rocha, 2022; Rocha, 2021; Tallman & Frank, 2018; Tallman, 2021). Instead, scholars have proposed that teachers' *ways of thinking* are critical to their ability to develop, advance, and convey coherent meanings to students (Carlson, O'Bryan, & Rocha, 2022; Carlson et al., in press; Musgrave & Carlson, 2017; O'Bryan & Carlson, 2016; Rocha & Carlson, 2019; Rocha, 2022; Tallman, 2015; Tallman & Frank, 2018). As one example, these researchers have proposed that quantitative reasoning is a critical *way of thinking* that supports teachers' in (1) developing coherent mathematical meanings, (2) conveying coherent and consistent meanings to students, and (3) designing activities that create opportunities for students to engage in quantitative reasoning (ibid).

The nature of a teacher's decentering actions has also been identified as a mechanism for advancing teachers' MMT and their ability to convey coherent meanings to students while teaching (Carlson & Bas Ader, 2021; Carlson et al., in press; Rocha & Carlson, 2019; Teuscher, Moore, & Carlson, 2016). In particular, researchers have argued that decentering or setting aside one's thinking to understand what students understand (Steffe & Thompson, 2000) is useful for explaining and characterizing teachers' responses and interactions with students while teaching (Carlson & Bas Ader, 2021; Carlson et al., in press; Rocha & Carlson, 2019; Tallman & Frank, 2018; Teuscher et al., 2016). As one example, Teuscher et al. (2016) propose that a teachers' ability to decenter is related to the nature of their interactions with students and the development of their mathematical meanings for teaching. Teuscher et al's., (2016) empirical study revealed that teachers' explanations advanced to become more conceptually oriented as the instructors made their speaking and listening to others a specific object of focus and reflection. In 2019, Rocha and Carlson corroborated Teuscher et al.'s (2016) proposal by providing empirical evidence of the reflexive relationship between teachers' meanings and their decentering actions. Rocha and Carlson's' (2019) analysis of an instructor's teaching revealed that the teacher's actions to decenter advanced her MMT by providing the teacher with more refined images of a student's ways of understanding the sine function.

It is relevant then to describe mechanisms for advancing teachers' MMT in ways that support their development of ways of thinking and decentering actions that support the instructor in developing images of students' thinking about an idea and leveraging those images while teaching. Carlson et al. (in press) have classified images of teaching that include ways of thinking about students' ways of learning an idea as a *way of thinking about teaching that idea*. In the next section I illustrate the alignment of these images with an instructor's instructional goals and actions.

Ways of Thinking about Teaching an Idea

The construct *Ways of Thinking about Teaching an Idea (WTTI)* was introduced by Carlson et al. (in press). These researchers leveraged Harel's construct of *way of thinking* to describe a "habitual form of reasoning that governs the application of a variety of specific mathematical schemes" (Thompson et al., 2014, p. 12). A teacher's *Way of Thinking about Teaching an Idea* (WTTI) includes:

1. How the teacher reasons about and understands the idea (the teacher's MMT which includes the ways of thinking the teacher engages in when reasoning about the idea) (Thompson, 2016).
2. The teachers' images (second order models) of students' thinking about and learning that idea (Steffe et al., 1983; Thompson, 2000).
3. The teachers' image of ways of thinking students may engage in to develop and refine their understanding of an idea (ways of learning an idea) (Carlson et al., in press; Silverman & Thompson, 2008).

Carlson et al., (in press) claim that a teacher's WTTI becomes more connected and refined as the teacher engages with students, while attempting to model students' thinking. Repeated efforts to do so typically results in refinements of a teacher's MMT and their images of students' way of learning an idea. The construct WTTI is useful for describing the relationship between teachers' MMT, their ways of thinking about learning an idea, and their actions while teaching. Researchers have demonstrated links between a teacher's MMT and how a teacher interprets and responds to students' questions (decenter) (Carlson et al., in press; Rocha & Carlson, 2019; Tallman, 2015, 2021). Based on these claims of the symbiotic relationship between a teacher's MMT and their decentering actions (Carlson et al., in press; Rocha & Carlson, 2019; Tallman, 2015, 2021), there is a need for research to investigate teachers' decentering actions and the impact of these actions on advancing teachers' MMT and ways of thinking about learning and teaching an idea.

An Example: A Way of Thinking about Teaching Sine Function

A teacher's WTTI is a status of their current meanings and ways of thinking about teaching and learning. As such, a teacher's WTTI can be more or less productive for teaching. Below I present an example of a precalculus teacher's current ways of thinking about teaching sine function. The data presented comes from a clinical interview with an instructor, Uri, who (1) was in his second semester of teaching pre-calculus using a research-based curriculum, and (2) regularly attended a weekly professional development seminar designed to support instructors in engaging in conceptually oriented teaching (Thompson & Thompson, 1996).

During the clinical interview, Uri was asked to describe how he wants students to reason about what the values in the table below (see Figure 2) represent. Uri responded by stating that he wanted students to understand that "sine is a function that relates an angle measure with the height [of the terminal point] above the horizontal diameter". Uri further conveyed that he wanted students to view the value of $\sin(\theta)$ as a means of representing the terminal point's height measured in units of the circle's radius. For example (see Figure 2), when prompted to explain the meaning of the point (337° , -0.391) Uri expressed that "the table of values is a snapshot, like if we were to pause an animation, at these instances in angle measure, then this [-0.391] would be the corresponding height above the vertical diameter in radius lengths."

As Uri described the meanings for $\sin(\theta)$ he wanted students to develop, he also conveyed that he wanted students to view sine as representing how the terminal point's height above the horizontal diameter varies with the measure of an angle. When Uri was asked if his prior teaching of sine function has informed his preparation for teaching the same topic during the current semester, Uri conveyed that his previous students' conception of the value of sine as a multiplicative comparison between the terminal point's vertical distance and the circle's radius supported his students' later learning of right triangle trigonometry. In particular, Uri expressed that his former students’ conception of the output of the sine function as a multiplicative comparison of two quantities’ values supported them in recognizing where the commonly applied trig ratios (SOHCAHTOA) come from.

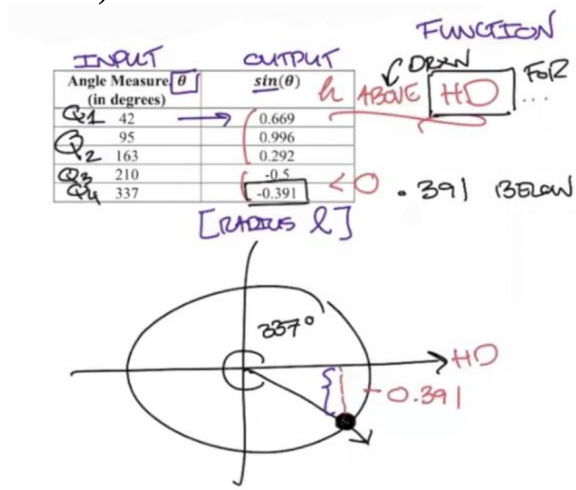

Figure 2. Uri’s Description of Sine Function

Uri’s description of the meanings he hopes to support students in developing for sine function provides a window into Uri’s way of thinking about teaching sine function. In this example, Uri expressed (1) a meaning for sine function that he intended to support students in developing, (2) ways of thinking (quantitative and covariational reasoning) that he anticipated would support students’ development of this understanding, and (3) images of student thinking and ways of learning that appeared to influence his instructional goals and actions.

Discussion and Conclusion

In the past thirty years, researchers have used the constructs Mathematical Knowledge for Teaching and Mathematical Meanings for Teaching to describe teachers’ knowledge base while teaching. While research on teachers’ knowledge base is vast, scholars’ use of these constructs varies greatly. The constructs MKT and MMT differ in the nature of the phenomena they describe. Researchers have used the construct MKT to describe a knowledge base as a target for teachers to achieve. In contrast, the construct MMT has been used to describe the current state of teachers’ mathematical understandings relative to teaching. While many researchers have advocated for more attention to teachers’ MMT, a sole focus on advancing teachers’ MMT is insufficient for explaining a teachers’ instructional planning and actions. Instead, a focus on a teacher’s *ways of thinking about teaching an idea* (that includes teachers’ MMT) may provide a more powerful lens for studying a teacher’s instructional goals and teaching of a specific idea. This report provides an example of how a teacher’s *ways of thinking about teaching* the idea of sine function are related to his instructional goals and actions. In particular, this report highlights the critical role a teacher’s mathematical meanings for teaching sine function and ways of thinking about learning and teaching sine function have in the instructor’s planning for teaching. The example provided in this report provides empirical support for Carlson et al.(in press) call for the design of professional development seminars focused on developing and advancing teachers’ WTTI. Seminars with this focus may support instructors’ enactment of teaching practices that focus on understanding and advancing student thinking and learning. As such, further exploration of the role of teachers’ WTTI on their enacted teaching actions and mechanisms for developing and advancing teachers’ WTTI are needed.

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Explicating the Role of Abstraction During Analogical Concept Creation

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Recent research has explored the potential for students to create new mathematical concepts by analogy with previously known concepts. However, there is more to be understood about the intricate processes of analogical reasoning that students leverage when creating new concepts. In this paper, we describe an initial theory explicating the role of abstraction during analogical concept creation. First, we introduce the abstracted domain to operationalize abstraction during analogical reasoning. We then describe two mechanisms of analogical concept creation through which abstraction may occur while reasoning about a ring-theoretic analogue to subgroups: (1) comparative analogy, and (2) inductive analogy. To broaden the interpretive scope of the theory, we further apply the mechanisms to scenarios involving meta-analogical reasoning and pedagogy. Implications for research and teaching mathematics with analogy are discussed.

Keywords: Abstract algebra, Abstraction, Analogical reasoning,

Analogical reasoning typically involves the comparison of two previously given or known concepts in search of relational similarity between the two (Gentner, 1983; Holyoak & Thagard, 1989). This forms the basis for many pedagogical analogies that are used in mathematics courses (Greer & Harel, 1998). For example, the concept of projection of a vector onto a general subspace is often presented with analogous references to spatial examples of a geometric vector projected onto a two-dimensional Euclidean plane, the goal being to ground students' knowledge of the abstract concept within a more accessible one.

While pedagogical analogies are often leveraged as a tool for reinforcing previously introduced concepts rather than creating new ones (e.g., Cobb, Yackel, & Wood, 1992), recent research has explored the potential for students to create new mathematical concepts, thereby expanding the scope of productive pedagogical analogies in undergraduate mathematics classrooms to allow for students to establish new mathematics for themselves. For example, it has been shown that students can explore, define, and create new concepts in ring theory by analogy with known concepts in group theory rather than simply reinforce their previous understandings (e.g., Hejný, 2002; Hicks, 2022; Stehlíková & Jirotková, 2002). This expanded view of pedagogical analogies in mathematics broadens the horizons for what pedagogical analogies can accomplish in advanced mathematics classrooms. However, to promote more productive pedagogical analogies in advanced mathematics, we must first better understand the intricate processes of analogical reasoning that students leverage when creating brand new (to them) concepts by analogy.

One avenue for deepening our understanding of analogical concept creation is to explore the role of abstraction within the process of analogical reasoning. Prior research in mathematics education has touched on the presence of abstraction in analogical reasoning (e.g., English & Sharry, 1996), although this research has not focused as heavily on cases in which students are generating new concepts or defining structures through analogy, focusing instead on developing conjectures to be verified (e.g., Lee & Sriraman, 2011). As such, the focus of this report is to develop an initial theory for explicating the role of abstraction during analogical concept creation. We are guided by the following overarching question: What is the role of abstraction during analogical concept creation?

Theoretical Underpinnings

The typical framework of analogical reasoning strives to capture very specific or 'perfect' representations of analogy rather than interpret potentially imperfect reasoning. In contrast, our proposed theory builds off the Analogical Reasoning in Mathematics (ARM) framework (Hicks, 2020) to interpret analogical activity that is specific to mathematics and may not align with usual or perfect cases of analogical reasoning. Analogies are viewed as series of mappings between a (typically) known source domain and a (typically) unknown target domain (Gentner, 1983), while analogical reasoning is the search for underlying structural relation between domains. A key component of the ARM framework is the adoption of the Actor-Oriented (AO) perspective (Lobato, 2012) to interpret analogical reasoning which allows for non-standard and idiosyncratic cases of analogical reasoning. During novel concept creation by analogy, immediate analogies are often impractical and instead require effort to refine the analogy. Thus, we draw upon the notion of *progressive alignment* (Gentner & Hoyos, 2017) to describe the process of developing an analogy through multiple analogical inquiries. In other words, we allow for the possibility for analogies to be incomplete at a given moment in time and consider them to be 'under construction' during analogical reasoning. Finally, for the purposes of this paper, we adopt Mason's (1989) notion of mathematical abstraction as a "delicate shift of attention" (p. 2), as well as acknowledging mathematical abstraction as a process closely linked to relational property recognition (e.g., Dreyfus, 1991; Ohlsson & Lehtinen, 1997; Piaget, 1985; Skemp, 1987). More specifically, Dienes (1967) indicated that "the essence of abstraction is to draw out common properties from different types of situations" (p. 201).

Developing the Theory: An Example of Student Thinking

To assist with presenting our proposed model of abstraction through analogical reasoning, we interlace the description of the general model with an example of analogical concept creation in abstract algebra. This particular example is based off of previously collected data of students' analogical reasoning in which the goal was to develop a ring-theoretic analogy to subgroups. Hence, the source concept is the subgroup concept, while the target concept is an analogous concept for rings.

To operationalize abstraction during analogical reasoning, we introduce the concept of the *abstracted domain*. We define the abstracted domain to be the domain to which the desired abstracted characteristics are mapped. Thus, the abstracted domain acts as a 'shell' of a concept or structure and forms the basis for future potential analogies. Figure 1 displays a possible abstracted domain for subgroups and subrings in which the notion of a 'sub-structure' would behave as a shell for future analogous concepts (such as subspaces of topological spaces.) The abstracted domain may vary in scope depending on the original concepts and domains being investigated.

The abstracted domain may not exist in every case of analogical reasoning. For example, a student may engage in a pure analogical comparison of two entities and construct a new concept without ever abstracting from the original structure by attending solely to surface level or superficial features. Instead, we claim that the abstracted domain exists only when abstraction activity arises during analogical reasoning. In alignment with ARM, we accept that the abstracted domain may be vague during its creation and may never be fully realized. Through progressive alignment, the abstracted domain may be refined over time as further inquiries are made. Our initial theory proposes two mechanisms through which the abstracted domain is developed: (1) comparative analogy and (2) inductive analogy.

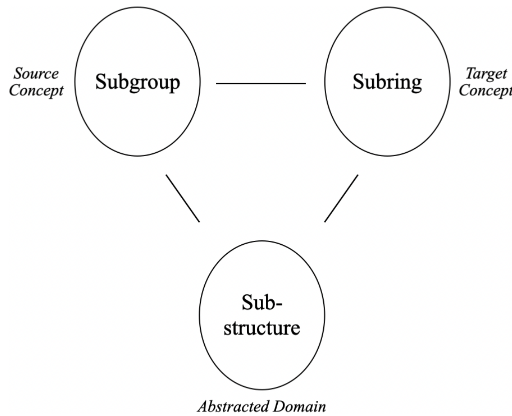

Figure 1. A possible abstracted domain associated with subgroup/subring.

Comparative Analogy

The first mechanism for developing the abstracted domain is *comparative analogy*. This process begins with a typical approach to analogical reasoning in which an analogy is either formed (or in the middle of being constructed) between the source and target domain. To illustrate this mechanism, we leverage an example of one student who had constructed an analogy to subgroups in the context of rings by directly mapping the subgroup structure and then making modifications to fit the ring context. After exploring the ring-theoretic analogy to subgroups, the student began to inspect his resulting analogy and reflect upon what he considered to be the key aspects of the two concepts of subgroup and the newly created concept of subring. During his reflection, some evidence of an abstracted concept appeared. In the following quote, the student is making observations about the subset similarity:

Student: My idea is just a set and a subset. Like that, it lives inside of it.
Interviewer: So you have this set and a subset idea? And that also carried over to this kind of context?
Student: Yeah, so we would say this would imply S is, maybe let me draw a little ring [draws subset symbol with a small "o" over it]. Like, subring.

Thus, after having developed the concept of 'subring' and directly comparing to subgroups, this student began to recognize a broader concept of a 'set and a subset' that 'lives inside of it' that could potentially describe a larger class of analogous structures. The left figure within Figure 2 displays the process of comparative analogy in the context of comparing the subgroup concept and the new subring concept to establish the abstract concept of 'sub-structure'. The box in Figure 2 indicates that it was the analogy between the source and target that produced the abstracted domain.

Inductive Analogy

While comparative analogy requires a comparison of two concepts to create the abstracted domain, our second mechanism involves the development of the abstracted domain before the target concept is established. We refer to this second mechanism as *inductive analogy*. In contrast to comparative analogy, inductive analogy may occur with only one known concept to begin with (e.g., subgroup), as well as a possible awareness of other contexts (e.g., the subject of ring theory.) Knowledge of the original concept is then leveraged to develop the abstracted domain. A target concept is then created by making inferences about how the abstracted domain may be mapped onto the target. Consider the following quote by another student who was in the process of creating the subring concept by analogy:

> Yeah, well that's pretty much the definition for groups, if we have some group G and some operation *. Then if H is a subset of G and H is a group under that same operation, then we say H is a subgroup of G. **So, I literally just took the exact same thing from group theory and applied it to rings**.

We view the last sentence as partial evidence that this student had identified the abstracted concept of a sub-structure based on the subgroup concept.[1] Afterward, the new sub-structure concept is then mapped to the context of rings to form what the student eventually described as a 'subring'. The visual on the right of Figure 2 displays the mechanism of inductive analogy wherein the source concept of subgroup is leveraged to create the concept of 'sub-structure' before subring. We note that it is possible that multiple source domains are leveraged as the 'base case' for the abstracted domain. In those cases, progressive alignment may occur through multiple instances of comparative analogy preceding the induction and the creation of the new target structure.

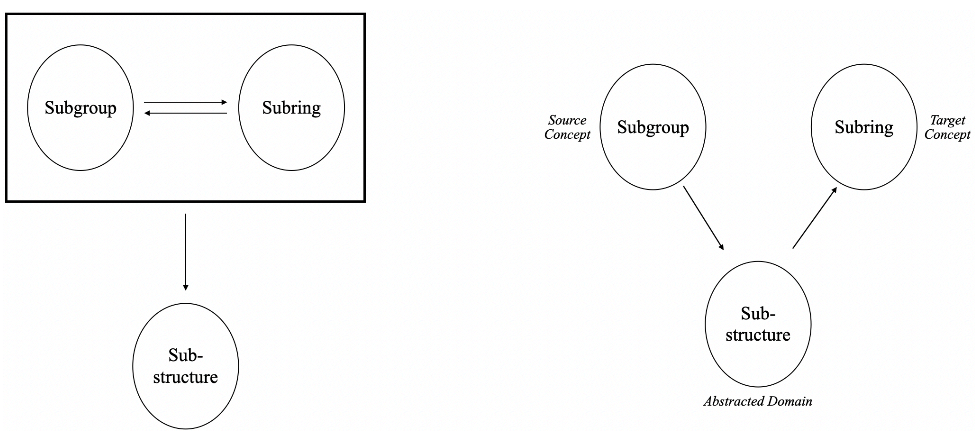

Figure 2. The mechanisms of comparative analogy (left) and inductive analogy (right).

In this section, we have introduced the mechanisms of comparative and inductive analogy to describe ways through which abstraction may occur during analogical concept creation. In the section that follows, we broaden the scope of this initial theory and apply it outside of the context of interpreting students' analogical reasoning between two structures in abstract algebra by looking more deeply at how this theory might inform purposeful analogical concept creation, and how this theory might be applied to pedagogy.

Further Applications of the Theory

Identifying Strategies for Purposeful Analogical Reasoning

In the absence of prior opportunities to reason by analogy in advanced mathematics, the students in the above examples may have been reasoning by analogy about advanced

[1] We choose this example to clearly display reasoning with one domain rather than comparing across two domains. However, we recognize that the student may have been carrying the subgroup concept itself into the ring context without having created the abstracted sub-structure concept.

mathematics in a manner that was brand new to them. Overtime, learners may develop *meta-analogical awareness* (Modestou & Gagatsis, 2010) and, by extension, learn to purposefully and strategically reason by analogy. In particular, the increased awareness of one's own analogical reasoning can widen the range of strategies for analogical concept creation. For example, purposeful reasoning by analogy is a known strategy used by research mathematicians when defining new concepts (Ouvrier-Buffet, 2015). In this section, we describe how the mechanisms of comparative and inductive analogy clarify what one case of purposeful reasoning might entail when defining a new concept. To show this, we describe a hypothetical example of a student purposefully defining topological subspaces by analogy with several known algebraic concepts.

Consider a student who possesses an understanding of several algebraic structures and has recently learned of the definition of a topological space (i.e., a set X along with a collection of subsets containing X, the empty set, and is closed under arbitrary union and finite intersection.) Their experience in algebra suggests that given any particular algebraic structure, one can construct an additional one. Thus, upon becoming aware of the context of topology, an inductive analogy can be invoked to begin considering the meaning of a topological subspace. First, the student understands that subgroup, subring, subalgebra, or algebraic sub-structures in general are modeled by taking a subset of the collection of elements with respect to the structure in some way (i.e., the subset must be closed under the algebraic operation(s) of the over-structure.) In this way, the student may have already created an abstracted domain relevant to algebraic sub-structures. The student may then inductively analogize the notions of both 'subset' and 'respect the structure' from algebra within topology to establish the concept of a topological subspace. For this mathematically experienced student, it may be straightforward to argue that because an algebraic substructure is modeled with subsets, a topological substructure should be modeled by subsets as well. However, the student can also begin investigating what it means for a substructure to 'respect the structure' in the context of topology. In this way, purposeful inductive reasoning has provided an opportunity for rich exploration for this student to begin thinking about how a topological subspace may be defined.

After having established some notion of topological subspace (whether it is complete or not from the student's perspective), suppose now that the student encounters the definition of a topological subspace in a textbook or in a class lecture. The student may now reason by comparative analogy while also being informed by the findings from the prior inductive analogical reasoning. As an algebraic structure is a set along with various operations and a topological space is a set along with a collection of open subsets, the student may try to draw an analogy between operations and collection of open subsets. While the subset analogy was perhaps trivial, the student may not be immediately sure of the analogy of the restriction of the operations within a topological space. The modeling of topological spaces purely in the language of sets may motivate them to understand n-ary algebraic operations as sets as well. Particularly, an n-ary operation could be modeled as a subset of the $(n+1)$-fold cartesian product of the underlying set, and the restriction to a substructure can be realized by considering the restriction to the substructure to be the sequences that only contain its elements. Thus, they are drawing a comparative analogy between a subset of the cartesian product of a set with itself and a subset of the power set of a given set. In negotiating a satisfactory definition of a topological subspace, they are drawing on comparing different kinds of sets: they are attempting to understand the notion of topological subspaces by drawing comparisons between the cartesian product of set and the power set construction.

Applications to Pedagogy

Students may engage in analogical reasoning in any classroom setting. While the prior examples linked our proposed theory to interpreting student thinking and analogical reasoning, our final example links our proposed theory to teaching and curriculum design. In particular, we assert that increased attention to comparative or inductive analogy can assist teachers with identifying affordances or obstacles to promoting analogies in specific ways as well as consider the timing for productively introducing an analogy. In this section, we consider an example in undergraduate linear algebra. An important analogy that helps link many of the concepts in this course is the three-way analogy between systems of linear equations, matrices (or matrix-vector equations), and linear transformations (Larson & Zandieh, 2013). Every system of n linear equations in m variables can be represented with a matrix-vector product using an $n \times m$ coefficient matrix (or an $n \times (m+1)$ augmented matrix), and vice versa. Similarly, every linear transformation $T\colon \boldsymbol{R}^m \to \boldsymbol{R}^n$ has a standard $n \times m$ matrix representation, and vice versa. This ability to transform between systems of equations, transformations, and matrices leads to many analogous theorems and properties in linear algebra, including the infamous Fundamental Theorem of Invertible Matrices.

One concept that is closely tied to this analogy is matrix multiplication, as matrix multiplication corresponds to compositions of linear transformations. Cook et al. (2018) conducted a textbook analysis on 24 introductory linear algebra textbooks, analyzing how matrix multiplication was presented in each text. They identified three overarching sequences for when and how matrix multiplication was introduced, and characterized these sequence rationales utilizing Harel's (1987) classification of linear algebra content sequencing. Pedagogically, the choice of which of these sequences is used for course design influences what analogical reasoning students engage in during class. Students' potential analogical reasoning in two of these sequences is provided. It is important to emphasize that neither sequence is considered better, but rather these examples demonstrate that pedagogical choice can influence students' analogical reasoning, and this framework provides a way to understand that reasoning.

Many of the common undergraduate linear algebra textbooks for large universities begin by introducing systems of linear equations and immediately transition into basic matrix operations, such as (typically in this order) addition, scalar multiplication, and multiplication (e.g., Larson, 2016; Poole, 2014). Textbooks with this structure often fall into the sequence where matrix multiplication is first defined using the matrix-matrix product AB determined by the dot products of row and column vectors, and the justification for the definition's rationale is postponed until much later in the course. Under this course design, after students have been provided with the definitions for matrix addition and scalar multiplication, it is possible that students may engage in inductive analogical reasoning, where these definitions induce the creation of an abstracted domain encompassing matrix operations as component-wise operations. Though this would lead to the 'incorrect' definition, it should be noted that this is reasonable analogical reasoning; hence, it should not be surprising that students are perplexed when the actual definition is presented.

Another common sequence for matrix multiplication includes first defining matrix multiplication with the matrix-vector product Ax as a linear combination of column vectors, which follows an introduction to systems of linear equations and linear transformations (e.g., Lay et al., 2015). Since the three-way analogy has already been introduced to the students, when some students see the full definition of the matrix-matrix product AB, they might construct a comparative analogy using any two (or all three) of those domains. In particular, some students may construct or refine an abstracted domain about 'how vectors are transformed.' In addition, prior to seeing this full matrix-matrix product definition, some students might engage in inductive analogy, hypothesizing what the actual definition is, being informed by the matrix-vector product definition. However, it is entirely possible that students will not engage in either of these forms of analogical reasoning.

Implications for Theory and Future Research

In this paper, we set out to describe an initial theory intended to explicate the role of abstraction during the process of analogical concept creation. To this end, we described two mechanisms through which abstraction may occur and illustrated several scenarios in which these mechanisms assist with interpreting or predicting possibilities of students' thinking. These mechanisms contribute to the literature on analogical reasoning in mathematics education, especially with respect to analogical concept creation. In particular, we greatly expanded the scope of possible scenarios presented by English and Sharry (1996) and Lee and Sriraman (2011) by both operationalizing and clarifying what abstraction during analogical reasoning might look like.

In addition to the examples presented in this paper, this theory can be applied to various other contexts, demonstrating even further potentially far-reaching implications. For instance, one can frame both problem-solving strategies (Weber & Leikin, 2016) and problem-solving analogies (English, 2004) in this theory. When an individual has solved two different problems with an analogous problem-solving strategy, there is a possibility that this individual will retroactively engage in comparative analogy, constructing an abstracted domain containing a 'problem-solving heuristic or strategy.' Moreover, an individual may also be in the process of solving a problem and is looking for ways to complete the problem. In this scenario, the individual could engage in inductive analogy, by reflecting on how a previous problem-solving technique could apply to the current problem, and constructing an abstracted domain for the general problem-solving strategy. This can also apply to proof-writing and proof-constructing strategies (e.g., Zazkis, Weber & Mejía-Ramos, 2015). Furthermore, researchers have demonstrated the value of example-based reasoning in mathematics (e.g., Lockwood et al., 2016; Lynch & Lockwood, 2019). It is possible that two or more examples give rise to comparative analogy, where the abstracted domain is an analogous property, concept, or new mathematical definition. Additionally, after an individual has meaningfully engaged with one example, and is about to engage with a subsequent example, this person may engage in inductive analogy. Prior to working out the details, the individual may construct an abstracted domain containing certain potentially relevant properties, which help the individual to predict how the example may work.

Though this paper has outlined the central components of this theory about the role of abstraction in analogical concept creation, and demonstrated its implications to various facets of mathematics education research, we have several future goals for expanding on and refining the theory. First of all, two mechanisms for this theory were introduced, comparative and inductive analogy; however, we acknowledge that these mechanisms do not always function entirely separate from one another. The example of the algebraically mature student engaging with topology explicates how one might engage in inductive analogy and then comparative analogy, and we would like to further delineate how these mechanisms interact. Second, we would like to more specifically explore how this theory is connected to theories of abstraction. In particular, we recognize that the construction of an abstracted domain aligns closely with Piaget's reflective abstraction (Beth & Piaget, 1966; Piaget, 1977, 2001). To what extent these two mechanisms entail reflecting and reflected abstractions versus empirical or pseudo-empirical abstractions is worth further exploration.

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Proving as cooperative communication: Using Grice's maxims and implicature to understand proof comprehension in undergraduate mathematics education

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In this exploratory theoretical analysis, we apply Grice's maxims for cooperative communication to make sense of the ways that proofs are written and understood. We argue that proving can sometimes be productively viewed as a form of cooperative communication. Grice's maxims place expectations on the prover and provide norms for how proofs should be written. The reader's expectation that the prover is adhering to these norms allow for the possibility of implicature—that is, the prover can imply statements to the reader that are neither explicitly said nor logically necessary. We use this analysis to better understand the practice of proving and to suggest a future line of empirical research.

Keywords: Linguistics; Grice's Maxims; Proof; Proof Comprehension

A central goal of research in undergraduate mathematics education is to design instruction to enculturate students into the process of proving. We would like students to use proof for the same reasons that mathematicians do (e.g., deVilliers, 1990) and to understand and learn from the proofs that they read (e.g., Conradie & Firth, 2000; Mejía-Ramos et al., 2012). Unfortunately, these goals are often not achieved in undergraduates' advanced mathematics courses, with many students emerging from these courses unable to read or write proofs successfully and, more generally, lacking a sense of what proof is about (e.g., Stylianides et al., 2017).

There has been substantial research into helping students understand what type of reasoning is permissible in a proof. For instance, there has been work into helping students realize that inferences must be based on deductive, rather than empirical, argumentation (e.g., Brown, 2014), to learn new proving techniques such as proof by induction (e.g., Harel, 2001), and about the nature of modus ponens reasoning (e.g., Dawkins & Norton, 2022). This research is important and has substantially increased our understanding of pedagogical issues related to proof. However, we argue that this work alone is not sufficient to enculturate students into the practice of proving. Knowing what types of reasoning is allowable in a proof underdetermines how proofs should be written and read. There are permissible moves in a proof that are almost never carried out in mathematical practice (e.g., introducing variables or making explicit assumptions that are never used again in the proof). There are formal violations of logic and syntax that frequently appear in proofs (e.g., abuses of notation). There are norms of proof writing that are independent of logic (e.g., not using the first-person pronoun "I" and not including information about the wrong turns made in the proving process). Some mathematicians have complained that their students are "tone deaf" with respect to proof and do not have a feel for the game (Weber, 2012). Elsewhere, we have speculated that part of enculturating students into proof is helping them learn the linguistic norms of proving along with the values that they uphold (Dawkins & Weber, 2017). These kinds of rules of the game that are not captured by the grammar or direct meaning of proof might be appropriately categorized as part of the pragmatics of proving. Pragmatics refers to the use and contexts of use. This is an understudied arena of proving practice, which might reflect how proving is often understood as decontextualized and depersonalized (Balacheff, 1988). We argue that proofs still communicate through shared rules.

The goal of this theoretical report is to use Grice's maxims for cooperative communication to better understand how proofs are written and read in mathematical practice (Grice, 1975, 1989). We first document that in mathematics education, proof is often treated as an adversarial form of communication. We argue that it would behoove mathematics educators to also view proof as a form of cooperative communication. Next, we review Grice's maxims for cooperative communication. Critical to this paper is Grice's notion of implicature: when we assume that a speaker is communicating cooperatively with us, we can infer many things that the speaker did not explicitly state. Then we explore the consequences of applying Grice's maxims to proof in mathematical practice and undergraduate mathematics classrooms, showing how proofs invite or even expect implicatures. There are ideas that one can from a proof that are neither explicitly stated nor logically necessary. Finally, we suggest a line of empirical research based on our theoretical explorations.

Proof as a cooperative form of communication

Proof as an adversarial form of communication in mathematics education research

Frequently, proof is framed as a semi-adversarial form of communication: the prover is trying to persuade a skeptic to accept a claim that the skeptic doubts and the skeptic will not (and should not) be swayed easily. Mason (1982) described proving as convincing an enemy and Volminik (1990) defined a proof as an argument that would convince a reasonable skeptic. More recently, Dutilh Novaes (2020) wrote that "a good proof is one that convince a fair but 'tough' opponent" (p. 50). When students are reading proofs, they are encouraged to take a skeptical stance, with Brown (2014) arguing that the capacity to retain doubt is critical to understanding proof. Similarly, Inglis (2022) argued that developing one's inner skeptic when considering proofs may be a critical part to becoming a mature mathematician.

Inglis (2022) referred to the dialog between a prover and their skeptical audience as "semi-adversarial" because neither the skeptic nor the prover engages in a "win-at-all-costs" strategy. A reasonable skeptic must grant sensible premises and not repeatedly ask for supporting evidence in bad faith. A reasonable prover should not use rhetoric to mislead the skeptic. Both the prover and the skeptic share the communal goal for identifying the flaw in a proof, should one exist, or otherwise verifying that the proof is airtight (Dutilh Novaes, 2020). Nonetheless, to achieve these shared aims, the reader is asked to take an adversarial role, only granting inferences if it is epistemologically necessary to do so. In short, according to the adversarial account of proving the audience of the proof of a theorem acts as a gatekeeper who checks that the proof passes a high standard of muster, lest a false theorem enter into their collection of mathematical facts (or, in the case of a referee, in the mathematical community's accepted statements).

Proof as cooperative communication

Mathematicians do not always view other mathematicians' proofs adversarially, or even with skepticism. Indeed, when mathematicians read proofs in the published literature, they often do not check proofs for correctness; they presume the proofs are correct (Mejía-Ramos & Weber, 2014; Weber & Mejía-Ramos, 2011). As one mathematician described things, "when I was reading a proof in a journal or a proof that was handed to me by a mathematician friend and they were pretty sure it was true, my assumption would be that the steps are probably correct and that I need to work hard to make sure I understand and can justify each step" (Weber, 2008, p. 448). The point here is that while proofs in the literature and proofs from trusted peers can contain mistakes, mathematicians often do not read the proof scouring for errors. They do not assume unclear statements in a proof are wrong, but rather work to justify these steps. We believe students are in a similar position when they read proofs in a textbook or hear proofs in a lecture.

Grice (1975, 1989) characterized communication as cooperative if the interlocutors are working together to actively achieve a particular goal. Proving can sometimes best be understood as a collaborative communication in which the prover is trying to help their audience see how their theorem can be reached from shared premises by a series of deductive inferences that their audience accepts as clear, indubitable, and not admitting exceptions. Of course, proofs in textbooks and in journals often contain inferences that are neither clear nor obvious. Hamami (2021) explained the situation as follows: The prover hands her audience a proof P. Some steps in P will not be clear to a reader. To address this, a reader will expand P to a longer proof P^*, such that each step in the proof is either an accepted statement or an immediate consequence of a known rule of inference. An accepted statement might be a definition, a result that is established elsewhere in the literature, or a statement that the reader had personally proven in the past. A known rule of inference might be an axiom or the application of a previously proven theorem.

As a brief clarification, while mathematicians sometimes seek out a proof as an end in itself (c.f., Paseau, 2011), they frequently use proof as a means to achieve other epistemic goals (deVilliers, 1990). Possessing the expanded P^* consisting of a series of indubitable inferences can help achieve epistemic aims of proof such as conviction and explanation highlighted by de Villiers (1990) and others (e.g., Hanna, 1990). For instance, if a mathematician harbored significant doubts about the truth of an assertion, seeing how the statement could be reached by a series of indubitable inferences could increase their confidence that the statement was true.

Grice's maxims and implicature

Grice's maxims

When interlocutors are engaged in a cooperative conversation, they are trying to achieve a shared goal. When a speaker offers a contribution to a cooperative conversation, Grice (1975, 1989) argued that the speaker is expected to follow the Cooperative Principle and make contributions that will efficiently facilitate the achievement of these goals. Grice offered four conversational maxims specifying how contributions should be made: The maxim of quality (be truthful); the maxim of quantity (offer the right amount of information); the maxim of relation (be relevant); and the maxim of manner (be clear). These maxims can be regarded as norms or obligations that constrain what contributions a speaker can make in a conversation (in the sense of Herbst et al., 2011 and Yackel & Cobb, 1996). When we apply these maxims of proof, these will yield norms about how proofs should be written (in the sense of Dawkins & Weber, 2017).

Implicature

In a cooperative conversation, a listener presumes that a speaker is following the Cooperative Principle in general and Grice's maxims in particular. Grice's (1975) key insight was that this enables the listener to make inferences that are neither explicitly stated nor deductively implied by the speaker's utterances. To illustrate, consider the following conversational exchange:

Man (speaking to a passerby on the road): My car has run out of gas.

Passerby: There is a gas station around the corner.

The explicit contribution from the passerby is the location of a gas station. Grice (1975) argued that the passerby is also making an implicit contribution, namely that (the passerby believes) the gas station is open. The reason this implication is possible is because the man assumes that the passerby is following the Maxim of Relation (and the passerby assumes this). The man and the passerby are working toward the shared goal of helping the man resolve his problem of finding fuel for his car. If the gas station around the corner were closed, the existence of the gas station would not be useful for achieving this goal. Therefore, by stating the location of the gas station, the passerby is implying that this is open. Grice (1975) referred to *implicature* as the act of making this type of implication and *implicatum* as what is being implied. As we view the situation with proof, Grice's maxims place norms upon the prover. The recipient of the proof makes the presumption that the prover is following Grice's maxims, which allows for the recipient to infer ideas that were neither explicitly stated nor logically necessary consequences of what appears in the proof. These ideas are implicatum. The shared assumption of cooperative communication allows the prover to convey implicata without explicitly stating them.

Grice's Maxims, Norms, and Implicature and proof

In this section, we present norms and their associated implicatures for proof presentation for each of Grice's Maxims. In our presentation, we will discuss norms that the maxims impose on the prover and the implicata that a reader may gain.

The Maxim of Relation

The Maxim of Relation asserts that contributions to a conversation should be relevant to the goals of the conversation. Any contribution in a proof P should be relevant to obtaining a chain of indubitable inferences that lead to the conclusion.

> *Norm of proof writing:* Any assumption, deduction, or variable introduced in a proof should be useful in the final chain of argumentation establishing the theorem.
> *Associated implicatum:* If a statement or variable appears in a proof, the reader can expect that it will be built upon at a later stage in the proof.

In other words, proofs do not contain digressions. The prover should not include "logical dead-ends" in a proof and the reader can expect that none will be included. Strictly speaking, from a formalist perspective, adding unused assumptions and including extraneous true statements or valid deductions in a proof does not render it invalid. Nonetheless, this is rarely done. One consequence is that when the prover makes an assumption in the proof, the prover is implying that this assumption will be useful for establishing the theorem. One consequence of the implicatum is that one can understand a proof by seeing why assumptions are necessary or how statements are built upon (e.g., Mejía-Ramos et al., 2012). This consequence only follows based on the expectation that the prover is following the norm above.

The Maxim of Quantity

Grice (1975) summarized the Maxim of Quantity by saying, "make your contribution as informative as required (for the exchange) [and] do not make your contribution more informative than is required" (p. 45). Applying this notion to proof, the prover is aiming to give the reader sufficient information in the proof P so the reader can produce an expanded chain of indubitable inferences P^*, but no more. This suggests some interesting norms and implicata.

> *Norm of proof-writing:* In a conditional statement, "If A, then B" or "Since A, then B", the truth-value of B is tracked by the proof value of A.

Associated implicatum: If "Since A, then B" appears in a proof, there is a general principle for how the truth of A necessitates the truth of B.

Grice (1989) discussed the nature of conditionals more generally. Suppose one asserts, "If John walks to the party, he will be late." Technically speaking, this statement would be true if the speaker knew that John would be late regardless of how he reached the party. However, making such as assertion would be violating the Maxim of Quantity, as the simpler statement "John will be late" would be simpler and more informative. Similarly, if the speaker knew that John would be driving to the party, the statement would be (vacuously) true, but making the assertion would be violating the Maxim of Relation, as John's timing if he were walking would not be relevant in this situation. Hence, when the speaker asserts that "If John walks to the party, he will be late," the implicature is that the speaker is certain neither of how John is getting to the party or whether John will be late, but the speaker believes that the truth of the antecedent implies the proof of the conclusion.

A similar analysis follows from statements such as "if A, then B" or "since A, then B" in a proof. A prover should not proffer such a statement if B could easily be seen in the absence of A, since it would be simpler and more informative to simply state B. Further, to understand the proof, the reader should try to find a chain of reasoning from A to B. There is evidence that this phenomenon is borne out in practice. Some mathematicians will reject statements like "if 13 is prime, then 1013 is prime" in proofs even though they are true because they are not informative. Some will infer that there is a relationship between 13 being prime and 1013 being prime (such as adding 1,000 to a prime number will yield a prime number) and reject the statement not because it is explicitly false, but because the implied warrant is invalid (c.f., Weber & Alcock, 2005). There is further empirical support for this statement. In pedagogical setting, mathematicians do not think it is appropriate to make a statement of the form "If A, B, and C, then D" unless A, B, and C are all relevant to the truth value of D (e.g., Lai et al., 2012). Presumably, this is because, if C was unnecessary, it would be simpler and more informative to state "If A and B, then D."

Norm of proof-writing: The prover will provide enough information in *P* for the reader to expand without extraordinary effort.

Associated implicata: New steps in the proof may require justification, but the justification should not be based on sophisticated ideas outside the proof or be exceptionally difficult.

Recall the passerby who told the stranded motorist that there was a gas station around the corner. The passerby probably would not instruct the man where to look for the gas station after he turned the corner if gas station was clearly visible. The passerby would only provide this information if the location of the gas station was not obvious (e.g., if it were behind a strip mall) and would not be found by a routine search of the area. Similarly, justifications for statements within a proof are only provided if the reader could not find them through ordinary effort. What is interesting here is that the implicature is based on what the passerby did *not* say. If the location of the gas station would not be obvious to the stranded motorist, the Maxim of Quantity would dictate that more direction would be provided. Since the passerby did not provide that information, the location of the gas station should be clear when the motorist walked to it.

There are a few remarks worth noting. First, the ability for the reader to justify steps within a proof *P* is clearly audience dependent, so how a prover follows this norm depends on the audience. A proof appropriate for a professional number theorist would not be appropriate for a student in a transition-to-proof course, and vice versa. Second, what the readers are capable of doing "without extraordinary effort" is a nebulous concept. At least in pedagogical settings, mathematicians seem to disagree on how much detail a proof should provide (Lai et al., 2012). Research has revealed that students do not appreciate that it is their responsibility to justify some of the steps in a proof (e.g., Weber & Mejía-Ramos, 2014). Nonetheless, when a prover chooses *not* to provide detail for how a particular step can be established, the prover is conveying information about the existence of a type of justification for the step—namely it is a justification that can be formed routinely without extraordinary effort.

In addition to the knowledge of the intended audience, we expect that the application of this maxim is also sensitive to the intended purpose within the proving context. If the proving context is focused on developing an axiomatic system or proving the properties of an axiom system, the prover may adhere to stricter rules regarding what the reader should have to expand on their own. This again portrays how the context of proving influences the rules at play in the cooperation between the prover and reader.

The Maxim of Manner

The Maxim of Manner asserts that speakers try to say things as clearly and orderly as possible. Suppose that you are asked to prove a statement of the form "A if and only if B". Suppose you have a subproof that A implies B, say $A \to A_1 \to B$, and you have a subproof that B implies A, say $B \to B_1 \to A$. You could, in principle, write a logically valid proof as follows.

1. Assume A
2. Assume B
3. Since A, A_1
4. Since B, B_1.
5. By (3), B
6. By (5), A
7. Since $A \to B$ and $B \to A$, A if and only B.

Of course, no one would write the proof in such a confusing manner.

> *Norm:* Proofs are ordered to make the chain of arguments clear.
> *Implicature:* The justification for how a new line follows in a proof will depend on previous statements in a proof.

Mirin and Dawkins (2022) have illustrated how when a proof involves a string of identities (i.e., showing A=D by showing A=B, B=C, and C=D), mathematicians are sensitive to the order of these identities. They will think norms are being broken and proofs are deficient if the identities are not ordered in the most sensible manner.

> *Norm:* Name objects in proofs according to convention to convey their role or status in the proof.
> *Implicature:* The name of an object will help the reader keep track of its role in the proof or its quantificational status.

The names assigned to mathematical objects is taken to be logically irrelevant. Nevertheless, there are strong norms for how things are usually named. For instance, input variables are frequently named x and bounds on output estimates are usually named ε. Quantification is often encoded using indices and superscripts. A reader who notes that a set indexed by the natural numbers would be justified in drawing the implicature that the set is countable or finite. It would also be strange to denote an object G^* if the variable name G was not used elsewhere in the proof. The superscript is only used to distinguish various objects bearing the label G. As ** stated, one would never give the name "6" to a group. Strictly speaking, these naming conventions bear no logical status and the proofs would remain valid were they not followed. Nevertheless, they are strong conventions of naming because the notation affords implicatures that aide readers in constructing the line of inference intended by the prover.

Discussion and future research

In this paper, we have argued that both in the advanced mathematical classroom and in mathematical practice, it is often productive to frame proof as a cooperative form of communication. We have asserted that Grice's maxims for conversation are useful in two respects. First, Grice's maxims suggest norms for proof writing that place expectations on the prover for how proofs are written. Second, the presumption that the prover is following these norms allow for implicature; the reader of a proof may infer things that are neither explicitly stated nor logically necessitated by the proof text being read.

We believe that this has pedagogical consequences. Much of advanced mathematical instruction (and undergraduate mathematics education research) focuses on the type of reasoning that is admissible in a proof. We agree that this is essential, but argue that this is not enough to fully enculturate a student into the practice of proving. We suggest more effort can be put into instruction as to norms about how proofs are written and how they can be read. There is an intimate relationship between the form of mathematical proofs and the proving practices they express. As Schleppegrell (2004) explained, "It is important for students to develop academic register options in different disciplines because particular grammatical choices are functional for construing the kind of knowledge typical of a discipline" (p. 137).

We also believe that our exploratory theoretical analysis suggests a further line of research. We have made a number of speculative claims about norms for proof writing and opportunities for implicature. Such claims can be empirically tested with breaching experiments by showing both mathematicians and students situations in which those norms were violated, or showing mathematicians and students situations in which our proposed norms were followed and asking them if they believed that the implicata likely held. Mathematicians' responses to such situations would reveal if our speculative claims were accurate and provide us with a richer sense of why these proving norms were operative. Differences between students' and mathematicians' responses could help us unpack one way in which students might not yet have a feel for the game of proving.

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(Re)defining Developmental Mathematics: A Critical Examination of the Research Literature

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Although "developmental math" is widely discussed in higher education circles, exactly what developmental math encompasses is often underdeveloped. In this theoretical report, we use a sample of highly cited works on developmental math to identify common characterizations of the term "developmental math" in the literature. We then interrogate and problematize each characterization, particularly in terms of whether they serve equity-related goals such as access to college credentials and math learning. We close by proposing an alternative characterization of developmental math and discuss the theoretical implications. We see this as a first step towards conversations about how developmental math could be conceptualized.

Keywords: Developmental Math; Equity; Time Capital; Postsecondary; College Level

Developmental math has been a regular focus of education research for decades. Understanding these courses is important as they disproportionately enroll students from historically marginalized backgrounds (e.g., Chen, 2016) and many students in these courses never complete their required math sequence (Bailey et al., 2010). We posit that despite the heavy focus on developmental math in higher education research and policy, the way that "developmental math" is defined or characterized in the research literature often runs contrary to equity concerns and begs the question of what developmental math actually is. Using definitions drawn from a comprehensive literature search, we consider how developmental math has been defined in the research literature, with the aim of problematizing some common characteristics of these definitions. We refrain from initially defining developmental math explicitly so as to let the term itself drive our sampling and analytical procedures. We devote the majority of this report towards developing a framework for understanding characteristics used to define developmental math and the different conditions each characteristic privileges. We close with a discussion about how we might define this term moving forward, and how this may impact equity goals.

Theoretical Framework: Approaching Equity from Credential vs. Learning Orientations

There is a growing recognition that developmental math and institutional structures around it are a consequence of and often reproduce structural inequalities in K-12 education (e.g., Larnell, 2016; Ngo & Velasquez, 2020). That is, the developmental population is *created* through often unexamined institutional norms and values. We aim to make the values and associated equity implications that are implied by various definitions of developmental math explicit, with the goal of proposing a way to move forward with how we understand and define developmental math.

Historically, developmental math was created to provide access to college math classes by providing instruction on content that has traditionally been considered prerequisite to advanced study, such as algebra, that students may not previously have had the opportunity to master, or that they have forgotten after a gap in enrollment between high school and college (e.g., Dotzler, 2003). Providing access to advanced study in math is motivated by two distinct, but related, equity goals: providing access to (a) college credentials (e.g., degree progression, retention), which we refer to as a *credential orientation*, and (b) math learning (e.g., procedural/conceptual knowledge), which we refer to as a *learning orientation*. Both orientations center equity, but have different outcomes of interest, even as they are interdependent (Figure 1). Considering only one at a time may have negative equity consequences. Problematically, research demonstrates that inequities exist in both developmental students' access to college credentials (e.g., Boatman & Long, 2018; Crisp & Delagado, 2014; Sanabria et al., 2020; Xu & Dadgar, 2018) and their access to rich and meaningful math learning or instruction (e.g., Givvin et al., 2011; Goldrick-Rab, 2007; Hammerman & Goldberg, 2003; Stigler et al., 2010; Webel & Krupa, 2015).

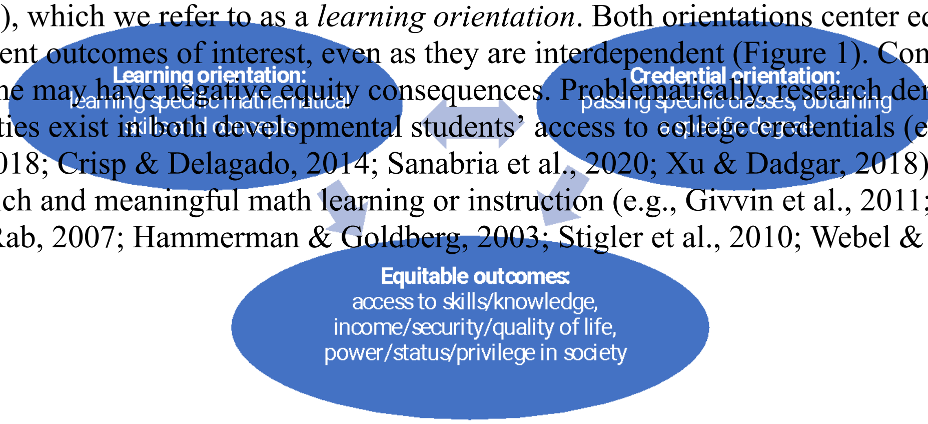

Figure 1. *Framework Relating Learning and Credential Orientations to Equity Approaches in Developmental Math*

Method

We conducted a comprehensive literature search for original reports related to developmental math. This work is part of a larger project. Here we focus only on how developmental math was defined or characterized in highly cited empirical reports or peer-reviewed journal articles.

Sample

To identify articles related to developmental math, we searched EBSCOhost for journal articles or reports published between 2000 and 2020 (inclusive) with abstracts that included the word "developmental" or "remedial", a word with the stem "math", and one of the following: college(s), universit(y/ies), post-secondary, postsecondary, or undergraduate(s). We removed duplicates, brief reports of full reports that were also included in the sample, and any articles that were not about developmental math instruction, classrooms, curricula, instructors, or students enrolled in U.S. colleges or universities. This resulted in 446 reports: 281 (62%) peer-reviewed journal articles and 168 (38%) non-journal reports. Of the 281 peer-reviewed articles, 66 (23%) were published in journals usually consumed by a math education audience. The remaining articles were published in journals with a more general education audience or that target an educational research subdomain, with the higher education audience the most prominent (47%).

Because we were interested in analyzing the most "influential" definitions in the literature, we used citation frequency as a rough proxy for a report's "influence" in order to determine which reports to code. Towards this end, in June 2022 we used Google Scholar to find the number of times each report in our sample had been cited (26 reports did not show up on Google Scholar, in which case we entered 0). We calculated the Annual Citation Rate (ACR; total number of citations divided by the report's age, in years) for each report. We formed our final subsample of influential reports by including any report that had the top 10 highest ACR and/or the top 10 total number of citations in one of the following categories: peer-reviewed journal article aimed at a math education audience, peer-reviewed journal aimed at a general education audience, and non-journal reports. We sampled from different groups to be able to contrast the definitions used by different stakeholders. There was overlap between articles with top ACRs and top overall number of citations, so the final sample included 36 records (8% of the total sample). The list of the 36 coded reports is available at Open Science Framework. Here we focus on two groups of reports within this sample: those geared at a math education audience and those not published in math education journals, hereafter referred to as "non-math education reports".

Analysis

Development of Coding Scheme. Using the constant comparison method (Lincoln & Guba, 1985), we developed an emergent coding scheme to capture the most common characterizations observed across the literature (Table 1). These were *Level* (e.g., described as "not-college-level"), *Credit* (e.g., described as "not-for-credit"), and *Content* (e.g., described in terms of the content covered or the course names). The three characterizations are necessarily intertwined, but capture different implicit orientations about developmental math, discussed in the next section.

Table 1. Coding scheme for characterizations of developmental math

Code & Description	Example
Level: Developmental math courses cover content that is not "college-level", sometimes described as secondary school level.	"Broadly speaking, the term 'developmental education' connotes a set of policies and practices designed for students who are underprepared to do college-level work in a given area. The goal of this experience is to give students the knowledge, skills, and habits that will help them be successful in the college-level version of the course (Bailey et al., 2016). The growing use of developmental education reflects an increasingly normative transition from high school to college, which while predicated on completion of secondary schooling, does not necessarily imply adequate preparation for what is deemed 'postsecondary' work." (Valentine, et al., 2017)
Credit: Developmental math courses are non-credit courses.	"With their open-door admission policy, community colleges serve a population with diverse needs and a wide range of skills. In order to prepare this diverse population for college-level courses, community colleges offer non-credit developmental courses in math, reading, and writing." (Ashby, et al., 2011)
Content: Courses cover specific math topics, typically substantially similar to second year school algebra or below.	"Remedial math includes basic arithmetic, pre-algebra, beginning algebra, intermediate algebra, and geometry. College-level math includes all courses that address topics of a skill-level equal to, or greater than, college algebra." (Bahr, 2008).

During the code development process, we noticed that sometimes characterizations of developmental courses were mentioned, but that developmental math was never explicitly defined. A lack of an explicit definition suggests the assumption that the reader has a shared understanding of what is being discussed, which may be problematic given the complex nature of developmental math. To capture such instances, we coded whether the characterizations were part of an explicit definition, an implicit definition (in which some characteristics of developmental math were described but no explicit definition was given), or whether there was no clear attempt to define developmental math (either explicitly or implicitly).

Coding Developmental Math Characterizations. Each report was coded by two coders. Initial agreement across all codes was between 81% and 92%. Further norming was undertaken to reach consensus for all codes. Codes for the characterization of developmental math were not mutually exclusive. Codes for the nature of the definition were mutually exclusive.

Results

Table 2 gives the coding distribution in the sample overall and for our subsamples of interest. "Not college-level" was the most common characterization of developmental math (92%), followed by mathematical content (67%) and then "not-for-credit" (53%). Characterization choice appears field related. Non-math education reports used "not-college-level" (100%) and "non-credit-bearing" (70%) characterizations more often than math education (77% and 23%, respectively). In contrast, math education research favored characterizing developmental math by content (85%) compared to non-math education research reports (57%). While "non-credit" and "not-college-level" characterizations might be presumed to be linked, these characterizations only co-occurred in 53% of reports. In addition, while the specific math content might be presumed to determine whether a course is "college-level", these characteristics did not always co-occur, especially in math education literature where 41% of papers that characterized developmental math in terms of content did not characterize the course as "not-college-level".

Table 2. Summary of characterizations of developmental math in research articles in the sample

	Overall		Math-education		Non-math education	
	n	%	*n*	%	*n*	%
College-level	33	91.7	10	76.9	23	100
Not for credit	19	52.8	3	23.1	16	69.6
Mathematical content	24	66.7	11	84.6	13	56.5
College-level & not for credit	19	52.8	3	23.1	16	69.6
College-level & mathematical content	21	58.3	8	61.5	13	56.5
All three	12	33.3	3	23.1	9	39.1
n	36		13		23	

In terms of implicit versus explicit definitions, 7 (19%) reports did not provide either an explicit or implicit definition; only 9 (25%) provided an explicit definition. The distribution of the nature of the definition was similar between non-math education and math education reports.

Problematizing Characterizations of Developmental Math

The characterizations of developmental math courses that we identified (level, credit, and content) align somewhat with the credential and learning goal orientations for developmental math discussed in the Theoretical Framework. However, the extent to which these characterizations measure the intended credential and learning goals is unclear. While there is much discussion in the literature about whether existing developmental math courses serve students, we could find no substantial discussion about whether or not existing *definitions* of developmental math serve these goals. Here we attempt to address that gap.

Level Characteristic

The most commonly used characteristic to describe developmental math was “not-college-level”, which often co-occurred with both “not-for-credit” and content characteristics (and by extension both credential and learning orientations). But what constitutes “college-level” math content was often left undefined and unexamined. Doing so invites a deficit framing of students in developmental classes by implicitly suggesting that developmental students lack necessary skills for engaging in college-level math work (e.g., Larnell, 2016). Indeed, characterizing developmental math courses as “not-college-level” suggests an assumption that students can only access college-level math by first repeating high-school course content; however, this assumption is problematic (e.g., Stigler et al., 2010) and unsupported by evidence.

Credit Characteristic

Characterizing developmental math courses as “not-for-credit” was particularly common in non-math education reports (and by extension, higher education research literature), which is consistent with a credential orientation that stresses degree progress. However, credits may not be a good measure of degree progress, and measuring credits in isolation de-couples equity from any direct relationship to learning outcomes. For example, there are many credit-bearing courses (e.g., precalculus) that carry credits but do not always “count” towards STEM degrees.

Previous scholarship has critiqued the non-credit characteristic of developmental courses for introducing stigma (e.g., Larnell, 2016) and not directly furthering degree progress (e.g., Logue et al., 2016). However, an oft-neglected issue with “not-for-credit” characterizations is that non-credit developmental courses have *inequitable time costs* for students. Students must still invest time in the material, regardless of whether credit is awarded. Simply attaching credits to all courses does not necessarily address this inequity. For example, co-requisite models often merge non-credit and credit-bearing math courses to produce a single for-credit course with additional non-credit hours attached (Meiselman & Schudde, 2021; Ran & Lin, 2022). While this may reduce the number of terms needed to access “college-level” math, it does not reduce the *time capital* students invest in the non-credit portions of the course. This exacerbates existing time inequities, as Black, Hispanic, female, and “non-traditional” students are more likely to both take developmental math (Chen, 2016) and to have disproportionately less time capital to invest in college (Wladis et al., 2018, 2021a, 2021b, 2022; Conway et al., 2021).

Critical structures, such as financial aid, do not give developmental students, or others with low time capital or high academic time demands, more time for college. Instead, higher time costs, which are the consequence of structural inequities, are borne by individual students (e.g., Wladis et al., 2018). Real systemic change requires an attempt to equalize time inequities, rather than focusing solely on credits, which requires shifting our definition of developmental math.

Content Characteristic

Characterizing developmental math courses as those that cover particular mathematical content was typically linked to whether that content was considered “college-level” or not. However, which courses signal the transition to “college-level” varies, ranging from Intermediate Algebra (e.g., Logue et al., 2016) to Calculus I (e.g., Hsu & Gehring, 2016). Sometimes “college-level” appears to be defined as “not-secondary-level”, but this is also contradictory, as many classes often considered “college-level” are regularly taught in high school: for example, courses above Algebra II (typically called Intermediate Algebra in college) almost universally carry college credit, yet 70% of students who enrolled directly from high school into college had taken a course above Algebra II in high school (IES, HSLS:09).

The content characteristic appears to be motivated by a learning orientation, yet learning objectives for these courses were systematically underspecified. Typically reports characterized content by the specific objects of study (e.g., linear equations) or specific course titles. But these characterizations tell us little about how students might be expected to engage with those mathematical objects. This is inconsistent with how courses are designated as "college-level" in other disciplines: for example, students might study Shakespeare in the secondary or postsecondary context, yet college Shakespeare courses are not typically classified as "developmental"[1]. This is presumably because it is not the specific *object* of study that determines the "level", but rather *how students engage with that object*. Mathematicians also do not classify "level" based on the objects of study, but by how one reasons about them: both first grade arithmetic and number theory focus on the integers, but at radically different levels.

Developmental algebra courses have typically taken the same teaching/learning approaches as 8th/9th grade Algebra I (Givvin et al., 2011; Grubb et al., 1999; Mesa et al., 2014; Stigler et al., 2010). However, college students are developmentally different from 14-year-olds: they have typically already passed an Algebra I class (e.g., Ngo & Velasquez, 2020); they have more sophisticated reasoning skills, independence, and life experience (e.g., Mesa et al., 2014); and they often excel in academic areas in which computational math skills are not a prerequisite. Yet, existing developmental algebra courses rarely leverage these strengths. Research suggests that developmental algebra students are capable of engaging in more rigorous reasoning, justification, generalization, and abstraction about core algebraic objects, without requiring extensive prerequisite courses in computation (e.g., Givvin et al., 2011). Thus, it is possible to offer no-prerequisite algebra courses at the college-level, if we re-define "college-level" as being measured by the *way in which students engage with those algebraic objects*. In fact, offering such courses is critical if we are to provide students access to college-level math: students need opportunities to develop the higher-order-math skills that will be necessary for upper-level math courses, rather than spending time repeating procedural processes that they already practiced in high school. This, along with other features of our problematization and interrogation of other definitional characteristics of "developmental math" in the literature, leads us to propose one potential new definition of developmental math.

New Definition of Developmental Math

In this section we present one potential new definition of developmental math, where developmental is no longer the complement to college-level, but rather a *subtype* of college-level. In particular, we propose to define developmental math as follows:

Definition: Developmental math courses in college are courses that (1) require **no extensive prerequisite knowledge of algebra** (or other computational skills beyond arithmetic), (2) provide students with **immediate access to college-level math** (defined based on the kinds of reasoning/justification, abstraction/generalization, and particular conceptions expected of students), and (3) provide students with **necessary time resources** for learning college-level math, based on their individual math time demands and access to time capital more generally.

This reconceptualization addresses several issues associated with current characterizations of developmental math. First, by positioning developmental courses around college-level skills that do not require algebra, this conceptualization is an asset-based approach that focuses on

[1] Special thanks to Carolyn James for the specific use of Shakespeare as a metaphor.

leveraging college students' strengths (such as their developmental maturity). Doing so also provides better access for *every* student to develop higher-order mathematical skills from the start, which are necessary to succeed in advanced math courses. Second, positioning these courses as college-level also eliminates stigma and practical problems associated with offering courses not-for-credit. Lastly, this conceptualization explicitly recognizes and addresses systemic *time inequities* that have been ignored by existing structures.

Reconceptualizing what developmental math *is* also requires reconceptualizing how success in developmental math is assessed. As previously discussed, measures of success require both a credential and learning orientation. Towards these goals, measures of success in a reconceptualized developmental math framework could include: 1) whether the courses meet the criteria stipulated in the developmental math definition above; 2) what students are actually learning in these courses (e.g., using concept inventories or other validated assessments [e.g., Carlson et al., 2010; Peralta et al., 2020; Wladis et al., 2018]); and 3) whether students are given the time resources needed to succeed in these courses (e.g., validated measures of time capital/demands [e.g., Wladis et al., 2018; Conway et al., 2021]). We note that if these three measures are met, we should also naturally see other positive outcomes (e.g., grades, persistence, general and STEM-specific degree progress and attainment). However, more traditional measures alone do not necessarily ensure that the first three measures are being met, and thus that equity in terms of access, learning, and time capital is being attained.

We recognize that our proposed definition requires a dramatic shift in how developmental math is implemented and thought about. If adopted, it also requires a renegotiation of undergraduate mathematics at large. This is not an easy proposition and would likely be the work of a generation of researchers and practitioners working together. However, substantial evidence suggests that existing definitions and implementations of developmental math are both inequitable and ineffective. We contend that dramatic reconceptualizations are necessary.

Conclusion

In this paper we have proposed to re-frame the definition of developmental math around two critical areas: (1) the extent to which developmental courses provide students access to true "college-level" math content (defined by the types of reasoning, abstraction and conceptions in which students engage, rather than computational skill), and (2) the extent to which developmental math courses provide students access to the time resources needed to address inequities in other resources (e.g., financial, prior educational access). These two foci of our revised definition align with the traditional learning and credential orientations, but in new ways that we have not seen represented in the literature. Our hope is that other scholars will build on this perspective, to further improve on our implementations of developmental math, so that it can better serve the equity goals developmental math was designed to support.

This new definition alone is not enough to solve all of developmental math's equity challenges. Research and advocacy are needed in many other areas, including: classroom and college climate, implicit bias, math anxiety and trauma, and specific teaching methods (e.g., Larnell, 2016; Martin, 2009; Mesa et al., 2014). This new perspective is just one important shift that we see as critical to more equitably serving developmental math students.

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What is College-Level Mathematics? A Proposed Framework for Generating Developmental Progressions in Mathematics up to and through College

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In this theoretical paper, our aim is to start a conversation about how "levels" in mathematics are operationalized and defined, with a specific focus on "college level". We approach this from the lens of developmental stages, using this to propose an initial framework for describing how learners might progress along a developmental continuum delineated by the kinds of reasoning/justification, generalization/abstraction, and types of conceptions that they hold, rather than by the particular computations learners are able to do, or the kinds of mathematical objects with which learners are engaging.

Keywords: college level; mathematical maturity; reasoning and justification; generalization and abstraction; conceptions

It is often assumed to be obvious whether a particular mathematics course is "college-level" or not; however, in practice, the transition point operationalized as "college-level" begins as early as Intermediate Algebra (Logue et al, 2016) and as late as Calculus I (Hsu & Gehring, 2016). In addition, determinations about which courses "count" as college-level are often based on syllabi that focus primarily on a list of computational skills on specific mathematical objects (e.g., linear equations, trinomials), rather than on how students reason with, justify, generalize or conceptualize mathematical ideas. Yet a conception of "level" that is driven more by the mathematical objects that are the focus of study, rather than *how* learners engage with those objects, contradicts many of the values of both mathematicians and mathematics educators about what high quality mathematics learning looks like. In addition, it further disadvantages students who may have strong higher-level thinking skills but who, for a variety of reasons, may not perform well on computational placement exams; currently, such students are often placed into non-credit bearing developmental courses that focus heavily on procedures, and which contribute to disparate impacts on college and economic outcomes (e.g., Bailey & Cho, 2010).

Mathematical learning is about more than just "content" (conceptualized as the specific mathematical objects of study or the particular procedures that students are expected to use): there are other skills, knowledge and practices that are important; yet much of this is left unarticulated in learning outcomes, and rarely used in the determination of course level. In this paper, we describe an initial framework for how we might define "college-level" mathematics, or more specifically, a spectrum of different "levels" of learning across the K-16+ mathematics curriculum as characterized by the kinds of reasoning, generalization, and conceptions that might describe developmental shifts or progressions in learning. This then includes a more focused discussion of how we might use such a framework to better articulate where the shift from K-12 to "college level" might occur in various mathematical domains.

Adolescent and Young Adult Development in Psychology and Neuroscience

It is known that the brain changes physically throughout the lifespan and that the last significant period of remodeling begins in adolescence and culminates in the early to mid-twenties. The parts of the brain most impacted by this last remodeling are those that control functions such as working memory, planning, and impulse control (Konrad et al, 2013). Based on this science, it is *developmentally appropriate* that college students should be able to interact with mathematical objects (e.g., algebraic expressions) in more sophisticated ways than younger students. Early educational literature, such as that of Piaget (1964), posited developmental stages for school-aged children that influence our expectations for what "grade level" means in subjects like math, reading, and writing. Psychology and neuroscience research acknowledge how understudied adolescents and young adults are and posit that much still remains to be learned about how brain development might impact behavior and learning (Blakemore, 2012; Shanmugan & Satterthwaite, 2016). In her survey of the field of adolescent brain imaging, Blakemore (2012) speculated about how changes in brain structure could make signal processing more efficient, which we speculate could have a direct impact on mathematics learning. This work in neuroscience compliments frameworks like that proposed by Erik Erikson (1994) for ongoing psychosocial development into and continuing through adulthood. Yet while it is known from neuroscience and developmental psychology research that college students differ developmentally from younger students, mathematics education frameworks have tended to ignore this when describing domains such as algebra that may be learned by students of widely varying ages. This paper seeks to explore how we might begin to conceptualize developmental stages as impacting how the same mathematical "objects" might be studied at different levels, with particular focus on what it might mean to do mathematics at the college level.

Proposed Mathematical Maturity Framework

In this paper we aim to problematize and redefine the term *mathematical maturity*. This term has been used in both research and practice to describe a kind of developmental progression like the one we hope to focus on here. However, this term has also been used in ways that are often vague and ill-defined; that provide deficit framings of students (e.g., "students can't take linear algebra before calculus because they don't have the mathematical maturity for the course"); and that describe binary destinations (e.g., students either have "mathematical maturity" or they don't) of what we conceptualize as a continuous life-long process of growth.

Mathematical maturity is a term used widely and often without formal definition within undergraduate mathematics education research and practice (Braun, 2019; Lew, 2019). In some instances, the completion of a specific course is used as an operational definition for the sake of a study, but even in those cases it is generally clarified that it is not the course *content* but a set of skills and increasing sophistication in how one approaches mathematics that is being referenced (Faulkner, Earl, & Herman, 2019; Lew, 2019). Two recent studies sought to determine how those using the term "mathematical maturity" define it. Faulkner et al (2019) interviewed engineering faculty and Lew (2019) interviewed mathematics faculty about their use of the term. Common definitions provided between these studies included many types of reasoning, generalizing, and conceptualizing, including the ability to: make connections across mathematical topics; use symbolic representations; relate different representations to one another and recognize when they describe the same phenomenon or relationship; choose between different representations for the purpose of solving problems; and understand if a solution to a problem makes sense.

Definition of Mathematical Maturity

Here we offer a new conceptualization of mathematical maturity via a framework, which we see as a starting point for describing how learners might acquire higher-level mathematical thinking skills and practices as they develop over time. We anticipate that this definition will evolve over time, but we present this as a first step, to start a conversation about what it means to learn mathematics at different "levels". A framework of this sort could then be used to generate new courses which could allow learners with missing "content knowledge" to nonetheless take a college-level, credit-bearing mathematics course that respects their different developmental stage compared to the age at which such content is traditionally introduced. For example, a course might require no algebra prerequisite, but allow students to engage with algebraic reasoning and justification at a higher level than would be expected in a typical K-12 algebra course.

We define ***mathematical maturity*** as a spectrum with no upper bound that describes the extent to which learners may acquire over time the ability to 1) reason and justify; 2) generalize and abstract; and 3) internalize particular conceptions of specific mathematical objects that occur throughout the curriculum. We conceptualize the process of developing mathematical maturity as a combination of physiological development and the outcome of particular mathematical experiences acquired over time. We now briefly present a framework for describing mathematical maturity which we have synthesized from existing research literature (Figure 1).

Increasing ability to reason and justify mathematically

Reasoning/Justification: To what extent are students expected to be able to reason (i.e., explain to themselves why/how something works), justify (i.e., communicate to others how/why something works), or prove (i.e., justify using more formal mathematical conventions accepted within a particular context)?

When reasoning, justifying or proving, what level of formality of language and convention is expected? (imprecise language vs. well-defined but informal language vs. formal mathematical terminology and/or symbols)

Increasing ability to generalize mathematical concepts

Generalization/Abstraction: How generalized is a learner's understanding expected to be (e.g., is the goal to understand a single example vs. a limited class of examples vs. a generic example)?
How explicit are students expected to be about the boundaries of the problem space? What kinds of connections between domains or representations are they expected to make?

Increasing ability to internalize conceptions of objects

Specific Conceptions/Concept Images: In a particular domain, which particular conceptions or concept images are learners expected to acquire (e.g., if a process vs. object transition, what is the specific object that is supposed to result from reification/encapsulation)?

Figure 2. Mathematical Maturity Framework for Describing Developmental Progression through K-16+ Mathematical Curriculum, Three Possible Dimensions

We contrast this approach with traditional conceptualization of "level" which have tended to focus on computations on particular mathematical objects as the primary feature which determines the "level" of a course (Figure 2).

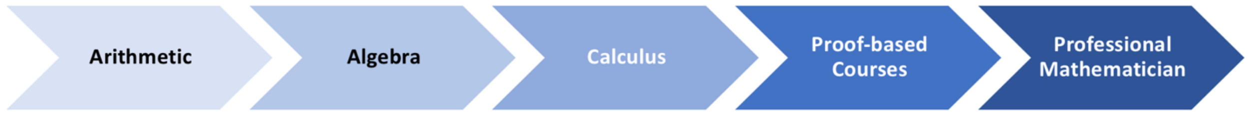

Figure 1: Informal model of progress through mathematics as currently conceived

In contrast, we consider which features are most relevant to determining the "level" at which the **same mathematical object might be learned at different points** in a students' K-16+ learning trajectory. We now describe each of the three dimensions of the framework in more detail.

Review of Literature from which the Proposed Framework was Drawn

Reasoning/Justification

One of the topics that is often discussed in the literature as a way of distinguishing whether students have "learned" mathematics, is the extent to which and ways in which students are able to reason or justify in mathematics. One of the major formal transitions in this area is when students are expected to generate mathematical proofs in college; substantial research has documented student difficulties with this. One explanation for this is that while students have experienced instruction focused on specific "content", students often do not come out of these courses with clear understanding of more general mathematical skills and practices, such as what constitutes mathematical proof (Selden & Selden, 2008). But formal proof simply describes one end of a much longer spectrum of skills, perspectives and practices. There have been extensive calls for students in K-12 to learn to reason and justify, long before the introduction of formal proof (e.g., National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010). And the processes of reasoning and justifying have been identified as critical mathematical skills that students may often not acquire during "standard" computational instruction (e.g., Mata-Pereira & da Ponte, 2017; Ball & Bass, 2003). Thus, reasoning and justification describes a core skill that is critical across mathematical domains.

Generalization/Abstraction

Another feature of mathematical "level" that often arises in the literature is the extent to which students are able to generalize about mathematical objects. To date, most of the research on generalization has been in the realms of early algebra (Carraher et al., 2008), pattern-forming (Amit & Neria, 2008), and linearity (Ellis, 2007) (see Ellis et al. 2022 for a more complete list). Notably, Ellis et al. (2022) is the first to consider students' generalizing activity across multiple domains and grade levels, ranging from middle school to undergraduates. Through their extensions of Ellis' (2007) taxonomy for categorizing types of generalization, they identified three main types: *relating*, *forming*, and *extending*. This framework enabled the researchers to identify what they considered to be generative (i.e., productive or useful) generalizing activity, and discuss how generalizing was both independent and dependent of the mathematical domain. We see this research area, and the RFE framework in particular, as having great potential for helping to illuminate our eventual taxonomy of delineating college-level mathematics.

Strongly related to generalization is the notion of abstraction. Abstraction may be defined as the processes that lead learners to grasp deeper understandings of mathematical structures, such as the underlying structure behind a vector space (Dreyfus, 2020, p. 13). Abstraction may also be thought of as a vertical reorganization of existing knowledge, or as a reconceptualization of information (as opposed to a de-construction). Many researchers have considered abstraction as part of a student's cognitive development, such as Piaget's ideas of empirical and reflective abstraction (Dubinsky, 2002), Thompson's processes and objects (1985), APOS theory (Asiala et al, 1997), Sfard's reification (1991), and Tall's structural abstraction (2013).

Conceptions/Concept Images

A third feature that often arises in studies of learners' progression through mathematical levels is the extent to which learners have particular conceptions about mathematical objects or concepts. One example that has been widely discussed is the transition documented by process-to-object theories (Sfard, 1991; Dubinsky, 1991; Gray & Tall, 1994), in which learners are theorized to conceptualize certain entities first as a process, and then later to reify/encapsulate that process into an object which can then subsequently be acted upon by even higher-order processes. For example, the expression $2x$ could be conceptualized as a process representing that 2 and x should be multiplied together. Later, a learner may conceptualize $2x$ as an object itself, representing the process of multiplying 2 by x or the result of multiplying 2 by x, without actually carrying out computation. Then $2x$ can be acted on by even higher-order processes, for example adding it to another object, $3x$, to obtain the result $5x$.

While students may switch back and forth between process and object conceptualizations, the ability to utilize an object conception is typically considered to be further along the developmental spectrum than using process conceptions alone (e.g., Sfard, 1991). Many higher-level mathematics courses also require object conceptions: for example, while arithmetic is rooted in a process conception of numerical computation, algebra requires that these same calculations be reified or encapsulated into objects (i.e., expressions/equations that can themselves be transformed using higher-order processes). Similarly, as algebra becomes more complex, students may be required to reify the process of the order of operations on algebraic expressions into subexpressions as objects (i.e., substrings of expression/equations that must be treated as unified objects); for example, this kind of higher-order structuring is necessary in order to perform function composition, u-substitution, or the chain rule in calculus. In fact, we can envision a larger progression in which one process is reified into an object, which is acted upon by higher-order processes which are themselves reified into an object, which is itself acted upon by even higher-order processes, etc. This progression has tended to be studied as individual shifts for one particular entity going from a process to an object, rather than discussed as a larger progression with many different shifts; however, original process-to-object theories precisely pointed out how reified objects became the focus of yet higher-order processes (Sfard, 1991), thus implying the existence of a larger progression containing many layers of more and more complex reified objects. Process to object views are likely not the only kinds of conceptual shifts that are expected of students as they progress through the mathematics curriculum; we present them here only as one example of how a key characteristic that determines the "level" of a mathematics course is the set of particular conceptions that learners are expected to internalize.

Brief Illustrative Example: The Distributive Property

In order to illustrate some of the affordances of the Mathematical Maturity Framework, we present one example of how this framework could be used to map out learning goals for the same object (the distributive property) at different stages of the K-16+ trajectory (Figure 3). The distributive property is first encountered in 3rd-5th grade, but is also the subject of study throughout the K-16+ curriculum. Currently, much focus is on which objects learners are expected to transform using the distributive property, yet research has documented extensive difficulties that students have in using the distributive property appropriately at many different levels (e.g., Malle, 1993; Schüler-Meyer, 2017). This may be because instruction often focuses on computation divorced from reasoning and justification. However, reconceptualizing the distributive property as a learning object by thinking about the types of reasoning, generalization, and conceptions students might use, could help us to shift our conceptions of how we determine the mathematical "level" of a particular course.

Algebra I (8-12th grade): Learners are expected to understand the symbolic representation $a(b + c) = ab + ac$ as a pattern in which a, b and c of the property represent objects (simple terms that are the product of a number and variable(s)). They are expected to conceptualize simple subexpressions such as px and qy as reified objects representing generic unknown numbers. The property is seen to hold because it represents two processes that produce the same numerical output for every possible numeric input from the domain into the expression which is being "transformed" by the property. They may or may not be expected to use an area model of multiplication to reason about or justify this idea, but they are not expected to prove the property, nor necessarily to generalize to algebraic objects with other forms.
Intro college-level algebra (lower-level undergraduate): Learners are expected to conceptualize the distributive property as a one-to-one mapping of specific reified subexpressions to a, b and c, respectively, in the property. Thus a, b, and c are seen as representing generic algebraic subexpressions and the specific reified subexpressions which are being mapped to variables in the property are seen as representing generic numerical values. The property is seen to hold because it is the process of replacing one expression with another equivalent one (substitution equivalence). The property is understood to be generalized to a generic number of terms, and students are able to describe this clearly but somewhat informally, and to justify why this is the case using an area model of multiplication. Reasoning and justification are expected, with well-defined language and some limited formal symbolism and terminology, but proof is not expected.
Abstract algebra (upper-level undergraduate or graduate): Multiplication and addition are conceptualized as abstract binary operations on an (often abstract) set, with a, b and c in the property representing generic set elements. Operations are defined axiomatically. The distributive law itself has been reified into an object: a property which a given set and pair of operations may or may not have. Learners are expected to prove, using formal mathematical terminology and symbolism, whether or not the left or right (or both) distributive properties hold for a given set with a given pair of operations (or for a larger class of sets with pairs of operations).

Figure 3. Examples of how levels of understanding of the distributive property might differ (as described by student learning goals) in high school versus lower/upper-level college, even when the objects which are the focus of the distributive property (algebraic expressions) are similar

The descriptions in Figure 3 are just one example of how we might describe learning outcomes which depict different levels of learning for a common mathematical "object". For students in the theoretical "intro college level" algebra class described in Figure 3, no extensive prerequisite proficiency in algebraic computation is required–however, once the learner starts working with the distributive property, higher-level reasoning/justification, generalization/abstraction, and reified objects are expected to be used and learned as a part of the curriculum. This is just one brief example of how using the Mathematical Maturity Framework as a tool for developing and describing learning in college-level mathematics may help us to shift our focus from computations on specific objects, to how students are learning to reason about, generalize and conceptualize specific key mathematical ideas as they progress along their mathematical learning trajectories up to and through college.

Conclusion

Our aim in presenting this framework is to shift our discussion of “college-level” mathematics (and levels in mathematics more generally) away from a focus on specific computations or particular mathematical objects, and towards a focus on reasoning, generalizing, and particular conceptual shifts. This reconceptualization can be particularly important from an equity perspective, since conflating computational skills with reasoning ability can be particularly detrimental to some of the most marginalized students. One example of this is developmental mathematics in college. The term “developmental” is often used to describe courses that are not “college-level”, but this definition is ill-defined in the research literature (Wladis et al, 2022) and may be circular (e.g., a course is developmental because it does not earn college credit and does not earn credit because it is not “college-level”). Most students in college are in fact re-taking mathematics which they already took in high school (e.g., 70% of those who attended college the year after graduating HS had already taken at least one math course above Algebra II [IES, HLS, 2009], and 52% of students who enroll in Calculus I in college have taken calculus previously (Sadler & Sonnert, 2016), yet what makes some of the courses that students repeat in college “college-level” and others not is unclear. We see this as a critical equity issue in mathematics: many college students (particularly those from more marginalized groups) are labeled “developmental” in college (with both stigma and practical barriers attached to these labels) because they are deemed “not ready for college-level work”, yet what it means to be ready for college-level work is not well-defined.

The transition from high school to college is also not the only transition point in mathematics learning in college that has been documented to be difficult for students. For example, many students struggle with the transition to proof in undergraduate mathematics (Selden & Selden, 2008). One reason the observed difficulty with many transition points into and through college mathematics may be that as a community we have not yet clearly enough articulated the specific goals of instruction, as defined in terms of particular types of higher-order thinking skills such as reasoning, generalization, and specific conceptions of mathematical concepts; nor have we adequately described on a larger-grained scale how we might expect students to progress through these developmental stages as they mature mathematically. This paper is an attempt to start a conversation about the potential of reframing “college-level” classifications based on specific high-level thinking skills, rather than organizing it around the specific mathematical objects to be studied or the particular calculations to be made. Our hope is that this will lead to more productive and equitable ways of teaching and assessing students in college, and across the K-16+ mathematics spectrum.

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Syntactic Reasoning and Cognitive Load in Algebra

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In the context of proofs, researchers have distinguished between syntactic reasoning and semantic reasoning; however, this distinction has not been well-explored in areas of mathematics education below formal proof, where student reasoning and justification are also important. In this paper we draw on theories of cognitive load and syntactic versus semantic proof-production to explicate a definition for syntactic reasoning outside the context of formal proof, using illustrative examples from algebra.

Keywords: Syntactic reasoning; Semantic reasoning; Cognitive load; Algebra

In this paper we outline a framework for analyzing student reasoning in mathematics, using the distinction between syntactic and semantic reasoning. This distinction has been a helpful framework for analyzing proof construction (Weber & Alcock, 2004) but has rarely been used as a framework for analyzing reasoning or justification in other mathematical domains where formal proof is not common. Even when syntactic reasoning is referred to directly in the context of proof, it is often conflated with manipulation of symbols without understanding. In this paper, we explicate a definition for syntactic reasoning outside the context of formal proof, presenting it as a form of reasoning that is distinct from rote "symbol pushing". We then use one example from school algebra to illustrate: 1) how syntactic and semantic reasoning play critical and complementary roles in this context; 2) how leveraging syntactic reasoning can reduce cognitive load; and 3) how preferences for syntactic or semantic reasoning approaches may relate to prior knowledge and schema. Our aim is to start a conversation about the potential affordances of more explicitly attending to syntactic reasoning in task and curriculum design and instruction.

Syntactic and Semantic Reasoning

In the literature on student proof construction, a number of studies have explored syntactic versus semantic reasoning during proof production. Weber and Alcock (2004) define syntactic proof production as drawing "inferences by manipulating symbolic formulae in a logically permissible way" and semantic proof production as using "instantiations of mathematical concepts to guide the formal inferences that [the prover] draws". In some work these categories are binary and assigned to a whole proof production, but in other work (e.g., Weber and Mejia-Ramos, 2009) syntactic and semantic reasoning are conceptualized as describing different steps in a students' reasoning process, where students may switch back and forth between different approaches; the latter is the approach we aim to take.

We also point out two key distinctions between our definitions of syntactic reasoning and those that have been used in proof literature. In particular, Weber and Alcock's (2004) definition of syntactic reasoning as drawing "inferences by manipulating symbolic formulae in a logically permissible way" is for us incomplete—we are not only interested in the result of a student's calculations, but also in their reasoning[1]. A second key distinction is that while Alcock & Inglis (2008) classify reasoning as syntactic if it takes place within the "representation system of

[1] We note that in other work, Weber (2005) distinguishes between *syntactic* and *procedural* proof production (the latter being based more on imitation of particular proof "templates" without necessarily understanding why they are valid)—this distinction is similar to, but different from our distinction between syntactic reasoning and symbolic manipulation that is not grounded in syntactic reasoning.

proof," we discuss a context in which syntactic reasoning can occur beyond proof production, which we call an "abstract symbolic system."

Abstract Symbolic Systems

We situate our definition of syntactic reasoning within *abstract symbolic systems*: self-contained systems in which the symbolic objects are what Tall et al. (1999) describe as *axiomatic objects*, or objects that arise "from specifying criteria (axioms or definitions) from which properties are deduced by formal proof" (p.239). This is also related to, but more narrowly defined than, what Goldin (1998) termed a **symbolic system**—a set of conventions and implicit or explicit axioms for the use of mathematical symbols which is:

1. **Abstract**, in the sense that it does not necessarily have any "real world" or alternative representations beyond what is specified by the axioms in the system.
2. **Self-contained**, in the sense that one has all the tools one needs already within the system to be able to identify or generate equivalent objects.

Our Definitions. *Syntactic reasoning* is reasoning which justifies mathematical work by referring back to conventions and axioms *within* the abstract symbolic system. *Semantic reasoning* involves justifying mathematical work by connecting it to representations or concept images *outside* the abstract symbolic system (e.g., the concept image of multiplication as area).

Distinguishing syntactic reasoning from "symbol pushing"

We have observed a tendency in the mathematics education community to frame syntactic reasoning as normatively undesirable, and semantic reasoning as normatively superior (see e.g., Easdown, 2009; Weber & Alcock, 2004). This is likely a reaction to approaches to teaching in which students are taught to manipulate symbols without connecting them to relevant underlying mathematical reasoning (e.g., Stacey, 2010). However, it is important to distinguish between syntactic reasoning versus symbolic manipulation that is disconnected from relevant logical reasoning.[2] We contend that the former is an essential component of mathematical reasoning, complementary to semantic reasoning, and necessary to manage the cognitive load of syntactically complex tasks. In our framework, syntactic and semantic reasoning are viewed as two essential and complementary components of mathematical reasoning.

For our definition of syntactic reasoning, it is not enough that a student manipulates symbols, even correctly—they must show evidence of reasoning within an abstract symbolic system. For example, a student may correctly transform with the distributive property in the following way: $2(3x + 5) = 2{\cdot}3x + 2{\cdot}5$. However, what reasoning this student has used is not clear from this work alone. Here are some examples of explanations that we would and would not consider to be syntactic reasoning (these examples are fictional, but have been written to mimic common forms of explanation that college students have provided in other empirical work):

1. ***Student 1 (neither syntactic reasoning nor semantic reasoning):*** *You take whatever is on the outside of the parentheses, and put it next to each thing inside the parentheses.*
2. ***Student 2 (syntactic reasoning):*** $2(3x + 5)$ *has the form* $a(b + c)$ *if we let* $a = 2$, $b = 3x$ *and* $c = 5$. *We know from the distributive property that* $a(b + c) = ab + ac$, *so we know that* $2(3x + 5) = 2{\cdot}3x + 2{\cdot}5$ *by substituting* 2, $3x$ *and 5 into the correct variables in the distributive property.*

[2] We use the modifier "relevant" because students who are "symbol pushing" often use complex or abstract forms of reasoning that are not connected to the relevant mathematical-logical justification.

3. ***Student 3 (semantic reasoning):*** *When you have two terms that are doubled, it doesn't matter if you add first and then double the result or double each term separately. Because either way you are still doubling each individual term.*

Student 1 describes a procedure but does not draw on any conventions or axioms of the abstract symbolic system in their justification, so this is not considered syntactic reasoning by itself. Student 2, in contrast, is drawing only on conventions and axioms given in the abstract symbolic system to justify their work. Unlike the others, Student 3 has drawn on meanings for multiplication and addition that go beyond what is given in the axiomatic system; this student is using other definitions of addition (e.g., as combining parts or terms) and multiplication (e.g., as doubling) to justify their work. Some readers may prefer the semantic explanation to the syntactic one or vice versa. We note only that both draw on distinct forms of reasoning, and that each approach directs cognitive resources towards something different. The syntactic approach directs resources in working memory towards parsing complex syntax so that a learner can view complex expressions as "having the form" $a(b + c)$. On the other hand, the semantic approach directs cognitive resources towards understanding *why* the distributive property can be derived from specific conceptions of addition and multiplication. However, the working memory resources necessary to process this semantic explanation may increase significantly as the subexpressions which represent a, b and c become progressively more syntactically complex. We likely do not expect students to justify the distributive property every single time that they use it, just as we do not expect students to semantically justify every sum or product every time that they calculate. Ideally, students would be able to employ *both* forms of reasoning when solving problems, switching *back and forth* between them strategically, in ways that both maximize their understanding of the underlying mathematics, and keep the cognitive load of a particular problem to manageable levels.

Cognitive Load and Strategic Selective Attention

There are several key features of cognitive load theory. Firstly, it characterizes learning as the process of encountering novel information, processing it in working memory, and then (to the extent that it is perceived as useful) encoding it in some way into long term memory (see e.g., Kalyuga, 2010). During this process of learning novel material, elements which are encountered during learning are encoded into mental schema[3], which vary in size and complexity based on the learner's expertise in a specific knowledge domain (de Groot, 2014; Ericsson & Kintsch, 1995; Sweller et al., 2019). The number of elements that can be held in working memory does not vary for "novices" vs. "experts" in a given domain; rather, it is the complexity of the mental schema that make up the individual elements which varies based on a learner's level of expertise in a particular area (de Groot, 2014; Ericsson & Kintsch, 1995; Sweller et al., 2019).

Thus, during the process of problem solving, how much cognitive load a particular task requires will vary based on the existing mental schema of the learner. A task that is perceived to have a dozen elements by a "novice" might be perceived as all fitting into a single mental schema for an "expert" in that domain (and thus from a cognitive load perspective, would take up only one element of working memory for the "expert"). In addition, the information encoded in mental schema may be automated, so that it is no longer processed consciously. This reduces the cognitive load for the learner, but also removes much of the work of solving problems from their conscious mental awareness (Cooper & Sweller, 1987; Sweller, 2011). This may have

[3] We note that the term "schema" is used differently in cognitive load literature and should not be conflated with more specific usage in mathematics education research (e.g., APOS theory [Dubinsky, 1991]).

consequences for both learners and instructors: for instructors who have automated particular complex schema, attempting to explain these processes which they no longer execute consciously can significantly increase their cognitive load during instruction (Lee & Kalyuga, 2014), and also make it difficult for instructors to accurately gauge the cognitive load that a “novice” (who does not have these automated schema) might experience from the same problem.

Strategic selective attention

Because syntactic and semantic approaches can have different implications for cognitive load, switching back and forth between these two approaches may be one way of strategically managing cognitive load within a single task. This could be described by theories of selective attention. *Selective attention* is a measure of the extent to which someone is able to filter out irrelevant information during the problem-solving process and focus only on the aspects of the problem that are salient to the task at hand (Broadbent, 1958; Treisman, 1964). Research has established that selective attention may be a key skill in mathematical problem-solving and is related to students’ working memory (see e.g., Arán Filippetti & Richaud, 2016; Campos et al., 2013). Most use of selective attention in the mathematics education literature focuses on it as a learner’s ability to filter out completely irrelevant information (e.g., on a one-step problem about red apples, ignoring information about green apples), but we focus on a related but slightly different aspect of selective attention, which we term ***strategic selective attention***: a student’s ability to *temporarily* ignore information that is not relevant to the *current* step in solving a problem (but which may be relevant at another step, or to interpreting the answer, etc.). For example, a student may temporarily ignore concept images that are helpful for reasoning semantically during a problem-solving step that is focused on syntactic reasoning (or vice versa), in order to lower their cognitive load.

Because strategic selective attention allows a student to temporarily ignore information that is not relevant to the current step, it narrows the amount of information that must be held and processed in working memory and can therefore be critical once mathematical problems become more complex. Thus, learning to work in valid ways with abstract symbolic systems by focusing on syntactic reasoning could be key to helping students reduce the cognitive load of many standard mathematics problems. To illustrate how strategic use of syntactic reasoning could help to reduce the cognitive load of mathematical problems, we present a few examples of how a standard algebra problem might be justified. We do not contend that any particular choice of when to reason syntactically versus semantically is right or wrong—this may vary for different people in different contexts.

Illustrative Example

As a starting point for discussion, we consider the following problem, which is a standard question in school algebra, typically introduced in 8th or 9th grade in the U.S.:

Example 1. Simplify $\frac{6x^2+2x}{2x}$ completely (assume $x\neq 0$).

We have chosen this example because it is fairly accessible, but student errors are nonetheless quite common, often involving invalid “cancelling” procedures, for example (Malle, 1993):

$$\frac{6x^2+\cancel{2x}}{\cancel{2x}} = 6x^2$$

Researchers have explained this invalid form of ‘cancelling’ in a variety of ways: generalizing from a limited set of examples where this heuristic holds to contexts where it no longer holds (Matz, 1982, p.26), and attending to visual similarities and patterns on the page (Erlwanger, 1973; Kirshner & Awtry, 2004). We note that if a student is performing this cancelling approach, they are *not drawing on important syntactic meanings within this abstract symbolic system*. For example, one critical syntactic meaning is that the numerator $6x^2 + 2x$ is a single unified object, and cannot be partially canceled, as is done in the work above. Another critical syntactic meaning is that canceling represents the replacement of a single fraction of the form $\frac{a}{a}$ with the equivalent expression 1 (assuming $a \neq 0$); it does not represent a "disappearance" of the objects being "cancelled".

A "semantic" reasoning approach

We now present a semantic reasoning approach, shown in Figure 1. However, we first note that no approach to this problem in its current form can be 100% semantic, because at a minimum, syntactic reasoning is necessary to read the symbolic expression. (For example, we need to know that when a number is written next to a letter, it means multiplication.) Therefore, because no approach justifying mathematics written using symbolic representations is 100% semantic, we write the word semantic in quotes for this overall example.

Step 1	$\frac{6x^2+2x}{2x}$ $= \frac{6x^2}{2x} + \frac{2x}{2x}$	**Semantic reasoning:** Fractions can be thought of as parts of a whole, where the top number represents the number of pieces and the bottom number represents the size of the pieces. So, if we want to split these two subexpressions into two separate fractions, it will have the same meaning as long as we use the same denominator for both fractions, using the denominator that was in the original fraction.
Step 2	$= \frac{(2x)(3x)}{2x} + \frac{2x}{2x}$	**Syntactic reasoning:** Because of the generalized associative/commutative property of multiplication, we can perform multiplication using any order or grouping (as long as only multiplication is involved). So $6x^2 = 6 \cdot (x \cdot x) = (2x) \cdot (3x)$.
Step 3	$= \frac{2x}{2x} \cdot (3x) + \frac{2x}{2x}$	**Semantic reasoning:** Dividing by a number is the same as multiplying by the reciprocal of that number because both dividing by c and multiplying by $\frac{1}{c}$ can be thought of as breaking the original number up into c-many equally-sized groups, and then taking the size of just one of those groups. **Syntactic reasoning:** Combining this semantic reasoning with the generalized commutative/associative property of multiplication, we can replace $\frac{(2x)(3x)}{2x}$ with $\frac{2x}{2x}(3x)$.
Step 4	$= 1 \cdot (3x) + 1$	**Semantic reasoning:** Dividing anything by itself will always be 1, because everything goes into itself only once, so $\frac{2x}{2x} = 1$ as long as x is not zero.
Step 5	$= 3x + 1$	**Semantic reasoning:** Multiplying anything by 1 is just like taking it one time, so it does not change it.

Figure 1: Using "Semantic" Reasoning

In Figure 1, the reader will notice the label "syntactic reasoning" for the two areas where the generalized commutative/associative property of multiplication is used. Students may be asked to justify this property semantically in specific cases, but are not expected to justify the general case semantically (i.e., that when multiplying any number of factors, neither the order nor the grouping matters). The generalized case is typically justified formally using proof by induction, which is by definition a syntactic justification that is not appropriate for most K-12 (or even college) students. Thus, most students who use the generalized commutative/associative property of multiplication are using it syntactically, as a stated axiom within the abstract symbolic system within which they are working.

Hidden forms of syntactic reasoning in our "semantic" example

Embedded within each step labeled 'semantic reasoning' is hidden syntactic reasoning which is necessary in order for the semantic reasoning to be connected to the symbolic representations. In Figure 2, we now fill in some of that reasoning, making the implicit more explicit for Step 1.

Step 1	$\frac{6x^2+2x}{2x}$ $= \frac{6x^2}{2x} + \frac{2x}{2x}$	**Syntactic reasoning:** In this expression, we can treat $6x^2$ and $2x$ at the top of the fraction and $2x$ at the bottom of the fraction each as a unified subexpression, because of the convention that the top and bottoms of fractions should be treated as unified subexpressions, and because according to the order of operations, exponents and multiplication come before addition. So, we can think of this as $\frac{a+b}{c}$ where a represents $6x^2$, b represents $2x$, and c represents $2x$. **Semantic reasoning:** Fractions can be thought of as parts of a whole, where the top number represents the number of pieces and the bottom number represents the size of the pieces. So, addition involving two subexpressions at the top of a fraction $(a + b$ $)$ represents addition of one number of pieces of the same size from another. So, if we want to split these two subexpressions into two separate fractions, it will have the same meaning as long as we use the same denominator for both, using the denominator in the original fraction.

Figure 2: First Step of Expanded "Semantic" Reasoning Example, with "Hidden" Semantic Reasoning Added

There are several interesting things to note about the expanded example in Figure 2. Firstly, in each step, the semantic reasoning is dependent upon some underlying syntactic reasoning. Although this reasoning is hidden, it is still needed to complete the problem. Thus, many semantic explanations provided to students during instruction and in curriculum may actually be more syntactic than they seem and may hide key parts of justifications from students.

Secondly, this explanation reads as quite long, because it combines the necessary syntactic reasoning with added semantic explanations which are, in a sense, justifications of the properties that need to be used when justifying syntactically (in a purely syntactic example these properties might be treated as axioms that do not require justification). Thus, there is increased cognitive load needed to provide both semantic and syntactic justification simultaneously. This increased load may be desirable in some cases and undesirable in others, depending on factors which will vary from one problem-solver to another, and from one instructional context to another.

The role of cognitive load in choices to implement syntactic vs. semantic reasoning

Many readers who are "experts" in algebra (e.g., algebra instructors, mathematicians) may prefer the short "semantic" justification in Figure 2 and feel that the expanded "semantic" justification in Figure 3 carries a higher cognitive load, because it contains significant amounts of "extraneous" information. For these experts, the syntactic reasoning that has been explicitly given and broken down in detail in the expanded explanation represents prior knowledge that they have already reified into unified schema and automated. Thus, unpacking this prior knowledge by breaking it down and making it conscious requires more cognitive effort.

On the other hand, for a problem-solver who lacks some of these schema, or who has not automated them (or has automated non-normative syntactic meanings), the additional information given in the expanded explanation may be helpful, or even essential, to understanding the “semantic” example. Including these additional details may make the “semantic” justification accessible in a way that it was not before. This contrast between how an “expert” and a “novice” might experience worked examples with additional explanatory information is similar to patterns that have been found in the research literature, in which additional explanatory information improved the performance of “novices” but slowed down “experts” who did not need it (the “expertise reversal effect”; see e.g., Kalyuga, 2007; Kalyuga et al., 1998). We note, however, that the cognitive load in the expanded “semantic” justification is still quite high. We see this expanded “semantic” justification as a useful tool for unpacking many of the different kinds of knowledge that are necessary for understanding the justification of the short “semantic” example; it is *not* intended to be presented as a useful example of how this might be taught to students.

We are left with the challenge of how to teach justification without overloading students’ working memory, even in cases where they have not yet acquired the necessary schema and automated syntactic knowledge to parse a short “semantic” justification. One option is to break down the complex network of interacting information into isolated elements which can be learned separately, each with a lower cognitive load individually. Studies that have employed this kind of approach in other contexts have shown that it can help students to learn complex interdependent types of knowledge which would have a too high cognitive load if learned all at once (see e.g., Pollock et al., 2002). Further research is necessary to determine the best methods.

Conclusion

In this paper we have described how syntactic reasoning could be defined for mathematical contexts that do not use formal proof. Our definition of syntactic reasoning is distinct from mere “symbol pushing.” It requires not just manipulation of symbols, but reasoning behind symbolic manipulation that draws on specific syntactic meanings of the abstract symbolic system in which the manipulations are being conducted. Thus, in our framework, semantic reasoning does not have to be present for productive and authentic mathematical reasoning to occur.

We have illustrated how syntactic and semantic approaches to problems may impact the cognitive load of particular tasks differently, particularly with respect to the existing schema of the problem-solver. We deconstructed one particular algebra task in order to illustrate many of the hidden ways in which semantic reasoning depends on syntactic reasoning as soon as algebra is written symbolically. This “hidden” syntactic reasoning may relate to the difference in cognitive load that “experts” and “novices” experience. “Experts” may have already automated much of the syntactic reasoning that they use, whereas “novices” may have no awareness of the role that syntactic reasoning plays. “Experts” may also be unaware of their own unconscious automated processes, which may perpetuate implicit, rather than explicit, handling of syntactic reasoning in curriculum and instruction. This suggests that to maximize rich understanding and minimize cognitive load for all learners, more research is needed to understand how syntactic reasoning can best be taught to learners with varied levels of prior knowledge, and to tease out how students and instructors can leverage strategic selective attention to switch between syntactic and semantic modes of reasoning.

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Reconceptualizing Algebraic Transformation as a Process of Substitution Equivalence

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In this theoretical paper, we describe how algebraic transformation could be reconceptualized as a process of substitution equivalence, and we discuss how this conceptualization affords mathematical justification of transformation processes. In particular, we describe a model which deconstructs the process of substitution equivalence into core subdomains which could be learned serially and then re-integrated, in order to make them accessible to students with lower prior knowledge in syntactic reasoning. Our aim in presenting this model is to start a conversation about what the core components of knowledge might be in order for students to reason about and justify algebraic transformation using symbolic representations.

Keywords: algebraic transformation; substitution equivalence; reasoning and justification; syntactic reasoning; cognitive load

Algebraic transformation has been identified as a core task of algebra (e.g., Kieran, 2004), yet students often struggle to transform algebraic expressions and equations correctly (e.g., Agoestanto et al., 2019; Dustin & Coleman, 2012). One reason may be that algebraic transformation is often taught procedurally. Without the connection to syntactic reasoning, students may not understand why certain transformations can be justified mathematically as preserving equivalence. For example, in the case of equations, students often do not realize that valid transformation produces an equation that has the same solution set as the original (e.g., Pilet, 2012, 2013). In this paper, we develop a theoretical model for how algebraic transformation could be reconceptualized as a process of substitution equivalence. Our model describes separate but related concepts necessary to use substitution equivalence to replace one expression or equation with an equivalent one during the problem-solving process. This model is the result of a decades-long design research experiment that we do not report on here; analysis of those data is the focus of other ongoing research (e.g., Wladis et al, 2022a, 2022b, 2022c, 2022d, 2022e). Instead, in this paper, our goal is to describe the theoretical model and its relationship to existing theory, including a discussion of its affordances in helping us to understand how learners might reason about and justify algebraic transformation.

Theoretical Framework: Generalizing Computational Versus Relational Views of the Equals Sign to Equivalence Relationships Preserved by Algebraic Transformation

Research on the equals sign distinguishes between whether students have a computational (the equals sign is a cue to compute what is on the left and put the answer on the right, which can lead to errors such as $2 + 4 = 6 + 2 = 8$) or relational (the equals sign represents a relationship between two equal quantities) conception of the equals sign (e.g., Stephens et al., 2013). We could similarly generalize this beyond the equals sign to any type of equivalence such as equivalent algebraic expressions or equations. For example, a ***computational view of equivalent algebraic expressions (equations)*** would describe a learner who sees transformation as a command to perform some sort of procedure on the expression (equation) to produce a resulting expression (equation), without realizing that there is an equivalence relationship between the original and resulting expressions (equations) (e.g., contributing to such errors as "cancelling" the $2x$ in the top and bottom of the expression $\frac{6x^2-2x}{2x}$ (e.g., Cunningham & Yacone,

2013) or to believing that $6x^2 - 2x$ could be interpreted to sometimes mean $6(x^2) - 2x$ and sometimes mean $(6x)^2 - 2x$, even though the two expressions do not produce equal outputs for most inputs of x). A ***relational view of equivalence of expression (equations)*** would view transformation as a way of replacing one expression (equation) with an equivalent one: thus, transformations are only permitted if they preserve equivalence (i.e., produce an equivalent expression [equation]). Further, reasoning about or justifying which transformations are allowed requires determining which preserve equivalence. The relational conception allows for justification of computation, whereas a purely computational conception does not. In particular, because the relational conception requires the notion of *replacing* one expression (equation) with an equivalent one, this is an example of substitution equivalence. Thus, algebraic transformation could be reconceptualized as a process of substitution equivalence, which could be viewed as a ***relational conception of algebraic transformation***. This conceptualization then also links the action of transformation directly to its justification: namely, whether it is equivalence-preserving.

A Motivating Example

In order to provide a concrete example of how algebraic transformation can be conceptualized as a process of substitution equivalence, we choose one common algebraic task:

Example 1. Simplify $\frac{6x^2-2x}{2x}$ completely.

We have chosen this particular task because student errors from employing a computational rather than relational view of transformation are common (e.g., invalid "cancelling" of $2x$); thus it is an illustration of some affordances of taking a relational approach. Here is one possible way that substitution equivalence could be used to begin to simplify this expression:

Step 1	$\frac{6x^2-2x}{2x} = \frac{6x^2}{2x} -$	$\frac{6x^2-2x}{2x}$ has the form $\frac{a-b}{c}$ (where $c\neq 0$), so we can use the property $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$ (where $c\neq 0$) by substituting $a = 6x^2$, $b = 2x$ and $c = 2x$ into the property, which gives us $\frac{6x^2-2x}{2x} = \frac{6x^2}{2x} - \frac{2x}{2x}$.
Step 2	$= \frac{(2x)(3x)}{2x} - \frac{2x}{2x}$	Because of the generalized associative/commutative property of multiplication, we can perform multiplication using any order or grouping (as long as only multiplication is involved). So $6x^2 = 6\cdot(x\cdot x) = (2x)\cdot(3x)$.

Figure 1. Example substitution equivalence justification for solution to Example 1

Components of a Relational View of Transformation

If a learner has no experience with reasoning syntactically, the details in steps 1 and 2 above would produce a cognitive load that is too high for a learner to understand or reproduce this solution. However, the explanations in steps 1 and 2 can be broken down in to more discrete knowledge elements (which we call subdomains) that could each potentially be learned separately, and then reintegrated later. By identifying these elements separately, we serve two goals: 1) this may allow us to better diagnose which specific conceptions are the cause of observed difficulties; and 2) this may allow us to teach substitution equivalence in smaller "chunks", limiting the cognitive load placed on learners. In Figure 2 we illustrate the model; we begin by describing each subdomain individually.

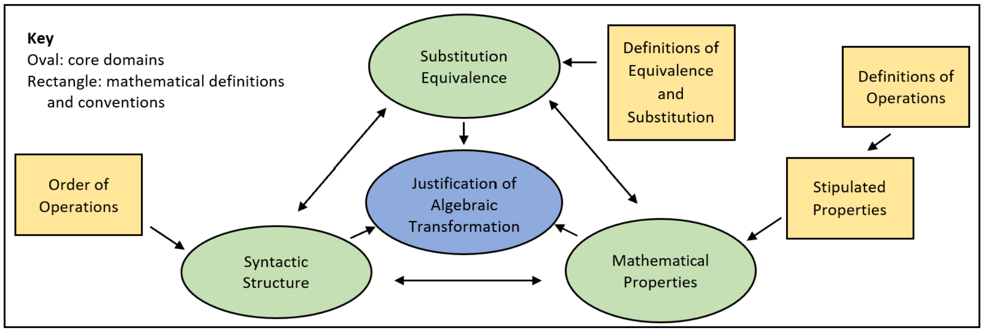

Figure 2. Model of Algebraic Transformation as Substitution Equivalence

Substitution Equivalence

To understand why $\frac{6x^2}{2x} - \frac{2x}{2x}$ can replace $\frac{6x^2-2x}{2x}$ in Step 1 above, a student must understand that any expression can be replaced with an equivalent one during the problem-solving process. Similarly, they must understand that if $a = 6x^2$, $b = 2x$ and $c = 2x$, then a, b and c can be replaced with $6x^2$, $2x$ and $2x$, respectively, in the property $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$, and the property will still be true because this is substitution of one subexpression for an equivalent one. During Step 2, they must also understand that $6x^2 = 6\cdot(x\cdot x) = (2x)\cdot(3x)$ means that the subexpression $6x^2$ can be replaced by the equivalent subexpression $(2x)\cdot(3x)$, and the resulting expression will be equivalent to the original one. All three of these processes are facets of the concept of ***substitution equivalence***: that one mathematical object can be replaced with another during the problem-solving process if and only if those two objects are equivalent (Wladis et al, 2022a).

Equivalence: Having a conception of substitution equivalence requires some underlying definition of equivalence, which in this case could simply be a stipulated insertion equivalence definition of expressions (Zwetzschler & Prediger, 2014, e.g., two arithmetic expressions are equivalent if they produce the same result for every possible combination of variable values). Or it could be a different definition of equivalence (e.g., a generalized equivalence relation definition). Regardless of the definition, in order to use substitution equivalence, a student has to recognize some key characteristics of the concept of equivalence more generally: that equivalence is a *stable relationship between two specific objects, based on some well-defined criteria* (Wladis et al, 2022a, 2022b). In other words, equivalence is a relationship (not a computation) between two objects (it requires two things to be compared) that is well-defined (it requires an unambiguous set of criteria for determining whether those two things are equivalent), and finally: two things are either equivalent or not, and they stay that way (two objects are not equivalent sometimes and not others; they don't "become" equivalent with transformation—rather, transformation reveals a pre-existing equivalence relationship).

Substitution: Using substitution equivalence also requires a more general definition of substitution, in which it is conceptualized as the replacement of one unified subexpression with an equivalent unified subexpression. For example, if a student only conceptualizes "plugging a number in" for a letter as substitution, it becomes difficult to talk about replacing $6x^2$ with $(2x)\cdot(3x)$ in the expression $\frac{6x^2}{2x}$ to generate $\frac{(2x)\cdot(3x)}{2x}$ as a process of substitution equivalence.

Thus, the domain of substitution equivalence describes the extent a learner can conceptualize equivalence as a stable relationship between two objects that meet well-defined criteria, and the extent to which the learner understands that one object can be replaced with another during the problem-solving process if and only if the two are equivalent, based on a stipulated equivalence relationship. This is more than being able to execute more complex substitutions correctly; it includes the ability to justify transformations because they preserve equivalence, by describing the particular equivalence relationship that is preserved by that transformation.

Syntactic Structure

Understanding the concept of substitution equivalence alone is not enough to have a relational view of transformations: for example, in Steps 1 and 2 in Figure 2, in order to perform substitution correctly, it is necessary to identify the correct unified subexpressions that can be substituted out or in. Another skill that is necessary to tackle the task above is the ability to parse the intended meaning of the syntax $\frac{6x^2-2x}{2x}$ by recognizing which substrings of the expression can be treated as ***subexpressions***, or which substrings of the expression could have brackets placed around them without changing the syntactic meaning of the expression (i.e., the expression would still represent the same operations on the same objects in the same order). In this example, this would mean being able to recognize that this expression has the following syntactic meaning as it is currently written: $\frac{(6(x)^2)-(2x)}{(2x)}$, and that $6x^2$ and x^2 are both subexpressions, but $6x$ is not.

Just as substitution equivalence can be seen as a more structural than computational approach (i.e., as a relational versus a computational view), understanding syntactic structure can also be seen as a shift from thinking computationally to thinking structurally. This is related to what has been observed by Sfard (1991) and others (Asiala et al., 1997 Dienes, 1969, Dubinsky, 1991, Gray & Tall, 1994), when they observe that an expression like $\frac{6x^2-2x}{2x}$ can be viewed as a process of squaring x, then multiplying 6 by that result, then separately multiplying 2 by x, then taking that result away from the first result, then dividing that result by the result obtained after multiplying 2 by x again. Or, it can also be seen as an object that is a reification/encapsulation of a process: the anticipated final result of the process described above (whether or not one has actually carried that process out). We focus on this slightly differently by focusing on how and whether a learner is able to identify *subexpressions as objects*. This requires more than simply a reification of the process of computation, but rather, a reification of the process of the order of operations: in the computational view, a learner would conceptualize the process of the order of operations as telling us that x must first be squared, and then 6 must be multiplied afterwards by the result; in contrast, in the structural view, a learner would conceptualize the order of operations as being reified into a fixed structure where x^2 is conceptualized as a unified subobject (i.e., a subexpression) which is a part of the larger expression $6x^2$.

We call the domain that includes knowledge of how to parse algebraic symbolic representations and to identify subexpressions ***syntactic structure***. This includes not just the ability to normatively interpret the symbols, but also the ability to link that interpretation to a normative justification (i.e., by explaining how the order of operations and other stipulated conventions dictate which substrings are subexpressions). Our definition of syntactic structure is closely related to the notion of *surface structure* as defined by Kieran (1989) and others in linguistics (e.g., Chomsky, 1966)[1], and is also related to Malle's *Termstrukturen*, or "expression structuring" (1993). We discuss this in detail elsewhere (Wladis et al, 2022c).

Using Mathematical Properties, or "Form Mapping"

Substitution equivalence and syntactic structure alone are not enough to justify the transformation work in Figure 1. To use the property $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$ (when $c \neq 0$) to determine that $\frac{6x^2}{2x} - \frac{2x}{2x}$ is equal to $\frac{6x^2-2x}{2x}$ in Step 1, a learner must do several things. First, they must map one-to-one each subexpression in $\frac{6x^2-2x}{2x}$ to each variable in the "form" $\frac{a-b}{c}$ so every symbol in $\frac{6x^2-2x}{2x}$ gets mapped to a symbol with the same syntactic meaning in $\frac{a-b}{c}$, and the mapping preserves the relative order of all the subexpressions and symbols in the expression. Second, learners must use the form $\frac{a}{c} - \frac{b}{c}$ to map the same subexpression to the same variable in $\frac{a}{c} - \frac{b}{c}$ as they did in $\frac{a-b}{c}$. Thus, the ***using properties (or form mapping)*** domain describes the extent to which a learner can construct one-to-one mappings from a symbolic representation to a mathematical property so that every symbol is mapped to a symbol (or syntactic convention) with the same meaning in the property, and each variable in the property is mapped to a subexpression (with the same variables mapped to the same or equivalent subexpressions).

As with the other two subdomains, this also involves a shift from a process to object view: the student must shift from conceptualizing the use of properties as plugging in one particular set of values (or variables) into the property, to thinking of the property itself as a canonical representation of a particular existing structure in the expression which they are attempting to transform. The form mapping required to use the property on more complex expressions requires that the student be able to think structurally about subexpressions as objects in the expression that they are trying to transform, as well as the relationship between these various sub-objects, and whether this is the same relationship as the relationship between various variables in the property. They must also have a relational view of equivalence, as the property must be conceptualized as a statement about the relationship between the original expression and the transformed result. A student might also reify the process of substitution into the particular form mapping object itself. There are many different objects which the student could conceptualize (the form mapping itself; the property as a canonical representation of structural relationships, etc.); the key difference is that the student is doing more than simply "moving around" symbols in the expression in an attempt to produce a pattern that "looks like" the property.

This domain includes not just the ability to use properties to correctly transform one expression or equation into an equivalent one, but the ability to reason or justify how a particular structural mapping of subexpressions and symbols in the expression/equation to various variables and symbols in the property allows us to make an argument about the equivalence of the original expression/equation and the resulting expression equation.

Learning subdomains serially

The subdomains of substitution equivalence, syntactic structure, and using properties are all deeply interconnected, and are all necessary in order to conceptualize algebraic transformation as substitution equivalence. But they need not be learned all at once; the cognitive load of such a

[1] Our use of the term *syntactic structure* should not be confused with Chomsky's use, which is different.

task is likely to be too demanding for learners with limited prior experience with syntactic reasoning. Thus, as a brief illustration, we demonstrate some ways in which aspects of these domains might initially be learned separately, or serially (and then later re-combined). In other contexts, this approach has been successful at improving student learning of complex ideas by reducing cognitive load (e.g., Pollock et al., 2002).

Syntactic structure and subexpressions

There are many ways that we could ask students questions that only draw on their knowledge of syntactic structure, and not require other types of complex and interrelated syntactic reasoning skills. For example, consider the following question, which limits the task not only to just identifying syntactic meanings, but also to identifying only *one* syntactic meaning at a time, significantly reducing the number of elements which must be held in working memory: "In the expression $\frac{6x^2-2x}{2x}$, what is being squared? Use the order of operations to justify your choice." In other research, we have found that many college students identify $6x$ as the base of the exponent instead of x (Wladis et al, 2022c), often because they have extracted their notions of which subexpressions "look right" based on experience, rather than reifying them from the process of the order of operations (even when they can recite the order of operations correctly, or use it to calculate correctly with numbers). This suggests that it may be essential to tackle syntactic structure individually, before proceeding to other syntactic reasoning skills which may be more complex and interrelated, and all of which depend upon a student first being able to identify the "right" subexpressions in an expression to be transformed.

It may also be necessary to ask students whether there is more than one right answer to this question. In our research, we have encountered students at many levels who have explained that an expression can have multiple correct meanings, where different meanings provided by the student are not equivalent (Wladis et al, 2022c). Thus, another component of this subdomain is discussing with students that all expressions must be well-defined, with one unambiguous meaning. Students may not realize that this is a core tenet of mathematics.

We note that determining whether a student has an object or a pseudo-object conception of the order of operations may be difficult to determine when looking only at "standard" problem contexts. Students have created pseudo-object mental schema precisely because they appear to mimic the subexpression structurings of expressions and equations that "work" during situations seen during instruction (e.g., Aly, 2022; Erlwanger, 1973). Often it only becomes obvious that students' justifications for choosing certain subexpressions are not mathematically valid when students are given more "non-standard" problems, or when students are asked directly how their choice of sub-expression relates to the order of operations (in our research, a common response, even from students in higher-level courses such as calculus was "it doesn't relate to the order of operations" [Wladis et al, 2022c]). Giving students explicit instruction in syntactic structure may act to mitigate this issue that has been observed elsewhere in the literature.

Substitution Equivalence

As with the syntactic structure subdomain, the substitution equivalence domain (and by extension equivalence subdomain) can be thought of as an element which could be learned separately, to reduce the cognitive load of learning syntactic reasoning all at once. For example, in this particular problem, it might be important to find out if a learner understands that replacing $\frac{6x^2-2x}{2x}$ with $\frac{6x^2}{2x} - \frac{2x}{2x}$ during step 1 is a process of replacing an expression with another equivalent expression. Some students may have a computational rather than relational view of equivalence of expressions or equations, where they see $\frac{6x^2}{2x} - \frac{2x}{2x}$ as the result of "doing something" directly to $\frac{6x^2-2x}{2x}$ and do not see an equivalence relationship between the two expressions (or may not even conceptualize equivalence as a relationship between two things, but rather as a process of computation) (Wladis et al, 2022a, 2022b). This can be seen particularly clearly when we look at the substitution equivalence that the student needs to recognize in order to perform step 2. We could, for example, limit the cognitive load of that step almost exclusively to the process of substitution equivalence if we asked: "Suppose that $6x^2 = (2x)\cdot(3x)$. Use this fact to replace the expression $\frac{6x^2}{2x} - \frac{2x}{2x}$ with an equivalent expression, and to explain why the new expression is equivalent to $\frac{6x^2}{2x} - \frac{2x}{2x}$." As long as a student has enough understanding of syntactic structure to know that the numerator of a fraction is always a subexpression, and they understand the notion of substitution equivalence, this information should be sufficient for them to be able to replace $\frac{6x^2}{2x} - \frac{2x}{2x}$ with $\frac{(2x)(3x)}{2x} - \frac{2x}{2x}$ and to explain why the two expressions are equivalent. This can later be combined with more robust knowledge of identifying subexpressions to combine the two domains of substitution equivalence and syntactic structure, after students have had opportunities to master key conceptions in each subdomain separately.

Using Properties/Form Mapping

As with the syntactic structure and substitution equivalence domains, the using properties/form mapping domain can also be thought of as an element that could be learned separately, in order to reduce the cognitive load of learning syntactic reasoning all at once. For example, in this particular problem a student could be asked: "Let $a = 6x^2$, $b = 2x$ and $c = 2x$. Then use the property $\frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}$ (where $c\neq 0$) to explain why $\frac{6x^2-2x}{2x} = \frac{6x^2}{2x} - \frac{2x}{2x}$." In this example, students need to have a basic idea of substitution equivalence, but they do *not* need to be able to identify the subexpressions of $\frac{6x^2-2x}{2x}$, as this has already been done for them. Thus, a task like this could be used to allow students to practice their knowledge in the domain of using properties, without yet requiring substantial knowledge of syntactic structure, and thus reducing the learner's cognitive load by allowing them to focus on fewer domains at a time.

After a leaner has had the opportunity to master substitution equivalence and syntactic structure, these separate conceptions that have been learned serially could be reintegrated with the using properties domain to complete questions like Example 1 without being given the specific values for *a*, *b* and *c*. Then, once the conceptions necessary to engage with these types of problems have been mastered, students could be given questions where they need to choose the particular property that could be fitted to the structure of a given expression (or equation); after that, they could be asked to select the property which serves a particular goal (e.g., producing an equivalent expression without parentheses, or with a particular form, etc.); and finally, after mastering each of these serialized tasks, they could progress to being asked to plan out the usage of a sequence of properties necessary to accomplish some larger goal.

Conclusion

In this theoretical paper we have aimed to identify and explore some necessary (but not necessarily sufficient) types of knowledge that are essential for students to be able to transform algebraic expressions and equations with understanding (by which we mean, to be able to reason about and justify these transformations in mathematically valid ways). We have framed this around the lens of substitution equivalence, with the aim of deconstructing complex knowledge structures into simpler component subdomains which could be learned serially before being reintegrated, to allow students with low prior knowledge in syntactic reasoning to build up this knowledge in ways that do not overload working memory. Our hope in presenting this model is to start a conversation about how we could more explicitly address reasoning and justification when teaching, and assessing learners' knowledge in, algebraic transformation.

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Professional Obligations in Mathematics Courses for Teachers

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This theoretical contribution draws on earlier work by Herbst and Chazan (2012; also Chazan et al., 2016) in which they describe the position of a mathematics teacher in an educational institution as accountable to stakeholders who issue four types of professional obligations. We propose an application and adaptation of that framework intended to address the case of instructors who teach undergraduate mathematics courses to future teachers. Considerations of not only the academic but also the professional ends of these courses are key in our application of the theory of obligations.

Keywords: instruction, professional obligations, professional preparation, undergraduate mathematics courses, mathematics courses for teachers

Understanding the work of teaching undergraduate mathematics courses for prospective elementary and secondary teachers is of interest to the RUME community (e.g., Hauk et al., 2017; Lai et al., 2019; Martin, et al., 2020; Yan et al., 2020). Separately, the professional obligations framework (Herbst & Chazan, 2012) has been found useful for RUME researchers to examine the decisions undergraduate mathematics instructors make (e.g., Bennett, 2022; Shultz, 2022; Shultz et al., 2022). In this theoretical contribution we offer an application and adaptation of the professional obligation framework to account for specific demands on the position of instructors of mathematics courses for teachers.

Improvement of Mathematics Courses for Teachers

Recommendations for improvement in the mathematical preparation of teachers, have posited that mathematics courses for teachers should include what researchers have called mathematical knowledge for teaching (e.g., AMTE, 2017; CBMS, 2012) for two reasons. First, research in mathematics education has documented how the mathematics needed for teaching includes types of knowledge that have not been commonly covered in university mathematics courses, even in those dedicated to teachers (Ball et al., 2001). Second, research in mathematics education has also documented that mathematical knowledge for teaching (MKT) can make a difference in the mathematical quality of instruction and in students' achievement (Hill et al., 2005; 2008). Given the prevalence of university-based teacher preparation in the number of entrants to the profession (Ronfeldt et al., 2014), it seems that if university mathematics courses for teachers could increase the MKT of new teachers (see Laursen et al., 2016), the improvement of mathematics courses for teachers could serve to improve the systemic capacity for mathematics teaching in K-12.

But because mathematics courses for teachers have been taught for decades (Ferrini-Mundy & Findell, 2010; Kilpatrick, 2019; Murray & Star, 2013), improvements in these courses need to

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consider and contend with extant curriculum, instruction, and assessment practices of those courses, all of which are anchored in institutional practices. Accounts of how individual instructors relate to such recommendations and how much they know about pedagogical content knowledge and the knowledge needed to teach teachers are valuable (Lai, 2019; Superfine & Li, 2014), but to make sense of how improvement could go, we also need accounts of the system in need of improvement (Bryk et al., 2015).

Some accounts of the system take the perspective of observers – for example noting the flow of people into K-12 teaching coming from university teacher education programs versus alternative certification (e.g., Ronfeldt et al., 2014) or charting the changes in MKT observed in courses of mathematics for teachers (Laursen et al. 2016; Pape et al., 2015). These observer accounts may help reformers understand what effects changes in university instruction may have in K12 instructional capacity. But that information by itself is unlikely to address some of the tensions instructors in these courses have to manage, which relate not only to their own knowledge and experiences (Lai, 2019) but also to the environments in which they work and the demands those environments place on them (Herbst et al., 2018). It is therefore important to understand the position and the roles of instructors of those courses. This involves not only understanding individuals in terms of what they are disposed to do but also the sets of systemic relationships and intact practices in which they are involved and that likely enable them to be so disposed (Bourdieu, 1990). To advance in this direction, the theoretical claim we make is that the professional obligations framework (Herbst & Chazan, 2012) can be used to describe how instructors of mathematics courses for teachers orient to their work.

The Professional Obligations of Mathematics Teaching

Chazan et al. (2016) describe the work of a teacher in terms of an institutional position–defined in relation to institutional stakeholders–and the various roles the teacher plays in the activities the institution expects them to engage. Mathematics instruction (narrowly defined as helping students learn a course of study) is a particularly important activity in which the teacher plays a role, but not the only such activity (e.g., mentoring youth is another such activity). The stakeholders Chazan et al. (2016) identify include (a) the Client–individual students and their advocates are stakeholders inasmuch as they are expected to benefit from schooling; (b) Knowledge–the accumulated knowledge and knowledge-producing practices of humankind, personified in scientific communities of knowledge producers, are stakeholders inasmuch as knowledge is to be disseminated and preserved through schooling; (c) Society–represented by community leaders, sponsors, and authority figures–is a stakeholder, putting at stake shared values, customs, needs, and goals of a society, community, or nation inasmuch as those stakes are to be interpreted, disseminated, and preserved for a new generation through schooling; and (d) Organization–the administrative, economic, and legal aspects of an organization are stakeholders inasmuch as school institutions need to abide by them. While these stakeholders are stakeholders of schooling and thus make room for instruction as well as other activities, they serve to specify the position of a teacher inside an educational institution, providing sources of justification for decisions that an individual teacher might make in the activities they engage in–particularly, in instruction. Chazan et al. (2016) identify these sources of justification as professional obligations, and they name them (a) the individual obligation (to students as individuals); (b) the disciplinary obligation (to the knowledge disciplines that are sources of the content of instruction); (c) the interpersonal obligation (to the society in which the class of students are to integrate themselves); and (d) the institutional obligation (to the various institutions that serve as environments for the activities in which the teacher plays a role). Those stakeholders structure the institutional space in which (mathematics) instruction takes place and individuals hired into teaching positions are socialized into roles in various activities in which it becomes natural for them to feel obliged in the ways that the four obligations describe. The obligations neither erase individual responsibility nor prevent individual agency; rather, they identify resources and constraints available to individuals to publicly warrant actions and decisions.

Those warrants for actions and decisions may be especially useful in the activity of mathematics instruction, especially as our field concerns itself with instructional improvement. Because instructors' agency needs to be co-opted in order to make some such improvements, it is important for improvement efforts not only to acknowledge the expectations teachers work under but also to identify resources that may support their participation in institutional argumentation around changes in practices. If instructional improvement requires instructors to act in ways that deviate from the expectations they or their students have coming into class, justifications may be needed. The professional obligations can help provide such justifications. Further, if recommendations for instructional improvement may seem to push teachers beyond what feels viable to them, the professional obligations may help practitioners negotiate those recommendations.

Because practical rationality and the obligations framework describe the position in which instructors are and the role they are expected to play in instruction (and because they are not a theory of action for improvement along a particular direction), they can identify sources of justification that instructors might use to argue both for and against change in any direction. As such, this framework can help researchers understand reactions against reform and it can help frame institutional discussions about improvement that acknowledge where different parties are coming from. However, practical rationality was originally developed to examine the position and the instructional role of mathematics teachers in academic contexts (that is, in the study of mathematics, be that in K-12 schooling or in mathematics courses for mathematics majors). The application of the framework to the mathematics instruction of professionals, who will use mathematics to do professional work requires some adaptation.

Expanding the Theory of Practical Rationality to Account for the Experiences of Instructors of Mathematics Courses for Teachers

We contend that in order to understand the position and the instructional role of those who teach mathematics courses for teachers, it is worth describing the instructional practice in which they are involved as contested by a tension between two different instances of the instructional triangle (Cohen et al., 2003): Each instructor of undergraduate mathematics courses for teachers is in a position from which they can be disposed to see the course they teach as both an undergraduate mathematics course and as a mathematics teacher preparation course.

Much work in mathematics education that attends to the work of teaching implements this attention by looking at the individual who does the teaching–for example, in terms of their beliefs and knowledge. In the case of mathematics courses for teachers, an important distinction among instructors runs along issues of academic preparation and professional identity: Individual instructors may have doctoral degrees in mathematics or in mathematics education, and, in some cases, they may not have doctorates; their professional identity may include having been teachers or not, doing mathematical research, doing mathematics education research, doing both kinds of research, or doing no research. Those contingencies may impact the way in which they take up their position (e.g., hired to teach mathematics courses for teachers vs. hired for other reasons and assigned to teach such courses). While we do not deny the importance of attending to those sources of individual differences, understanding how they matter in the way individuals play their role in instruction and how they recognize the various obligations is beyond the scope of this paper. In proposing an adaptation of the obligations framework from the theory of practical rationality, we are deliberately bracketing out those sources of individual differences to describe the set of resources and constraints that support the position in which all instructors of mathematics courses for teachers are hypothesized to be and arguing that it is more complex than that of instructors of mathematics courses for academic and scholarly preparation (e.g., mathematics courses for mathematics majors). We sketch this adaptation below and indicate what further research could be done to gather evidence of how these obligations operate.

The Institutional Obligation on Instructors of Mathematics Courses for Teachers

The institutional environments in which one finds mathematics courses for teachers include not only the mathematics departments that offer these courses and the colleges which employ the instructors but also the teacher education programs that admit the students and require them to take these courses. At least in the United States, these tend to be separate organizations within universities. All those organizations hold the instructor accountable in some way. Some of those mechanisms of accountability may be common across organizations, unspecific to the courses, and inscribed in rules and policies (e.g., work contracts, class schedules). Mathematics departments also hold instructors accountable (e.g., expectations for office hours and syllabi). Teacher preparation programs are likely to exercise this accountability through expectations that become visible when they are violated (e.g., education advisors letting teacher education students fulfill alternative requirements when they fail mathematics courses for teachers, or students themselves feeling empowered to confront mathematics instructors on matters of content choice when they don't see its relevance for future teachers). We contend that the location of mathematics courses for teachers in institutional environments that include these various organizations makes for instructors' institutional obligation to be more complex than that of instructors of mathematics courses for mathematics majors. We suggest that research could endeavor to elicit from instructors narratives of episodes in which their instructional decisions were shaped by explicit interactions with teacher education personnel or expectations they have internalized from prior interactions with teacher education faculty or students. Research could elicit from mathematics department chairs and from teacher education coordinators the expectations their respective institutions have for these courses. What experiences do teacher education programs expect these courses to provide their students and how do they convey these expectations to mathematics instructors? Are those expectations different among mathematics courses for elementary and secondary teacher preparation? How do those expectations play out in recruiting and evaluating instructors of those courses?

One strategy that mathematics departments have employed to ensure attention to the needs of teacher education programs is to employ individuals with degrees in mathematics education or who participate in mathematics education professional communities to teach mathematics courses for teachers. Research could document the diversity of episodes in which instructors with different professional profiles are called to attend to the institutional obligation as well as how the departmental discussions on curriculum, pedagogy, and assessment in mathematics courses for teachers incorporate attention to the expectations from teacher education programs. Research might also document how teacher education program discussions incorporate and process information from instructors and students of mathematics courses for teachers (e.g., about the quality of students' mathematical work or the difficulty of courses). While it seems auspicious that mathematics departments might employ faculty interested in and prepared for mathematics teacher education in teaching these courses, we cannot assume that a change of instructor will eliminate any need for institutional accountability on the part of these instructors. Instead, it seems important to describe how this institutional accountability is experienced by instructors with different preparation and to gather case knowledge on how instructors handle institutional influences on their instruction.

The Individual Obligation on Instructors of Mathematics Courses for Teachers

The obligations framework proposes that students, as the clients of education institutions, hold teachers accountable to serve them as individuals. The case of students in mathematics courses for teachers is similar insofar as they are undergraduate college students: Instructors are obligated to attend to them as whole persons, with cognitive, emotional, and physical needs and goals. As young adults, students are likely to be able to advocate for many of these needs themselves rather than through their parents. But for instructors of mathematics courses for teachers, the individual obligation seems more complicated. We suggest that the instructor may also need to be accountable to their students as future professionals, attending to other aspects of their individuality that the students themselves may not be as ready to advocate for. These include aspects of the professional identity of a teacher that will be important for the individual presentation of self of these novice teachers when they commence their professional life. Instructors' accountability to students as clients may compel them to see their students not only as individuals with present needs and desires but also as future professionals with future needs.

Research could elicit instructors' narratives to find out what aspects of students' future professional identity emerge in the course of instruction as compelling opportunities for instructors to make individual accommodations. Personal characteristics often part of moral character (e.g., self-control or intellectual honesty), individual beliefs (e.g., about mathematical ability), and skills (e.g., voice projection) might be included and that instructors may recognize opportunities to build students' sense of professional identity when characteristics like those emerge in the context of classroom work. Research should especially attend to how instructors negotiate with prospective teachers the need for them to learn material that might seem difficult or uninteresting to them at the moment (during college years) and whether and how instructors' recognition of an obligation to students as future professionals serves them in such negotiation.

The Interpersonal Obligation on Instructors of Mathematics Courses for Teachers

The obligations framework proposes also that society is another stakeholder of the work of teaching and the source of a professional obligation for instructors to conduct their classes in ways that steward social values and goals. Insofar as an undergraduate class includes young adults that have to interact with each other in socially productive ways and that will incorporate themselves into the social world after college, it is likely that instructors will feel compelled to steward some social values and expectations that apply to everybody (e.g., taking turns in conversation, respecting the personal space and property of others). But insofar as the undergraduates will be teachers, some of those values and expectations may concern the role that teachers play in society. A salient one is a sense that society needs teachers for its children (Beckmann, 2011) and that these teacher cadres need to represent the diversity of society (e.g., Bristol & Martin-Fernandez, 2019; Frank, 2019). This has implications for instructors with regards to the ways that their courses and the grades they assign in those courses tend to serve as a gateway not only for individual students to get their degrees but also, and especially, for society to receive its teachers. Students' interactions with instructors and their course material can also affect students' continued interest in the profession of teaching. The interpersonal obligation suggests societal needs that should compel instructors not to alienate students from mathematics or discourage their desire to teach mathematics in K-12 contexts.

We suggest that research could better unpack how instructors encounter and respond to opportunities to attend to the interpersonal obligation. What are some ways in which instructors' obligation to recruit and form teachers that can serve society is and can be displayed? How has it been ignored?

The Disciplinary Obligation on Instructors of Mathematics Courses for Teachers

The disciplinary obligation as proposed by Herbst and Chazan (2012) alludes to mathematics instructors' obligation to represent the discipline of mathematics when they teach in classrooms. One could expect this disciplinary obligation to be present also for instructors of university mathematics classes, including mathematics courses for teachers. For example, because textbooks and lecture notes are presentations of course content rather than disciplinary communications (as journal articles are), it is expected that this didactical transposition of knowledge (Chevallard, 1991) may depart from the norm in mathematical communication (e.g., sometimes stating a proposition without providing proof or providing a definition that goes beyond necessary and sufficient conditions). Instructors might show recognition of their obligation to the discipline of mathematics if in the context of such pedagogical practices they pointed out to students that the statement could be proved even if the proof is beyond the scope of the course or that some of the stipulations of a definition are implied by others.

But in addition to an obligation to the discipline of mathematics, Knowledge as a stakeholder can be expected to hold instructors of mathematics courses for teachers accountable in other ways. The progress made by mathematics education research in the last few decades added to the fact that this scholarship has become more widely known among and practiced by college mathematics instructors (such as RUME participants), suggest that instructors might recognize a disciplinary obligation toward the knowledge generated in mathematics education research. This scholarship includes at least two types of knowledge worth noting, both contained within the notion of mathematical knowledge for teaching and illustrating the interdisciplinary nature of mathematics education research. The first is specialized knowledge of mathematics or mathematically specific knowledge needed for teachers to engage in tasks of teaching mathematics. As Ball et al. (2008) have argued and illustrated, this is purely mathematical knowledge that reveals itself as needed in the context of tasks like creating problems for students or analyzing the mathematical errors students make in problems. The second is empirical knowledge about students' understanding of mathematics and the ways in which students might respond to instructional strategies; that is, elements of pedagogical content knowledge. We distinguish these two types of knowledge because their truth status results from different types of inquiry, one more deductive, founded on mathematical reasoning, and the other inductive, founded on scientific reasoning. We suggest that research could elicit from instructors narratives of episodes in which their awareness of research on mathematics teacher knowledge has compelled them to deviate from, qualify, or add to the content they teach.

Furthermore, the professional practical knowledge of mathematics teachers, which is not warranted solely by mathematical or scientific reasoning but also informed by policy and educational politics, may contribute yet a third source of accountability to knowledge for instructors of mathematics courses for future teachers. Documents such as the Common Core Standards (National Governors Association, 2010), exercise a transdisciplinary accountability similar to that of the disciplinary mathematical knowledge or the interdisciplinary knowledge from research in mathematics education. For example, while a mathematical concept (e.g., geometric transformations) might be well defined in different ways, a Standards document may implement a particular choice of definition to be used in K-12 settings and that choice may serve as a justification for the instructor of a geometry course for teachers to make instructional decisions (e.g., not to adopt the definition provided in the textbook, but to change it to match that of the Standards). Similarly, professional organizations such as NCTM have at times advocated for including the history of mathematics with its teaching as means to show students that mathematics is a human activity; such recommendations add transdisciplinary resources and constraints to instructors. The investigation of the instructors' position needs to account for the various knowledge sources of the disciplinary obligation. Because these disciplinary sources come from different origins and epistemologies, it seems important to inventory not only the ways they present themselves as opportunities or as constraints to instructors, but also to document how instructors handle them when they might not align with each other.

Conclusion

In their approach to improvement science, Bryk et al. (2015) argue that a needed step toward improvement is to secure an understanding of the system in need of improvement–how the system works–from the multiple perspectives that are called upon to work toward improvement. Instructors are important in improvement not only because their agency can (and needs to) be co-opted to realize improvement but also because their perspective on how the system works can help anticipate what may happen with those improvement efforts (including their own). Instructional improvement thus requires not only relying on instructors as agents of change but also incorporating the rationality (in the sense of sensibility or sense-making) of instructors as a vital component of improvement efforts. We contend that incorporating this rationality means seeking ways to reconcile (a) what instructors are disposed to accept as part of their roles in instruction and how they perceive their position in the higher education system and (b) how their position in a system of power relations and institutional regulations conditions them to be so disposed (Bourdieu, 1990). We argue that at the college level, where traditionally instructors of record have had much authority to decide what to do, it is important to understand not only who the instructors are and what they know or believe, but also how they see the system in which their effort is inscribed and that avowedly needs to be improved. In our discussion of the obligations as they apply to instructors of mathematics courses for teachers, we elaborate on how each of these obligations might split into different branches. Better understanding empirically how these various branches of professional obligations actually show up for instructors of mathematics courses for teachers can help in the quest of improving the mathematical preparation of teachers. Eventually, measures can be constructed to assess the extent to which individual instructors recognize the different aspects of each obligation as hypothesized above. Further, similar kind of layout for empirical inquiry could guide the study of instruction in mathematics classes for other professionals (e.g., engineers, nurses).

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The role of derivatives across multivariable and vector calculus instruction using multivariational reasoning and students' interpretations of matrix equations

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Multivariable and vector calculus (MVC) encompasses a wide array of different types of functions and the extension of numerous ideas from univariable calculus to MVC contexts. In this paper, I reflect on some of the more recent work on student thinking and learning of derivatives in MVC settings by leveraging some of the recent work on students' multivariational reasoning and student thinking and learning of linear algebra topics. I conclude with examples of ways to conceptualize derivatives as linear transformations to support students in progressing towards a unified notion of the derivative concept throughout MVC instruction.

Keywords: student thinking, multivariable calculus, derivatives, partial derivatives

Students in the United States are first introduced to the notions of multivariable and vector-valued functions in a multivariable and vector calculus (MVC) course. As a lead up to studying vector-valued functions the course introduces vectors, vector operations, graphing in three-dimensional coordinate systems, partial and directional derivatives, multiple integration, line and surface integrals, vector fields, the gradient operator, and the fundamental theorems of vector calculus (Stewart, 2011). Each of these concepts is important, but I am concerned that students perceive MVC as a collection of disconnected ideas and shortcuts (Harel, 2021), as opposed to an interconnected web of meanings. I am curious about ways students' understandings of MVC ideas are interconnected and how they extend their understanding of rate of change and accumulation functions (P. W. Thompson et al., 2019) to MVC contexts. In this paper, I want to reflect on ways to use current research in mathematics education to draw attention to how nuanced students' ways of thinking about MVC topics can be. To demonstrate my intention, I isolate my focus to the various uses of partial derivatives detailed across recent math education literature and reflect on how to make connections between students' reasoning about partial derivatives and their meanings for their symbolizations of linear approximations for different functions. I draw on co-variational and multivariational reasoning (Thompson & Carlson, 2017; Jones, 2022), in relation to an altered version of a theoretical framework from linear algebra (Andrews-Larson & Zandieh, 2013; Zandieh & Andrews-Larson, 2019).

A Review of the Literature on the Teaching and Learning of Partial Derivatives

Over the last five years, there has been an increase in mathematics education research investigating students' understandings and ways of thinking about multivariable calculus topics. These topics include: multiple integration (Jones & Dorko, 2015), ways of thinking about scalar and vector line integrals (Jones, 2020), students' treatments and conversions of graphical registers for gradient vectors (Moreno-Arotzena et al., 2021), manipulatives for supporting students in graphing surfaces (Kang et al., 2020; Wangberg, 2020; Wangberg et al., 2022), students' covariational reasoning about partial derivatives (Mhkatshwa, 2021), and students' multivariational reasoning (Jones, 2018, 2022). The various ways that partial derivatives are seemingly used by students and researchers are described in Fig. 1. Each use of partial derivative described in Fig. 1 is nuanced, but heavily emphasizes contexts of functions of two variables.

Interpretation	Description of interpretation	Related articles
Partial derivatives as slopes of tangent lines to traces	Intersect the graph of the function $z = f(x, y)$ with a plane $x = a$. Find the coordinate triple $(a, b, f(a, b))$ and draw the tangent line in the y direction contained in the plane $x = a$ at the given point. The slope of that tangent line is the value of the partial derivative of f with respect to y $f_y(a, b)$.	(Martínez-Planell et al., 2017)
Directional derivatives as slopes of a tangent plane in a particular direction	Intersect the graph of the function $z = f(x, y)$ with the plane $z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$ at the coordinate triple $(a, b, f(a, b))$. The change in depth (variation in z) measured with respect to the horizontal displacement (the length of the direction vector $\vec{u} = \langle \Delta x, \Delta y \rangle$ which is parallel to the xy plane) is the value of the directional derivative $D_{\hat{u}} f(a, b)$.	(Wangberg et al., 2022) (Martínez-Planell et al., 2017; McGee & Moore-Russo, 2015)
Partial derivatives as rate of change functions	Given an amount function $z = f(x, y)$ the rate of change of z with respect to y as y varies through a small-enough interval containing $y = b$, mentally holding the value of x fixed at $x = a$, is the value of the partial derivative $f_y(a, b)$.	(Mhkatshwa, 2021)
Partial derivatives as components of the gradient vector	Given a multivariable function $z = f(x, y)$, the gradient is a vector pointing in the direction of the greatest directional derivative $\|\|\vec{\nabla} f(a, b)\|\| = D_{\hat{u}} f(a, b)$ whose tail is located on the contour plot of f at (a, b). The x component of the gradient is the partial derivative of z with respect to x and the y component of the gradient is the partial derivative of z with respect to y so that $\vec{\nabla} f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle$.	(Moreno-Arotzena et al., 2021)
Partial derivatives as entries in the Jacobian matrix	Given a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, the linear transformation, represented by the Jacobian matrix J, produces the best, local, linear approximation for f when $x_1 = a$ and $x_2 = b$. The components of matrix J are all of the first-order partial derivatives of the components of a given function, where each row is the transpose of the gradient of each component function.	(Harel, 2021)

Fig. 1: Interpretations of partial derivatives across recent math education research literature

Some of the earlier mathematics education research on MVC instructional design investigated students' responses to tasks within cycles of a Genetic Decomposition from APOS (action, process, object, scheme) theory which also leveraged Duval's theory of semiotic registers (Trigueros Gaisman & Martínez-Planell, 2013). The Genetic Decomposition for teaching partial derivatives of two-variable functions emerged from characterizing in-class activities for teaching two variable functions (Martínez-Planell & Trigueros Gaisman, 2019) and partial derivatives and directional derivatives through explicit lecture on 3D slopes (Martínez-Planell et al., 2017). McGee's tactile graphing manipulatives reinforced the notion that explicit attention to 3D slopes may provide students with a way to measure amounts of vertical change in terms of an amount of horizontal change, corresponding to the value of the directional derivative of a given two-variable function (McGee et al., 2012, 2015; McGee & Martínez-Planell, 2014; McGee & Moore-Russo, 2015). Students reasoning about 3D slopes (in the x and y direction respectively) and geometric conceptions of constant rate of change (as slopes) provided them with ways to reason about the values of partial derivatives graphically in 3D space. The research detailing the instructional implementations of the 3D acrylic surfaces of Wangberg and colleagues' also supported the notion that students reasoning about multivariable rates of change in graphical contexts require ways to envision measuring amounts of vertical in terms of the corresponding horizontal change to conceive of the values of directional derivatives as the slope of the surface in a given direction (Wangberg, 2020; Wangberg et al., 2022). Reasoning about surfaces as something to be traveled upon is extremely useful, however care should also be taken to ensure students make connections between tactile approaches to graphing surfaces and also reasoning about how the shape of the graph is a record of how the values of three quantities change together (Moore & Thompson, 2015; Weber & Thompson, 2014).

There are fewer research articles investigating how students conceive of vector-valued amount and rate functions. Jones investigated how students think about scalar and vector line integrals (Jones, 2020). Researchers have also investigated the ways students think about vectors and vector operations in the context of linear algebra (Lee, 2022), or physics (Dray & Manogue, 2006; Roche, 1997). However, there is a need for research experiments investigating how students think about and learn calculus ideas related to the derivatives of vector-valued functions, (i.e., divergence, curl, Green and Stokes' theorem, etc.). The research literature in physics education has also addressed ways to support students in using partial derivatives, and the relevant notation, to reconstruct Maxwell's equations of thermodynamics (Cannon, 2004; J. R. Thompson, 2006; J. R. Thompson et al., 2012). Some physics education researchers have leveraged Zandieh's derivative framework to determine the efficacy of tasks and instructional interventions in thermodynamics courses (Bajracharya et al., 2019). Further, Roundy and colleagues discussed how instructional issues may emerge when different disciplinary practices impact students' conceptions of the use of partial derivatives (Roundy et al., 2015). For instance, mathematicians and physicists appear to leverage partial derivatives in different ways in their disciplines. It would be prudent to continue the dialogue between math and physics instructors at each institution to determine ways to support students in shifting or extending their understanding of partial derivatives flexibly across the STEM disciplines.

Theoretical perspective

Before elaborating on various interpretations and uses of partial derivatives for different function types, I am going to discuss students' conceptions of different types of functions. MVC courses encompass many different types of functions sorted into at least two groups: (1) functions with multiple independent quantities and (2) functions with multiple dependent quantities (including vector-valued outputs). When discussing functions with more than one independent quantity, I will say *multivariable* functions (i.e., $z = f(x, y)$ or $w = g(x, y, z)$ as opposed to *univariable* functions (i.e., $y = f(x)$). To distinguish between functions with multiple dependent quantities and a single dependent quantity I will say *vector-valued* and *scalar-valued* respectively. It is highly unlikely that most students have encountered multivariable functions, or the corresponding notation, by the time they enroll in MVC. Some students may have encountered vectors and vector operations in the context of physics or linear algebra (LA). In Calculus I and II, most students have encountered univariable function notation, input/output tables, and memorized information about the graphs of various function families, along with a thorough introduction to derivative and integral techniques. However, it appears less likely that most student have been supported in reflecting on the idea of univariable functions (across various representational contexts) as sets of invariant relationships relating the values of two quantities whose values vary in tandem with the realization that, at any moment, the value of the independent quantity determines exactly one value of the dependent quantity (P. W. Thompson et al., 2017; P. W. Thompson & Carlson, 2017).

The literature reporting on researchers' investigations and theories about student thinking related to the function concept encompasses several decades of experimental results and theory (Breidenbach et al., 1992; M. P. Carlson, 1998; Monk, 1994; Oehrtman et al., 2008; Sfard, 1992; Zandieh et al., 2017). While there have been some studies investigating student thinking about functions of two variables (Kabael, 2011; Martínez-Planell & Trigueros Gaisman, 2019; Yerushalmy, 1997), the math education theory of student thinking about multivariable and vector-valued functions is comparatively less developed (Rasmussen & Wawro, 2017). I am genuinely curious about how students make sense of the sheer number of different *types* of functions covered in a single semester of MVC instruction (i.e., multivariable functions, parametric equations, vector-valued functions, vector fields, linear and non-linear transformations), in relation to their understanding of calculus ideas related to each function type (e.g., the divergence and curl of multivariable vector-valued functions graphed as vector fields).

Mathematics education researchers might ask research questions related to the various ways students conceive of different types of functions (e.g., action vs. process views of functions of two variables, a structural vs. operational views of linear transformations, covariational or multivariational views of parametric functions). Students' conceptions of univariable functions likely impact their understanding of multivariable functions, and eventually more general mappings such as linear transformations (Trigueros Gaisman & Martínez-Planell, 2013; Yerushalmy, 1997; Zandieh et al., 2017). Addressing the ways students think about and generalize their understanding of the function concept across their undergraduate mathematics course sequence is an important consideration for research and instructional design. Zandieh, Ellis, and Rasmussen (2017) analyzed students' conceptions the function concept cutting across the students experiences from high school algebra to linear transformations in linear algebra. I am curious about how students' notion of function, unified or otherwise, interacts with their conception of derivatives across various MVC contexts. Investigating students' *stable ways of thinking* about functions (P. W. Thompson et al., 2014) seems like a necessary first step to investigating how students conceptualize multivariable rate functions and partial derivatives more generally. Leveraging theories to investigate how students think, generalize, and symbolize their conceptions of functions, and their derivatives, is also worthwhile and relevant to the current trajectory of mathematics education research.

To that end, Jones recently expanded his multivariational framework (Jones, 2018, 2022) based on some of the work emerging from Carlson et al.'s (2002) covariational reasoning framework and Thompson and Carlson's (2017) variation and extended covariation frameworks. Jones described three types of multivariation as well as a framework describing students' multivariational reasoning (MR) mental actions. Jones defined three types of multivariation: (1) independent multivariation (i.e., functions of the form $z = f(x, y)$), (2) dependent multivariation (i.e., situations like the ideal gas law $PV = kT$), and (3) nested multivariation (i.e., composite, or parametric functions $y = f(x)$ where $x = g(t)$). Jones' multivariation and MR framework seem viable for building models of student thinking to determine how students engage in quantitative reasoning (P. W. Thompson, 1993, 2022) about multivariable and vector-valued functions. Further, modeling student thinking to determine how they coordinate the values of multiple quantities changing together is a viable lens for investigating how students conceptualize multivariable rates of change and, eventually, multivariable rate functions.

For example, suppose a student possesses a process view of the function $f(x, y) = xy^2 - xy$ where $z = f(x, y)$ (independent multivariation). The student mentally fixes the value of $x = a$, through decomposing the situation into *isolated covariations*. By considering how the value of z

co-varies with y as the value of y varies from -2 to 2 (holding the value of x fixed), the student can rely on previous ways of thinking about graphs of univariable functions. The student coordinates their mental image of the 2D graphs of $z = ay^2 - ay = ay(y-1)$ for particular values of a with images of the various traces situated in the planes $x = a$ pictured in $\mathbb{R}^3$. The student may reconceive of the situation and anticipate the shape of the surface as an emergent process through mentally coordinating the changing shape of the traces as x varies from -2 to 2. This is exactly the hypothetical learning trajectory (HLT) proposed by Thompson and Weber via the *sweeping out* metaphor (Weber & Thompson, 2014).

Mentally fixing the value of $x = a$ corresponds to the *reducing into isolated covariations* MR mental action and sequentially reconceiving of x as a varying quantity corresponds to the *coordination of multiple simultaneous changes* MR mental action, and perhaps, eventually, *smooth continuous multivariation*. This series of Jones' set of MR mental actions highlights one of the ways a student may come to graphically quantify the partial derivative of the function $f(x, y) = xy^2 - xy$ with respect to y, holding the value of x fixed at $x = a$, by associating their images of each trace of the surface which emerged through their initial conception of how the values of y and z *co*-varied together. The linear approximation for this particular trace (corresponding to the intersection of the graph of $z = f(x, y)$ and the plane $x = a$) emerges by assuming a constant rate of change (specifically $f_y(a, b)$) relating amounts of variation in the value of y away from $y = b$, and the corresponding amount of variation in the value of z emerges from pretending there is a constant rate of change relating amounts of variation in the values of y and z, i.e., $\Delta z \cong f_y(a, y)\Delta y$ or $z - f(a, b) \cong f_y(a, b)(y - b)$. Providing students with the opportunity to conceptualize partial derivatives as imagined constant rates of change due to variation in the value of y, holding the value of x fixed, may undergird why explicit discussions 3D slopes have proven helpful to students when reasoning about the values of partial derivatives as the slopes of tangent lines to the trace of f (Martínez-Planell et al., 2017).

Zandieh's (1997; 2000) derivative framework demonstrated how nuanced students' conceptions of derivatives can be. Each of the three process-object layers (ratio, limit, and function) encapsulates the *reification* of a particular process into a more static, *structural* conception of that layer (Sfard, 1992). The columns of Zandieh's framework encompass various contexts in which the framework can be used to depict the various aspects of a student's conception of derivatives. As Zandieh (1997; 2000) noted in the articulation of her derivative framework, each of the process-object pairs (the rows of the framework) correspond to a particular component of the derivative concept. The ratio, limit, and function layer may have their uses in particular contexts, but each aspect of the derivative concept is important to the overall conceptual structure. Students may have conceived of aspects of the derivative concept as contextual, or comprising the entire concept (Zandieh & Knapp, 2006). For example, in graphical settings, students may claim derivatives are slopes of tangent lines, but in a related rates task, a student may claim derivatives are just rate of change functions, without recognizing an implicit graphical connection to their previous statements about slopes of tangent lines. Investigating the ways students extend their understanding of derivatives to multivariable contexts is non-trivial. However, Zandieh's derivative framework is viable for articulating some of the nuanced aspects of the derivative concept in MVC. One such aspect of derivatives in MVC contexts, that is rarely discussed, is the conception of *derivatives as linear transformations*, as represented by the Jacobian matrix (Gravesen et al., 2017; Harel, 2021).

Zandieh, Ellis, and Rasmussen addressed the ways that students interpret linear transformations in relation to their high-school conceptions of function (Zandieh et al., 2017).

Zandieh et al. (2017) noted that students' concept images of function (from $\mathbb{R} \to \mathbb{R}$) may be separate from their conception of linear transformations as matrix equations. However, as students progress through the semester, their concept images become more interwoven to shift towards a unified notion of the function concept. I hypothesize that direct discussions of linear transformations are necessary to support students in extending their understanding of derivatives, more generally, towards a unified notion of the derivative concept. Zandieh and Andrews-Larsen discussed the various ways students interpret matrix equations (Larson & Zandieh, 2013), and extended their framework to include more nuanced interpretations of systems of linear equations and matrix equations (Zandieh & Andrews-Larson, 2019). I have outlined theoretical students' interpretations of the various roles that partial derivatives play in producing linear approximations for two types of functions in Fig. 2. Again, the various types of derivatives present in MVC may perpetuate students' apparent conception that MVC topics are seemingly disjoint and highly contextual. I hypothesize that centering the idea of derivative as a linear function, or transformation, which produces the best local, linear approximation of *any* type of function will help support students in conceiving of a more unified notion of derivative and thus leading towards coherence of MVC instruction in the minds of students.

Interpretation of $J_f \Delta\vec{x}$	**LT example:** $f: \mathbb{R}^2 \to \mathbb{R}^2$, $y_1 = P(x_1, x_2)$, $y_2 = Q(x_1, x_2)$	**MVC example:** $f: \mathbb{R}^2 \to \mathbb{R}$, $z = f(x, y)$	**Verbal description of each interpretation**
Linear Combination Interpretation **(LC)**	***Linear combination of amounts of change:*** $\begin{bmatrix}\Delta y_1\\ \Delta y_2\end{bmatrix} \cong \begin{bmatrix}\frac{\partial y_1}{\partial x_1}\\ \frac{\partial y_2}{\partial x_1}\end{bmatrix}\Delta x_1 + \begin{bmatrix}\frac{\partial y_1}{\partial x_2}\\ \frac{\partial y_2}{\partial x_2}\end{bmatrix}\Delta x_2$ $\Delta\vec{y} \cong \frac{\partial\vec{y}}{\partial x_1}\Delta x_1 + \frac{\partial\vec{y}}{\partial x_2}\Delta x_2$	***Coordination of amounts of change:*** $\Delta z_x \cong f_x(a,b)\Delta x$ $\Delta z_y \cong f_y(a,b)\Delta y$ $\Delta z \cong \Delta z_x + \Delta z_y$	Approximation emerging from coordinating the amount of variation in the value of the vector or scalar-valued dependent quantity due to variations in the value of each independent quantity separately.
System of Equations Interpretation **(SE)**	***System of linear approximations:*** $\Delta y_1 \cong \frac{\partial y_1}{\partial x_1}\Delta x_1 + \frac{\partial y_1}{\partial x_2}\Delta x_2$ $\Delta y_2 \cong \frac{\partial y_2}{\partial x_1}\Delta x_1 + \frac{\partial y_2}{\partial x_2}\Delta x_2$	***Linear approximation:*** $\Delta z \cong \frac{\partial f(a,b)}{\partial x}\Delta x + \frac{\partial f(a,b)}{\partial y}\Delta y$ $z \cong \frac{\partial z}{\partial x}(x-a) + \frac{\partial z}{\partial y}(y-b) + f(a,b)$	Approximation for each dependent quantity (as a component of a vector-valued output) due to simultaneous variation in the value of each independent quantity.
Dot product interpretation **(DotP)**	***System of dot products:*** $\Delta y_1 \cong \vec{\nabla}P(a,b)\cdot\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}$ $\Delta y_2 \cong \vec{\nabla}Q(a,b)\cdot\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}$	***Dot product:*** $\Delta z \cong \vec{\nabla}f(a,b)\cdot\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}$	Approximation for each dependent quantity as emergent from a geometric relationship as projection or based on the angle between the gradient vector and a direction vector.
Transformation interpretation **(LT)**	***Approximating a non-linear transformation:*** $\Delta\vec{y} \cong J_f\Delta\vec{x}$ $\begin{bmatrix}\Delta y_1\\ \Delta y_2\end{bmatrix} \cong \begin{bmatrix}\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2}\\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2}\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}$	***Inner product:*** $\Delta z \cong \vec{\nabla}^T f(a,b)\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix}$ $\Delta z \cong \begin{bmatrix}\frac{\partial z}{\partial x} & \frac{\partial z}{\partial y}\end{bmatrix}\begin{bmatrix}dx\\ dy\end{bmatrix}$	Approximation of the output space via the linear transformation represented by the Jacobian matrix.

Fig. 2: Theoretical student interpretations of linear approximations altered from Zandieh & Andrews-Larson (2019). The geometric context column has been omitted due to space limitations.

Based on some of the uses of partial derivatives pictured in Fig. 1 and the relevant math education literature on student thinking about partial derivatives, I altered Zandieh and Andrews-

Larson's framework for students' interpretations of matrix equations to anticipate various ways students could interpret and symbolize derivatives in the context of constructing linear approximations for different types of functions. The table in Fig. 2 could be used to anticipate students' conceptions and symbolization for scalar and vector-valued functions with more than two independent or dependent quantities (e.g., multivariable, scalar-valued functions with three or four independent quantities). While the table in Fig. 2 only elaborates on linear approximations for functions involving independent multivariation, the interpretations can be applied to situations involving nested multivariation through the chain rule and conceptualizing products of matrices as compositions of linear transformations (Harel, 2021).

The MR mental action of *reducing to isolated covariations* seems necessary to conceive of linear approximations while coordinating the amount of variation in the value of the output vector or scalar-valued dependent quantity (z) due to separate variations in the value of each pair of independent quantities (x_1 and y_1 and x and y respectively). Coordinating the amount of total variation, or at least an approximation of this amount of variation, emerges from the *coordination of multiple simultaneous changes* MR mental action. The LC and SE interpretations more readily highlight the *reducing to isolated covariations* and *coordination of simultaneous changes mental actions* due to the emphasis on coordinating net changes in the values of the dependent quantities due to separate amounts of variation in the values of each independent quantity.

Conclusion

One way to continue to build on the current state of the mathematics education research literature on the teaching and learning of MVC topics is through coordinating research on student thinking about linear algebra concepts in conjunction with burgeoning theories about student thinking within the context of MVC instruction (Cobb, 2007). One approach is to leverage an altered version of Zandieh and Andrews-Larsen's (2019) framework elaborating students' interpretations of matrix equations in the context of constructing linear approximations for multivariable and vector-valued functions with Jones' (2022) elaboration of MR to anticipate ways students can interpret and symbolize derivatives in MVC. The ways in which students conceive of multivariable and vector-valued rate of change functions necessarily depends on their conceptions of amount functions. Future research studies should seek to coordinate students' symbolization activity with their associated meaning in graphical and non-graphical contexts (Tyburski et al., 2021). Triangulating students' meanings and stable understandings of vector calculus ideas will help to extend the research field's understanding of how students learn and interpret vector calculus ideas and notation.

In this manuscript, I have summarized the recent work on the ways students use partial derivatives in MVC. It should be restated that there are different *types of functions* present in a MVC course, but there are also numerous ways students could conceptualize them; as one example is the diverse approaches to coordinating variation in the values of numerous quantities. As Zandieh, Rasmussen, and Ellis noted, students' conceptions of the function concept may vary widely depending on the context. My hope is to gain insights into the nuanced ways students interpret, use, and symbolize partial derivatives when creating linear approximations.

Future studies could extend the theory I have presented here by applying it to students' written work in MVC contexts. This report is largely theoretical and based almost entirely on my experiences working with students enrolled in a conceptually oriented MVC course for two academic years in conjunction with my research with the IOLA research group.

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Theoretical Considerations for Designing and Implementing Intellectual Need-Provoking Tasks

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The idea of intellectual need (IN) has received much interest from instructors in trying to design tasks that engage students in impasse-driven learning. However, we argue that the literature on IN is currently insufficient for supporting the careful design and implementation of tasks meant to provoke IN. In this paper, we examine two shortcomings: (1) What exactly IN can be created for, and (2) How an instructor might support students in navigating the experience of resolving the confusion and constructing the targeted meanings. For the first of these, we describe the category error of thinking of producing IN for a "topic," and use the idea of conceptual analysis to suggest a way to address this shortcoming. For the second, we bring in control-value theory to explain what an instructor might attend to in order to ensure that the disequilibrium stays productive and does not lead to frustration and disengagement.

Keywords: Intellectual Need, Task Design, Constructivist Pedagogy, Epistemic Emotions

Introduction and Literature Review

Educators and researchers have long described the positive impact of "desirable difficulties" and "failure-driven scaffolding" on student learning (e.g., Dewey, 1938; Kapur, 2015; Sinha et al., 2020). Studies have shown that experiencing cognitive incongruities (e.g., Graesser et al., 2005), working on problems prior to instruction (e.g., Kapur & Rummel, 2012), and experiencing impasses (e.g., VanLehn, 1988; Kapur, 2014) are associated with improved learning and constructive reasoning. Graesser and D'Mello (2012) proposed that these results align with Piaget's (1971, 1985) work on cognitive development—specifically, the idea of disequilibrium, in which the impasses students encounter prompt them to revise their cognitive schemes or construct new ones to accommodate for novel stimuli and experiences.

In the context of learning mathematics, Harel (e.g., 2013), described the role played by cognitive disequilibrium in the learning process in terms of the construct of the *intellectual need*, which is the perceived need to resolve "a perturbational state resulting from an individual's encounter with a situation that is incompatible with, or presents a problem that is unsolvable by, his or her current knowledge" (p. 122). Since its introduction, intellectual need has been a widely-cited principle of instructional and curriculum design (e.g., Burger & Markin 2016; Caglayan, 2015; Foster & de Villers 2015; Koichu, 2012; Leatham et al., 2015; Rabin et al., 2013; Zazkis & Kontorovich, 2016). The idea of intellectual need has been enthusiastically endorsed by some instructors. In their planning, these instructors begin with a target mathematical concept ("[x]") and then begin their lesson planning by asking something like: "If [x] is aspirin, then how do I create the headache [i.e., the intellectual need]?" (Meyer, 2015).

Despite its widespread use in the research literature, there is little guidance about how to design and implement effective intellectual need-provoking (IN-P) tasks. Weinberg and Jones (2020, 2022) proposed a theoretically-grounded framework for IN-P task design, but identified issues in need of further clarification, elaboration, or theoretical justification in both their own framework and the research literature on intellectual need generally. First, Weinberg and Jones (2020) noted that the research literature has not clarified what a student can experience intellectual need *for*, an issue that we believe is at the origin of subtle but significant epistemological contradictions in IN scholarship. Second, Weinberg and Jones (2022) highlighted the importance of attending to the role played by students' affective states as they engaged in mathematical activity, but their framework did not clarify how to account for these states in IN task design or implementation.

On its face, it is possible for one to interpret the idea of IN as suggesting that an instructor need only design a task in which their students are required to confront a problem they do not yet possess the intellectual tools to solve. However, we believe this view orients instructors to develop tasks that might stimulate interest and motivation, but fall short of promoting students' constructive mathematical reasoning. In this theoretical paper, we address these gaps in the research literature by presenting an example of a task designed to provoke intellectual need, highlighting some aspects that, upon further inspection, potentially fall short of engendering this cognitive state, and synthesizing additional theoretical perspectives to ameliorate these gaps.

An Example

We begin by presenting an example of a typical task, some variation of which is commonly included in curricula and instruction to motivate the concept of a derivative. For our example we chose to follow the presentations in some widely-used textbooks[1], viewing textbook presentations as potentially reflecting the authors' instructional intentions. This is supported by the authors' descriptions of their texts as "a teaching tool for instructors and … a learning tool for students" (Stewart et al., 2021, p. xi). Suppose a calculus instructor expects their students to learn about the concept of a derivative and considers how to create an intellectual need for the topic. If they follow the exposition of some of the textbooks we examined (e.g., Larson & Edwards, 2018; Hughes-Hallet et al., 2017; Stewart et al., 2021), they will likely begin by posing either "the tangent problem" (i.e., how can we compute the slope of a tangent line?) or "the velocity problem" (i.e., how can we compute instantaneous velocity?). If an instructor adopted the later approach, they might facilitate an informal discussion of velocity and, typically, define "average velocity" as an *arithmetic*—as opposed to *quantitative—operation* (i.e., division of the change in position by corresponding elapsed time) (Thompson, 1990). Then, the instructor might pose a problem or example in which they demonstrate an inability to perform this computation, either due to insufficient information or an indeterminate form (i.e., "0/0"). The instructor using this approach might highlight the conundrum that "we are dealing with a single instant of time, so no time interval is involved" (Stewart et al., 2021, p. 81) and suggest computing the average velocity over successively smaller time intervals. Then, the instructor might represent the situation using a graph with a point representing the time of interest , and add a second point on the graph with arbitrary coordinates. The instructor then might propose that students imagine that the second point approaches the first point (e.g., Stewart et al., 2021; Hass et al., 2019). Alternatively, the instructor might follow slightly different presentations, such as suggesting that students choose points closer together (e.g., Larson & Edwards, 2018; Herman & Strang, 2020), or that they imagine "zooming in" (e.g., Boelkins et al., 2018; Hughes Hallett et al., 2017).

When the instructor first proposes the problem with finding an instantaneous velocity, the students ostensibly do not possess the intellectual tools to solve this problem. Consequently, an instructor might be tempted to view this initial task as having provoked intellectual need (in their students) for the concept of instantaneous rate of change, and view the definition of the derivative at a point as having resolved this intellectual need. However, as we will argue below, there are additional considerations that are essential for understanding if IN was actually

[1] We examined Stewart et al. (2021), Larson & Edwards (2018), Hughes Hallet et al. (2017), Hass et al. (2018), Herman and Strang (2020), and Boelkins et al. (2018).

produced and what, exactly, it was produced for. Furthermore, the instructor might have only made their students experience confusion, without necessarily helping them navigate the experience of resolving this confusion, and constructing a justification for the necessity of derivative at a point.

DNR-Based Instruction in Mathematics

The concept of intellectual need is one of many theoretical constructs organized within an elaborate framework called DNR-based instruction in mathematics (Harel, 2008). The DNR theoretical framework is grounded in several constructs, the most germane of which (for the present paper) we report here. The triad of mental act, way of understanding, and way of thinking is central to the concepts domain of the DNR framework. Mental acts are the basic applications of cognition to our experiential world; they encompass behaviors such as interpreting, proving, conjecturing, inferring, justifying, explaining, generalizing, and predicting. Harel (2008) defined the cognitive constructs way of understanding and way of thinking and described their respective relations to mental acts:

> A person's statements and actions may signify cognitive products of a mental act carried out by the person. Such a product is the person's way of understanding associated with that mental act. Repeated observations of one's ways of understanding may reveal that they share a common cognitive characteristic. Such a characteristic is referred to as a way of thinking associated with that mental act (p. 490).

We regard an individual's way of understanding a mathematical idea as synonymous with their meaning for the idea (Thompson et al., 2014). We interpret meanings in terms of Piaget's notion of a scheme as an internalized organization of actions, images, and operations constructed and refined through distinct forms of abstraction (Tallman, 2021).

Harel's mathematics premise frames mathematical knowledge as a union of historically institutionalized ways of understanding and ways of thinking. Additionally, Harel (2008) expressed the the reciprocity between these distinct forms of mathematical knowledge in the duality principle, which states, "Students develop ways of thinking through the production of ways of understanding, and, conversely, the ways of understanding they produce are impacted by the ways of thinking they possess" (p. 899). The *knowing premise* asserts,"Knowing is a developmental process that proceeds through a continual tension between assimilation and accommodation, directed toward a (temporary) equilibrium" (Harel, 2008, p. 894). Relatedly, the *knowing-knowledge linkage premise* states, "Any piece of knowledge humans know is an outcome of their resolution of a problematic situation" (Harel, 2008, p. 894).

Commensurate with these theoretical premises, *intellectual need* (IN) is a psychological state–with cognitive and affective entailments–resulting from an individual's interpretation of a problematic situation, the resolution of which requires the individual to extend, modify, or create knowledge structures. Unlike confusion, intellectual need results from an individual's conception of a problematic situation in a way that orients them to engage in the constructive activity required to resolve the situation. IN is thus energetic in that it stimulates, rather than impedes, generative mathematical reasoning.

Harel distinguished different categories of IN. A need for *certainty* motivates deductive reasoning; it is a need to prove, or remove doubts. A need for *causality* is a need to identify the cause of some phenomenon. A need for *computation* involves a desire for efficient quantification. A need for *communication* refers to an obligation to persuade others that an assertion is true. Finally, a need for *connection and structure* entails a commitment to identify similarities and analogies and to abstract unifying principles.

Issue #1: Intellectual Need for What?

There is a subtle category error lurking behind a statement like, "I want students to experience an intellectual need for derivative." Replace "derivative" in this sentence with "constant rate of change," "radian measure," "polynomial rings," "Lyapunov functions," and the category error remains. The issue is this: students cannot experience an intellectual need for a mathematical *topic*. A student can, however, experience an intellectual need to engage in the actions that form the experiential basis of a particular *meaning*, or way of understanding, that they expect might enable them to resolve a problem *they* have conceptualized.

We previously described the experience of intellectual need as a stimulant for generative mathematical activity, *not* a state of confusion that paralyzes reasoning and induces an affective state of interest, analogous to the curiosity provoked by having witnessed a compelling magic trick. These affective states result in the student being receptive to *absorbing* (to use a term consistent with Meyer's aspirin-headache metaphor) the mathematical concepts communicated to them, or the skills demonstrated for them, by a knowledgeable expert. This type of affective experience is nearer to how Harel described affective *need*, indisputably useful but certainly not identical in its affordances for supporting students' learning as IN. So where exactly is the category error in aspiring to support students' experience of IN for derivative (or any other mathematical "topic")? Intellectual need cannot, on the one hand, be conceptualized in a manner consistent with Piaget's idea of *equilibration* of cognitive structures–and radical constructivist epistemology generally–while on the other hand promote an exclusively affective state where learning is a product of a *fundamentally empiricist process* (i.e., perception). Theoretical coherence requires that both the inducement of IN and the learning afforded by its experience be a product of the actions, operations, abstractions, and generalizations *of the learner*.

Intellectual need is thus a need to engage in the precise actions that must be internalized and organized to form targeted ways of understanding (i.e., cognitive schemes) through *pseudo-empirical* and *reflecting abstractions* (Piaget, 2001). A meaning, or understanding, exists only in the mind of an individual, and thus consists of an elaborate organization of idiosyncratic images, actions, and operations (diSessa, 1988; Tall & Vinner, 1981; von Glasersfeld, 1995). Successfully engendering IN therefore requires an instructor to be cognizant of the mental actions and conceptual operations entailed in the meanings they want to support or necessitate (Tallman, 2021; Tallman & Frank, 2020). Without such awareness, an instructor cannot design problematic situations that require students to engage in, and abstract from, these actions.

Solution to Issue #1: Conceptual Analysis. Thompson (2008) described conceptual analysis as the process by which an instructor or curriculum designer clarifies the ways of understanding they intend to support, and elucidates how these understandings depend on and contribute to developing particular ways of thinking. In other words, a conceptual analysis is an articulation of "what students might understand when they know a particular idea in various ways" (Thompson, 2008, p. 43). Conducting a conceptual analysis involves (1) unpacking what it means to conceptualize a mathematical idea in a specific way, and (2) explicating the cognitive activity necessary to construct the targeted understanding. Conducting conceptual analyses is therefore essential to an instructor's development of tasks that effectively necessitate the actions that form the basis of the ways of understanding (i.e., schemes) an instructor or curriculum designer expects students to construct.

Applied to the archetypal tasks described above for motivating students' learning of derivative at a point, a conceptual analysis would specify what it means to understand instantaneous rate of change and clarify how this understanding depends on particular conceptions of constant and average rate of change. Moreover, the role of quantitative and covariational reasoning in the structure and development of the targeted understanding would be made explicit. A conceptual analysis would highlight the distinction between two of the meanings that are implied in the example above. One such meaning—a quantitative meaning—involves conceptualizing average rate of change as a proportional relationship between changes in the measures of distance and time and the successive approximation process as involving thinking of these quantities as continuously covarying and measuring this covariation. In contrast, a second meaning—a geometric meaning—involves thinking of the distance-time pairs as represented as points on a graph of the function, thinking of average speeds as the "slantiness" of lines connecting the two points, and then thinking of the "approaching" process as suggesting a pattern in the "slantiness." These two meanings can be coordinated, but this coordination is automatic. Though it is far beyond the scope of this paper to provide an example of such a conceptual analysis we encourage the reader to reflect on the experiences that might be required to support these distinct students' understanding of each of these meanings of instantaneous rate of change.

Issue #2: Managing and Leveraging Students' Experiences

Suppose that the instructor was successful in provoking intellectual need for a particular meaning of the derivative at a point—for example, the quantitative meaning described above (i.e., the limiting value of successive approximations of instantaneous velocity by average velocities over successively smaller amounts of change in time). The research literature on intellectual need is largely silent on what the instructor needs to attend to and how they should act to guide students through the process of constructing the relevant schemes that will enable them to resolve the disequilibrium. In particular, the students are in a state of cognitive disequilibrium and might react negatively to the confusion that is associated with being in such a state. In this section, we adapt theoretical perspectives from the research literature on student affect to highlight these issues and propose aspects of the class' activity that can guide instructors' attention and action.

Control-Value Theory and Epistemic Emotions. Numerous researchers (e.g., Mandler, 1975; 1999; Morton, 2010; Stein & Levine, 1991; Tallman & Uscanga, 2020) have highlighted the critical relationship between the emotions students experience, their engagement with a task, and their subsequent cognitive activity. Pekrun (2000; 2006) proposed a theory of the emotions that students experience in academic settings: the students' emotions are influenced by their perception of the extent of their *control* over their activity and its outcomes, and their emotions are also influenced by their *values*—their perception of the importance of the activity and outcomes. Muis et al. (2015) found that students' experiences of emotions in mathematical settings were closely related to the extent to which they valued mathematics and the degree to which they felt a sense of agency over their engagement with the class' pedagogical practices and of the mathematics they were learning.

Pekrun and Stephens (2012) classified academic emotions into achievement emotions (such as anxiety and contentment); topic emotions (e.g., how the individual feels about the mathematical topics); social emotions (such as pride and shame); and epistemic emotions, which arise when "the object of their focus is on knowledge and learning" (Muis et al., 2015, p. 173). Epistemic emotions arise from "knowledge states involving discrepancy, incongruity, or conflict between cognitive schemas" (Nerantzaki, 2021, p. 2). These emotions include surprise, curiosity, enjoyment, confusion, anxiety, frustration, and boredom (Pekrun et al., 2016) and have been explored by numerous researchers (e.g., Efklides, 2017; Loewenstein, 1994; Mandler, 1975, 1984; Metcalfe et al., 2017; Touroutoglou & Efklides, 2010; Vogl et al., 2020).

We anticipate that curiosity and confusion are the epistemic emotions most closely linked to experiences of intellectual need, though IN is not reduced to students' experience of these emotions, as previously argued. Curiosity is "the complex feeling and cognition accompanying the desire to learn what is unknown" (Kang et al., 2009, p. 963) and it arises when a student experiences a gap in their knowledge (Loewenstein, 1994; Metcalfe et al., 2017); Litman (2008) proposed that epistemic curiosity motivates students to learn new ideas, and, thus, appears to be closely related to the role of intellectual need in the DNR framework. Similarly, confusion occurs when students are unable to assimilate new information into their cognitive schemas and must engage in accommodation (D'Mello et al., 2014); thus, confusion appears to play an important role in students' experiences of intellectual need. However, while curiosity tends to uniformly support students' participation and learning (e.g., Arguel et al., 2019), confusion can either support or hinder student engagement (e.g., Lodge et al., 2018), and D'Mello et al. (2014) distinguished between what they called "productive" and "unproductive" confusion.

Solution to Issue #2: Monitoring and Leveraging Epistemic Emotions. To think about how an instructor can attend to students' epistemic emotions, we can organize the students' engagement with an activity through a model of affective dynamics, such as the one proposed by D'Mello and Graesser (2012) in Figure 1:

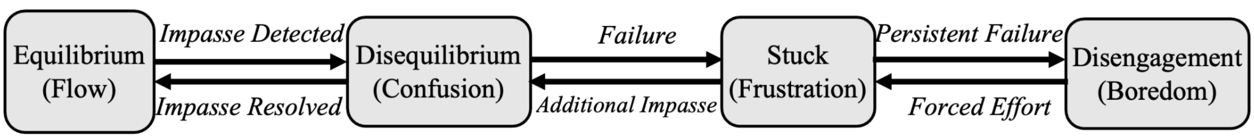

Figure 1. D'Mello and Graesser's (2012) model of affective dynamics

When students begin working on the instantaneous velocity task, the teacher needs to understand the students' perceptions of their roles in mathematics classrooms—a component of their values about mathematical ways of thinking—to help students experience an impasse. When the students recognize their inability to compute the instantaneous speed *and* can articulate the reason for this inability, they shift to a state of disequilibrium; as suggested by D'Mello and Graesser's (2012) model, this shift corresponds with epistemic emotions of either curiosity or confusion (or both).

At this point, as the instructor begins to help students begin the process of articulating and constructing meanings for the targeted schemas, they need to carefully monitor the students' perception of control over their cognitive activity and the targeted outcomes, and also need to remain cognizant of the students' values to help align the activity with those values. In particular, to prevent students from experiencing persistent failure that leads to boredom, the instructor needs the students to experience confusion productively by monitoring their experiences of failure and providing support to move students back into a productively confused state (unfortunately, D'Mello and Graesser (2012) were not able to identify the specific actions that stimulated this transition in affective states).

After working within a state of disequilibrium, the instructor needs to help their students resolve their disequilibrium in a way that leads them to, by Harel's (2008) knowing-knowledge linkage principle, generate the target mathematical meanings for instantaneous velocity. This requires that the teacher helps the students articulate the meanings they need to construct, thereby enabling them to believe that it is possible to resolve the impasses they are experiencing. In the case of instantaneous speed, this could entail helping students articulate the quantities of distance and time involved in the situation, focus on the amounts of change of these quantities, imagine these quantities as varying and co-varying, and then focus on the relationship between the quantities. During this process, the teacher needs to, first, continue to align the class activity with the students' perception of their control over the outcomes of the activity and, second, to help align the act of constructing meaning for instantaneous velocity with the students' values.

Discussion

In this paper, we set out to provide theoretical clarifications on key ideas in designing IN-provoking tasks. In particular, we focused on what IN can be experienced for, what experiencing IN should lead to in terms of student actions, and how to guide students through the disequilibrium toward the resolution of the problem

If the concept of intellectual need is to be consistent with its theoretical premises and epistemological foundations, students' experience of it must induce the precise actions that are abstracted and internalized to form the substrate of cognitive schemes (i.e., *ways of understanding*). Intellectual need is a need for *action*–it sets in motion a constructive process by provoking the experiential basis of knowledge structures, in a manner consistent with Piaget's notion of equilibration and, crucially, the cognitive mechanisms by which equilibrium is established. While affective needs are of value in mathematics education, experiencing them alone does not necessarily place students on a trajectory to constructing targeted ways of understanding. In this paper we distinguished intellectual need from other types of experiences that stimulate interest, provoke curiosity, or foster motivation. We clarified the relation between intellectual need and to constructivist epistemology so that mathematics educators might be better positioned to engender this productive experience through thoughtful task design and implementation. We proposed Thompson's (2008) notion of conceptual analysis as the essential process that positions instructors and curriculum designers to effectively create intellectual need-provoking tasks.

Although the research literature includes numerous examples of tasks that are designed to provoke intellectual need, there has been little guidance or theoretical framing about how an instructor should engage their students after the task has been presented. The disequilibrium that results from the stimulation of intellectual need is associated with particular affective states, and there is a critical relationship between these states and the students' activity. The perspective of control-value theory allows us to frame this activity to identify and help students manage these affective states by focusing on the students' perceptions of control of their activity and coordinating this with the value students place on their activity and its outcomes. In particular, by focusing on epistemic emotions can help instructors identify particular affective states and their likely antecedents so that the instructor can help students articulate the meanings they need to construct, successfully resolve their disequilibrium, and leverage intellectual need to support student learning.

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Decentering and Interconnecting as Professional Skills in the Preparation of New College Mathematics Instructors

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Making progress in justice, equity, and diversity in post-secondary teaching and learning requires systemic change. The development of novice instructor professional knowledge is a critical subsystem of the undergraduate mathematics education system. Novices play key roles in instruction and have the potential to play key roles in change efforts later in their careers. Yet, there is little in the way of theory to support research and development in this area. In other fields, professional development that engages novices in building skill at self-sustaining, generative change as professionals is the ground in which agency for change is seeded and nurtured. We describe two dimensions of professional skills for interacting with ideas and people: decentering and interconnecting. In this report, we explore and illustrate the role of these dimensions in professional development for novice college mathematics instructors.

Keywords: professional development, novice instructors, TAs, decentering

Research in the undergraduate mathematics education (RUME) community has generated findings, materials, and programs that have shaped today's post-secondary mathematics instruction. Changes that leverage this research often occur as a result of efforts by "change agents"– people who notice and seize upon opportunities for improvement who work with others to enact change. Some things are known about the work and characteristics of change agents in undergraduate education (e.g., Froyd et al., 2008; Henderson et al., 2011). However, despite their key roles in undergraduate education, much remains to be learned about how someone becomes a change agent. The work of the RUME community has advanced the teaching and learning of undergraduate mathematics through curricula informed by research, innovations in assessment, programs to support instructors and many other things. For such efforts to reach more students, we need to ensure that future faculty have the knowledge, skills, and dispositions to be change agents and to successfully leverage the RUME work as they initiate and sustain change.

Twenty-first century approaches in professional learning about college teaching have included opportunities for novices to imagine and build purposefully towards future-self goals – addressing questions like "How will I define my professional success as an instructor? How will I measure that and be accountable for it?" If we also want instructors to consider and answer the question, "How will I contribute to innovative change in my professional community?" this needs to be a focus of professional learning opportunities we offer. Thus, we need to consider how professional learning can support novices to become effective instructors now while also positioning them as the next generation of those who act as agents of positive change.

Take any example where a department seeks to implement a program innovation and questions arise: What needs to be happening for new instructors so they can teach in the targeted ways? And, later, when a person graduates from Innovative U. and lands a job at Status Quo College, how does that person – who is a relative novice in college mathematics instruction and departmental politics – initiate and maintain productive exchanges with others to improve teaching and learning? The answers to both questions include building social and management skills for interacting with structures and power constraints in various ways (Elizondo et al., 2020). For change that disrupts the academic mathematics status quo, skills must be grounded in an understanding of the norms and values of the status quo, how those are different from and similar to new norms and values (e.g., those anchored in justice and anti-racism), how to enact change from the former to the latter, and how to determine the nature of the success of the change (and start a new cycle of change based on it).

Change requires non-linear, multi-dimensional, and context-sensitive efforts. To succeed as a change agent, one needs awareness and knowledge about subsystems, understanding and anticipation about how those subsystems connect, interact, and influence one another, and skill at seizing opportunities to generate new learning from each professional encounter. Creating equitable, inclusive mathematics learning opportunities for students requires an analogous set of skills and knowledge. As part of a larger effort (Hauk & Speer, forthcoming), this report presents two theory-driven dimensions of professional learning and illustrates how they apply to the design of professional development about teaching for novice instructors. We argue that these dimensions (decentering and interconnecting) serve today's novice instructors both to be better equipped as instructors and to navigate future demands and function as change agents in tomorrow's departments and universities[1].

Decentering as a Professional Skill

Around the world, mathematics courses and programs in most post-secondary institutions are built on an instructor-centered model (PCAST, 2012). This approach has been effectively self-sustaining for many decades. The practice of lecturing has been passed on from generation to generation of college teachers through personal classroom experiences and through graduate school training with curricula that preserve lecturing as the status quo. Now, however, it is clear that an instructor-centered approach is not universally effective or appropriate (see, e.g., Abell et al., 2018; Bressoud et al., 2015; Freeman et al., 2014; Laursen et al., 2014). That is, instructor-DEcentered methods are needed.

Research on instructor-decentered approaches has paralleled research about equity in post-secondary teaching. Both point to the value of student-centered methods. It is worth noting that explicitly, at this moment in the educational research and practice communities (and more broadly) there is not a well-defined, crisp, and shared definition of equity (Aguirre, et al. 2017; Gutiérrez 2012). Beyond "fairness," equity is evidenced by the absence of disparities: membership in a group that has been historically disadvantaged or oppressed is not correlated to access to opportunities, attainment of educational outcomes, or achievement of life goals. Our use of equity is in the spirit of this definition.

The apprenticeship of observation is powerful. People tend to teach the way they were taught (Lortie, 1975). It is important for novice instructors to experience teaching that models and provides touchpoints in their efforts to teach differently (e.g., more equitably). For example, professional learning opportunities can be offered in ways that model instructor decentering. Thus, novices can refer to how they have recently been taught, in professional development, to contrast with the power-culture-driven, instructor-centered experiences that likely make up the bulk of their histories as learners.

What is Decentering?

Successful implementations of the kinds of instructional practices described as equitable call for teaching that elicits and utilizes student contributions and student choice (Jacobsen et al.

[1] Note: This report is not a primer on how to design professional learning about teaching for novice instructors (for that, see, e.g., Bragdon et al., 2017; College Mathematics Instructor Development Source [CoMInDS], 2021; Council of Graduate Schools, 2021; Deshler et al., 2015; Saichai & Theisen, 2020 and references therein).

2012). The expectation emerging from research and practice is that instructors facilitate discussions to which students contribute their thinking and voices. This kind of instructional decentering is, at its most basic, the act of seeing from someone else's point of view and has historical roots in the work of Piaget (1955). It entails the instructor engaging with students as a participant in interaction, rather than as the center of interaction.

In decentering, instructional attention is on uncovering, understanding, and expanding on what students know and do to include novel, non-standard, and standard mathematical ideas and methods (Carlson et al., 2007; Rahman, 2018; Teuscher et al., 2016). Being self-aware and facilitating self-aware learning by students are the focus (instead of attention and authority vested largely in the instructor). Decentering requires attention to other people as (potentially) different from oneself, noticing nuances in similarity and difference between one's own views or experiences and those of others. In its most developed forms, decentered instructors bridge across similarities and differences in formulating in-the-moment responses to situations.

An important step in building skill at noticing how the thinking of students is similar to or different from an instructor's own is creating the opportunity in one's classroom to hear and see student thinking. Decentering depends on a variety of individual instructor factors (e.g., self-knowledge, goals, orientations, beliefs, psycho-social challenges).

Novice instructors need to learn how to create, maintain, and manage classroom environments where undergraduates are participants in student-centered ways. This includes instructors learning about many things from a student-centered perspective, such as content, curriculum, and assessment (Bok, 2009), communication and interaction (e.g., related to classroom authority or socio-political factors, Gutiérrez, 2009, 2013; Winter & Yackel, 2000), as well as how to learn in and from instruction itself (Speer & Hald, 2008). Learning to elicit student thinking and learning how to shape instruction based on that thinking is the foundation on which generative change is built (Franke et al., 1998).

Development of Decentering Skills

Professional development can provide opportunities for instructors to build skill in decentering, along with other facets of cross- or intercultural competence. There is a developmental continuum for decentering: from an ethno-centric view that everyone is like me to an ethno-relative view, that any person (including me) is like and unlike every other person in identifiable and valuable ways (Bennett, 2004).

Some common components in professional development create opportunities for instructors to experience and learn about decentering (e.g., have novice instructors attempt the mathematics tasks from the course and come together to discuss them). Current research and development in RUME continues to provide ideas for the professional preparation of novices that include opportunities to model decentered instruction, from the CoMInDS collection (2021) to specific professional guidance on particular practices (e.g., cooperative learning using group-worthy tasks; Reinholz, 2018). We contend that if novices develop and use skills in decentering (and interconnecting, discussed next) in their work as instructors of mathematics they will be better prepared to leverage those skills in their future work as change agents.

Interconnecting as a Professional Skill

While decentering is awareness from within the perspectives of others, interconnecting uses meta-awareness to make connections across perspectives and contexts. This can occur at many levels and grain sizes. Such linking is essential in developing and nurturing coalitions, an essential component of local and systemic change (Kotter, 2012). A mathematics class is a foundational opportunity for coalition. Many other structures and groups rely on and influence it. Figure 1 offers one way of seeing the relationships among people and instructional structures.

RUME reports describe many instructional practices, what happens inside the disk labeled INSTRUCTION in Figure 1. Some reports describe content aspects (Figure 1, MATH), others cognition by students (individually or jointly, as represented by the arrows in Figure 1). A few reports address what happens from the perspective of another layer in Figure 1, the region labeled PROFESSIONAL DEVELOPMENT. For those involved in *providing* professional development to new instructors, this report itself is a contribution to the outermost region in Figure 1, LEADERSHIP DEVELOPMENT.

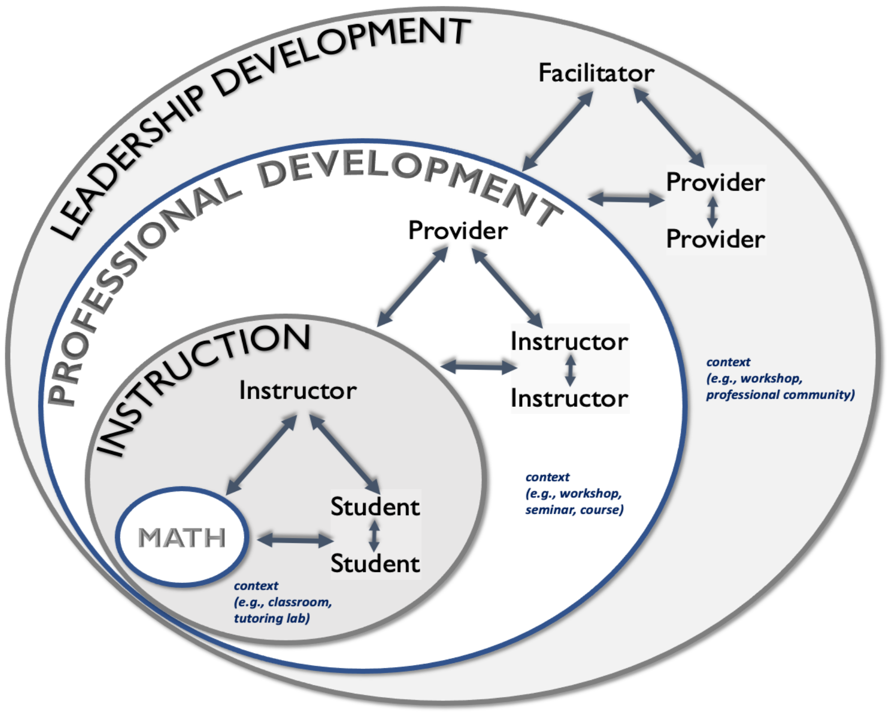

Figure 1. Model of the nested, self-similar systems of learning in mathematics (based on Carroll & Mumme, 2007).

Interconnecting: Examples

As an example, consider interconnecting rooted in the concept of derivative (MATH in Figure 1). If an instructor is aiming to help students learn about the idea of a derivative, instructional goals are influenced by what the instructor understands about students' knowledge of slope, ratio, and change. Interconnecting by the instructor involves noticing how students' conceptions may support or constrain the way learning progresses. That is, the instructor considers what is happening at the Students ←→ Students node in Figure 1, where the kind of thinking brought to mind for students might include m in $y = mx + b$, previous experience with unit rate, experiences that discretize change (e.g., compare slope at point A to slope at point B), or treating change as covariational. Instructors develop skill at interconnecting by knowing these student conceptions as well as the dynamics of communicating about them in a multi-contributor, student-centered, context (the arrows in INSTRUCTION). Instructors also consider and link to students' thinking in selecting formats (e.g., group work on some problems, viewing a pre-class video for a topic). Instructors connect across and prioritize the mathematical and contextual factors, to decide what is instructionally useful. Novices learn about these interacting connections in and from their teaching and through professional development.

When someone embarks on providing professional development to support instructors to learn to teach, they are taking on the challenge of thinking at another layer out: a meta-meta-awareness of connections is required. Providers of professional development need to think about how to have novice instructors interact with ideas of mathematics teaching (and each other) in ways that will help those instructors think about how to support students to interact productively with mathematical ideas (and each other). Continuing the derivative example in the PROFESSIONAL DEVELOPMENT region of Figure 1, for the Provider there are additional considerations about what instructors will need to know and do if they are to create the desired learning opportunities for students. For example, identifying learning goals might occur in various professional activities (e.g., as a step in lesson design, or as a provided component in a set of curricular materials). Novice instructors also may need opportunities to learn about, have practice with, and connect across the mathematics of limit, limit quotient, and derivative along with instructional approaches that may be particularly effective (e.g., students working on a group-worthy task in which limits, rates, and limits of ratios are compared and contrasted). This, too, may occur through various professional learning tasks that are designed for use with novice college instructors (e.g., in an interactive activity, or in a teaching guide).

Also needed by novice instructors– as they notice and structure their understanding of a web of interconnections – is guidance about how to orchestrate mathematical and instructional ideas while decentering. This includes building knowledge about working in racially, ethnically, and linguistically diverse classrooms (e.g., a reading about student funds of knowledge and how to leverage those in teaching; González, et al., 2011) as well as creating and maintaining socio-mathematical and social norms in the classroom (e.g., a professional learning activity about what to do the first day of class to begin setting norms). These all make up the "content" of the professional development (in Figure 1, this includes the MATH, information about students and instructors as well as context knowledge from research and practice about the interaction arrows in INSTRUCTION back and forth among MATH, instructor, and students). Also part of the information that novice instructors interconnect is the learning about teaching they encounter in hallway conversations with colleagues and other informal interactions (Latulippe, 2009).

In particular, interconnecting includes instructors thinking about students' thinking about mathematics. In an analogous fashion, those who provide professional learning opportunities must concern themselves with an additional level of interconnecting: Providers think about how instructors are thinking about how students are thinking about mathematics.

Development of Interconnecting Skills

Like decentering, skill at interconnecting develops from an ethno-centric to ethno-relative orientation (Hauk et al., 2015). It may begin in a self-focused denial of differences (e.g., no connections are needed since it is only MATH that matters). Further development of skills for interconnecting will go through a phase characterized by a tendency to polarize, to focus on mathematics as disconnected from human interaction (e.g., there is one best or right way to solve every problem). From there, development moves to a search for universals, connections that are compressed into a single process, but not multiple interconnected processes (e.g., there are "objective" or "mastery based" ways that are universally applicable to assess *all* students, and grades become the essential element of interaction for instructor and student, disconnected from students' mathematical funds of knowledge). With time and intentional development, one can learn more about mathematical ideas, contexts, and human interactions and reflect on teaching with greater attention to relational details (e.g., learning about implicit bias and suddenly noticing it in every word problem in the text), but how to use this knowledge to improve opportunities to learn, classroom climate, and interactions with others remains elusive. At its most developed, interconnecting is adaptive— networks of people and their interactions can be anticipated (enough) that teaching serves the needs of the people, and networks of people, in and outside the room (Hauk et al., 2014).

Interconnecting is important for change agents because they need to know how people, policies, and perspectives function in and across interacting systems (e.g., in and beyond those shown in Figure 1). The knowledge from that is extremely valuable when advocating for a change in one part of a (sub)system: one can anticipate how change in one place will cause or necessitate change in another.

Data-Driven Interconnecting and Decentering

It is clear from the variety in approach and the nature of successes in the research and practice literature that collecting and examining data for decision-making is valuable (Laursen, 2019). One method for planning for and examining the success of change that has been used effectively in RUME is the four frames or "lenses" model (Bolman & Deal, 1991; Reinholz & Apkarian, 2018). This model is one way of describing what is interconnected in change efforts. It involves considering the evidence of change in terms of **Structures:** rules, policies, procedures, management; **Power:** resource allocation, formal and informal seats/sources/sinks of power; **People:** demographics, experiences, needs; **Symbols:** meaning and culture, rituals and habits, stories, sensemaking.

End-of-term grades are only one form of readily accessible local data (like hearing from only one student in the classroom). In taking a systems approach to change, useful data for determining need and success are generated in intentional and inclusive ways, from across diverse stakeholder groups. Identifying stakeholder groups happens when change agents decenter, look outside themselves and the voices of the status quo.

More can be learned from collecting and analyzing types of data that are largely absent in existing reports: instructor and implementation data. Now is the time to interconnect across contexts. Examples of how to do that as part of the professional development of novice instructors include activities in which novices and/or providers:

- Gather observational data about the nature of classroom questions and answers (American Association for the Advancement of Science, 2013).
- Data-mine learning management systems for evidence of equitable and inclusive instruction (e.g., an audit of time/contributions to discussion fora broken down by student demographics, or a review with feedback to instructors of course sites in learning management systems using a rubric; Baldwin et al., 2018).
- Conduct surveys of instructors about practices and instructor interactional experiences of teaching – including experiences of racialized or sexualized or gendered interactions (Sue et al., 2011), repeat these types of data gathering, analysis, and reporting in and through the professional development experiences, where instructors are the learners.

Some tools for such data collection exist already (e.g., Laursen, 2019). However, next steps include moving into data gathering at the broader levels in Figure 1 and a group-level decentering that includes identifying and inviting outside experts to support connected knowledge growth about each other as thinkers and doers of instruction (Henderson et al., 2011; Reinholz, Stone-Johnson, et al., 2020). Such expertise will help programs create professional learning opportunities that are informed by research on equitable and just teaching development (including ideas learned in other disciplines, e.g., in biology, Gormally et al., 2016).

Generating the Future

As in any field, organizational change requires a multi-threaded effort across personal reflections, development, cycles of professional preparation, implementation, evaluation, administrative-level accountability, within and across-institution continuous improvement work, and broad policy efforts to support large scale change within and across subsystems (Deszca et al., 2019). Although individual humans play key roles in change, some have found it productive to view organizational change focused on interaction, foregrounding relationships among people as they communicate with each other and interact with organizational structures, symbols, and power (Elizondo et al., 2020; Kotter, 2012; Reinholz, Rasmussen, et al., 2020; Slemp et al., 2021). Students can only benefit from the efforts of the RUME community if people residing in mathematics departments can initiate and help sustain organizational change.

Many members of the undergraduate mathematics community have taken on roles as change agents, responding to needs and shaping efforts to improve teaching and learning. There may be additional benefits to the community if, in parallel with these efforts, we aim to develop and refine theory. The history of mathematics education includes many examples of the productive interplay between empirical research and theory development (e.g., studies of problem-solving and theory about meta-cognition, studies of teachers and theory about mathematical knowledge for teaching). Having additional theory-development around the characteristics, roles, knowledge, and skills of effective change agents would certainly be welcomed. That can further inform the theory-building efforts we have begun to lay out in this report.

We have asserted that decentering and interconnecting are key to success as a change agent. We based this assertion on examination of what organizational change entails. The research and practice communities will likely benefit if these claims are investigated in conjunction with organizational change efforts. Parallel efforts to create theory and enact change can lead to the accumulation of knowledge which can, in turn, inform the next cycle of efforts.

It is worth noting that change efforts can fail because people do not know about, or do not know to, pay attention to interconnected-ness and the complexity that comes from decentering. For example, it can be fatal to a change effort to focus attention on one thing (e.g., "all that is needed to reform how this course is taught is changing the textbook" or "student-centered instruction will fix the problem" see, e.g., Henderson, 2011). If we design and test ways of helping novices develop skill at decentering and interconnecting in their mathematics classrooms, that creates a foundation for further design and theory development for analogous skills used more broadly to contribute to change efforts in their departments.

As is true in the broader literature, collaboration by a group of change agents, not all of whom are mathematicians, is valuable (Laursen & Austin, 2020; McShannon & Hynes, 2005; Saichai & Theisen, 2020; Theobald et al., 2020). A corollary of acquiring skill at decentering is that it prepares one to be a participant in a collective effort (e.g., with a classroom full of students, with a department full of colleagues, with a cross-professions team). With attention to decentering and interconnecting, the next generation of change agents will be equipped to participate in the collective action required for future change.

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Balancing the Unbalanceable: A Theoretical Perspective for Teaching and Learning in Undergraduate Mathematics Classrooms

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I propose a theoretical model for teaching, learning, and assessing mathematics learning in the undergraduate mathematics classroom combining participatory and acquisition views of learning. The model, Balanced Learning Needs Framework, was created from an organization of the 10 learning needs proposed by Sfard in 2003. I align the model with different forms of classroom instructional techniques used in an undergraduate Calculus I classroom. Included with the organization of the learning needs, this conceptualization has the potential to impact both teaching and assessment methods in undergraduate mathematics classrooms.

Keywords: participatory learning, acquisition learning, Calculus I, assessment

Many theories and frameworks exist to help researchers try to understand the phenomena of teaching, learning, and assessment within mathematics education (Radmehr & Drake, 2019). Dubinsky and McDonald (2001) stated how: "models and theories in mathematics education can support prediction, have explanatory power, be applicable to a broad range of phenomena, help organize one's thinking about complex, interrelated phenomena, serve as a tool for analyzing data…" (p.275). In this paper I provide a framework designed for a larger study that was used to investigate learning in an undergraduate Calculus I course. This proposed framework helped to organize the researcher's thinking about learning (Dubinsky & McDonald, 2001) in both social and individual classroom. Two main learning theories guided the construction of the proposed framework: situated and constructivist. Together, the theories and framework guided each aspect of construction of the Calculus course: setup, course design, classroom instruction, and assessment of learning.

Theoretical Background

Sfard (1998) metaphorically described students' development of knowledge under situated learning theory as participatory and under constructivist learning theory as acquisition. I consider a combination of the two learning theories (using their metaphors) along a continuum (Figure 1). This combination of situated theory as participatory learning and constructivist theory as acquisition learning is the theoretical foundation for proposed framework. Vygotsky's sociocultural view of learning is placed in the middle of the continuum. Although this view of learning does place importance on the social process of learning, Vygotsky still considered the internalization of knowledge in the learner's mind (Sfard, 2003).

While creating the framework, I reflected on both metaphors for learning, and considered knowledge as something that is both acquired in the individual learner's mind and co-created from participation within a community of practice. The largest difference between the two perspectives is regarding the emphasis and need of social interaction and where knowledge is constructed (i.e., in the mind of the individual learner or in the community). The framework is created with the notion of holistic learning where an individual both acquires knowledge (cognitive-acquisition perspective; Cobb & Bowers, 1999; Sfard, 1998) and constructs knowledge through participation within a community of practice (situated-participatory perspective; Cobb & Bowers, 1999; Sfard, 1998; Lave & Wenger, 1991).

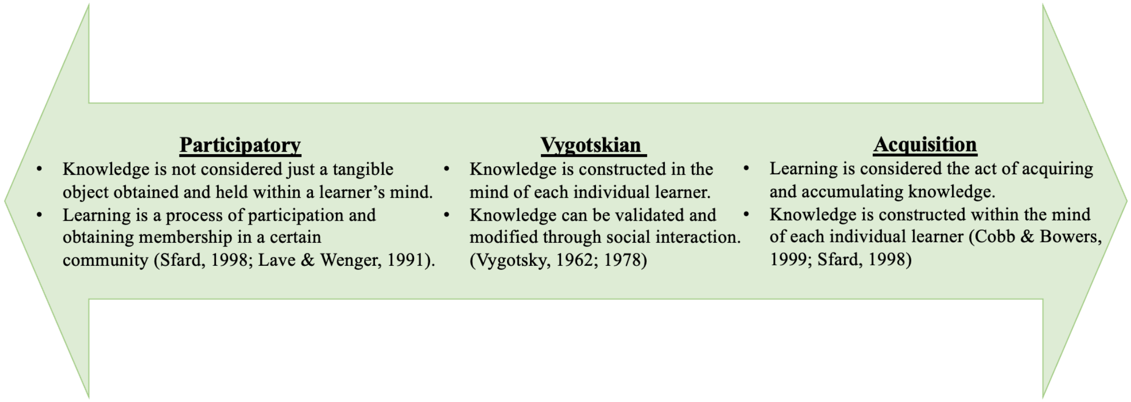

Figure 1. Participatory-acquisition Learning Continuum.

Alignment of Learning Needs

Expanding upon Sfard's (2003) presentation of the 10 needs of mathematics learners, I constructed an organized model to represent these learning needs (LN) within the situated (participatory) and constructivist (acquisition) learning theories. This organization is a representation of how the LN can be systematized to enhance teaching and learning for undergraduate mathematics, as well as focusing on equitable assessments of learning. Each of the 10 LN corresponds to knowledge creation either within the mind of the learner or within the community of learners.

We can align the respective LN with the ideas of learning and knowledge to either an individual unit (i.e., acquisition) or a group unit (i.e., participatory). That is, each need may be internally driven (i.e., for internal knowledge construction) or socially driven (i.e., for participation within a community), or a combination of both. The majority of the needs as defined by Sfard (2003) are primarily driven either only by one or the other. I classify the LN into three main categories: internal needs, social needs, combination needs.

Internal Needs

There are four internal LN that each student has in order to construct mathematical knowledge: a need for *meaning*, *structure*, *repetitive action*, and *difficulty*. Constructivist views of learning attend to the internal needs of the learner and consider knowledge creation within the mind of the individual learner. Either Piagetian or Vygotskian, or both, perspectives of constructivism influence each of these needs.

All learners have this need to make *meaning* of the world around them and communicate their *meaning* of the world to others (Bruner, 1997; Sfard, 2003; Vygotsky, 1962). The creation of *meaning* is an internal process used by all learners to understand and make sense of the world around them (Piaget, 1953; Sfard, 2003). This view of *meaning* as internally constructing knowledge about the world around them and assessing one's internally constructed knowledge aligns with the acquisition view of learning.

The need for *structure* incorporates both the Piagetian notion of reorganizing mental schemes when acquiring new information and Vygotsky's hierarchical organization of new concepts (Bruner, 1997; Sfard, 2003). The need for *structure* incorporates what students already know (i.e., previous or prerequisite mathematical knowledge) and connects this knowledge with new ideas and concepts (Sfard, 2003). This view of *structure* is an extension of the need for meaning and builds on one's internally constructed knowledge, and therefore also aligns with the acquisition view of learning.

Through the process of *repetitive action* with mathematical ideas learners can reorganize their internalized knowledge (i.e., structure) to add the new mathematical object into their current knowledge schema (Piaget, 1953). This view of *repetitive action* is an extension of the need for structure, as a way of organizing new meaning, and rectifying one's understanding (Sfard, 2003) which considers knowledge to be something that is internally constructed. Therefore, *repetitive action* aligns with the acquisition view of learning.

The final internal need is the need for *difficulty*. Sfard (2003) stated that "true learning implies difficulties" (p. 366), what these difficulties are, however, is not the same for all learners. Therefore, learning requires a level of *difficulty* unique to each learner depending on previous constructed mental schema (Piaget, 1953). This corresponds to Vygotsky's Zone of Proximal Development (ZPD) in which each student has already built a certain level of ability and learning occurs inside of their ZPD with assistance (Vygotsky, 1978). This view of *difficulty* incorporates one's internally constructed knowledge and cognitive abilities, which aligns with the acquisition view of learning.

Social Needs

There are four social LN that each student has in order to construct mathematical knowledge: a need for *social interaction*, *verbal-symbolic interaction*, *well-defined discourse*, and *sense of belonging*. As with the internal LN, the social needs may be met with different means for each learner, and each are further described below.

Social interaction can be broadly defined as an exchange of ideas between an individual and another person via written or verbal communication (Sfard, 2003). Vygotsky (1962) suggested that *social interaction* is an essential part of learning especially for one to obtain a conceptual understanding of objects or ideas. *Social interactions* provide the space for students to challenge their preexisting notions about mathematical ideas. Lave and Wenger (1991) suggested that *social interaction* is essential for one to show knowing by doing within a community (i.e., situated learning theory). Therefore, the need for social interaction aligns with the participatory view of learning.

Social interaction and being able to communicate one's ideas and knowledge require learners to use both *verbal* (i.e., talking and listening) and *symbolic* (i.e., reading and writing) representations of knowledge and thoughts (Sfard, 2003). This idea that knowing comes from having a word or symbol is grounded in discursive psychology (e.g., Harre & Gillet, 1995). Sfard (2003) described the importance of either a *verbal* or *symbolic* use to give meaning as "one just cannot construct the meaning of a concept before introducing a word or a symbol with which one can think about that concept. The sense of *understanding* [emphasis added] then develops through the use of the word or symbol" (p. 374). Therefore, *verbal* and *symbolic representations* are a need of learning mathematics because these representations provide the tools for social interactions to occur, and aligns with the participatory view of learning.

The need for *well-defined discourse* follows from the needs for social, verbal, and symbolic interactionism. *Discourse* in mathematics becomes well defined when sociomathematical norms centered around what is considered mathematical argumentation are established in the classroom (Yackel & Cobb, 1996). Engagement in mathematical argumentation requires students to have intellectual autonomy (Yackel & Cobb, 1996). Specifically, students would need to understand their own "mathematical capabilities" (Yackel &

Cobb, 1996, p. 473) as they judge and decide what constitutes a mathematical argument of another student. This notion of *discourse* in the classroom among students includes both social interaction and the use of verbal or symbolic representations of knowledge, and aligns with the participatory view of learning.

A *sense of belonging* is a human need to feel valued as a member of a community. A student's *sense of belonging* can be defined as their "sense of being accepted, valued, included, and encouraged by others (teachers and peers) in the academic classroom setting and of feeling oneself to be an important part of the life and activity of the class" (Goodenow, 1993, p. 25). Ultimately, if students do not feel they belong in a field of study, classroom, or group they are less likely to continue or engage with that group (Cheryan et al., 2009; Strayhorn, 2019), and thus makes this is an important need that must be addressed for all mathematics learners (Sfard, 2003). This need to belong aligns with the participatory view of learning because it includes a community or members outside of oneself.

Combination (Internal and/or Social) Needs

The last two LN either fall in the middle of the proposed continuum between (i.e., significance and relevance) or span (i.e., balance) both the participatory and acquisition metaphors for learning. First, *significance and relevance* as one LN is placed in the middle of the continuum. This need is what can drive the motivation for the learner (Sfard, 2003). If we consider the ideas from Piaget where new knowledge can only grow out of existing knowledge, then learners must internalize the *significance* of what they are learning as they reorganize their mental schemes (Sfard, 2003). If we consider instead the importance of social interaction in learning, the *significance and relevance* of new knowledge can come through social interactions and participation within a community (Lave & Wenger, 1991). Because the *significance and relevance* can be formed/made either internally by one's own drives and thoughts, or influenced by one's peers in their learning community, this LN is placed in the between the internal and social needs.

The last LN may seem obvious now and yet may be the hardest one to meet: the need for *balance*. This essential LN both corresponds to the requirement for the learning environment to maintain a balance between the participatory and acquisition metaphors for learning; as well as attempting to balance the other nine LN for each student. A further description of how the balance need is deciphered and represented is provided in the following section.

Balanced Learning Needs Framework

The framework presented (Figure 2) arranges the 10 LN along a continuum of two main metaphors for learning (i.e., participation and acquisition). This continuum is represented as a balance scale, where the *balance need* is represented as the lever, holding the remaining nine needs, placed upon a fulcrum that can pivot between the two learning metaphors. I chose this representation to demonstrate the importance of both social and individual LN. The idea is that if only one side of the balance is met in the learning environment, then learning is hindered for the students. The arrangement of the needs toward the participatory or acquisition side was done meaningfully; the vertical placement, however, is not meant to imply any order among those needs placed on each side. The placement on the left (or right) only implies these constructs *lean toward* the participatory metaphor (or acquisition metaphor) for learning.

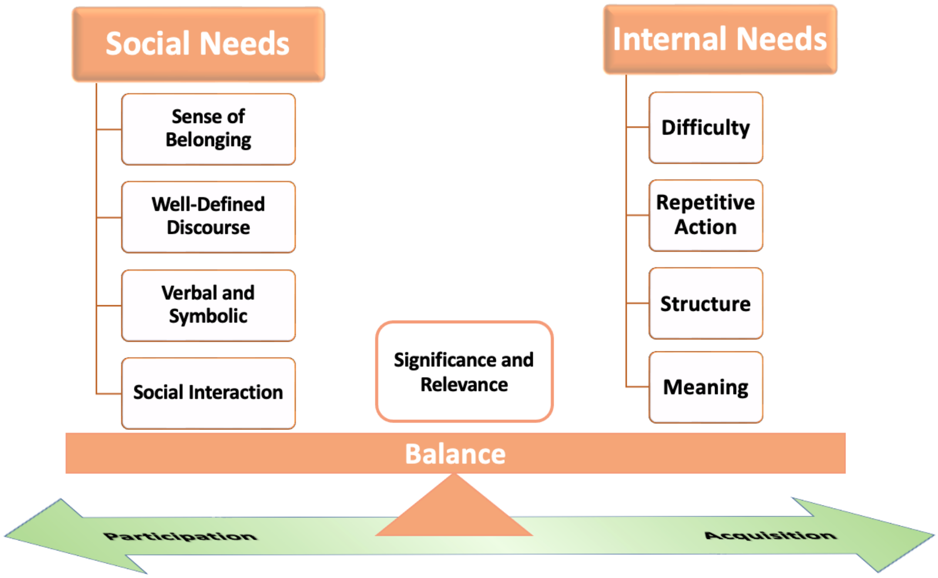

Figure 2. Balanced Learning Needs Framework

The author posits that the Balanced Learning Needs (BLN) framework presented here is a multidimensional construct with interrelated elements that impact student learning in either social or lecture settings. The framework can be utilized by academics to reflect upon the learning environment within their undergraduate mathematics courses and the implications for instructional and assessment practice. In the next section, I provide an example of how different classroom instructional methods from a Calculus I course that align with each learning need.

BLN Framework with Classroom Instruction

Sfard (2003) suggested "to meet learners' multifarious needs, the pedagogy itself must be variegated and rich in possibilities. The learning individual is a complex creature with many needs that must all be satisfied if the learning is to be successful" (p. 384). Educators must help create learning environments that allow students the opportunity to meet each need through equitable instruction (Sfard, 2003). Therefore, equitable instruction refers to the instructional techniques used to support the development of the 10 needs for *all* students and promote a balance between the needs, the metaphors of learning, and the students. For instance, collaborative group work is an equitable instructional practice that, if implemented with fidelity, students are given opportunities to explain their thinking and justify their mathematics using both verbal and written methods (Perry, 2018).

Often, it is difficult to connect theories of teaching and learning with the actual implementation of classroom activities and assessments. In Table 1, I provide an outline of how classroom activities and instructional practices meet each of the LN in the BLN framework. The course components described in Table 1 were implemented in a Calculus I classroom that also included individual and group testing methods. I provide a rationale for the assessment methods in the next section. Note, in the table below, *lecture* refers to instructor-centered classroom moves (i.e., instructor reviewing prerequisite material or concepts), without student-to-student engagement. Additionally, lecture may include whole class discussions and clarifications of mathematical language and symbols.

Table 1. Learning Needs Matched to Course Instructional Techniques.

Learning Need	Course Component	Description of how the component of the course was designed in an attempt to meet each learning need
Meaning (Internal)	Lecture and web-based homework	Allows students to work through problems and build their own internal meaning.
Structure (Internal)	Lecture and web-based homework	During lectures, instructor attempts to build structure in lecture by identifying connecting concepts from student's previous knowledge. Homework allows students to extend and organize new information to their previous knowledge.
Repetitive Action (Internal)	Lecture and web-based homework	Individual practice time in lecture and the homework time on the computer allow the students multiple problems to practice.
Difficulty (Internal)	Lecture and web-based homework	Instructor designs lecture and homework problems with a certain level of difficulty to allow the students to productively struggle when building their knowledge.
Social Interaction (Social)	Group Setting and web-based homework	Student-centered activities during group work require students to talk through problems and social interact to complete the learning activity. Students could choose to either work alone or with a peer on the homework allowing for more possible time for social interaction.
Verbal and Symbolic (Social)	Lecture and Group Setting	During lecture, after students initially explored a concept during group work, instructor provides the students with the formal terminology and symbolic representations of the concepts. During group work, students needed to communicate with one another using both verbal and symbolic representations for the mathematics.
Well-defined Discourse (Social)	Lecture and Group Setting	During lecture, the instructor sets the sociomathematical norms and models mathematical discourse that is needed for the students to engage in mathematical argumentation. During group work, the students establish the group's norms and engage in discourse.
Sense of Belonging (Social)	Lecture and Group Setting	During the lectures, instructor attempts to encourage all students to participate in whole class discussions and feel that they could share any idea, thought, or question. During group work, instructor monitors each group to ensure each student felt comfortable: (1) with their group members, (2) to voice their ideas, and (3) encouraged to share their ideas with their group.
Significance and Relevance (Combination)	Lecture, Group Setting, and web-based homework	Because this need is driven by the learner and can be met either while working alone or in a group setting, this need can be met for any learner within any one or all of the course components.
Balance (Combination)	The course as a whole	The instructor attempts to maintain balance by providing equal opportunity for each need to be met in their designed course component.

An instructors view of where knowledge is constructed largely impacts the environment of the classroom and how information is presented to students. Although student-centered instruction has been shown to increase student learning outcomes (e.g., Bressoud, 2011; Freeman et. al, 2014), traditional lecture is still a prominent form of instruction in undergraduate calculus courses (Bressoud et al., 2015). Note in the table above, both lecture and student-centered activities were used in the classroom as instructional techniques. As both aspects of classroom instruction can help students construct their mathematical knowledge. Additionally, web-based homework was included as an instructional technique, as there has been a large increase in the use of web-based homework systems in undergraduate mathematics education (Serhan, 2019).

Discussion

The creation of the BLN framework came from both anecdotal and empirical evidence from students' formative and summative assessments in Calculus I classrooms. Many tests in a mathematics classroom assess what is "easy" to measure and not what is "worth" measuring (National Research Council, 1993), especially at the undergraduate level. Consider the following four main purposes of assessments: an individual student's proficiency of the subject matter in knowledge and skills; an individual student's performance after working within a group; group effectiveness and productivity; and students' interpersonal skills (Webb, 1995). In mathematics classrooms, however, students are usually formally tested only on individual expertise of the subject matter knowledge. The testing of individual knowledge and skills helps perpetuate the competitive nature of learning mathematics. Although competition can also be a motivation for learning, many women in mathematically intensive areas of study can feel isolated in competitively focused environments. Therefore, if a learning environment is created with an attempt to balance the LN of each student using the BLN framework, then a shift in assessment also needs to be considered. For instance, increased use of student-centered teaching strategies creates a mismatch between the learning and assessing contexts: students are asked to work in groups to learn together cooperatively, but are tested individually and potentially competitively. If we consider assessing knowledge in the space it is constructed (i.e., assessing the constructed knowledge on the lever at both sides of the fulcrum) then group testing may be a necessary addition to undergraduate mathematics courses.

Limitations and Future Research

I recognize that the proposed BLN framework being enacted in a classroom might differ depending on the pedagogical beliefs of the instructor. The framework was created from the perspective of one researcher, and still requires further empirical testing and validation. I have begun to explore this framework on the social side using group testing environments. Preliminary results did indicate how a lowered sense of belonging for female students impacted group performance on a group exam (Quinn, 2021). That is, their sense of belonging decreased and they underperformed in the group setting. Additionally, measuring the needs in a collaborative learning environment verses traditional lecture environments may help to show the *balance* needed between the pedagogy and learning environments in the classroom to help all students learn. Future testing of the framework will include individual and group testing methods to measure the internally constructed knowledge (i.e., individual exams) and socially constructed knowledge (i.e., group performance).

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Building Instructional Capacity Across Difference: Analyzing Transdisciplinary Discourse in a Faculty Learning Community focused on Geometry for Teachers Courses

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In response to enduring methodological tensions in analyzing collaborative discourse, we detail an assemblage of argumentation analytic methods that can support research about faculty learning communities interacting across substantive differences. Drawing on our research with a cross-institutional faculty online learning community, we use data to show how theories from discourse analysis, systemic functional linguistics, and argumentation modeling can be operationalized to support researchers in brooking methodological tensions, including framing argumentation as the topic of or a resource for investigation and considerations of collaborative discourse as both process and content. Our methodological findings illustrate an example of this operationalization, highlighting analysis of transdisciplinary, collaborative discourse in a community composed of instructors of college geometry courses required for pre-service teachers. We share possible uses for this methodological approach vis-a-vis research about the professional work of undergraduate mathematics education and pre-service teacher preparation.

Keywords: Geometry, Argumentation, Professional Development

In explaining the exigencies of continued research about discourse in mathematics education, Sfard (2015) argued that researchers ought to investigate communication not merely as a means for learning, but as “the centerpiece of the story—the very object of learning” (p. 249). Such is our interest in faculty members’ collaboration around instructional improvement, in their discourse is a product of their collaboration. However, as Sfard (2014) and others have detailed extensively, research about discourse in mathematics education is theoretically and methodologically diverse, and so does not rely on a common set of assumptions, conceptual models, or analytical tools (Niss, 2007; Ryve, 2011). In our own research with a faculty online learning community (FOLC) of college mathematics instructors, we encountered the need for methodological resources to support analysis of instructors’ discourse—specifically, methods that would align with our goal of understanding the nature of the arguments among a diverse community of practitioners. This need is not unique to our research team. In an extensive review of research about instructors’ collaborative discourse, Lefstein et al. (2020) found that “the field would benefit from greater coherence between theoretical perspectives and research designs” (p. 11). In mathematics education research, we see evidence of such coherence from scholars using argumentation theories and models to analyze classroom discourse (see, e.g., Conner et al., 2022) and teachers’ knowledge and beliefs (see, e.g., Nardi et al., 2012). In turn, our methodological approach contributes careful consideration and practical examples of analyzing argumentation in undergraduate mathematics instructors’ collegial discourse.

In this paper, we detail a qualitative methodological approach that supports the investigation of undergraduate mathematics instructors’ cross-disciplinary knowledge resources—including synthetic geometry knowledge and mathematical knowledge for teaching geometry. We demonstrate this approach in the context of analyzing faculty members’ collaborative discourse geared toward the development of a curricular resource for instructors of college geometry courses required for pre-service secondary mathematics teachers. With attention to relevant methodological disputes, we illustrate how an assemblage of social semiotic methods—guided by Toulmin's (2003) and Gilbert's (1997) argumentation models—can support analysis of instructors' collaborative discourse across disciplinary and institutional differences.

Relation to the Literature: Argumentation Analysis in Mathematics Education

Discourse analytic research in mathematics education commonly adopts a fundamentally social constructivist stance towards human thought and knowledge (Sfard et al., 2001). That is, researchers assume that knowledge is socially and culturally, rather than individually, constructed; is dynamic rather than static; and is inherently contextual (Palincsar, 1998, p. 354). Unlike formal logic-based argumentation models, informal and quasi-logical argumentation models and theories involve inductive or diffuse disputes in which participants can "explore positions flexibly" (Nussbaum, 2008, p. 349) and in which different "ways of representing the world" can become "premises" supporting (a potential plurality of) conclusions (Fairclough & Fairclough, 2012, p. 86-87). In the data we share from our study with a FOLC, mathematics instructors' interactions are discursive activities situated in their development of a curricular resource. In many ways, the group's discourse represents their efforts to build a plane while they fly it (so to speak)—with the destination being instructional improvement. A review of research shows methodological challenges when conceptual fidelity requires researchers to consider this kind of collaborative discourse in which argumentation is more exploratory in nature.

Taylor (2001) named key tensions that researchers face in discourse analysis work: (1) investigating discourse as the topic of study versus analyzing discourse as a means of (or resource for) investigating something else and (2) fore-fronting the process or content of discourse. Conceptualizing discourse as process means focusing on the interactivity of discourse, including questioning the functions or effects of talk. In contrast, conceptualizing discourse as content typically focuses on the recurrence of themes, ideas, or other elements in the corpus of text. While these tensions exist across discourse analytic approaches, we identified how they are instantiated in argumentation analyses related to mathematics education research and research about educators' collaborative discourse.

Mathematics education research that analyzes argumentation often focuses on deliberate classroom interactions, such as patterns of teacher and student participation in formal (logical) argumentation. Such scholarship often frames argumentation as the topic of study. For instance, Forman et al. (1998) described their aims in analyzing transcriptions from a mathematics lesson as "understanding …the socialization of argumentation in her mathematics classroom…[and] to provide teachers and teacher educators with a detailed picture of argumentation in this classroom (p. 529). We also found several examples of researchers studying students' argumentation in the context of proof activities, illustrating further scholarly focus on argument as the main topic of investigation (see Harel & Sowder, 1998; Knipping, 2003; Rodd, 2000; Weber & Alcock, 2005). Still, many of these same researchers (and others) have related their findings to other phenomena, suggesting argumentation is a resource for engaging concepts relevant to social constructionist notions of learning. For instance, Krummheuer (2007) analyzed students' argumentation and theorized that their contributions reflected their learning autonomy.

Research about teachers' professional knowledge and learning communities has used argumentation as a resource to investigate other educational phenomena—most notably, teachers' beliefs, values, and knowledge (e.g., Conner & Singletary, 2021). Scholarship like this has demonstrated the need for researchers to analyze argumentation as both topic and resource. For instance, Nardi et al. (2012) showed that teachers' arguments could be understood in terms of mathematical accuracy and other professional concerns (including pedagogical, curricular, and personal considerations).

Mathematics education research in our review often elided distinctions between argumentation as content and process. We posit that the mathematical nature of logical argumentation and the cultural diffusion of mathematical thought into more popular and informal modes of argumentation (Keitel, 2006) mean that, in mathematics education research, the process and content of argumentation can both be objects of study vis-a-vis the same phenomenon (e.g., students' proving work in Hollebrands et al., 2010). Similarly, social constructionist frameworks can complicate distinctions between the content and process of argumentation. Researchers who have investigated the quality of collaborative discourse as a matter of process as much as content have faced this challenge (Lefstein et al., 2020). For instance, scholars noted that facilitators can help provide the *content* of expertise (Horn & Kane, 2015) and facilitate social *processes* (Kintz et al., 2015). We conclude that field-specific conceptions of mathematics and learning must inform these kinds of methodological distinctions.

Conceptual Framework & Operating Theories

Theoretical underpinnings of our research guide our attention to how and why undergraduate mathematics instructors draw on diverse knowledge resources to engage in collaborative argumentation. We discuss our study, including the community, its members, and its formation, in this report's research methodology section. We focus on the theories and models of argumentation we have operationalized to facilitate analysis premised on treating discourse as both topic and resource and content and process. In particular, we discuss Toulmin's (2003) six-part model of argumentation and Gilbert's (1997) theory of coalescent argumentation.

Toulmin (2001) averred that analyzing argumentation should balance attention to the social situation of the argument and attention to its artifacts (i.e., texts). Gilbert (1997) wrote that "we need to shift the focus…from the *artifacts* that happen to be chosen for communicative purposes to the *situation* in which those artifacts function as a component" (p. 46). In both views, analyzing argumentation involves the particularities of the situation, including participants' motivations, goals, and positioned relationships. In bringing these two theories together, we do not want to reduce Toulmin's and Gilbert's robust bodies of work to narrow facets or conflate shared theoretical implications with common theoretical assumptions. Instead, we consider how their theories complement each other—allowing for dual-attention to argumentation as both (a) topic and resource and (b) content and process.

Toulmin's model includes six parts: claim (conclusion), data (evidence or grounds for the claim), warrants (justifications explaining the relationship between the data and the claims), backings (beliefs or evidence underlying warrants' logic), qualifier (claim's degree of certainty), and rebuttals (conditions that would make the claim untenable). Not all arguments fit this framework, as others have noted (Ellis, 2015; Schwarz, 2009). Still, as Inglis et al. (2007) wrote, Toulmin's framework is "less concerned with the logical validity of an argument, and more worried about the semantic content and structure in which it fits" (p. 4). That is, the Toulmin framework is useful for decomposing everyday reasoning, even if partially. On the other hand, coalescent argumentation captures how participants may join their justifications to achieve collaborative aims. Of note, Gilbert's theory directs attention to multi-modal, goal-oriented, and position-based argumentation (Gilbert, 1994; 1997; Godden, 2011)—a conceptualization compatible with Toulmin's larger theory of argumentation. Extending Toulmin-esque theories of informal argumentation, Gilbert suggested that models of argumentation could include warrants of varying sources. Warrants from other sources (what Gilbert calls "multi-modal" participation), include the emotional, the visceral (e.g., "That's a sensitive subject for me"), and the kisceral[1] (e.g., "It strikes me as the right thing to do"). Attention to multiple modes increases possibilities for coalescence (e.g., two participants may not agree on the logical validity of a claim, but they might find coalescence around a claim for which one finds logical warrant and the other finds emotional warrant). Gilbert (1997) also theorized that people engage in argumentation with multiple goals, including task goals and face goals (see Gilbert, p. 67-68). Thus, we understand resolution may be oriented toward a shared aim (e.g., the development of a curricular resource) *and* that participants have various other goals and priorities (e.g., maintaining relationships).

Engaging in participants' complex positions is a key to using the tools of coalescent argumentation (Gilbert, 1997). So, beyond modeling individual micro-arguments, coalescent argumentation can help us understand how participants relate micro-arguments to their diverse knowledge resources, institutional contexts, and sustained community engagement. We draw on Toulmin's model to support investigation of the topic and content of FOLC members' multiple arguments. When argumentation does not resolve in a single conclusion or involves more diffuse discursive engagement with elements of argumentation, we draw on Gilbert's (1997) theory of coalescent argumentation. Based primarily on these two theories, we developed an approach to support our analysis of participants' micro-arguments (argumentation as topic and content) and situating those micro-arguments in disciplinary, institutional, and community-specific contexts and relationships (argumentation as resource and process). This approach allows us to analyze participants' multiple, dynamic, and sometimes contradictory or unrelated positions and to understand a mode of argumentation in which members of the FOLC neither invalidate participants' starting positions nor disengagement from reasoning together. In our methods and results sections, we detail how we operationalized the theory of coalescent argumentation to interpret argumentation that initially included two oppositional positions—and out of which a third position emerged and coalesced them without invalidating their logic.

Research Methodology

Our research occurs in the context of an FOLC that focuses on "Geometry for Teachers" (GeT) courses (i.e., college geometry courses required for prospective secondary mathematics teachers). Within the FOLC, 15 GeT instructors formed a working group to discuss student knowledge outcomes. Working group members' exchanges have included argumentation concerned with the state of the American college geometry course, which Grover and Connor (2000) had identified as having a "wide diversity" of course elements and a lack of consistent curriculum or instruction across institutions (p. 47).

While GeT courses are typically taught out of mathematics departments, these courses are usually taught by mathematicians or mathematics educators. The 15 members of the FOLC working group are a balance of instructors who identify as either mathematicians or mathematics educators. Some GeT courses are exclusively taken by future teachers, while others are electives

[1] Gilbert (1994) defined the kisceral mode as a "the mode of communication that relies on the intuitive, the imaginative, the religious, the spiritual, and the mystical" (p. 10). People often draw upon kisceral reasons in their everyday argumentation, and Gilbert noted (2011) that kisceral experiences can often be recast as strictly rational. Elaborating on the kisceral mode of argumentation, Gilbert (2011) gave the example of Euclid's Fifth postulate as a subject of knowledge that has often elicited "a *feeling* about what was right, what made sense, and what fit" (2011, p. 164).

for mathematics majors. In some cases, the GeT course is the only one that covers geometry content. Unlike many other college mathematics courses (e.g., Calculus), there is no broad consensus about what should be included in a GeT course, as there are many different types of GeT courses (Grover & Connor, 2000; Venema et al., 2015). To improve secondary geometry instruction, the FOLC working group realized it would be useful to identify a set of essential student learning objectives (SLOs, hereafter) for inclusion in GeT courses. *Essential* means the identifying content knowledge that all prospective secondary geometry teachers should learn in the course. They created this list of SLOs to help other GeT course instructors, especially new ones. The working group met monthly for the first year and a half and biweekly for the past six months to develop, elaborate, and steward the SLOs. The instructors' work is framed by their shared commitment to developing consensus—and SLOs that reflect that consensus—while navigating significant differences in their preparations, institutional and departmental affiliations, and disciplinary orientations. Our study of this group's work aims to understand the professional reasoning that occurs across such a situational landscape.

Results: Operationalizing a Methodological Approach

In this section, we show how a methodological approach drawing on both Toulmin and Gilbert provides a basis for conducting argumentation analysis without a single conclusion and towards a common goal. The analytical process has two major parts: (1) modeling micro-arguments and (2) mapping and analyzing position interactions, including coalescence.

Modeling Components of Micro-Arguments

One of the primary goals of the FOLC's working group has been to write SLO narratives that are detailed enough for GeT instructors outside the FOLC to use. The process of drafting these narratives starts with a subgroup of two to three people writing an initial draft narrative for each SLO and bringing it to the whole group for discussion and revision; this process repeats with the second draft and so on. The data in this section comes from a one-hour whole group meeting in March 2021, where the group reviewed a revised draft of the narrative of an SLO devoted to the role of definitions in mathematical discourse. As context: In a previous meeting about the first draft of the narrative, there had been a lively discussion about what constitutes non-Euclidean geometry, and whether they needed to define Euclidean and non-Euclidean geometry for their readers—as they themselves had varying definitions of these two types of geometries. Our example highlights exchanges between three participants: Miriam[2], Royce, and Michael. Over the course of the discussion, Miriam, Royce, and Michael engage in argumentation around issues related to teaching the nature of definitions in GeT courses.

To analyze the interactions and relationships between the arguments in the meeting(s), we first applied Toulmin's (2003) extended model of argumentation to meeting transcripts. We coded participants' turns of talk to identify claims (i.e., any conclusions they offered). Then, we coded turns of talk for contributions to arguments in service of a claim (or counter-claim). Each claim and its related elements (e.g., data, backing) represented a micro-argument that was putatively connected to the arc of the group's inquiry. Because we coded transcripts of verbal conversations, this process also involved looking for linguistic indicators of argumentation (e.g., "because"), including the use of modal qualifiers (e.g., "probably"). One example micro-argument consisted of the following elements, made by a participant called Miriam: (*claim*) the SLO narrative text uses too many examples of non-Euclidean geometry; (*data*)

[2] Pseudonyms are used for the participants in the study.

examples from the SLO text; (*qualifier)* the issue is context-specific, given the group's aims in writing the SLOs; (*warrants*) explaining that too many non-Euclidean examples might seem to promote a non-Euclidean approach to the course; and, finally, (*backings*) citing twice that she feels worried (a *kisceral* justification) and that her GeT course would show the need for the text to be less weighted toward non-Euclidean examples (a *logical* justification).

While completing the Toulmin (2003) diagramming of the transcript data helped us map the (quasi-)logic of the micro-argumentation, we also wanted to understand connections to the group's larger processes and individual and shared contexts for their work. We understood the group as sharing the goal of developing the SLOs. Gilbert (1997) detailed coalescent argumentation beginning from identified opposing sides (e.g., pro, con) of an "avowed disagreement" (p. 104). Thus, we identified disagreements participants had in direct relation to the SLO narrative, allowing us to thematically group elements of micro-arguments according to their indicated positions vis-a-vis a common dispute. Through this analysis, we identified three positions (i.e., thematic clusters of arguments): (1) the SLO should use examples of Euclidean geometry; (2) the SLO should use examples of non-Euclidean geometry; and (3) the SLO should explain the relevance of teaching the nature of definitions to pre-service teachers.

Mapping & Analyzing Position Interactions, Including Coalescence

In Gilbert's (1997) theory of coalescent argumentation, argumentation is not necessarily dialectical, because dialectical exchanges resolve in affirming the validity of some arguments and the invalidity of others. Toulmin's (2003) model of argumentation is well-suited for representing more dialectical exchanges, including collective participation (Krummheuer, 2007; Nardi et al., 2012). Diagramming micro-arguments illuminates participants' practical reasoning. Analyzing coalescence across the arguments illuminates the multi-modal rationales disparately maintained by various participants and that ultimately facilitate consensus. Figure 1 illustrates three parts of the argumentation occurring between Miriam, Royce, and Michael.

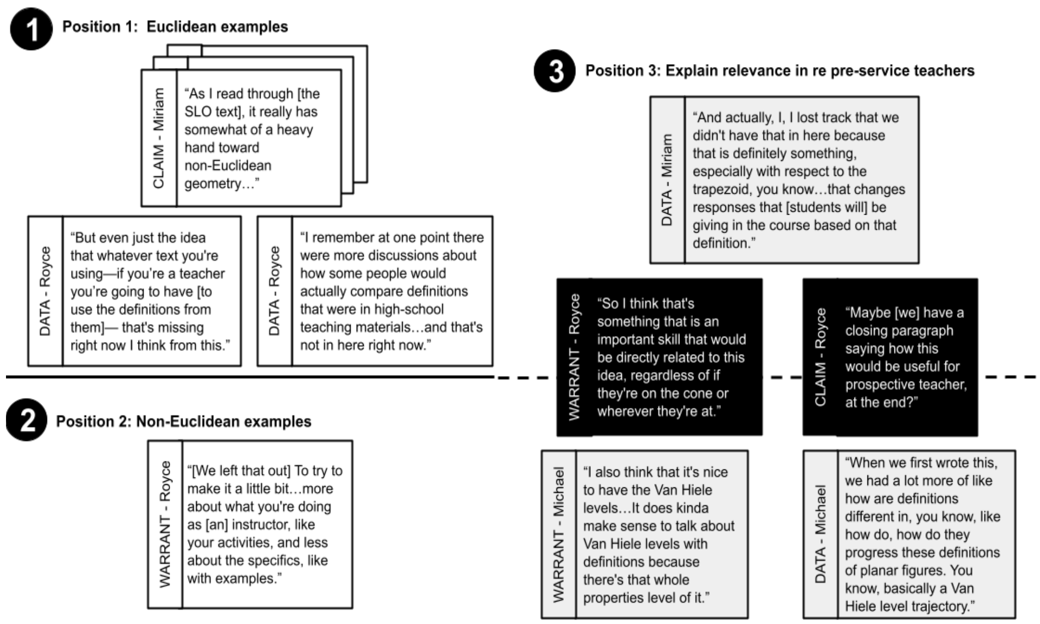

Figure 1. Modeling Phases of Argumentation Leading Toward Coalescence

Position 1 is constituted by micro-arguments advocating for using examples from Euclidean geometry. It includes Miriam's previously-described argumentation. It also includes two pieces of data that Royce offered in response to Miriam. Royce also offered a warrant that would support Position 2 (advocating for using non-Euclidean examples). However, he also gave a warrant that could support Position 1 *or* Position 2: "…that's something that is an important skill that would be directly related to this idea, regardless of if they're on the…." Here, Royce amplified Miriam's point that an example about defining a circle on the surface of a cone does not necessarily communicate the same set of skills she is saying a Euclidean geometry example might *and* offered justification for why non-Euclidean examples would still suffice. This warrant serves as the first indication of potential coalescence. Miriam responds to Royce's warrant by using it to agree with Royce's suggestion that they add additional text detailing the relevance of teaching the nature of definitions for pre-service teachers. This suggestion—a claim that represents Position 3—thus gains Miriam's support and allows her to maintain the rationale she employed in arguing Positions 1. It also allows Michael to accommodate his own rationale. Specifically, Michael identifies that Royce's data—a reference to structuring the narrative of examples in a way that would align with Van Hiele levels—is reason to support Royce's suggestion. Notably, Miriam and Michael's continued participation *could* align with the previous arguments made vis-a-vis Position 1 and 2, but they built coalescence around a third position that allowed them to maintain elements of the rationales underlying Positions 1 and 2.

Applications to/Implications for Teaching Practice or Further Research

Thus, we claim that Gilbert's (1997) theory of coalescent argumentation is complementary to Toulmin's (2003) modelling of arguments. We identify coalescent argumentation's suitability for theoretical and empirical investigations of mathematics instructors' professional discourse in situations that support developing consensus without invalidating participants' individual rationales. We operationalize tenets of coalescent argumentation, including using the Toulmin framework to model multi-modal argumentation, bounding arguments by identifying positions (i.e., thematic clusters of micro-arguments), and analyzing interactions across positions to identify possible coalescence. We suggest that the theoretical implications of the three principles of coalescent argumentation—that argumentation is multi-modal, position-based, and goal-oriented—may have wide-reaching applications in mathematics education research.

Scholars in the RUME community have advocated for qualitative researchers to align their research questions and methods (Melhuish & Czocher, 2022), and for more research about cross-disciplinary knowledge resources in undergraduate STEM instruction (Speer et al., 2020). Lefcourt (2020) noted that, while there has been "increasing attention…on making undergraduate courses for mathematics teachers relevant to the work of teaching," it is uncertain how such efforts can occur with larger degrees of "scope, scale and impact" (p. 830). Even though we are explicit about our methodological approach's conceptual boundaries and limits, we share it to support conceptually-aligned research in undergraduate mathematics education that examines questions about cross- and trans-disciplinary knowledge resources and instructors' practical applications of these knowledge resources in their mathematics teaching.

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Investigating Student Perceptions towards Mastery-Based Grading in Precalculus

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In this study, mastery-based grading was implemented in three sections of precalculus taught at two institutions. We investigated student perceptions towards this grading system by analyzing math autobiographies written by the students at the beginning of the semester as well as responses to surveys that were distributed at three points in the semester. In this paper, we present our results and discuss the implications to help support students during mastery-based courses.

Keywords: mastery-based grading, precalculus, standards-based grading

Mastery-based grading is not new, but it has begun to receive more attention from undergraduate mathematics educators who are looking for a different way to assign grades that will both accurately reflect their students' progress in the course and temper some of the anxiety and negative attitudes towards mathematics that their students may possess. In this study, we investigated students' (N=21) perceptions of a mastery-based grading scheme being used in a semester-long precalculus course. The use of mastery-based grading is new for many students, so they may need more support to adapt to it. Our results give indications of where this support is needed and in what forms.

Background

Since there are many forms that mastery-based grading can take, we will begin by defining our method of mastery-based grading. We adapted the ideas of Nilson (2015), first developing a set of learning objectives for a precalculus course, then creating a direct link between the objectives in the course and a student's overall grade. In general, students are graded based on how many of the course learning objectives they have mastered. Mastery of the learning objectives can be assessed in a variety of ways, for us this was done using tests, but a key feature of the system is that students do not receive partial credit for partial understanding, rather they must show the grader that they have full proficiency with the objective (Collins, et al., 2019). While students do not earn partial credit, a key element in mastery-based grading is that students are given multiple attempts to demonstrate proficiency on the learning objectives. In addition to the learning objectives, we also developed additional requirements including homework completion and projects to include in the criteria for each grade.

The increase in implementation of mastery-based grading in undergraduate mathematics courses has brought with it an interest in studying the grading scheme and reporting on its effects and outcomes specifically in those contexts. Benefits have been noted in these settings, such as improving students' attitudes towards mathematics, increasing students' confidence in their mathematical abilities (Stange, 2018), and giving the students more control over their final grade (Williams, 2018). Additional benefits were seen by Harsy and Hoofnagle (2020), who found that students felt that their final grade more accurately reflected their knowledge as opposed to grades in a traditional grading environment. Since students know the requirements for each grade from the first day of class, they are able to set goals and make choices to study objectives they feel will allow them to achieve those goals.

There is also evidence that the emphasis that mastery-based grading often puts on reattempting mastery can alleviate stress associated with tests (Stange, 2018) since the reattempts give flexibility for students to continue working on an objective over time. However, there may also be drawbacks to this in terms of student motivation; for example, students may be tempted to put off learning objectives or not take first attempts as seriously as they would in a traditional environment (Weir, 2020). One traditional purpose of mastery-based grading and learning is to allow each student enough time to master the content for the course (Bloom,1968). However, a course typically has a set ending date, limiting the time available. Instructors will need to think deeply about how they can support their students to set appropriate goals and strategies for success. To do so, it would be useful to understand more about the students' perspectives, which is the goal of this study.

Research Question

The results discussed in this paper are part of a larger study investigating mastery-based grading within college-level precalculus classes conducted by the authors. For the purposes of this paper, we will be discussing our findings related to the following research question: How do students' opinions of mastery-based grading change throughout the semester?

Methodology

This study collected data from three sections of precalculus taught at two different institutions in the same semester. One section was offered as a 3-credit course at a small, private, liberal arts college in the northeast United States (Institution 1) while the other two sections were offered as 4-credit courses at a medium, private university in New England (Institution 2).

Course Structure

The grades for students in the three sections were based on their proficiency on 27 learning objectives as well as their homework completion and their completion of up to two projects. Proficiency on the learning objectives could be shown on five test days, four during a regular class period and one in place of a final exam. Within the 27 objectives were 7 core objectives, which needed to be mastered to pass the class. The questions used for the tests were common across the three sections, as were the homework questions and projects. The base letter grade depended solely on proficiency on objectives and project completion. Students needed to complete one or two projects to earn a B or A in the class, respectively. After the base letter grade was decided, homework scores impacted whether the letter grade would have a plus, minus, or neither attached to it.

Data Collection

The data for the larger study consists of both quantitative and qualitative data, collected through student responses to three surveys, math autobiographies written by the students at the beginning of the semester, homework scores, and test responses and scores. To address the research question above, we use the qualitative data from the surveys and math autobiographies.

Each survey contained various closed- and open-response questions. In this paper, we focus on the analysis of responses to those questions relating to the students' views on the grading used in the class. The first survey was completed in the first two weeks of the semester and students responded to the open-ended question, "What are your first impressions of the grading system in this class?" In the second survey, sent to students after they had taken two tests, the students responded to the open-ended question, "At this point in the semester, what are your impressions of the grading system in this class?" A third and final survey was sent at the close of the semester, and students responded to four open-ended questions in which they discussed their ideas about the grading structure in the class, those questions being:

1. In preparation for your tests, how did you choose which learning objectives to attempt?
2. Do you believe the grade you will receive in this class accurately represents how much you have learned? Why or why not?
3. Do you think your overall grade would have been better, worse, or the same in this class if you had been graded traditionally? Why?
4. What are your impressions about the grading system used in this class?

Math autobiographies were written by students at the beginning of the semester describing their past experiences with mathematics and their expectations for the current course.

Participants

Students in the three sections of Precalculus were emailed an invitation to participate in each survey. Participating in at least one survey indicated the student's consent to being a participant in the study, as indicated by filling out a consent form on the survey. A total of 21 students agreed to participate in the study, twelve on Survey 1, thirteen on Survey 2, and ten on Survey 3. Out of the 21 participants, seventeen submitted math autobiographies but only eight of those referred to the grading used in the class. There were 8 participants from Institution 1 and 13 from Institution 2. To protect our participants' identities, throughout this paper you will see participants referred to by code. Participants at Institution 1 are labeled using the letter N followed by two digits (ex. N20), whereas those at Institution 2 are labeled using the letter R followed by two digits (ex. R01).

Data Analysis

The math autobiographies and survey responses to open ended questions were analyzed using an inductive thematic analysis (Patton, 2002). To investigate the students' opinions about mastery-based grading at different points in the semester we focused on the students' initial perceptions, perceptions after taking two tests (midsemester), and their perceptions at the end of the semester.

To analyze the initial impressions about the grading scheme we considered the responses from math autobiographies and from the first survey. To analyze the midsemester impressions that students had about the grading scheme, we analyzed their responses to the question from Survey 2 that was discussed above. Finally, to analyze the students' opinions about the grading system at the end of the semester we considered the responses to the four open response questions on Survey 3.

Discussion of Results

This section will briefly describe the categories discovered through the analysis of initial impressions, midsemester impressions, and final impressions that students had about the grading scheme.

Initial Impressions

There were four categories that emerged during our analysis of the responses to Survey 1 and the math autobiographies.

Confusing. Responses showed that several students saw the grading system as confusing or strange to them. For example, N20 wrote the grading scheme was "confusing but might be beneficial". These responses were not surprising since for many students this was the first class in which they had experienced mastery-based grading.

Straight-forward. There were a few students who thought the grading was clear. N50 said, "I love how straight forward, and easy the grading is". Similarly, R09 emphasized the clarity of the learning objectives, saying, "I think they are really helpful in keeping me focused. Nothing feels jumbled together, I can clearly see what path I need to take."

Retries. We also saw that students were excited about the ability to retry objectives that they had not shown proficiency on with their first attempt. R12 mentions, "I feel like it is way less stressful for us new incoming students who are use to just having to fail if we do bad on a test … with the way you grade our tests I get to redeem myself with questions I may get wrong which is very comforting." This was an aspect of the grading that students seemed to grab on to as a positive aspect of the method, although there was some hesitancy from R06 who said, "I think it's cool that I can retry topics at my leisure but I hope I don't mess up on a lot so it snowballs into a disaster."

Pacing. Students also mention that the grading scheme will give them more freedom and/or independence to complete objectives at their own pace and that there is less stress due to the pacing of the class. N50 says, "I see myself doing well in this class because I'll be able to go at the pace I want … I'll be able to complete [objectives] on my time."

Midsemester Impressions

Through our analysis of the responses to Survey 2, three categories emerged. Two of the categories were similar to two found in the initial impressions, so we chose to use the same name and definition for them.

Confusing. Even at the midsemester mark, after the students had taken two tests, there were still students who found the grading scheme confusing or strange. However, it seemed that there was some improvement in understanding. R10 said, "I'm very confused by it still, but looking at each individual grade I've received helps me calculate what my overall grade is."

Retries. Students continue to mention the benefit of being able to retry objectives. R10 says, "I love the fact that you can attempt each core objective until you receive the grade you want."

Stress. A new category that emerged spoke to the students' perceived stress and anxiety levels in the course, with some students feeling that the level of stress was heightened by the grading scheme and some feeling that it was lessened. From R09, "It makes the class feel more intense, like we have less room for error, and over all, just a really stressful and weird experience." With an opposing view, R11 says, "I really enjoy it. I think that because of how the tests work and how we're able to retry problems, my anxiety is much lower for this class."

End of Semester Impressions

Four categories emerged through our analysis of the four open-response questions from Survey 3. One of the categories, Stress, uses the same name and definition as was found in the midsemester analysis since the ideas were similar.

Stress. A couple students mentioned opposing views on their perceived stress and anxiety levels due to the grading scheme. R01 said, "Objective based learning didn't help me change how I study, but rather gave me huge amounts of anxiety as my failings kept weighing on me" whereas R02 said, "I think it showed that it is up to us to work hard but also it is okay if not every day is our best day because we get another shot. It took a lot of the anxiety down."

Strict. Students spoke more specifically about the requirements in the grading system, most likely due to the additional questions included in the final survey. Some students felt that the grading structure was too strict and that they were marked wrong for too small of errors along with not having enough opportunities or ways to show their understanding. R09 felt that "there were small mistakes [they] made from calculation or missing one step of a five part problem where in any other class [they] would have gotten some type of credit for it because [they] clearly understood how to do the problem." Others mentioned they wanted the projects to count differently, such as R06 saying, "Through doing the projects and homework I have done so much more and the projects will not even be factored into my grade unless I have everything done [that is necessary for an A or B]."

Effort. As we see in some of the examples for Strict, students also discussed that their effort in the class has affected their feelings toward the grading system. Several students felt that the effort they put in for the course was not reflected in their grade, as we saw from R06 above. However, there was also a student, N04, who mentioned that because they always felt they had more chances to complete objectives, they did not put in enough effort: "If it was graded traditionally I think my grade would've been higher, because more pressure levied upon me every week meant that I wouldn't have procrastinated until the last minute to do everything."

Core. Several students mentioned the core objectives in their responses. This was often relating to how they chose which objectives to study for, stating that they focused on the core objectives first, but we also had students say that they believe the expectation of and focus on mastering the seven core objectives caused them to miss out on other objectives. R09 said, "Most of the class still needs core objectives on the final and are at a high risk of failing." It is worth noting that of the 18 participants that did not withdraw from the course, only 7 had completed the core objectives prior to the final, so the concern about core has some merit for our sample.

Implications

We believe that investigating students' perceptions of mastery-based grading is important to understanding how we can support them in our courses. By considering three different points in the semester, we were hoping to gain insight into how support can adapt to the changing concerns of students. Our results indicate that there are some concerns that remain present throughout the semester, while others may emerge later after the students have been able to experience more of the consequences of the grading. Understanding these concerns is useful in planning for a course, especially for instructors who are new to this grading style.

For example, while emphasizing the opportunities for students to retry objectives can alleviate some of their concerns early in the semester, it may be beneficial to have specific recommendations about how to take advantage of these opportunities so that the pressure is not just pushed to the end of the course or due date. This agrees with findings by Matz, Derry, Bennett, LaRose, and Hayward (2022), who found that the use of behavioral nudges can lead to better outcomes for students. In their study, personalized email reminders were sent prior to assessment due dates. Other ideas could be to have class discussions about study strategies, hold individual meetings to plan for future assessments, or to structure the course with some intermediate deadlines.

These results represent a portion of the data we collected. In addition to the math autobiographies and survey responses, we also collected tests, homework scores, projects, and grades. We hope the RUME community can provide insights on how to further explore these results and make connections to the other data sources.

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Ongoing Efforts to Develop an Assessment Tool for Students' Participation in Defining

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There has been a push to develop curricula that engages students in various mathematical practices (e.g., defining). As more curriculums of this type arise, there is a greater need for assessment tools for students participating in these practices. We see Modeling Eliciting Activities (MEAs) as one answer to this need. In this study we discuss how the six design principles of MEAs can be used to (re)design an assessment tool for students' defining activity.

Keywords: Modeling Eliciting Activities, Assessment, Defining, Classroom Practices

One common goal of many curriculum designers is to engage students in mathematical practices such as defining, conjecturing, and proving. This engagement can be a means to support students in learning about mathematical concepts (Larsen, 2013; Lockwood & Purdy, 2019; Oehrtman et al., 2014; Swinyard, 2011) and/or as a means to increase students' participation in these practices themselves (Larsen et al., 2022; Rasmussen et al., 2005; Zandieh & Rasmussen, 2010). We have engaged students in mathematical practices for both reasons in our ongoing efforts to design inquiry-oriented curriculum for Introduction to Proof courses.

We have faced a practical problem in our efforts: the lack of an appropriate assessment tool to reveal students' participating in mathematical practices. While there are relevant concept inventories (e.g., Melhuish, 2019; Mejía-Ramos et al., 2017), these take an acquisition lens on learning (Sfard, 1998) to measure students' understandings of important concepts or skills rather than a participation lens. We also found relevant observational tools (Melhuish et al., 2021; Melhuish & Thanhesier, 2017); yet, using these would require one to analyze classroom data to get a sense of whether students' participation changed overtime.

An assessment tool that measures students' participation in mathematical practices would be useful for us as researchers to measure the efficacy of our materials on student participatory learning. Additionally, the need for an assessment tool has been echoed by the instructors who use our curriculum materials; they want their tests and quizzes to reflect what they do/teach in class. This study aims to make progress on this goal for the case of assessing students' participation in defining - a practice that our curriculum materials are specifically engineered to promote. In doing so, we hope to open the discussion about how such an assessment could be adapted to assess students' participation in other classroom mathematical practices (Cobb & Yackel, 1996; Rasmussen et al., 2015). In what follows, we introduce Model Eliciting Activities (MEAs), which informed our design and revisions of the assessment.

Conceptual Framework

Lesh and his colleagues (2000) describe MEAs as tasks designed so that students' work "reveal explicitly the development of constructs (conceptual models) that are significant from a mathematical point of view" (p. 362). These tasks emphasize communication and reflection by asking students to explain their reasoning, thinking, and strategies including revisions and refinements. Teachers and researchers have used these artifacts to assess students' conceptual understanding (e.g., Carlson et al., 2003), problem-solving strategies (e.g., Chamberlin & Moon, 2005), and metacognition (e.g., Kim et al., 2013). In our study, we aim to add to this literature by using MEAs as a tool to assess students' participation in classroom practices, and specifically, we are designing an assessment tool that elicits a model of students' defining activity.

The six principles to guide the creation of productive MEAs with guiding questions (adapted from Carlson et al., 2003, pp. 472-473 to fit our defining context) are: (1) *Model Construction* (Does the task immerse students in a situation in which they are likely to confront the need to develop their defining activity?), (2) *Construct Documentation* (Will the task require students to reveal their defining activity?, (3) *Reality* (Will students make sense of the situation by extending their own knowledge and experiences?), (4) *Model Shareability and Reusability* (Does the model provide a general model for analyzing this type of defining activity?) (5) *Effective Prototype* (Is the situation simple?), (6) *Self-Assessment* (Does the task promote self-evaluation on the part of the students?). This study aims to answer: How can the MEA principles guide the design of an assessment tool for students' participation in defining?

Methods

Data for this study comes from an NSF-funded projected (ASPIRE in Math IUSE #1916490) that aims to create modular inquiry-oriented introduction to proof curricular and instructor support materials. One goal of the curriculum is to support students in participating in defining. A typical defining task sequence first engages students in a situation in which an informal idea of a concept emerges. Then students create and negotiate (non-)examples of the concept and then the teacher and students work together to refine their informal idea into a formal definition (see Vroom, 2020). We expected that our course would support students to strategically use (non-)examples when creating and testing their definitions, and to use those (non-)examples to strengthen their understanding of the concept to support them in formalizing their informal ideas.

Our primary goal with the assessment was to measure how students' participated in the sort of defining that our curriculum materials supported. The assessment had two main parts. First, the students were asked to define two concepts in a similar manner as our curriculum materials. To do so, students were asked to read an informal description of a concept, come up with (non-)examples, and write a formal definition. The two made-up concepts on the assessment were called Mirror-point Functions (described as "functions defined on a domain D that have at least one input-value that is the same as the corresponding output-value") and Repeat Functions (described as "functions where no matter what input you consider, that input maps to the same output value as the input that is one unit to the left of it"). We had no expectations for a "correct" definition as our focus was on the students' defining activity. The second part of the assessment was a "strategy guide" task that asked students to reflect on and articulate how they engaged in the previous defining tasks. We anticipated that the students' responses would provide us insights into how/if the students engaged in defining in such a way that was supported by the curriculum.

Creating and refining the assessment was a cyclic process in which we designed (or redesigned) the assessment, tested it with students, and analyzed the students' assessment responses and conducted follow-up interviews. We administered our assessment in three community college courses over the course of a year. For each course we administered the assessment during the first week of the course and then again during the last week. After each administration, we invited students to a follow-up semi-structured interview to discuss their responses. This resulted in a total of 18 pre-assessments with 6 follow-up interviews and 16 post-assessments with 7 follow-up interviews. Our data analysis focused on how, if at all, the assessment satisfied the six MEA principles based on what the students expressed on the assessment and follow-up interviews. We used this analysis to redesign the assessment before the next administration. In this report, we primarily focus on the second part of the assessment (the strategy guide task) since it provided us the most insight into how students participated in defining activity. The most updated version of the task is given in Figure 1. We will refer to this figure (and it's labeled parts) in the next section.

> **Strategy guide.** Notice that in the previous two sections you were given a name for a concept along with an informal description of the concept. Then in part B you were asked to write a definition. Now your job is to write a strategy guide that would help other students do what you just did on their homework! This strategy guide should explain exactly what the students should do (and how) to create a definition. Imagine they are given some NEW concept and an informal description and then they are asked to write a definition. Fill in the bulleted list below (add more bullets if needed) to write this general strategy guide. b. c.
>
> *After creating examples and non-examples of a given concept, you will be asked to write a formal definition of the new concept on part B. In order to write this definition, you should…* a.

Figure 1. Latest version of "how to write definitions" strategy guide prompt

Results

In this section we discuss how our assessment meets each of the six MEA principles and describe refinements that were made to address issues we found during our implementations.

Model Construction Principle

Given that we wanted to assess students' defining practice, designing a task in which they would create a model of that practice was critical. We addressed that need (and this principle) with the strategy guide task, where students were asked to create a guide for writing a formal definition for any concept. To do so, the students had to generalize their defining activity and thus, construct a model of their practice. In our first version of this task, students were informed that the strategy guide would be used to prepare students to write definitions on their exam and that "*They will not know what they will be asked to define [...] in advance.*" We found that this prompt was confusing for students, especially about what we meant by a "strategy guide". Some students opted to skip the task altogether while one student attempted the task but adding the note "I'm still a little unclear about this portion". To be more explicit about the desired model, we refined the task by adding a sentence starter (see Figure 1a).

Construct Documentation Principle

Presenting students with the opportunity to create a model of their activity (by creating a strategy guide to address the previous principle) was only part of the battle. We also needed to design the strategy guide task in such a way that the students explicitly revealed their defining activity. We wrote the strategy guide prompt in such a way that would encourage students to reflect back on their activity on the previous tasks to not only generalize it (as part of the Model Construction Principle) but also to document that generalization (as part of the Construct Documentation Principle). For instance, the first version of our prompt stated "*This strategy guide should explain what exactly the students should do (and how) to create the definition requested in part B of the example exam items.*" In the previous tasks, "Part B" was where students were asked to create a formal definition of the concept. By directly pointing to that part of their tasks, we hoped students would reflect on and document that aspect of their activity.

As part of the response to the students' initial confusion described in the previous subsection, we edited the prompt to make it clearer that we wanted students to create a guide that reflected their activity in the previous tasks by expliciting stating to reflect on what they did as they defined the two given concepts on the assignment, see Figure 1b. Not only did we see the refined prompt as supporting students in creating a general model of defining activity (addressing the

Model Construction Principle) but it also made it clearer that the model should reflect their activity when completing the previous tasks. This edit appeared successful. For instance, one student wrote for their first two steps in their guide “1) Look at the differences between examples and non-examples, and 2) Look at the similarities between examples.” During the student’s follow-up interview, they confirmed that this was reflected in what they did to create definitions, saying “it did yeah, that's what I use here.”

Reality Principle

In our original design of the assessment, we framed the strategy guide as a guide for a friend who would be taking an exam in which they would define some new concept. Students at this stage of their mathematics career have substantial experience studying for and taking mathematics exams. Thus, we felt that students could “make sense of the situation based on extensions of their own personal knowledge and experience” (Lesh et al., 2000, p. 367). While the revisions of the assessment described above showed promise, we kept running into a problem that could often be tied back to the context of the task: students provided general test taking strategies rather than strategies that reflected their defining activity. For instance, one students’ guide listed “1. remember the difference between for all there exists and there exists for all. 2. practice defining familiar terms such as linear function. 3. become comfortable with making mistakes, you probably won’t come up with your formal definition on your first try.” The fact that this guide was written about preparing for an exam rather than defining was confirmed in a follow-up interview where this student said in reference to his second point “That isn't so much a thing I used here. It's a thing that I’d use if I was getting ready for a test….”

After encountering several responses similar to the one above, we reflect on how the exam context failed to meet the reality principle and how we could refine it. We determined that students might struggle to see the exam context as realistic for their defining activity since they most likely have never been put in a position during an exam where they are asked to engage in such an exploratory task. Instead, we realized that the task would be better suited in a “homework” context as it aligns more closely with the situation in which students would be doing this kind of activity. We reflected this context in Figure 1b as it suggests students will be in a similar situation as them.

Model Shareability and Reusability Principle

In order to meet this principle, the model students are asked to develop needs to be general enough that it can be used by someone other than the writer itself (i.e., shareable) for other scenarios than the one that motivated its creation (i.e., reusable). Our assessment works to meet these requirements by having students write a strategy guide *for another student* and by asking that the strategy guide be written so that it can be used for *any* concept. We found that framing the task as writing a guide for a peer seemed natural for students. This was perhaps unsurprising since other scholars have successfully written Model Elicit Activities with a similar framing (e.g., Carlson et al., 2003; Vroom, 2020). Additionally, we found that our previously discussed edits to the prompt also were effective in supporting students to create a general model that can be reused when defining other concepts (see Figure 1b and 1c).

Effective Prototype Principle

As mentioned, the assessments first engaged students in formally defining two concepts: mirror-point functions and repeat functions. We saw these two tasks as “effective prototypes” of defining activity since students would have knowledge of various function properties at this point and thus should be able to make sense of a new property. Additionally, there is no one "right answer"; we acknowledge students would likely have different interpretations of the given informal descriptions. At the same time, we aimed to write the descriptions so that the different interpretations would be rich enough that students would be able to come up with various (non-)examples to support their defining activity. We encountered a problem when it came to our initial description of mirror-point functions. Initially, we had not clarified the "at least one" property in the description, this led to various students interpreting the statement to mean *all* input-values needed to equal the corresponding output value. With this interpretation only one type of function fits the description (i.e., $f(x) = x$ for all x in the domain) making the task trivial. As such, we added the "at least one" note to enrich the context and encourage a wider variety of (non-)examples.

Self-Assessment Principle

To create a rich model that reflects the descriptiveness and transparency that we are searching for in an assessment, students need to be able to assess themselves whether their model was adequate. One way we addressed this need was by asking students to create a strategy guide for *any* concept. This provided criteria for which students could assess their model: can this guide be used to any concept a student might encounter? More explicitly, we addressed this principle in the follow-up interviews by asking students "How does your strategy guide reflect what you did when you wrote your claims about mirror point functions and repeat functions?" With this question, the students were encouraged to assess how/if their strategy guide reflected their defining activity (another criteria of their guide). We found this question to be particularly revealing as it often identified when there were discrepancies between the students actual defining activity and what they had written in the guide. Given its revealing nature, we hope to incorporate a similar question in future versions of the guide in hopes that it supports students to refine their guide to more accurately reflect their activity.

Discussion

Through working with students we have found that MEAs seem to be a productive first step for developing an assessment tool that documents the extent to which students participate in the sort of mathematical practices that our curriculum materials aim to support. We found that the MEA principles supported some effective refinements, including more explicitly articulating what kind of model we expected the students to create and document in the strategy guide (i.e., with the use of sentence starters, emphasizing the need for generality, and being explicit about reflecting on their activity) and by changing the context of the task from an exam to homework so that the activity was more realistic to the students.

Our experience also generated a number of questions that we will pose to the audience for discussion. Specifically, we encountered that one limitation of using MEAs is that rather than assessing students' activity directly, we are assessing their ability to reflect on and articulate their activity, which is a higher-order task (Wheatley, 1992). With that, we wonder: 1) What kind of insights might we gain from a "high bar" assessment of this type? What are ways we can mitigate this challenging aspect of the assessment?

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Investigating Norms in a Professional Learning Community

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Research has shown that support from collaborative professional development communities benefits faculty introducing student-centered instruction to their classrooms (Henderson, Beach, & Finkelstein, 2011; Speer & Wagner, 2009). While the RUME community has some understanding of how professional development groups can be effectively formed to support teachers (e.g., Kelley & Johnson, 2022)*, further research is needed. In this study, I investigate the norms established within one such professional development community: an online working group designed to support teachers' implementation of an inquiry oriented abstract algebra curriculum. Preliminary results indicate that participants established norms related to working on tasks, contributing ideas and responding to each other's contributions, and sharing authority.*

Keywords: Instructional change, professional learning communities, sociopedagogical norms

In response to calls for reform in undergraduate mathematics education in the United States, various modes of student-centered instruction have been developed. To support faculty in implementing student-centered curricula, professional development resources like curricular support materials (e.g., Larsen et al. 2013) and summer workshops (e.g., Andrews-Larson et al. 2016) have been developed. This study focuses on online working groups (OWG) that supported mathematics faculty learning to teach inquiry-oriented abstract algebra (Larsen et al. 2016). Prior research has suggested the importance of community building in professional development groups (e.g., Clark et al. 2008, van Es, 2009, Kelley & Johnson, 2022), but the mechanisms by which such communities develop need more exploration. Thus, this case study aims to answer the research question: what community norms develop in online professional development groups?

Literature Review and Theoretical Framework

Inquiry-oriented instruction (IOI) is a type of student-centered instruction aimed at supporting students as they work individually and collaboratively on carefully designed mathematical tasks (Rasmussen & Kwon, 2007). As students inquire into new mathematical ideas, instructors inquire into students' mathematical reasoning, seeking to reveal and leverage students' informal conceptions toward more formal mathematics. Because IOI emphasizes classroom social interactions as grounds for the reinvention of mathematical concepts, it is crucial for teachers and students to negotiate social and sociomathematical norms that foster meaningful learning opportunities within the classroom community (Gravemeijer, 2020). While there has been some research regarding norms in inquiry-oriented (IO) classrooms (e.g. Rasmussen et al. 2005, 2009, 2010; Serbin et al. 2020), recent developments in instructional support for IOI necessitate further investigation into normative behaviors in professional development groups. Primarily, I am interested in understanding the social norms established by Teaching Inquiry-Oriented Mathematics: Establishing Supports (TIMES; NSF Awards: #1431595, #1431641, #1431393) online working group participants as they developed a professional learning community in their online meetings.

A professional learning community (PLC), as defined by Kruse, Louis, and Bryk (1995), is focused on cultivating peer interactions between teachers to improve teaching and learning. A

PLC has five central characteristics: reflective dialogue, focus on student learning, interaction among teacher colleagues, collaboration, and shared values and norms. Similar to Clark et al. (2008), the emergence of OWG social norms was investigated, viewing the OWG as a PLC. Social norms are defined by Cobb & Yackel (1996) as a person's "beliefs about [their] own role, others' roles, and the general nature" of classroom activity (Cobb & Yackel, 1996, p. 178). These beliefs are developed collaboratively between the instructor and students as they interact in the classroom and "characterize regularities in communal or collective classroom activity" (Cobb & Yackel, 1996, p.178). Serbin et al. (2020) documented norms related to working on mathematical tasks, students sharing contributions, and students responding to others' contributions in inquiry-oriented abstract algebra (IOAA). Norms surrounding students contributing to group discussions included sharing contributions, explaining their reasoning, and explaining difficulties they experienced. Finally, norms about responding to others' contributions in group discussions involved asking clarifying questions, being nonjudgmental, and providing productive feedback.

Because professional development literature suggests that modeling the type of instruction expected from participants is an effective strategy (Elmore, 2002), the focus here was on whether norms documented in IO classrooms were present in the OWG. OWG participants often worked through mathematical tasks, discussed their pedagogical reasoning, and provided feedback on each other's instructional practices, a clear parallel to the IO classroom activities documented by Serbin et al. (2020). According to Clark et al. (2008), just as student behavioral norms can be negotiated in a classroom, participant behavioral norms can be negotiated in a PLC. Dick et al. (2018) termed established expectations for participation when teachers talk with their colleagues about instruction *sociopedagogical* norms, taking direction from Cobb and Yackel's (1996) conceptualization of behavioral norms which are specific to the community's content area. Hence, such a conceptualization can be broadened to describe the rules and expectations for the facilitator and participants' behaviors within the OWG as they develop a community of practice, drawing parallels between the social norms previously documented in IO classrooms and those observed in this study.

Sharing authority has also become a central theme of discussion in research on student-centered instruction (e.g. Ball, 1993; Cobb et al. 1992). Given that some classroom activities parallel OWG activities, it is reasonable to suspect that norms surrounding sharing authority may also arise. Because one of the primary goals of IO approaches to teaching and learning is creating a community of mathematical discourse, instructors often avoid resting the validity of claims on "teacherly authority", rather aiming to use their authority in the classroom community to encourage norms surrounding discussion, reasoning, and mathematical argument (Ball, 1993). That is, teachers may take advantage of their ability to direct the future actions of the class with the goal of encouraging students' epistemic authority, which concerns how participants treat themselves and others as more or less knowledgeable rather than concerning the actual depth of individuals' knowledge (Byun et al. 2020). Because the OWG was composed of teachers with varying levels of expertise with implementation and had clear parallels to IO classroom activities, the notion of sharing epistemic authority as a normative social behavior was adopted.

Context and Methods

This investigation takes place within the context of a broader project, Teaching Inquiry-Oriented Mathematics: Establishing Supports (TIMES; NSF Awards: #1431595, #1431641, #1431393), a research and development project focused on supporting large-scale implementation of several inquiry-oriented mathematics curricula at the undergraduate level. To investigate best practices in bringing IOI to scale, TIMES utilized three forms of instructional supports: curricular support materials, summer workshops, and online working groups. The curricular support materials provided additional online resources to teachers. The summer workshops provided a three-day intensive session where participants became acquainted with the instructional tasks and curricular resources. Finally, the online working groups met for one hour each week during the semester following the summer workshop to work on instructional practices and discuss implementation issues with the goal of providing ongoing, content-specific support.

My primary focus is an online working group (OWG) that met for 14 weeks in the fall semester of 2016. The members of the OWG were engaged in implementing the IOAA curriculum designed by Larsen et al. (2016), a research-based, student-centered curriculum designed to engage undergraduate students in reinventing fundamental concepts of group theory. The group consisted of two instructors currently implementing the curriculum, one instructor preparing to teach in the following academic quarter, and one facilitator with previous experience teaching IOAA. The teachers were introduced to the curriculum materials in the summer workshop and met synchronously online via Google Hangouts once per week to reflect on the curriculum and implementation with a facilitator. Members of the OWG were regularly asked to share reflections on their instruction and investigate the curriculum tasks from a mathematical perspective. These online meetings were recorded and sections were transcribed for retrospective analysis. Social norms were analyzed as they developed throughout these meetings, using a priori codes developed by Serbin et al (2020) and emergent codes from the analysis related to sharing authority. Taking direction from Clark et al.'s (2008) conditions for documenting the development of norms, it was inferred that a behavior was normative to the OWG if it was commonly enacted without prompting, participants did not challenge someone enacting that behavior, and/or participants challenged someone when they did not comply with that behavior.

The OWG consisted of three instructors, Elena, Laura, and Roger, and one facilitator, Mickey. Mickey facilitated the OWG as a former TIMES fellow, having participated in the previous year as a new IOAA instructor. Elena, Laura, and Roger had experience teaching abstract algebra, but had not used IOAA. Because Roger's institution operated on a quarterly schedule, he was unable to teach the course synchronously with Elena and Laura. It is worth noting, however, that Roger was an active contributor in most discussions, and that the OWG continued to meet with Roger after their fall semester classes concluded to support his instruction during the winter quarter.

Results

Throughout the semester, Mickey indicates implicitly and explicitly that he is trying to model IOI within the online working group, making comments like "I hope I'm modeling what I would like to do in the classroom," and even describing the function of the OWG using language commonly applied in IO curricula design: "We're generating teacher thinking, building on teacher thinking, and developing a shared understanding, and then bringing it into the formal world." This supports my investigation of documented classroom social norms developing within the OWG. I found that participants enacted social norms related to working on tasks, sharing and responding to others' contributions, and sharing authority. Illustrative examples are presented below.

Working individually first.

Mickey explicitly stated the expectation for participants to work on tasks individually first with phrases like, “Everybody, take a minute or two to think about it,” and “I want you to take a few minutes before you answer this [question].” After this, participants thought to themselves and worked quietly until prompted to share their thoughts. In one instance, after Mickey had posed a question to the group, Elena began to respond immediately and Roger interjected, “I thought we were thinking to ourselves.” Roger challenging Elena when she did not give others time to think independently serves as evidence that this was an established social norm in the OWG.

Explaining reasoning.

Mickey also encouraged the group to share their ideas, beginning most meetings by inquiring into the teachers’ perspectives and continually asking for their contributions. He also positively reinforced this expectation by emphasizing key ideas, repeating them to the group or saying phrases like, “Can you say that again? We’re going to write it up here on the board.” Encouraging each other’s contributions was commonly enacted by the rest of the group throughout the semester. For instance, after Laura shared some thoughts on a clip of her teaching, Elena responded, “Tell me a little bit more about your thought process.” This is also evidence for the development of norms around participants explaining their thinking. Mickey explicitly stated the expectation for participants to explain their reasoning, saying “I’m going to press my teachers to give reasons for their teaching actions.” This behavior seemed normative, since it was encouraged by the facilitator and enacted without prompting by the rest of the group.

Sharing difficulties.

A similar type of contribution involved participants sharing the difficulties they experienced with curriculum tasks. Mickey would often open a meeting by saying, “So, tell us about your week – what went well, what didn’t?” to encourage participants to share their struggles. For the last 11 meetings, teachers regularly shared about their difficulties without being prompted. In many instances, this contributed to rich conversations surrounding pedagogical moves and inquiring into students’ thinking. After working on a curriculum task about group isomorphism concepts, Roger shared that he “struggled with the additive notation,” explaining that he switched to multiplicative notation with the goal of preventing students from being confused. Elena and Laura then both shared their experiences with developing shared notation in the classroom, which suggests that the norm of sharing difficulties often supported group discussion.

Showing support.

The fact that Elena and Laura responded to Roger by explaining their experiences is also evidence of norms about providing productive feedback. In particular, the norm that participants should be supportive and share their strategies was often enacted throughout the semester. Responding to others’ struggles by sharing strategies was first encouraged by Mickey after Laura described her difficulty with graded work, asking Elena and Roger to discuss their approach to assessment in IOI. Later, it was evident that providing productive feedback was a norm when Elena explained some difficulty she was having with a student during discussion, prompting the group to respond by asking explicitly for their suggestions. It also became normative for participants to encourage each other – when Roger explained that he was overwhelmed by the pacing of the course, Laura responded, “You can do it, you can, you’ll get there!” and proceeded to suggest different strategies to help Roger stay on track.

Sharing epistemic authority.

Mickey also promoted participation by using his deontic authority to encourage participants' epistemic authority. Often, when a participant asked Mickey a direct question, he would redirect it to the entire group, saying phrases like, "So that [question] is for everybody, I can answer later," and "I think this [topic] merits at least a three-to-five-minute discussion. I'll add a disclaimer that I have no idea how to handle that situation." Sharing epistemic authority became a social norm throughout group discussions. On occasion, Mickey would get excited about a topic and monologue for a few minutes. The group nearly always became quiet and tended to not respond to Mickey's monologues, which I view as evidence of participants challenging Mickey for breaking the norm of sharing authority. Because Mickey was challenged for not enacting this behavior, it can be thought of as a social norm within the OWG.

Table 1. A description of online working group norms.

Category	Norm Name	Description	Prototypical Quote
working on tasks	working individually first	participants engage with content independently before discussing with the group	"Take a few minutes before you answer this [question]."
	explaining reasoning	participants explain their thinking	"Tell me a little bit more about your thought process."
sharing and responding to contributions	sharing difficulties	participants discuss their teaching challenges	"I struggled with the additive notation."
	showing support	participants provide encouraging feedback	"You can do it; you'll get there!"
sharing authority	sharing epistemic authority	facilitator redirects discussions away from his expertise toward group discussion	"So that [question] is for everybody, I can answer later."

Discussion and Questions

The fact that the OWG possesses the characteristics of a professional learning community occurred mostly by design; teacher colleagues were expected to engage in reflective dialogue focused on pedagogical strategies and student learning. The shared norms they established around working on tasks, sharing and responding to each other's ideas, and sharing authority are relatively unsurprising in terms of professional development. However, the fact that these norms parallel the norms documented in IO classroom communities is notable. One norm not previously established in the literature that could be explored further was that of sharing conversational space, since participants regularly noted that they wanted to avoid dominating a discussion and prompting others to respond. In addition to further exploration of norms, this research could provide a basis for designing facilitator training in how to support and intentionally develop productive norms in PLCs.

Questions for the audience:

1. Are there other norms from professional development literature relevant to the OWG?
2. Other aspects of this community were not captured in these norms (e.g., comradery) – how can these aspects be documented and analyzed?

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The Influence of an Experiential Learning Social Justice Class on Undergraduate Students' Beliefs About Mathematics and Mathematics Teaching

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In this study, we explore the influence of an experiential learning social justice math class on undergraduate students' beliefs about mathematics and mathematics teaching. Twenty-three undergraduate students participated in this ten-week long class. Students reflected on their learning by responding to weekly journal prompts and designed a learning product for the high schools. Students' journal entries and project design will be analyzed to answer three research questions. How does the class influence students' beliefs about mathematics? How does the class influence students' perceptions about who does mathematics? How does the class influence students' understandings of how to teach math equitably? Preliminary results suggest the following: Students' views of mathematics expanded from a focus on application and problem solving to include intrinsic values of mathematics such as beauty and exploration. Students' perception of mathematicians became more inclusive. Students' understanding of equitable teaching varied.

Keywords: Experiential Learning, Social Justice, Reflection, Beliefs

Introduction

For over two decades, there has been a call for undergraduate mathematics to become a "gateway not a gatekeeper" (Douglas, 2017; Bryk & Treisman, 2010; Cooper, 1996). Mathematics educators have been identifying equitable teaching methods (Ching & Roberts, 2022; Martin, 2019; Hand, 2012) and focusing on the teaching of mathematics for social justice (Branson & George, 2022; Su, 2020; Gutiérrez, R. 2018; Leonard et al., 2010). As equitable teaching practices are gradually being established, promoted, and practiced in undergraduate mathematics (Leyva et al., 2022), there is an opportunity for mathematics departments to go deeper. In particular, the authors designed an experiential learning class to delve directly into issues of mathematics education and social justice.

In this study, we consider three research questions about the impact of this class on the undergraduate students who participated.

1. How does the experiential learning social justice class influence undergraduate students' beliefs about mathematics?
2. How does the experiential learning social justice class influence undergraduate students' perceptions about who does mathematics?
3. How does the experiential learning social justice class influence undergraduate students' understandings of how to teach math equitably?

Literature Review

In this review, we first present literature related to beliefs about mathematics, beliefs about who does mathematics, and equitable math teaching. These correspond to three themes common to research on equitable mathematics teaching: positioning of mathematical power, psychological approaches to teaching, as well as specific pedagogical approaches (Yolcu, 2019).

We then highlight literature related to the role of experiential learning courses in deepening undergraduates' understanding of social justice issues.

Beliefs about mathematics

Ernest (1989) and Muhtarom (2019) highlight three philosophies of mathematics which are prevalent in the teaching of mathematics: beliefs: instrumentalist, Platonist, and problem solving or constructivist. At the lowest level, instrumentalist belief is that mathematics is a "useful, but unrelated collection of facts, rules, and skills" (Ernest, 1989, p.8). The Platonist view is that mathematics is an unchanging product of interconnected structures and truth which is invented, but not created. The problem solving or constructivist view is that mathematics is a dynamic field open to revision and connected to other disciplines.

The shift of mathematics away from being a "gatekeeper" to a "gateway" into STEM fields represents a shift from a deficit orientation to asset orientation (DiGregorio & Hagman, 2021; McGee et al., 2021). This language shift also represents a change in thinking about mathematical power. Rather than demonstrating power in achievement to reduce the education gap, there is a movement to "rehumanize" mathematics in ways that empower all students to benefit from and contribute to mathematics (Gutiérrez, 2018; Ching & Roberts, 2022). A narrow view of mathematics limits the diversity of people who do mathematics. In *Mathematics for Human Flourishing*, Francis Su (2020) argues that mathematics meets the intrinsic human desire to build virtues: virtues such as exploration, play, truth, beauty, justice, struggle, and power.

Beliefs about who can do mathematics

Representation is important for students to identify as someone who can do mathematics (Oyserman & Lewis, 2017; Boaler & Greeno, 2000). Productive learning mindsets are also key to helping students identify as someone who can do math (Casad et al., 2018). Having a mindset of relevance is related to the idea that mathematics is relevant to the individual learning it (Romero, 2019). A mindset of belonging is about learning mathematics within a welcoming community (Romero, 2015). Growth mindset is the idea that intelligence can be developed through effort rather than being an innate ability; in a classroom this means mistakes are opportunities to grow and asking questions deepens understanding (Latterell & Wilson, 2022; Sun, 2018; Dweck, 2014; Boaler, 2013).

Equitable Math Teaching

Equitable math teaching is student centered. Equitable teaching methods includes active learning (Burke et al., 2020), flexible structures (Levya et al., 2022) and grading systems (Fernandez, 2021), and a focus on high expectations (Jamar & Pitts, 2005) cultural relevance (Bartell et al., 2017), and student engagement in which problem solving is key (Samson, 2015; Hake et al, 2003; Suh & Fulginiti, 2011). Along with teaching in ways that promote social justice, equitable practice also involves math that teaches social justice (Gutstein & Peterson, 2005). As we search for ways to teach math in ways that promote social justice, we consider five basic questions: *Who*, *What*, *Where*, *Why* and *How* we teach (Branson & George, 2022).

Experiential Learning and Social Justice

Experiential learning and reflection are critical components of deepening understanding of social justice. Experiential learning provides a tangible way for students to grapple with the abstract concepts of social justice (Pugh, 2014). Online service experiences have also been shown to contribute to students' understanding of social justice (Ahmad & Gul, 2021). Group discussions and individual reflections help students synthesize and analyze the issues within the context of their teaching experience (Jacob, 2006). It is also thought that reflection is a key piece of developing effective tutoring strategies (Johns & Burks, 2022).

Context

The authors designed an upper division undergraduate math class, *Mathematics Education and Social Justice* which examines issues of social justice surrounding mathematics education in U.S. public schools. Topics included problem solving, pedagogy, power, meaning, struggle, and freedom. Class time was split between student-led discussions of the readings and small group problem solving tasks. The immersive experience involved tutoring precalculus onsite at local high schools whose student body was an underrepresented population; due to the pandemic, seven of the twenty-three undergraduates tutored online through Zoom. The experiential learning social justice component cultivated social justice, civic life, civic perspective, and civic engagement. The tutoring component helped the undergraduate students grow in their ability to understand student thinking and developed ways to strengthen high school students' understanding of precalculus content. Three overarching themes were presented.

- What is math?
- Who does math?
- How can we teach math more equitably?

Mathematics Education and Social Justice was first taught in the ten-week long spring quarter of 2022 at a mid-sized private university on the west coast. The twenty-three students enrolled in the class included math, computer science, engineering, finance and economics majors. Weekly readings included chapters from Mathematics for Human Flourishing (Su, 2020) supplemented by readings about pedagogy and equity in mathematics. Students responded to weekly journal prompts to reflect on their tutoring experiences within the context of the readings and discussion. For their final project, the undergraduate students designed a learning product which could be used by teachers in the local high schools where they tutored.

Methods

We will use the constant comparative process to analyze the journal entries (Yin, 2009). First, the authors will identify incidents, or idea threads, which demonstrate any of the three main ideas: "What is math?", "Who does math?" and "How can we teach math more equitably?" To reach an agreement on the incident, we will identify incidents together for two randomly selected transcripts and then do so independently for the remaining entries. We will then meet to assess agreement on the incidents. Note that a single incident may contain two main ideas. For example, the idea thread from Student 4, Journal 1 in Table 1 addresses both the ideas: "What is math" and "Who does math." This thread illustrates the belief that math is useful for the student's major; it also indicates the belief that some people have a natural talent for math and that effort can help a student succeed in school mathematics.

Once the idea threads for the week one journal entries are identified, we will develop a code book using inductive and deductive processes. In the idea thread from Student 4, Journal 1, the a priori code applied to "What is math" will be instrumentalist, because this student sees math as being useful to their major. We will also develop emergent codes if the "What is math" idea threads when one of the three a priori codes do not apply. For example, the idea thread from Student 3, Journal 1 in Table 1 indicates that the student thought of math as memorization.

Table 1. Sample codes

Parent Code	Sub Code	Example
What is math	Instrumental	Once I got to high school, I started putting a little more effort into math, mostly because I knew my grades were going to matter for applying to colleges. The real switch, however, came about halfway through high school when I started looking at what major I wanted to pursue. From that point on, I realized the importance of math for my life and aspirations. (Student 4, Journal 1)
Who does math	Innate gift Effort for grades	
What is math	Memorization	I felt like multiplication tables were too hard. I had a difficult time focusing in order to memorize them and it took a lot of effort from my parents and myself in order to work through that difficulty. (Student 3, Journal 1)

We will follow a similar process to address the second research question. A priori codes will be based on the beliefs that mathematics intelligence is an innate gift or achieved through effort. In the incident above from Student 4, Journal 1, the a priori code applied will be an innate gift because they are "naturally good" at math. Emergent codes will be developed as different views of mathematicians arise in the idea threads. That same incident will have an emergent code describing "effort for grades" as that is different from effort for growing math intelligence.

We will follow a similar protocol for the third research question. A priori codes for teaching math will be developed around Su's (2020) ideas to teach math for human flourishing through exploration, problem solving, and play. After coding journal entries, we will develop a rubric to analyze the final project in terms of those same equitable math teaching practices.

Upon completion of the codebook, we will divide remaining journal entries between the authors to identify and re-code the idea threads. We will then engage in a two-step process to assess reliability. First 100% of each author's journals will be given to the other author to ensure all idea threads in the journals were captured and identified accurately. We will then calculate the percent agreement between the authors. The second process of reliability assessment will be seeking agreement on codes that were applied to each idea thread. Again, 100% of each coder's journals will be exchanged and the percent agreement calculated.

To address the first research question, we will examine general trends across the beliefs about mathematics. To address the second research question, we will examine patterns across beliefs about who does math. To address the third research question, we will connect student's understanding of equitable teaching with different student characteristics, such as college major.

Preliminary Results

While the full data analysis has yet to be completed, these are some preliminary findings related to each of the three research questions.

How does the experiential learning social justice (ELSJ) class influence undergraduate students' beliefs about mathematics? Rather than change students' beliefs about mathematics, the ELSJ learning experiences expanded students' beliefs about mathematics. At the beginning of the quarter, most of the undergraduate students described mathematics as memorized facts, procedures, or a tool; in contrast, the four mathematics majors tended to see math as problem solving and logical thinking. By the end of the quarter, most students moved from seeing mathematics as a set of rules to a set of ideas and a way of thinking. In addition to seeing math as a toolkit, students became aware of the intrinsic value of mathematics and see mathematics as a way of "understanding the world." Most students moved forward along the continuum from an instrumentalist to a problem-solving view of mathematics.

How does the experiential learning social justice class influence undergraduate students' perceptions about who does mathematics? Students' experiences throughout the quarter also expanded their perception about who does mathematics. Undergraduate students, who saw themselves as innately good at math, grew to recognize that through persistence and effort, anyone can develop math skills. Students saw that invisible labels through 'grouping' could encourage or discourage that persistence and effort. Students began to see that all students have the potential to be mathematicians. In their last journal entry, one business major enthusiastically noted that anyone could be a mathematician, "even me!"

How does the experiential learning social justice class influence undergraduate students' understandings of how to teach math equitably? Students' journal entries gave evidence to their thoughts about teaching math equitably, such as high expectation, relationships, motivation and engagement, and such; the final projects revealed the depth of their understanding. Some groups were able to identify ways to provide higher order thinking and problem solving; this aligned with equitable teaching. Some other groups focused more on basic concepts and rote learning, which tend to perpetuate inequities.

Implications and Future Research

The reflections in this course influenced students' thinking about what is math, who does math, and how to teach math equitably. In person tutoring gave undergraduate students a chance to get to know underrepresented high school students, to build a joint learning community with them, and to experiment with what types of teaching strategies helped the students learn.

These undergraduate students scratched the surface of learning what equitable math education looks like. We need to continue to study equitable teaching practices at all levels: elementary through graduate school. At the undergraduate level, we can cultivate an open mindset about mathematics, an awareness of the disparities that exist, and highlight equitable practices such as building learning communities among students from different backgrounds.

The mathematics learning center is a natural place to cultivate that awareness (Reinholz et al., 2020). Tutors have a unique opportunity to develop a relationship with their students, to understand their students' thinking, to encourage a growth mindset, to engage students and build on their individual mathematical strengths. Raising tutor awareness of social justice issues and having tutors try out different teaching methods is one way to build a diverse learning community. Creating a sense of belonging in the math learning center is a critical first step. From there we can explore other teaching methods to promote equitable math teaching and make a difference in the mathematical future of the students who come.

Questions for Audience

1. Two key components of the class are the immersion experience and the reflection piece. What questions do you have about either of these?
2. What questions do you have about the specific reflection prompts?
3. How do these preliminary results impact what you will do in your own classes?

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A Literature Review Across Discipline-Based Education Research: What is a Theoretical Framework and How is it Used in Education Research?

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Extant literature has emphasized the importance of education research being theory-based. To this end, many studies have an explicit "theoretical framework" section that describes the theoretical assumptions that inform the research. Nevertheless, there is large variation in the extent to which studies incorporate theory in the research process. This work describes a literature review conducted across discipline-based education research, focusing on the role of theory in research. We focus on a snapshot of theory use by analyzing studies published in 2021 (n=589) across biology, chemistry, engineering, mathematics, and physics education research journals. Preliminary analysis focuses on trends related to the presence of a theoretical framework and reflecting on how the theory is used in the research process. Our goal is not to evaluate the quality of research, but rather describe how theory is used to inform future work and open a cross-disciplinary dialogue about theory.

Keywords: theory; cross-disciplinary; research questions; methodologies

Research on teaching and learning within science, technology, engineering, and mathematics (STEM) has increased considerably as a collection of fields called discipline-based education research (Henderson et al., 2017; Singer et al., 2012; Talanquer, 2014; Trujillo & Long, 2018). Discipline-based education research (DBER) fields each reflect an interdisciplinary agenda, drawing from educational psychology, learning sciences, and STEM disciplinary fields to seek evidence-based knowledge and practices that improve our understanding of teaching and learning (Henderson et al., 2017; Slater, 2011). To this end, DBER consistently draws research methodologies from educational psychology and the learning sciences, such as case study, phenomenology, grounded approach (Charmaz, 2006; Denzin & Lincoln, 2008), and quantitative analyses (Petscher et al., 2013). Nevertheless, each DBER field has their own discipline-based perspectives with shared goals, norms, behaviors, and identity. The shared goals, norms, behaviors, and identity of the DBER fields are what make them individual communities of practice (Lave & Wenger, 2004; Rynearson, 2015; Wenger, 1998).

CoPs have boundary objects, understood as artifacts, documents, terms, and concepts that serve as an interface between boundaries of domain knowledge (Lave & Wenger, 2004; Pawlowski & Raven, 2000; Star & Griesemer, 1989). CoPs can communicate with one another through brokering, which includes activities by individuals that facilitate transactions and flow of domain knowledge (Lave & Wenger, 2004; Pawlowski & Raven, 2000; Star & Griesemer, 1989). When brokers use boundary objects (i.e., boundary object brokering), different CoPs can achieve convergence by creating shared information systems among different CoPs (Pawlowski & Raven, 2000). As a CoP, the field of mathematics education has a shared mission of encouraging quality research in undergraduate mathematics and its application to teaching practice (*RUME Charter*, 2010). Practices that bound the community together include contributing and attending conferences, publications in journals (e.g., *Journal for Research in Mathematics Education* and *Educational Studies in Mathematics*), and creating evidence-based practices used for instruction in K-12, undergraduate, and graduate spaces (Yığ, 2022).

Researchers interested in finding unifying concepts across mathematics education have explored how different epistemological perspectives have shaped emergent research (Hannula et al., 2004; Lester, 2005). Here, epistemological perspectives are summarized as theoretical frameworks, which use models for learning to define the constructs under investigation (Hannula et al., 2004). Theoretical frameworks in practice shape how research is designed, collected, analyzed, and shared (Lester, 2005). For example, recent literature reviews in mathematics education reveal that theoretical frameworks that underpin emotion- and motivation-based constructs are foundational for learning, but are currently undertheorized (Schukajlow et al., 2017) and theories in general are heavily fragmented across mathematics education research (Bikner-Ahsbahs & Prediger, 2014). Future themes for mathematics education research call for an emphasis on understanding the role of theoretical frameworks that maintain the validity and replicability of the field (Bakker et al., 2021).

In other DBER fields, theoretical frameworks are also inconsistently used and are often undertheorized within studies (Bussey et al., 2020; Lohmann & Froyd, 2010; Reinholz & Andrews, 2019). Thinking about how theoretical frameworks are organized across DBER fields can act as a transformative force in STEM higher education (Reinholz & Andrews, 2019). Importantly, theoretical frameworks can be used to connect multiple domains of knowledge that enable needed connection amongst the currently siloed STEM higher education landscape (Reinholz & Andrews, 2019). As DBER researchers, our role is to connect CoPs by using theoretical frameworks to facilitate the flow of knowledge across STEM disciplinary domains. Thus, we endeavor to understand how theoretical frameworks are used across DBER fields to support researchers framing mathematics education research and STEM education research broadly. The following project is a literature review study that draws on n=589 research articles published in 2021 across DBER journals with the highest impact factors in math, physics, chemistry, engineering, and biology. The project seeks to open a dialogue across DBER fields, focusing on the general question: *How do DBER research articles use theoretical frameworks in their work?*

Methods

Data Collection and Analysis

For the purposes of our literature review, we defined theoretical frameworks epistemically as, "a system of ideas, aims, goals, theories and assumptions about knowledge; about how research should be carried out; and about how research should be reported that influences what kind of experiments can be varied out and the type of data that result from these experiments" (Bodner & Orgill, 2007). We strategically chose this definition because (1) it was meant to cover the breadth of DBER fields; (2) it allows for flexibility while maintaining shared understanding; and (3) it emphasizes the role in overall research design.

The initial challenge with this literature review was to narrow the scope sufficiently to make this project feasible. This work is part of a larger project intended to contextualize the use of theory within the broader DBER landscape. For the purposes of this conference report, we focus on a narrow timeframe to draw connections across discipline-based education research fields. In particular, we report on education research studies published in 2021, with an emphasis on discipline-specific research journals (biology, chemistry, engineering, mathematics, physics), as opposed to journals that focus broadly on STEM education research. In the case where there were multiple discipline-specific education research journals, the journals for our sample were selected based on journal impact factors, as well as input from the relevant community regarding journal quality (Williams & Leatham, 2017). It is important to note there are limitations in using journal impact factors to measure research quality (Cameron, 2005), but their prevalence in academia provides a useful metric for gauging the likelihood that a community would be familiar with the journal. Our sample focused on research studies, that is, papers that describe the collection and analysis of data, contextualized within prior research; this excluded non-research papers such as editorials and commentaries. The final sample was n=589 research studies, collected from two journals for each community (except for physics). The journals sampled are in summarized in Table 1.

Preliminary analysis involved characterizing whether each study in the final sample had an explicit framework section (*Framework* vs. *No Framework)*. Here, the unit of analysis was the publication itself. In most cases, this was located toward the beginning of the manuscript with a title heading involving some variation of phrases such as "theoretical framework", "theoretical perspectives", "theoretical framing", etc. Characterization first involved the two authors coding all the chemistry papers in-tandem, requiring 100% agreement (Campbell et al., 2013). An independent coding team of two researchers applied this analysis to the chemistry sample, resulting in 93% agreement and a Cohen's Kappa of 0.85. Following this, one researcher coded the biology and engineering papers, and the other researcher coded the mathematics and physics papers. Further analysis focused on analyzing the studies within the *Framework* category to note how the theoretical framework was used throughout the different sections of the paper (i.e., research question; methods; findings; conclusions and implications); at the time of submission, this process has been applied to the chemistry studies only, with the intention of scaling to the remaining sample, with particular attention given to chemistry, physics, and mathematics trends for the RUME community.

Preliminary Results

For this report, we focus on providing a general overview of how many studies across the sample had an explicit theoretical framework, followed by providing an example of how a framework was used across a single study. Summarized in Table 1, there was large variation in whether studies had an explicit theoretical framework section, even within a specific discipline (e.g., engineering – 57% of studies had a framework in *Journal of Engineering Education*, but only 12% of studies had a framework in *International Journal of Engineering Education*). Claims regarding the generalizability of these values are limited and it remains to be determined how these values relate to the consensus of specific communities and the journal context. Notably, the lack of journal guidelines related to the inclusion of a theoretical framework help explain the low values for *Biochemistry and Molecular Biology Education* (3%) and *International Journal of Engineering Education* (12%); however, *Physical Review Physics Education Research* (37%) and *Educational Studies in Mathematics* (63%) have higher values despite also not having journal guidelines related to the inclusion of a theoretical framework.

Table 1. How many discipline-based education research studies published in 2021 had an explicit theoretical framework section?

DBER Field	Journal	Impact Factor	Framework
Biology	*CBE-Life Sciences Education*	3.365	25%

N=128	**N=68**		
	*Biochemistry and Molecular Biology Education** **N=60**	1.160	3%
Chemistry **N=110**	*Journal of Chemical Education* **N=45**	1.385	62%
	Chemistry Education Research and Practice **N=65**	1.902	55%
Engineering **N=173**	*Journal of Engineering Education* **N=42**	1.569	57%
	*International Journal of Engineering Education** **N=131**	1.280	12%
Mathematics **N=76**	*Journal for Research in Mathematics Education* **N=17**	3.676	53%
	*Educational Studies in Mathematics** **N=59**	2.402	63%
Physics **N=102**	*Physical Review Physics Education Research** **N=102**	2.412	37%
*Indicates journal submission guidelines do not mention theoretical frameworks/theory.			

In addition to characterizing whether or not a study had a framework, we are interested in how studies use frameworks as part of the research process. Focusing on the chemistry papers, we noted that 83% of the studies integrated the theoretical constructs throughout the different sections of the manuscript, which is consistent with our framing of how theory should influence study design (Bodner & Orgill, 2007). To illustrate, we provide an example research study published in *Chemistry Education Research and Practice* (Parobek et al., 2021). The study focused on students' reasoning related to a representation used in chemistry, a reaction coordinate diagram. Part of the challenge with this representation is that it looks like a *graph* (y-axis is *potential energy* and x-axis is *reaction coordinate*), but it is only used as a quantitative graph (i.e., a function) in upper-level physical chemistry courses. In most cases a reaction coordinate diagram is used as a qualitative tool where the x-axis does not have a physical meaning; see Parobek et al. (2021) for an in-depth discussion of reaction coordinate diagrams.

Relevant to our literature review, Parobek et al. (2021) integrated multiple frameworks, discussed in the "theoretical considerations" section of the manuscript: resources framework (Hammer et al., 2005); actor-oriented model of transfer (Lobato, 2003); graphical forms (Rodriguez et al., 2019); value-thinking and location-thinking (David et al., 2019); these frameworks draw across learning sciences, chemical education, and math education. Moreover, the theoretical constructs were well-integrated throughout each section of the paper, illustrating the guiding nature of the frameworks. For example, the research question adopts the language of the frameworks: "What mathematical *resources* do students *transfer* to make inferences about the points and trends on a reaction coordinate diagram?" (p.699, emphasis added). As part of this, the authors make it clear that the specific mathematical resources of interest are *graphical forms* and *location/value-thinking,* which influenced prompt development and data analysis:

> "These diagrams [representations used in interview prompts] were constructed with an awareness of existing graphical reasoning frameworks that highlight the various ways that students may interpret points and regions along a graphical trend … Iterative cycles of inductive and deductive coding … Each graphical reasoning code … is first designated according to the graphical reasoning frameworks that served as a lens for crafting these categories in the unique context of RCDs." p.701-702

Lastly, in the Findings and the Conclusions/Implications sections the authors contextualized the data within the previously discussed theoretical frameworks and reflected on what the theory suggests regarding how instructors can support students. In addition to exemplifying theory use, we assert this paper is important because of how it illustrates the ways in which communities of practice are connected through the theoretical framework, not only because of cross-cutting concepts and skills (e.g., energy and graphical reasoning), but through the epistemological assumptions reflected in theoretical frameworks used by different fields.

Conclusion

We assert theoretical frameworks currently act as boundary objects within mathematics education research field as well as DBER fields broadly, which contribute to the existing fragmentation and under-theorization characterized by scholars (Bikner-Ahsbahs & Prediger, 2014; Schukajlow et al., 2017). DBER researchers can participate in boundary object brokering by acting as brokers to support the flow of interdisciplinary knowledge through their valid and reliable use of frameworks. We are currently analyzing the data to elicit themes related to the larger dataset, with the intention situating chemistry, physics, and mathematics within the broader DBER context because of their relevance to the RUME community. Our goal is to use these data and findings to initiate a broader conversation about theory and its use in education research. For example, it is worth considering whether *every* study needs a framework and what a framework might (or might not) contribute to a specific study. Moreover, we need to reflect on the nature of different theoretical frameworks; in particular, the analytic power of frameworks – it may be difficult to design a study and operationalize some frameworks. Lastly, it would be helpful to discuss the process of developing a theoretical framework, especially in relation to established approaches such as grounded theory (Charmaz, 2006).

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A Precalculus Student's Use of Slope in Finding Linear Equations and Linear Approximations

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This article reports mathematical understanding of one of the precalculus students who participated in a teaching experiment. The experiment was designed to help students strengthen the understanding of the concept of rate of change and linear equations to prepare them for the calculus concepts: tangent line equation and linear approximation. Before the experiment, the student understood slope mostly as an algebraic ratio and could not use the slope to determine the y-intercept of the line. With the intervention, she strengthened her knowledge of slope as both algebraic and geometric ratios and used her knowledge to construct linear equations. In addition, she was able to use slope to determine linear approximations of function values in various functions. The results suggest that when provided with appropriate tasks and prompts, precalculus students can be better prepared for their upcoming calculus concepts.

Keywords: Slope, Linear equation, Tangent line, Linear approximation

Do precalculus students adequately understand linear equations for their upcoming calculus courses? In my teaching experience, I found that most calculus students struggled to find the equations of tangent lines or linear approximations. I initially thought that their struggles stemmed from their insufficient skills or understandings related to derivatives. However, after realizing that many students with correct derivatives could not construct tangent line equations, I speculated that their struggles were also due to their lack of understanding of linear equations. So I conducted a survey with precalculus students on two conversion problems from graph to equation and confirmed that students indeed had difficulty constructing a line equation unless the *y*-intercept could be easily observed from graph. Thus, in order to help them develop understanding of slope and linear equations, I designed and researched a teaching experiment (Steffe & Thompson, 2000) with a research question: How do precalculus students understand linear equations and in what ways understanding linear equations helps them construct equations of tangent lines or approximate function values? In this report, I share my preliminary findings and seek for suggestions and inputs for the future direction of research.

Theoretical and Concept Background

Slope as Algebraic and Geometric Representations

Representations, especially algebraic and geometric, play a central role in mathematics curriculum and instruction. Yet many students do not understand mathematical concepts by connecting representations or have difficulty connecting representations in mathematical problem solving (Dufour-Janvier, 1987; Even, 1998). Among the concepts requiring the connections of representations, rate of change has attracted a lot of research (such as Carlson et al., 2002; Nagel et al., 2019; and Thompson & Carlson, 2017). I here describe the rate of change in linear equations—slope—, using the framework by Nagle, Martínez-Planell, and Moore-Russo (2019).

Nagel et al. (2019) categorizes the concept of slope within the APOS framework (Arnon et al., 2014). Slope as an algebraic ratio is the formula, $\frac{y_2-y_1}{x_2-x_1}$. At an action stage of slope as an algebraic ratio, an individual knows the formula as a memorized fact and can compute the value with the ordered pairs provided. At the transition level (from action to process) of slope as an algebraic ratio, she understands that slope is independent of any two points and may also imagine obtaining the equation y-y_1 = m(x-x_1) or $y = mx+b$ without explicitly doing it. Yet she may understand slope only in algebraic aspect and has no geometric referents to justify the formula, $\frac{y_2-y_1}{x_2-x_1}$.

In comparison, slope as a geometric ratio is the rise over the run in a geometric figure. At an action stage of slope as a geometric ratio, an individual may see slope as a static image and do a simple counting to find slope. At this stage, she has no understanding that the numerator changes as the denominator changes; as such, she may struggle with the sign of slope. At the transition level (from action to process) of slope as a geometric ratio, the individual knows that slope is independent of the size of the triangle being used. As such, she may use slope to determine relationships of properties or to describe the behaviors of a line, such as increasing or decreasing.

When an individual reaches the transition level of slope as both algebraic and geometric ratio as well as understands slope as a parameter in $y = mx+b$ by freely moving between algebraic and geometric ratios, she reaches a process stage of understanding of slope. With a process conception, she could imagine conversions of slope among arithmetic, algebraic, and geometric representations, without explicitly doing the conversions (Nagle et al., 2019).

Tangent Line and Linear Approximation

When a function f is differentiable at a point, $x = x_1$, one can determine its tangent line to the curve with the formula, $y - f(x_1) = f'(x_1)(x - x_1)$. Unfortunately, neither the derivative nor the tangent line equation itself involved in the formula is easy for students. Many students do not conceptually understand that $f'(x)$ is the limit of the average rate of change (Asiala et al., 1997; Orton, 1983; Thompson, 1994). Neither do they understand the structure or meaning of the tangent line, making errors such as claiming the slope of the tangent line as $f(x_1)$ (Orton, 1983) or the tangent line as the derivative of $y = f(x)$ (Amit & Vinner, 1990). Further, many individuals may not be much familiar with point slope form in which the formula is presented. According to Nagel and Moore-Russo (2013), preservice secondary teachers rarely hold the point slope form in their concept map of slope. While 11 of 19 teachers included the slope intercept form, only 3 included the point slope form in their concept map.

There are two common ways to find a linear approximation. The first is to find the tangent line equation using the formula $y - f(x_1) = f'(x_1)(x - x_1)$ and to determine $y = f(x_2)$ by substituting $x = x_2$ into the equation. The second is to find the slope of the tangent line at $x = x_1$, $f'(x_1)$, find dy as $f'(x_1)dx$, and add dy to $f(x_1)$, yielding $f(x_1) + dy$ (see Figure 1). The cognitive processes involved in the two ways are quite different, although they may be intrinsically the same—as $y - f(x_1) = f'(x_1)(x - x_1)$ can be converted to $y = f(x_1)+f'(x_1)(x - x_1) = f(x_1) + dy$.

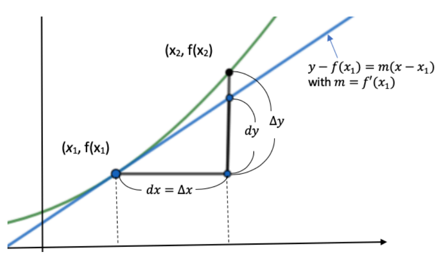


Figure 1. Linear approximation.

Methodology

The participants of this study were three precalculus students at a small state university in the southeast region of the US. I asked precalculus instructors to recommend their students who are active in classroom. I sent an invitation email to all 15 students recommended by their instructors. Three students responded to my email and participated in this study. Student 1 was a local high school student and had an Algebra 2 course at her high school prior to precalculus. Student 2 was a sophomore in biology and had a college algebra course at our university prior to precalculus. Student 3 was a freshman in computer science and had an advanced mathematics course at a high school in southeast Asia where English was the primary language of instruction.

Each of the three students were interviewed for a total of 5-7 sessions, with each session lasting about one and a half hours. The interviews were conducted in two stages. At the first stage—for the first one or two sessions—I used a form of semi-structured clinical interview (Ginsburg, 1997) to assess their level of understanding. The interview questions were designed to investigate their understanding of various forms of linear functions and quadratic functions as well as their ability to use rate of change in constructing function equations.

At the second stage—for the remaining sessions—I used a form of teaching experiment (Steffe & Thompson, 2000) to intervene their understanding, in particular, to strengthen their understanding of rate of change as well as to have them use slope in constructing equations and finding linear approximations. I began interviews with pre-designed tasks. As the interviews progressed, the tasks were modified and more tasks were created based on their responses.

The analysis of data began at the onset of the interviews. After each interview, I viewed the video recordings and prepared a time-coded written record of the content, including rough transcriptions of interesting episodes. At the same time, I developed codes based on the research questions, the framework of the study, and students' understandings observed in the data. During this initial stage of analysis, I came up with hypotheses on students' understanding and difficulties described in this report. In the next stage, I plan to review data by constantly comparing and revising codes in the spirit of constant comparative method by Glaser and Strauss (1967), which will help me confirm, rebut, or revise the hypotheses I made at the first stage of analysis.

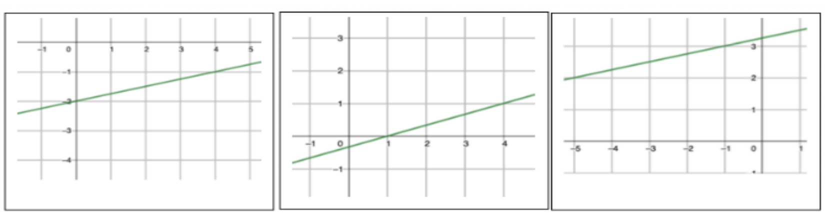


Figure 2. Linear equation task in pre-intervention.

Preliminary Findings

The results presented here are for Student 1 mostly from the interview and written data on the problems in Figures 2 and 3. The first graph of Figure 2 shows an integer valued y-intercept, but the second and third graphs do not. Figure 3 is three linear approximation problems designed at the level of precalculus or algebra. The problem 2 was provided with graphs of the function and the tangent line. Problems 3 and 4 were not provided with any graphs. Before given these problems, students were explained that a tangent line is almost identical to a curve at a given point and that the tangent line can be used to approximate function values.

Line Equations

Slope as an algebraic ratio and the Cartesian Connection. When constructing an equation for line graphs in Figure 2 during the assessment (at the first stage of the interviews), Student 1 chose two lattice points through which the line passes and found the slope of each line by calculating $\frac{y_2-y_1}{x_2-x_1}$ (action as an algebraic ratio). For the first graph, she read the *y*-intercept (0, -2) from the graph and successfully used the slope intercept form to determine the equation, $y = \frac{1}{4}x - 2$. For the second and third graphs, she wrote $y = \frac{1}{3}x + b$ and $y = \frac{1}{4}x + b$, respectively, but incorrectly determined the value of *b* as -0.3 and 3.3, respectively. When asked how she determined *b*, she said she guessed the values based on the locations of the *y*-intercepts.

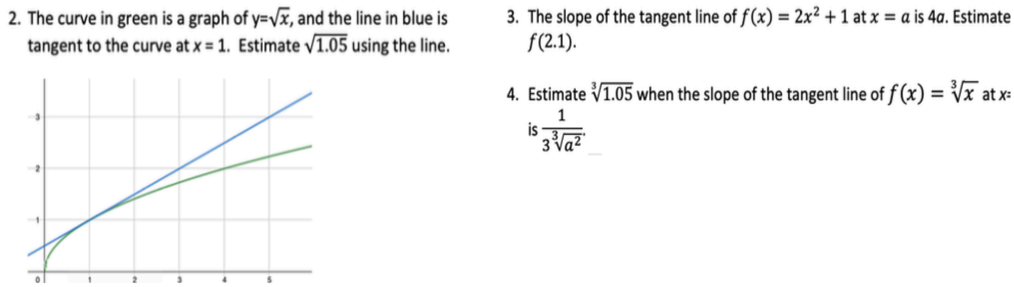


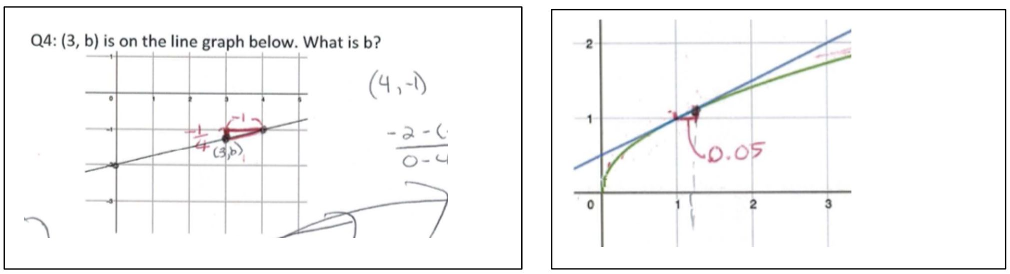
Figure 3. An example of linearization problem in teaching experiment.

During the intervention, I asked her if the *y* = 3*x*+1 graph passes through the point (2, 7). She answered yes because 3(2) + 1 = 7, showing her understanding of the Cartesian Connection—"a point is on the graph of the line L if and only if its coordinates satisfy the equation of L" (Moschkovich et al., 2013, p. 73). I also asked her what the value of *h* was when the point (4, 1) was on the $y = \frac{1}{3}x + h$ graph. She wrote $1 = \frac{1}{3}(4) + h$ and successfully solved for h, confirming her understanding of the Cartesian Connection in the problem context. I then asked her to determine *b* in the third graph of Figure 2, which she struggled prior to the intervention. She wrote $2 = \frac{1}{4}(-5) + b$ by using the point (-5, 2) from the line graph and claimed that *b* = 13/4. This suggests that students may have difficulty activating their understanding of the Cartesian Connection when the whole graph is presented without a reference of a specific point.

Transition level as algebraic and geometric ratios. When asked to find the value of *b,* with (3, *b*) on the graph in Figure 4a, she picked two points, (4, -1) and (0, -2), found the algebraic ratio, $\frac{-2-(-1)}{0-4}$, and simplified it to ¼ (action as an algebraic ratio). She then wrote $\frac{b-(-1)}{3-4} = 1/4$ and solved for *b*, showing the understanding that slope between any two points on the line is constant (the transition level of slope as an algebraic ratio). When asked what the change in *y* is if 1 is the change in *x*, she said ¼ without any hesitancy or work, confirming her understanding of slope as an algebraic ratio in the transition level. When I asked her to use the idea on the graph to determine the value *b*, she drew a triangle using points, (4, -1) and (3, b), wrote -1 and -1/4 for the horizontal and vertical segments (Figure 4a), and found *b* as -5/4 after subtracting 1/4 from -1, showing a transition level of slope as a geometric ratio. I then asked her to use her geometric strategy to find the equations of the second and third graphs in Figure 2. She successfully found equations for both. For the third graph, for example, she came up with $y = \frac{1}{4}x + \frac{13}{4}$ by writing *b* as 3+1/4 with no drawing, by increasing Δy of ¼ to the y-coordinate of (-1, 3), hinting her understanding of slope as a process.

Linear Approximation

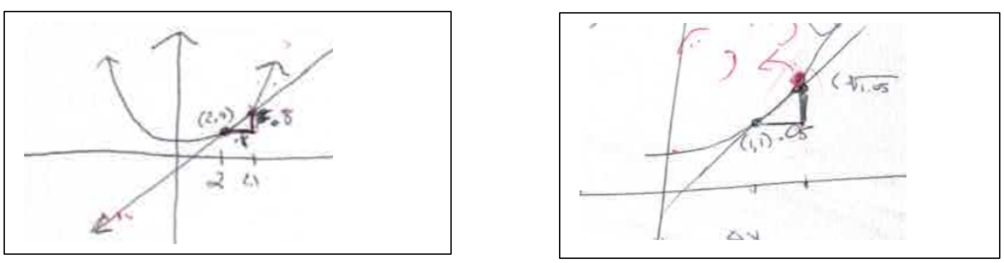
When asked to estimate $\sqrt{1.05}$ using the given tangent line (problem 2, Figure 3), she first determined the slope of 1/2 from the line graph and then the value of *b* based on the point (1, 1), by thinking $\Delta y = -\frac{1}{2}$ when $\Delta x = -1$, without drawing (the transition level of slope as an algebraic ratio). She then substituted x =1.05 into the tangent line equation, $y = \frac{1}{2}x - \frac{1}{2}$, and found the value of y for the approximation of $\sqrt{1.05}$ (the Cartesian Connections). When asked if there was any other way to find the estimate, she drew a triangle (see Figure 4b), found $\Delta y =$ .025 by setting up ½ = $\frac{\Delta y}{0.05}$, and added 0.025 to 1, showing the transition level of slope as a geometric ratio and possibly a process stage of slope.



(a) (b)

Figure 4. Slope in linear equation and its use in approximation.

For both problems 3 and 4 (in Figure 3), she first attempted to represent the functions in graphs. For $y = 2x^2+1$, she correctly drew a graph and its tangent line (Figure 5a) and found an estimate of $f(2.1)$ by performing a similar work to her work in the problem 2 in Figure 3. For problem 4, however, she asked for my help for the $y = \sqrt[3]{x}$ graph as she did not remember the graph. When asked if the shape of a curve mattered in the previous approximation problems, she said no and randomly drew a smooth curve along with its tangent line at $x = 1$ (Figure 5b). She then successfully estimated $\sqrt[3]{1.05}$ by doing a similar work to the previous problems.



(a) Estimate of $2(2.01)^2+1$ (b) Estimate of $\sqrt[3]{1.05}$

Figure 5. Use of slope in approximation.

Discussion and Future Research

As for the other two students, Student 2 showed at most an action conception of slope as both algebraic and geometric ratios throughout the intervention. Student 3 initially showed the transition level as an algebraic ratio, with little conception of slope as a geometric ratio. With the intervention, he developed a transition level as a geometric ratio. The results suggest that tasks such as those used in this study can strengthen student understanding of slope and linear equations that is necessary for the concepts of the tangent line and linear approximation. However, for students who profoundly lack the understanding of slope and linear equations, such as Student 2 who remained at the action stage despite intervention, more research is needed to investigate their underlying problems.

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Learning Assistants: The Assets of Relationships in Mathematics Classrooms

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Near-peer mentor models to support students in mathematics are becoming increasingly used at institutions across the United States. One model, the Learning Assistant model, has been implemented at Crossroads State University. Using Yosso's Community Cultural Wealth framework, this paper discusses the qualitative impact this model has on students, learning assistants, and the instructor of a College Algebra class. Based on findings, additional qualitative research is suggested to further investigate the Learning Assistant model.

Keywords: Learning Assistants, Near-Peer Mentors, Departmental Change

The use of near-peer mentors has become more common among universities and colleges in the US in the past 15 years (Otero et al., 2010b; Sellami et al., 2017). Many different efforts are being used across the nation to combat STEM attrition at the college level, such as co-requisite models, active learning initiatives, and curricular reform (Battersby et al., 2015; Dawson et al., 2015; Estrada et al., 2016; Theobald et al., 2020). Near-peer mentors have been used as a way to help implement active learning and curricular reform, while also seeming to offer some unique benefits both socially and academically to support students. Near-peer mentors thus serve as a topic of interest when considering systemic change within mathematics departments (Barrasso & Spilios, 2021; Sellami et al., 2017). While near-peer is not a well-defined term, and different schools use various models to implement the use of near-peer mentors, typically these near-peer mentors are fellow undergraduate students who have demonstrated the skills to function as mentors in certain classes to support other undergraduate students.

Crossroad State University, a four-year Hispanic serving institution (HSI) in the southwestern United States, utilizes a variation of one such model, called the Learning Assistant (LA) model, to support students in introductory mathematics courses. In this model started by CU Boulder, students who have successfully passed a course can apply for a learning assistant position for that course to serve as an additional resource to students in the classroom (Otero et al., 2010a; Otero et al., 2010b; Top et al., 2018). These LAs receive pedagogical training and then work with instructors in classrooms to help facilitate discussions and promote student learning.

A body of research shows several improvements to student performance related to the implementation of the LA model. Studies have found that the LA model has helped increase the implementation of cognitively demanding tasks in the classroom, as well as improved student performance on higher order cognitive skills assessments (Sellami et al., 2017; Webb et al., 2014). Other studies have demonstrated increases in active learning and improved collaboration in the classroom, as well as improved student satisfaction in these STEM courses (Campbell et al., 2019; Sellami et al., 2017). Additionally, studies have found that the implementation of the LA model has helped to reduce DFW rates (i.e., the percentage of students who withdraw, or receive a D or F grade in the course) and close performance gaps for under-represented minority (URM) students (Alzen et al., 2018; Sellami et al., 2017). While many benefits have been measured quantitatively with the LA model, there is sparse literature to qualitatively describe the phenomena that are occurring from implementing this model.

Qualitative research is of interest to better understand *how* the LA model is making a difference. For instance, while it is valuable to know that studies have found LAs useful in supporting URM students, further study to know *why* this is the case can serve to further improve the model and support students. Where Crossroads State University's student population is composed of over ⅓ URM students, studying the qualitative impact of the LA model in Crossroad State's Mathematics and Statistics department is especially relevant. With this in mind, in this pilot study I set out to address this question: *What are the perceived assets of LAs as described by the students and instructor who work with learning assistants, and the LAs themselves?*

Analytical Framework

Yosso (2005) outlines the framework of *community cultural wealth*, which takes a more holistic look at the assets, or capital, that an individual brings with them into a given situation, assets which are often overlooked by societal norms. She articulates why these assets are overlooked through the lens of Critical Race Theory (CRT), explaining that Whiteness prescribes the norms of cultural capital, thus overlooking other types of capital that individuals, and especially those who are minoritized, carry. Yosso (2005) outlines six dimensions of community cultural wealth, or six types of capital, that minoritized individuals especially use to "to survive and resist macro and micro-forms of oppression" (p. 77). These six dimensions are 1) aspirational capital— the ability to maintain hopes and dreams despite facing real barriers, 2) linguistic capital—the communication skills gained through various uses of language and style, 3) familial capital— cultural knowledge nurtured from *familia*, 4) social capital— networks of people and community resources, 5) navigational capital— the knowledge and skills to maneuver various institutions, and 6) resistant capital— the skills fostered through opposition which help challenge the status quo. Where mathematics has long been a space of oppression (Kozol, 2005; Martin et al., 2010), Yosso's (2005) community cultural wealth framework serves as an appropriate lens to view some of the ways that students bring various assets with them into mathematics classrooms to help themselves and others persist through the gate-kept field. Especially from a qualitative research standpoint, this framework as the potential to bring to light data which would likely be overlooked in quantitative studies, providing rich insight into the LA model that so far has not really been explored.

Methods

Over the course of a semester, I conducted four class observations in two sections of a College Algebra course. All names used are pseudonyms. Elena, a Hispanic female senior studying mathematics, was the undergraduate learning assistant in one section, and Jesse, a white male junior studying engineering, was the undergraduate learning assistant in the other section. Both sections were taught by a graduate student with prior college mathematics teaching experience named Megan, a white female studying mathematics. Based on observations, both class sections were racially diverse, with a seemingly even distribution of male-presenting and female-presenting students.

At the end of the semester, I conducted a semi-structured focus group interview with three students (1 White female presenting student, 1 Asian male presenting student, and 1 Hispanic male presenting student) from the College Algebra sections I had observed. Additionally, I conducted a semi-structured interview with Elena (LA), Jesse (LA), and Megan (instructor). I had three different protocols—for students, LAs, and the instructor—each of which asked questions which allowed participants to elaborate their experiences working with (or being) an LA during the semester. The observations I had conducted earlier allowed me to ask more personalized follow up questions during each interview based on interactions between students, LAs, and the instructor I had seen during the semester. From these interviews, I generated memos of the salient ideas shared in response to each question, and then used a priori coding of Yosso's (2005) community cultural wealth framework to analyze the memos.

Findings

I will now discuss the findings across each of the six dimensions of cultural capital wealth, articulating how the experiences of working with (or being) LAs were perceived by students, LAs, and the instructor.

Aspirational Capital

Aspirational capital was described both by LAs and the instructor in the form of supporting the students and LAs, respectively, in their own trajectories as students. Jesse shared, "I might not do the most, but I want to show students that I want them to succeed," recognizing that while he might not be able to provide every support to help students, he wanted students to know that he was indeed there to support them and help see them through to succeed in the class. Elena described her love of helping students have "aha" moments in class, articulating her belief in their capabilities to succeed through the class.

Megan talked about enjoying the opportunity of working with LAs because she recognized it as a resource to help support these LAs in their own student careers. She expressed enjoying being able to help them grow and succeed by providing this mentoring opportunity to students. For both the LAs and the instructor, their recognition and belief in the students for whom they were supporting was a form of aspirational capital that benefited the class.

Linguistic Capital

There was one instance discussed of linguistic capital by students. They shared how LAs in the class could explain mathematical ideas in different language from the instructor, which was something they benefited from. Where mathematics is often described as a language, having various ways to discuss mathematical concepts is indeed a nod to linguistic capital. LAs are able to bring different perspectives into the classroom and articulate them in ways that students relate to, allowing for more opportunities for mathematical understanding.

Familial Capital

Familial capital was not discussed during the interviews, because the focus of the interviews was about an academic setting. Thus, questions did not focus on outside community where familial capital might have been mentioned.

Social Capital

Social capital was perhaps the most mentioned form of community cultural wealth during interviews, being discussed by students, LAs and the instructor. Students discussed how LAs would make small talk with them at the beginning of class, getting to know them on a more social level. Students expressed that this helped them to feel more comfortable with the LAs, giving them the courage to also approach them about mathematics questions during class. LAs also were aware of this phenomenon, and noticed students being more comfortable to approach them over the course of the semester.

LAs discussed specific examples of noticing how students at times seemed apprehensive to ask questions to the instructor, but were willing to call the LA over to ask questions they had about the math. As Jesse put it, he was aware that LAs "have a better influence on you as a student." Another form of social capital that LAs discussed had to do with the network that was formed with other LAs. Elena and Jesse both described the support they received from their LA peers in their LA roles, as well as the social support as students in general navigating their own courses.

The instructor was also aware of the connection that LAs had with students, being an easier authority for students to approach during class. Megan articulated this social capital by saying, "it's valuable to have a second person [in the class] that is potentially in a little bit more of a mediation kind of middle zone," describing the power dynamics that students might feel when approaching the instructor compared to a fellow undergraduate student. Megan also talked about the LAs being a great support to her, and someone to bounce ideas off. LAs were also a support and a network to the instructor, providing valuable insight about students and ideas for how to navigate class.

Navigational Capital

Both the instructor and the LAs discussed the ways that LAs supported both students and themselves in navigating the role of being a university student. The instructor discussed that because she had not experienced being a student at the university, she did not have the insight to provide her students related to questions regarding things like where to take an exam, where to get accommodations, or where to go for study help. Megan discussed the LAs as a major strength to the class because they were a few years ahead in their studies and were familiar with the various university systems that younger students often had questions in navigating.

The LAs also discussed on an emotional level understanding what it was like being a student at the university and being able to relate to students in this way while also giving them advice for how to navigate the systems of the university. LAs also discussed how being in LA positions had helped them to develop better skills to succeed in their own careers as students. Elena discussed improved time management skills from balancing being an LA and a student, and Jesse discussed developing more efficient study skills through helping model them to students he supported as an LA. Additionally, Elena discussed how taking this job had connected her to network of new faculty that were opening doors for her future career. Across these comments, it is very clear that these LAs both provided and developed navigational capital.

Resistant Capital

One student and both LAs discussed forms of resistant capital that LA positions brought to the classroom, and to the lives of the LAs, respectively. One student shared that this was his second time taking College Algebra because he had failed the previous semester and struggled a lot with the material. He happened to have Jesse as an LA both semesters. The student discussed that he felt that Jesse was aware his unique circumstance and would regularly check in on him. He described feeling grateful that Jesse wanted to help him persist and succeed this time around, despite his struggles. These small efforts on Jesse's part made a big impact on the student.

The LAs also talked about developing resistant capital in taking up these roles. Jesse talked about having imposters syndrome when first starting as an LA, but that this job had helped him learn how to make mistakes and learn from them—something he was not used to from his engineering courses working with other upperclass students. Elena said that she has "worked smarter" as a student from the skills she has gained being an LA. Both LAs discussed how taking up these positions had helped them to develop more confidence in themselves as students, and that they saw skills learned as an LA showing up in their own studies as students, helping them to persist through higher level STEM courses.

Discussion

These findings demonstrate a variety of assets that the LA model has afforded multiple groups of people associated with mathematics classes at Crossroads State University. When asked whether they would prefer to take a future math class with or without LAs, one student said, “I would obviously take the one with the [LA]”, and the other students shared similar sentiments. Megan said that given the choice, she would also choose to teach in the future with LAs rather than without them. Elena and Jesse both expressed how being an LA has afforded them both with meaningful personal growth and skills that continue to benefit them in their journeys as students. The LA model, from these interviews, seems to have a qualitatively positive impact on students, learning assistants, and instructors involved with the program.

Many of the assets described seem to relate to the *relationships* that are formed among LAs, students, and instructors. Where traditionally introductory mathematics courses are often taught as a lecture-style course (Webb et al., 2014) where an instructor relays information for students to passively absorb, with less opportunities to form relationships during class time, the LA model adds a more human element to the mathematics. While several studies have demonstrated improved quantitative gains, these findings perhaps point towards a major reason for these benefits—the opportunity for more personal relationships to be formed allows for additional capital to be utilized while teaching and learning mathematics. These findings matter because these are not the data that would necessarily be tracked to understand the value of implementing roles like learning assistants within a department. Yosso’s (2005) framework reveals several important assets that these LAs bring to the department that would not typically be measured.

Conclusion

This study demonstrates assets that LA roles can bring to a mathematics department, namely social and navigational capital among others. Further research should study the implementation of learning assistants within the department (i.e., professional learning and training models), and to investigate how this system can be improved. Additionally, it would be of value to understand the impact of learning assistants on STEM persistence by doing some type of ethnographic study over time by following students who have engaged with learning assistants during their careers, as well as following students who have taken up the role of LAs to see their trajectories. These are a few of many potential angles of qualitative work to better understand the benefits of near-peer learning models on the teaching and learning of mathematics.

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Levels of Reflection in Strength-Based and Weakness-Based STEM Faculty Peer-Observation

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The Teaching TRIOS project aims to promote strength-based faculty peer-observation of teaching for professional development. In this project, we present preliminary work from a comparative case study of two teaching TRIOS groups, one largely strength-based and the other primarily weakness-based. We seek to understand the levels of reflection facilitated by a strength-based and weakness-based approach to observation through the lens of the Onion Model for levels of change (Korthagen, 2004). Preliminary results indicate that a strength-based approach may promote more frequent reflection upon deeper aspects of oneself, such as beliefs, identity, and mission.

Keywords: Faculty Development, Peer Observation, Teaching TRIOS, Core Reflection

Many of our RUME colleagues are working to understand how to impact university instruction, at the level of the faculty (e.g., Yoshinobu et al., 2022), graduate teaching assistant (e.g., Ellis et al., 2019; Reinholz, 2017; Rogers et al., 2020), or through broad departmental change (Reinholz et al., 2020; Smith et al., 2021). In our work, we contribute to and extend this literature by providing an in-depth look into the discourse between university faculty when they are reflecting together upon one another's instruction, attending to the tone and content of their reflections. This research is in direct response to Esterhazy et al. (2021), who provided a call to action encouraging researchers to not only explore what works in collegial-level faculty development (CFD) but also how it works. In particular, they mentioned that the field is in need of greater understanding of "the actual interactions and the content of feedback provided during concrete [collegial faculty development] situations" (p. 255). Within this preliminary report, we explore the content of feedback provided during Teaching TRIOS debrief sessions, a Time-Sensitive (T), Reciprocal (R), Inclusive (I), Operative (O), and Strength-Based (S) instantiation of CFD that utilizes peer observation and has been used within the mathematics department, and more broadly across various STEM departments, at Middle TN State University (Bleiler-Baxter et al., 2021).

Within TRIOS teams, groups of three interdepartmental faculty observe one another's instruction and then meet to debrief. The goal is to unpack the strengths of the instructor observed during that class session. Unlike many traditional forms of peer evaluation, where the *observers* are positioned as coaches, the TRIOS model is intended to position the *instructor* as the coach and as an expert in their classroom setting. What we have noticed, though, is that some teams are more effective than others at debriefing in a truly strength-based way (i.e., attending to and unpacking the strengths of the instructor rather than identifying weaknesses or offering areas for improvement). In this research, we explore two such TRIOS teams, one team of mathematicians that tended toward weakness-based reflection, and one team of biologists that tended toward strength-based reflection. We ask the following research questions:

1. In what ways and upon what content do undergraduate-level faculty reflect within TRIOS debrief sessions that utilize a strength-based approach?
2. In what ways and upon what content do undergraduate-level faculty reflect within TRIOS debrief sessions that utilize a weakness-based approach?

Theoretical and Analytical Framing

Providing negative, yet honest, feedback on a teaching session can lead to uncomfortable moments and cause considerable anxiety (Esterhazy et al., 2021). Through this study, we offer a converse hypothesis rooted in *positive psychology* (Seligman & Csikszentmihalyi, 2000): focusing faculty feedback on the strengths of the instructor can facilitate a deeper, more vulnerable and positive level of reflection and promote professional development.

The Onion Model for levels of change (Figure 1; Korthagen, 2004) describes six levels at which teachers may be influenced to change their classroom practices, and therefore provides us with a lens to qualitatively view instructor reflection. Within this model, reflection can range from the outer, external layers such as *environment*, *behavior*, and *competencies* to the deeper, core layers of *beliefs, identity,* and *mission.* Reflection centered on the core levels can create room for deeper introspection and learning (Korthagen & Vasalos, 2005).

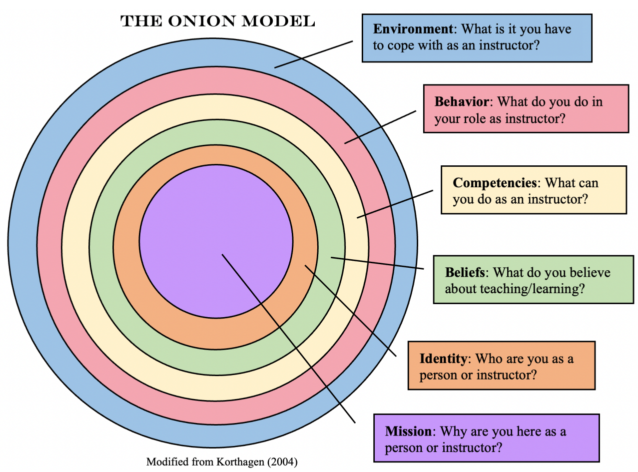

Figure 1. The Onion Model for levels of change, modified from Korthagen (2004).

Methods

Context and Data Collection

The study presented here is part of a state-level grant, awarded during the 2020-2021 academic year. The overarching objectives for the grant were to (1) support faculty in becoming more aware of and responsive to varied backgrounds, learning styles, and cultures of learners in STEM courses, (2) promote reflection among STEM faculty with respect to inclusive pedagogy, and (3) spark cultural change within STEM departments with respect to a focus on inclusion. We began the professional development (PD) program in Fall 2020 with nine faculty participants, three from Biology, three from Chemistry, and three from Mathematics. One of the faculty participants from the mathematics department was unable to participate in the program after the Fall semester due to a significantly increased teaching workload during Spring 2021, so the mathematics trio became a duo during the Spring 2021 semester.

One of the main levers of change in this program, especially with respect to goal 2, was a peer observation of classroom instruction initiative called Teaching TRIOS. In this manuscript we explore the Teaching TRIOS debrief sessions for the biology team (Betty, Bridget, and Brian) and the mathematics team (Margaret and Moby), which occurred during Spring 2021. We selected these two teams to highlight as cases in this manuscript because they illustrate clearly a strength-based approach (biology team) and a weakness-based approach (mathematics team) to debrief discussions, and hence provide us with insight into our research questions.

As this project was conducted during the COVID-19 pandemic, all TRIOS debrief sessions were conducted and recorded via Zoom. We did not collect recordings of the participants' class sessions, which were the topic of observation and discussion during the TRIOS debriefs. The six TRIOS debrief transcripts (three from biology, B1, B2, and B3, and three from mathematics, M1, M2, and M3) were edited, cleaned, and separated into talk turns. We define a talk turn to be the duration of one participant speaking until another participant speaks a complete word (i.e., interjections such as “hmm” did not separate a talk turn, but “yes” or “no” did). Transcript files were uploaded into Google Sheets to organize, classify, and analyze data.

Data Analysis

Our data analysis employed two layers of qualitative coding by talk turn. First, we coded each transcript to determine a quantitative description of the use of strength- or weakness-based approaches by each TRIOS team. Then, to compare the levels of reflection present within each TRIOS team, we coded the transcripts through the lens of the Onion model (Korthagen, 2004).

Strength-based and weakness-based reflections. Two members of the research team independently identified segments from the transcripts that demonstrated a strength or a weakness-based approach. After independently coding one transcript from each the biology and mathematics TRIOS teams, these researchers met to discuss their understanding of a strength-based and weakness-based reflection and developed the following definitions.

In a ***strength-based reflection****, an asset perspective on instruction is offered, often paired with a genuine interest in learning about instruction. These reflections are typically educative in nature.* For example, Brian (B2) commented on Betty’s risk-taking in the classroom,

> I applaud you for it, because what you did, I think, for some people would be considered risky. Where, it's like, it's easier just “tell them what they need to know, give them multiple choice exam don't take risks”, and it's like, but that is not the world of how we need to develop people, and so I really applaud you for what you're doing. Yeah, I mean, my notes are so much for myself of, like, “how can I adapt these?” Because this pushes me to go in ways I hadn't thought about so I'm, I really… overall just really well done.

In a ***weakness-based reflection***, *a deficit perspective on instruction is offered, often paired with interest in offering an alternative approach to instruction. These reflections are typically evaluative in nature.* For example, Moby (M1) commented on Margaret’s teaching, “Right there would have been another spot. Because you corrected [student] when maybe the class should have corrected [student].”

The same two researchers coded each transcript independently for strength- and weakness-based reflection and then met to discuss disagreements and reach consensus. Any talk turns where consensus could not be reached were reviewed by five team members and discussed to reach agreement. Each transcript finally received a sum total for the number of strength- and weakness-based reflections present in the transcript. In order to compare the transcripts, we then scaled the number of strength- and weakness-based codes each to the total number of strength- or weakness-based codes.

Levels of reflection: the Onion Model. We used the levels identified in the Onion model (Korthagen, 2004) to categorize faculty talk-turns as reflection upon environment, behavior, competencies, beliefs, identity, and/or mission. The entire research team coded two transcripts to refine our code descriptions for each of the six levels. The revised codebook was used by two researchers to evaluate the transcripts independently. Following the independent coding of each transcript, these two team members met to determine consensus. Any talk turns where consensus could not be reached were reviewed with five team members and discussed to reach agreement. Each transcript received a sum total for the number of codes from each category present. In order to compare the transcripts, we then scaled the number of codes per category to the total number of Onion Model codes.

Preliminary Results

Strength- and Weakness-Based Reflections

As stated above, we calculated the proportion of codes within a single transcript that demonstrated either a strength- or weakness-based reflection during the TRIOS debriefs for the biology trio and mathematics duo (Figure 2). Confirming our original impression, the biology trio largely embraced the strength-based reflection approach to the TRIOS debrief sessions, and although the mathematics duo illustrated more weakness-based reflections than strength-based reflections in their first two debriefs, their third debrief demonstrated growth in the use of a strength-based perspective.

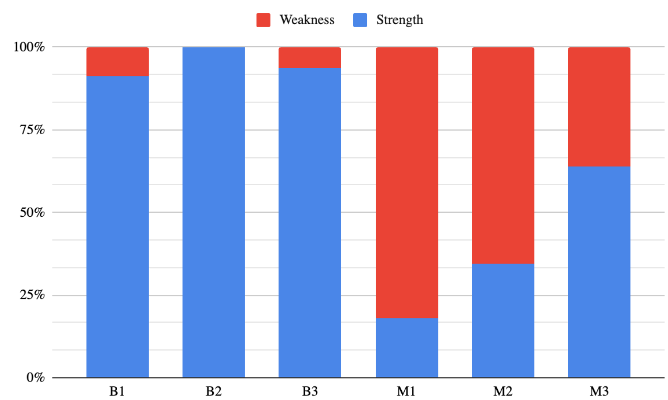

Figure 2. Proportion of codes demonstrating a strength- or weakness-based reflection by TRIOS debrief manuscripts for the biology trio (B1, B2, and B3) and mathematics duo (M1, M2, and M3).

This apparent change in approach is not surprising as all participants were reminded of the strength-based facet of the TRIOS program during whole-project faculty learning community sessions. It is also evident that the mathematics duo made fewer weakness-based reflections during each debrief. The third mathematics duo debrief (M3), in which 64% of coded talk turns included a strength-based reflection and 36% of coded talk turns included a weakness based reflection, was considerably shorter than other debriefs at 163 talk-turns compared to an average of 282 talk-turns across all six transcripts. Further, M3 still offered more weakness-based reflections than any of the biology team transcripts.

This observation regarding M3 along with the aggregate proportions of types of reflections for both the biology trio (95.3% strength and 4.7% weakness) and the mathematics duo (33.6% strength and 66.4% weakness) led us to consider the biology trio as an example of a group using a largely strength-based perspective and the mathematics duo as an example of a group using a largely weakness-based perspective to frame our analysis of the levels of reflection present throughout the debrief sessions.

Levels of Reflection in TRIOS Debriefs

The analysis of the TRIOS debrief transcript revealed a slightly different distribution of levels of reflection present in the biology trio's and mathematics duo's discussions (Figure 3). Although for both teams, the largest area of reflection was behavior and the smallest area of reflection was mission, the strength-based biology trio had a higher proportion of reflections regarding their beliefs (18.31%), identity (4.92%), and mission (1.37%) than the weakness-based mathematics duo (12.19%, 1%, and 0.5%, respectively). This indicates that it is possible a strength-based perspective may lead to deeper conversations and facilitate safe environment that facilitates reflection upon the most core features of oneself.

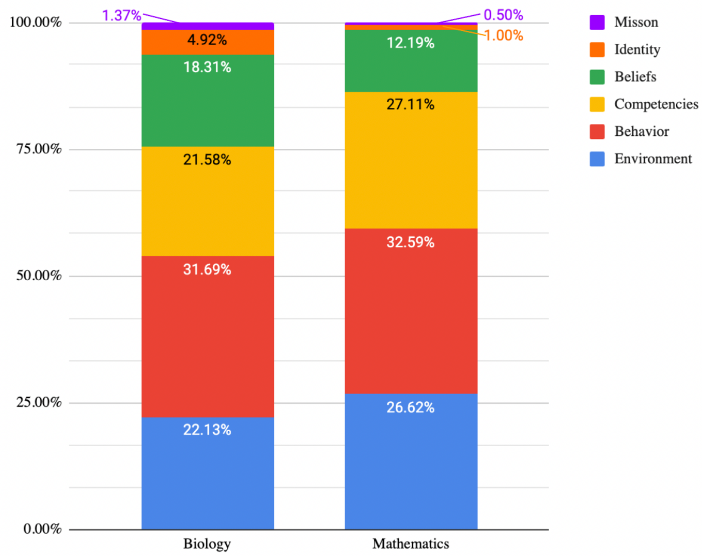

Figure 3. Proportion of levels of reflection across total Onion Model codes aggregated by biology (B1, B2, and B3) and mathematics (M1, M2, and M3) debrief transcripts.

Continued Research Plans

We have currently completed our coding and have explored the quantitative proportions of levels of reflection present in the two team debriefs, but we also plan to identify exemplary episodes for each level of reflection and consider the context of how a strength- or weakness-based reflection framed the subsequent conversation. This continued work will qualitatively respond to our hypothesis that focusing faculty feedback on the strengths of the instructor can facilitate a deeper, more vulnerable and positive level of reflection to promote professional development.

We also noted during our research team discussions a qualitative difference between certain strength-based and weakness-based reflections. Although we defined strength-based reflections to be *typically educative in nature* and weakness-based reflections to be *typically evaluative in nature*, we noted instances in which participants offered evaluative strength-based reflections. Future work distinguishing between evaluative and educative reflection may help us better respond to why a strength-based perspective alone may not promote deeper reflection.

Acknowledgments

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Proof Without Claim: Problem Packaging

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We present a case study of a student engaging in "proof without claim" tasks. In these tasks, students are presented with proofs in which the proposition being proven (its claim) is removed. Students are asked to ascertain what this missing claim is. This brief report serves (1) to illustrate how our novel proof without claim tasks can provide insight into how students engage with proofs and (2) as an existence proof that some students conceive of proofs as packaging for calculational math problems.

Keywords: Reasoning and proof, Undergraduate education, Preservice teacher education.

There is a significant body of research on the teaching and learning of proof-based mathematics (Stylianides et al., 2017). A lot of this literature has taken a deficit approach by highlighting various challenges that students face in proof-oriented content. This includes difficulties with constructing proofs (Iannone & Inglis, 2010; Weber, 2001), reading proofs (Dawkins & Zazkis, 2021, Inglis et al., 2012), translating novel arguments into proofs (Zazkis et al., 2015, Zazkis & Mills, 2017), and determining whether a given proof is valid (Selden & Selden, 2003, Alcock & Weber, 2005). Although it is useful to have a thorough list of challenges that students face, it is also important to understand the roots of these challenges.

In undergraduate mathematics, there is a transition between calculation-based courses and proof-based courses (Moore, 1994). A number of mathematics departments implement courses specifically to ease this transition (David & Zazkis, 2020). So, given the differences between proofs and calculations, the transition-to-proof courses also marks a transition in how students interact with mathematics, and by extension, what they might believe mathematics is. Since students have vastly more experience with calculation-based mathematics when they begin proof-based courses, it is reasonable to make sense of some nuances in students' approaches as an extension of their experience in calculation-based coursework (Dawkins & Zazkis, 2021). Importantly, such experiences might color students' engagement with proofs. Studying these conceptions has the potential to lead to meaningful improvements to proof education.

Through utilizing novel tasks, our work addresses the following question: How do students understand the relationship between a proof and what is being proven (its claim)? Addressing this question sheds light on how student thinking about proof might be influenced by past experiences with calculation-based mathematics. In the particular case study discussed here, we show evidence that at least some students view a proof as a container for a calculation problem. That is, students view proofs as essentially the same kind of task they encountered in their previous calculation-based classes, with its component sentences dressing up the calculation. This single case study is insufficient for making causal claims about the relationship between students' conception and their prior experiences with calculation-based course work. However, we do believe that conceptions transferred from those prior experiences are the most straightforward explanation for this phenomenon.

Task Design, Subjects, and Methods

We investigate this topic by creating and implementing the novel idea of *proof without claim* tasks. These tasks involve providing valid proofs while removing any explicit statements (claims) about what is being proven. Immediately prior to the proof is a statement "We will prove that" followed by a blank space, whereas after the proof is a statement "We have proven that" followed by another blank space. Students are prompted to fill in their interpretation of what should be in these blank spaces (Figure 1).

We developed six of these tasks in order to give insight into how students read and interpret proofs. This includes (1) how students determine what text goes in the blank spaces, and the related issue of (2) whether what they write in one blank space differs from what they write in the other. Our tasks included typical logical structures of claims, and they have employed basic proof strategies, such as proofs of universal claims via arbitrary instantiations, proof by counter-example, and constructive existence proofs.

The participant, Maria (pseudonym), was, at the time of the interview, an undergraduate preservice secondary mathematics teacher. She had successfully completed a transition-to-proof course. The interview lasted two and a half hours and can be categorized as an exploratory task-based clinical interview (Clement, 2000).

Results

Our data indicate that Maria views a proof as a package for three different components: the beginning (first blank), the middle (text provided between blanks), and the end (second blank). She brings a calculational orientation (Thompson et al., 1994) toward each individual part; the beginning describes what calculations or math problems are to be performed, the middle carries them out and serves as a holder for the constituent computations and equations, and the end presents the answer or result of these problems or computations. In other words, she appeared to conceptualize proofs as records of answer-getting problems. See Figure 1 for a typical example of Maria's work. Although there were six tasks involved, we use Figure 1 heavily as a typical example of Maria's work (due to space limitations).

The First Blank Box

In filling in the blank "claim" boxes, Maria indicated that she viewed the proofs as math problems where the "problem" is the first claim and the "solution" is the second claim. The reason we write quotations around "claims" is that, with the exception of a single task out of six, only once did Maria write in the first box what *we* would consider to be a claim. Her understanding of the first "claim" box as a repository for putting a math problem or prompt is evidenced by her writing of commands such as "find", "find out", and "prove". In other words, rather than writing propositions as her claims, she wrote instructions for what mathematical task to perform. For example, in the task disproving the statement "$2^{2^k} + 1$ is prime for all natural numbers k" (see Figure 1), Maria wrote "find f(k)" within the box for the first claim. In one task, she used the command "prove" in the first claim box. While "prove" is not the same as "find", we note that it shares the aspect of being a command; it is an instruction that might be given to a student to solve a math problem. Interestingly, in two different tasks, she listed "given" information in the first claim box. In one of these tasks, these "given" statements were listed in isolation. We note that, despite not including explicit commands, there is a sense in which writing the "given" information is similar to writing a command; both are pieces of information or symbolic prompts to be operated on or worked with towards a "solution". In other words, something is "given" in that it is information with which one will perform a task of finding an answer.

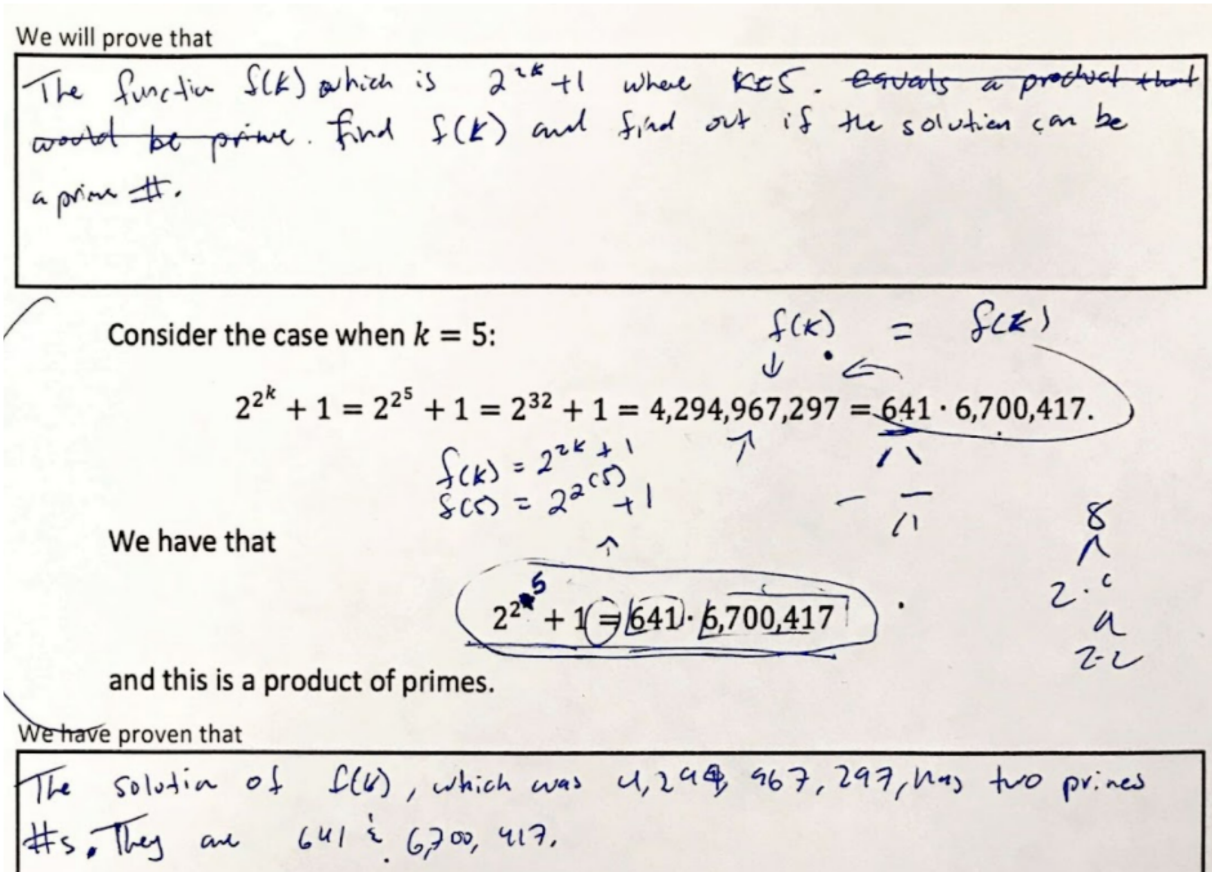

Figure 1. A typical example of Maria's work

The Second Blank Box

Interestingly, Maria did not write the same content in the second claim boxes as she did in the first claim boxes. Instead, in the second claim boxes, she wrote what we would call propositions. We can compare this with what she tended to do in the first claim box, which was to write an instruction or a calculation problem. Despite ostensibly being a proposition, when viewed in relation to the rest of the task (in particular the content of the first claim box), we understand Maria to be presenting the answer to the "question" or "problem" posed in the first box. Consider, for example, Maria's response to the task shown in Figure 1. While she wrote a command in the first claim box to "find $f(k)$" and to "find if the solution is a prime number", in the second claim box she wrote what her "solution" to $f(k)$ is and noted that it "has" two primes; in other words, the first claim box is for a question or prompt, and the second claim box is for a solution or answer. We see a similar structure in other tasks. For example, in a *proof without claim* task that proves the upper half of the unit circle has area $\frac{\pi}{2}$, Maria reports that the "derivative [sic] for the function $f(x) = \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$ is $\frac{\pi}{2}$" in response to her first blank box command to "Given the function $f(x) = \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$, find the derivative [sic] dx/dy". Another example of this structure is seen in a task that proves the quadratic formula (written with different symbols than the way it usually appears). In the first blank she inscribed the instruction “The equation $x^2 + px + q = 0$ has zeros, Please feel free to prove this how you think is best.” In reply, her second blank box read “The equation $x^2 + px + q = 0$ has two zeroes which are x = [...].” Maria correctly interpreted the last line of the proof, where x is isolated, as a statement of the zeroes of the function. Yet, taken together, the ways she inscribed the first and second blank boxes supports the idea that Maria was thinking of proofs as records of calculation problems.

The Middle

While Maria used the first claim box to write down the “problem” or command and the second claim box the “answer” or result, the middle served as a location for the computations leading to the answer. In the middle of the quadratic proof, Maria spent around five minutes attempting to verify the two algebraic steps—completing the square and rearranging terms. Throughout the study, Maria gave much attention to intermediary steps; her attention was on the particulars of the calculation rather than on the claim.

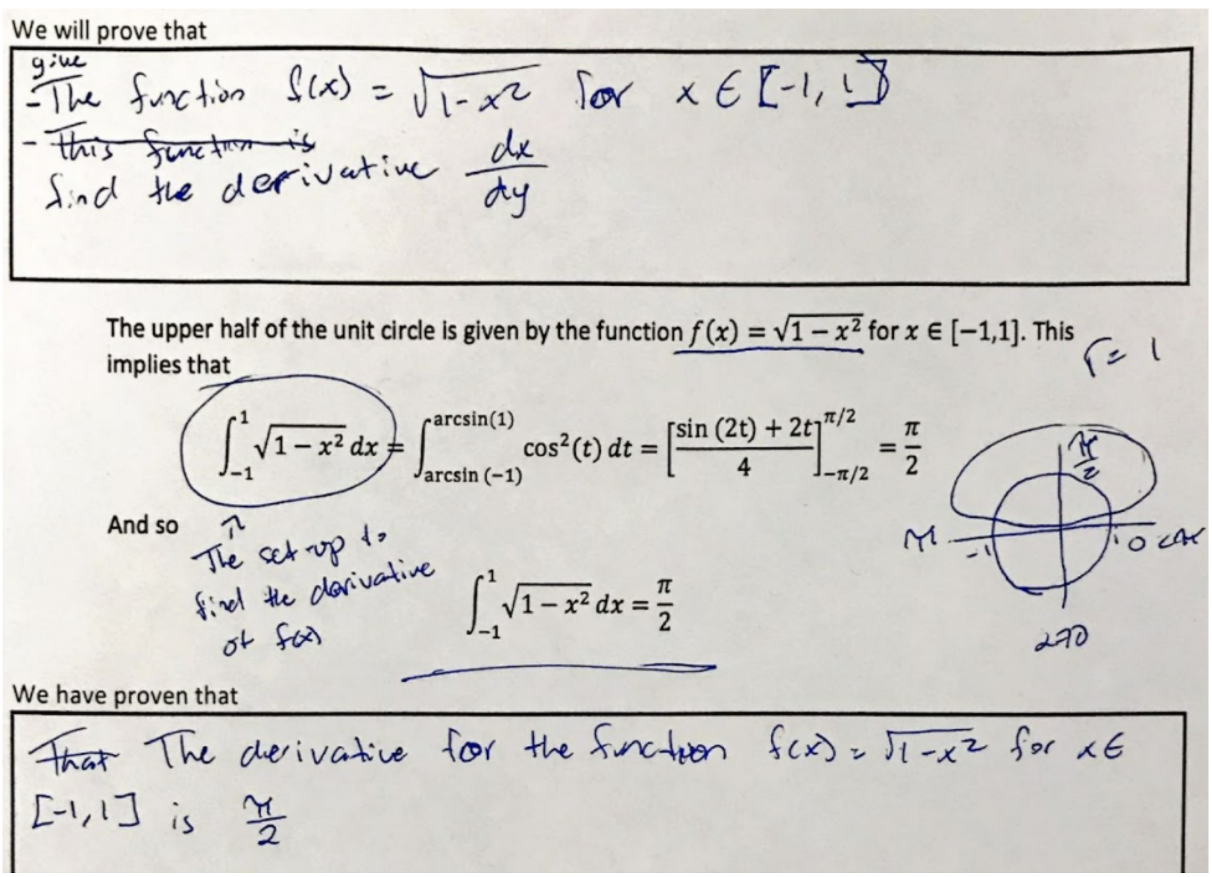

Figure 2. Another typical example of Maria’s work

Figure 2 (above) shows some inscriptions Maria made on the calculus task. This attention given to the intermediary steps is consistent with Maria’s overall view of a proof as packaging a calculation problem; the middle part is what she saw as the work to solve the “problem” being posed. Note that she circles the leftmost integral and inscribes below that it is “The set up to find the derivative [sic] of $f(x)$". In the first blank box she inscribed an instruction to find the derivative, and in the middle she indicates that she demonstrates that the way to start such a calculational-oriented problem is by writing the integral. In other words, her attention to intermediary steps can be understood as her focusing on the "problem" (the computation) instructed by the first box.

Figure 1 also supports our interpretation that the middle serves as a holder for the constituent computations. In drawing three arrows to 4,294,967,297 and labeling it "$f(k)$", Maria indicates that this is the $f(k)$ the proof is having her "find". Below that, she inscribes that "$f(k) = 2^{2^k} + 1$" and that "$f(5) = 2^{2^5} + 1$". Her other command from the first blank box, to "find out if the solution can be a prime number" is explored when she circles 641 and 6,700,417. From the middle of the task she extracts the numerical value of $f(k)$: "$f(k)$ is 4,294,967,297". This then allows her to present her finding as an "answer" in the second blank box and comment on it ("it has two primes"). Yet, Maria was not satisfied with her response and stated that it "has no meaning whatsoever" in relation to her transition-to-proof course. In reflecting on the task in Figure 1, Maria remarked that the mathematics presented in that task was not at all relevant to the mathematics she was learning in proof-based math—"I don't know what this has to do with proofs". For Maria, the task was at the pre-algebra level, because it only involved addition, multiplication, and a few exponents – she was attending only to the calculations when judging the proof's mathematical sophistication. Since this particular proof involved only pre-algebra calculations, she held this in conflict with the notion that proofs are part of higher-level mathematics. This supports our interpretation that what she saw as the mathematics of the proof was the calculational content in the middle, further supporting our interpretation that Maria saw the proofs as records of calculation problems.

Discussion and Future Directions

We reported exploratory findings on an undergraduate preservice teacher who recently completed a transition-to-proof course. This particular student made sense of these tasks by conceptualizing the first claim as a statement of a "problem" to be solved, the last claim as the "answer", and the middle as a location for calculations connecting top to bottom. Hence, ascertaining the claim amounted to determining to which "problem" these calculations correspond. This technique of *proof without claim* gave us insight into how this student conceptualized proofs. We believe prior calculation-based coursework to be a reasonable explanation for the phenomena highlighted in this preliminary work. Hence, future directions can expand the use of these tasks to a larger audience and more systematically explore the connections between such conceptions and students' prior course work.

Furthermore, it may be worth expanding interviews to include comparison tasks, where interviewees are presented with both a container for calculation-type response (e.g. Maria's work) and one for a normative response (e.g. a mathematician's) to a *proof without claim* task. Interviewees' responses to these comparison tasks have the potential to reveal additional nuances of students' conceptions of proof. In particular, since there is no blank claim box that needs filling, there is potential to confirm that some students truly believe the problem-calculation-answer format to be a legitimate proof format and perhaps more desirable than the normative response.

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“When are we ever going to need abstract algebra?”

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Instructors who prepare prospective teachers should provide opportunities for them to make connections between Abstract Algebra and secondary mathematics. This study focuses on an instructor who guided prospective teachers to connect properties of algebraic structures with equation solving procedures by evoking their intellectual needs (Harel, 2013) for computation, structure, causality, and communication. The students resolved such intellectual needs by using ring properties as tools to justify their solutions to equations. We characterize the intellectual needs the instructor evoked, along with her pedagogical moves used to evoke them during class discussion. We discuss how we will extend this analysis and provide implications for teaching.

Keywords: Abstract Algebra, Equation Solving, Prospective Teachers, Intellectual Need

Understanding abstract algebra has the potential to support mathematics teachers in their various teaching practices (e.g., Baldinger, 2018; Murray & Baldinger, 2018; Murray et al., 2017; Serbin, 2021; Wasserman & Stockton, 2013; Zazkis & Marmur, 2018; Zbiek & Heid, 2018). However, this potential is often not realized (e.g., Cofer, 2015; Ticknor, 2012; Wasserman, 2017; Zazkis & Leikin, 2010). For teachers’ understanding of advanced mathematics to be useful in their teaching, they need to first make connections between the advanced and secondary mathematics that change their understandings of the content they teach (Wasserman, 2018). One content area in which prospective secondary mathematics teachers (PSMTs) can connect abstract and secondary algebra is equation solving (Wasserman, 2016). Understanding how the properties of algebraic structures are used while solving equations can help PSMTs develop a deeper structural understanding of the often-rote procedures used to solve equations (e.g., Murray & Baldinger, 2018; Serbin, 2020; 2021). This connection between algebraic structures and equations was the focus of a unit in a Mathematics for Secondary Teachers course, which is the unit of analysis for this study. The course instructor guided students to connect abstract and secondary algebra by evoking various types of intellectual need (Harel, 2013) that required students to find a new “tool” they could use to solve equations in $\mathbb{Z}_{12}$ and $\mathbb{Z}_5$. We investigate how the instructor evoked these intellectual needs, which led students to identify ring properties as “tools” to use to meet those needs. This study addresses these research questions: (a) What types of intellectual need did the instructor evoke as she implemented the equation solving task sequence during class? (b) Which pedagogical moves did the instructor use to make the need intrinsic to students?

Literature Review

Equation solving is a fruitful point of connection between abstract and secondary algebra (Wasserman, 2016). By the time PSMTs take an Abstract Algebra course, they are familiar with procedures used to solve equations, including canceling, simplifying, factoring, and using the zero-product property. Learning about the properties of algebraic structures can help PSMTs develop a deeper understanding of why those equation solving procedures work (e.g., Serbin, 2021; 2022). Wasserman (2014) demonstrated how attending to the assumptions underlying algebraic manipulation used in equation solving helped teachers understand the axioms and properties of algebraic structures. Murray and Baldinger (2018) similarly showed how teachers developed a deeper understanding of the properties used in equation solving and how they were connected to algebraic structures. Equation solving has also been used as a context for students to build formal reasoning about algebraic structures from their intuition. Cook (2015) showed how students could reinvent ring axioms by attending to properties that allow them to solve equations in $\mathbb{Z}_5$ and $\mathbb{Z}_{12}$. Cook (2014) also showed how students discerned properties of integral domains and fields by reasoning about zero-divisors and multiplicative inverses while solving linear equations. Overall, equation solving has been documented as a useful context in which students can connect abstract algebra and secondary algebra. Guiding PSMTs to make these connections is a fruitful approach for helping them better understand the content they teach.

Theoretical Background

This study leverages Harel's (2008) *necessity principle*, which suggests that students learn mathematics when they perceive an intellectual need for it. Harel (2020) defined an *intellectual need* as "A behavior that manifests itself internally with learners when they encounter an intrinsic problem—a problem they understand and appreciate" (p. 274). When students encounter a problem intrinsic to them, their intellectual need for something is evoked, which can be satisfied by finding a resolution. Harel gave teachers the responsibility of making intellectual needs intrinsic to the students. Harel (2013) found five categories of intellectual needs: (1) *need for certainty*, (2) *need for causality*, (3) *need for computation,* (4) *need for communication*, and (5) *need for structure*. The need for certainty refers to a person's desire to prove whether a conjecture is true or false occurs when students are asked to write proofs. The need for causality is the need to explain the cause that makes an assertion to be true. The students' need for computation can be evoked by the instructor when asking them to solve an equation, as it is described as an individual's need to quantify. A teacher can evoke students' need for communication by formulating spoken language into algebraic expressions and formalizing concepts. The need for structure is a need for organizing knowledge into a logical structure. Harel (2013) proposed the necessity principle can be used in instruction by teachers setting an intellectual need for a particular concept, translating this need into a set of questions, designing a task sequence that allows students to investigate these questions, and helping students identify the concept that is present in their solutions to these tasks. Our study focuses on an instructor's implementation of a task sequence in which she evoked students' various intellectual needs.

Methods

Study Context, Data Collection, and Data Analysis

Data were collected in a senior-level undergraduate Mathematics for Secondary Teachers course at a large research university in the US. This course covered topics from secondary algebra and precalculus that were connected to group and ring axioms and properties. Data were collected in the second unit of the course, which addressed the ring properties used while simplifying expressions and solving linear and quadratic equations in $\mathbb{Z}$, $\mathbb{Z}_5$, and $\mathbb{Z}_{12}$. The data include video and audio recordings of the first three class days of the instructional unit on equation solving. Each class was 75 minutes long. The course was taught by Dr. Grey (pseudonym), a mathematics professor and mathematics education researcher. For each clip of a task implementation, we summarized the task discussion and identified the tools students needed to do the task. We used deductive coding (Miles et al., 2013) to classify the type of intellectual need that was evoked by the instructor, using an a priori list of codes based on Harel's (2013) types of intellectual need: (a) need for structure, (b) need for computation, (c) need for causality, (d) need for communication, and (e) need for certainty. We also used inductive coding (Miles et

Task	Tool Needed for Task	Intellectual Need	Pedagogical Moves Used to Provoke Intellectual Needs
1. What are some sets that have more than one operation?	Awareness of sets with more than one operation	• Need for structure	• Juxtaposing different sets
2. What do you notice about the properties, structure, or rules for $\mathbb{Z}_{12}$?	Awareness of the properties of sets with two operations	• Need for structure • Need for causality • Need for communication	• Asking students to justify their work by using ring axioms • Juxtaposing different sets
3. Find solutions x in $\mathbb{Z}_{12}$ that satisfy the equations and tell which property/ axiom you use on each step. a) $x+3=2$ b) $5x=4$ c) $2x=0$ d) $x+1=2+2x$ e) $3(x+2)+1=4$	a) Additive inverse and additive identity b) Multiplicative inverse and multiplicative identity c) Repeated addition and use of additive inverses d) Additive inverse of cx e) Distributive property	• Need for communication • Need for computation • Need for causality	• Asking students to justify their work by using ring axioms • Challenging student idea to prompt them to revise their work • Directing students' attention to what properties were involved in cancellation. • Establishing shared terminology of mathematical properties • Revising students' written work to make it more precise/explicit
4. Find solutions x in $\mathbb{Z}_5$ that satisfy the equations and tell which property/ axiom you use on each step. a) $2x=0$ b) $3(x+2)+1=4$ c) $x+1=2+2x$	a) Multiplicative inverse b) Cancellation rule c) Distributive property and zero-product property	• Need for communication • Need for computation • Need for causality	• Asking students to justify their work by using ring axioms • Challenging student idea to prompt them to revise their work • Directing students' attention to what properties were involved in cancellation. • Establishing shared terminology of mathematical properties • Juxtaposing different sets
5. Find solutions x in $\mathbb{Z}$ that satisfy the equations and tell which property/ axiom you use on each step. a) $x+1=2+2x$ b) $x+4=2$ c) $2x=0$ d) $4(x+2)=4$	a) Zero-product property b) Additive identity and additive inverse c) Zero-product property d) Distributive property, zero-product property, and cancellation rule	• Need for communication • Need for computation • Need for causality	• Asking students to justify their work by using ring axioms • Directing students' attention to what properties were involved in cancellation. • Establishing shared terminology of mathematical properties • Juxtaposing different sets • Revising students' written work to make it more precise/explicit
6. Will even integers behave any different then integers? Does $4(x+2)=4$ have an answer in $2\mathbb{Z}$? What were we not allowed to do and what does that mean?	Compare the properties of $\mathbb{Z}$ *and* $2\mathbb{Z}$ (existence or non-existence of multiplicative identity)	• Need for communication • Need for computation • Need for causality	• Asking students to justify their work by using ring axioms • Directing students' attention to what properties were involved in cancellation. • Establishing shared terminology of mathematical properties • Juxtaposing different sets
7. a) Complete table comparing rules for $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ *and* $2\mathbb{Z}$. b) Define a ring.	a) Properties for $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$ b) Introduce the definition of ring using properties in $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ *and* $2\mathbb{Z}$.	• Need for structure • Need for communication	• Asking students to justify their work by using ring axioms • Directing students' attention to what properties were involved in cancellation. • Establishing shared terminology of mathematical properties • Juxtaposing different sets

Figure 1. Summary of Intellectual Needs and Pedagogical Moves Used to Evoke These Needs on Tasks

al., 2013) to code the instructor's pedagogical moves used to make the intellectual need intrinsic to students in that video clip. We wrote analytic memos (Maxwell, 2013) that included justifications of our coding and explanations of how Dr. Grey's pedagogical moves evoked an intellectual need and made the need intrinsic to the students. We compared the coded video segments to identify patterns and relationships among the various intellectual needs evoked and the instructor's pedagogical moves used to evoke those intellectual needs.

The Instructional Task Sequence

Dr. Grey used an adaptation of Cook's (2012; 2015) task sequence for the guided reinvention (Freudenthal, 1991) of rings, which leveraged student reasoning about ring properties used to solve equations. Dr. Grey introduced the task sequence (see Figure 1) by comparing sets with one operation to sets that have two operations. Having previously learned about groups, the instructor began by evoking the students' awareness of sets with two operations by asking about the properties of these $\mathbb{Z}_5$ and $\mathbb{Z}_{12}$. She restricted the operations students could use to solve equations in $\mathbb{Z}_5$ and $\mathbb{Z}_{12}$ to only addition and multiplication (mod 12 or mod 5), so the tools students needed to use to satisfy their evoked intellectual needs were group/ring axioms or properties. Dr. Grey asked students to justify the algebraic manipulations in their equation solutions by identifying each of the properties used at each step. Students identified properties of additive inverse, multiplicative inverse, and identity. The instructor used these equation tasks to show students that not all properties hold for all sets. As the tasks progressed, students recognized that they needed additional tools to solve equations, such as the distributive property, cancellation rule, and zero-product property. Once they learned the properties in $\mathbb{Z}_5$ and $\mathbb{Z}_{12}$, the instructor presented equation solving tasks in $\mathbb{Z}$ and $2\mathbb{Z}$. Students then completed a table where they listed the properties of $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$ that they had found while solving equations. This led to the final task where they used the properties in the comparison table to define a ring.

Results

We identified four types of intellectual need evoked through the tasks: the need for structure, computation, causality, and communication. An overview of each task used during class, the tools needed for the task, the intellectual needs evoked by the instructor, and the instructor's pedagogical moves used to evoke those intellectual needs is presented in Figure 1.

How the Instructor Evoked Students' Need for Structure

Dr. Grey evoked students' need for structure at both the beginning and end of the task sequence. She made this need for structure intrinsic to the students by *juxtaposing different sets* together and having students identify similarities and differences in their structure. She posed task 1 (see Figure 1), in which she juxtaposed sets with one operation to sets with two operations to have students recognize a difference in the properties of those two kinds of sets. The students were aware of the structure of groups, so this task led them to see a need to understand the structure of sets with more than one operation. Dr. Grey further used this pedagogical move of juxtaposing sets in task 7, which prompted students to compare the rules used to solve equations in the different sets, $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$. This task allowed students to organize the properties they have learned thus far into a table to visualize which properties held for each of the sets. Dr. Grey's pedagogical move of juxtaposing sets evoked students' intellectual need for structure by organizing/listing the axioms they found for algebraic structures $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$ through equation solving. The students' intellectual need for structure prompted them to identify the axioms/properties these sets have in common that they later used to define a ring.

How the Instructor Evoked Students' Need for Computation

Dr. Grey's goal for the equation solving tasks was to evoke students' need for computation. She made this need intrinsic to them was by *asking students to justify their work by using ring axioms*. In Task 3, she asked students to find solutions to equations in $\mathbb{Z}_{12}$. She asked the students to write every property they had used to solve and why they used it. This pedagogical move allowed students to identify the properties needed to solve the equations. Once students had their solutions and explained their reasoning about them, the instructor evoked students' need for computation by *challenging a student's idea to prompt them to revise their work* by asking them follow-up questions. For example, in Task 3, the instructor prompted students' thinking by asking "What would happen if we could only use addition and multiplication?", when one of the students used division in their solution to an equation in $\mathbb{Z}_{12}$. Dr. Grey also evoked this need for computation by *revising students' written work to make it more precise/explicit*. In tasks 3d and 5d (see Figure 1), she provided a different approach to solving the equations that evoked students' thinking about how to approach these problems and how they could reduce their steps. Lastly, the instructor made this need for computation intrinsic to the students by *directing their attention to where properties can or cannot be used.* She made sure students were aware of how some properties do not hold for all sets. In task 4c (see Figure 1), Dr. Grey explained how the zero-product property holds in $\mathbb{Z}_5$ but not in $\mathbb{Z}_{12}$. Not being able to use this property evoked a need for students to find another way to perform the computation.

How the Instructor Evoked Students' Need for Causality

A need for causality was evoked by the instructor during several tasks. This intellectual need became intrinsic to the students by *asking them to justify their work by using ring axioms.* The task sequence consisted mostly of equation solving tasks, where students needed to show and justify their work in each step. These tasks evoked this intellectual need because students needed to explain why they could go from one line of the equation to the next. During Task 4c (see Fig. 1), students needed to explain why $-1(x+1)=0 \rightarrow x+1=0$ when solving equations in $\mathbb{Z}_5$. This need for causality was met by introducing the name of the property they had implicitly used: the zero-product property. Dr. Grey then evoked students' need for causality by *directing students' attention to where properties can or cannot be used* by asking them if the zero-product property would hold in $\mathbb{Z}_{12}$. This directed students to reason about why or why not this property would hold in another algebraic structure. Altogether, the instructor evoked students' intellectual need for causality, so students could explain why they could use certain ring axioms to solve an equation in a particular set and why some of the properties did not hold for other sets.

How the Instructor Evoked Students' Need for Communication

Dr. Grey evoked a need for communication in students and made this need intrinsic to them by *establishing a shared terminology of mathematical properties*. She asked students to name the properties used to solve equations in $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$, which evoked students' need to communicate the names of these properties. This need was satisfied by class's establishment of a shared terminology. For example, Dr. Grey changed a student's expression of "moving 8 to the other side" to use the language of "adding the additive inverse of 8 to each side of equation" in Task 5d (see Figure 1). This shared terminology helped students use precise mathematical language to communicate their justifications of their equation solutions and helped them create new definitions. The need for communication prompted students to give explicit names to the ring axioms they used to solve equations. Aggregating these named properties and comparing whether they held for $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$ allowed the class to define a ring on Task 7b.

Discussion and Conclusion

Engaging PSMTs in reasoning about the properties of rings used in equation solving can help them better understand connections between secondary and abstract algebra. A Mathematics for Secondary Teachers course instructor led students to make this connection by evoking students' intellectual needs (Harel, 2013) for structure, computation, communication, and causality on tasks involving equation solving in $\mathbb{Z}_5, \mathbb{Z}_{12}, \mathbb{Z}$ and $2\mathbb{Z}$. We identified the pedagogical moves that the instructor used to evoke students' various intellectual needs, which included: (a) asking students to justify their work by using ring axioms, (b) directing students' attention to what properties were involved in cancellation, (c) establishing shared terminology of mathematical properties, (d) juxtaposing different sets, and (e) revising students' written work to make it more precise/explicit. These findings provide a pedagogical implication for mathematics instructors who prepare PSMTs. Instructors can use Dr. Grey's pedagogical moves to make intellectual needs intrinsic to their students. Evoking students' intellectual needs that can be met by using ring axioms seemed to be a beneficial way to guide students to connect abstract and secondary algebra in this study. We plan to continue this analysis by identifying the intellectual needs and pedagogical moves used by this instructor in subsequent class days as the class transitioned from solving linear and quadratic equations to solving equations involving function composition. We pose the following discussion question to the audience: How can we further explore the relationships between the intellectual needs and pedagogical practices used to evoke the needs?

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“What makes it eigen-esque-ish?”: Eigentheory Development in a Quantum Mechanics Course

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Eigentheory concepts are central in mathematics and physics; they serve multiple functions, such as symbolizing physical phenomena and facilitating mathematical computations. The words associated with eigentheory develop and vary over time (e.g., eigenvector, eigenstate), as do the associated symbols (e.g., Ax = λx, H|En> = En|En>). The focus of this study is how the concept of "eigen" develops over time for a quantum mechanics classroom community. We analyze the public displays of form-function relations used in the classroom community (Saxe, 1999). In this preliminary report, we summarize the forms and functions identified in the first 19 class sessions and share progress on our analysis of shifts in the uses of forms and functions over time.

Keywords: eigentheory, linear algebra, quantum mechanics, form-function, classroom analysis

Eigentheory concepts are central in mathematics and physics. They serve multiple functions, such as symbolizing associated physical phenomena and facilitating mathematical computations. Physics students reason about eigentheory concepts in Linear Algebra courses where eigen-equations often first look like $A\boldsymbol{x} = \lambda\boldsymbol{x}$, with $n \times n$ matrix A, $\boldsymbol{x}$ in $\mathbb{R}^n$ or $\mathbb{C}^n$, and λ is a scalar; they may also encounter $T(\boldsymbol{v}) = \lambda\boldsymbol{v}$, where T is a linear operator on vector space V. Students further reason about eigentheory in quantum mechanics, where eigenvalues of various Hermitian operators represent possible measured values of corresponding observables. Furthermore, physics students learn Dirac notation, in which eigenequations take on forms, such as $S_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$ and $H|E_n\rangle = E_n|E_n\rangle$, and function to convey information related to spin and energy, respectively. In quantum mechanics, it is “eigen, eigen, eigen all the way” (Shankar, 2012, p. 30).

Several verbal, symbolic, and written forms can be associated with the same eigentheory concept, and many of these forms can serve different functions (Saxe, 1999) in both math and physics. These form-function relations can develop during class as the community’s common ground is negotiated through verbal or written communication. This development for eigentheory concepts in a Quantum Mechanics course is the focus of this study. We pursue the following research question: *How does the concept of "eigen" develop over time for a quantum mechanics classroom community?* To do so, we leverage a form-function analysis (Saxe, 1999), analyzing the public displays of form-function relations used in the classroom community. We identify each form that was used during whole-class discussion and infer the function of the form. So far, we have identified forms and functions from the first 19 class sessions. Our current analysis involves examining how shifts in the uses of these forms and functions occurred over time.

Literature Review

There is a growing body of literature on student understanding of eigentheory and teaching innovations that support student reasoning about eigentheory (e.g., Altieri & Schirmer, 2019; Beltrán-Meneu et al., 2016; Bouhjar et al., 2018; Gol Tabaghi & Sinclair, 2013; Karakok, 2019; Manogue et al., 2001; Plaxco et al., 2018; Pollock et al., n.d.; Salgado & Trigueros, 2015; Serbin et al., 2020; Wawro et al., 2019b). There are many conceptually complex aspects to a deep understanding of eigentheory. For example, interpreting the symbols $A\boldsymbol{x} = \lambda\boldsymbol{x}$ can involve recognizing the matrix-vector product $A\boldsymbol{x}$ as the same mathematical object as the scalar-vector product $\lambda\boldsymbol{x}$ (Thomas & Stewart, 2011). It can also involve conceptualizing eigenvectors as those $\boldsymbol{x}$ which are stretched by matrix A (e.g., Henderson et al., 2010; Sinclair & Gol Tabaghi, 2010). These correspond to a relational meaning of the equals sign (Knuth et al., 2006) and a functional interpretation of $A\boldsymbol{x} = \boldsymbol{b}$ (Larson & Zandieh, 2013), respectively, and have also been documented with quantum mechanics students (Wawro et al., 2019a; Wawro et al., 2020b).

Physical science students face additional complexities in reasoning about eigentheory, given that the mathematical objects and symbols relate to physical referents (Rodriguez et al. (2018). Researchers have suggested that interpreting mathematical symbolic expressions in terms of physical phenomena is a nontrivial endeavor for students (e.g., Caballero et al., 2015; Her & Loverude, 2020; Serbin & Wawro, 2022). Physics students face challenges in interpreting eigentheory symbols because those symbols and their word forms take on additional meanings in quantum mechanics contexts (Dreyfus et al., 2017; Gire & Manogue, 2011; Pina et al., 2022; Wawro et al., 2020a). For example, Wawro et al. (2020a) found that as physics students interpreted the meaning of the symbols in the eigenequations $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $S_x|+\rangle_x = \frac{\hbar}{2}|+\rangle_x$, a student was unsure how to resolve the disconnect between their geometric interpretation of the first equation and their quantum mechanical interpretation of the latter equation. Pina et al. (2022) also demonstrated different meanings students had for an eigenequation. They identified three different symbolic forms (Sherin, 2001) that physics students had for a position operator eigenequation that share a symbol template but have different conceptual schemata: a transformation which reproduces the original, an operation taking a measurement of state, and a statement about the potential results of measurement. These studies illustrate the complexity associated with interpreting eigentheory symbols and the varied meanings these symbols convey.

Theoretical Background

Through his anthropological work on the cultural development of mathematical ideas, Saxe (1999) created a framework for investigating the form-function relations created by individuals and a community over time. A *form* is a verbal, symbolic, graphical, or physical representation that takes on mathematical meaning. As individuals engage in activity or communication, they tailor forms to serve certain functions in activity, thereby establishing form-function relations. *Functions* are defined as the "purposes for which forms are used as individuals structure and accomplish practice-linked goals" (Saxe, 1999, p. 20). Forms can be adapted to serve several different functions, and functions can be served by different forms. For example, the form $y = 5x$ can function to conveying a constant rate of change or exhibit covariation of two variables, and the forms e and 1 can both serve the function of symbolizing a group identity (Plaxco, 2015).

Saxe and colleagues leverage this framework for analyzing the "reproduction and alteration of a common ground of talk and action over lessons in classroom communities" (Saxe et al., 2015, p. 71). Common ground refers to "shared knowledge of word meanings and norms for communication" that "enables successful communication and coordinated action" (Saxe & Farid, 2021, p. 8). Common ground is created as community members "produce and interpret displays of mathematical thinking, making use of representational forms (linguistic, graphical, gestural) to serve communicative and problem-solving functions" (Saxe et al., 2015, p. 4), is generated in interaction, and is at best taken-as-shared by community members. To identify how forms serve certain functions and how form-function relations shift over time, we perform microgenetic and ontogenetic analysis of the concept of "eigen" over a period of 19 class sessions.

Microgenesis is the process by which individuals construct representations by tailoring forms to serve functions that accomplish goal-directed activity (Saxe et al., 2015). This often occurs in public displays during class and contributes to the alteration of a common ground. Individuals are enabled and constrained by the common ground as they create new form-function relations. They can use familiar forms to serve new functions or recruit new forms to serve existing ones. Saxe et al. (2015) referred to the use of familiar forms or functions as continuity and the use of new forms or functions as discontinuity. *Ontogenesis* is characterized by shifts in microgenetic displays. An ontogenetic analysis involves an "analysis of continuities and discontinuities as individuals reproduce and alter form-function relations" (Saxe et al., 2015, p. 13).

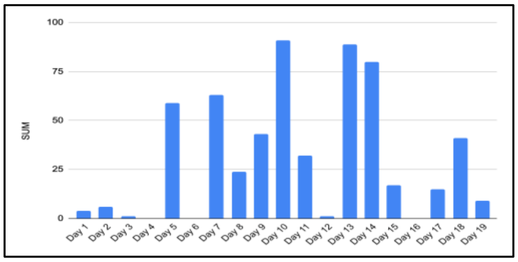
Methods

The data analyzed in this paper come from an in-person, senior-level Quantum Mechanics course taught in a medium, public, research-active university in the northeast US. The semester-long, 3-credit course met three times a week for 50 minutes per class. The first nine weeks (23 days) of class sessions were video recorded, with a focus on capturing the professor and whole-class discussions. The professor was an experienced quantum mechanics instructor and physics education researcher. The course had 17 students and used McIntyre et al. (2012) as its textbook. The data sources were video recordings and associated transcriptions. Only exchanges that occurred with the entire class (as compared to small groups) were analyzed.

We imported transcripts into MaxQDA, which is a qualitative and mixed methods data analysis software (Kuckartz & Rädiker, 2019). We inductively coded (Miles et al., 2013) of both transcript and screen captures of slides and boardwork. We coded together while watching the videos, discussing our codes, and resolving inconsistencies as needed. We coded the instances in which "eigen" was explicitly used, as well as instances implicitly related to eigentheory. The implicit uses occurred when words such as state, value, or $\hbar/2$ were used in ways compatible with eigen-related interpretations. Our coding system had a nested organization according to form. At the primary level, main forms were separated according to what was characterized by "eigen" (e.g., eigenstate, eigenbasis). The second organization level separated main forms into the specific forms that existed in the data. For instance, the form "eigenvector - verbal" was assigned when "eigenvector" was said, the form "eigenvector - written" when "eigenvector" was written on the board or on a slide, and the form "$|+\rangle$ - symbolic" when the symbol $|+\rangle$ was used to convey eigenvector meaning. The final level of coding corresponds to the function that we interpreted the form to have when the form-function relation was detected. For example, any of the aforementioned eigenvector forms could have associated functions such as being an element of a basis that diagonalizes a spin operator or being part of the computation for determining the probability of measuring a certain spin value. If an instance involved more than one form – for instance, both saying and writing "eigenvector" – the same function was assigned twice; this resulted in two form-function pairs for that instance. We started preliminary ontogenetic analysis by examining how forms, functions, or form-function pairs were used throughout the data.

Preliminary Results

At the time of this preliminary proposal submission, we have coded the whole-class discussions of 19 consecutive class sessions for a total of 576 instances in which a form-function relationship for the concept of "eigen" was communicated. See Figure 1a for the distribution of those instances over the days. We have a small amount of remaining refinement to do (codes from Day 6 and some of Day 9 need to be added, Day 10 needs a final confirmation). We will complete this refinement prior to the conference and will integrate it into our presentation.

(a) 

(b)

Main Form	Number of unique form-function pairs in the secondary form category			Number of unique form-functions pairs in the main form category
	Written	Verbal	Symbolic	
Eigenvalue [188]	13	60	32	105
Eigenvector [73]	15	27	9	51
Eigenstate [229]	23	59	36	118
Eigenequation [65]	5	13	40	58
Eigenbasis [13]	0	8	1	9
Eigen(misc) [8]	0	8	0	8
	Total unique form-function pairs			349

Figure 1. Bar chart indicating the frequency per day of the instances in which a form-function pair for the concept of "eigen" was communicated (a), and summary of the various types of form-function pairs (b).

The form-function pairs are summarized in Figure 1b according to the six main forms of eigenvalue, eigenvector, eigenstate, eigenequation, eigenbasis, and miscellaneous uses (e.g., eigen-esque-ish, eigen math, eigen-ness). The numbers in the square brackets, which sum to 576, indicate the total number of times a form-function pair occurred in the data. The middle three columns organize unique form-function pairs with specificity regarding the form. For example, there were 32 different functions that were associated with a symbolic form accompanying "eigenvalue." For instance, the symbol a_n could be the form used to symbolize the eigenvalue in an eigenequation, the eigenvalues along the diagonal of a diagonalized matrix, or the expectation value of a measurement corresponding to an eigenstate; these are three of the 32 functions for the form "a_n - symbolic" within the Eigenvalue row / Symbolic column. On the other hand, the form "E_n - symbolic" could also pair with these functions in the energy context, such as symbolizing the nonzero entries of a diagonal energy Hamiltonian; that form-function pair is a fourth of the aforementioned 32 form-function pairs. In total, there were 349 unique form-function pairs.

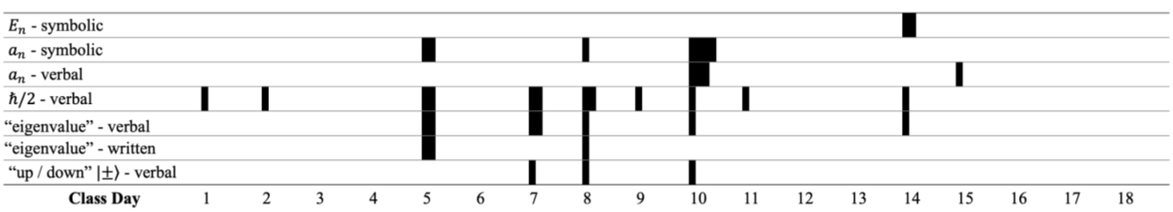


Figure 2. The dispersion of forms with the same function of *being the result of a measurement*.

Functions Associated with Several Different Forms

We performed a preliminary ontogenetic analysis by documenting the instances that different forms were used to serve the same function and determining the dispersion of those forms over time throughout the 19 class days. For example, conveying *the result of a measurement* (Fig. 2) and *a structure inherently tied to an operator* (Fig. 3) were two commonly-used functions, each associated with several forms in our dataset. The dispersion of the forms that served the function of *being the result of a measurement* is shown in Figure 2, and the dispersion of the forms that served the function of *being a structure inherently tied to an operator* is in Figure 3. The width of the black bar conveys the number of times the form occurred on that day (e.g., in Figure 2, the a_n symbolic form occurred twice on day 5, once on day 8, and four times on day 10).

We identified seven forms that were each associated with the function of *being the result of a measurement* (see Figure 2). This function was conveyed during ten of the class days, spread between days 1 and 15. Furthermore, in one class session (day 10), the class community used five of the seven different forms to serve this same function (see Figure 2). This illustrates how the class community associated the different eigenvalue forms of "a_n - symbolic," "a_n - verbal,"

"$\hbar/2$ - verbal," and "eigenvalue – verbal," and the eigenstate form of "up/down $|\pm\rangle$ - verbal" with the same function all in the span of one class session. Furthermore, we identified 15 forms that were each associated with the function of being *a structure inherently tied to an operator* (see Figure 3). By this function, we mean that the speaker was conveying some sort of structural quality imbedded in or inextricable from the operator. For example, a student claimed that certain given states "are eigenstates of the operator." This conveyed, in the "eigenstate – verbal" form, that eigenstates belong to or are a property operators, an inherent relationship between eigenstate and its operator. This function was conveyed in ten of the class days, spanning from day 5 to day 18, which illustrates a wide dispersion of the use of this function during class. On both day 7 and day 14, there were five different forms used that were all associated with this same function. Overall, these findings from the preliminary ontogenetic analysis illustrate how the class community used several different forms to convey the same function, even within the same class day. The recurrent use of the same functions over time demonstrates the continuity in the class community's use of those functions. The varied forms associated with the same function gives evidence to discontinuities present in the community's communicative displays over time. The class members altered existing form-function relations as they tailored different forms to serve the same function. Our remaining ontogenetic analysis will involve additional fine-grained examination of change over time; for instance, how the eigenequation itself changed form across contexts (e.g., abstract, spin, energy, position) but what threads of continuity exist and were leveraged throughout the semester as the classroom engaged with quantum mechanics.

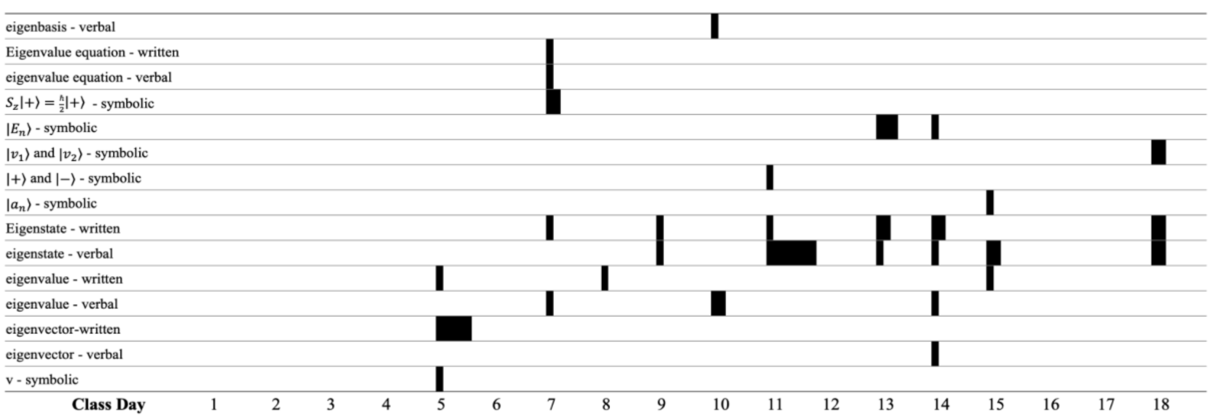


Figure 3. The dispersion of forms with the same function of *being a structure inherently tied to an operator*.

Conclusion

In this study, we are investigating a quantum mechanics class community's development of eigentheory over a span of 19 class days. We performed microgenetic and preliminary ontogenetic analyses (Saxe, 1999) of form-function pairs that the class used to communicate about eigentheory concepts. We plan to continue this analysis, as well as examine how these form-function relations are reproduced and altered over the remaining class days in our data set that address content related to wave functions. At the conference, we will engage the audience with discussion questions related to the pros and cons for various axial coding choices and the grain size of our codes, as well as solicit feedback specific to working with longitudinal classroom data and communicating about data related to both mathematics and physics.

Acknowledgments

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Zone Theory Exploration of Teaching to Foster Mathematical Creativity in a Coordinated Calculus

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We report on experiences of two Calculus 1 instructors who implemented teaching practices geared towards fostering students' mathematical creativity. While teaching a coordinated Calculus 1 course, these instructors participated in weekly professional development sessions on such teaching practices. Our preliminary analysis of participants' experiences describes how they navigated between the coordinated course structure and the added teaching practices discussed in the professional development. We utilize a zone theory lens to gain insight into participants' navigations in their professional environments, observing tensions between existing zone of proximal development and the zones of free movement and promoted action complex.

Keywords: Calculus, coordination, instructional practices, mathematical creativity.

The inclusion of mathematical creativity as a learning outcome in mathematics courses and programs has been discussed in numerous research studies and curriculum-standard documents (e.g., Askew, 2013; Levenson, 2013; Schumacher & Siegel, 2015; Silver, 1997; Sriraman, 2009). Despite these calls, the growth of explicit curricular work on creativity in mathematics courses, and professional development on teaching to include mathematical creativity at the tertiary level, has been slow. Recent work at the tertiary level has been limited to pre-service teachers (e.g., Bicer, 2021), mathematics majors in mathematics seminars (e.g., Mayes-Tang, 2020) or upper-division courses beyond Calculus (e.g., Omar et al., 2019). Thus, our overarching goal is to challenge dominant ways of teaching undergraduate mathematics by adding a focus on students' mathematical creativity and explicit teaching strategies to foster it.

In this preliminary report, we first describe our work with Calculus instructors. We discussed strategies to implement Calculus tasks geared towards fostering mathematical creativity in a semester-long online Professional Development (we refer to this professional development as C-PD here). Our project focused on Calculus 1 as this course plays a critical role in students' persistence in science, technology, engineering, and mathematics (STEM) programs (Rasmussen et al., 2019) and creativity is often overlooked in this course (Ryals & Keene, 2017). We also share results of preliminary analysis of data from the qualitative case study designed to explore the lived experiences of Calculus 1 instructors' implementation of creativity-based tasks (El Turkey et al., 2020) and teaching strategies to foster their students' mathematical creativity within a coordinated Calculus course. Our focus on coordinated Calculus 1 course was intentional - course coordination is considered a driver for instructional change (Rasmussen et al., 2019), and we believe it is timely to explore individual instructors' experiences within such structures. The research question of this study is: What are the experiences of Calculus 1 instructors in a coordinated Calculus course when utilizing teaching strategies to foster mathematical creativity?

Conceptual Framework

We situate our work on fostering students' mathematical creativity within the Developmental theoretical perspective (Kozbelt et al., 2010). The primary assertion of this theory is that creativity develops over time, and the main focus of investigation is a person's process and actions as opposed to the product created by them. Within this theoretical framing, we consider mathematical creativity as a process of offering new solutions or insights that are unexpected for the student with respect to their mathematics background or the problems they could have seen before (Savic et al., 2017). Fostering mathematical creativity with this conceptualization requires instructors to take intentional actions to engage students in the creative process of problem solving (Sriraman, 2005, Cilli-Turner et al., 2019).

In Tang et al. (2022), we examined teaching actions that students reported as contributing to their sense of creativity in their Calculus 1 course during which the instructors participated in C-PD. The student-reported teaching actions focused on designing and implementing creativity-based tasks using inquiry-oriented actions, in addition to holistic actions attending to students' affect. These teaching actions centered around students' lived experiences in these Calculus 1 courses, whereas in this paper, we shift our focus to the instructors' lived experiences implementing creativity-based tasks and teaching strategies in a coordinated setting.

We utilize a theoretical model proposed by Goos and colleagues (Goos, 2009, Goos et al., 2007; Goos & Bennison, 2019) to examine teacher learning and development, which applied and adapted concepts from Valsiner's (1997) zone theory. As this model centers its constructs in socio-cultural theory of teaching, it provides opportunities to examine our participants' experiences in their context (i.e., one institution's culture) and through their interactions with other C-PD participants while trying to utilize the C-PD materials.

Goos et al. (2007) describe three zones, each of which focuses on different aspects of teacher learning and development: the *Zone of Proximal Development*, the *Zone of Free Movement*, and the *Zone of Promoted Action*. Adapted from Vygotsky's (1978) work, the *Zone of Proximal Development* (ZPD) refers to the possibilities for developing new teacher knowledge and beliefs on content and pedagogy. The *Zone of Free Movement* (ZFM) focuses on "constraints and affordance within the professional context" (Goos et al., 2007, p. 26). It explores the environmental and contextual factors that allow or hinder certain actions. For example, teachers' views of curriculum design and assessment methods, and their connections to institutional culture and so forth are part of the ZFM. The third zone, the *Zone of Promoted Action* (ZPA), "represents the activities, objects, or areas of the environment in respect of which an individual's actions are promoted" (Goos & Bennison, 2019, p. 408). For example, in a department that focuses on active-learning strategies (i.e., institutional norm), an instructor may utilize teaching practices learned in a professional development on inquiry-based learning, which represents ZPA.

Overall, Goos et al. (2007) observed that these zones complement each other in the effort to describe teacher's learning and development. Goos et al. state, "For teacher learning to occur, professional development strategies [ZPA] must engage with teachers' knowledge and beliefs [ZPD] and promote teaching approaches that the individual believes to be feasible with their professional context [ZFM]" (p. 26). Also, the ZFM and ZPA are stated to be "dynamic and inter-related and form a ZFM/ZPA complex" (Goos & Bennison, 2019, p. 408) which plays an important role on learning and development of teaching strategies. For our study, this zone theory approach provides a structure to better understand the experiences of participants' teaching to foster mathematical creativity in a coordinated course environment.

Methods

We report on two instructor-participants at a university in the Northeastern United States. This works falls within a larger research project investigating students' mathematical creativity and the ways in which it can be fostered through teaching of Calculus I. Even though the larger research project consisted of 3 cohorts with 14 instructors total, we focused on only two participants, as they were in the same cohort and at the same institution in which the course was coordinated.

The two instructors chose their pseudonyms, and we also report their self-identified pronouns, genders and race: Farah Nurhalizah (They, Male, Asian) and On the 1s and 2s (She, Female, Black). These instructors formed the case for this study. At the time of the C-PD, Farah was a full-time instructor and On the 1s and 2s was an assistant professor, and both were in their first year teaching at this particular institution. Both participants had taught Calculus 1 more than 5 times previously with 6-10 years of teaching experiences.

Both instructor participants attended weekly online professional development with members of the research team to discuss the different components of the project: creativity-based task design and implementation; implementation of a reflection tool for formative assessment (Karakok et al., 2020); pedagogical actions; and attending to students' affect. All participants were asked to implement two common creativity-based tasks in their courses and design at least four additional tasks to implement. Participants could decide how to implement these tasks according to their setting. During the C-PD, they discussed implementation strategies with the research team and other instructor participants. We collected various data: Calculus teaching artifacts from prior semesters, video recordings of all C-PD sessions, participants' entrance tickets to weekly C-PD, video recordings of teaching of creativity-based tasks, and end-of-semester interview with participants. In this report, we share some of our initial analysis of video-recorded interview data from each participant. We started our data analysis with the interview data to gain insights into participants' experiences to identify critical moments or experiences before engaging with other data sources. For this initial coding (Saldaña, 2013) as the first cycle in our analysis, the first two authors read through the interview transcripts to get analytic leads to explore the lived experiences of teaching Calculus while attending to the C-PD. From this coding, we started our second cycle with a deductive approach to structure the observed experiences through a zone theory with ZPD, ZFM, and ZPA constructs. By the time of the conference, we aim to complete interview data analysis and analyze participants' entrance tickets and conversations during C-PD sessions with ZPD, ZFM, and ZPA constructs to describe their experiences.

Preliminary Results

We first note that both participants implemented only two creativity-based tasks, due to interruptions in teaching modality caused by the COVID-19 pandemic in the middle of the semester.

Farah Nurhalizah (They, Male, Asian)

Farah described their teaching Calculus 1 experience through its "highly coordinated" structure. They referred to this structure as a new experience for them in which "you have the same exams, the same homework, and the same worksheets even." Coordination was the most significant feature of their environment (ZFM) and Farah seemed to believe that it hindered their access to their already-established ways of teaching Calculus. In addition, Farah thought "[i]t took a while for me to get used to it… And I think students could feel that I wasn't like totally buying into the structure."

As we examined Farah's reported experience, we noticed that their negotiation between the recommended focus on conceptual understanding from the coordinator (ZFM) and the newly added focus on creativity from C-PD (ZPA) created a tension with their existing beliefs on teaching Calculus (ZPD). When asked how they could have adjusted pre-designed worksheets to incorporate fostering creativity tasks, Farah stated they would have "underemphasized the conceptual understanding as much…And…sort of do more creativity instead of conceptual." For Farah, conceptual understanding questions on worksheets required "theoretical foundation," definitions, and some proofs, which the course did not design to deliver them. For their section of the course, Farah "snuck in…the epsilon-delta definition," that they thought they were not supposed to do. The inclusion of definitions and some proofs for Farah was about being "intellectually honest," and relates to their existing beliefs of mathematics and teaching mathematics (ZPD).

As Farah tried to reconcile their existing teaching Calculus practices with these pre-designed worksheets and the coordination structure, they identified some aspects of the worksheets that could potentially foster mathematical creativity (ZPA). Incorporating their experience with C-PD on features of creativity-based tasks (El Turkey et al., 2020), they mentioned "some of the worksheet problems…had like some open-endedness" to them. Farah further noted that not all students got to these questions nor was there an emphasis to discuss or do these questions in class (ZFM). However, Farah thought that there was a possibility to foster creativity if they had a chance to discuss these questions. These observations highlight the dynamic and inter-related relationship of ZFM and ZPA in Farah's experience (ZFM/ZPA complex) (Goos, 2009). Furthermore, Farah's proposed method of using such worksheet problems through more flexible class discussion speaks to their developing ZPD on teaching Calculus 1 to foster mathematical creativity and navigation through the tension they experienced between their existing ZPD and the professional environment (ZFM).

On the 1s and 2s (She, Female, Black)

On the 1s and 2s's experience teaching Calculus 1 in a coordinated Calculus to foster creativity highlights explicit aspects of the ZFM/ZPA complex. Speaking about her experience teaching coordinated calculus (ZFM) and participating in C-PD (ZPA), she thought her teaching did not necessarily change in the current course, but this C-PD experience changed "how I think about calculus and encouraged me to include more…creativity tasks" in her future courses.

For On the 1s and 2s, one of the important differences between the coordinated worksheets (ZFM) and the creativity-based tasks (ZPA) was the goals of these tasks. She identified one of the goals of the creativity-based tasks as "what different ways can you think about this problem" or "what different ways can we as a group think about this problem" whereas the worksheet problems tended to have a goal of "can you get this answer?" With this noticing, and her interactions with Farah and other participants of the C-PD (ZPA), she was able to change some elements of the professional environment (ZFM). For example, they (all Calculus instructors, including Farah and the course coordinator) used one of the creativity-based tasks in their weekly check-in assessment. This task asked students to create a function where the limit exists at -1, but not continuous. Even though all instructors did this task in the same format on an assessment, On the 1s and 2s followed up with students, after the assessment, with a whole-class discussion on their solutions. As she believed her class "is a life that grows and changes and you have to adjust," explicit discussion in C-PD on how to design and implement creativity-based tasks provided her opportunities to envision how to navigate in her environment (ZFM).

Even though she and Farah were teaching their own sections of the same coordinated Calculus 1 course, her experience with the worksheets or the coordinated structure (environmental constraints) mostly interacted with the ZPA to form a ZFM/ZPA complex that informed her evolving ZPD. These ZFM/ZPA negotiations played a role in On the 1s and 2s's teaching experience for mathematical creativity.

Conclusion and Questions

Coordinated courses have opportunities to create coherent experience for students taking Calculus 1 and subsequent courses at the same institution. For instructors, coordinated structure could provide several opportunities: collaboration among faculty with shared objectives, professional development, and building of common tools and resources. Furthermore, Williams et al. (2022) claim that course coordination can be "a vehicle for shifting departmental culture" (p. 126). However, some instructors may feel loss of autonomy and flexibility in their teaching. This loss of autonomy relates to 'coordinated independence' construct, which is "intended to embrace how in-step elements of a calculus program work together with elements that allow for individual autonomy" (Rasmussen & Ellis, 2015 p. 114). As many institutions use or aim to use coordination for Calculus courses, it is important to explore the coordinated independence construct. This construct could allow us to understand the barriers and opportunities to create a coherent course design for students.

In this study, we examined instructor's experience of implementing teaching practices from a professional development in a coordinated structure. Our preliminary analysis results relate to how two instructors experienced "coordinated independence" in slightly different ways through a ZFM/ZPA complex. For Farah, this complex created a tension with their existing ZPD. Farah's existing ZPD was more aligned with the ideas promoted in C-PD (ZPA) than the coordinated calculus structure (ZFM). On the other hand, On the 1s and 2s identified certain aspects of the C-PD (ZPA) to make changes in her professional environment (ZFM) and had a vision to do more in her teaching practices (developing ZPD). These experiences, with their differing tensions and negotiations, may explain formally why it is difficult for teachers to incorporate creativity into their classrooms, particularly in a coordinated setting. We must acknowledge the unique nature of the setting in our analysis – COVID-19 forced many to use pedagogical tools that were not in their ZPD at all. In our future work, we will continue to explore these areas as they relate to the coordinated settings and instructors' developing ZPD.

We are interested to learn from audience:

1- What strategies do instructors use when they experience tensions between their zones of development?
2- What strategies do professional development facilitators need to adopt when instructor participants experience such tensions?

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"Well, it's obvious": Students' Experiences with Mathematical Microaggressions

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Mathematical microaggressions (MM) refer to the subtle ways in which mathematical authorities use language, behavior, and assumptions that communicate negative messages to students that they do not belong in mathematics (Su, 2015). This study analyzes the reflections of 173 undergraduate mathematics students who were asked to reflect on an article (Su, 2015) about MMs. Findings show that students experience different types of MMs, including microslights, microinsults, and environmental microaggressions. Our results indicate that female students were more likely to report experiences with MMs than male students. This research supports the need to investigate this phenomenon further and to develop initiatives at departmental and institutional levels to encourage more inclusive spaces in math classrooms.

Keywords: Mathematical Microaggressions, Microslights, Microinsults, Sense of Belonging

Microaggressions (MA) have been characterized as the intentional or unintentional indignities that communicate "hostile, derogatory, or negative" messages that target a person and/or their marginalized group (Sue, 2010). Within the realm of education, the prevalence and effects of MAs have been studied in relation to students' sense of belonging. Sense of belonging, in a university context, refers to the perceived social support and the experience of mattering or feeling cared about, accepted, respected, valued by, and important to the classroom and campus community (Strayhorn, 2018). Recent scholarship has revealed that MAs are prevalent in classrooms and inhibit students' sense of belonging, especially for students coming from marginalized groups (Solórzano et al. 2000; Yosso et al., 2009). As an example, Lewis et al. (2021) reported that a higher frequency of racial MAs significantly predicted lower overall sense of belonging for students of color, and similar connections between MAs and sense of belonging can be found in studies centered on women (McCabe, 2009).

Microaggressions have been well reported as widespread in academic spaces and detrimental to student outcomes. Within STEM departments, which already face issues in representation from marginalized communities, reports of MAs are distinctive. Students of color, especially Black students, report racial MAs from STEM instructors, advisors, and peers (Lee et al., 2020). Further work shows that the effects of MAs can be magnified in STEM spaces. Leyva and his colleagues (2021) report how discouraging classroom practices, many of which can be characterized as MAs, had discriminatory impacts on women and students of color. Marshall et al. (2021) noted that in the context of STEM, MAs hinder students' academic success and ability to participate in the academic pipeline, especially for students historically excluded from the sciences on the basis of race or ethnicity. MAs directly impact students' sense of belonging in STEM, and more needs to be done at an institutional level to better support student retention and sense of belonging within STEM fields (Lee et al., 2020; Rainey et al., 2018).

Our paper is framed through the subtle layers of microaggressions and how they can be received by mathematics students (Marshall et al., 2021; McKenna et al., 2021; Su, 2015). As an extension from discussions of racial microaggressions, Francis Su (2015) used the term *mathematical microaggression* (hereby labeled as MM), which refers to the subtle ways that mathematical authorities (such as instructors, classmates, or textbook authors) communicate that one does not belong in mathematics. Su offered examples such as “It is obvious/clear/trivial that…”, “The rest is just algebra,” and “There’s a trick for doing this” and elaborates that such comments can convey negative messages towards students (e.g., their knowledge is lacking, their questions are unwelcome, their potential in mathematics is limited). In order to highlight the undeliberate intentions in which MMs are often exchanged in math classrooms, he used the term *microslights*, which are defined as unintentional comments or language in which the aggressor is often unaware could be exclusionary. Accounting for more intentional and/or aggressive MMs from instructors toward students, McKenna et al. (2021) define *microinsults* as “intentional, subtle snubs” that can “demean one’s racial heritage or identity,” which we extend in the case of MMs to cover abrasive exchanges that targets students’ mathematical identity and perception of ability. Marshall and her colleagues (2021) define *environmental microaggressions* (EM) as relating to MAs that exist in the environment in which a person exists, not directly received by an individual. We adapt Marshall et al.’s (2021) definition of EMs to include any MM that exists in the experience of learning but is not inflicted by any one person to another person directly. For example, an EM could be witnessing peers understand a topic quickly or a professor moving quickly through a mathematical explanation. Because a student understanding a topic quickly is not done purposely to harm another person, we categorize this MM as subtly different than microslights or microinsults, which are more directed towards individuals.

Our study focuses on the MMs that mathematics students reflect on experiencing throughout their learning. This paper focuses on the following three research questions: 1) How do mathematics students describe experiencing MMs? 2) What types of MMs do students experience? 3) How do students’ experiences with MMs vary by gender?

Methods

This study took place at a large public university on the West Coast of the U.S., designated as a Hispanic-serving Institution. Data for this paper were collected between Fall 2019 and Spring 2022. The sample included 173 participants enrolled in calculus I (47 students in two sections) or abstract algebra (126 students in four sections), taught by the same instructor (Author 4). Course modalities included both in-person and virtual instruction. These classes incorporated inquiry-based learning and active learning and provided a significant amount of time for student collaboration and discussion during class instruction. Based on institutional data of the 173 students, 42.8% were female and 57.2% were male. 50.9% of the students were Latine[1], 3.5% were Black, 18.5% were AAPI, 13.3% were white, one student was Native American, 6.9% were undocumented[2], 3.2% two or more races, and 4% where race was unknown.

Throughout the course, students were asked to submit reflection assignments. In one of these assignments, students were asked to read Su’s (2015) article on MMs, to reflect on the paper, and to describe if they have ever been made to feel like they do not belong in mathematics. Students could reflect on any moments in their math experience. The instructor created this assignment as part of a series of assignments for the purpose of creating a more inclusive classroom space for collaborative learning. The 173 reflections form the data set for this study; each reflection was given an identifier that included the course and students’ gender, race/ethnicity, and major.

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[1]*Latine* is a gender-neutral replacement of the term Latino. It has been used as a more linguistically natural alternative to Latinx or Latin@ for Spanish-speakers (Celis Carbajal, 2020).
[2] We note that “undocumented” is not a race or ethnicity, however, the university uses this category.

The data were analyzed using constant comparative methods (Corbin & Strauss, 2008). The first round of coding utilized deductive coding. All de-identified reflections were initially coded by all authors to identify microaggressions that students experienced in STEM, specifically mathematical, racialized, and gendered microaggressions. For purposes of this paper we focused on MMs and therefore analyzed these data in the second round of coding. First, we coded for *Who Received a MM* with four sub-codes emerging: i) student has directly received/experienced an MM, ii) student has never received/experienced an MM, iii) student has not directly received/experienced an MM but can imagine how it could affect others, and iv) student has inflicted an MM on another person (reflections could be coded for more than one sub-code). Next, for those students who indicated directly receiving an MM, we coded for *Type of MM discussed* and *Who/what inflicted the MM*. Finally, we coded for *Pushback to the concept of MM*. Pushback could include a student stating that MMs were not real or a student explaining away the MM by claiming that they themselves lacked discipline or knowledge or should be a better student. A third round of coding commenced where authors grouped types of MM directly received and labeled each as a microslight (Su, 2015), a microinsult (Mckenna et al., 2021), or an EM (Marshall et al., 2021). After each round of coding, the authors cross-coded a subset to verify validity of application of codes. Authors met to discuss any disagreements of codes.

Findings

From the 173 student reflections, 69.9% indicated having directly received a MM, 4% have never directly received a MM, 20.8% could imagine how MMs can negatively affect others, and 6.4% indicated that they have inflicted MMs on others. Students who directly received an MM (N=121) indicated 11 distinct types of MMs (see Table 1); 65.3% of students indicated a microslight, 16.5% indicated a microinsult, and 19.8% indicated an EM.

From the 121 students who received an MM, most indicated having received microslights, with 40.5% of students indicating the MM “It’s obvious/trivial/easy”. Representative of other entries, Edwin, a Black male student in abstract algebra said, “I heard…a professor say that a proof to a lemma was obvious. I didn’t ask about it, although I didn’t understand it, because I didn’t want to look dumb in front of my peers for not seeing an obvious proof”. Other students described similar experiences. Subsequently, 14% of students reported being told “You should have learned this before”, and 9.9% reported hearing “The rest is just algebra” or not being given enough wait time. Typical of “No wait time” examples, Lianne, a white female student in abstract algebra described experiencing professors say, “Does anyone have any questions?” with a very short pause before moving on. “Sometimes it takes a few seconds to build up the courage to ask a question, but when the teacher moves on too quickly it makes you wonder, ‘How does everyone else understand but me?’”. The remaining three microslights represented less than 10% of student responses and will not be discussed here. 16.5% of the students reported hearing the microslights that Su (2015) wrote in his paper, but did not specify the type.

Table 1: Frequencies by type of mathematical microaggression students reported directly receiving (N=121)

Type of Mathematical Microaggression	Frequency (%*)
Microslight	**79 (65.3%)**
It’s obvious, trivial, easy	49 (40.5%)
You should have learned this before	17 (14%)
“The rest is just algebra”	12 (9.9%)
No wait time	12 (9.9%)
Ignoring/dismissing questions, “see me after class”	8 (6.6%)

“The concept is not that bad”	6 (5%)
There is no such thing as a dumb question	6 (5%)
Microinsult	**20 (16.5%)**
Belittle/shame student	12 (9.9%)
Not believing in a student, “You’re not good enough”	8 (6.6%)
Environmental	**24 (19.8%)**
Seeing others get it quickly	18 (14.9%)
Professor moves too quickly/Skips Steps	7 (5.8%)
MM not specified	**20 (16.5%)**
Other	**11 (9.1%)**

*Note that totals are greater than 100% as some students reported more than one MM

Students reported experiencing microinsults less often than microslights; 70% of microinsults were reported by students in abstract algebra. Students discussed instructors belittling them or shaming them (9.9%) or not having an instructor believing in their ability to do mathematics (6.6%). For example, Reese, a Latino male student, felt belittled during class when he wrote a problem that he struggled with on the board and recounted the instructor’s response. “When she got to the problem she said, ‘I’m not even going to do this problem because it’s so easy, and if you need help on this problem you’re in the wrong class’”. This made Reese feel embarrassed and less intelligent than the other students in his class. Students described two ways of experiencing EMs; 14.9% of students indicated feeling like they did not belong when they saw other students understanding the material quickly while 5.8% of students described feeling excluded when instructors skipped steps or moved too quickly through the material.

Students who directly received a MM indicated receiving them from professors (63%), K-12 teachers (11.5%), peers (16.5%), and the textbook (6.6%). Alice, a female Asian American calculus student, explained how she experienced receiving MMs during class group work:

> There were times where I was in group work settings and I would ask a question and all the members would know how to do it. They would say, this is “obvious” and teach me briefly. It made me think that I am not qualified enough to be in this class because I was not able to learn my basics well enough.

Eleven students admitted to being a microaggressor toward a peer who was struggling with course material or students they were tutoring. All but two of these students were male. As an example, Douglas, a Latino male abstract algebra student mentioned, “I feel like [MMs are] very important as I’ve said some of these things to students I have tutored in the past. I didn’t personally consider the perspective of students who may be experiencing these same topics which I’ve been practicing for 3 to 4 years”. For most of these students, their reports of being microaggressors at a previous time were combined with reflections of their own teaching philosophy, newly informed by the potential negative effects of their previous comments.

Of the 173 reflections, 17 students (9.8%) demonstrated pushback against the discussion of MMs. Reasons behind pushback varied but largely arose from three perspectives toward MMs: i) MAs are unintentional and do not warrant emotional response, ii) MAs are an opportunity to call out a student’s shortcomings, and iii) discourse regarding MAs is a politicized diversion from addressing lack of effort. An example provided by Garrett, a white male calculus student, exemplified how MMs are an excuse for students to maintain poor effort:

> I feel that the use of microaggressions as an excuse to why a student feels like they are failing a math class is concerning. I feel that too many students do not put in the effort to

learn how to do math and refuse to ask questions which results in students blaming so called "microaggressions" on their failure.

All but one of the 17 students who demonstrated pushback to the idea of MMs were majors within the colleges of science and engineering. Five of the 17 students who displayed some pushback against MMs were female. We should note that these five students focused on the unintentional nature of MMs as a reason to push back and argued that they themselves may have been “too sensitive” when reacting to the MM.

There is evidence to show other findings when comparing gender. Female students more frequently experienced MMs than male students; 78.4% of female students (N=74) reported experiencing an MM compared to 63.6% of male students (N=99). Female students indicated higher experiences with microslights and EMs. A notable discrepancy occurs specifically with reports of microslights, where 52.7% of the female students reported having experienced microslights, compared to the 40.4% of male students. Male and female students indicated receiving microinsults at similar rates (12.1% compared to 10.8%). It should be noted that all students who indicated that they had never experienced a MM were male; none were female.

Discussion

Our study showed that a large proportion of students have directly experienced a MM. Most students experienced the “It’s obvious/trivial/easy” microslight. This microslight may have been discussed more than others as Su (2015) lists this in his paper as the first example of a MM. It is particularly notable that female students reported experiencing MMs more than male students. We believe that it is important for instructors to attend to the way these microaggressions acutely affect women. Findings indicate that the nature of MMs are carried on through students in their interactions with others. Microinsults are more severe forms of MAs, and even though they were less frequently referenced by students in their reflections, we need to consider how microinsults affect all groups of students. While students reported experiencing most MMs from instructors, students did indicate receiving them from peers, which is an important factor to consider when attempting to create a safe and welcoming classroom climate while utilizing active learning teaching techniques.

Types of MMs experienced align with previous findings, specifically that students do feel the impact of seemingly innocuous comments often found in mathematics classrooms (Leyva et al., 2021; Su, 2015). These MMs can directly impact a student’s sense of belonging in STEM, and is very impactful on marginalized students (Rainey et al., 2018), specifically women. This study points out that more research is needed to understand the ways both students and instructors experience and internalize MMs. It also suggests potential for including MMs in professional development for college math instructors, specifically examples of MMs and how they are received by math learners. We hope to learn from the RUME community ways that they acknowledge MMs with their students to form a more inclusive learning environment, as well as different ways to interpret our data.

While this study focused on understanding and categorizing the types of mathematical microaggressions students have experienced, we have begun preliminary analysis of the data in other ways. First, we continue to analyze the MM data for any differences by participant race and gender. While the Su (2015) paper focused on mathematical microaggressions, some student discussed racialized or gendered microaggressions within their reflections. Future analysis will explore the ways students talk about these important microaggressions and how they impact sense of belonging in math classrooms.

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What Are Instructors Doing to Initiate Refinements of Students' Definitions?

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A key feature of inquiry-based instruction is to develop formal mathematics by directly building on student thinking and contributions, which often requires engaging students in refining their work. In our experiences we have noticed that it is challenging for instructors to support this refining activity and so we see a need to better understand what instructors can do in-the-moment to initiate refinements. In this study, we investigated how three instructors across four courses guided their students through the refinement of one definition. We identified three different moves that these instructors used to support the definition refinements: (1) suggest an edit with implicit mathematical reasoning, (2) suggest an edit with explicit mathematical reasoning, and (3) ask a question to create a problematic situation. The first two moves focused on the refinement itself whereas the third move focused on a problem to motivate a need for a refinement. We discuss future research directions and implications of our work.

Keywords: Defining, intellectual need, teacher moves

A key feature of inquiry-based instruction is to develop formal mathematics by directly building on student contributions (Kuster et al., 2018; Laursen & Rasmussen, 2019; Rasmussen & Kwon, 2007). For instance, in many inquiry-based classrooms students are guided to reinvent formal definitions. To work with the students' contributions, the teacher supports students in refining their definition drafts so that their definition is equivalent to a standard definition given in textbooks. This is our intention with the curriculum materials that we are developing as part of a larger research project (ASPIRE in Math NSF IUSE #1916490). Our task sequences engage students in defining (Zandieh & Rasmussen, 2010) by first engaging them in a task that evokes a concept image (Tall & Vinner, 1981), then students and the instructor generate and negotiate examples and non-examples of the concept, and lastly they work together to refine an informal description of the concept towards a formal definition (see Vroom, 2020).

It is documented in the research literature that it is challenging for teachers to build on student thinking in general (Andrews-Larson et al., 2019; Johnson & Larsen, 2012; Speer & Wagner, 2009) and we have noticed this to be especially the case when refining students' definitions. In particular, students sometimes offer definition drafts that are inconsistent with mathematical norms and conventions since students are not only learning about the concept itself but also are in the midst of learning about formal mathematical language (Vroom, 2022). At the same time, formal mathematical language is highly technical and filled with nuances, but functions to articulate precise mathematical meanings (Halliday, 1978; Schleppegrell, 2007). Thus, we see that it is a non-trivial endeavor to support students to refine their definitions in such a way that they more effectively communicate the students' ideas to a broader mathematical audience.

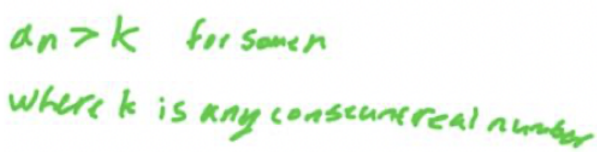
This study focuses on investigating what instructors do to support students in refining their definitions in such a way that supports them to abide by mathematical norms. Such an investigation will directly support our ongoing research project by informing instructor support materials for using our curriculum. This study will also add to the research literature on inquiry-based instruction by providing insights into how instructors can guide students to refine their drafts of definitions and potentially statements and proofs as well.

Theoretical Grounding

Instructors who teach with the ASPIRE curriculum materials are positioned as brokers (Rasmussen et al., 2009; Vroom, 2020; Zandieh et al., 2017) holding membership in the classroom community as well as having relevant knowledge about mathematical norms and concepts. In earlier work, Vroom (2020) described the role of a broker as they engage students in mathematical activity (e.g., defining) in which they aim to construct some product (e.g., a definition). In the case of defining, the broker engages in a cyclic process in which they iteratively aim to (a) make sense of a students' draft of a definition and its intended meaning in relation to how the mathematics community might interpret the draft and (b) support the students to edit the draft (in the case that it is not likely to be interpreted as the students intended). To support the edits, the broker can strategically share mathematical norms or build on the students' thinking, often creating an intellectual need (Harel, 2008; 2013) for refinement.

Harel claims that students should have an intellectual need to learn what we intend to teach them, where an intellectual need refers to a problematic situation that motivates the construction of the piece of knowledge (Harel, 2013). Thus, an instructor who aims to create intellectual needs for definition refinements would focus efforts on creating problematic situations for students. In the case of definition refinements, we see an intellectual need as referring to a problem with the draft that motivates both an edit and an understanding for why such an edit might resolve the problem. For example, the students in Vroom's (2020) study offered Figure 1A as their definition for a sequence that had "no upper bound". While explaining the meaning of the definition, the student described how it fit for the example in Figure 1B. To do so, he first sketched a horizontal line at k and then pointed out that he could find a sequence term that was greater than k (referring to the sixth term in Figure 1B). The teacher-researcher then attempted to create a problematic situation by pointing out that when the student described the definition with the example, he *first* introduced k but that his definition introduced n first and k last. Here, the teacher-researcher attempted to bring light to a problem: the structure of the definition did not match the structure of how he described his intended meaning with the picture. The teacher-research then explained that the student's meaning was clear when they explained it with the picture and suggested editing the definition so that it better fit his description. The student then altered the definition so that the variables were introduced in the same order as he introduced them with the picture, giving some evidence that the student experienced an intellectual need for refinement.

A.



$a_n > k$ for some n
where k is any constant real number

B.

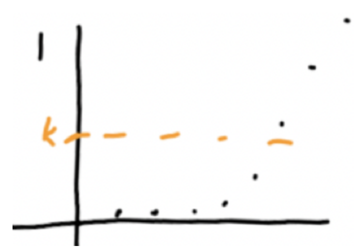


Figure 1. Students' draft of "no upper bound" definition.

Vroom's (2020) conceptualization of the broker emerged from a successful case study of a broker with a pair of students in a teaching experiment setting. This case was successful in the sense that the teacher-researcher worked with the students to refine the students' mathematical statements so that (1) the statements could be interpreted by the larger mathematical community as the students' intended and (2) the teacher-researcher seemed to create problematic situations that motivated definition refinements. Because the framework emerged in a laboratory setting, we were curious how (if at all) instructors in whole-class settings worked with students in a similar way. Specifically, in this study we aim to answer: what are instructors doing to initiate refinements of students' definitions?

Methods

For this study we investigated how three instructors across four courses guided their students through the refinement of one definition. The three instructors taught Introduction to Proof courses using curricular materials developed as part of the ASPIRE in Math project. One lesson of this curriculum focuses on mathematical language where students are guided to define several concepts and learn about the formal mathematical language that they used to do so. For this study, we focused on the instructors' moves during one of those defining tasks. In this task, students create a formal definition of a sequence they term "Eventually Constant". Students typically describe this sort of sequence as one that at some point gets "stuck" or "becomes constant". While this can be defined formally in several ways, one common formal definition is "A sequence x_n is eventually constant if there exists an $N \in \mathbb{N}$ such that for all $n > N$, $x_n = x_N$." After students develop a shared informal description of the concepts (e.g., a sequence that will get stuck at some point), students work in small groups to draft a definition. Then, students are guided to refine their draft(s) in a whole class setting. We use data from four courses, three of which were university courses and one a community college course. All four courses were conducted remotely with synchronous meetings over Zoom and with students using Google Docs during their group work. Data was collected using screen recordings of the class sessions, and small group work was captured by a researcher entering a breakout room to record the students' work. The focus task occurred over one day of the classes.

We began our analysis by watching the video of each of the defining sessions (four instructional days from the four classes). During this viewing we identified each moment that the instructor made some move to initiate a refinement of the definition. For each move we made note of the definition draft that was being refined and asked ourselves "What did the teacher do/say/ask during this time?" and "Do we see this as an attempt to create a problematic situation? Why or why not?", documenting our answers. We then documented what (if at all) refinement was made as a result of this move. As we documented each move, we constantly compared to previous moves to make note of any similarities or differences. In total we identified 15 teacher moves from the three instructors and through this constant comparison we categorized the 15 moves into three types: (1) Suggest an edit with implicit mathematical reasoning, (2) Suggest an edit with explicit mathematical reasoning, and (3) Ask a question to create a problematic situation.

Results

In what follows, we describe the three categories that we identified for what instructors were doing to motivate a refinement.

Suggest an edit with implicit mathematical reasoning

We identified instances from all three instructors where they suggested an edit with implicit mathematical reasoning (eight total instances). This type of move happened when the instructor highlighted a refinement that seemed (to us) to have mathematical value but the instructor did not explicitly state its value to the students. For instance, in one whole class discussion, an instructor highlighted a group's draft of their definition with the property: "there exists an a such that $x_a = x_{a+b}$ for all $b \in \mathbb{N}$". She read the property out loud and asked "what type of number does a have to be?" and added "we should probably indicate what type of value a is". After a student quickly responded "a natural number", the instructor agreed and edited the property by replacing "there exists an a" with "there exists an $a \in \mathbb{N}$". We see this instance of the instructor suggesting a particular edit that communicated that a is a natural number presumably because it is normative to not only quantify variables but also indicate what set the variable belongs to. This reasoning was left implicit since the instructor only emphasized the edit ("indicate what type of value a is") rather than also highlighting why this edit was advantageous.

Suggest an edit with explicit mathematical reasoning

We identified six instances (from two instructors) in which instructors suggested an edit with explicit mathematical reasoning. This type of move happened when the instructor highlighted a refinement and explained why the refinement had mathematical value. For example, one instructor was in a breakout room with two students who wrote the following draft definition in their Google Doc: "The sequence Xn is eventually constant if there exists X_m in X_n such that X_m=X_n for every n>m." The instructor said:

> "so the one thing I would say is usually when you declare a variable with 'there exists' or 'for alls' with sequences a lot of times you'll just do the subscript only. Like when you have X_m in X_n. [...] Here you say X_m in X_n, usually we don't do that, the reason why is that it sort of abuses the notation a little bit. On the one side you're thinking of a specific term sequence and on the other side you're treating it as a set. So that's a little bit of an icky way to do it."

We see this move as suggesting a particular edit ('there exists m in N' instead of 'there exists X_m in X_n') and the instructor explicitly shared the reasoning for that edit. In particular, the students' draft used sequence notation in two different ways: X_m represented a sequence term and X_n represented the sequence itself.

Ask a question to create a problematic situation

We identified only one instance in which an instructor asked a question to create a problematic situation with the students' drafts. In this instance, the students were in breakout rooms tasked with drafting a definition in a shared Google Doc. One group's first attempt read: "The sequence Xn is eventually constant if, eventually, there is a number in the sequence that repeats itself infinitely many times." The instructor added a comment to their definition that said: "Does this allow something like 1, 2, 1, 2, 1, 2, 1, 2… where 1 and 2 are repeated infinitely many times?" We see this as attempting to create a problematic situation because he highlighted a non-example of a sequence for which the property was true. Specifically, the sequence 1, 2, 1, 2, 1, 2, 1, 2… has two numbers (1 and 2) that repeat themselves infinitely many times but this sequence is not eventually constant since it never "gets stuck". This problem could motivate a need to refine the property so that it does exclude all non-examples.

Discussion

We found that the instructors in our study implemented three different moves to support definition refinements for the case of Eventually Constant Sequences. Two of these moves focused on the refinement itself (*suggest an edit with implicit mathematical reasoning* and *suggest an edit with explicit mathematical reasoning*) whereas the third move focused on a problem to motivate a need for a refinement (*ask a question to create a problematic situation*). We do not intend to claim that one move is always preferable. For instance, it might be appropriate for an instructor to suggest an edit with implicit mathematical reasoning when, for instance, the reasoning has been established and taken-as-shared within the classroom community. However, we conjecture creating problematic situations with drafts of students' definitions to be advantageous in many cases. Guided by Harel's (2008; 2013) theoretical perspective, we anticipate that focusing on the problem a refinement can solve rather than the solution itself could better support students in understanding the nuances of the formal mathematical language.

This study is an ongoing project and our results here motivated some additional questions that we wish to further explore. To start, we wonder about the context of the task. In our experience we have seen that students often create an informal definition of the Eventually Constant Sequence concept that structurally parallels a formal definition. Thus it is often the case that any needed refinements are less substantial or more stylistic. This context could explain why we did not see many instances of the last category (*ask a question to create a problematic situation*) especially in comparison to Vroom's (2020) case study. Hence, we wonder whether different defining situations afford more opportunities to create problematic situations. We are also aware that the type of move that an instructor might use to initiate a refinement could be influenced by previous classroom activities. For instance, as we mentioned above, an instructor could decide to keep the mathematical reasoning implicit since it had been established and taken-as-shared within the classroom community. With that in mind, we think it is important to also pay attention to the order the instructors use the moves and what, if any, previous norms had been established. We plan to explore these questions in future research.

Our results also suggest some practical implications. A majority of the instructor moves fell into the *suggest an edit with implicit mathematical reasoning* category. While we have discussed that this move could be appropriate in some cases, this should not be the only move instructors make. As such, when it comes to designing support materials for our curriculum, these results suggest to us that we need to better support instructors to have explicit discussion about the mathematical reasoning for edits. This will be particularly important in the defining tasks that occur early in the task sequence when relevant mathematical norms and conventions have not yet been discussed. We also see this ongoing work as contributing to efforts beyond our own materials, and in particular, we hope that this study will provide insights for inquiry-based instructors who aim to refine students' definitions, conjectures, or proofs.

Acknowledgments

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Developing Network Modeling and Analysis Methods for First-Year Mathematics and STEM Student Course-Taking Sequences

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Student success is typically measured by grades in coursework and 6-year graduation rates (York, Gibson, & Rankin, 2015). The purpose of this exploratory data analysis is to develop statistical methods that will provide a deeper understanding of student success, specifically success in first-year mathematics (FYM) prerequisite courses as well as the STEM courses they serve. We consider the many possible student course-taking sequences (paths) as a complex network system. We model the paths that students take with a network of vertices (courses) and edges (links between courses) by using 10 years of registrar data to generate vertex and adjacency matrices, and associated graphical representations (e.g., heat maps and network graphs). We also will conduct relevant exploratory data analysis to determine the importance of certain mathematics courses and transitions to overall student success.

Keywords: student success, mathematics prerequisites, modeling, statistics

Introduction

Student success is typically measured by grades in coursework and 6-year graduation rates (York et al., 2015). Regarding the latter, we know that student placement and success in mathematics courses are some of the primary factors that mitigate student time to degree (Aiken, Bin, Hjorth-Jensen, & Caballero, 2020). In addition, students who have STEM interests and place into lower-level mathematics courses take longer to complete their degrees. Furthermore, they are more likely to change their majors or drop out completely (Klingbeil, 2013).

The long-term purpose of this research is to gain a deeper understanding of student success, specifically success in first-year mathematics (FYM) prerequisites as well as the STEM course they serve. To accomplish this, we consider the many possible student course-taking sequences as a complex network system. We model student course-taking sequences as paths through a network graph of vertices (courses) and edges (links between courses taken in consecutive academic terms). Next, we generate a vertex matrix (courses) and adjacency matrix (edges that link courses) from registrar data of student course taking and generate associated graphical representations (e.g., heat maps and network graphs). Lastly, we will utilize statistical analysis of networks to determine the importance of certain mathematics courses to overall student success.

In this paper, we first provide a brief background of network analysis followed by the prior use of network representations to analyze the complex system of undergraduate coursework. Second, we outline our long-term research goals and contrast our research with the prior research about using networks to analyze undergraduate coursework. Third, we present our current research results in relation to our long-term research goals. Lastly, we discuss our future research agenda in more detail.

Background

In this background section, we define what a network is, provide a brief overview of the history and uses of network analysis, and culminate with the use of networks to model undergraduate courses. Networks, represented as graphs of vertices and edges, have emerged as useful tools in the modeling and analysis of complex systems across a diverse range of applications. Vertices represent the objects of study and are illustrated by the nodes on the graph.

Edges are the links between two vertices and are illustrated by the lines or arrows connecting two vertices. Edges represent the connection or flow between the objects of study.

Networks are useful tools in a wide range of research. Kolaczyk and Csardi (2014) discussed the history and expanse of network analysis, summarized in this paragraph. Due to advances in computer science in the 1950's, the use of network analysis became prevalent in areas such as transportation, allocation of resources, and distribution of products. In this initial phase, a few sociologists also utilized networks to characterize interactions in social groups. Then, interest in network modeling and analysis increased significantly in the 1990's due to the expansion of the internet and associated social media platforms. In addition, network modeling and analysis has proved useful in research areas such as computational biology (Mason & Verwoerd, 2008), engineering (Chen, 1997), finance (Abrams, Celaya-Alcala, Baldwin, Gonda, & Chen, 2016), marketing (Webster & Morrison, 2004), neuroscience (Farahani, Karwowski, & Lighthall, 2019), political science and public health (Goodin, Moran, & Rein, 2008). Two factors have contributed to the proliferation of network modeling and analysis: the increased interest in the study of complex systems, and the ability to collect, store, and maintain large data sets. These two factors prompted the development of statistical methods to analyze large, complex, connected data sets.

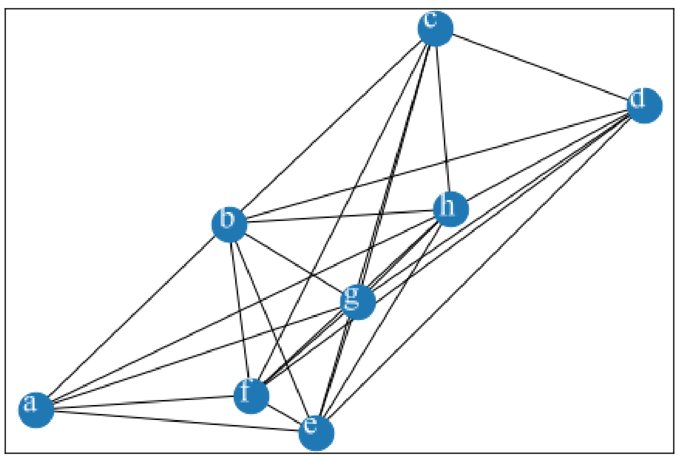

Figure 1. Network graph, G=(V,E) with vertices, V={a,b,c,d,e,f,g,h} and edges, E, that link any two vertices.

Recently, network analysis has been utilized as a method to examine the complex system of undergraduate coursework. For example, Aldrich (2015) and Ren et al. (2021) represented the intended sequences of courses using a directed network of vertices and edges in which each vertex represents a course and each edge represents a directional link between courses. To generate the course networks, university course catalogues (Aldrich, 2015) and program guides (Ren et al., 2021) can be used. In the context of undergraduate coursework network, the number of vertices represented the number of courses, and the number of edges represented the number of links between pairs of courses. They also discussed the general topology of the network by outlining the number of vertices and edges.

Long-Term Research Goals

Here, we present our long-term research goals to provide a context for our current research and results. In our research, we will utilize similar statistical methods as Aldrich (2015) and Ren et al. (2021), but our generation of the coursework network differs significantly. We define the *undergraduate course networks* presented in the referenced curriculum research as *intended* *course sequences* because those networks were generated by web-scraping data from a university course catalogue (Aldrich, 2015) and from a dental school program guide (Ren et al., 2021). In contrast, we define the *undergraduate course-taking network* we generate as the *enacted course sequences* because the network is produced from student course-taking data from a university registrar.

We currently have written a Python program to analyze registrar data of FYM course-taking, and we will expand this program to include STEM courses and General University Requirement Quantitative Reasoning courses. In our current research, we have generated a vertex matrix and adjacency matrix. Our long-term goal is to use such a vertex matrix and adjacency matrix to generate the *undergraduate course-taking network,* which will be a directed weighted network graph. In this *undergraduate course-taking network*, each vertex will represent a course and each edge will represent a directed link between a pair of courses. Each vertex will be weighted by the number of students taking that course and will contain a pie graph of the grade distribution for each course. Each edge will be weighted by the number of students who take linked pairs of courses in a sequence of terms. After generating the *undergraduate course-taking network*, we will conduct statistical analysis to determine different centrality measures of a network (e.g., degree centrality, out-degree centrality, betweenness centrality), which is discussed further in the Future Research section.

Research

To produce the *undergraduate course-taking network*, we utilized the Python programming language to generate a vertex matrix of the number of students who took each course (see Table 1) from student FYM course-taking data from a university registrar. Note that these courses are not necessarily taken in numeric order (see Figure 2). In addition, we utilized the Python programming language to generate an adjacency matrix in which each cell represents the number of students who took the sequence of the pair of courses referenced in the row (course going from) and column (course going to) (see Figures 3 & 4).

Research Questions

1. How can we represent the total number of students who take each course in a useful and flexible representation?
2. How can we represent the sequence of student course-taking in a useful and flexible representation?

Analysis

The analysis of the network of mathematics course-taking sequences is presented as follows:

1. Student course-taking is represented by a vertex matrix in which the total number of student enrollments for each course is tallied. Additional student summary data may be included in this matrix (e.g., grade distributions).
2. We generated an adjacency matrix from 10 years of registrar data. In the adjacency matrix, each cell consists of the number of students who took the sequence of the pair of courses referenced in the row (course going from) and column (course going to).

This adjacency matrix can be used in combination with the associated vertex matrix to generate a directed weighted network graph of student course-taking sequences.

Results

Our results thus far are represented in the Vertex Matrix (Table 1), Course Adjacency Matrix with Heat Map (Figure 3), and the Log Transform Adjacency Matrix with Heat Map (Figure 4). The log transform here is given by $\log_{10}(n+1)$, where n is the original number. The vertex matrix consists of a sample of FYM courses of interest and the student enrollment in those courses over a 10-year period. Note that course numbers do not necessarily indicate the prerequisite sequence. For example, a student who passes Math-90 can proceed to take Math-110 (Figure 2).

Table 1: Vertex Matrix.

Course Number	Course Name	Enrollment
090	Introductory Algebra	679
100	Quantitative Reasoning	3203
110	Functions and Algebraic Methods	5660
115	Precalculus 1	5750
116	Precalculus 2	2096
120	Calculus 1	3473
140	Business Precalculus	2334
141	Business Calculus	3270
200	Statistics	2008

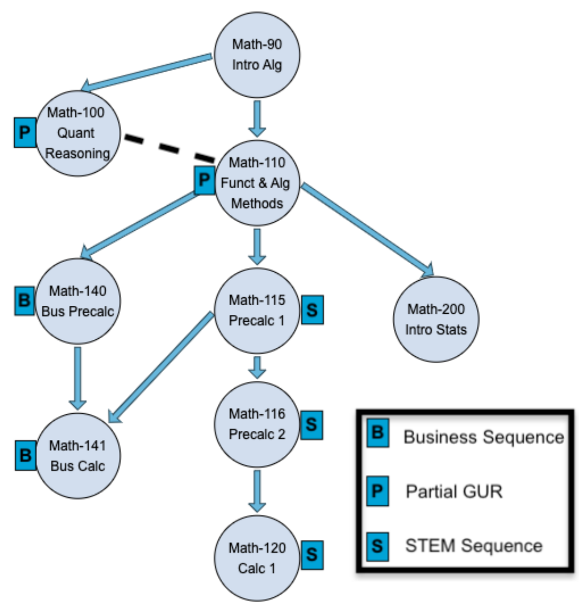

Figure 2: Intended Course Sequence.

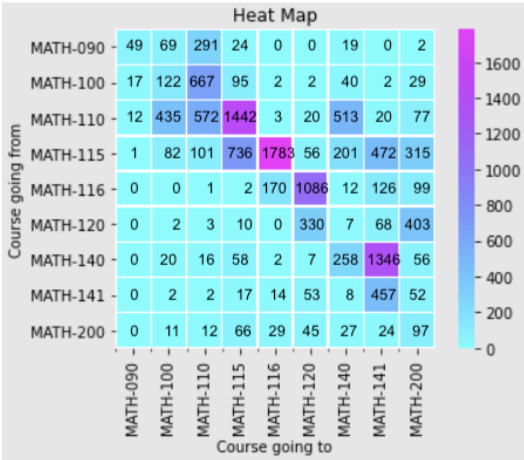

Figure 3: Course Adjacency Matrix with Heat Map

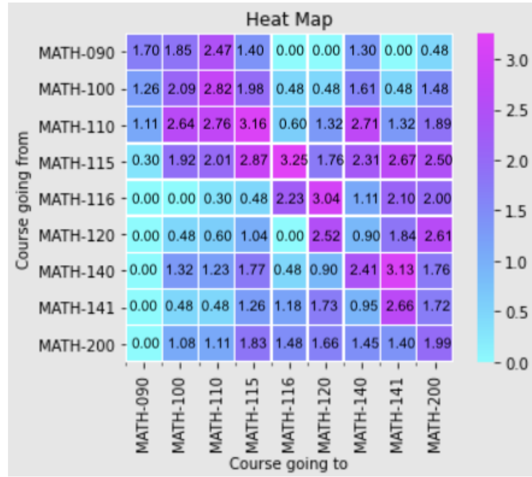

Figure 4: Log Transform of Adjacency Matrix with Heat Map

In the adjacency matrix (Figure 3), each cell consists of the number of students who took the sequence of the pair of courses referenced in the row (course going from) and column (course going to). For example, from prior data, we know that Math-110 has a higher DFW rate, so we may want to examine common course-taking sequences of those students. If we examine the row of cells indicating pairs of courses beginning with Math-110, we can determine the highest enrollment sequence pairs are [Math-110, Math-115] followed by [Math-110, Math-110]. In cell [Math-110, Math-115], we can see that 1442 students followed the STEM sequence, taking Math-110 as a prerequisite to Math-115. In cell [Math-110, Math-110], we can see that 571 students repeated Math-110, resulting in a loop in the network. Due to space limitations, we leave further interpretation to the reader and for the conference presentation.

Discussion

Our study builds on prior research that analyzed *undergraduate course networks* of *intended course sequences* generated from university course catalogues and programs of study. In contrast, we are developing methods to analyze the *undergraduate course-taking network* of *enacted course sequences* because the matrices that will be used to generate the network are produced from student course-taking data from a university registrar. The methods for generating and analyzing a directed weighted *undergraduate course-taking network* of *enacted course sequences* from actual student course-taking are the major finding of this research. Thus far, this study illustrates FYM course-taking as a vertex (course) matrix containing the number of students enrolled in the course and is useful in identifying courses with the highest enrollment. In addition, this study illustrates FYM course-taking as an adjacency matrix (transitions between pairs of courses). The adjacency matrix is useful in determining the most common course pairs taken in a sequence, which can be used to determine the emphasis of course content. The vertex matrix and adjacency matrix are generated as a first step in creating the *undergraduate course-taking network.*

Future Research

In our future research, we will conduct a network analysis on the *undergraduate course-taking network.* We will employ statistical techniques utilized by Ren et al. (2021) and Aldrich (2015) to produce the following measures of centrality: degree centrality, out-degree

degree centrality, and betweenness centrality. In the context of an undergraduate course network, these three measures of centrality signify the following: (i) degree centrality indicates the importance of a course in relation to all courses to which it is connected, (ii) out-degree centrality indicates the importance of courses in relation to those courses that follow, and (iii) betweenness centrality indicates the importance of the course in relation to the flow of information between other pairs of courses. For all three centrality measures, degree centrality, out-degree centrality, and betweenness centrality, the higher the measure, the more important the course is to an undergraduate coursework network. The degree centrality of vertices is calculated by the number of both incoming and outgoing edges to each vertex (Ren et al., 2021). On the other hand, the out-degree centrality of vertices is computed by the number of outgoing edges from each vertex (Aldrich, 2015). Lastly, the betweenness centrality is given by the extent that a vertex is situated between other pairs of vertices (Aldrich, 2015; Ren et al., 2021).

This ongoing network analysis will allow us to determine important courses in course sequences, to analyze student success in courses, and to prioritize courses in need of reform. These research techniques have the potential to apply across different departments, programs, and colleges of the university, as well as the university as a whole.

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Establishing a Shared Vision of Effective Pedagogy to Sustain Networked Reform

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In this preliminary report, we share results from an ongoing project related to the professional development of K-12 math and science teachers in Washington state. The goal of this work is to establish and sustain a network of STEM teacher leaders who can support teachers statewide in using effective STEM pedagogy. In the project's first phase, we surveyed 290 stakeholders, including administrators, support specialists, teachers, and other community/industry partners, in order to identify attributes valued in a STEM teacher leader. Initial findings from this mixed-methods study indicated that STEM stakeholders prioritize teacher leaders' abilities to foster the incorporation of integrated, community-based STEM projects and culturally responsive pedagogy in theirs and their peers' teaching. We seek input from the RUME community on the project's second phase, in which we will study the efficacy of a particular change model in facilitating cross-institutional systemic change toward the adoption of STEM best practices.

Keywords: STEM teacher leaders, network, professional development of K-12 teachers

Networks are an established vehicle for effecting pedagogical change (Lewis, 2015). Networked communities exist in a variety of STEM education contexts; such diversity merits the investigation of a variety of network models that may contribute to large-scale pedagogical reform in STEM. For example, the Academy of Inquiry Based Learning provides a network for individual instructors seeking to adopt inquiry-based pedagogical practices in their classrooms (Yoshinobu et al., 2022), while the SEMINAL project established a network of university mathematics departments pursuing the implementation of active learning strategies in Precalculus through Calculus courses (Smith & Funk, 2021). The overarching goal of our WA-STELLAR project (NSF DUE #1950332, #2150054) is to grow and support a sustainable network of K-12 STEM instructional change agents (teacher leaders) across Washington state. Through this project, we will support the development of teacher leaders to mentor other STEM teachers in using effective STEM pedagogy (Objective 1); we will then support the systemic change efforts of the teacher leaders via network activities that promote collaboration among a range of STEM stakeholders (Objective 2).

Education networks vary in their motivations and goals, their participation and leadership structure, and their processes for catalyzing organizational change. Regardless of localized complexities, effective network collaboration is predicated on the following eight features: (1) focusing on ambitious student learning outcomes linked to effective pedagogy; (2) developing strong relationships of trust and internal accountability; (3) continuously improving practice and systems through cycles of collaborative inquiry; (4) using deliberate leadership and skilled facilitation within flat power structures; (5) frequently interacting and learning inwards; (6) connecting outwards to learn from others; (7) forming new partnership among students, teachers, families, and communities; and (8) securing adequate resources to sustain the work (Rincón-Gallardo & Fullan, 2016). In this report, we share preliminary results identifying a *stakeholder-informed* vision of "ambitious student learning outcomes linked to effective STEM pedagogy"

(Feature 1; Project Objective 1). In particular, we recognize that professional development associated with this project's second phase will only be effective if the aforementioned vision is truly *shared* amongst the network's constituents. Subsequently, we discuss implications for achieving project Objective 2 and conducting further research during this second project phase.

Theoretical Framing

This project is grounded in communities of practice theory (Wenger, 1998). Schools or districts form individual communities of practice. Through the WA-STELLAR network, we (the project leadership team) and teacher leaders act as *brokers* (Wenger, 1998) between these communities by translating, aligning, and facilitating coordination of perspectives, creating the possibility for collective meaning-making. Teachers' instructional strategies are simultaneously influenced by obligations to the discipline, themselves, others, and the institution or school (Herbst & Chazan, 2012). As such, changing practice is operationalized as "both building on existing instructional situations and breaching with some of their norms" (p. 611). With teachers from schools and districts across Washington state, the WA-STELLAR network can serve as a tool to breach existing norms while accounting for existing conditions. The network can unify strategies for supporting student success by leveraging the strengths of all members.

Methods

Research Questions and Design

The following research questions guided the first phase of this project (Objective 1) in order to determine what attributes WA-STELLAR network stakeholders value in a STEM teacher leader, and what program components the stakeholders view as likely to develop these attributes:

RQ 1: *What are principles and practices that advance STEM education and that a STEM Leadership program can support?*

RQ 2: *Who is the target audience for a STEM Leadership program?*

RQ 3: *What are the skills, certifications, and responsibilities necessary for an effective STEM leader, thus establishing guidelines for program curriculum?*

Utilizing a mixed-methods approach, we first surveyed 290 STEM stakeholders from four key groups: administrators, STEM support specialists, STEM teachers, and other STEM community and industry partners. Follow-up emails and phone interviews were conducted with seven survey participants, with four additional survey participants attending in a focus group. This report focuses on the first and a portion of the third research questions, as these results offer insight into how we might establish shared vision for "effective STEM pedagogy," an *essential feature* (Rincón-Gallardo & Fullan, 2016) needed to sustain the work of the WA-STELLAR network.

Participants and Sampling

To answer the research questions from the perspective of STEM stakeholders in Washington state, we developed and administered a survey online via Qualtrics. Stratified convenience sampling was employed to collect survey data. Members of the Apple STEM network, members of the South Central Washington STEM network, members of the Washington state Science Teachers Association, and graduates of our undergraduate STEM teacher preparation program were contacted until a minimum of 100 responses were received from STEM teachers and at least ten responses were received for each of the three other subgroups of stakeholders. In total, 290 participants consented to be part of the study and completed at least part of the survey. Their positions are summarized in Table 1.

Table 1. Summary of survey participants' educational positions.

Educational Position/Role	Count
Administration (District Superintendents, School Principals)	13
STEM Support Specialists (Curriculum and STEM Specialists, Coaches)	18
STEM Teachers (Math, Science, Technology, Computer Science)	208
STEM Partners (STEM community and industry advisors, Other)	51

Quantitative and Qualitative Data Sources

The skills and responsibilities included in quantitative survey questions were based on the characteristics of effective STEM leaders identified in literature on the topic (The Aspen Institute, 2014; National Council of Supervisors of Mathematics, 2008; Sublette, 2013; Tanenbaum, 2016; Toncheff, 2020). Prior to administering the survey, we incorporated feedback from other STEM educators to establish face validity (Gravetter & Forzano, 2012). In this preliminary report, we focus on results from two questions pertaining to establishing a shared vision of effective STEM pedagogy.

Qualitative data was collected to supplement and triangulate quantitative results from the survey and consisted of the following three data sources. At the end of the survey, participants were given an opportunity to share their thoughts in an open-ended question: *What would you like to tell us about STEM education? You can explain responses, add new ideas, or ask questions*. Follow-up phone and email interviews were conducted with seven survey respondents including representatives from all categories of stakeholders. Finally, we organized one focus group with four survey respondents including three STEM teachers and one STEM coordinator.

Results

Thematic Analysis of Qualitative Sources

In addition to triangulation of three data sources, we used analyst triangulation methods to strengthen the credibility of inquiry (Patton, 2002). Two of the authors independently analyzed qualitative sources, identifying repeated themes within and across participant subgroups. The two researchers then met to compare findings and understand any inconsistencies, generating a final list of four themes; three are related to a shared vision of effective STEM pedagogy:

1. All stakeholder subgroups stated STEM content needs to be integrated across disciplines.
2. All stakeholder subgroups identified the importance of promoting equity and diversity in STEM education. Several responses suggested meaningful community partnerships and project-based learning might help address equity issues.
3. From a curriculum standpoint, we need rich and engaging activities as well as meaningful applications connecting to industry.

Participant responses related to these themes are integrated into the next results section.

Quantitative Survey Results

In question one of the survey, participants were asked to rank a list of statements in order of importance for advancing STEM education. Table 2 provides average rankings of each statement for each stakeholder subgroup of participants. To determine the overall ranking, each stakeholder subgroup was treated as equally weighting the overall ranking. Thus, the ranking provided in each row of the 'Average' column is an unweighted average of the four numbers to its right.

Table 2. Statement rankings in order of importance for advancing STEM education (1=most important, 5=least).

Statement	Rank	Average	Admin	Support Specialists	Teachers	Partners
A. STEM projects should be integrated into all STEM courses.	1	1.89	1.85	1.71	2.14	1.86
B. STEM education principles are best advanced by hiring effective math, science teachers.	2	2.14	2.08	1.93	1.91	2.62
C. STEM education should focus on Career and Technical Education.	4	3.32	3.08	3.50	3.37	3.33
D. STEM education principles are best advanced through programs with industry, community partners.	3	3.15	3.46	3.29	3.13	2.71
E. STEM education principles are best advanced through extracurricular programs.	5	4.51	4.54	4.57	4.46	4.48

For all stakeholders, the two most important principles/practices for advancing STEM education were integrating STEM projects into all STEM courses (Statement A) and hiring effective math and science teachers (Statement B). These findings were further triangulated with qualitative results, in that all stakeholder subgroups mentioned that STEM content needs to be integrated across disciplines. In particular, as one STEM coach/coordinator expressed, “The trend and long-standing practice of trying to do each one of these [disciplines] exclusive of the other is the biggest hurdle we face.” Community partners, CTE support, and extracurricular programs help foster and grow STEM learning experiences; however, effectively implemented, integrated STEM projects must be the foundation of STEM education. Any STEM program must have integrated, project-based learning at its center. Additionally, STEM partners advocated for ‘integrating community projects, participatory science, and industry partners into schools.’

In question four of the survey, participants were asked: *How important are the following responsibilities for a STEM leader?* Table 3 provides a weighted average for each stakeholder subgroup and each responsibility. Each participant’s response was assigned one of the following weights: Extremely Important → 1; Moderately Important → 0.5; Not at all Important → 0. A score close to 0 indicates the subgroup views the feature as less important. A score closer to 1 indicates the subgroup views the feature as more important.

The most highly valued STEM leader responsibility was utilizing culturally responsive teaching practices. This finding was also substantiated in open-ended responses, where participants voiced a need for promoting diversity and equity in STEM education. For instance, one of the STEM coach/coordinator participants expressed “how important it is for students to see professionals they identify with in STEM professions. In order to draw more diversity into the STEM field, we need more STEM leaders with diverse backgrounds. Programs should promote diversity.” One STEM teacher felt that “STEM education that is hands-on problem solving has a better chance of bridging gaps with equity issues and schools that struggle with cultural responsiveness, better than most any other field.”

Table 3. Weighted averages for different stakeholder subgroups addressing question four.

Responsibility	Admin	Support Specialists	Teachers	Partners
Lead Professional Development in Designing and Teaching Problem-based Instruction	0.78	0.76	0.67	0.72
Initiate and Administrate Community-based STEM Projects	0.73	0.71	0.62	0.73
Coach colleagues in effective STEM Teaching	0.84	0.97	0.76	0.82
Coordinate STEM education district curriculum and programs	0.75	0.88	0.72	0.78
Lead professional development in integrating engineering practices for STEM courses	0.80	0.91	0.68	0.70
Utilizing culturally responsive teaching practices	0.90	0.94	0.79	0.87

Additionally, administrators, STEM support specialists, and community and industry partners identified initiating and administrating *community-based* STEM projects as an important responsibility of STEM leaders. However, STEM teachers felt support from their district and other STEM educators is essential to assist STEM leaders in both using STEM projects in their own classrooms and becoming STEM champions of district and community-wide projects.

Discussion and Implications

These results evidenced the need to redesign our (previously) traditional, content-focused master's degree in mathematics to train teacher leaders (Objective 1) in a way that better aligns with the values of the larger collective of STEM stakeholders. In our talk, we plan to provide a brief overview of the novel graduate STEM Leadership program we designed to serve as a *network hub* (Bryk et al., 2015) for the WA-STELLAR network.

As we move toward the next phase of our project (Objective 2), we will implement the Keck/PKAL change model (Elrod & Kezar, 2016) to facilitate the work of the WA-STELLAR network. This model has been used in higher education settings to coordinate multiple stakeholders and has eight stages, the first of which is: *establishing a shared vision for improving STEM student learning outcomes and success*. This next phase of our project will be guided by the research question: *What is the efficacy of the Keck/PKAL model in facilitating cross-institutional systemic change toward the adoption of STEM best practices?*

Our intended questions for the RUME audience concern ways in which we can leverage preliminary results to *maintain* a vision of effective pedagogy throughout implementation of network activities that is shared by all network stakeholders. We are interested in identifying potential opportunities/barriers to implementation that might impact the efficacy of our implementation of the Keck/PKAL model.

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Introductory Calculus Instructional Practices Around Student Prior Knowledge

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Calculus is a pivotal course for postsecondary students and serves as a gatekeeper to advanced mathematics courses and STEM majors. This study considers the frequency with which instructors report engaging in practices that may help to strengthen introductory calculus students' prior knowledge. Using survey responses (N=136), this study reports on results from a set of questions that measures these practices. Results illustrate that instructors may use various practices to address students' prior knowledge in their introductory calculus classroom. Additionally, instructors provided examples of practices they use in response to student prior knowledge that are not explicitly considered in the literature. This implies that a closer look at what is happening in the classrooms may be key in understanding the current state of how instructors are dealing with student prior knowledge in introductory calculus.

Keywords: Introductory Calculus, Instruction, Student Prior Knowledge

Introductory calculus is an essential course for many students; it can bridge the secondary-tertiary transition. As such, it is often a culmination of students' K-12 knowledge and a gatekeeper to more advanced mathematics (Bressoud, 2015; Rasmussen et al., 2019). Despite its importance, students continue to struggle to make it through the course in postsecondary settings (Bressoud et al., 2013). One issue facing instructors at the postsecondary level, many of whom have been trained as mathematicians and not as teachers, is the varying levels of prior knowledge that students have when they enter the classroom. One reason for the difference in students' prior knowledge might be the various paths to get to postsecondary introductory calculus (i.e., having seen calculus in high school or not) (Sadler & Sonnert, 2018; Schraeder et al., 2019). Research has shown that student understanding of prerequisite knowledge connects to student success in calculus (Rasmussen et al., 2019); therefore, it is important to understand how instructors engage in practices that support the development of students' prior knowledge. Studying instructional practices that may strengthen student prior knowledge can help researchers understand more deeply practices that might be leveraged in the classroom to help more students succeed.

Theoretical Background

Introductory Calculus Instruction

Despite research on the impacts of and barriers to success in postsecondary calculus, mathematics instruction at the postsecondary level remains comparatively unexamined (Speer et al., 2010). Some researchers have looked at the instructional format in calculus, finding that calculus classes typically feature lecture and demonstration (Biza et al., 2016; Larsen et al., 2015). Other researchers have shown that instruction in introductory calculus classrooms may be more varied. For example, while lecture continues to be the most common mode of instruction for postsecondary mathematics, recent arguments suggest that this mode can help to model mathematical discourse (Viirman, 2021). In addition, Bressoud and Rasmussen (2015) found that while university instructors considered themselves fairly traditional (i.e., they thought that calculus students learned best from clear, organized lectures), they were open to trying other instructional approaches. Finally, several studies have pointed towards instructional practices in calculus courses that move away from lecture; recent work in math and related STEM fields has shown that university faculty do attempt to combine content- and student-centered practices and that these practices increase student achievement (Bressoud & Rasmussen, 2015; Hovey et al., 2019; Rasmussen et al., 2019; Tang & Titus, 2002).

Prior Knowledge & Introductory Calculus

In my dissertation study, I frame prior mathematics knowledge as the mathematical reasoning and skills students bring into the classroom from their previous mathematics classes or experiences. I draw here from Tobias' (1994) definition of domain knowledge: "domain knowledge deals with familiarity with general information in an area, even though it may not be specifically referred to in a particular passage" (p.39). A key difference between my definition of prior mathematical knowledge and Tobias' definition of domain knowledge is that I choose to include the idea of 'mathematical experiences' that may fall outside mathematics content. I borrow the concept of mathematical experiences as part of prior knowledge from Campbell et al. (2020), who put forth "how students understand core ideas based on their own lived and cultural experiences" (p. 68) as their definition of prior knowledge. By combining these definitions of prior knowledge, I can consider students' specific mathematics content exposure and how this exposure happened (e.g., procedural instruction in algebra courses). I also conceptualize prior mathematics knowledge as distinct from prerequisite knowledge because not all prior mathematics knowledge is important for calculus understanding.

One reason that instructional support of student prior knowledge is important is that it allows students to connect old and new ideas, creating opportunities to make sense of them. Neumann (2014) argues that student prior knowledge is brought to the surface in encountering new ideas. Furthermore, learning occurs when this prior knowledge and new ideas are worked through together. One way to understand where opportunities for prior knowledge and new ideas to be worked together is to examine instructional practices.

Building on Neumann's (2014) claims, Campbell et al. (2017) developed a framework to study how postsecondary courses might cluster by pedagogies and included instructional attention to supporting prior knowledge in their pedagogical characterizations. They found that both lecture and active learning strategies can be used to support student prior knowledge. Campbell et al. (2020) explain that while strides have been made in postsecondary institutions to have student-centered classrooms (which may promote discussions and use of prior knowledge), instructor attention to prior knowledge in college classrooms is still uncommon. This line of inquiry provides evidence that support of prior knowledge (though perhaps rare) does occur in postsecondary classrooms and that there may be observable instructional practices among those who consider prior knowledge in their instruction. I further this line of inquiry by asking the following research question: *How and to what extent do introductory calculus instructors report engaging in practices that promote students to make mathematical connections to their prior knowledge?* Results that answer this question help us better understand if and how these practices are being put into use by introductory calculus instructors around the U.S.

Methods

Participants and Procedures

The data for this study were collected using a survey distributed to postsecondary instructors at institutions across the United States in Spring 2022. The population of interest for this survey included full-time instructors at 4-year universities with a Carnegie classification of R1, R2, Doctoral, or Arts & Sciences Focused Baccalaureate who have taught introductory calculus at least twice. The choice to sample full-time instructors at these types of institutions was informed by research on where most students take introductory calculus (Blair et al., 2015). After excluding responses that did not participate past questions about demographic information, there was an analytic sample of 142 respondents. I received 42 responses from R1 instructors, 32 responses from R2 instructors, 39 responses from Doctoral institution instructors, and 29 responses from instructors at Baccalaureate Arts & Sciences focused institutions. In addition, 61 participants identified as female, 79 participants identified as male, 1 participant identified as non-binary, and 1 participant chose not to disclose gender identity. 13 participants identified as Asian, 3 participants identified as Black or African American, 117 participants identified as White, and 8 participants chose to not identify their race. In addition, 70 participants identified as Tenured Faculty, 15 as Tenure-Track Faculty, 57 participants as other full-time faculty. Finally, 23 participants indicated that they had taught introductory calculus 2-5 times before the survey, 32 had taught the course 6-10 times, and 87 had taught it more than 10 times.

Survey Construction

The survey data used here was intended to capture how often instructors engaged in practices that have been theorized to strengthen student prior knowledge. I developed a set of questions to measure this construct, drawing from the C-SAIL Mathematics Teacher Survey (California Version, 2017; Edgerton & Desimone, 2019). As written, this survey did not specifically measure practices that engage student prior knowledge. However, it was appropriate for this purpose because the questions were structured and validated to ask about instructional practices. I selected specific items based on theoretical grounding in prior literature. For example, I chose items asking about discourse in the classroom, teaching foundational skills, and presentation of multiple strategies (Chappell & Killpatrick, 2003). As the C-SAIL Mathematics Teacher Survey was written primarily for use in K-12 classrooms, I adjusted wording for the calculus and postsecondary contexts when necessary. It included 12 items measuring frequency on a 5-point Likert scale that indicated instructors engaged in these practices: Never, Once or Twice a Semester, Once or Twice a Month, Once a Week, or Almost Every Class Session. A higher score indicates that an instructor engages in the practice more often.

Additionally, data presented here was gathered from a single open-ended question at the close of the survey, "How, if at all, is your instruction affected by the mathematical understanding students bring when they *come into* your class at the start of the semester?" This question was intended to gather information on any instructional practices that my survey questions may not have caught.

Analysis

Analysis for this set of survey data occurred along three lines. First, de-identified data from the participants was used to validate the survey. Second, the survey data were analyzed to understand which practices instructors used more frequently. For these analyses, only data from instructors who had completed the full set of questions was considered (N=136). Finally, qualitative data was open-coded (Corbin & Strauss, 2014), to understand additional practices espoused by instructors; 127 instructors responded to this question.

Results

Reliability

Internal consistency of the items was analyzed using a correlation matrix and Cronbach's alpha. Inspection of the correlation matrix showed that all items except for one ("How often do you have students solve practice problems using explanations or solution strategies you have modeled in class?") had correlation coefficients greater than 0.3. The one item with no correlation coefficients greater than 0.3 was dropped from the analysis, leaving 11 items for analysis. These 11-items had a high level of internal consistency, as determined by Cronbach's alpha of 0.81.

Survey Response Instructional Practices

Overall, participants in this sample reported that they engaged in some form of instructional practice that may strengthen student prior knowledge. I calculated a composite score for each participant by averaging their item scores. I then calculated the mean across the participant sample. The mean was 3.77, indicating instructors in my sample participate in some of these instructional practices between once or twice a month and once a week. Additionally, there were practices that instructors were engaging in less or more (see Table 1). For example, instructors reported assessing students in prerequisite skills the least, with a mean of 2.93 (nearly once or twice a month). They also reported most often building calculus knowledge by connecting it to prior knowledge, with a mean of 4.42 (between once a week and every class).

Interestingly, the lower reported practices seem to align with a more procedural approach to strengthening student prior knowledge (e.g., reteaching or simplification of problems). In contrast, the higher reported practices involved a less procedural approach (e.g., engaging in discourse, connecting calculus content to students' experiences). Interestingly a one-way ANOVA showed no significant difference between instructors at different institution types for each item or across items. This implies that regardless of institution type instructors are engaging in similar practices.

Table 1. Means for Instructional Practices for Student Prior Knowledge.

"On average, across the span of a semester, how often do you engage in the following practices in your introductory calculus course?" (N=136)

Practice	Mean	Std. Deviation
Build calculus content by connecting it to prior knowledge	4.42	0.80
Correct or refine students' use of language when they are presenting their ideas	4.14	0.91
Provide opportunities for students to discuss their mathematical reasoning in class	4.20	1.08
Assess students in procedural skills taught prior to calculus	2.93	1.26
Reteach content using simpler numbers if students are struggling	3.47	1.15
Teach foundational skills students have not yet mastered	3.29	1.01
Have students apply prerequisite mathematics skills in the context of calculus concepts	4.40	0.82
Simplify the context of word problems by re-writing them	3.07	1.36

Present multiple solution strategies for a given problem or problem type	3.84	0.83
When a problem can be solved multiple ways, use strategies that students most often use	4.07	0.93
Connect calculus to students' experiences	3.68	1.01

Short Answer Responses

During the open coding of the short-answer responses to the question, "How, if at all, is your instruction affected by the mathematical understanding students bring when they *come into* your class at the start of the semester?" several themes emerged about how introductory calculus instructors perceive changes to their instruction based on their understanding of students' prior knowledge. Initially, this coding prompted me to divide the responses into five categories: instructors who said that their instruction was affected and gave me specific examples, instructors who said that their instruction was affected but gave me no or vague examples of instruction, instructors who said that it did not affect their instruction, instructors who gave an unclear response, and instructors who gave a student-focused response. Many (71 out of the 172 responses), instructors provided examples of instruction they engaged in response to students' prior mathematical knowledge. Although many of these align with the surveyed practices, a few new ideas came up. For example, explicit review was a common response. One respondent explains, "I assume most students need a "reminder" of the prerequisite topics. If they ask [or] seem to need more I do more review as part of my in-class explanations of problems." In addition, many instructors seemed to anticipate and build review into their plans for the course. This indicates that instructors can articulate practices they engage in to activate and connect to student prior knowledge and that they may do so in different ways than the literature has previously noted.

Discussion

These results are emerging as part of a more extensive dissertation study but indicate that introductory calculus instructors are mindful of the need to activate students' prior knowledge in the course. Additionally, they report engaging in practices that might help students strengthen their prior knowledge during introductory calculus instruction. Further, they might be engaging in more and less procedural instruction to support students (e.g., reteaching prerequisite skills versus engaging students in discourse). This finding is in line with previous research (Hovey et al., 2019; Rasmussen et al., 2019). Qualitative data supports the idea that instructors can identify and describe practices they engage in to respond to students' prior knowledge. Future research is needed to understand exactly how these practices are implemented and which are most accessible or provide the best student outcomes. Work in this vein can help the field move towards an incremental improvement approach (e.g., Litke, 2020) to better support calculus students without asking current instructors to overhaul their entire instructional approaches to introductory calculus.

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Characterizing College Instructors' Attention During Peer Observations

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Research has highlighted that actively involving students during instruction can lead to positive outcomes for students. However, college mathematics instructors may need support to develop the knowledge and skills necessary to effectively implement this type of instruction. This study looks at how college algebra instructors in a grant-supported professional learning community (PLC) focus on different aspects of their own and others' teaching. We leverage the instructional triangle as an analytical framework to characterize the foci of participants' observations. We analyzed PLC meetings where participants reported on specific aspects of each other's observed classes. Our analysis revealed that instructors each had a primary focus that drove their observations. We anticipate these different foci will inform future PLC meetings and lead to new questions about instructor thinking, and to continued development of the instructional triangle.

Keywords: Instructional Triangle, Peer Observations, Professional Learning Community

Research has repeatedly emphasized that teaching that actively engages students can lead to improved learning outcomes and conceptual understanding for students (e.g., Eddy & Hogan, 2014; Freeman et al., 2014; Kogan & Laursen, 2014; Laursen et al., 2014; Theobald et al., 2020). However, lecture remains a common form of instruction in many college mathematics classes (Apkarian et al., 2021; Stains et al., 2018). As such, it is important to support college mathematics faculty in developing the knowledge and skills necessary for teaching in a way that centers students and their thinking and moves beyond lecture.

This study is part of a larger ongoing funded project of collaborative instructional improvement at the university level, aiming to support the implementation of evidence-based instructional practices that actively engage students with the course content (funding information blinded for proposal). The project is a professional development experience for the instructors of all sections of a university-level college algebra course at a single institution. As part of this project, instructors work as a group to implement Continuous Improvement cycles (Berk & Hiebert, 2009) to develop and facilitate lessons on particular course topics. This process closely mirrors Lesson Study in that instructors work together to develop lessons, observe each other's lesson facilitation by watching video recordings, and then reflect on and revise the lessons (Dick et al., 2022). Instructors chose to create and use Desmos Classroom activities as a way to incorporate active learning in their teaching. Although instructors differed in how they used these activities (with some soliciting and leveraging student thinking more than others), they all gave students time to work individually or in small groups before discussing the activities in class. Notably, no instructors felt that they needed to lead students through the activities themselves.

This study centers on the discussions surrounding participants' observations of their peers' teaching. These observations were conducted in the style of a video club (e.g., Sherin & van Es, 2005), with each instructor watching the recording of another instructor's class and choosing clips to highlight for the whole group. Our research aims to answer the following question: What do college mathematics instructors notice when observing each other's teaching?

Theoretical and Analytic Frameworks

Teacher noticing informed the conception, data collection and analysis of this study. Noticing occurs when a teacher identifies important instances in a teaching situation, and then works to make sense of them (Jacobs et al., 2010; Sherin & van Es, 2005). Research on teacher noticing has highlighted that expert teachers tend to focus on noteworthy events, provide interpretive comments, and make connections between student thinking and pedagogical moves. In contrast, teachers with less developed noticing skills focus on general impressions and provide descriptive or evaluative comments about what was observed (van Es, 2011).

For this study, we used the "instructional triangle" as our analytic framework to capture what instructors were attending to as they observed one another's classes. Cohen and Ball (2001) argued that instruction consists of "interactions among teachers and students around content, in environments" (p. 122). These three components (teacher, students, content) are often represented as the vertices of the Instructional Triangle, with the edges of the triangle representing the relationships between each of these elements.

Recent research examining instruction at the college level has leveraged the Instructional Triangle to analyze instructors' reflections and discussions during an online working group centered on the teaching of Abstract Algebra using inquiry-oriented instruction (Kelley & Johnson, 2022). Specifically, Kelley and Johnson (2022) categorized instructors' comments as focused on the instructor, students, or mathematical content, and then connected this to instructors' roles during discussion in the working group. They found that over the course of the working group that individual instructor's foci shifted in different ways, but both instructors demonstrated a shift from a focus on content to a focus on the relationship between content and students. Our research aims to characterize instructors' foci when observing another instructor teaching a target lesson that was collaboratively developed, and then to use what we learn to inform how to support shifts in participant discussion for future observations.

Methods

The participants were four college algebra instructors at a large university who met weekly in a professional learning community (PLC) to share and discuss video clips of classroom observations and revise course curriculum for future semesters. For this study, we first recorded each instructor teaching an online lesson on algebraic properties. The PLC facilitators (who are the project's principal investigators) then asked participants to watch the video-recording of another participant teaching this lesson and select clips that caught their attention before the next meeting. Participants were paired, each watching their partner's recorded lecture in order to highlight specific aspects of each other's teaching. Participants were asked to consider things that would help improve the lessons for future classes and to identify any other interesting aspects of the teaching. At the next PLC meeting, participants showcased what they noticed from recordings of each other teaching this lesson, following a video club format. We then repeated this process for the lesson on fractions and the lesson on factoring. We recorded and transcribed the PLC meetings for analysis.

We utilized the Instructional Triangle (Cohen & Ball, 2001) to generate and assign codes to both the selected video clips and the participants' observations immediately following each video clip. Our aim was to gain insight into which aspects of a clip the participant was focusing on. We coded segments for the participants' focus on mathematical Content (C), Teachers (T), Students (S), and the interactions between the three: Teacher-Content interactions (CT), Student-Content interactions (CS), and Teacher-Student interactions (TS). We also differentiated between participant focus on teacher-student interactions about specific content, such as an instructor answering a content-related question (Teacher-Student Content interactions; TSC), and teacher-student interactions that were not explicitly related to content, such as discussions about classroom procedure or the instructor validating a student's contribution to the class discussion (Teacher-Student Non-Content interactions; TSN). Another addition we made to our codebook was creating the Content-Teacher-Student (CTS) code, which captured the participant tying all three corners of the instructional triangle together at once. This code differs from TSC by emphasizing the ways in which all three vertices interact at once, with each vertex engaging with the other two equally, while TSC highlights interactions between students and teachers where they are focusing on each other rather than engaging with the content individually.

The first and second author independently coded the transcripts, and then met to discuss all coding decisions. After reconciling all coding decisions, we calculated the frequency of codes that arose from each participant's comments about the video clips. By analyzing these frequencies and trends, we were able to identify focus profiles for each of the instructors.

Results: Participant Focus Profiles

Our data analysis allowed us to identify three distinct focus profiles demonstrated by the four participants, including a profile focused on Non-Content Teacher-Student interactions, a profile on Content-Teacher interactions, and a profile focused on Content interactions. In this paper, we provide a brief summary of each of the three profiles based on the participants' discussion from the algebraic properties lesson. Table 1 shows the frequency of specific codes assigned to each participants' discussion during the PLC meeting, highlighting what they noticed while observing the algebraic properties lesson.

Table 1. Frequency of Codes for Each Participant's Discussion on the Algebraic Properties Lesson

Participant		C	CS	CT	CTS	S	T	TSC	TSN	Total
Alex	*n*	0	2	0	2	1	3	0	15	23
	%	0.0	8.7	0.0	8.7	4.4	13.0	0.0	65.2	100.0
Nicholas	*n*	0	0	1	0	0	4	0	11	16
	%	0.0	0.0	6.3	0.0	0.0	25.0	0.0	68.7	100.0
Ivy	*n*	3	1	4	2	1	0	1	5	17
	%	17.7	5.9	23.5	11.8	5.9	0.0	5.9	29.4	100.0
Shay	*n*	0	0	10	1	1	1	0	4	17
	%	0.0	0.0	58.8	5.9	5.9	5.9	0.0	23.5	100.0

Focus on Non-Content Teacher-Student Interactions

Two participants, Nicholas and Alex (pseudonyms), tended to focus on Non-Content Teacher-Student interactions (TSN) as more than 65% of their discussion about the video clips were coded as such. Interestingly, both Nicholas and Alex appeared to have a secondary focus on the Teacher category, with 25% of Nicholas' and 13% of Alex's contributions receiving Teacher codes (T). Because of this, we characterized Nicholas and Alex as primarily concerned with the ways in which teachers conduct themselves in the classroom, and the interactions they have with students that are not related to specific pieces of content.

We find it interesting that these participants were so similar in their foci, because the other participants had very different foci in their discussions about the observations. We hypothesize that similar priorities and strategies between Alex and Nicholas' teaching may have contributed to the similarity of their foci during the video club meetings. For example, Alex spoke a lot about Nicholas' (and other participants') approach to gathering buy-in from students, saying, "He said, 'You know, no matter how we feel about the group work, there's a reason we're doing it.' He's… providing that… explanation of why we should do the group work [for students]." Alex also responded to Shay's comments about a clip of Ivy's teaching, saying, "I kind of just agree with everything that was said. I like, a lot the way that [Ivy] is kind of candid with her students. I think that helps encourage the buy-in. She just tells it like it is."

Similarly, Nicholas spoke about Alex's (and others') efforts to motivate students and the response from students those efforts garnered. Responding to a clip of Alex's teaching, Nicholas commented, "I liked that he actually gets his students to talk and discuss." Nicholas also responded to a clip of Ivy's teaching that Shay presented by saying:

> I think in general, even if we feel like we were on rinse and repeat, the more we say, 'This is why we're doing this this way. This is what we're trying to accomplish.' I think… it's enough throughout the semester to get the buy-in from the students … I think [Ivy] handled that perfectly - I always get the pushback of, 'Well, you're not teaching, you're not explaining something.' Well, we are. We're interacting and we're doing it together.

This focus on the interactions between teachers and students, and how those interactions can encourage students to participate fully in class, was a major component of Alex and Nicholas' contributions to the discussion.

Although Nicholas and Alex tended to focus a similar amount on Non-Content Teacher-Student interactions, we did identify some differences between Nicholas and Alex based on what else they focused on during the discussion of video clips. Nicholas' focus on Non-Content Teacher-Student interactions and Teacher comments accounted for 93.75% of his contributions, while Alex's focus on the same two categories accounted for only 78.26% of his. The remaining contributions from Nicholas fell under Teacher-Content Interactions (CT), while Alex's were spread between CS, CTS, and S. Note all of these codes involve interactions with students, which highlights that Alex was also concerned with the students' experience. This difference between Nicholas and Alex may be influenced by their standing within their instructional team. Nicholas is a course coordinator, so his primary focus on instructors may follow from that. Alex is a younger instructor, with preparation in secondary mathematics teaching and a graduate background in mathematics education research. We hypothesize that this may contribute to his focus on the experience of the students.

Focus on Content-Teacher Interactions

The second profile we identified is demonstrated by Shay (pseudonym), who was primarily focused on Content-Teacher interactions, with 59% of her observations focusing on the relationship between the instructor and the content they presented. Shay was secondarily focused on Non-Content Teacher-Student Interactions, with 24% of her contributions receiving that code.

A large portion of Shay's focus on Content-Teacher interactions was directed inward, reflecting on her own teaching. For example, after observing Ivy teach the algebraic properties lesson, Shay said:

> That's something that I never thought of … I'm like, when was the last time I actually did … teach the properties? And I'm like, it's been a really long time. So being able to see how … [Ivy] talked about these two [properties] was really nice.

We characterized Shay as Content-Teacher Focused because of this emphasis on her own and other instructors' relationship or interaction with the content they teach.

Focus on Content Interactions

Our third profile that we identified was characterized by Ivy's (pseudonym) primary focus on content and the ways in which individuals in the classroom interacted with said content. In particular, over 65% of Ivy's contributions were anchored to content, with the Content code consisting of 18% of her comments, Content-Teacher-Student interactions as 12% of her comments, and the codes Teacher-Student Content interactions and Student-Content interactions codes accounting for 6% each of her comments. Ivy's focus can be illustrated by this quote:

> Students got answers of 16, 8, 10, and 1, and [Shay] was asking for input … 'Would you mind sharing how you got your answer?' And that was after they had already established that 16 was the correct answer. So students actually did start answering and saying that this is what they did.

Because Ivy's focus is on the content in relation to both the instructor's words - "Would you mind sharing how you got your answer?" - and on the students' engagement with the content - "Students actually did start answering and saying this is what they did" - we characterize Ivy's contributions as being Content-Focused. We distinguish this from the Content-Teacher Interactions profile described above because Shay was focused on content and instructors, whereas Ivy was focused on both instructors' and students' relationship with the content.

Discussion

The diversity of participant foci in our results offers the opportunity to look into and pose questions about why certain instructors had the focus they did. We saw Alex and Nicholas both had a focus on Non-Content Teacher-Student interactions in their observations. This makes us wonder if they may have taken this observation opportunity to look for ways to further develop their pedagogical skills, specifically in relation to their interactions with students. This begs the following questions for further study: 1) Does this focus come from a perceived lack of those skills or from a perceived strength in those skills?, and 2) How does this impact the instructors' approach to soliciting and leveraging student thinking in class?

Further, during our coding and analysis of the transcripts we noticed a consistent focus of pedagogical observations relating to the use of technology, such as Desmos and Zoom. The prevalence of these observations leads us to consider the potential of adding technology as a fourth vertex to the instructional triangle in our analytic framework, especially for classes conducted online. Further investigation on the interactions between the three elements of the instructional triangle and technology could highlight how the mechanics within Desmos can be used to elicit student thinking in different ways. Finally, we intend to explore further the ways that instructors differ in their pedagogical and content priorities and their interactions in order to better shape professional development opportunities within the project and better support PLC interactions and discussions in the future.

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A Survey of Programs Preparing Graduate Students to Teach Undergraduate Mathematics

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In the last decade there has been a concerted effort to improve the preparation of graduate students for teaching undergraduate mathematics. Now, it is important to gauge the current state of the national landscape of graduate teaching assistant (GTA) professional development for teaching (PDT) and to assess the current needs of the community. To this end, a census survey was conducted in Fall 2021 of Ph.D-granting mathematics departments in the US. The survey followed up and extended a similar survey conducted in 2014. Some survey questions were duplicated, while others probed more deeply into the depth and extent of the GTA PDT being offered. In this paper we compare some results from the surveys, comment on how the landscape of GTA PDT has changed, and highlight areas that deserve additional attention moving forward.

Keywords: teaching professional development; graduate student instructors; teaching assistants

The need to improve success, retention, and graduation rates of undergraduate students in science, technology, engineering and mathematics (STEM) is clear (e.g., Holdren & Lander, 2012; Zorn et al., 2014). Efforts of education researchers have amassed evidence of the significant roles that instructional practices play in shaping learning outcomes for such students (e.g., Freeman et al., 2014; Laursen et al., 2014). Students' experiences in undergraduate classes also impact their views of mathematics, interest in persisting in their studies, and likelihood of completing a STEM major (Hake, 1998; Seymour & Hewitt, 1997; Seymour & Hunter, 2019). Moreover, findings from education research have illuminated disparities in how instruction can impact students who arrive in our courses with different sets of experiences and backgrounds (e.g., Ellis et al., 2016; Laursen et al., 2014; Reinholz et al., 2022).

Fortunately, the education research community has generated findings to guide improvement in the teaching of undergraduate mathematics (e.g., Bressoud et al., 2015; Laursen, 2019). Of note are impacts of active, engaged student learning approaches on achievement, retention, and equity. Professional societies and other groups have amplified these findings and provided guidance and resources for implementation (e.g., Abell et al., 2018; Laursen, 2019). However, for students to benefit, instructors need professional development for teaching (PDT). It is now quite common for departments to offer PDT to graduate teaching assistants (GTAs) (Ellis, Deshler & Speer, 2016). Yet there is still much to learn about the PDT that is offered and how programs have changed over time. To explore this, we focused on two research questions: 1) What are the characteristics of current GTA PDT programs in Ph.D-granting mathematics departments? And 2) Have those characteristics changed since 2014 and if so, in what ways?

Data Collection

Description of the 2014 Survey

In 2014, a survey was sent to all 341 graduate-degree granting mathematics departments in the US. The survey was conducted jointly by two projects of the Mathematical Association of

America (MAA). The first, College Mathematics Instructor Development Source (CoMInDS), provides support to those who prepare GTAs for their teaching responsibilities, including workshops and access to materials for instructor development. The second project, Progress through Calculus (PtC) aimed to further understand characteristics associated with students' success identified by the Characteristics of Successful Programs in College Calculus project.

The survey items related to GTA PDT were designed jointly by members of the PtC and CoMInDS teams. The survey included both multiple-choice and open-ended items designed to provide insights into characteristics of programs and to reveal the interests and needs related to GTA PDT. Department chairs were encouraged to have local experts respond to portions of the survey they were most knowledgeable about. Specifically, coordinators and providers of GTA PDT were considered ideal respondents for survey questions related to GTA PDT. The survey had a 75% response rate from Ph.D-granting institutions.

Results characterized GTA PDT programs offered in mathematics departments across the US (see Apkarian et al. (2017) for a report of the full 2014 survey). A cluster analysis of these data revealed nine models of GTA PDT that varied with respect to the activities included, and the amount and timing of the PDT (Bragdon et al., 2017). Additional analysis indicated that only 19% of departments considered their GTA PDT programs to be preparing GTAs "very well," and 33% of departments reported there was room for improvement in their programs (Ellis, Deshler, et al., 2016; Rasmussen et al., 2019). One area that was identified as needing further attention was program evaluation (Ellis, Deshler & Speer, 2016; Speer et al., 2017).

Description of the 2021 Survey

To understand how things have changed since 2014, and get additional information, a survey specifically about GTA PDT (based on the 2014 design) was distributed in Fall 2021 to Ph.D-granting institutions; the response rate was 69.5%. Again, department chairs were asked to have local experts complete it. Some questions were identical or like the questions asked in 2014 while others probed more deeply to better understand the nature of the PDT. New questions were added about the providers of PDT and how they are chosen for their positions. Mixed-methods analysis of the data is ongoing.

Results: Understanding the current landscape of GTA PDT

Here we highlight some notable similarities and differences between findings from the 2014 and 2021 surveys based on our descriptive and qualitative analysis. Both surveys generated data about roles GTAs play in undergraduate instruction and whether they participate in PDT. All respondents to the 2021 survey reported that graduate students assisted with instruction and over half reported that most, if not all, GTAs serve as instructors of record at some point in their graduate education. GTAs' roles in instruction have largely remained the same since 2014 (being graders, tutors, recitation leaders, instructors of record, and assisting with in-class instruction).

There was, however, a substantial increase (from 77% in 2014 to 88% in 2021) in the percentage of Ph.D-granting institutions that reported **all** of their GTAs participate in department-based PDT. PDT may be offered in the department, outside the department within the institution or outside of the institution. Currently, over half (57%) of responding institutions reported that over 90% of the PDT was offered within the department and only 7% report that less than half of the PDT is offered within the department.

There has been a notable change in the formats in which GTA PDT is offered. A majority of departments now offer term-long courses or seminars, short workshops or orientations, and additional occasional workshops or seminars. Table 1 shows changes from 2014 to 2021. Open responses indicate that some departments have changed their program's structure to include credit-bearing courses, regular teaching seminars, and have adjusted placement of GTAs in their teaching roles. New in 2021, institutions were asked to choose from a list of topics GTAs typically learn about in their formal preparation for teaching. Table 1 highlights the eight most frequently selected topics addressed in the PDT.

Formats offered	2014 (n=111)	2021 (n=118)	**Most addressed topics**	2021 (n=114)
Term-long course or seminar	60%	75%	University and or dept. policies	86%
Multi-day workshop	34%	48%	Grading strategies	78%
Short workshop or orientation (1-4 hours)	24%	79%	In-person classroom management	76%
Occasional seminars or workshops	16%	73%	Active learning	76%
One-day workshop	13%	39%	University resources for students	70%
Online modules	Not asked	43%	Equity and diversity in the classroom	65%
Other	13%	16%	Facilitating group work	64%
			Academic integrity concerns	64%

Table 1: Findings from survey questions: Which of the following formats are typically included in your GTA preparation program? Which of the following topics do GTAs typically learn about in their formal preparation for teaching? (Mark all that apply.)

The 2021 data allow us to better describe the community of providers of GTA PDT than was possible with the 2014 data. The providers are a mix of non-tenure track and tenure track faculty where a majority are non-tenure track. When choosing or assigning providers of GTA PDT, departments favored individuals who were willing to take on the role and were considered excellent teachers. Answers from the non-required open response questions indicated that other considerations included whether faculty work with GTAs in another capacity, like supervising or coordinating courses (40% of responses) or are involved in the graduate program (20% of responses). 25% of responses indicate that faculty were hired specifically with this responsibility.

Both surveys asked respondents to identify what additional supports are needed for providers to improve GTA PDT. In the 2021 survey, we asked "What resources would be most helpful to you, as a provider, in strengthening your GTA teaching preparation program, if desired?" Respondents were given six choices and asked to mark all that apply. Some of the results from Table 2 can be compared over time. One of the choices was "Tools for evaluating effectiveness of GTA teaching preparation"; 67% of respondents identified this as a need in 2014 and 80% of respondents checked that box in the 2021 survey. Respondents identified a critical need for the community to develop models, instruments, and methods to better understand the extent to which PDT efforts are effective and successful. This result suggests that a major effort in future work needs to be in developing tools and protocols for effective assessment of GTA PDT programs.

Most helpful to strengthen GTA PDT programs	2014 (n=91)	2021 (n=104)
Tools for evaluating effectiveness of TA teaching preparation	67%	80%
Research-based information about best practices in TA teaching preparation	74%	78%
Online library of tested resources	48%	72%
Professional development for TA teaching preparation staff	51%	53%
Online workshops for GTAs	Not asked	45%

Table 2: Findings from question: What resources would be most helpful to you, as a provider, in strengthening your GTA teaching preparation program, if desired? (Mark all that apply.)

The desire for the "Online library of tested resources" also stands out in Table 2. The CoMInDS project has created just such a library of materials to use in facilitating PDT (available via connect.maa.org). The CoMInDS Resources Suite contains materials for use in pre-term orientation sessions, semester-long seminars, periodic workshops and more. Resources, including assignments, activities, readings, syllabi, etc., are submitted and reviewed through an editorial process as the repository continues to grow. Given the existence of these resources, one interpretation of the survey finding is that the CoMInDS Resources Suite is being underutilized. Another interpretation is that folks would like a larger or differently accessible online library, or perhaps a different set of resources entirely.

Limitations and Future Directions

In the 2021 survey, 96% of institutions responded "yes" to the question "Do graduate students in your department participate in any formal preparation for their teaching responsibilities?" In any survey with less than a 100% response rate one should consider the effects of unit nonresponse bias (Bose, 2001). A response rate of 69.5% for a survey of this kind is very high. However, intuitively, one might suspect a higher response rate from departments with programs than from those without. It is likely that fewer than 96% of the non-responding institutions have a GTA PDT program, though it is difficult to quantify.

Another concern was whether there was a "CoMInDS summer workshop bias." At these multi-day workshops (running each summer 2016-2021) participants designed or revised GTA PDT programs. The workshops have served 203 participants from almost half (87) of the Ph.D-granting mathematics departments in the U.S., in addition to some masters-degree granting institutions (Bookman & Speer, 2021). Institutions that sent participants to a CoMInDS summer workshop might have been more likely to respond to the survey than those that did not. 73.6% of workshop-participating institutions responded compared to 65.4% of non-participating institutions. A 2x2 chi-square test revealed no statistically significant differences at the .05 level in those response rates ($p=0.223$). In future work, we may examine effects of differences between 'early' respondents to 'late' respondents and the effect of item non-response.

In both surveys, we collected data about the formative and summative feedback given to GTAs about their teaching and about how well providers feel their program does at preparing new GTAs. Analysis of these questions is ongoing, however the mixed-methods analysis of the 2014 survey data revealed that most programs used student evaluations of teaching (SETs) to evaluate the teaching of graduate students in their department and there were limited efforts undertaken to evaluate the effectiveness of the programming (Speer et al., 2017). From preliminary analysis of the 2021 survey data, it appears that SETs are still the most common method of formally evaluating GTAs' teaching (91% of institutions reported doing so).

As of 2021, graduate students in 88% of Ph.D-granting mathematics departments have opportunities to participate in PDT that is offered by their department. Although that certainly indicates that many GTAs are receiving preparation for their teaching-related responsibilities, programs vary in terms of enrollment. As a result, we do not know what percentage of all mathematics graduate students are participating in PDT. If the large Ph.D-granting programs are represented in the 88% then person-level participation may be even higher. However, if the 88% primarily includes small-enrollment programs, then the actual person-level participation rate may be lower. Knowing more about the size of the programs represented in the response sample could inform strategies used to increase the reach of efforts to departments that are currently not offering programs. On-going analysis will further examine this issue via other data provided on the survey as well as publicly available data from the CBMS survey of programs.

We anticipate that further analysis of responses to open-ended prompts will provide additional insights. Several comments highlighted major changes that have been implemented to GTA PDT programs, including hiring a GTA PDT provider and/or dedicated course coordinators. Course coordinators may be well-positioned to take on the role of providing PDT for GTAs (Martinez et al., 2021), and future analysis of the survey data could shed light on the ways coordinators are complementing existing GTA PDT programs.

Conclusions

The mathematics community has responded to calls for increased attention on professional development for those new to teaching. Now, 88% of departments offer PDT to graduate students whereas in 2014 only 77% did so. The scope and content of those programs has also changed and now better reflect what is known from research about effective design of PDT. The K12 education community has recognized that development of teaching practices and knowledge requires sustained engagement (Blank & de las Alas, 2009). In 2014, 60% of departments offered a term-long course or seminar and today, 75% of departments do so. Such courses or seminars may represent as little as 10-15 contact hours (for a 1-credit course) or as much as 30-45 contact hours (for a 3-credit course). This falls short of the 100 hours that is regarded as necessary for impact (Banilower et al., 2006), however, when taken together with other changes we see, the shifts since 2014 are encouraging.

We also see that current programs are focusing on issues of importance to the broader mathematics community (e.g., active learning, collaborative groupwork, and diversity/equity/inclusion, see Table 1). We lack comparison data but the presence of such topics in the majority of programs suggests that those involved in program design are well-informed about current needs of the mathematics community and findings from education research.

Those involved in GTA PDT continue to request tools to better understand the impact of their programs on teaching and learning. In 2014 this was expressed by a majority (67%) and today, 80% say this is something they need. While some of this increase may be a result of trends about assessment and evaluation seen throughout the mathematics community, the finding points to the need for ways of examining the impact of PDT on instructors (which also requires tools for assessing and evaluating instructional practices). This creates opportunities for increased collaboration among those with expertise in PDT programs for GTAs, research on the teaching and learning of undergraduate mathematics, and program evaluation.

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Professed Beliefs of a Teaching Assistant of an Introductory Proof Course

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It is known that teachers' beliefs influence how they filter new knowledge and understanding of pedagogies and can directly impact their classroom practices. Moreover, the context can shape beliefs, and thus it is imperative to examine the beliefs of teaching assistants (TAs) of introductory proof courses because they play important roles in supporting student learning at this crucial mathematical junction. This preliminary study explored the professed beliefs of Lisa, a second-year, mathematics doctoral student. We qualitatively analyzed her responses in a semi-structured interview - with a particular emphasis on exploring her beliefs about teaching and beliefs about introductory proof courses. Overall, she believed that introductory proof courses pose new challenges and demands for undergraduate students. We saw connections to her beliefs about how TAs should deliver content and support students. Critical examination of TAs' beliefs can inform how mathematics departments prepare graduate students to be teaching assistants.

Keywords: Teaching Assistants, Beliefs, Introductory Proof Courses

Introduction

In the present preliminary study, we examined the professed beliefs (Speer, 2005) of a teaching assistant (TA) of an introductory proof course. Introductory proof courses mark a critical junction in the academic trajectories of many undergraduate students pursuing mathematics degrees. Students are confronted with significant challenges in proof-construction (e.g., Moore, 1994; Weber, 2001), because the demands associated with proof-construction are unlike those of courses such as calculus and linear algebra. Graduate students who serve as TAs play important roles in supporting student learning during this transition to upper division mathematics courses that heavily rely on proof-construction competencies. It is imperative that we better understand what mathematics TAs believe about students, learning, and teaching at this important junction. These beliefs can not only impact their current instructional practices (e.g., Pajares, 1992; Goldin et al., 2009), but they can also filter how TAs learn about and develop their pedagogies (Brousseua et al., 1988) to teach upper division coursework.

Prior work has illustrated how teachers of different domain areas (e.g., STEM, humanities, etc.) have described and exhibited differences in beliefs about aspects of their instruction (e.g. Patterson et al., 2016; Roothooft, 2019), and we posit that the specific content areas within the mathematics discipline (e.g., calculus, linear algebra, etc.) can uniquely influence what TAs believe. To date, there is limited work that is specific to examining the beliefs of TAs of introductory proof courses. As we move forward, findings from studies of this nature can inform how higher education institutions and mathematics departments can better support graduate students' professional development and teaching practices in proof-based courses. If we are to positively develop these dimensions of teaching and instruction, it is important that we first identify and examine the beliefs that graduate students bring into these courses and develop as they serve as teaching assistants or early instructors of record. The research question that guided this study was: How did a mathematics teaching assistant discuss their beliefs about teaching and mathematics in an introductory proof course?

Framework

We drew on a categorization of professed teacher beliefs (Calderhead, 1996; Carter & Norwood, 2010; Torff, 2011) to frame the present exploratory study. Research on teaching and educational psychology have previously described beliefs as being professed or attributed (Calderhead, 1996; Putnam & Borko, 2000). Speer (2005) stated, “professed beliefs are defined as those stated by teachers, while attributed beliefs are those that researchers infer based on observational or other data” (p. 361). We consider the beliefs of the focal TA as primarily professed beliefs because we as researchers did not observe their classroom practices nor inferred beliefs from observed actions. We built upon the work of Speer (2001) who previously connected teaching assistants’ beliefs to their teaching practices in reform-oriented calculus courses. Speer described and categorized teaching assistants’ *belief profiles* to encompass 1) what the teaching assistants believed about their students, 2) what they believed about teaching, 3) what they believed about learning, and 4) what they believed about mathematics.

Other scholars have similarly categorized beliefs as a way of identifying and assessing teachers’ professed beliefs (e.g., Carter & Norwood, 2010; Voss et al., 2013). Although beliefs are extremely interconnected, categorizing them is helpful in making sense of an individual’s broader belief system. Voss et al. (2013) provided a rationale for why researchers often classify and categorize teacher beliefs, writing: “These detailed taxonomies are helpful and necessary when it comes to addressing theoretical questions about specific components or painting a coherent picture of belief systems” (p. 252). In this preliminary study, we focus on two primary dimensions: beliefs about teaching and beliefs about mathematics. More specifically, we sought to identify how beliefs about teaching and mathematics are related to introductory proof courses and the extent to which the focal TA’s beliefs about her teaching and mathematics is shaped by the course content and her perceptions of the demands of introductory proof courses.

Method

Purposeful sampling (Miles et al., 2020) was used to recruit graduate students of an introductory proof course at a Minority-Serving Institution in California. It was part of a larger study about transfer students’ experiences in a department intervention involving a set of courses to help transfer students transition to a four-year university. The focal introductory proof course, Math A, was one of these courses. Participants were mathematics doctoral students who were TAs for the course during the Fall 2020 quarter. We focus on Lisa (a pseudonym) as a case for this preliminary study because she was revelatory (Yin, 2016) about nuanced professed beliefs regarding teaching and mathematics that were also related to the context of the introductory proof course. She self-identified as female and was a 2nd year doctoral student pursuing a degree in mathematics. Moreover, at the time of data collection, Lisa had two quarters of experience as a TA for the introductory proof course. The research team qualitatively analyzed her responses to a semi-structured interview protocol (Rubin & Rubin, 2011). The following questions guided our categorization, coding, and analysis of Lisa’s professed beliefs: (1) What does Lisa believe about her teaching practices and role as a TA? (2) What does Lisa believe about mathematics and introductory proof courses? We pair-coded the interview transcript, discussed our codes, and wrote memos throughout this preliminary analysis of themes in her professed beliefs. Code charting (Saldaña, 2021) using matrices was used to organize the categorization of her beliefs and this afforded insight into themes across the various dimensions of her beliefs.

Findings

We present initial findings highlighting themes in Lisa's professed beliefs about introductory proof courses and her role in teaching and supporting student learning. Lisa expressed some beliefs unique to the context of and circumstances around introductory-proof courses (e.g., it is a course mainly for mathematics majors, it is the gatekeeper to upper division courses, etc.). She also expressed beliefs related to characteristics and key practices of what she believed makes a successful TA of such courses.

Beliefs about Teaching

Lisa expressed beliefs related to her role as a content-deliverer and as someone who is a supporter of her students. Regarding the dimension of her teaching related to content-delivery, she recognized the constraints of her role as a teaching assistant and believed that she should reinforce the material already presented by the professor - but make it more personable and approachable for the students. She described the importance of being "less scary" than the professor and recognized the impact that instructors' demeanors can have on student learning and engagement. She said:

> I guess, I kind of view it as like I'm someone to reinforce the material that the professor is doing. So just like it can be more personal, reinforce ideas, because they're very abstract when they're first given to students. And then, just to be someone who is less scary than the professor. My job is just to be approachable.

Lisa frequently returned to this notion of being personable. She reflected on how she introduced herself to students at the beginning of the semester. Her goal was to convey the message: "I care about you as a person and I'm going to take the time to try and show it." She believed TAs should be clear about their intentions to support students and that this should be explicitly communicated. Another key belief associated with teaching and content delivery was her belief that a TA should present material in a way that is digestible and understandable for students, especially with material they had not seen before. She said, "I think knowing how to present the material in a way that you know someone who's just seen upper division math for the first time can do it, can understand it." Lisa understood that many students struggle in making sense of proof-construction and believed that TAs should strive to make material accessible and understandable to all students. This included actions such as exhibiting patience, breaking down complex concepts, and scaffolding students' proof construction by first showing easy examples, and moving to "hard interesting problems." She recalled noticing a difference in students' engagement and interactions when she expressed that she cares for them. She said, "I tried really hard to like, start off being with like, 'Okay, I care about you as a person and I'm going to take the time to try and show it to you.' and I think that worked well." Overall, Lisa believed that a TA needs to make the content relatable and accessible while also providing support and demonstrating care for students.

Beliefs about Mathematics and Introductory-Proof Courses

In examining her beliefs about teaching, we saw that there were some connections to her beliefs about the mathematical difficulties associated with introductory proof courses. Lisa primarily expressed beliefs about the nature of these courses - specifically that they mark a junction in the content, problem-solving, and sense-making demands that students encounter in these courses and beyond. She believed that constructing mathematical proofs are unique from the mathematics that students have previously seen and that this poses challenges that many students may not yet have faced. She said, “They’ve never been into, like a proof-writing, class everything's really different.” Moreover, in reflecting on her position as a teaching assistant for the Math A course, she expressed a concern and critique about the focus of introductory proof courses. Lisa believed that the emphasis of such courses is on the correctness of the final proof, rather than a deeper understanding of the conceptual and logical underpinnings of constructing proofs. She commented,

> I think my biggest thing about Math A is there's such, still such an emphasis on like the correct answer in Math A. But like ideally, from my point of view, [the] Math A goal should be: “Do you know how to write a proof?” So I think a lot more emphasis on like writing proofs well, and like having students actually get graded feedback.

She suggested that providing students with feedback, rather than just a grade would help attend to the concern and overemphasis on correct answers and proofs. Lisa believed that introductory proof courses should better prepare students to construct proofs – not just in rote and procedural ways, but rather with a deeper understanding that can support them as they continue to more advanced mathematics.

Discussion

Overall, we observed that Lisa placed importance on being approachable as a TA and engaging in practices that allow all students to access the mathematics content. Lisa believed that TAs should make content digestible and understandable and that the emphasis of Math A should be on supporting students’ deep and conceptual understanding of proof-construction. Altogether, we can infer that she believed that affect-oriented practices that attend to students’ motivation and fostering positive perceptions of TAs can support student learning in introductory proof courses. Wiesman (2012) previously found that some teachers believed that students’ motivation in the classroom are impacted by teacher actions and that demonstrating care can have positive impacts on academic achievement. This idea is echoed in Lisa’s reflections on how she should foster safe classroom environments where students feel supported and cared for. Also, Lisa’s consideration of students’ perceptions of TAs and providing more socio-emotional support is consistent with aspects of the work of Beijaard et al. (2000) who characterized some teachers as *pedagogical experts*, or those who base their sense of selves as teachers on supporting students’ social and emotional development. Ultimately, our preliminary findings of Lisa’s beliefs are consistent with existing research on how beliefs may be categorized (e.g., Ernest, 1989; Speer, 2001; Carter & Norwood, 2010; Voss et al., 2013) but are still interconnected and influence each other (e.g., Pajares, 1992) to form larger belief systems.

We also recognized that some of the professed beliefs were explicitly connected to introductory proof courses, which supports the notion that beliefs are informed by the mathematical contexts and domains. For example, her belief that proof-construction poses challenges and introduces a level of cognitive demand many students have not yet faced (i.e, a belief directly related to the specific course she was a TA for) was a significant part of her reflections on why she should be more personable and approachable to students. We also saw preliminary findings illustrating how course contexts can impact beliefs. We may expect a TA in a precalculus course and a TA in an introductory-proof course to have nuanced beliefs about mathematics and teaching that are reflective of their different experiences and content areas. Lastly, we also highlighted Lisa’s belief regarding the nature of introductory proof courses and their overemphasis on rote proof-construction and the correctness of the final answer. This belief can have implications for how Lisa develops her own pedagogies and practices as a TA and a potential faculty later in her career because it can serve as a lens through which she adopts or rejects new pedagogies. She may make the effort to combat the traditional emphasis on correctness and procedural understanding and make conscious effort to develop practices that better support deeper learning.

Limitations and Implications

One limitation of the study is that we only focused on the beliefs of one TA, which did not allow for a comparison of professed beliefs across multiple participants. Future work could perhaps examine more participants (i.e., more TAs of introductory proof courses) or more TAs across multiple courses (i.e., examine TAs of calculus, linear algebra, introductory proof courses, etc.) Secondly, a key limitation of research on beliefs is the extent to which participants can convey and articulate their beliefs. Future research should examine other dimensions of TAs' beliefs (e.g., beliefs about how students learn) or observe how TAs facilitate their sections to identify any connections between observed practices and their professed beliefs. A richer understanding of TAs' beliefs, uniquely situated in specific courses, has important implications for practice and research. It can inform department-level training, coursework, and other professional development opportunities for graduate students, and this can expand our understanding of the belief systems of individuals involved in undergraduate mathematics instruction.

Conclusion

We consider research on TAs' beliefs an important, yet under-researched, dimension of undergraduate mathematics education, especially in courses like introductory proof courses which are foundational to students' success in upper division coursework. Our preliminary findings highlight connections between a TA's beliefs about introductory proof courses and mathematics and her beliefs about teaching, and this sheds light on the complexity and interconnectedness of beliefs. A richer understanding of the beliefs of TAs of these courses can also be resources for mathematics departments because they afford insight into students' experiences from the perspective of those who often, more frequently, interact with students and see firsthand the challenges they encounter. With many graduate students continuing on to become faculty and may later teach proof-based courses, critically examining their beliefs when they serve as teaching assistants can inform how institutions and mathematics departments prepare them for careers as post-secondary educators.

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Student Confidence and Certainty in Comprehension of a Proof by Induction

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Researchers typically utilize response correctness to interpret student proficiency in proof comprehension. However, metacognition has not been simultaneously analyzed alongside correctness to determine student competency in proof comprehension, yet it offers important information about performance behavior. The primary objective of this study is to investigate the accuracy of student confidence and certainty levels at local and holistic aspects of proof comprehension regarding a proof by induction. Students were given a three-factor proof comprehension assessment at the beginning and end of an undergraduate transition-to-proof course that collected student confidence, correctness, and certainty at each tier of an established proof comprehension framework. Results of this study highlight a critical distinction between high and low performers' metacognition throughout the host course. One outlying assessment item especially illuminates additional considerations for future application of metacognition in proof comprehension research.

Keywords: Proof comprehension, Proof by induction, Confidence and certainty, Metacognition, Multidimensional competency assessment

1 Introduction

The present study investigates the interplay between students' confidence and certainty (states of metacognition) with response correctness within proof comprehension assessments. The discussions presented in this paper operate on the following definitions of confidence and certainty. *Confidence* refers to an individual's level of belief that they *can provide* a correct response to an assessment item pertaining to a given set of clearly defined criteria, whereas *certainty* refers instead to an individual's level of belief that they *have provided* a correct response to an assessment item. Confidence and certainty are differentiated by their context and timing at which they are measured. Confidence, measured prior to viewing assessment items, is a self-assessment for anticipated performance, while certainty is a self-assessment of past performance measured after providing a response to an assessment item.

1.1 Metacognition and Multidimensional Knowledge Assessment

Metacognition includes any way of monitoring of cognition (Kitchener, 1983) and thus the term has been applied with a wide variety of interpretations. Commonly, metacognition is categorized into two broad areas – metacognitive knowledge and metacognitive skills (Veenman & Spaans, 2005). Metacognitive knowledge involves one's beliefs about the interactions of person, task, and strategy knowledge (Flavell, 1979), whereas metacognitive skills refer to the "procedural knowledge that is required for the actual regulation of and control over one's learning activities" (Veenman & Spaans, 2005). Confidence and certainty as previously defined address a subcategory of metacognitive knowledge (relating person knowledge to task knowledge) wherein an individual self-monitors their cognitive performance.

Knowledge is information which is both true and justified (Hunt, 2003). Student certainty has been well studied for providing response justification during assessment (Gardner-Medwin and Gahan, 2003; Ghadermarzi *et al.*, 2015; Hunt, 2003; Snow, 2019). The authors of the present study have investigated simultaneous evaluation of confidence, certainty, and correctness (the

"3Cs") within an undergraduate linear algebra course and established corresponding accuracy indices for identifying outlying student metacognitive behaviors and providing new insights for assessment item analysis (Preheim *et al.*, 2022; Preheim *et al.,* 2023). The 3C assessment structure and analysis has yet to be used to investigate student competency in proof comprehension.

1.2 The Role of Confidence and Certainty in Proof Comprehension and Proof Writing

An individual's confidence and certainty in proof comprehension may impact decisions which occur throughout proof writing tasks wherein they must make strategic choices on how to proceed (*e.g.*, deciding which method of proof to initially employ, which strategy to use to satisfy a particular case, *etc.*). For example, consider a student who does not feel confident in their comprehension of proof by contrapositive as a technique. When posed with the task of proving "if n^2 - $(n\text{-}1)^2$ is not divisible by 8, then n is even," this student might shy away from a proof by contrapositive and be led to attempt a direct argument (which would be more tedious). Similarly, an individual's certainty in their proof comprehension has the potential to influence their willingness to move to a next step in a proof-writing task or impact their conviction in the truth of the mathematical statement being proved.

While the role of proof in establishing mathematical conviction (*i.e.*, belief in the truth of a mathematical statement) has been discussed and investigated by researchers such as Weber or Mejía-Ramos (Weber & Mejía-Ramos, 2011; Weber *et al.*, 2014; Weber & Mejía-Ramos, 2015; Weber *et al.*, 2022), student confidence and certainty regarding proof comprehension has not been thoroughly investigated. Mejía-Ramos *et al.* (2011) developed a model by which to construct assessment items for proof comprehension assessments (PCAs) in undergraduate mathematics. This proof comprehension framework (PCF) identifies seven aspects of proof comprehension separated into two tiers: local and holistic (Mejía-Ramos *et al.*, 2012). Assessment items at the local level are intended to assess an individual's comprehension of a small number of statements contained within a proof. At the holistic level, items assess an individual's understanding of the proof as a whole (Mejía-Ramos *et al.*, 2012). Student metacognitive behavior at each level of the PCF has not been empirically investigated to measure student competency or assessment item validity.

The objective of this study is to investigate students' confidence and certainty levels and their accuracies at local and holistic aspects of proof comprehension regarding a proof by induction. Distinctions between high and low performers' metacognitive behaviors from the beginning to end of the host course are also presented and discussed.

2 Methods

2.1 Course Information and Assessment Structure

The introduction to abstract mathematics course which hosts the focused student population in the present study serves as the transition-to-proof course for undergraduate students at North Dakota State University. This course focuses on proof-writing tasks and proof comprehension amidst various topics in abstract mathematics. 3C data was collected via two assessments per section from 36 students in four sections of the course.

The PCA used in this study assesses comprehension of a proof by induction that every third Fibonacci number is even. This multiple choice PCA developed by Mejía-Ramos *et al.* (2017) which adheres to the PCF (Mejía-Ramos *et al.,* 2012) was adapted to include 3C assessment methodology. This same 40-minute in-person assessment was administered near the start of the course (the initial assessment) as well as near the end of the course (the post assessment).

Each assessment item from the PCA was mapped to one PCF subitem. Immediately preceding each assessment, student confidence levels were collected for pertinent PCF subitems. The prompt requested students to indicate their level of confidence in each PCF subitem (4=High confidence, 3=Somewhat high confidence, 2=Somewhat low confidence, and 1=Low confidence). In the first semester of collected data, students provided their confidence levels prior to viewing the proof or assessment items. During the second semester, students were given five minutes to read the proof prior to providing confidence levels.

Once the confidence self-assessments were completed, students were given an assessment packet which contained the 12 multiple-choice questions. Following each multiple-choice question, a corresponding certainty prompt (similarly structured to the confidence prompt) instructed students to record their certainty in their response before moving on to the next item.

2.2 Accuracy Indices and Metacognitive Accuracy Plots

An individual student response yields the following three values: c_1 = confidence level, c_2 = correctness (1 for correct and 0 for incorrect), and c_3 = certainty level. Two similarly defined functions were developed to measure the alignment of confidence with credit (AccConf) and the alignment of certainty with credit (AccCert). The formulas for each are shown in Figure 1.

$$\mathrm{AccConf}(c_1, c_2, c_3) = \begin{cases} 1 & \text{if } \mathrm{c}_2 = 1 \text{ and } c_1 = 3 \text{ or } 4 \\ -1 & \text{if } \mathrm{c}_2 = 1 \text{ and } c_1 = 1 \text{ or } 2 \\ -1 & \text{if } \mathrm{c}_2 = 0 \text{ and } c_1 = 3 \text{ or } 4 \\ 1 & \text{if } \mathrm{c}_2 = 0 \text{ and } c_1 = 1 \text{ or } 2 \end{cases} \qquad \mathrm{AccCert}(c_1, c_2, c_3) = \begin{cases} 1 & \text{if } \mathrm{c}_2 = 1 \text{ and } c_3 = 3 \text{ or } 4 \\ -1 & \text{if } \mathrm{c}_2 = 1 \text{ and } c_3 = 1 \text{ or } 2 \\ -1 & \text{if } \mathrm{c}_2 = 0 \text{ and } c_3 = 3 \text{ or } 4 \\ 1 & \text{if } \mathrm{c}_2 = 0 \text{ and } c_3 = 1 \text{ or } 2 \end{cases}$$

Figure 1. The metacognitive accuracy formulas AccConf and AccCert. A weighted average of these accuracy values produces the accuracy indices AIConf and AICert.

Accuracy values of 1 indicate accurate confidence/certainty while accuracy values of -1 indicate inaccurate confidence/certainty. A weighted average of accuracy values across responses was computed for each assessment to generate an accuracy index for confidence (AIConf) and an accuracy index for certainty (AICert). For this weighted average, responses reporting confidence/certainty levels 1 and 4 were weighted at three times the weight of those corresponding to levels 2 and 3. Metacognitive accuracy plots were produced to display the transitions in AIConf and AICert from the initial to post assessment among all students, high performers (students in the top 25th percentile of course standing), and low performers (students in the bottom 25th percentile of course standing).

3 Results

Among all students, AIConf increased from the initial assessment to the post assessment for all local items except item Q4 (Figure 2a). Among high performers, AIConf increased by more than 0.20 from the initial assessment to the post assessment for ten of the twelve items (excluding Q4 and one item with near maximum accuracy) (Figure 2b). Among low performers, AIConf and AICert decreased from the initial to post assessment for all holistic items (Figure 2c). Furthermore, Q4 showed substantial decrease in both AIConf and AICert among low performers (Figure 2b) and substantial increase in AICert among high performers (Figure 2c).

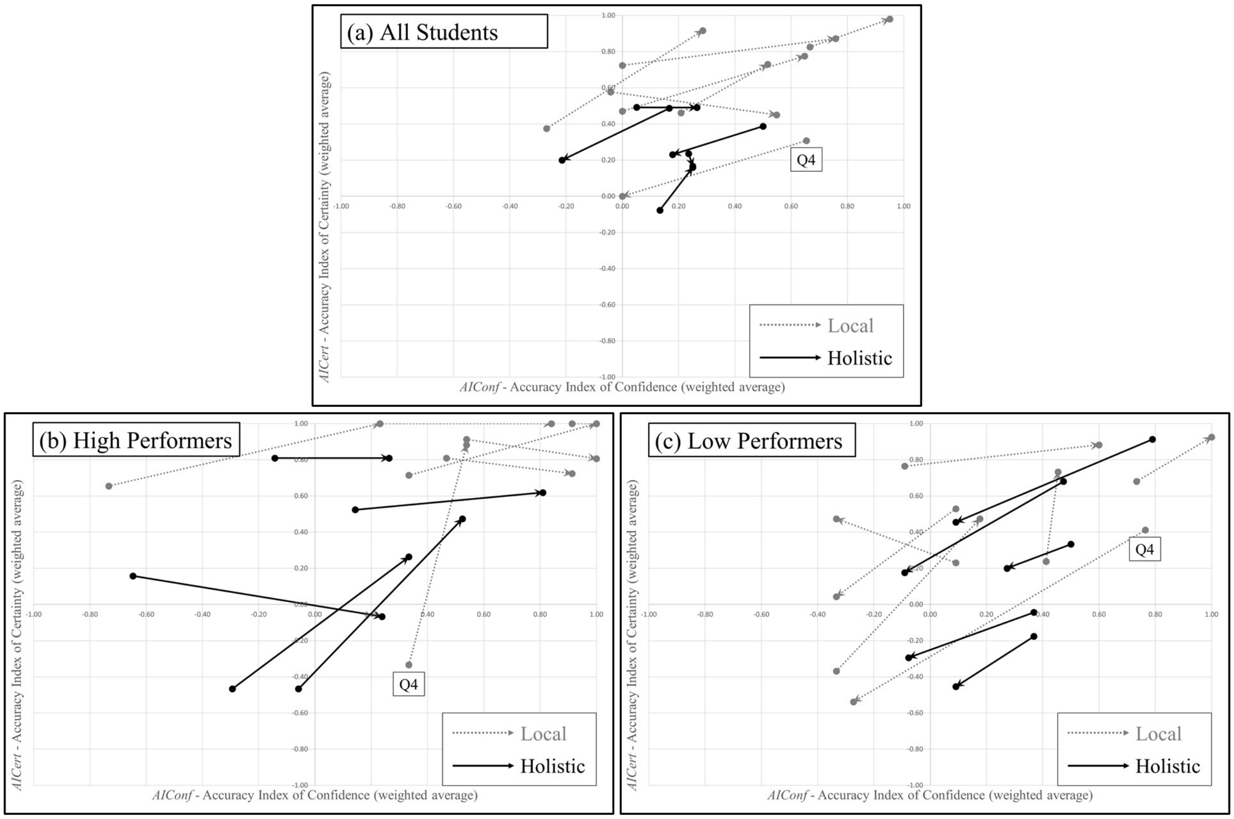

Figure 2. The transitions of AIConf and AICert for each assessment item from the initial to the post assessment are plotted as arrows on a graph of AICert vs AIConf. Items addressing local aspects of the PCF are represented with gray dotted arrows while those addressing holistic aspects are represented by black solid arrows.

Reported confidence and certainty levels demonstrated significant ($\alpha<0.05$) increase from the initial to the post assessment for all and eleven of the twelve items, respectively. Certainty levels were higher than confidence levels on average among questions pertaining to local aspects of the PCF for both the initial assessment and the post assessment. However, certainty levels among questions pertaining to holistic aspects of the PCF were lower than confidence levels during the post assessment. On average AICert was higher than AIConf among both assessments for all items, local items, and holistic items. Average confidence and certainty levels as well as correctness scores increased from the initial assessment to the post assessment among all items, local items, and holistic items. Though average AIConf and AICert among questions addressing local aspects of the PCF increased from the initial to the post assessment, both decreased among questions addressing holistic aspects of the PCF (Table 1).

Table 1. The average confidence level, certainty level, score (correctness), AIConf, and AICert on the initial and post assessments among all items, local items, and holistic items.

		Confidence	Certainty	Score	AIConf	AICert
All	Initial	2.26	2.63	0.58	0.19	0.44
	Post	3.07	3.14	0.67	0.37	0.50
Local	Initial	2.56	2.92	0.70	0.17	0.53
	Post	3.26	3.40	0.75	0.53	0.67
Holistic	Initial	1.95	2.21	0.40	0.22	0.30
	Post	2.88	2.77	0.54	0.15	0.25

4 Discussion

Transitions from the initial to the post assessment highlight a critical distinction between questions addressing local aspects of the PCF and those addressing holistic aspects. Increased accuracy of confidence regarding local aspects of the PCF indicates that students' increase of confidence levels at the local tier (Table 1) was accurate to their performance. Though questions at the holistic tier also saw an improvement of confidence levels and score (Table 1), the varying shifts in students' accuracy of confidence (Figure 2a) suggest that the increase of confidence was not necessarily indicative of an increase in performance. This may suggest that students feel more comfortable reading mathematical proof by the end of the course, but it does not guarantee they are more competent in comprehending proof at the holistic level.

A further distinction between local and holistic aspects is made when comparing the shifts of AIConf and AICert among high versus low performers. The results shown in Figure 2 (b and c) demonstrate that while high performers exhibited increased accuracy of confidence among holistic aspects of proof comprehension, low performers experienced a decrease in metacognitive accuracy among holistic aspects of the PCF. These preliminary findings suggest that training student metacognition with respect to proof comprehension may be beneficial for improving student performance in abstract mathematics. Further investigation of this idea is merited.

3C data identified outlying metacognitive behavior on assessment item Q4 which would not have been detectable by correctness-only assessment methods. Qualitative evaluation of the question prompt and multiple-choice options indicates that while the prompt seems to address a local aspect of the PCF, the correct option requires holistic understanding of the proof. As these PCA items originated from open-response questions (Mejía-Ramos *et al.*, 2017), this discrepancy is not particular to the multiple-choice structure. Moreover, the most selected incorrect option includes a correct description of the proof's procedural structure followed by an incorrect conceptual justification for the final claim of the proof. This outlying behavior therefore supports an observation by Piatek-Jiminez (2004) that procedural knowledge and conceptual knowledge of proof by induction are not necessarily reliant upon each other.

5 Conclusions, Future Directions, and Questions for the Audience

The 3C assessment distinguished metacognitive behavior at the holistic level between high- and low- performing students. Development and use of methods for enhancing students' metacognitive accuracy in proof comprehension could improve student performance in abstract mathematics. Future directions of this research are focused on employing the 3C assessment methodology to improve students' metacognitive accuracy among holistic aspects of proof.

Adapting proof comprehension assessments to utilize 3C assessment methodology allows for the detection of interesting outlying metacognitive behavior. Investigating such outliers can aid in revising assessment items, generating new assessment items, and guiding directions for continued research. For example, the investigation of outlying behavior observed in this study made the distinction between conceptual and procedural knowledge observable. To more comprehensively analyze proof comprehension within the PCF, future studies might consider methods to further distinguish conceptual and procedural knowledge and study the 3C methodology in other related and unrelated domains.

The authors conclude this proposal with some intended questions for the attending audience:

1. How might 3C data be used to guide assessment item revision or PCA construction?
2. How might metacognitive behavior differ if PCAs utilized open-response questions?
3. Are there other methods of proof which you expect would generate critically different metacognitive behavior than that demonstrated in the present study?

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Identifying Instructional Autonomy Approaches in Calculus and its Connection with Student's Perceived Classroom Climate

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Calculus courses are an integral component for STEM degree attainment yet are historically attributed to high attrition rates and lowering student confidence. Although prior research identified coordination of instruction as a successful characteristic of Calculus programs, there is a need to further examine the role of Instructor Autonomy (e.g., decision making on content and instruction) and its potential impact on student perceptions of the climate. We analyzed data from two national studies of introductory math courses focusing on instructor (N=321) reports of course decision making and student (N=14,488) reports of climate. Preliminary results indicated a great deal of variability in Instructor Autonomy across the Precalculus to Calculus 2 sequence within our sample and even within a department. Narrowing our focus to Calculus 1, variability in Instructor Autonomy persisted and student perception of climate was not unified; however, courses that utilized team-based decision making promoted higher student perceptions of climate.

Keywords: course coordination, calculus, instructor autonomy, student voice

Research indicates Calculus 1 is effective in lowering student's confidence, their enjoyment of mathematics, and their desire to continue in a discipline or future course where mathematics is needed (Bressoud & Rasmussen, 2015). Globally, Calculus 1 courses are associated with high failure rates and attrition (Cavalheiro & Grebot, 2021; Hagman et al., 2017). In the United States, Calculus 1 averages a 22 percent DFW rate, the percentage of students who receive a grade of D, receive a grade of an F, or withdraw from the course (Rasmussen et al., 2019)—about one out of every five students. As such, efforts have increased towards the recruitment and retention of students (Calleros & Zahner, 2021; Patrick & Borrego, 2016) through the improvement of introductory math courses.

From 2009 to 2015, a national study of college Calculus 1 courses was conducted to measure what characteristics of Calculus 1 influenced student success. The Characteristics of Successful Programs in College Calculus (CSPCC) study aimed to improve retention rates for students in Science, Technology, Engineering, and Mathematics (STEM) majors (Bressoud & Rasmussen, 2015). The CSPCC study launched a national survey (n=212) to two- and four-year institutions and conducted case studies of 16 institutions identified as having successful programs in Calculus. The study identified seven characteristics of successful Calculus programs: 1. Regular use of curricular and structural modifications; 2. Attention to the effectiveness of placement procedures; 3. **Coordination of instruction, including building communities of practice**; 4. Construction of challenging and engaging courses; 5. Use of student-centered pedagogies and active-learning strategies; 6. Effective training of graduate teaching assistants; 7. Proactive student support services.

As identified by the CSPCC study, coordination of instruction is a characteristic of successful Calculus programs. Course coordination, as defined by Rasmussen and Ellis (2015), consists of two features, uniform course elements and regular instructor communication. Uniform course elements are elements of a course that are the same for every student across a multi-section course (e.g., textbook, topics, common homework, common tests). Regular instructor communication are moments when instructors contact and support each other regarding aspects of the course. With these two features, course coordination aims to reduce variation across multiple sections of a course by providing consistent experiences (Rasmussen & Ellis, 2015). When it comes to the uniform elements implemented and the frequency of communication with instructors, there is not one blueprint for implementation (Apkarian et al., 2019).

Methods

Data from this analysis comes from two national studies of college Calculus, the Progress through Calculus (PtC) study and the Student Engagement in Mathematics through an Institutional Network for Active Learning (SEMINAL) study. PtC cataloged departmental efforts in improving student success and documenting the effectiveness of implementation efforts (Rasmussen et al., 2019), while SEMINAL researched systematic change in mathematics departments towards the use of active learning in introductory mathematics courses (Smith et al., 2021). Both studies conducted in-depth case studies at 20 universities and implemented the X-PIPS-M Survey suite (Apkarian et al., 2019) for instructor, student, and teaching assistants (both graduate students and undergraduate students) across several academic terms.

Through a case study approach with constant comparative methods, we explored and evaluated instructor autonomy and student reports of the perceived classroom climate. The PtC and SEMINAL data sets have three categories of courses namely, Precalculus, Calculus 1, and Calculus 2. All institutions were coded according to their institutional characteristic as a Historically Black College or University (HBCU), as a Hispanic Serving Institution (HSI), or as a Predominantly White Institution (PWI) to both anonymize the institution as well as provide another way to compare institutions and trends. In total we had instructor and student data from 20 institutions with a total response of 321 instructors and 14,488 students. For the 9 institutions that had teaching assistants there were a total of 328 responses.

Instructional Autonomy

To analyze instructor autonomy, we looked at the instructor responses to the Likert scale questions on who was making the decisions about the course content and the instructional approach. Specifically, instructors answered the questions;

1. "How are most decisions about <u>course content</u> (e.g., syllabi, exams, homework, pacing, grading) made for the course? Clarify if you wish." and
2. "How are most decisions about <u>instructional approach</u> (e.g., use of clickers, group work, active learning) made for the course? Clarify if you wish."

with answer options of *1 ("I make most decisions"), 2 ("I am part of a team that makes most decisions"), or 3 ("Someone else makes most decisions")*. To better help understand the data, we developed a classification table to categorize the level of autonomy instructors had in their course, see table 1 (Instructor Autonomy). To classify a site using the Instructor Autonomy table we looked per site per instructor autonomy question. For the two questions we noted which answer choice had the highest percentage of instructor response per site. Given the exploratory nature of this analysis, categorization based on highest percentage allowed for further exploration and reflected the lived reality of instructors but does fail to capture the level of agreement between instructors. Then we combined the classification of the highest percentage of responses for the approach and the content to get the overall instructor autonomy per site for the course. For example, if HSI 1 had 60% of instructors say someone else decides content and 52% of instructors say they are part of a team that decides the approach, the site would be classified as

"Collective Approach Top Down Content". Due to 1 institution having too much variability and low response rates we chose to exclude PWI 14 from the analysis leaving us with 19 institutions. For analysis, after determining the instructor autonomy classification we compared and contrasted across sites, course levels, and within institutional characteristic groupings.

Student Classroom Climate

To evaluate the perceived classroom climate from the student's perspective we looked at three 5-scale Likert questions asking the student to describe the overall climate. Questions asked about the intellectual engagement, academic rigor, and atmosphere the student perceived in the Calculus courses as seen below,

"How would you describe the overall climate within [Calculus 1]?

1. Excluding and Hostile (1) _ _ _ _ _ Including and Friendly (5) [Atmosphere]
2. Intellectually boring (1) _ _ _ _ _ Intellectually Engaging (5) [Intellectual Engagement]
3. Academically Easy (1) _ _ _ _ _ Academically rigorous (5) [Academic Rigor]"

For analysis we compared and contrasted institutions across the average of the three climate questions. We then looked for possible trends between the site's instructor autonomy classification and the student's perceived climate by use of the institutional characteristics. Through grouping institutions by their institutional characteristics we also examined the student data to note any trends within a particular group of institutions that could then be compared to the other institutional characteristics groupings.

Preliminary Results

Initial analysis of looking across course level (Precalculus, Calculus 1, and Calculus 2) revealed that as the sequence of Calculus progressed there was more instructor autonomy. 35% of all Precalculus instructors stated they choose their content whereas 44% of Calculus 1 instructors and 49% of Calculus 2 instructors stated they choose their own content. With decision on approach 67% of Precalculus instructors stated that they choose their own approach whereas 73% of Calculus 1 and 80% of Calculus 2 instructors stated they choose their own approach. With this trend of seeing more individual choice as the Calculus sequence progressed, we still witnessed variation within a course level on the type of instructor autonomy. Meaning that in our data set, Precalculus courses had a larger variety of instructional autonomy approaches (10 variations), compared to Calculus 1 (8 variations) and Calculus 2 (8 variations). It is worth noting that of the 19 analyzed institutions there are 16 institutions with Precalculus, 19 institutions with Calculus 1, and 18 institutions with Calculus 2 responses. Therefore, while Precalculus had the fewest institutions represented, it had the largest number of different instructor autonomy types.

Analysis of looking across courses within an institution determined that most institutions (15 of 19) have differing categories of instructor autonomy across their Calculus sequences. From those 15 institutions, six institutions had the same categorization for two of their courses with most of the same categorization being seen in the Calculus 1 and Calculus 2 courses - highlighting how Precalculus can often be disconnected from the course sequence. The remaining 9 institutions had no categorizations in common across their Calculus sequences. For the remaining four institutions that have the same categorization of instructor autonomy, three of those institutions only had data from two of the courses in the sequence, meaning only one institution (HBCU 1) had the same level of instructional autonomy across the entire sequence.

Given the variation in instructor autonomy within an institution across course level we decided to move forward with our analysis of just Calculus 1 courses across the 19 institutions.

We began the deep dive into Calculus 1 by categorizing the instructor autonomy (see Table 1), noting that there were many institutions that had a split highest percentage on who was deciding the content in the department.

Table 1. Categorization of Instructor Autonomy Approaches of Calculus 1 Courses

		Decision on Instructional Approach		
		I decide	*Team Decides*	*Someone Else Decides*
Decision on Course Content	*I decide*	Full Individual Autonomy- **PWI 1, PWI 5, PWI 11, PWI 7, HSI 1, HSI 3, HBCU 1**	Collective Approach Individual Content- **PWI 9**	
	Split Individual and Collective	Individual Approach Split Individual and Collective Content- **PWI 4, PWI 12, PWI 15**		
	Team Decides	Individual Approach Collective Content- **PWI 3, PWI 8, HSI 2**	Team Collective Autonomy- **HSI 4**	
	Split Collective and Top Down	Individual Approach Split Collective and Top Down Content- **PWI 10**		
	Someone Else Decides	Individual Approach Top Down Content- **PWI 6, PWI 13**		Top Down Highly Coordinated- **PWI 2**

By organizing the institutions into the Instructor Autonomy table, we can see that the majority of Calculus 1 instructors are getting to choose their own instructional approach but there is a lot of variety in who is deciding the content of the course. We see that 11 of the 19 institutions are experiencing a form of coordination with who is deciding content as there is another voice of a team or more of a top down coordination happening even within the split categories. To better understand what was happening in these varied Calculus 1 courses we analyzed student data about their perceived climate within the classroom.

For the climate questions on atmosphere, intellectual engagement, and academic rigor we looked at the means and standard deviations for instructor autonomy categories. Recall these climate questions are all based on a 5-scale Likert style question with 5 being representative of a more friendly atmosphere, more intellectually engaging, and more academically rigorous. Notice that the individual approaches seem to lead the student perception of the atmosphere to be lower (i.e. less friendly) and are exacerbated by also having top down content, as seen in the Individual Approach Top Down category. The highest ranking of the classroom atmosphere is in the Team Collective Autonomy approach where instructors are working in a team to decide the content and approaches in the classroom. We also see a similar pattern for the engagement question, highest ranking of engagement in the Team Collective Autonomy and lower in the individual approaches with an exacerbated effect when individual approach is paired with a top down content. Within the rigor question there is a less clear pattern which may be a result of the student interpretations of the questions and thus requires further analysis. It is noteworthy that we are not seeing a more unified student experience in the top down courses as the standard deviations are within the same relative range across all categories of courses.

Table 2. Student Perceptions of Climate by Instructional Autonomy Approaches

Instructional Autonomy Classification	Number of institutions	Atmosphere	Engagement	Rigor
		Mean (SD)	Mean (SD)	Mean (SD)
Full Individual Autonomy	7	4.13 (1.01)	3.57 (1.24)	4.07 (.99)
Individual Approach Split Individual and Collective Content	3	4.24 (.91)	3.59 (1.17)	3.93 (.96)
Individual Approach Collective Content	3	4.23 (.92)	3.61 (1.21)	3.62 (1.14)
Individual Approach Split Collective and Top Down Content	1	4.15 (.99)	3.59 (1.20)	3.90 (.99)
Individual Approach Top Down Content	2	3.95 (1.02)	3.49 (1.16)	4.11 (.98)
Collective Approach Individual Content	1	4.03 (.95)	3.57 (1.18)	4.03 (.955)
Team Collective Autonomy	1	4.48 (.80)	4.07 (1.05)	3.70 (1.04)
Top Down Highly Coordinated	1	4.06 (1.06)	3.40 (1.24)	3.74 (.99)

Discussion

From our investigation we found that when looking across the Calculus sequence there is a great deal of variation happening across the 19 institutions, with greater variability in Precalculus, and variability within an institution. It was seen in the student data that there was the same level of variability of reported climate even within the top down courses, but team collective approaches may support the highest levels of reported climate. These findings seem to support prior research on the efficacy of course coordination as a successful characteristic of Calculus programs (Rasmussen & Ellis, 2015). Additional analysis is needed to examine the perceptions of the students who are moving through an institution's Calculus sequence where each level of course has a different category of instructor autonomy. Lastly, we would like to examine institutions that do have the same level of instructor autonomy throughout to explore the instructor perceptions as well as the student experience.

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Student Reasoning about Definite Integrals on a Calculus Concepts Assessment

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Assessing student understanding of calculus concepts is of interest to RUME researchers and practitioners alike. This paper details the initiation of the development phase of a new instrument to assess student understanding of the central concepts in the introductory collegiate calculus course. We discuss a framework for defining "conceptual understanding" and make transparent our process for selecting and refining robust tasks to measure it. We provide discussion on interview responses that reveal how students may attempt to rely on memorized procedure and, as a result, discredit productive conclusions. Finally, we outline and seek discussion on future work to continue development of valid and reliable questions to assess the broad range of productive understandings in calculus.

Keywords: calculus, conceptual understanding, assessment

Introduction

Although notable strides have been made by researchers such as Epstein (2007) who created the Calculus Concept Instrument (CCI), no curriculum-independent concept inventory for Calculus I has been validated and found to conform to accepted standards of educational testing (Gleason, Thomas, Bagley, Rice, White, & Clements, 2015; American Educational Research Association, 2014). The purpose of our study is to initiate the development of a valid and reliable instrument to assess students' understanding of central concepts in an introductory college calculus course for Science, Technology, Engineering, and Mathematics (STEM) majors. Our goal is to make our process and criteria transparent so that (a) researchers and educators can make informed choices about what portions they want to include in their research and (b) others in the community can contribute to the instrument in a modular way. Our initial research focuses on interpretations of accumulation and area for definite integrals.

Literature Review and Theoretical Framing

Graphical representations of definite integrals as signed area under a graph are ubiquitous in introductory calculus courses. Research on students' understanding of integrals shows that while area interpretations are widely adopted by students, they are often misapplied and hinder further productive quantitative reasoning (Jones, 2015; Sealey, 2006). Past research has also shown that an adding-up-pieces conception of a definite integral is critical for successfully modeling and interpretation especially in STEM applications (Meredith & Marrongelle, 2008; Von Korff & Rebello, 2012; Jones, 2015; Oehrtman & Chhetri, 2015). Sealey (2014) documented the nature of such student understanding paralleling the mathematical structures of products, sums, and limits in the formal definition, and Jones (2015) labeled the application of such reasoning as a Multiplicatively-Based Summation (MBS). Simmons and Oehrtman (2017) extended this analysis to Quantitatively-Based Summation (QBS) in which the basic models used to construct the parts to be added in the definite integral are not necessarily conceived as a product of the integrand and change in input.

Tallman, Reed, Oehrtman, & Carlson (2021) found that the productive meanings assessed on calculus exams at colleges and universities in the U.S. are sparse. The concept of a definite integral as area is not always supported by quantitative reasoning. Students often approach integration as simply finding the area of a planar region without considering an underlying meaning of accumulating a quantity. Tallman et al. (2021) identified key features of assessment tasks that improve their potential to assess students' productive understandings including that the items should require students to (a) coordinate interpretations, (b) provide explanations or justifications, (c) make comparisons, or (d) draw inferences. Such features require students to make strategic decisions in the problem-solving process rather than only select and apply a memorized procedure.

Methods

After reviewing the literature on student understanding and learning of definite integrals, we identified clusters of particularly consequential meanings and reasoning, based on comments in the literature and our own judgment as teachers and researchers. We selected those involving accumulation and area for our first round of developing assessment items. We collected variations of conceptual questions from the literature and available homework and exams then selected and refined seven of these questions that we felt best addressed the meanings and reasoning identified in the literature. We then developed interview protocols which included contingencies for anticipated student responses.

We conducted and video-recorded one-on-one interviews with 12 students at three institutions. These students were either completing a first-semester calculus course or currently enrolled in a second-semester calculus course. The interviews were approximately 90 minutes in duration and consisted of the seven open-ended questions with ample space for students to show their work. Each interviewer followed the protocols and encouraged students to elaborate their meanings and reasoning. In cases with unexpected responses, the interviewer probed for additional reasoning or allowed the student's reasoning to take its course then provided additional information or suggestions to see how they would proceed. We created detailed summaries of the students' work with notes about potential underlying meanings and reasoning. The entire research team then met to discuss the notes and create a model of productive understandings and the meanings with substantial evidence in the data.

Results

In this paper we discuss and contrast two pairs of tasks. The first pair (Tasks 1 and 2) requires an MBS and a QBS conception of the definite integral, respectively. The second pair (Tasks 3 and 5) illustrates students' conceptions and reasonings about area when interpreting integrals given a graphical representation.

Task 1

Carl was driving on the highway when he saw a sign that said ROAD ENDS 250 FEET. He slammed on his brakes and came to a stop in 4 seconds. Below is a table of data from his truck's computer while he was braking.

Time (seconds)	0	0.5	1	1.5	2	2.5	3	3.5	4
Speed (ft/s)	98	96	92	84	74	60	43	23	0

Given the table, what is your best estimate for the total distance Carl traveled before coming to a stop?

Most students approached this problem heuristically, drawing the graph of a function and finding the area under it. They often reached a correct conclusion using memorized procedures and associations rather than expressing any meaningful quantitative reasoning. Some applied a simple unit analysis to verify their computations but did not support this analysis with associated quantitative reasoning. Others, with prompting, did eventually invoke some quantitative meanings, which however did not alter their approach or answer.

Student 1: Multiply…feet per second…times one second. How much the area is underneath that [graph]…that's how many…feet.

$$0.5(\cancel{98}+96+92+84+74+60+43+23+\cancel{10})$$

Student 2: The speed, feet per second times second is just feet. All of this times .5.

Student 3: Speed is feet a second and time is just measured in seconds, so if I multiply time times speed, it can obviously give me a distance. Then obviously, the value under the curve is the entire…estimate.

Task 2

A train travels along the monorail at a theme park and is programmed to change speed according to where it is on the track, to make turns, stops, etc. The train passes a sign that says the next stop will be in 30 meters. The speed of the train is $v(x)$ meters per second at a point x meters along the track from the sign. As it approaches one stop, its speeds at 5-meter increments are given in the table below.

Distance from sign (m)	0	5	10	15	20	25	30
Speed (m/s)	8	6	5	4	3	2	0

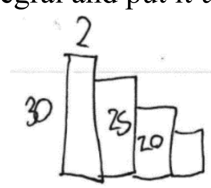
Given the table, what is your best estimate for how long it takes the train to stop from 30 meters out?

Some students guessed the correct integral using unit analysis. Without the underlying quantitative reasoning that is necessary in a problem such as this, however, they were unable to make accurate associations and generate the appropriate Riemann sum. This discrepancy is partly due to their perception that the answer should be the area under whatever curve they have.

Student 1: [manipulates formula and tries switching axes and using slope] Slope is rise over run. You could take the integral and put it under 1. You can just set distance as the *y*..

$T = \frac{D}{S}$

$\text{Slope} = 1/T$



$$2(30+25+20+15+10+5)$$

Student 2: [using non-quantitative unit analysis and manipulating ratios]: I'm multiplying distance times speed and that's meters times m/s…but distance times speed does not equal…time. If 8 is to 1, then what is that to 5 over *x*.

$m \cdot m/s$

$\frac{m}{m/s}$ $\quad m \cdot \frac{s}{m}$

$\frac{8}{1} = \frac{5}{x}$ $\quad 8x = 5$ $\quad 0.625\,s$

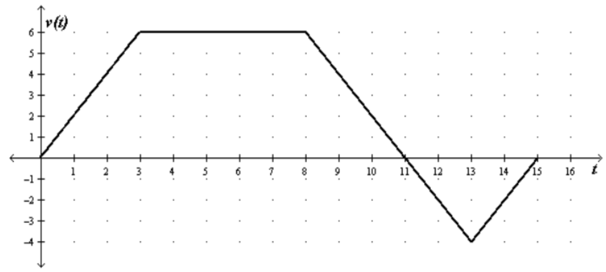
$0.625 + 5/6 + 1 + 5/4 + 5/3 + 5/2 = 7.875\,s$

Task 1 vs. Task 2

Most students approached Task 1 procedurally without any quantitative reasoning about (a) the distance-velocity-time relationship or (b) the structure of a definite integral. This type of problem invoked an instant association with computing a Δt, multiplying by function values, adding the results, areas of rectangles on a graph, area under the graph, and an integral representing area and/or distance. However, there was no connection between these associations and meaningful quantitative reasoning. In contrast, Task 2 required students to reason quantitatively in the context of the problem rather than simply use formula manipulation and/or unit analysis. Those who successfully completed the task, developed a QBS interpretation that enabled them to break up the scenario into pieces of time and create an appropriate Riemann sum. Further, these correct answers differed from their initial procedural approach.

Task 3

Assume that the graph of $v(t)$ shown below represents the velocity of a car (in meters per second) that is traveling along a straight road. Positive velocity corresponds to traveling straight north.



1. How far is the car from its starting point after 15 seconds? Explain.
2. How far does the car travel over the 15-second interval? Explain.
3. Write down integrals that correspond to the numerical answers for #1 and #2.
4. Compute $\int_0^{15} v(t)dt$.
5. Is there an interval on which $\int_0^{15} v(t)dt \int_a^b v(t)dt = 0$? If so, where, and what does it mean in terms of the context of the car?

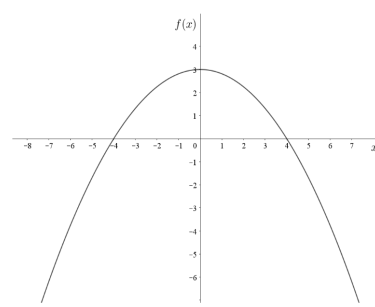
Draw a graph of position vs. time and relate it to each interval of the velocity vs. time graph.

Most students were able to interpret the movement of the car correctly. However, when finding displacement or distance traveled, many relied on memorized procedures with no productive understanding of the situation.

> *Student 1:* I'm not even really reading the words. I'm just seeing graphs. Find area.
> *Interviewer:* Does it matter that velocity is the *y*-axis and time is the *x*-axis?
> *Student 1:* No. That helped, I suppose.
> *Student 2:* Why would I set up an integral when I could just like use shapes?
> *Student 3*: I just found the difference of those areas.
> *Interviewer:* What led you to compute areas?
> *Student 3:* That was just my go-to…start finding the areas. I just remembered doing it from other problems.

Task 5

Use the graph below for this item. For each pair below, determine which one is larger or if the quantities are equal. Explain how you know.



a) $\int_0^2 f(x)dx$ _____ $\int_0^4 f(x)dx$ (fill in the blank with <, >, or =)

b) $\int_3^4 f(x)dx$ _____ $\int_4^6 f(x)dx$ (fill in the blank with <, >, or =)

c) $\int_0^4 f(x)dx$ _____ A left-hand Riemann sum on the interval from *x*=0 to *x*=4. (fill in the blank with <, >, or =)

Determine if each expression below is positive, negative, or zero. You do not need to evaluate the expression. Explain how you know.

d) $\int_{-4}^{4} f(x)dx$ e) $\int_{-3}^{3} f'(x)dx$ f) $\int_{3}^{1} f(x)dx$

This question revealed flawed reasoning that can result when students lack a meaningful basis for their understandings. Imagistic reasoning led some students to misinterpretations.

Student 1: [explaining his correct answer (>) on part (b)] 4 to 6 isn't even above the *x*-axis, I don't think it makes it negative. It's still area. I just put that because since 3 to 4 was above the *x*-axis, it'd be positive, and this 4 to 6 wouldn't really count in my mind…so I just discredited it. If you do think about if it keeps going below the *x*-axis, it would be…infinite. If you're just thinking about it being below this graph, it keeps going down.

Student 2: [explaining his incorrect answer of "0" on part (d)] Assuming it's a symmetrical parabola, the negative area [points to left of *y*-axis] is equal to the positive area [points to right of *y*-axis], so they cancel each other out.

Task 3 vs. Task 5

The use of imagistic and associative reasoning enabled some students to answer question 3 correctly but caused serious interpretive problems in question 5. For example, student 1 (above) conceptualized the area below the curve extending to negative infinity yet obtained the correct answer to question 3 (with bounded regions). Student 1 seemed to think that all area is positive and "discredited" the area that was below the *x*-axis.

Discussion

Although we may be tempted to be complacent about students' memorized associations and procedures, these questions reveal that when not backed by productive understandings of the underlying mathematical concepts, they are merely arbitrary rules. Without a meaningful basis, students are likely to invoke highly inappropriate variations of these procedures alongside or even displacing appropriate ones.

We reviewed Tallman et al.'s (2021) analysis of productive meanings assessed on calculus exams in the United States. Results showed that the exams generally required low levels of cognitive demand and students were rarely required to demonstrate or apply their understanding of concepts. Therefore, the need arises for exams to contain discerning questions to assess students' quantitative conceptions. To facilitate this assessment, students must be required to model the accumulation of quantities using a localized adaptation of a basic model relating the multiple relevant quantities. They must also explicitly connect definite integrals with the approximations generated from the resulting Riemann sums.

Ultimately, we aim to develop an instrument that will (a) identify aspects of productive understandings that are both broadly and effectively adopted by students to make sense of calculus content, (b) evaluate broad-scale interventions which seek to identify instructional practices that lead to productive understandings of calculus content, and (c) convey important characteristics of productive understandings to the general mathematics faculty in order to illuminate effective and ineffective instructional practices in calculus.

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Examining the Interplay Between Self-Efficacy Beliefs, Emotions, Stress Mindset and Achievement in Different Assessments in University Mathematics

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There is currently little research that examines the interplay between various affective constructs, particularly within the context of mathematics assessment. This longitudinal study seeks to understand how student assessment-related affect changes across a university mathematics course and seeks to identify latent or explicit factors that play a role. We collected data on achievement emotions, self-efficacy, stress, and stress mindset across three time points, in addition to measures of academic performance, specifically on different forms of assessment. Without accounting for interactions, preliminary results considering the constructs individually demonstrate an increase in negative and decrease in positive affect. However, incorporating path analysis suggests that engagement on low-risk summative assessment has the potential to promote self-efficacy. As a consequence of this research, we hope to pinpoint suitable interventions that may have a positive influence on affect and achievement. The disentangling of these relationships will have practical implications for course design and assessment structure.

Keywords: assessment, affect, higher education

Social-cognitive theory (SCT) posits that personal affective factors, such as beliefs and emotions, have reciprocal relationships with learning behaviors and educational environments (Bandura, 1997). From this perspective, understanding assessment-related affect is critical to the design and delivery of assessments conducive to student achievement. Based on findings that different assessments elicit different affective responses (Riegel & Evans, 2021) and that frequent, low-risk assessment may enhance self-efficacy (Evans et al., 2020), we identified two broad research objectives: (1) To gain a deeper understanding of assessment-specific affective constructs, including why and how they change during a mathematics course, and (2) To gain a holistic view of the relationship between affect and assessment through examining students' assessment-related *affective field*, defined as, "the bundles of affective factors involved in particular situations in their intraplay" (Schindler & Bakker, 2020). This research particularly seeks to consider relationships between student assessment self-efficacy, emotions, and stress mindset in both an online quiz and final exam, amongst other measures, such as gender and achievement. We chose these affective variables as their relationships and interplay have been underexplored in the context of mathematics assessment. We hope to formulate generalizable implications for assessment practice.

Literature Review

Achievement Emotions in Assessment

Achievement emotions are emotions experienced by learners pertaining to achievement activities and outcomes (Pekrun, 2006). Positive emotions have generally been shown to correlate positively with engagement, attention, motivation, and performance, while negative emotions correlate negatively with these constructs (Mega et al., 2014; Peixoto et al., 2017; Pekrun et al., 2002; Reeve et al., 2014; Schukajlow & Rakoczy, 2016). Most research on achievement emotions with respect to assessment has focused on test-anxiety, with studies generally demonstrating a negative correlation between test-anxiety and achievement (e.g., Lang & Lang, 2010; Steinmayr, et al., 2016; Zeidner 2014). Little attention has been paid to other emotions in this context. When considering online assessment, studies have generally indicated that students experience less stress and anxiety (Cassady & Gridley, 2005; Dermo, 2009; Engelbrecht & Harding, 2004; Stowell & Bennett, 2010), as well as more positive and fewer negative emotions (Daniels & Gierl, 2017; Riegel & Evans, 2021). In line with SCT, Pekrun (2000) argues that repeated experiences of emotion in recurring scenarios, such as assessment, can cause cognitive appraisals to be bypassed and lead to the emotion becoming habitualized. Research has shown there are reciprocal effects between emotions and achievement (Ahmed et al., 2013; Pekrun et al., 2017; Pekrun et al., 2019). More studies are needed to understand how emotions in assessment change and interact with other affective constructs over time.

Assessment Self-Efficacy

Self-efficacy is a construct in SCT defined as, "the beliefs in one's capabilities to organize and execute the courses of action required to produce given attainments" (Bandura, 1997, p. 3) and is a known predictor of achievement (Pajares & Graham, 1999; Zimmerman, 2000). Experiences of success support the development of self-efficacy while experiences of failure hinder it (Usher & Pajares, 2009), indicating that success or failure in one assessment may impact students' beliefs about future assessments. Few studies consider assessment-related self-efficacy, despite Bandura's (1997) proposal that self-efficacy measures must be both context and content specific. Since summative assessment is the foremost way tertiary institutions gauge students' progress (Iannone & Simpson, 2011), it is vital for research to understand and operationalize student assessment-related self-efficacy.

Stress Mindset

Crum et al. (2013) present a new construct of stress mindset, defined to be the extent of one's beliefs that stress has enhancing or debilitating consequences. The few existing studies in education indicate significant relationships with student affect and performance (Jenkins et al., 2021; Keech et al., 2018; Kilby & Sherman, 2016; Riegel et al., 2021). A further study by Crum et al. (2017) concluded that adopting an enhancing view of stress may be beneficial by altering how individuals respond in stressful scenarios over time. Research demonstrates that self-efficacy associates positively with positive emotions and negatively with negative emotions (Luo, et al., 2016; Pekrun et al., 2004; Pekrun, et al., 2011; Usher & Pajares, 2009). No studies have yet incorporated stress-mindset as a moderator for these relationships.

Methods

Setting and participants

The participants in this study were students enrolled in semester two of 2020 at a major New Zealand university in a second-year service mathematics course. The course featured frequent online quizzes to be completed any time between each lecture. These were designed to be low-risk through the use of multiple-choice questions, 30 minute completion windows, two attempts with instant feedback, and keeping best 27 out of 31 quizzes throughout the 12 weeks term. They amounted to 13 percent of the final grade. The course also had a test and a two-hour, invigilated, 50 percent final exam. After data cleaning, 277 of the 410 students enrolled at the start of the semester had consented to their data use and completed all three surveys. This study was approved by the University of Auckland Ethics Committee (approval number 024710).

Data collection and analysis

The research design is a repeated measures design with no control group. As part of the coursework, there were three ten-minute questionnaires administered online (via Qualitrics), each worth 0.1% of the final grade. During the twelve-week semester they were administered in week one (Q1: before any assessments), seven (Q2: after the mid-semester test), and eleven (Q3: before the final exam), and student achievement measures were collected throughout the semester. Participants were removed from the analysis if their responses demonstrated sufficient evidence of straight-lining or the survey was less than 50% completed. Missing data was inserted using EM-imputation. All analyses used version 27 of SPSS and its AMOS programme.

Measures

Achievement Emotions Questionnaire (AEQ). All three questionnaires included an adapted version of the test-related section of the AEQ (Pekrun et al., 2011), with statements pertaining to achievement emotions experienced before (χ^2/df = 2.09, CFI = .92, RMSEA = .06, n = 18) and during an exam (χ^2/df = 2.72, CFI = .90, RMSEA = .08, n = 18). Our adapted version consists of statements measuring anxiety, enjoyment, hope, and hopelessness on a Likert scale from 1 (*Strongly Disagree*) to 5 (*Strongly Agree*).

Measure of Assessment Self-Efficacy for Quiz (MASE-Q) and Exam (MASE-E). All three questionnaires included the newly developed MASE-Q (χ^2/df = 2.38, CFI = .99, RMSEA = .07, n = 7) and MASE-E (χ^2/df = 1.47, CFI = .99, RMSEA = .04, n = 7) (Riegel et al., 2022), designed to assess the participant's beliefs in their *comprehension and execution* and *emotional regulation* abilities while studying and during an assessment. Responses to statements were measured using a slider scale from 1 (*Cannot do at all*) to 100 (*Highly certain can do*). The participants were randomly presented with two assessment scenarios, a low-stakes online quiz and a high-stakes final exam to respond to the MASE-Q and MASE-E, respectively.

Stress and Stress Mindset Measure (SMM). To measure assessment stress, participants were asked "How stressful do you perceive this mathematics QUIZ/EXAM to be?" following each scenario. They responded on a nine-point scale from *Not stressful at all* to *Extremely stressful*. Using the scales developed by Crum et al. (2013), we included an adapted general version of the SMM in the first questionnaire (χ^2/df = 2.67, CFI = .97, RMSEA = .08, n = 7) to obtain an overall measure of students' perception of stress. Additionally, an adapted SMM was designed to measure participant stress mindset in relation to the quiz (measured in Q2 and Q3) and exam scenarios (measured in Q1 and Q3). The scales consist of three stress-is-enhancing statements and four stress-is-debilitating statements. Participants responded on a five-point scale (*Strongly Disagree* to *Strongly Agree*).

Achievement. Student prior achievement was recorded as a self-reported prerequisite grade. We also collected performance results, which contributed to their final grade (quizzes, tutorial participation, assignments, test, and exam).

Demographic Information. We collected self-reported gender, ethnicity, and major.

Preliminary Results and Discussion

Several analyses have already been conducted using this data, including the validation of the MASE-Q and MASE-E (Riegel et al., 2022). Riegel et al. (2021) found stress mindset to be significant and achievement emotions dominant in predicting exam self-efficacy. As understanding the consequences of the pandemic became crucial, Riegel and Evans (2022) included new variables and found engagement with online quizzes, prior achievement, positive affect, and wellbeing were predictors of how students evaluated the impact to their learning.

We now seek to use the longitudinal data to understand how different affective constructs interact and individually change over time for students, as well as investigate students' affective fields in assessment. To address the first part of this goal we ran repeated measures ANOVAs and *t*-tests, which largely suggested a significant increase in negative and decrease in positive affect across the semester (Table 1). This raises interesting questions around 'trait' affect, which we hope to unpack, but also fails to illuminate why these constructs change. To investigate further, we employed path analysis. We first examined the unexplored construct of assessment self-efficacy, incorporating the effects between and within assessment and self-efficacy.

Table 1. Repeated measures ANOVAs and t-tests for affective variables.

		1	2	3					
	Range	Mean (*SD*)	Mean (*SD*)	Mean (*SD*)	*df*	*F*	η^2	*t*	*d*
Self-Efficacy									
Quiz (CE)	(0,100)	71.81 (15.42)	68.33 (18.22)	66.23 (19.43)	1.87	18.28**	.06	-	-
Quiz (ER)	(0,100)	69.56 (18.95)	65.81(20.42)	64.77 (20.00)	1.83	11.00**	.04	-	-
Exam (CE)	(0,100)	66.69 (15.98)	63.46 (18.31)	59.14 (18.84)	1.98	38.02**	.12	-	-
Exam (ER)	(0,100)	65.09 (18.44)	62.27 (20.32)	58.63 (20.04)	1.92	20.87**	.07	-	-
Emotions (Before)									
Anxiety	(1,5)	3.37 (0.75)	3.41 (0.78)	3.43 (0.76)	1.95	1.40	.01	-	-
Enjoyment	(1,5)	2.93 (0.74)	2.92 (0.77)	2.82 (0.77)	1.92	4.51*	.02	-	-
Hope	(1,5)	3.30 (0.65)	3.17 (0.70)	3.07 (0.71)	1.96	19.3**	.07	-	-
Hopelessness	(1,5)	2.36 (0.78)	2.62 (0.87)	2.78 (0.85)	1.99	47.62**	.15	-	-
Emotions (During)									
Anxiety	(1,5)	2.89 (0.90)	3.02 (0.89)	3.15 (0.88)	1.97	18.75**	.06	-	-
Enjoyment	(1,5)	2.97 (0.73)	2.91 (0.79)	2.87 (0.76)	1.99	3.11*	.01	-	-
Hope	(1,5)	3.28 (0.67)	3.14 (0.75)	3.09 (0.76)	1.96	12.41**	.04	-	-
Hopelessness	(1,5)	2.39 (0.67)	2.59 (0.80)	2.80 (0.88)	1.98	53.10**	.16	-	-
Stress									
Exam	(1,9)	6.48 (1.70)	-	6.87 (1.58)	-	-	-	-3.69**	.22
Quiz	(1,9)	-	5.02 (2.08)	4.94 (2.02)	-	-	-	0.63	.04
Stress Mindset									
Exam (Debilitating)	(1,5)	3.10 (0.68)	-	3.19 (0.70)	-	-	-	-1.96*	.12
Exam (Enhancing)	(1,5)	3.12 (0.79)	-	3.02 (0.76)	-	-	-	1.96*	.12
Quiz (Debilitating)	(1,5)	-	2.95 (0.76)	2.98 (0.74)	-	-	-	-0.59	.04
Quiz (Enhancing)	(1,5)	-	3.18 (0.81)	3.14 (0.76)	-	-	-	0.81	.05

**Note.* ***p* <.001, *<.05; CE = comprehension and execution, ER = emotional regulation.

Acceptable models are presented in Figure 1. Anticipated reciprocal effects between exam self-efficacy with a similarly high-stakes test are seen. However, quiz *comprehension and execution* self-efficacy does not demonstrate a reciprocal relationship with quiz achievement following the test. Moreover, for quiz *emotional regulation* self-efficacy, there is no reciprocal relationship with performance. These findings may suggest that the relationship between self-efficacy and performance is weaker in low-risk assessments, which elicit less of an emotional response, and the effects are overwhelmed by the affective influence of a high-stakes assessment.

Generally, we see simultaneous effects between different forms of assessment self-efficacy, suggesting performance in one assessment may indirectly influence self-efficacy in another. In the first half of semester, exam self-efficacy more strongly predicts quiz self-efficacy. However, in the second half of the semester (after the mid-semester test) quiz self-efficacy is a stronger predictor of exam self-efficacy. This in part indicates that repeated experiences on low-risk assessments can influence the efficacy of students going into an exam, though high-stakes assessments during the semester can overwhelm these effects. The quizzes were designed so participating students were likely to receive full marks, meaning performance on quizzes largely reflects student engagement. Thus, it is plausible to suggest that through incorporating the intervention of regular quizzes, educators can indirectly promote exam self-efficacy and achievement. To further support this hypothesis, we intend to conduct a cross-lagged panel analysis so to determine directions of influence between the types of assessment self-efficacies.

Questions

We have presented some of the results of our preliminary analyses and seek to discuss with the attendees of RUME the following questions:

- We can perform similar path analyses to the ones presented here for other variables collected to look at affective relationships with achievement, but how might we retain the interplay between affective variables with our limited sample size?
- Certain affective constructs (e.g., beliefs) are considered slow to change. How can we use our data to provide insight into our understanding of “trait” or “global” affect?

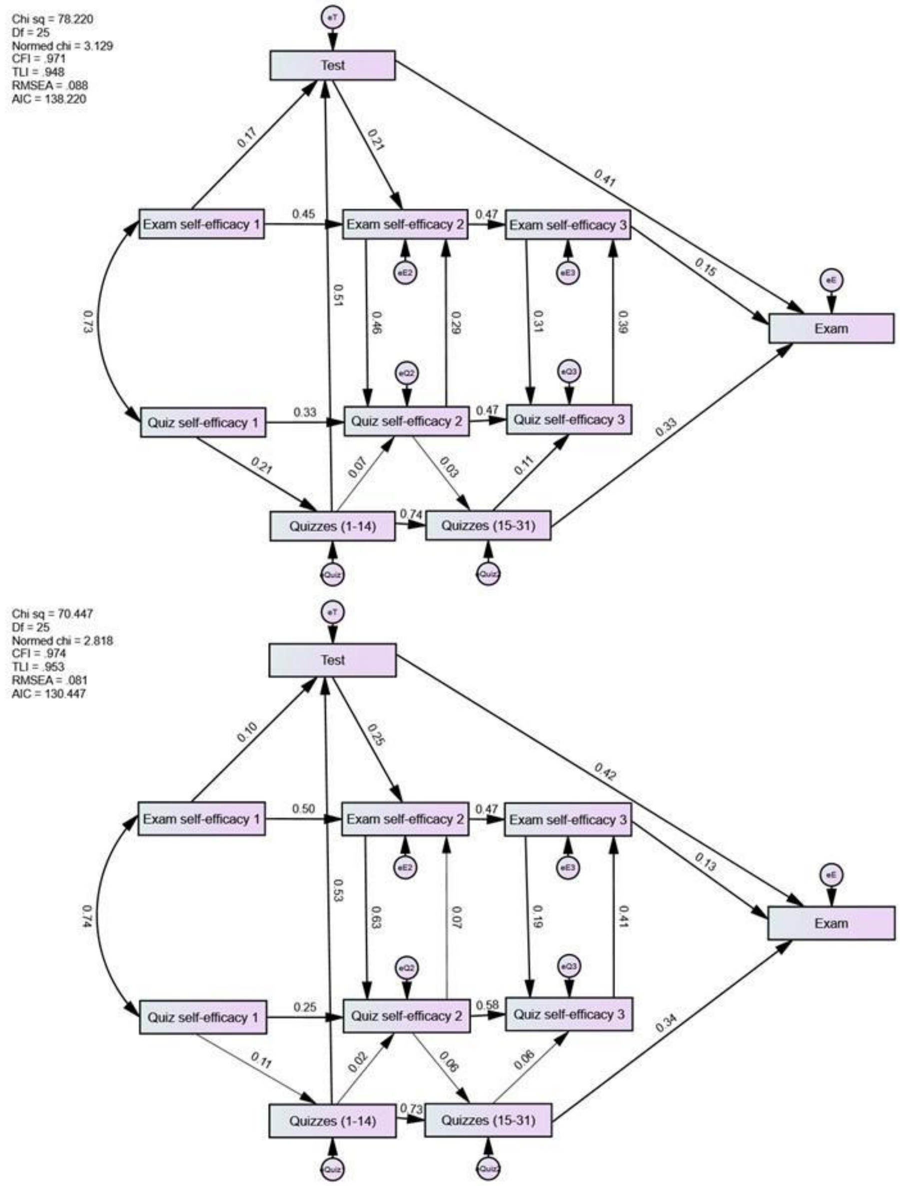

Figure 1. Path analyses of comprehension and execution (top) and emotional regulation (bottom) self-efficacy.

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Changing the Conversation: Building Success in a Community College Mathematics Classroom

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Calculus remains a major barrier of entry into STEM for students without strong mathematics backgrounds. This study looks at one implementation of the STEM Core model, which provides an accelerated pathway through developmental math classes along with extensive support to get students calculus ready in one year. Viewing this program as a community of practice around math learning I examined course completion data and conducted interviews to understand how the model is impacting student success. Furthermore, I seek to understand what aspects of the model matter most to the students. Initial results of student success rates in the course sequence show significant positive outcomes, with nearly 65% of STEM Core students reaching Calculus I in one year. Interviews with students reveal how their success goes beyond course completion and the importance of the relationships they built with their instructors and one another.

Keywords: Community College, STEM pathways, Developmental Mathematics, Calculus

Within the context of science, technology, engineering, and mathematics (STEM) many studies have shown the reality of the "leaky pipeline" with students dropping out of STEM majors along the way to completing a degree or dropping out of school (Flynn, 2016). Historically, many faculty in these disciplines promoted this culture as they thought it was a filter removing students who would not be capable of succeeding, but this is not the case (Seymour & Hewitt, 1997). Unfortunately, the students who drop are not weaker academically but do have backgrounds already underrepresented in STEM fields and cite the realities of a "chilly climate" as a significant factor in why they step away from finishing their STEM degrees (Jorstad et al, 2017). Addressing the issues creating the leaky pipeline and students not persisting in STEM degrees is crucial to improving broader issues of diversity in STEM.

Research has shown the importance of teachers and effective teaching in students' academic outcomes (Burroughs et al., 2019; Christe, 2013; Goe, 2007; Goh & Fraser, 1998; Sonnert et al., 2015). This research spans grades K-16 and all subject areas. Unfortunately, undergraduate mathematics often serves as a barrier for students attempting to pursue STEM degrees (Bressoud et al., 2015). Improving mathematics instruction and classroom environments is an important part of addressing the leaky pipeline and equity in STEM (Boaler & Selling, 2017).

Public two-year colleges educate roughly 35% of undergraduate students in the United States (Community College Research Center, 2022). They educate a disproportionate share of minority, first generation and low-income students. Community colleges are an important starting point for many students entering the college pipeline and 49% of students completing bachelor's degrees had attended community college at some point in the previous 10 years (National Student Clearinghouse Research Center, 2017). While this is an important context of study because of its large share of the undergraduate student population in the US, it is also an understudied context (Mesa et al., 2014a; Fletcher & Carter, 2010).

To work on improving the environment of STEM classrooms in community colleges we need to know both what effective teachers are doing in their classrooms and how students are responding to their instruction. Related to that it is necessary to understand the social environment of these classrooms and what students are learning.

Theoretical Framework

Of greatest interest for this work is understanding the mathematics classroom learning environment in this school. The environment is impacted by broader structures of the NSF funded STEM Core program of which it is a part, but it is also uniquely shaped by the instructors, staff and environment of this particular school.

This work is situated in the idea of learning as a community of practice (Wenger, 1999). The STEM Core model changes the mathematics learning environment for the students to engage them not only in the classroom as an individual, but in all aspects of their life and as a part of a community. With a cohort model, students spend a full year with consistent instructors who know them and each other. The student cohorts become a community with genuine relationships forming among the students themselves and the teachers and staff who support them. With the funding provided by STEM Core students also have access to a dedicated student support specialist.

This study is conducted in partnership with the mathematics course instructor and seeks to better understand what is working with this implementation of the STEM Core model. The research questions for this work are:

RQ1: How is this implementation of the STEM Core model affecting student success rates?

RQ2: What do student interviews reveal about the most impactful aspects of the program over time?

Success is defined by the institution as obtaining a C or better in a course. This definition in assessing the impact of STEM Core on academic outcomes for students aligns with requirements for taking follow-on courses and meeting transfer requirements. However, in this work success is also defined more broadly. Student persistence in the face of failure or adversity along with taking up identities as knowers and doers of mathematics, engineering or computer science are also markers of success for this researcher and the administrators of the program.

Literature Review

In 2012, Ann Sitomer and colleagues (2012) laid out a research agenda for mathematics education researchers specifically focused on community colleges. They acknowledged that there is much overlap with the long history of mathematics education research in K-12 classrooms and with the more recent research in undergraduate mathematics education (RUME) work, but they also note ways in which community colleges and the math learning they encompass are distinct. In a research commentary written a few years later Mesa and colleagues (2014a) also called for the need to expand research specifically in the math classrooms of community colleges. They note the difficulty and limitations of transferring the findings of work done in one setting (ie K-12 classrooms or 4-year colleges) to another (ie community colleges) as while there may be some similar characteristics there will also be many differences. In their commentary they also explicitly note the lack of research conducted specifically within classrooms and focusing on instruction. The research in classrooms that did exist tended to be conducted by practitioners and focused on particular interventions and more in line with sharing "what works" with other practitioners and not producing generalizable knowledge. They described the state of math education research in this setting as having, "the same feel of disorganized guerrilla warfare" (Mesa et al., 2014a, p. 179). Community colleges are constantly under pressure to improve student outcomes and are often looking for a quick fix. While instructors want to do better for their students, there is a lack of an organized research agenda to help them know the best ways to do so. In laying out their research agenda they reiterated the same four areas highlighted earlier by Sitomer and colleagues (2012).

Since the writing of these two commentaries additional important work in understanding community college mathematics instruction and learning has been done. A large video study of community college instructors sought to connect how they talked about instruction to their actual instructional practices in the classroom (Mesa et al., 2014b). While this study provided a significant amount of insight into instruction the authors note the need for more studies focusing on what students are learning, “what do students learn and understand as they experience particular teaching approaches” (Mesa et al., 2014b, p. 146). A further study of community college faculty teaching developmental math that sought to apply the framework of mathematical knowledge for teaching resulted in a proposal for the need to add an amendment of “the Caring Map” to make sense of the concept at the community college level (Nabb & Murawska, 2019). Studies using quantitative longitudinal methods have considered the variance of students’ pathways through developmental mathematics and what drives varied student outcomes (Howell & Walkington, 2022). However, community college mathematics students themselves, especially their individual experiences, remain understudied.

Methods and Data Sources

This study uses past-performance data from 2015-2019 on student success rates in Algebra II through Calculus III, success is defined as obtaining a C or better in the course. It should be noted all course completion data is from before the implementation of AB705 in California which dis-incentivized colleges from offering developmental mathematics classes and led to a restructuring of the course sequence at this college along with changing the placement system for students. Updated course completion data is not yet available.

These course records are supplemented with student interviews (n=10) to understand what students viewed as the most salient aspects of their experience with the program. The students interviewed were alumni of STEM Core who had completed the one-year program up to four years prior. Volunteers were sought through messages on a facebook group for alumni in the program. Due to this sampling methodology the views expressed by students in the interviews may be biased but they still provide useful insight into what mechanisms might be driving the program success rates revealed by institutional data. Student demographic characteristics matched roughly with the overall demographics of the program, 8 students were male identifying and 2 were female identifying. The students also held a diversity of racial and ethnic identities. Further details of students’ demographic characteristics are not included to protect student privacy and confidentiality.

During the roughly 30-minute interviews students were asked about many different aspects of the program. By interviewing students from multiple cohorts of students the researcher sought to better understand the longitudinal impact of this program for students after their first year of classes and when the intensive support structures of the program were no longer present.

Preliminary Results

Research Question 1

The first 5-years of the STEM Core program at this institution has produced results that show significant increases in student success in the courses from developmental math through calculus. Table 1 shows the results for cohorts of students in the program from four cohorts. As stated earlier in this paper success is defined as passing the course with a C or better. The number of enrolled students for a given course may exceed prior course numbers because students may take the course multiple times before obtaining a C or better and students are able to join the STEM Core cohort at any point in the pathway as long as they have met the prerequisite. Of students who start the STEM Core program with Algebra II 64% complete Calculus within one year compared to only 16% percent of students statewide (Mejia, Rodriguez, & Johnson, 2016).

Additionally, preliminary data shows that students in the STEM Core have higher success rates in English courses 77% versus 71% for the institution, even though English is not a part of the program. This difference indicates that the impact of the STEM Core model goes beyond helping students perform better in their STEM classes but may be helping them become better students overall.

Preliminary data also shows that success rates, while still high, decrease for courses that come after the program. Specifically, Calculus II success rates are significantly lower for students in their first attempt at the course but data shows that students who are not successful in their first attempt are taking the course again. The overall success rate for Calculus II for STEM Core students was 40% versus the overall institutional rate of 64%. However, a deeper look at the data revealed an interesting pattern. The results for one cohort show the success rate listed as 48% because 22 students passed in 46 attempts. However, this STEM Core cohort only included 29 students who had completed Calculus I, thus 76% of the STEM Core students who completed Calculus I also completed Calculus II, but some required multiple attempts.

Table 1: Shows student success rates in math courses in the institution as a whole and in the STEM Core cohort.

Course	**Typical success rates (3-year average)**	**STEM Core students**
Algebra II	56.4% (n=3311)	93.8% (n=145)
Trigonometry	60.9% (n=1542)	95.7% (n=164)
Precalculus	75.7% (n=1209)	89.7% (n=156)
Calculus I	65.8% (n=1376)	79.4% (n=136)

Research Question 2

Interviews with students revealed that nine of the students identified as still being involved in STEM either in their educational or career trajectory. Nine of the students had attempted calculus II with eight completing the course. One student dropped the course the first time and intends to retake again in the future and four of the eight who completed the course required more than one attempt to do so.

Students' discussion of their various experiences in calculus II shared a common theme of the drastic drop in the support they felt from their prior year taking calculus as a part of the STEM Core program. One student shared how her STEM Core instructors had warned their class not to expect the same level of support in future classes and worked to prepare them. Three of the students mentioned going back to their STEM Core instructor to seek advice and support. He encouraged them that failure was okay and that they would ultimately succeed.

The importance of the community they had found in the STEM Core program was a common theme among interviewees. They talked about how they trusted the faculty and staff in the program and built relationships with their classmates that extended beyond the classroom. One student mentioned that he was currently at a 4-year university and rooming with friends from the STEM Core program.

Another common theme in the interviews was how the program helped students connect with internships that helped lead to other jobs. These students who generally came from low-income families and were the first in their family to attend college appreciated how the STEM Core program provided guidance on the importance of internships and how to apply to them. They shared how these real-world connections have helped set them up with practical experiences and networks that they can leverage as they move forward in their careers.

Discussion

For students entering college at a developmental mathematics level the research says their chances of succeeding in a STEM major are very small (Bullock et al., 2017), but this work interrogates that 'truth.' It reveals through the success of students in one program how a model that supports students as whole people and builds community can lay the groundwork for long term success.

An important point to note about this program is that it is not 'remedial.' Any student who has an interest in pursuing a STEM degree can be admitted into the program. The main, initial appeal of the program to most of the students interviewed was that it would help them complete their math courses faster. This program leverages students' motivation and desire to succeed instead of labeling them as in need of remediation.

The instructors and staff of the STEM Core program studied in this work are incredibly dedicated and experienced. Their work may not be possible to replicate in all circumstances, but this work should lead institutions to consider whether implementing cohort models, emphasizing community building and considering wider supports for students might lead to higher success rates.

Finally, from a theoretical perspective this work could be seen as an empirical example of the mathematical knowledge for teaching (MKT) revised framework offered by Nabb & Murawska in their 2020 paper. They shared how understudied the domain of community college classrooms is and looked at the MKT framework that had arisen from K-12 instructional research, but they added one important area based on interviews with instructors: caring. The instructors and staff in this program care deeply for their students and that care is clear to the students in the program. While the students did learn content, the much more important messages they received was about their capabilities as students and that while they might struggle along the way they had a path to success.

Questions and Points for Audience Discussion

This work has served as a pre-study to a more in-depth study that will be undertaken during the upcoming 2022-23 academic year. I will be conducting a full ethnographic study of a cohort participating in this program and will attend and observe the mathematics classroom throughout the program. I am interested in what questions this work raises for audience members that I can consider in more detail in the longer study and what additional insights could be gleaned from further analysis of the interview data already collected.

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Implementing Digital Graphing Activities in College Algebra: Two Instructors' Views of Benefits and Challenges

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We address the transformation of instruction in entry-level college mathematics courses, such as College Algebra. Our research question is: What do instructors view as benefits and challenges when implementing novel digital graphing activities in College Algebra? We report on a case study of two instructors who implemented the activities during both semesters of one academic year, drawing on instructors' individual interviews at the end of each semester. The instructors viewed it as beneficial to implement these activities as part of a community. They also found the activities' focus on reasoning helpful for their students. They found integrating the new activities with existing online learning management systems challenging at times, and they wished their students were more engaged when implementing activities asynchronously. Overall, the challenges were not roadblocks, and the benefits outweighed the challenges. We conclude with discussion and implications for research and practice.

Keywords: undergraduate education, instructional activities, communities of practice, technology

The transformation of instructional practices in introductory college math courses, such as College Algebra, is important for students' persistence in Science, Technology, Engineering, and Mathematics (STEM) degrees (e.g., Freeman et al., 2014; Henderson et al., 2011; Herriott & Dunbar, 2009). Yet, instructional materials in such courses tend to privilege a status quo of finding numeric answers at the expense of promoting reasoning (Mesa et al., 2012). When instructors implement digital activities designed to press against a status quo of answer finding, there can be tension regarding the status of the activities within the course (Olson & Johnson, 2022). For this preliminary report, we investigate the question: What do instructors view as benefits and challenges when implementing novel digital graphing activities in College Algebra? We report on a case study of two instructors, Riya and Carol (pseudonyms), who implemented activities in both semesters of one academic year in conjunction with their participation in a faculty learning community (FLC) (Cox, 2016).

College Algebra is an entry-level undergraduate course at many U.S. institutions. The Committee on Undergraduate Programs in Mathematics (2015) has recommended revisions to College Algebra, including rethinking the class to better support students as logical and quantitative thinkers. However, change has been slow to develop (Tunstall, 2018). Working with innovative digital activities can make room for instructors to question conventions in their curricular materials (Sinclair et al., 2020) and deepen their understanding of mathematical relationships (Moore et al., 2019). By incorporating novel digital graphing activities into College Algebra, we work to affect change in the course. Our case study provides insight into instructors' views on the benefits and challenges of implementing such activities while discussing implications for research and practice.

Background

To explain how instructors may make changes to their instruction via implementing novel digital activities while participating in an FLC, we draw on Wenger's (1998) Community of Practice (CoP) theory. From this perspective, practice is not something people hand down from one group to another; it is ongoing and continually evolving. One way to engage instructors in new practices is to make room for negotiation between the core members of a community and those along the periphery (Wenger, 1998). This way, instructors can "dip their toes" into practice without demanding full participation. For example, instructors may participate by listening to others implement the activities or trying out a new digital activity in their class.

FLCs are common in higher education (Cox, 2016; Kezar et al., 2018). They can form when instructors connect through a common scope of practice and come together to learn and share about a concept or process over time. Communities of Transformation (CoTs) are a special type of FLC. Kezar et al. (2018) define CoTs as "communities that create and foster innovative spaces that envision and embody a new paradigm of practice" (p. 833). CoTs include three aspects: an idea to challenge the status quo, the space to carry out practices, and a group with which to sustain those practices (Kezar et al., 2018). For example, focusing on reasoning rather than finding answers is one way to challenge the status quo in early undergraduate mathematics courses. Interacting with CoTs allows instructors to have space to conduct new practices with colleague collaboration and support.

Methods

Our case study (Yin, 2016) stems from a larger, National Science Foundation-funded project spanning multiple institutions. The project intends to address the overemphasis on finding the correct answer in U.S. undergraduate mathematics education by prioritizing mathematics reasoning. The project aims to transform instruction in College Algebra via instructors' implementation of digital graphing activities, *techtivities,* that emphasize students' reasoning over answer-finding. We report on a case of two College Algebra instructors, Riya and Carol, who participated in the project for one academic year.

Techtivities

The techtivities are digital graphing activities developed in the free, Desmos platform (Desmos, n.d.). Each techtivity starts with animation, such as a "Cannon Man" propelled out of a cannon and then parachuting back to the ground. Students explore the change in two attributes identified in the situation (e.g., height from the ground and total distance traveled) and create a Cartesian graph relating the attributes. They compare their graph to a computer-drawn image and graphs generated by their classmates and reflect on their observations. Then, students sketch another Cartesian graph representing the same relationship between attributes but with the attributes on different axes.

Data Collection

Instructors participated in the project while receiving stipends on a semester-by-semester basis. In their first semester of implementing the techtivities in their College Algebra courses, instructors attended four professional development (PD) sessions via videoconference to accommodate participants from multiple institutions. During these sessions, participants explored features of the techtivities, discussed strategies for implementation, and reflected on the role the techtivities played in their instruction. Co-investigators of the larger project held PDs roughly once per month. After the first semester, instructors met in small group CoT meetings at their institution to debrief their implementation of the techtivities and share related ideas.

At the end of each semester, instructors participated in 30-minute, individual, semi-structured interviews conducted by video conference. In an instructor's first semester of participation, the interview had four topics: benefits and challenges of teaching College Algebra, benefits and challenges of participating in the project, instructors' views of students' interactions with the techtivities, and the instructors' perceived impact of the techtivities on their teaching. In subsequent semesters, the interview revisited benefits and challenges and student interactions, as well as three additional topics: instructors' views on what constitutes a techtivity and how the techtivities fit within their course, instructors' collaborations with others about the techtivities, and instructors' takeaways from working on the project. Graduate research assistants produced verbatim transcripts for each of the instructor interviews.

Analysis

Instructor interviews were our source of data, analyzed using a modified form of open coding (Corbin & Strauss, 2008). We entered the analysis process with two broad codes, "benefits" and "challenges." By a benefit, we meant something an instructor perceived to be good, helpful, or enjoyable. By a challenge, we meant something an instructor perceived to require thought, skill, or innovation to address.

Whitmore and Knurek led the coding and data analysis. They began by reading the transcripts and watching the videos. Then, they identified excerpts (uninterrupted speech turns from instructors), which they coded as benefits or challenges for the instructor. For each excerpt, they wrote a short field note explaining how they viewed the excerpt to represent a benefit or challenge. To vet the codes, they met in pairs to agree and brought the codes to the larger author team, who weighed the codes against the evidence in the interview and refined the themes emerging from the excerpts and codes.

Results

Our analysis revealed four themes related to benefits and challenges. Instructors' views of the benefits related to the techtivities outweighed the challenges they faced. Benefits included their participation in an instructor community and the value of focus on reasoning. Challenges included integrating the new activities with existing learning management systems and engendering student participation in asynchronous settings.

Benefits

Participating in an Instructor Community. Riya and Carol acknowledged the value of meeting regularly with other instructors while implementing the techtivities. In her second interview, Riya described structuring their small group CoT meetings around discussions about instruction.

> Riya: In terms of me as a faculty, we've been doing something different this year by like visiting each other's classes to check how these go with other instructors. And our schedules were in conflict, so we really didn't get the chance, but we all recorded and met after that. We discussed the recording and giving kinds of answers also about the set of questions, what was our challenges, what we like, what we faced, how things were going.

> And also, you see like the same video from a different instructor perspective, that was really helpful.

Carol found the CoT meetings to be a space where she saw other instructors' practice. She stated, "It was nice to see how other instructors, well, it was just Riya so far, but it's nice to see how they implement this." Their comments pointed to the ongoing nature of developing new practices, as put forward by Wenger (1998). Their implementation of the techtivities was something that continued to develop, not something the research team handed down for them to replicate.

Focusing on Students' Reasoning. Riya and Carol saw benefits for their students as well as themselves. In their view, the focus on reasoning over answer-finding was a positive aspect, which Carol stated that she appreciated in both interviews. In her second interview, Carol shared that this focus was something she wanted to expand in her teaching.

> Carol: I really liked the techtivities, and I would try to continue to have the students focus more on the reasoning and what they think rather than whether they get the right answer, so I'm hoping to maybe do this longer, um, incorporate this into my course, to get them to think versus just trying to find the right answer.

Riya and Carol felt that the techtivities impacted their students' mathematical thinking. In the first interview, Riya described how her students' responses gave her evidence of their learning and engagement.

> *Interviewer*: Do you think that the techtivities impacted your students' math thinking?
>
> *Riya*: Yeah, sure. I am sure about that one. Most of the answers they were saying that they "didn't think about that." Y'know, or "We didn't know that there is such a relation." or "That's very nice." So, so they were likely engaged with what they have learned from the new skills.

Riya and Carol's comments provided evidence that they were comfortable making room for students to think and hear each other's insights from their work on the techtivities. Notably, their comments suggested that the broader project's focus on reasoning is something that they valued as part of their instruction.

Challenges

Navigating multiple online platforms. Implementing the new activities meant that instructors needed to navigate both the Desmos platform and their learning management system, Canvas. Sometimes the challenges were specific to the platform, and sometimes they involved integration within the system. During her second interview, Riya talked about difficulties she experienced when trying to switch between techtivities on the Desmos platform. She stated: "Sometimes it's very hard to go back to, to go back it takes a while to go back to the techtivity you want, because I was doing two techtivities. So I cannot go from one to another quickly." Both instructors were new to using the Desmos platform, and the research team worked to help them navigate the online venues, both with facilitation guides and online modules.

The research team created a Canvas module for instructors to support different delivery formats. The first semester, Carol had a major challenge integrating the module into her Canvas course. Her upload inadvertently overwrote her course settings. She was quite frustrated and thought about quitting the project. Yet, she appreciated the research team's help troubleshooting the problem and decided to stay on with the project. She reflected on this in her second interview, stating, "Well, the challenge is initially, you know, that I had problems with, with the Canvas site. There was just a technical problem that I had, but it was something that I had last semester." Carol's and Riya's challenges pointed to the complexities of implementing new digital activities. Not only had they needed to learn the activities, but they also needed to learn to navigate and connect within different online platforms.

Encouraging student participation with asynchronous implementation. Both instructors had autonomy in implementing the techtivities in their courses. When the activities were asynchronous, even in face-to-face courses, instructors found it challenging to encourage student participation. During her second interview, Carol talked about decreased student participation when she assigned the techtivities asynchronously through Canvas.

> Carol: They were always lower for the ones that I assigned through Canvas, where I just tell them, here, this is what you need to do. And I'm just wondering whether there's anything I can do to get them to actually do the ones through Canvas.

Riya's comments also supported her value of synchronously implementing the techtivities when possible. She said in her second interview, "They worked really good this semester, better than last semester. I think the plan is to have the time for all of them to be implemented in the class so we can discuss them in more detail with the students." Riya and Carol's comments spoke to the utility of having the techtivities be something more than an "add-on" to the course, as recommended by Olson and Johnson (2022).

Discussion/Conclusion

We investigated a case of two instructors' views of benefits and challenges when implementing the novel techtivities in their College Algebra classrooms. While the instructors encountered challenges, they were not roadblocks to their implementation. A key benefit was instructors' participation in small group CoT meetings, in which they could discuss and reflect on their practices.

Our case study aims to illuminate how implementing novel digital activities can engender instructional transformation in College Algebra. We offer three emerging contributions. First, instructors' participation in a community, beyond just a researcher-led PD, is crucial for instructors to develop agency in their practice and to allow for new approaches to take hold. When the community is also a CoT (Kezar et al., 2018), instructors have a space to develop new practices to push back against the status quo of answer-finding and to promote students' reasoning in courses such as College Algebra. Second, the status of the new activities makes a difference in students' participation (Olson & Johnson, 2022). Riya and Carol found it difficult to encourage student participation when implementing the activities asynchronously within a synchronous course. Third, implementing new digital activities involves navigating new online platforms, and instructors need to have space to learn that navigation.

Our analysis is ongoing. To further develop the case of Carol and Riya, we will analyze their small group CoT meetings to learn more about how their interactions support their evolving practices. Then, we will triangulate those analyses with evidence from their classroom practices.

Questions for Audience

1. What are your experiences implementing digital activities (or active learning elements) into early undergraduate mathematics courses? What were the benefits and challenges?
2. How do you see FLCs, specifically CoTs, contributing to instructional transformation in undergraduate mathematics?

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Physics Student Understanding of Divergence and Curl and Their Constituent Partial Derivatives

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This work is part of a broader project to investigate student understanding of mathematical ideas used in upper-division physics. This study in particular probes students' understanding of the divergence and curl operators as applied to vector field diagrams. We examined how students reason with partial derivatives that constitute divergence and curl of the vector field diagrams. Students' written responses to a task on derivatives, divergence, and curl of a 2D vector field were collected and coded. Students were generally successful in determining the sign of some of the constituent derivatives of div and curl, but struggled in one case in which components were negative. Analysis of written explanations showed confusion between the sign, direction, and change in the magnitude of vector field components.

Keywords: Partial Derivatives, Divergence, Curl, Vector Field Diagrams

Introduction

Many physical quantities, such as force and momentum, are represented with vectors. For several topics in physics, e.g., interactions in gravitation and electricity and magnetism, it is useful to define a vector field: a vector quantity is assigned to every point of a subset of space. Vector fields can be represented in different ways, with field lines, an array of arrows, or a symbolic expression like $\hat{V} = ay\hat{\imath} + bx\hat{\jmath}$. Students are introduced to vector fields in introductory courses, typically in the contexts of electric and magnetic fields. These vector fields vary in space, and vector calculus provides several ways to describe this variation, including the gradient, divergence, and curl of the vector field. Several significant physical quantities are associated with vector derivatives: Maxwell's equations for electromagnetism describe relationships between the divergence or curl of electric or magnetic fields and other physical quantities, and the fields themselves can be expressed in terms of derivatives of scalar and/or vector potentials. While most students have not encountered vector calculus the first time they study vector fields, those who go on to major in physics and electrical engineering will use these ideas extensively in a junior-level course in electricity and magnetism. Students encounter vector field representations for electric and magnetic fields in an electromagnetism course, and research involves electric and magnetic fields and even gravitational fields. Students are expected to reason with symbolic equations but also with vector field representations of divergence ($\nabla . \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y}$) and curl ($\nabla \times \vec{V} = det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$). We expect students to look at the vector field and find the appropriate components and see how the proper components are changing as we move to certain directions.

Several previous studies in physics education research (PER) have examined student understanding of divergence and curl in post-introductory courses (Baily & Astolfi, 2014; Bollen et al., 2015; Gire & Price, 2012; Singh & Maries, 2013). Generally these studies have involved a two-dimensional representation of a field as an array of vectors. Singh and Maries (2013) reported that even though graduate students successfully calculated divergence and curl of the vector fields, they were unable to interpret the div and curl of vector field plots. Baily and Astolfi (2014) and Bollen et al. (2015) repeated the previous study with different diagrams, reporting that around 50% of their students could correctly determine divergence and curl given the vector field diagrams. Bollen et al. (2015) also qualitatively studied students' responses, categorizing them into three approaches: description based, concept based, and formula based. For example, determining div or curl based on a description of its meaning was categorized as description based. If students used concepts like flux or the "paddle wheel" to determine div or curl, respectively, it was categorized as concept based.

While previous studies have focused on the divergence and curl, the classroom experience of one author of this study suggested that reasoning with the partial derivatives that constitute these operations, e.g., $\frac{\partial V_x}{\partial x}$ and $\frac{\partial V_y}{\partial y}$ for divergence or $\frac{\partial V_x}{\partial y}$ and $\frac{\partial V_y}{\partial x}$ for curl in Cartesian coordinates, might be one element of the challenges faced by students. Previous studies have asked students to determine the sign or value of the divergence and/or curl for a given field diagram, but there has not been as much focus on their constituent derivatives.

Prior PER studies have documented student difficulties with partial derivatives, often in thermal physics contexts (Bajracharya & Thompson, 2016; Thompson et al., 2006). Student understanding of derivatives is studied widely in RUME. Zandieh (2000) developed a theoretical framework for student understanding of derivatives, which was extended by Roundy et al. (2015) to include partial derivatives. Wangberg and Gire (2019) investigated student understanding of partial derivatives of scalar fields represented as surfaces using Zandieh's framework.

The derivatives in the expressions for divergence and curl have the additional complication that they are derivatives of vector components. In these partial derivatives, V_x refers to the x-component of the vector field, so $\frac{\partial V_x}{\partial x}$ is the partial derivative with respect to x of the x-component of the vector V. Extracting information about the derivatives from a vector field diagram involves multiple steps. Existing frameworks are restricted to derivatives of scalar functions, and need to be extended to deal with the derivatives of vector quantities. While prior frameworks have offered insights into student reasoning, they do not account for functions of multiple variables, nor for the vector nature of the derivatives.

We set out to develop and implement tasks that used similar questions to prior studies investigating div and curl for vector field diagrams, but with added explicit questions about the constituent partial derivatives. The goal is to begin to answer the following research questions:

- To what extent can students determine the sign of the constituent derivatives of divergence and curl given a vector field diagram?
- To what extent are student responses to tasks focused on the signs of constituent derivatives related to success in determining the sign of divergence and curl?

Methods

Written data were collected at two universities in sections of Mathematical Methods for Physics, post-introductory courses for physics and engineering physics majors and minors that are intended to cover the advanced mathematics students will encounter in upper-level theory core courses like Electricity and Magnetism or Quantum Mechanics. All students (N=32) had completed introductory sequences in both physics and calculus, and the data were collected in the course after instruction on vector calculus. One campus serves a diverse student population in the southwest, the other is a predominantly white institution in the northeast. Responses from the two universities were similar and are thus combined and reported together in this paper.

In each course, the students had considered similar representations of vector fields in class and answered questions relating the features of a field to its divergence and curl. In one university, students had completed a research-based instructional tutorial; in the other, these questions were presented as a whole-class discussion. After instruction, the questions shown in Fig. 1a were posed on a course midterm exam. Students are shown a 2-d field representation and asked to determine the signs first of the divergence and curl, then of the constituent derivatives.

In the broader study, different versions of this task were used. In this report, students responded to a task asking first about the derivatives and then about div and curl. With this sequence, we hoped to see whether the reasoning for div/curl included derivatives or if other reasoning would emerge.

The coding process began with general codes for both correctness of the sign and the correctness of reasoning provided by students. After the lead author generated the initial codes, other members of the team independently coded several student responses to refine the coding scheme. Answers without explanations (i.e., "$\frac{\partial V_x}{\partial x}$ is zero") or explanations that were not clear enough to be understood were coded as unclear reasoning.

Figure 1b (not given to students) shows which components students are expected to use to determine the sign of $\frac{\partial V_x}{\partial x}$.

A slice of **a vector field V** (for z=0) is shown. Assume that the field has no components in the z-direction (into and out of the page) and that other slices for other values of z would look the same (i.e., ignore z components or direction).

A. For each of the quantities below, state whether the quantity is *positive*, *negative*, or *zero*. Show or explain briefly (stating any assumptions you are making).

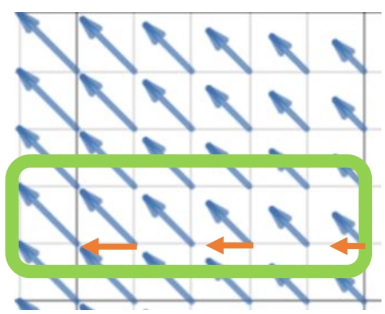
The (z component of) the curl of the field V

the divergence of the field V

B. Indicate whether the following derivatives are *positive, negative,* or *zero.* Show or explain briefly (stating any assumptions you are making).

$\frac{\partial V_x}{\partial x}$ in the region $\qquad$ $\frac{\partial V_y}{\partial x}$ in the region

$\frac{\partial V_x}{\partial y}$ in the region $\qquad$ $\frac{\partial V_y}{\partial y}$ in the region



a $\qquad$ b

Figure 1. (a) The research task asked to probe students' reasoning about div and curl. (b) The figure for the task, with components of the vector field students are expected to examine to determine $\frac{\partial V_x}{\partial x}$.

Results and Discussion

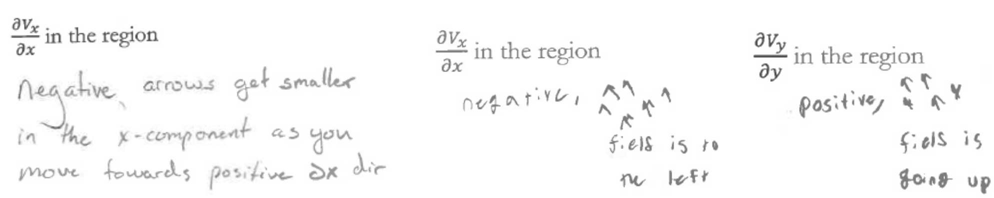
The numbers of students that correctly determined the constituent derivatives and the distribution of correctness of student reasoning are shown in Table 1.

Only 4 of 32 (13%) students answered all parts of the question (all derivatives, divergence, and curl) correctly, suggesting that the set of questions was especially challenging. Most students successfully identified the constituent derivatives in three out of four cases: for $\frac{\partial V_y}{\partial y}$, $\frac{\partial V_x}{\partial y}$, and $\frac{\partial V_y}{\partial x}$, success rates were over 75%. For two of these derivatives the vector field component was not changing in the indicated direction, and for the third it was positive and decreasing.

In contrast, only 21% of the students correctly determined that $\frac{\partial V_x}{\partial x}$ was positive with correct reasoning. Because this derivative was the most challenging, we examined the reasoning required in some detail. The first step was identifying the appropriate components; Figure 1b shows the components that students were expected to examine to determine $\frac{\partial V_x}{\partial x}$. Analysis of student responses showed that most (<90%) used the correct components to determine $\frac{\partial V_x}{\partial x}$. For the given vector field diagram, the absolute value of V_x is getting smaller with respect to the x-axis, but due to the direction of V, the change in V_x (∂V_x) is positive. Written responses did not show evidence of explicitly attending to subtracting the two components. Instead, students wrote about the trends in the magnitude of the components, whether that component was increasing, decreasing, or staying constant when you move in a certain direction. In one incorrect student responses, shown in Figure 2a, the student wrote "arrows getting smaller" and seemed to associate this with the resulting negative value for $\frac{\partial V_x}{\partial x}$. This response, while incorrect, included some correct reasoning. Responses that associated the component with the appropriate direction but reversed the sign were coded as correct reasoning with a sign mistake. However, this student wrote about change in magnitude rather than change in component.

Table 1. Performances and distribution of student reasoning for $\frac{\partial V_x}{\partial x}, \frac{\partial V_y}{\partial y}, \frac{\partial V_x}{\partial y}$, and $\frac{\partial V_y}{\partial x}$ of the diagram. Correct answers for the derivatives are given in parentheses. (C: Correct, I: Incorrect).

N=32	*Constituent derivatives for divergence*				*Constituent derivatives for curl*			
	$\frac{\partial V_x}{\partial x}$ (+)		$\frac{\partial V_y}{\partial y}$ (0)		$\frac{\partial V_x}{\partial y}$ (0)		$\frac{\partial V_y}{\partial x}$ (−)	
Reasoning	*C*	*I*	*C*	*I*	*C*	*I*	*C*	*I*
Correct	7	17	21	2	18	2	19	1
Incorrect	1	4	0	1	1	3	3	2
Unclear or none	3	0	4	4	6	2	1	6



a b c

Figure 2. (a) Incorrect response to $\frac{\partial V_x}{\partial x}$, reasoning about the decreasing arrow length to justify the response. (b) and (c) Example responses of students relating derivatives with the direction of the vector field component.

Previous studies conducted in introductory physics courses reported student difficulties in subtracting negative vectors when given as a graphical representation, even though students were very successful enacting the procedure of subtracting when vectors are provided in an equation with coordinates (Barniol & Zavala, 2014; Susac et al., 2018). Our results suggest similar difficulty among students in this more advanced context.

While students had good success with many of the constituent derivative tasks, only 38% of students determined the correct signs for divergence and curl with correct reasoning, as shown in Table 2. For the divergence, most of the responses coded as incorrect signs with correct reasoning stem from finding $\frac{\partial V_x}{\partial x}$ as negative, as described above. Student responses included several alternative forms of reasoning for divergence, such as inferring flux from the vector field diagram, or identifying changes relative to a perceived "source" of the arrows.

There was also no clear relationship between success on derivative task and on the curl task. Many of the students coded with incorrect reasoning for curl answered that the z component of curl was zero and referred to the problem statement that the vector field had no z component. This may reflect a misinterpretation of the question or a misunderstanding of the relationship between components of the vector field and those of the curl. A previous version of this question did not include this text and more students did find the curl correctly for the diagram.

Table 2. Student performances for divergence and curl of the diagram (C: Correct, I: Incorrect).

N=32	*Divergence*		*Curl*	
Reasoning	*C*	*I*	*C*	*I*
Correct	12	10	12	3
Incorrect	1	6	1	13
Unclear or none	2	1	1	2

Another small set of student responses show confusion between the direction of the vector field component and the change of the vector field component, as shown in Figure 2b and 2c. Similar confusion between a quantity and its change or rate of change has been widely reported in both mathematics and physics contexts at the introductory level (Meltzer, 2004; Trowbridge & McDermott, 1981). Our data show examples in a more advanced population, suggesting the persistence of this confusion.

Conclusions and Future Work

Our results were consistent with previous studies that reported that divergence and curl are challenging for students. For this task and population, students were largely successful in determining the sign of the constituent derivatives of div and curl. Incorrect responses showed confusion between the sign, direction, and change in the magnitude of vector field components; this confusion seems reminiscent of previous findings in both introductory physics and mathematics classes.

Determining the constituent derivative incorrectly did result in an incorrect divergence sign, but correct responses on constituent derivatives were not sufficient for success on divergence and curl. Students seemed to confuse components of curl with components of the field itself.

Explicit attention to the derivatives in expressions for divergence and curl seems to be a fruitful direction for future research and curriculum development. We plan to collect additional data, including student interviews, to further investigate the understanding of these derivatives as well as of divergence and curl, including relationships between quantities. We also intend to re-examine existing data through the lens of covariation (Carlson et al., 2002) and to examine whether it is possible to extend existing frameworks to vector derivatives.

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Why Mathematicians 'Fully Understand': An Exploration of Influences on Their Understanding

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One of the primary mathematical practices of many professional research mathematicians is engaging with highly advanced mathematical concepts in their research. As mathematicians engage with these concepts in their work, they may or may not 'fully understand' them, which could include not being able to deduce why each step in a proof logically follows from the next. This preliminary study investigates what reasons professional mathematicians give for not 'fully understanding' mathematical concepts in their own work. One mathematician researching coding theory was interviewed, which utilized one of the participant's own research journal articles as a launching point for discussion. Various reasons for the participant's conceived amount of understanding were observed, including authoritative reasoning and academic pressures. Implications for future research on professional mathematicians are discussed.

Keywords: Advanced Mathematical Thinking, Professional Mathematician, Understanding

Introduction

Professional research mathematicians are constantly engaging with highly-abstract, advanced mathematical concepts in their academic endeavors, especially if they study pure mathematics. Nevertheless, even with the heavily abstract nature of the mathematics, mathematicians are successful at answering mathematical problems and publishing new research. Therefore, it is worth investigating how professional mathematicians engage with and understand highly-abstract, advanced mathematical concepts. Some research has investigated how mathematicians understand mathematical concepts (e.g., Flanagan, 2022; Shepherd & van de Sande, 2014; Sinclair & Gol Tabaghi, 2010; Soto-Johnson et al., 2016; Wilkerson-Jerde & Wilensky, 2011). Despite this, there is certainly a need to further analyze the intricacies of how mathematicians understand these concepts, and specifically what factors influence how mathematicians engage in this understanding. This preliminary research study aims to address this need.

Literature Review and Conceptual Framework

The study of mathematical understanding has been addressed in a variety of ways, including procedural versus conceptual understanding (Hiebert and Lefevre, 1986), intuitive, relational, symbolic, and logical understanding (Byers & Herscovics, 1977; Skemp, 1976, 1987), and the Pirie-Kieren dynamical theory of understanding centered on how one experiences growth in understanding mathematical concepts (Pirie & Kieren, 1989, 1994). Additionally, distinguishing between different types or levels of understanding has appeared in the research on proof, such as the distinction between semantic and syntactic proof productions (Weber & Alcock, 2004) and the need for strategic knowledge (Weber, 2001). Furthermore, research has demonstrated that professional mathematicians sometimes choose to not 'fully understand' all the mathematics they work closely with. Fully understanding a mathematical concept is impossible to define precisely, but a demonstration of a lack of full understanding could include choosing to use a theorem in a proof without knowing why the theorem is true, or accepting a proof is valid without being able to deduce why each step logically proceeds from the previous one.

In the context of proof-reading and validation, there is strong evidence that professional mathematicians will accept a proof as true or valid without using deductive reasoning (e.g., Weber et al., 2014; Weber et al., 2022). For example, Weber (2021) provided evidence that even set theorists, who work closely with the building blocks of mathematics, are sometimes content without being able to construct certain details of their derivations. In addition, professional mathematicians have demonstrated that they sometimes use authoritative reasoning when reading and validating proofs (e.g., Inglis & Mejía-Ramos, 2009; Mejía-Ramos & Weber, 2014; Weber, 2008; Weber & Mejía-Ramos, 2011). Harel and Sowder (1998) defined an authoritarian proof scheme to be when the source of an individual's truth claims come from a place such as a teacher or a textbook. Instead of teachers or textbooks, mathematicians sometimes appeal to the authority of reputable journals and famous mathematicians.

Research Questions

Two primary research questions guided this research study. The first question captures the broad purpose of the research, while the second pertains to the specific goals of this study.

1. (Broadly) How do mathematicians successfully work with and understand highly-abstract, advanced mathematical concepts?
2. (Specifically) What reasons do professional mathematicians give for why they believe they 'fully understand' or do not 'fully understand' the mathematical concepts they use in their own work?

Methodology

To address the above research questions, a one-hour, semi-structured Zoom interview was conducted with one professional research mathematician. This mathematician, Callie (pseudonym), was a female algebraist who specializes in coding theory. She was specifically chosen for this study because her area of research was tangentially related to my own mathematical expertise, which would allow for me to better understand the mathematics she discussed in the interview. This interview was video-recorded, and the audio was transcribed for data analysis. Furthermore, this interview consisted of three phases. Callie was first asked some basic questions to better understand her mathematical background. Afterwards, we discussed specific components of one of her own research journal publications. Questions were centered around aspects of the article where she referenced mathematics from other sources, to help assess whether she believed that she understood the mathematics she was using from other sources that she did not publish herself. The final phase consisted of questions directly pertaining to the research questions, two of which are provided below for reference.

1. Are there any times where you believe that you have encountered a new advanced or highly-abstract mathematical concept where you were able to successfully work with it, but did not 'truly understand' the concept?
2. Have there been any incidents where you read a research paper relevant to your own research where you utilized some of the mathematics in the paper, but did not understand why it was true?

The interview transcript data was subsequently coded in two phases. The initial phase of coding looked for general codes pertaining to the research questions. These initial codes were then refined and grouped together into more precise codes, which were applied to the transcript in a second phase of coding. After final adjustments were made, these codes were organized into four themes, which will be discussed in the following section.

Results

The results from the data analysis will be presented as a thematic analysis (Braun & Clarke, 2012) of the reasons that Callie provided for having certain levels of understanding for the mathematical concepts she uses in her own work. The coding process revealed four major themes, which are (1) authoritative reasoning, (2) external pressures, (3) academic influences, and (4) intellectual factors. Each of these themes will be addressed individually, with supporting evidence provided from the interview.

Authoritative Reasoning

One pattern that appeared frequently in the data was authoritative reasoning. This consists of when Callie stated that she believed or accepted something was true because it came from a trustworthy or authoritative source, such as a reputable journal. In the case of appealing to the authority of journal articles, Callie stated that "I also think that because of the peer review process, and all of that, typically I feel confident that results that have appeared in the literature are correct. You know sometimes that's based on journal reputation, right?" Moreover, in addition to the peer review process providing Callie with confidence in the validity of results, she expressed that if a result is important, then likely enough people have looked over the result to verify its correctness. Regarding a particularly important result in her field, she mentioned, "And so it's had lots and lots of eyes on it, so there's something that comes from people believing that this is true and knowing that other people have checked it…it's received a lot of attention." Another facet of authoritative reasoning came in the form of believing results were true because certain people were involved. When discussing an area of her own field of research where she did not understand all of the results, she stated that this is "an example of where we're looking to use something and we trust that the people who have done the work have done it correctly."

External Pressures

Another theme that came out of the interview data with Callie was external pressures. This theme consists of when her choice to try to understand a mathematical concept was directly influenced by external factors, such as time constraints due to academic duties. The most common type of external pressure was time constraints. Callie's busy schedule and its influence on her ability to take time to deeply understand all the mathematical concepts in her work was summed up when she stated:

> And you know now, there's just not the time to do that. I mean I have my own students I have to worry about. I also have an administrative role that's practically a full-time job, in addition to the faculty member role. And so, so that's part of it, it's the time.

Moreover, she emphasized that "it'd be hard to confirm every single thing that we use and actually create something new ourselves at the same time." In actuality, highly advanced mathematics can take time to understand, and there is a lot of it being produced. According to Callie, this makes it unrealistic to produce new results and 'fully understand' all related results.

Another type of external pressure is the natural influence of what the current research community believes is important. For example, Callie described how she and her colleagues were looking to find a sufficient condition for a mathematical result, even though they were not entirely sure what they were looking for. However, they did that because they "knew it would be well-received by the research community" if they could find it. Additionally, she mentioned that her research pursuits have shifted because she wants to "stay current." Specifically, she stated:

> The field has moved a bit, and so I kind of like to move a little bit too in order to stay current, you know, in terms of applications and things that people actually care about,

> like for funding agencies or journals...because I'm having to pivot. I shouldn't say having to; I'm choosing to pivot based on some internal motivation to follow external factors.

Even though these are reasons why a mathematician might choose to pursue certain research questions, they also demonstrate that there are academic pressures that influence what one chooses to focus on and try to better understand.

Lastly, Callie shared how her level of understanding on certain mathematical concepts has likely been affected by the availability of resources now versus when she was starting her career. In particular, now there are a whole host of relevant journal articles that we can access on the internet with minimal effort. However, twenty years ago, Callie described in contrast to having twenty-five online articles, “If I had gone to the library and I only have two papers related to a subject…I’m going to really pour into those in ways that I wouldn’t do with the twenty-five.” This comment is especially interesting because it highlights that the extent to which mathematicians choose to understand mathematical concepts for their own research may be changing due to technological advancements, which would need further research to verify.

Academic Influences

A third pattern that emerged from the data is academic influences. This consisted of when Callie discussed anything influenced how the actual mathematical results would be used or the nature of the mathematical content itself, including what field of study that the mathematical content came from. Callie explained how the field of study influences what she chooses to ‘fully understand’ by stating, “And in order to follow all that reason, I mean it might be a series of many many papers, you know, traversing other fields. So, it might be outside of what would be easily doable without help.” This means that sometimes it may be difficult to understand the mathematics you are using, if you need to use advanced concepts that are outside of your area of expertise. Moreover, it is also possible that the field of research you engage in might influence the breadth of what you are able to understand, as suggested by Callie. She stated that “Different research areas are different as well…Coding theory’s evolved a lot in the last twenty years. And so that means lots of different types of things go into my research program.” She contrasted this with a mathematician who focused deeply on one idea for the entirety of his career, where he may deeply understand everything that he uses in his own work.

Another factor that appeared in the data was the influence of one’s stage of career or current mathematical experience. Callie described how she believes she is “more comfortable now not knowing things than she once was,” and questioned whether that is a “natural career progression” or not. Lastly, Callie expressed instances where the mathematical purpose of a concept for a journal article influenced how much effort she put into understanding it. Generally, she stated, “I use material from the literature and assume that it’s correct, and I think that how far down that we go into that depends on what we’re trying to do with the material." In the context of the journal article from the interview, she mentioned:

> You know there are times when you know, maybe you need to reference something and, you know, the results are there or you can see them, but you don't have full awareness of the supporting evidence. It certainly happens a lot, but in this case, you know we were trying to really nail down this value, and so I had spent some time with the techniques in that paper.

This sentiment was reoccurring in our discussion of her journal article, because she said that she understood the important references well but did not fully read or understand some of the references that were only provided for context or background.

Intellectual Factors

The final theme from the interview data was intellectual factors, which were any mentions of mental or intellectual influences on one's understanding, including Callie's mental capacity and how reasonable a mathematical statement appears. For instance, Callie expressed how the reasonability of a mathematical statement influences how much effort she needs to put into deeply understanding the mathematics.

> So sometimes my knowledge and awareness might come from hearing a talk about it and feeling like "Okay those techniques make sense; it seems reasonable; that's all I need." Other times, you know, it might be deeper confirmation if you feel like "Well that sounds surprising. I didn't think something like that could be true."

It is possible that when a mathematical statement appears reasonable, that this is influenced from other mathematics you know. In this case, one might believe that the statement is probably true based on other mathematical expertise or intuition. In addition to reasonability, Callie mentioned how she only has so much mental capacity to understand mathematical facts. She discussed this phenomenon by stating that most mathematicians cannot know everything in their field, and verbalized, "But to know the whole field is just not, at least for me, possible. I don't think it's possible for most people." These remarks provide support that mathematicians may not be able to 'fully understand' everything they work with, even if they would like to.

Discussion

The above thematic analysis provided supporting evidence that professional mathematicians do not always 'fully understand' the mathematical concepts they use in their work. Moreover, there are a plethora of reasons for why mathematicians have certain levels of understanding for these concepts. However, despite this, this study was preliminary in many ways and has some limitations. Most noticeably, this study only consisted of one mathematician researching coding theory. To further support these claims, it is important to research other mathematicians in similar ways, to see if Callie's experiences are shared among other mathematicians. A second limitation was that the chosen journal article was written over fifteen years before the interview. Due to this, some of Callie's remarks may have been misremembered, and she was not able to share as much detail about aspects of her producing the paper. In turn, much of the data came from the third phase of the interview. I believe that utilizing mathematicians' journal articles for interview data collection is valuable, but further data collection is certainly needed.

Moreover, the results from the analysis support the claims of the aforementioned literature that mathematicians appeal to authoritative reasoning and have reasons for not 'fully understanding' certain mathematical concepts they use (e.g., Inglis & Mejía-Ramos, 2009; Weber, 2008). In addition, this study contributed a concrete example of a mathematician who demonstrated numerous reasons for the understanding she had. However, further research is needed to delineate these various reasons why mathematicians do not 'fully understand' concepts. Is it mostly because of external and academic factors? Or is it also due to mathematical factors, such as using the concept in rich, mathematical "inventising" (Pirie & Kieren, 1994; see also Hadamard, 1945) or that the concept is too advanced or abstract for the individual? The above analysis provided some evidence that these could all be at work, but further research needs to be conducted. Lastly, the evidence in this study has implications for the teaching of advanced mathematics to students. If professional mathematicians are influenced in various ways on what they choose to 'fully understand,' we should not be surprised when students are similarly influenced. Instead, focus should be given to helping students learn to navigate what mathematical concepts should be 'fully understood' and what could be postponed until later.

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Student Strategies Playing Vector Unknown Echelon Seas, a 3D IOLA Videogame

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We present preliminary results of students' strategies playing Vector Unknown: Echelon Seas [VUES], a 3D videogame intended to support student reasoning about vectors. Our team designed VUES by drawing on theories from Inquiry-Oriented Instruction (IOI), Game-Based Learning [GBL] and Realistic Mathematics Education [RME]. VUES builds from a prior 2D game by giving players vectors with 1, 2, or 3 components, depending on the level. We use codes from our team's prior analysis (Mauntel et al, 2020) to analyze strategies in the 3D game. Early results show that students develop similar strategies during 3D gameplay as other students developed while playing the 2D game. However, we have also found new strategies that we did not witness with 2D gameplay, requiring us to extend our coding scheme. Further, early results emphasized the need for design changes to the 3D game to better support players' progress.

Keywords: Linear Algebra; Mathematical Play; Inquiry-Oriented Instruction; Game-Based Learning; Realistic Mathematics Education

The importance of Linear Algebra in undergraduate STEM students' degree progression is well-established (Stewart et al, in press). In this on-going design-research project, we have iteratively implemented and re-designed a browser-based video game intended to support student experiences with vectors before entering (and early during) a first course in Linear Algebra. Our team has drawn on a combination of design principles from Game-Based Learning (Gee, 2003; Gee, 2005; Gresalfi & Barnes, 2016; Williams-Pierce & Thevenow-Harrison, 2021), Inquiry-Oriented Instruction (Rasmussen & Kwon, 2007; Kuster et al, 2018), and Realistic Mathematics Education (RME; Freudenthal, 1991; Gravemeijer, 1994) to iteratively refine and develop the videogame (Zandieh et al, 2018; Mauntel et al, 2019; Mauntel et al, 2020; Mauntel et al, 2021). We have iteratively re-designed this video game to better support players' algebraic and geometric understanding of linear combinations of vectors and vector equations. In this paper, we present the newest iteration of the video game, discuss early results of clinical interviews with undergraduates playing the video game, and discuss what implications these results have for future iterations of this game and educational game design more generally.

Literature Review and Theoretical Framing

Game-Based Learning (GBL) is a growing area of education research that shows promise for supporting meaningful gains in student thinking outside of high-stakes classroom environments (Gee, 2003; Gee, 2005; Gresalfi & Barnes, 2016). With this project, our team set out to blend best practices from GBL and mathematical curriculum design theory to create a video game based on the Inquiry-Oriented Linear Algebra curriculum (IOLA; Wawro et al, 2013). Our game design leverages the key idea of using vectors as modes of transportation in the Magic Carpet Ride task from the IOLA curricular materials, which use the curriculum design theory of RME (Gravemeijer, 1994; Freudenthal, 1991). Using vectors as modes of transportation provides students and, now, video game players with an experientially real setting for using and understanding linear combinations of vectors. Our team has also incorporated theoretical framing from the Inquiry-Oriented Instruction (IOI) literature (Rasmussen & Kwon, 2007; Zandieh et al, 2017) to inform how student/instructor roles translate into designing a video game interaction.

The current version of the video game, Vector Unknown: Echelon Seas [VUES], takes place in a 3-dimensional environment with the player controlling a pirate avatar who runs between stages and puzzles. In this paper, we focus on Stage 4 of VUES, which we designed as an extension of the original 2-dimensional Vector Unknown game (VU). VUES Stage 4 is subdivided into three different difficulty levels, all of which are played via the same controls and format, but differ based on the number of components in the vectors used (i.e., whether the vectors in the level have 1, 2, or 3 components) and each difficulty level consists of 6 puzzles in which the player helps the pirate aim a grappling hook at an anchor. The player uses the vectors to create a linear combination equal to a goal location by dragging vectors (represented as column vectors in a given collection of cards) into a vector equation and using sliders to adjust scalars in front of each vector. To provide just-in-time feedback (Gresalfi & Barnes, 2016; Williams-Pierce, 2019), the game calculates the appropriate linear combination of those vectors, shows (1) a geometric representation of that linear combination consistent with a "tip-to-tail" representation of linear combinations and (2) a numeric representation of the vector equation with the calculation on the right of the equal sign, labeled as "Answer".

Figure 1. Images of gameplay in Vector Unknown Echelon Seas.

Using the earlier iteration of the game, Vector Unknown, we characterized the strategies that participants used to play the game (Mauntel et al, 2020, 2021). These strategies focused on two main components of the game, geometric and numeric. Geometric strategies related to the generated geometric display of the linear combination of two vectors. Numeric strategies involved focusing on a vector equation. There were four core strategies each with a numeric and geometric variant. ***Focus on one vector*** involved selecting a vector, scaling it to be as close to the goal location as possible, and then choosing another vector to reach the goal. ***Focus on one coordinate*** involved choosing an x- or y-coordinate in the goal vector and adjusting the scalars on each of the two vectors to first match one coordinate, and then adjusting as necessary to reach the goal. When a student employed ***quadrant-based reasoning*** they chose vectors to reach the goal based upon the quadrant of the goal (geometric) or by the signs of the goal (numeric). ***Button-pushing*** was characterized by adjusting scalars and switching vectors rapidly as a trial-and-error strategy. While not officially characterized as a strategy, participants also tended to choose vectors that had a zero to help them solve the puzzles. Our current work seeks to identify whether and how these identified strategies might emerge as students play the newer version of the game and also what new strategies we might encounter students using.

Methods

One member of our team conducted clinical interviews with five students from a small, four-year undergraduate university in the American Southeast. Each interview lasted between 60 and 120 minutes and consisted of the participant playing at least four puzzles in each difficulty setting as time allowed. Three of the participants had completed an introductory course in linear algebra, but had not yet played any version of the video game. Two participants had. Recently completed a Calculus I course and had not yet taken any course in Linear Algebra. Interviews occurred virtually over Microsoft Teams. Players shared their screens and maximized the window in which they played the game. Each interview was recorded and transcribed using the embedded features of MS Teams. The interviewer guided the participants through gameplay with written prompts and asked follow-up questions to encourage participants to explain their reasoning and to clarify strategies and rationale as they played the game. After data collection, the authors met to discuss analysis methods and narrowed our focus on using one participant's video for an initial coding pass. Our initial analysis is driven by the research question: *How might student strategies while playing VUES be similar or different from the strategies we identified with students playing VU?*

In the next section, we will discuss some of the results from our data collection, exemplifying them with key excerpts from our interview with Kyah (pseudonym, pronounced "KAI–yuh"; a woman, who is African American, majoring in mathematics). We will first briefly discuss some of Kyah's responses during gameplay on Level 2 and explain how we use the codes from prior analysis (Mauntel et al, 2021) to categorize her activity. We then provide examples from Kyah's Level 3 gameplay to highlight how some of her 2-D strategies persisted into her 3-D gameplay. We will then highlight aspects of her 3-D gameplay that did not fall into our existing categories. We expected such strategies would exist and, thus, will necessarily extend our existing taxonomy to include new types of activity that students engage in in the newer iteration of our game.

Results

Kyah spent the first 22 minutes of the interview working through five puzzles of Level 1, which uses vectors from $\mathbb{R}^1$. Across these levels, Kyah correctly articulated numeric strategies for using multiplication and addition to reach the goal and described a general solution for how she could find more than one possible correct solution when asked. When Kyah moved on to Level 2, the 2-dimensional version of Stage 4, the interviewer prompted her to notice the visual aspect of the game which they referred to as blue, red, and green lasers. Kyah at first was surprised that "a whole three different lasers" appeared. She added, "We just need one, right." The interviewer confirmed that "the green is the same as the Final Answer." Kyah then realized "oh, so the green needs to be on the white line." Although Kyah primarily used numeric methods in five puzzles of Level 2, she did explicitly return to the idea of matching the green line to the white line later when playing Level 3.

The majority of Kyah's Level 2 (vectors from $\mathbb{R}^2$) gameplay involved the strategy ***focus on one coordinate***. This worked especially well for her when she had a vector with a 0 in the second coordinate. In this case she could scale a vector with a non-zero second coordinate until its second coordinate matched the second coordinate of the "Goal" vector. Then she could add the vector with the 0 in its second coordinate and scale it until the first coordinate of the "Answer" vector matched the "Goal" as well. For example, Kyah solved Level 2, Puzzle 4 using the linear combination (4)<-1,3> + (-5)<1,0> = <-9,12> by first scaling <-1,3> to get <-4, 12> and then adjusting the scalar on the <1,0> vector. This is consistent with the ***focus on one coordinate*** strategy we identified from analyzing student gameplay in VU (Mauntel et al, 2021). In Puzzle 5, where no vector choice had a 0, Kyah was unable to implement this exact strategy, though she did attempt to modify it. She articulated this clearly when asked, "I'm looking at the Final Answer. I always aim to go for the bottom number first for some reason. Try to get the bottom number … [then] adjust to match the top." It was not until the interviewer said, "I'm looking at the lasers," that she changed her focus, "Oh! I forget about the lasers. … I wasn't even thinking about the lasers.” She did agree that the lasers could help. She and the interviewer then worked together, combining geometric and numeric strategies to solve Level 2, Puzzle 5.

When Kyah transitioned to Level 3 (vectors from $\mathbb{R}^3$), she was presented with the vectors: <1,3,1>, <-1,1,1>, <-3,-2,0>, and <-1,2,2> to reach a goal of <-7, 7, 5>. Similar to level 2, we coded Kyah’s initial strategy as Numerical Focus on One Coordinate. She moved the vector <1,3,1> into the equation and adjusted the scalar to 1, stating “I was looking at 7s first. I was thinking…oh wait. Actually, I think I should look at the bottom first. I’m going to go with the 5 first.” She adjusted the scalar to 5*<1,3,1> to match the z-coordinate of the “Goal”. Kyah then said, “[I] need the top number to be negative,” and chose the vector <-1,1,1> matching the sign with -7 in the x-coordinate. We coded this as Numerical Quadrant-Based Reasoning because she chose a prospective vector based upon the sign of the location she wanted to travel. She adjusted the scalar on <-1,1,1>, but was unable to get the first pair of vectors to work.

After this, the interviewer suggested that Kyah rotate the camera. Once she tried this, Kyah focused more on the geometric aspects of the game, especially the “lasers” that represented the scaled vectors and their linear combinations. She then adapted the strategy that she used in Level 2 and described her goal by saying “the green line is supposed to go on the white line”. Because the prior version of the videogame did not include geometric representation of the resulting linear combination, we did not have a code for this strategy. After describing her intentions, Kyah adjusted the scalars until the green line looked like it overlapped the white line (Figure 2, left). She then rotated the camera view and discovered that the view “can be very misleading” (Figure 2, right). This highlights an important difference between Levels 2 and 3. Solving a puzzle on Level 3 with a geometric strategy requires viewing the vectors from a variety of viewpoints and coordinating these viewpoints to make sense of the linear combination. In this moment, Kyah used this to determine if it was possible to find a solution with these two vectors.

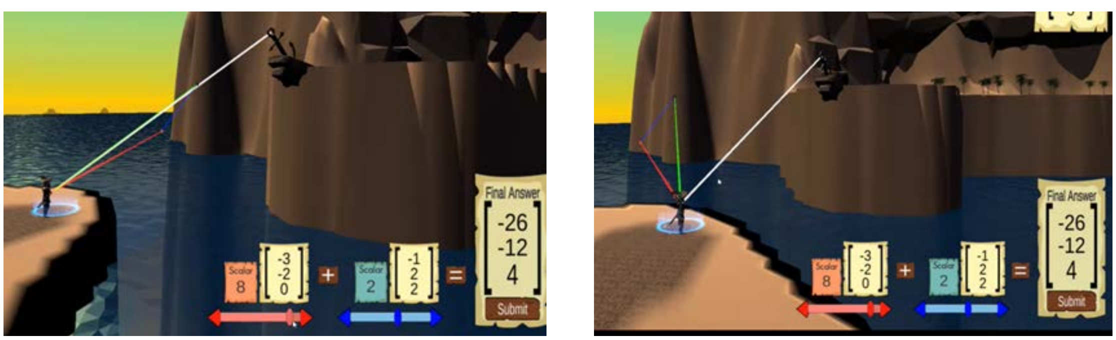

Figure 2. Kyah’s “close” linear combination on Level 3 (left) and the same combination with camera moved (right).

Kyah adjusted the scalars several more times and then switched vectors repeatedly with the other two in the list. She arrived at the vectors <1,3,1> as the red vector and <-1,1,1> as the blue vector she tried to figure out how the red and blue vectors affected the green vector. After playing her conclusion was that “the red one is just shifting left and right” and “the blue, that is going to shrink it [the green vector].” From this we can infer that the Kyah’s strategy is to try to shift and shrink [or expand] the green line until it lands along the white line. We call this new strategy geometric shrinking and shifting. Kyah experimented with this strategy, but was unable to find a solution and returned to a more numeric strategy.

Kyah had the vector <-3,-2,0> as the red vector and tried several vectors for the blue coordinate. The interviewer asked asked how having a zero in z-coordinate for the red vector affected the coordinate choice for the blue vector. Kyah realized that the scalar for the blue vector had to result in a 5 for the z-coordinate since the red vector contributed nothing to z-coordinate. Kyah tried multiple vectors setting the blue scalar to 5 for the vector <-1,1,1> and then adjusted the scalar on <-3,-2,0> to see if it would work. After trying several scalars for <-3,-2,0> she kept the vector <-3,-2,0>, swapped <-1,1,1> for <1,3,1>, adjusted the scalar for <1,3,1> to 5 so the z-coordinate would work, and finally adjusted the scalar on <-3, -2, 0> to -10 and then increased it to 4 and found that number worked. While Kyah explored geometrically, her final solution was the result of returning to a focus on one coordinate and combining it with a vector with a zero entry in the final coordinate. In the next level there was a vector with two zeros in it, and Kyah employed a similar strategy to reach the goal position by selecting the vector with two zeros, selecting another vector, adjust it to reach the z-coordinate, and then adjusting the vector with two zeros to get the final solution. Thus by this point, Kyah was employing a strategy similar to using vectors with zeros in several coordinates.

Discussion

First, we note that Kyah used several strategies consistent with those we observed students using with previous versions of the game, including Quadrant-Based Reasoning and choosing vectors with zeros in order to Focus on One Coordinate (Mauntel et al, 2021). Further, Kyah extended strategies from Level 2 to Level 3. However, when prompted to notice the geometric aspects of the game, she employed a new strategy, Green Line on White Line, but needed to develop a sense of how to assess whether this strategy was successful and how to manipulate the vectors to achieve this goal. Rotating the camera view allowed her to determine if the strategy was working. This points to a core difference between VU and VUES: adjusting the camera allows a player to solve puzzles in Level 3 of VUES geometrically that was not necessary for players to solve levels in VU because the earlier game only used vectors in $\mathbb{R}^2$. Kyah also found the most success using numeric strategies. We hypothesize this may be due to a key difference between VU and VUES, specifically how the game generates the vectors on any given level. In VU, players could always reach the goal as long as they selected a pair of vectors that were not scalar multiples of each other. In VUES, four vectors are provided and the player is only guaranteed that one pair of the vectors can reach the goal. This means that players could choose linearly independent vectors that do not reach the goal with the available integer scalars. This makes geometric strategies that were employed frequently during VU less effective in VUES.

Questions

We will continue to analyze Kyah's and the other student's gameplay. During the conference presentation, we anticipate presenting more robust findings from these data, including analysis of other participants' strategies and more focused coding of 3-D game strategies. We expect that the presentation will prepare audience members to engage with the following questions:

1. How might these results inform our game design and GBL design theory?
2. What theoretical and methodological approaches could help us analyze how players move the camera and interact with the computer program's 3-D environment?
3. How does the shift from vectors in $\mathbb{R}^2$ to vectors in $\mathbb{R}^3$ within VUES inform how students use the camera?

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Creativity and Subjectivity: Two Researchers' Poetic Re-Presentations of Critical Student Perspectives of Mathematical Success

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Dismantling notions of objectivity in the qualitative research process necessitates engaging research subjectivity in meaningful and explicit ways. Poetic transcription methodology allows for an entry point into conversations around research subjectivity. In the context of a critical exploration into students' definitions of their own mathematical success, the poetic transcription results of two researchers are comparatively analyzed to highlight structural and thematic (mis)alignments in researchers' poetic interpretations of students' definitions of mathematical success. We emphasize both the scholarly value and uncertainty that the results of such a comparison produce and discuss how future work might build on these comparisons and on our understanding of what poetic transcription can bring to mathematics education.

Keywords: Methodology, Student Success, Subjectivity

Within qualitative research processes, researchers themselves are the "primary research instrument" (Corley, 2020, p. 1023). This positioning makes clear that researchers and their collection, framing, analysis, and presentation of qualitative data are deeply intertwined. One way this manifests is in recent academic discussions on how identity impacts mathematics education scholarship, and the acknowledgement that mathematics education is *not* culture-free nor is the knowledge it produces disjoint from socio-cultural and socio-political notions of power and oppression (Gutiérrez, 2013; Ladson-Billings, 1997; Martin et al. 2010).

By resting on the research axiom that neutrality is fundamentally nonexistent within qualitative mathematics education research, space is opened to explore ways to make explicit the discussion of subjectivity. Corley (2020) writes that "the goal for researchers should be to write in such a way that challenges mainstream knowledge claims and that values the subjective, contextual, relational nature of both reality and inquiry" (p. 1024). In this work, we aim toward this goal through comparatively analyzing the results produced by our independent engagement with poetic transcription methodology.

Poetic transcription draws from the literary tradition of a *found poem*, in which excerpts from existing written media are juxtaposed to create poetry with distilled meaning. In poetic transcription, participants' words from interview or focus group transcripts are pulled in such a way as to honor and reflect the participant's meaning and narrative style (Glesne, 2007). Prendergrast (2009) emphasizes that these poetic transcriptions are built from the participants' words but filtered and "re-presented" through the lens of the researcher by referring to these poems as "participant-voiced poems." Via the artistic embodiment of experiential knowledge, poetic transcription can present nuances of qualitative data in a way that the quoting and analysis of traditional prose may not, particularly when viewed through a lens of feminist theory (Faulkner, 2018) and critical, justice-oriented qualitative research (Fernández-Giménez et al., 2019; Keith & Endsley, 2020).

Poetic transcription is utilized in meaningful ways in educational spheres (see, for example, Thunig & Jones, 2020; West & Bloomquist, 2015; Mercer-Mapstone et al., 2019; Vannini & Gladue, 2008). It has a particularly rich history in regards to re-presenting the narratives of Indigenous participants through its existence as an "embodied meaning-making practice that is consistent with more holistic Indigenous approaches to teaching and learning which engage the heart, mind, body, and spirit" (Madden et al., 2013, p. 215; see also Ambo, 2018; Thunig &

Jones, 2020). In mathematics education spaces specifically, poetic transcription has been used to understand the structure of students' mathematical thinking (Staats, 2008), interrogate the intersection of theological beliefs and beliefs about mathematics (Norton, 2001), and examine mathematical identity in the face of "low attainment" labels (Helme, 2022). The poems constructed in the context of this work centered around re-presenting narratives of mathematical success shared with us by three first-year undergraduate women students who also identified as first-generation students, Pell-grant eligible students, and/or women of color.

Poetic transcription as a methodology lends itself well to a critical exploration of marginalized students' experiences of mathematical success because it (a) promotes "evocative critique and resistance of the status quo" (Faulkner, 2018, p. 23) and (b) lends itself to "topics that lead into the affective experiential domain" (Prendergast, 2009, p. 546). Students' own conceptualizations of their mathematical success align with both attributes; centering students' voices in the defining of their own mathematical success both provides critique of traditional neoliberal measures through implicit or explicit misalignment (Tremaine, 2021), and student definitions and narratives of student success are deeply affective in that they are evidentially embodied and emotional (O'Shea & Delahunty, 2018).

The two authors of this work engaged independently in a parallel poetic transcription process using the transcript data from three participants, resulting in three pairs of poems. In an investigation into both the use of the methodology and how it allows for rich conversation regarding the student participants' experiences and definitions of mathematical success, we pose the following research question: In what ways do the poetic interpretations of two researchers, when constructed through a prescribed poetic transcription methodology, align and misalign, and what do these (mis)alignments teach us about this data analysis method?

Methods

The data used in this analysis are three sets of two poems, each set containing one poem constructed by Tremaine and one poem constructed by Hagman through the poetic transcription process depicted in *Figure 1.* As poetic transcription is inherently a creative process, there is no one *right* way to engage in this methodology– the methodology illustrated by *Figure 1* draws from the work of Prendergast (2009) and Glesne (1997), as well as Burdick (2011)'s notion of tandem poetic transcription, and is intended not as a definite guide to poetic transcription, but as a description of process undertaken by the authors. This particular method of poetic transcription consisted of three phases: Phase 1 (Initial Parsing of the Data), Phase 2 (Poetic Juxtaposition), and Phase 3 (Member Checking).

Poems were constructed using qualitative interview data from three participants- Ada, Isabel, and Kenzie- as collected and transcribed by the authors in January 2021. Ada, Isabel, and Kenzie are first-year, Pell-grant eligible women students and were asked in an interview context to (a) describe an experience in which they felt successful in mathematics and (b) define what *student success in mathematics* meant to them, with conversational room to elaborate on responses.

The analysis below, as it is preliminary, identifies themes in both meaning and style that align and misalign between the two researchers' poems. We then discuss the significance of this alignment and misalignment as it relates to poetic transcription as a methodology and in research more broadly, with the intent of bringing others into this conversation as this methodology (and what it means to do research) is continually refined.

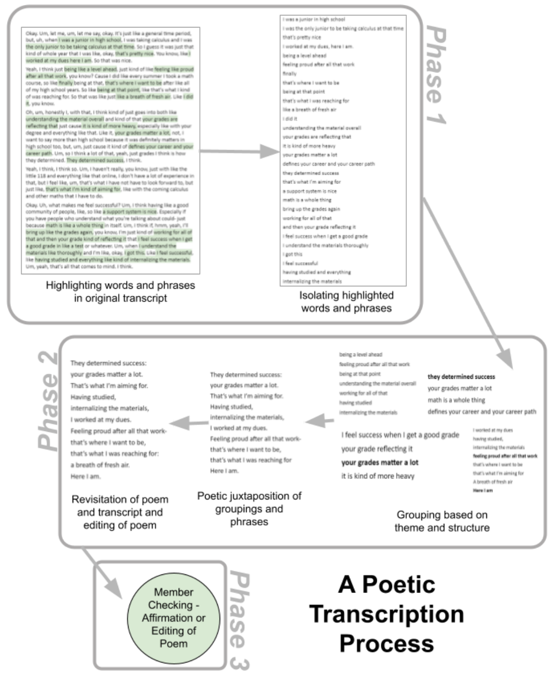

Figure 1. The poetic transcription process engaged in by both Tremaine and Hagman.

Analysis

In *Table 1*, we present the poems constructed by Tremaine and Hagman from each participant's data and invite the reader to contemplate what they tell us about the participants' definitions of mathematical success. We then follow the poems with a comparative analysis, diving more deeply into the analysis of Kenzie's poem and providing brief overviews of the others to account for succinctness in submission.

Table 1. Poems written by Tremaine and Hagman which draw from each participant's data.

Kenzie	
<u>Tremaine</u> They determine success: your grades matter a lot. That's what I'm aiming for. Having studied, internalizing the materials, I worked at my dues. Feeling proud after all that work- that's where I want to be, that's what I was reaching for: a breath of fresh air. Here I am.	<u>Hagman</u> Every summer I took a math course, that's what I was reaching for: being a level ahead, feeling proud, like a breath of fresh air. I did it. Just grades is how they determine success. Your grades matter a lot. It is heavy: your degree, your career, your career path,

	everything.
Isabel	
<u>Tremaine</u> It’s two different intelligences: being able to do well in school and being able to actually use what you’ve learned. Some people just enjoy doing math for the sake of math; I really enjoyed being able to know how much fabric to get. Being able to practically apply your math skills- this is what math was meant to be used for.	<u>Hagman</u> It’s two different intelligences there: There’s being able to do well in school and there’s being able to actually use what you’ve learned in school. Some people just enjoy doing math for the sake of math… I really like it when I can use math to do something: I have math skills to know how much fabric to get. To be able to calculate how much fabric you need- it’s successful because you’re using it for something.
Ada	
<u>Tremaine</u> I’ve had a lot of struggles in math. I’ve also had a lot of successes. We all fail, we’ve all been there. I had to take a step back and be resilient; work around it; put in the effort; learn critical thinking; I could pull through. I know how to do this. Cross it off a checklist? That’s not what college is. Further your education, get something positive out of the experience. I was still successful because I didn’t give up.	<u>Hagman</u> I was killing it. Getting nineties on my tests, I did so good on their finals. Running home to my parents- bragging to my friends- Hey, look, look at my scores, hahaha. Grades are important, but at the end of the day I’ve learned from college that grades aren’t everything. In a college math classroom, it’s more about learning something and getting something out of it. I need to get something out of it and still care about my grade, but not let that grade define me as much. At the end of the day, it’s not all about grades. Now it’s like, okay, I’m growing up.

As the reader is no doubt aware of after reading the poems, the set contains a myriad of alignment and misalignment, both in phrase, theme, style, and structure. In each pair of poems, we see stylistic differences between the authors' poems which emphasize different aspects of the students' interviews: a key distinction we see is the use of more abstract descriptions of student success versus more concrete descriptions of moments of success. The impact of this stylistic difference varies by set of poems, with the least differences for Isabel and most for Ada. Here we analyze Kenzie's poems to exemplify these differences and provide brief additional analyses of Isabel's and Ada's poems.

We see that several phrases from Kenzie's transcript were poetically juxtaposed in both authors' poems, such as the highlighting of how "they determine success", as well Kenzie's "feeling proud" and experiencing "a breath of fresh air" upon achieving her goals. However, one can see thematically that Hagman attends more closely to Kenzie's concrete experiences that led to mathematical success (in this case, taking a math course every summer), while Tremaine utilized more abstract imagery to illustrate how Kenzie discussed her experience of success. The reader may have also noticed Hagman's use of two stanzas, versus the single stanza used by Tremaine. In Tremaine's poem, this "they" entity is presented as having more influence on Kenzie's own perspectives and experiences by nature of being a part of a unified poetic narrative; the two stanzas utilized by Hagman present this entity as something which Kenzie discusses but perhaps does not have as direct an influence on her own experiences, as the experience of working hard to get "a level ahead" in mathematics is separate from what "they" define success as. Further, Tremaine and Hagman both highlight structural parallelism in Kenzie's narrative; Tremaine juxtaposes "that's where I want to be / that's what I was reaching for" and Hagman juxtaposes "your degree / your career / your career path." Thematically, both center on Kenzie's hard work in mathematics as central to her definition of success, while simultaneously painting nuanced interpretations of her transcript via poetic style and structure.

We see similar phrase and structure (mis)alignments in Isabel's set of poems. Interestingly, lines 1-6 in both poems draw from the same components of the transcript. Hagman splits their poem into two stanzas, allowing for a clear distinction between the two intelligences alluded to by Isabel, while Tremaine maintains a single stanza, reinforcing continuity of the overall theme found in both poems of the exaltation of mathematical applications as a means of mathematical success. Meanwhile, the most notably different set of poems are those constructed from Ada's narrative. Both in theme and in phrase choices, these two poems paint two different (albeit complementary) pictures of Ada's ideas of success. Tremaine's poem focused on the imagery and action phrases Ada used to describe her perseverance, attributing this perseverance as immensely relevant to how she was describing success. Hagman attended more to Ada's concrete experiences in mathematics, including "getting nineties on her tests." Both authors emphasize in their poems the salience of Ada's acknowledgement that she "needed to get something out of [her mathematics courses]"–something more than a good grade–to feel successful. We are left pondering these differences in what they tell us about students' own definitions of student success, and welcome this as a discussion point during the presentation.

Discussion

The comparison of two researchers' results when engaging with poetic transcription methodology enables multiple perspectives on how these students defined their mathematical success. A multitude of poetic interpretations of the data provides opportunity for nuance that could not have been captured by a singular poem. Further, it makes space for discussion around the ways in which participant data can be varyingly interpreted, necessarily incorporating discussion of the researchers' perspectives on data presentation. This is especially evident with the poetic interpretations present in Ada's poetry, in which each poem emphasizes differing components of her narrative of mathematical success. Glesne (1997) describes the creation of a *third voice* when engaging in poetic transcription—one which is constructed from both the participant's and researcher's experiences. Future work would more closely and critically interrogate our own *third voices*—how our experiences and identities as researchers, and what of the participant's own experience we drew from, enabled these two irreplicable poetic interpretations.

We also acknowledge the tensions brought forth in our experiences as researcher-poets; while poetic transcription offers an opportunity to work toward "healing wounds of scientific categorization and technological dehumanization" (Glesne, 1997, p. 214), it also maintains the hegemonic power of the researcher, and thus the potential to privilege perspectives that are already dominant in research spaces. In addition to explicit acknowledgement of this tension, other poet-researchers have illustrated ways in which to centralize participants' poetic voices in the process, either through researcher-participant co-construction of poetry (Levinson, 2020), tandem research poetry (Burdick, 2011), or with the researchers' own transcripts as data from which to engage in poetic transcription (Mercer-Mapstone et al., 2019). While not a possibility in the context of this study, future work with poetic transcription could entail comparison of multiple researchers' poetic interpretations with poetic interpretations written by the participants themselves to further highlight the subjective nature of research and deepen understanding of the research topic at hand.

We also find it important to emphasize the joy that engaging in this particular methodology brought to us as researchers; seldom is the researcher's experience of researching brought into academic discussion, but we find it particularly relevant here due to the inherent emotional nature of poetic transcription, and its evidenced potential to grow our own research practices (Romero, 2020). We find ourselves excited to discuss implications of the comparison of poetry and appreciative of the opportunity to deeply engage with our participants' perspectives in a poetic, artful way.

In addition, future work with this data will employ computational analysis to rigorously compare the poems, attending to differences in style, affect, and the use of concrete imagery (Kao & Jurafsky, 2012). This analysis will add to our understanding of the application of poetic analysis as a qualitative methodology for research in undergraduate math education, as well as the potential benefit of creating multiple poems by multiple authors to gain a richer analysis. In engaging in this comparison, we build upon prior scholarly work which challenges the notion of objectivity in mathematics education research and invite others to join the discussion of both the value of subjectivity and arts-based research to our field.

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Connections Pre-Service Teachers Make Between Social Justice and Mathematics

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This study explored pre-service teachers' (PSTs') initial experience connecting mathematics and social justice. This work is part of an on-going action research project on improving mathematics PSTs' experiences in their methods courses, specifically focusing on culturally responsive mathematics teaching. Research stresses the need to prepare teachers who can teach a diverse student body, are mindful of differences in student backgrounds and, are cognizant of their own biases (Mark & Id-Deen, 2020; Brown et al., 2019; Gallard, Mensah, & Pitts, 2014; Johnson & Atwater, 2014; Lemons-Smith, 2013; Leonard, 2008; Lewis, Pitt, & Collins, 2002). PSTs developed lesson plans balancing both mathematics and social justice goals. Lesson plans were analyzed to find initial connections PSTs made between mathematics and social justice.

Keywords: Pre-service teachers, mathematics for social justice, mathematics teacher education.

A focus on equity and social justice is needed in mathematics education (NCSM, 2019; AMTE, 2020; Kokka, 2020). Research stresses the need to prepare teachers who can teach a diverse student body, are mindful of differences in student backgrounds and are cognizant of their own biases (Mark & Id-Deen, 2020; Brown et al., 2019; Gallard, Mensah, & Pitts, 2014; Johnson & Atwater, 2014; Lemons-Smith, 2013; Leonard, 2008; Lewis, Pitt, & Collins, 2002). Teachers must be able to look beyond their classrooms and see their role as a teacher in a wider context. They must recognize their responsibility to prepare responsible citizens who can use mathematics to change the world (Aguirre & Zavala, 2013).

Existing frameworks can help mathematics teachers develop practices in line with this call. Examples of these frameworks include standards-based mathematics instruction (National Council of Teachers of Mathematics, 2000), complex instruction (Cohen, 1994; Cohen & Lotan, 1995; Lotan, 2006), culturally relevant pedagogy (Ladson-Billings, 1994, 1995a, 1995b), and teaching mathematics for social justice (TMSJ; Gutstein, 2003, 2006). All these frameworks build on each other and emphasize teaching for understanding (Rubel, 2017). However, even with the existence of these frameworks it is challenging to teach novice teachers about pedagogy that connects social justice, culture, and mathematics (Gutstein, 2018). Teachers need to be proficient in making connections between mathematics and community, as well as developing their students' awareness of the world (Gutstein, 2018; Freire, 1970). To prepare such teachers it is necessary to determine what skills they would need, and develop ways to cultivate these skills (Gutstein, 2018). To fill this need, the study focused on pre-service teachers' (PSTs') experience in a mathematics methods course. The guiding research question was: What initial connections do mathematics PSTs make between social justice and mathematics?

Theoretical Perspective

Teaching Mathematics for Social Justice (Bartell, 2013; Gutstein, 2006) or Critical Mathematics (Frankenstein, 1983; Gutierrez, 2002) is pedagogy that seeks to develop students' conscientização (Freire, 1970). This means helping students understand content and become academically proficient, develop an awareness of inequities around them and begin to take action (Ayers, 2009; Kokka, 2019). This kind of teaching requires teachers to set two kind of goals, mathematics pedagogical goals and social justice pedagogical goals (Gutstein, 2006).

Teachers need to be mindful of both these goals as they work together to guide their students' development of conscientização (Gutstein, 2006). Mathematics pedagogical goals include, reading the mathematical world, traditional mathematics success, and conceptual understanding of mathematics/learning how to use mathematics as a tool (Gutstein, 2006). To achieve the mathematics pedagogical goals, teachers can seek guidance from standards frameworks (e.g., national or state standards), textbooks or standardized assessments (Gutiérrez, 2002; Kokka, 2020). On the other hand, social justice pedagogical goals focus on, grasping sociopolitical situations, taking action towards change, and developing positive identities (Gutstein, 2006). These identities include both cultural and social identities and are of particular importance for students who are not part of the dominant culture (Gutstein, 2006; Kokka, 2020).

In addition to goals, teachers must develop and implement tasks that help students use mathematics to learn about their cultural context, such as, a task comparing students' own neighborhood to those with different socio-economic status (Kokka, 2019). When implemented well, such tasks allow students to see the power of mathematics to learn about local social issues (Yang, 2009). While powerful, these tasks are employed in tandem with a district/textbook curriculum, and are used only 15% to 20% of the time (Gutstein, 2006). It is challenging for teachers to develop and implement mathematics tasks that develop students' conscientização (Bartell, 2013; Gregson, 2013; Brantlinger, 2013; Rubel et al., 2016; Kokka, 2020), and as mentioned earlier, it is crucial to develop supports for them (Gutstein, 2018).

The goal of this study was to learn from mathematics PSTs' experience developing lesson plans that balanced both mathematics and social justice goals. PSTs' developed lesson plans and made videos implementing them. Their lesson plans were analyzed to find the initial connections they made between mathematics and social justice.

Research Methodology

This study employs action research (Sagor, 2000) methodology to develop supports for PSTs inside mathematics methods courses. In particular, the goal of this project is to develop PSTs' teaching philosophy and teaching practice connected to culturally responsive mathematics teaching (CRMT) and TMSJ. The study took place at a mid-sized university in the Mid-Atlantic region of the United States. Participants in the study were twenty-three PSTs enrolled in a middle school mathematics methods course during Fall 2021 and Spring 2022 semesters.

PSTs were asked to develop lesson plans guided by the CRMT lesson analysis tool (Aguirre & Zavala, 2013). The CRMT tool was developed to help PSTs contextualize the mathematics they teach and to help them see their role as teachers who prepare their students to engage in the society (Aguirre & Zavala, 2013). It includes 8 dimensions: (1) intellectual support; (2) depth of student knowledge and understanding; (3) mathematical analysis; (4) mathematical discourse; (5) communication and student engagement; (6a) academic language support for ELL; (6b) use of ESL scaffolding strategies; (7) funds of knowledge/culture/community; and (8) social justice. The PSTs were required to include both mathematics and social justice standards (Learning for Justice, 2016). Open coding (Strauss & Corbin, 1998) was used to analyze the lesson plans and learn about the connections PSTs made between mathematics and social justice.

Results

PSTs favored some social justice standards in their lesson plans over others (please see table 1). For example, standard 12 was used most frequently, *students will recognize unfairness on the individual level (e.g., biased speech) and injustice at the institutional or systemic level (e.g., discrimination)* (Learning for Justice, 2016). This standard was connected to the following mathematics topics, evaluating and simplifying algebraic expressions, equivalent fractions, graphing linear functions and identifying the slope and y-intercept of a linear function, identifying the components of the coordinate plane, histograms, rational numbers, and solving linear equations. In general, students favored Probability and Statistics over other content strands. This was followed by Patterns Functions and Algebra, Numbers and Number Sense, Measurement and Geometry, Computation and Estimation, and Expressions and Operations.

Table 1. The most commonly used social justice standards

Standard	count	SJ Standard
12	7	Students will recognize unfairness on the individual level and injustice at the institutional or systemic level.
6	6	Students will express comfort with people who are both similar to and different from them and engage respectfully with all people.
9	6	Students will respond to diversity by building empathy, respect, understanding and connection.
1	2	Students will develop positive social identities based on their membership in multiple groups in society.
8	2	Students will respectfully express curiosity about the history and lived experiences of others and will exchange ideas and beliefs in an open-minded way.
10	2	Students will examine diversity in social, cultural, political, and historical contexts rather than in ways that are superficial or oversimplified.

Here, I share examples to show connections that PSTs made between social justice standards and mathematics. In a 6th grade lesson, one PST connected, Measurement and Geometry to social justice standard 12. The lesson focused on identifying the coordinates of a point, and graphing ordered pairs in a coordinate plane. The PST planned to have students graph sets of ordered pairs. Each ordered pair would represent a residential unit, with the origin of the coordinate plane representing a hospital. Students would analyze the location of the housing units and their proximity to the hospital. They had prepared to ask questions such as: *Compare and contrast the consequences of living closest to the hospital and furthest from the hospital?* and *If you could move the hospital anywhere on the coordinate plane, where would you move it and why?*

The PST stated that they wanted their students to discuss, the various advantages and disadvantages of living in a specific location. Standard 12 focuses on recognizing unfairness either on a personal or a systemic level and the PST had planned for their students to investigate the map of their local area to find essential resources (i.e., hospital, police station, school) with the goal to discuss systemic injustice and raise students' awareness.

Another PST developed a 7th grade lesson plan, using social justice standard 6, *students will express comfort with people who are both similar to and different from them and engage respectfully with all people* (Learning for Justice, 2016). The PST connected this standard to solving two-step linear equations. They started the lesson with a problem frequently found in standard mathematics textbooks (please see figure 1). Following this problem, they planned to ask questions to foster discussion focusing on both mathematics and accessibility. Their questions for the mathematics discussion were, *What is the cost dependent on? How is this different for members vs. non-members? How does the membership fee affect the total cost? How* *can we represent this problem using algebraic terms? What does each part of the equation tell us? How will we know when having a membership is a better deal?* These questions would foster

The local amusement park sells summer memberships for $50 each. Normal admission to the park costs $25; admission for members costs $15.

a) If Darren wants to spend no more than $100 on trips to the amusement park this summer, how many visits can he make if he buys a membership with part of that money?
b) How many visits can he make if he does not?
c) If he increases his budget to $160, how many visits can he make as a member?
d) And how many as a non-member?
e) Based on what you found above, after how many trips would having a membership be a better deal? Why?

Figure 1. A Problem about student recreational activities.

discussion about what the algebraic equations represent and the meaning of each part of the equation. In addition to the mathematics discussion question, they had prepared questions for what they called an "accessibility and implications discussion." These questions were, *Are amusement parks widely accessible? What determines who is able to go to amusement parks and who isn't? What are other forms of entertainment/recreation that are not widely accessible? Why is this? What forms of entertainment/recreation are widely accessible? How do these differ from those that are not widely accessible?* The PST explained that their goal was for their students to study and explore the accessibility of different forms of recreation. They shared that students should become aware that some students might be able to spend more time and money on recreational activities than others. Their lesson focused on awareness of similarities and differences between peers.

In the last example, a PST developed a 7th grade lesson plan using social justice standard 9, *students will respond to diversity by building empathy, respect, understanding and connection* (Learning for Justice, 2016). They connected standard 9 to histograms, stem-and-leaf plots, line plots, and circle graphs. The goal of their lesson was for students to learn, *How do histograms allow us to interpret data? How can you use intervals, tables, and graphs to organize data? Why do refugees flee their own country? What opportunities do refugees receive when they come to the United States? What age are the majority of refugees?* At the beginning of the lesson, the

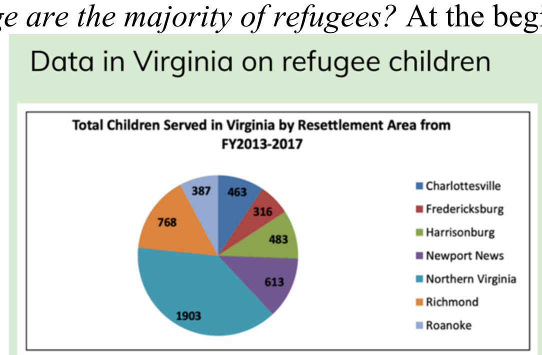

Figure 2. A graph depicting refugee data.

PST had planned to ask the students if they had any pre-existing knowledge on refugees and why refugees flee to other countries, and to have a class discussion. After the initial discussion, the PST had planned to show a video to the class on, *What does it mean to be a refugee?* (TED-Ed, 2016) followed by asking the following questions: *Why do refugees leave their home/country?*

What is the difference between migrants and refugees? What is a refugee camp? What data could be used on refugees to make a histogram? If I wanted to create a histogram on the age ranges of refugees, what would I label my x and y axis?

Following a discussion, the PST had planned to share graphs depicting state data on refugee children (please see figure 2 for one such graph). The PST had prepared the following questions for their students to answer, using data from the graphs: *Based on the histogram data what is the highest age range of refugees and what is the lowest? Why do you think so many refugees are young? How do histograms show the differences in distributions of data? List a couple of challenges that refugees could encounter at school? What are some actionable steps that you could take to help refugees in your local community? What is something new you learned today about refugees?* This lesson plan had the potential to have the students learn about mathematics as well as develop awareness. It is noteworthy that lessons such as these that focus on statistics lend themselves well to connections between mathematics and social justice.

Implications for teaching practice and future research

The process of developing lesson plans allowed the PSTs to research social justice topics that would connect to a middle school population, and they began to make connections between specific contexts and mathematics. PSTs also challenged themselves to develop questions that would lead to discussions in their classrooms. Such discussions would focus on unpacking the mathematics content and helping students become aware of the world around them. This exercise had the potential to help PSTs develop their own awareness of the world, potentially leading to their development of conscientização (Freire, 1970).

It was a challenging task for the PSTs to balance mathematics and social justice goals in a lesson. This finding is in line with existing research, as learning to balance mathematical and social justice goals is difficult and a long term pursuit (Harper, 2019; Bartell, 2013). Successfully balancing the two goals means being mindful during planning so that a focus on social justice does not take away from students' development of mathematical proficiency (Moses & Cobb, 2001; Harper, 2019). The PSTs began to experience this tension and would need further support to develop their own understanding of social justice in a mathematics classroom.

Teaching for social justice is a long journey that requires teachers to connect their teaching to contexts that are relevant to them and their students (Cochran-Smith, 1999). This process takes time, perseverance, and reflection (Darling-Hammond, 2002; Bartell, 2013). Teachers who begin this journey must understand that it is not a short-term goal (Gutiérrez, 2009), as their thinking and practice will develop and grow with practice and reflection (Bartell, 2013). As teachers develop an understanding of teaching mathematics for social justice, the changes are reflected in their teaching goals and teaching practice (Bartell, 2013). For PSTs it is particularly important to reflect on their teaching experience to develop a sense of who they are as teachers, this identity is necessary for them to employ pedagogies that connect culture, social justice, and mathematics (de Freitas, 2008, Leonard et al., 2010). Experiencing the challenge of connecting mathematics and social justice goals, allowed the PSTs to start thinking about teaching for social justice. At this early stage, most of their lessons aimed to have their students become aware of their selected context. The PSTs did not plan to have deep conversations about why the issues existed or to discuss actionable steps that can be taken to address the issues. With continued support PSTs can not only focus on awareness but also learn to discuss possible solutions so their students can take action and feel empowered. Such practice is necessary for PSTs to engage in repeatedly, for them to become experts (Nasir et al., 2008). Future research will focus on guiding PSTs to develop their identity as teachers who connect culture, mathematics, and social justice.

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Physics Students' Interpretation of Matrix Multiplication

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Most upper-division physics courses use mathematics beyond that encountered in the calculus sequence. Matrix multiplication and eigentheory are used by physics students in courses including classical mechanics, optics, and quantum mechanics. While there have been recent investigations of student use of linear algebra in quantum mechanics, no prior work has examined how physics students approach this mathematics in other contexts. In this paper, we investigated the extent to which student interpretations of matrix multiplication could be characterized by those from Larson and Zandieh (2013): linear combination, system of equations, and transformations. We have examined student responses to written questions and used the results to develop an interview protocol to examine how physics students interpret matrix equations using the three interpretations to classify their responses. Results are shown from an interview sample of six undergraduate physics majors.

Keywords: Linear Algebra, Mathematical Reasoning, Physics

Introduction

It is well established that upper-division physics courses have increased mathematical difficulty and that students in these courses encounter a variety of mathematical topics beyond calculus, including linear algebra and ordinary and partial differential equations. From linear algebra, matrix multiplication and eigentheory are used most in physics. The most well-known application is in quantum mechanics, but matrices are also used in other courses including classical mechanics and optics. Prior work on student knowledge of linear algebra in physics contexts has primarily focused on quantum mechanics (e.g, Wawro et al., 2020), which brings into play a set of conceptual and representational issues that may or may not reflect student understanding of the underlying mathematics. For this project, we have chosen to look at student interpretation of matrix multiplication in a broader array of topics, including some acontextual mathematical tasks as well as contexts drawn from classical mechanics and optics.

Theoretical Framework

Several models have been proposed to describe student use of mathematics in physics. In each, successfully executing the mathematical procedure in question is only one element of success. Redish and Kuo (2015) proposed a framework for student usage of mathematics in science courses describing stages of modeling, processing, interpreting, and evaluating. For the specific case of upper-division physics courses, Wilcox et al. (2013) proposed the ACER framework 'to guide and structure investigations of students' difficulties with the sophisticated mathematical tools used in their physics classes.' In this framework, students activate the appropriate mathematical tool in addition to constructing a model, executing the math, and then reflecting on results. For earlier work, we were guided by the physical-mathematical model developed by Uhden et al. (2012). This model describes three skills used at the boundary of math and physics: mathematization (converting a model into mathematical expressions), technical operation (executing relevant mathematical procedures), and interpretation (translating a mathematical result into physical interpretations and implications). In a previous study, we characterized student written responses to a number of tasks, and coded responses for each. Technical operation was seldom the most challenging for students; rather, more than half of the students had difficulty in converting a physical model into a matrix equation (mathematization) and relating eigenvalues to system behavior (interpretation) (Her & Loverude 2020).

Given that interpretation of matrix multiplication is both a critical skill and a source of student difficulty, we focused the next portion of the study on how students interpret matrix multiplication. Larson and Zandieh (2013) described three mathematical interpretations of the matrix equation A**x**=**b** among students in linear algebra courses: Linear Combination (LC), System of Equations (SOE) and Transformations (T). In previous work, Larson (2010) described two mathematical procedures to matrix multiplication, 'Matrix Acting on a Vector' (MAOV) and 'Vector Acting on a Matrix' (VAOM) which can be connected to (T) and (LC), respectively. The three interpretations have distinct conceptual and procedural interpretations [see Table 1].

*Table 1. A brief description of each interpretation from Larson & Zandieh (2013) for the matrix equation, A**x** = **b**.*

Linear Combination (LC)	**System of Equations (SOE)**	**Transformations (T)**
• Vector **x** components act as weights that modify the column vectors of matrix A • Associated with 'Vector Acting on a Matrix' (VAOM)	• **x** is a solution that coordinates the values for **b** with matrix A as coefficient entries	• Matrix A operates on input vector **x** where vector **b** is the modified output vector • Associated with 'Matrix Acting on a Vector' (MAOV)

Their work in the context of mathematics courses led us to consider the usage of these interpretations among physics students applying matrix multiplication in upper-division physics. In informal conversations with physics instructors, most expressed a strong preference for a transformation perspective, perhaps due to the use of matrix multiplication in quantum mechanics. This guided the formation of the following research questions, the first of which is addressed in this preliminary report:

To what extent do physics students use each of the three interpretations from Larson & Zandieh, and do they express preference for any of the three?

What relationship, if any, does the preference for interpretations have to solutions to physics tasks using matrix multiplication and eigentheory?

Methodology and Data Analysis

This project involves qualitative research on student learning. Data collection involved individual interviews with undergraduate physics students ($N = 6$) at a public university serving a diverse student population. All students were physics majors; one was a double major (computer engineering) and one had switched from engineering to physics. Their math and physics backgrounds were similar with some variations; in particular, three of the six students had taken a linear algebra course. All of the students had previously taken a course titled 'mathematical methods in physics,' intended to provide support in more advanced mathematics for later upper-level physics core courses. The mathematical methods course included a brief (2-3 week) introduction to matrix multiplication and eigentheory. Four of six student had subsequently completed a quantum mechanics course in which matrix multiplication and eigentheory were used frequently. The interview sample included three men and three women; three were white, two Latinx, one Asian.

Due to COVID-19, our standard routine of conducting interviews in-person had to be modified for an online setting. The first three interviews were recorded and conducted through Zoom, and the remaining three were in person. The interview protocol was designed to collect information on how physics students think about matrix multiplication and analyze the bridge between their mathematical interpretations and their physical interpretations. The interview consisted of nine questions that were separated into three sets. The first set were acontextual tasks intended to explore how students interpreted matrix multiplication without any physical context [see Figure 1] and were influenced by prior RUME studies (Henderson et al., 2010). The second set focused on reflection and rotation matrices along with the eigenequation. The last set of questions included three different physics concepts that used matrix multiplication and the eigenequation (optics, coupled blocks on springs, and a quantum spin operator), though some students did not complete all of these tasks. The first set of tasks was used to probe which of the interpretations, if any, were articulated by students Questions 1 and 2 provided examples of matrix multiplication and a matrix equation and asked students for their interpretation; question 3 provided three expressions associated with (LC), (T), and (SOE), respectively, and asked students to describe how, if at all, each expression represented the matrix equation.

Portions of the interview transcripts were coded iteratively by two members of the research team. Interview coding was in part emergent and in part driven by prior research. The primary focus of our analysis procedure was to investigate the students' interpretation

Question 1	**Question 2**	**Question 3**
Consider the expression $\begin{pmatrix} 2 & 1 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. How do you think about this expression?	Consider the matrix equation $\begin{pmatrix} 3 & 5 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$. First, how do you interpret this?	$\begin{pmatrix} 1 & 2 \\ 8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ For each of the following expressions, explain how if at all, it represents the matrix equation. (i) $x\begin{pmatrix} 1 \\ 8 \end{pmatrix} + y\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ (ii) $\begin{pmatrix} 1 \cdot x + 2 \cdot y \\ 8 \cdot x + 4 \cdot y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ (iii) $x + 2y = a$ $8x + 4y = b$

Figure 1. The first set of interview tasks.

rather than assessing 'correctness'. The initial analysis was based on the interpretations described by Larson and Zandieh (2013); student utterances were coded as (LC), (T), or (SOE). The grammar of student utterances was part of the analysis; e.g., transformation interpretations were often characterized by transitive verbs like 'manipulated,' 'changed,' or 'rotated.' MAOV and VAOM were coded and associated with (T) and (LC), respectively. The transcripts were also examined for evidence of interpretations that were not accounted for by any of these codes.

Initial analysis suggested that additional refinements would be required. Some student responses seemed to shift depending on whether they were primarily procedural or conceptual. This led to creating a parallel set of codes in which responses were identified as being procedural (relating to the method used to evaluate the mathematical expression, e.g., 'the first row with the column of the second matrix'), conceptual (how they make sense or meaning of the expression or outcome, e.g., 'change a vector from one initial state to a final state'), graphical, or none of these. An utterance might be coded as, say, both procedural and LC. These distinctions mirrored analysis in Larson and Zandieh (2013) as well as other recent work (Serbin et al., 2020). The researchers worked to resolve differences in coding and adjusted the codes appropriately; one example of this process is described in the Results section. The analysis process consisted of grouping terms and sections of the interview transcripts and examining the resulting data set for trends and patterns. Responses were further examined for either, on one hand, consistency of interpretation or, on the other, fluid, changing interpretations across different tasks.

Results

Table 2 presents a summary of our analysis. Student descriptions of matrix multiplication were generally consistent with the three descriptions (T), (LC) and (SOE). Nearly all student statements were readily aligned to one of the three.

There was some degree of preference for the Transformation interpretation (T), with some variation. All six students incorporated the (T) interpretation at some point. Three of the six students consistently interpreted matrix multiplication, in a variety of tasks, in terms of the transformation interpretation and the associated MAOV description, with few or no codes for (LC) or (SOE). This was a clear signal throughout these three interviews, and was often accompanied by explicit rejection of, or confusion with, equations in Question 3 associated with other interpretations. Three of the six students showed signs of more fluid interpretations, with all articulating (LC) and/or (SOE) interpretations, though again all shifted to (T) at some point during the interview. Most of the students offered both procedural and conceptual support for their preferred explanations, but one of the six remained focused on procedural knowedge and offered little in the way of conceptual interpretation.

Below we present extended examples of responses given by students in different categories: student A, a consistent user of T interpretations, and student C, whose responses were coded as supporting all three of the interpretations at various parts of the interview.

Table 2. A summary of student backgrounds and responses. LA/QM refers to whether students had completed linear algebra and quantum mechanics, respectively. Conc/Proc = conceptual and procedural.

Student	A	B	C	D	E	F
Major(s)	Physics	Physics	Physics (formerEng)	Physics	Physics +CompEng	Physics
LA / QM	Y/Y	Y/Y	Y/N	Y/Y	Y/Y	Y/N
Initial / Final Interpretation	T / T	T / T	LC / T	T / T	SOE / T	LC / T
Stable/Fluid	Stable	Stable	Fluid	Stable	Fluid	Fluid
Conc/Proc	Both	Proc	Both	Both	Both	Both

Student A

Student A was coded as consistently using a (T) interpretation throughout the interview. For the first two questions, Student A immediately described characteristics (T) interpretations, e.g., using terms 'manipulated' and 'transformation matrix': "So uh I see this as we got some of vector quantity here and it's being manipulated by some sort of transformation matrix."

Student A's responses further suggested a strong association of the (T) interpretation with the procedure of matrix multiplication. On question 3 he immediately chose the second equation and commented that choices (i) and (iii) were difficult to parse conceptually. He was unsure whether these were even mathematically equivalent, stating that (iii) was 'divorced from the whole process of matrix multiplication.' After some discussion, student A performed procedures to reproduce the expressions, and once he was satisfied that all three were representations of the matrix equation, he reversed course somewhat and stated that choice (i), the (LC) interpretation, was the best representation. However, his conceptual description in support of choice (i) was coded as a (T) interpretation as he referred to 'transforming' the elements of the vector and 'what the matrix is doing,' rather than the weighting of column vectors associated with (LC). Indeed, he interpreted all three expressions in terms of (T), though the intent was that each expression mapped to one of the three interpretations.

Student A: "…number (i) seems like it is more representative…it represents what the matrix is doing more than the other two. So the fact that this [points to first column of matrix A] is transforming that [points at x] and this [points to second column of matrix A] is transforming that [points at y] kind of represents the idea of the matrix multiplication a bit more than—a lot more than this [option (ii)] and more than this [option (iii)] I would say. Because we see a component of the matrix acting on sort of a scalar."

Student C

Coding student C's responses could be challenging, in part because she used unconventional terminology. For example, in her response to question 1, she explained that while the knew the accepted terminology, she preferred to refer a 2 by 2 matrix as a "2 by 4." She gave responses consistent with each of the three interpretations at different moments in the interview, offering both procedural and conceptual explanations. In her initial procedural response, she showed multiplying the rows of the matrix with the column vector, consistent with MAOV. However, her subsequent conceptual description was coded as being consistent with VAOM and thus (LC). In question 2, she described the matrix equation as: "$\begin{pmatrix} x \\ y \end{pmatrix}$ is modifying $\begin{pmatrix} 3 & 5 \\ 2 & 7 \end{pmatrix}$ where $\begin{pmatrix} x \\ y \end{pmatrix}$ would be some arbitrary modification being performed to get an arbitrary outcome $\begin{pmatrix} a \\ b \end{pmatrix}$." The vector 'modifying' the matrix suggests VAOM, but written work was consistent with MAOV.

Ultimately, Student C concluded that choices (i) and (iii) were possible representations of the matrix equation but was ambivalent because she could not fully track their procedures. She identified choice (ii) as 'no representation' and rejected its validity altogether.

Conclusion and Implications for Future Work

Analysis reveals some patterns that may have implications for instruction as well as future research efforts. Three of the six subjects showed a strong and consistent preference for the (T) interpretation. While each of the choices in task 3 was intended to support one interpretation, these students provided (T) interpretations for all three. The other students all used (T) at some point but offered more fluid responses, shifting among the interpretations. The preference for (T) interpretations may reflect a difference in disciplinary usage, as recent experiences with matrix multiplication came in physics courses that likely emphasized the (T) interpretation above others.

While the sample is small, two additional patterns are of note. First, the three students who had the most consistent (T) interpretations had all completed two semesters of quantum mechanics, and two of the three who shifted among interpretations had not yet completed the course. Quantum mechanics instructors and texts frequently use language like 'the operator acts on the vector', which seems to be consistent with the (T) interpretation. Second, the two students with engineering backgrounds were both among those who used other interpretations; this may suggest additional disciplinary influences.

Analysis is ongoing and later publications will include analysis of later parts of the interviews including physical contexts, as well as written problems developed based on these interview tasks. Informal discussions with instructors for relevant physics courses suggest a marked preference among at least these physics instructors for the (T) interpretation, possibly suggesting a further avenue of inquiry.

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"Hey, I am doing it!" Developing the Mathematics Identity of Latin* Students in a Specifications Grading Calculus Course

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Calculus courses should embrace innovative pedagogies that promote Latin students' mathematics identity development. One such pedagogy is specifications grading, which might foster students' growth-mindset and perseverance in mathematics. In this study, we examine the mathematics identity development of two Latin* students' in Calculus courses with specifications grading, and we compare this development to that of a student not enrolled in a specifications grading course. We compare the changes in the students' mathematical competence and the changes in their sense of recognition as a doer of mathematics. We discuss how aspects of the specifications grading pedagogy can contribute to students' mathematics identity development.*

Keywords: Alternative Grading, Specifications Grading, Mathematics Identity, Calculus

For many, mathematics is a formidable academic challenge, especially at the college level (Roska et al., 2009; Weisburst et al., 2017). Indeed, despite recent significant gains in Latin* students pursuing mathematically-oriented degrees (Crisp & Nora, 2012), still remaining is a significant underperformance of Latin* students in entry-level mathematics college courses (Otero et al., 2007; Solórzano et al., 2013; Sparks & Malkus, 2013). Even though Latin* students have the capability of succeeding in college-level mathematics (Cole & Espinoza, 2008; Crisp & Nora, 2010), the mathematical under-preparedness experienced by many Latin* college students has even led researchers to designate Calculus as a *gatekeeper course* (Norton et al., 2017; Pyzdrowski et al., 2013; Suresh, 2006), or a course that (un)intentionally impede many Latin* and other underserved students from pursuing mathematically-oriented degrees and careers (Convertino et al., 2022). Despite these hindrances, however, research has demonstrated that fostering mathematics educational spaces in which students develop a stronger sense of self, or *mathematics identity*, significantly impacts their mathematics perseverance and success (e.g., Authors, in press; Matthews, 2020; Miller-Cotto et al., 2020; Morales & DiNapoli, 2018).

To contribute to the ongoing efforts for more equitable educational experiences for undergraduate Latin* students, especially in gatekeeper mathematics courses, we must shy away from pedagogies that rely on archaic dispositions about how people learn. Instead, we must opt for pedagogies that foster stronger mathematical identities by adopting an asset-based orientation towards all learners, including Latin* students. For that reason, we report our pilot efforts in implementing *specifications grading* Calculus I courses in Spring 2022, a pedagogy based on the idea of viewing learning as a process achievable by everyone rather than a result obtained only by a few (Nilson, 2015). This study was guided as follows:

1. How does the mathematics identity development of Latin* students enrolled in a specifications grading Calculus I course compare to the mathematics identity development of Latin* students enrolled in a non-specifications grading course?
2. What aspects of specifications grading foster students' mathematics identity development?

Literature Review and Theoretical Background

Mathematics Identity and its Constructs

Adams (2018) gives a clear definition of *mathematics identity* as "the dispositions and deeply held beliefs that students develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics in powerful ways across the context of our lives" (p. 121). Furthermore, Cribbs et al. (2015) offer an explanatory model for mathematics identity by focusing on students' self-perceptions in three constructs, namely mathematics interest (i.e., how a student perceives their interest in mathematics), recognition (i.e., how a student perceives their external recognition, or lack thereof, as mathematics doers), and competence/performance (i.e., how students perceive their comprehension and performance of mathematics). As shared by Cribs and colleagues (2015), self-perceptions about these three constructs significantly informs students' mathematics identity, and this has the capacity to influence students' goals and participation in mathematics, including the degree of choosing, or not, their future careers in STEM. Therefore, it is crucial for educators to understand and empower students' mathematics identity, especially among marginalized student populations, and let that guide their classroom interactions and practices so that students' self-perceptions in relation to mathematics can be positively impacted (Boaler & Greeno, 2000; Cribbs, et al., 2015).

Specifications Grading

We argue that specifications grading as a pedagogical framework provides us with an educational space that can foster stronger mathematical identities. For instance, instead of awarding points for assessments, students are given specifications on assignments, or learning objectives, and these are assessed via a more asset-oriented process that measures progress rather than correctness. Students are also provided with ample opportunities to revise their work based on instructor feedback and resubmit it until they meet the learning objective at a satisfactory level. In other words, specifications grading adopts a growth-mindset orientation towards learners, believing that all learners have the capacity to overcome setbacks in their learning trajectories. As a result, specifications grading has been shown to provide students with a sense of agency as they obtain a greater control over their mathematical trajectories, not to mention with a more authentic and equitable reflection of their proficiency in their respective courses (Boesdorfer, et al., 2018; Katzman et al., 2021; Stange, 2016; Talbert, 2017; Tsoi et al., 2019). Empirical research in this regard, however, remains scarce (Beatty, 2013; Post, 2017; Toledo & Dubas, 2017;). The gap is even more notorious when we consider the positive impact that a specifications grading pedagogy can have on Latin* students' mathematical identity development. For that reason, we aimed to explore the effects that a specifications grading-Calculus I course could have on Latin* students' mathematical identities.

Methods

We collected data at a Hispanic-Serving Institution in the southwest. In Spring 2022, our institution had 14 sections of Calculus I with approximately 500 students. We implemented the specifications grading course design to 9 sections with approximately 300 students. We divided the content of the course into 29 learning objectives that were assessed throughout the semester. Our assessments consisted of group worksheets, exams, and online homework. The mastery of every learning objective was tested on both a worksheet and an exam. If a student failed to show mastery of a learning objective on a worksheet on the initial submission, we allowed the student to resubmit. If a student failed to show mastery of a learning objective on an exam, the student was provided with opportunities to retest. Retesting sessions were available weekly after the first exam until the end of the semester. In our traditionally-graded Calculus I sections, a student's grade was based on quizzes, group work, exams, and homework with no opportunities for re-assessments. All 14 sections were expected to teach the same content.

To compare the mathematics identities among our Latin* students, we adapted Cribbs et al.'s (2015) well-established, 12-item mathematics identity survey instrument. Following a 5-point Likert-scale, it not only quantifies the mathematics identity of an individual, but also the three subconstructs of mathematics identity: self-perceived interest, recognition, and competence/ performance in mathematics). Additionally, all individual scores were scaled between one (1) and (5) to facilitate its comparisons within and across students. This meant that each student who took the survey had four different scores representing each construct, all within a 1 – 5 scale. The surveys were given during the second week of classes and on the third-to-the-last week of classes, respectively. In all, the survey was taken by 218, mostly Latin*, students enrolled in Calculus I. Out of the total, 119 enrolled in one of the 9 courses that were implementing specifications grading. These students were mostly 1st generation college students pursuing an Engineering, Biology, or health professional-related major.

Eight survey participants volunteered to participate in an individual follow-up interview. We focus on three of these participants for this preliminary report. To exemplify the most diverse types of cases studies, a common methodology in case study research design (Seawright & Gerring, 2008), we selected three cases of Latin* students, Katie, Oscar, and Carlos (pseudonyms) who demonstrated low to mid-range mathematics identity scores in the pre-survey. Katie and Oscar were enrolled in a Calculus I course using specifications grading, while Carlos was enrolled in a Calculus I course without specifications grading. Individual semi-structured interviews (Bernard, 1988) were conducted virtually with each of the three participants. The interview protocol was designed according to the components of Cribbs et al.'s (2015) model for mathematics identity. Follow-up questions were asked as needed for clarification and elaboration. These interviews were recorded and transcribed.

We analyzed the interview transcripts by using inductive coding (Miles et al., 2013) to classify the themes present in the participants' interview responses. We performed axial coding to organize our codes according to the constructs of interest, competence, and recognition. We then considered the results of the overall survey responses to inform our decision regarding the focus of further qualitative analysis. In doing so, we found that students in classes with specifications grading had higher scores on competence/performance and recognition than those not enrolled in such courses. Therefore, we chose to focus our qualitative analysis on the three students' perceptions of their competence/ performance and recognition in doing mathematics.

Results

Spider-web charts of the participants' mathematics identity scores measured on the surveys are in Figure 1, with mathematics identity, competence/performance, interest, and recognition scores plotted on each of the four axes. The grey region on the charts represents the students' pre-survey scores and the teal region the post-survey scores. These charts illustrate the change in Katie's, Oscar's, and Carlos's scores on the pre- and post-surveys. Katie and Oscar, who participated in a specifications grading course, both demonstrated large increases of one to three points in every measured score of mathematics identity and its subcomponents. Carlos, who did not participate in specifications grading, had small increases in his mathematics identity, competence, and interest scores, but his recognition score did not change. In what follows, we discuss the results from the qualitative analysis of participant's interview responses, focusing on their perceptions of their competence and recognition in being a doer of mathematics.

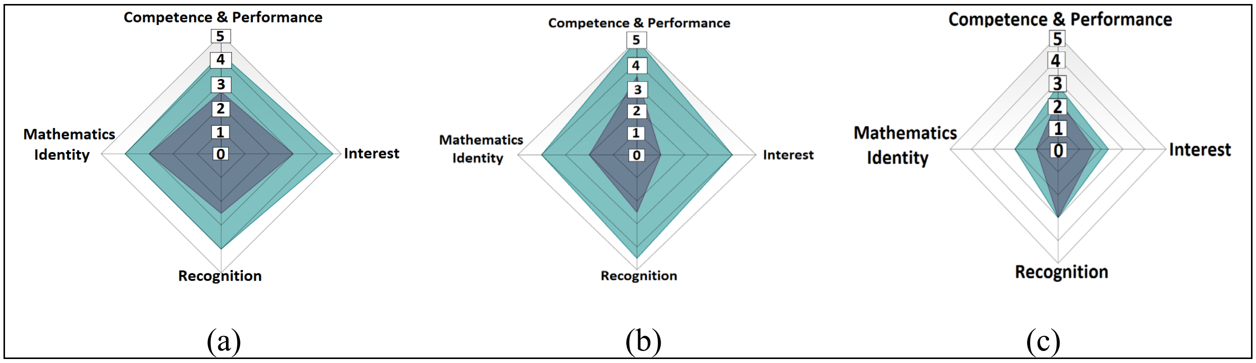

Figure 1. Spider charts demonstrating changes in Katie's (a), *Oscar's* (b), *and Carlos's* (c) *pre- and post-survey scores on measures of mathematics identity, competence, interest, and recognition.*

Students' Competence/Performance in Doing Mathematics

Katie and Oscar reported struggling in their prior mathematics courses: "I [Katie] had an overwhelming plate on my workload, and so I would either fall behind or just keep struggling with that part of the work. I kinda felt alone, I guess, back then… I had to drop the class." Similarly, Oscar referenced his prior Calculus course experience as "a dooming failure," sharing the negative feelings that failing had on him,

> "You're just like, 'Well I'm never going to finish. This is not working,' and you know, all those red X's and stuff like that, it's just puts you down man. It puts you down. You're like 'No I don't want to do this anymore. It's not worth it.'"

In all, they reported feeling overwhelmed and unmotivated to "deal with" their failures. Furthermore, Oscar referred to traditional grading as "either you pass or you don't pass," highlighting how traditional grading acts as gatekeepers for many.

Once in the specifications grading Calculus I course, however, Katie and Oscar demonstrated higher confidence in their competence in doing mathematics in their post-course-survey scores and during the interviews. As shared by Katie, "I am confident with myself…I know what not to do and what I'm capable of doing on my own." Oscar also described his change in confidence as, "You know I thought…I'm never going to pass Calculus. Yes, you can. You can pass it if we put hard work and dedication to it. It is reachable." To this, Oscar attributed his change in confidence to seeing his progression of passed learning objectives: "You start seeing that you're making progress. That motivates you, and that amplifies your performance and gives you more confidence…it gives me confidence like 'Hey, I'm doing it!'…It's a boost to your confidence in your performance to do math." Carlos, on the other hand, self-reported a decrease in his confidence from high school to his traditionally-graded Calculus I: "I'm doing these college courses, and they're more difficult, and there's not that much time… I'm struggling more to really grasp at it… specifically in math where I've been struggling the most."

Overall, Katie and Oscar, who were in a specifications grading course, demonstrated increased confidence in their competence in doing mathematics. Oscar, for example, attributed this change to aspects of specifications grading. On the other hand, Carlos, who was not in a specifications grading course, self-reported a decrease in confidence, which he attributed to his struggle with the fast-paced Calculus course that moved on to the next topic before he grasped the prior one.

Students' Recognition as a Math Person

Katie and Oscar both increased their measured recognition scores between the pre- and the post-survey, while Carlos showed no change in his recognition score. In the interviews, we further investigated their perceptions of whether their family, friends, and mathematics instructors recognized them as a math person. All three participants claimed that their friends and family recognized them as a math person, and they gave anecdotal evidence about helping their friends with doing mathematics problems. Carlos expressed frustration that his family and friends "definitely" perceived him as a math person: "It's frustrating because… I know that I'm *not* that much. But unfortunately, they keep this illusion, and I'm forced to be pushed down this path that they think I am on, but I'm not." Carlos did not perceive himself as a math person and was frustrated that others did because he thought he could not meet their expectations.

After taking the course with specifications grading, Katie and Oscar reported that their instructors recognized them as a math person. Katie explained, "I think [my instructor] gives me praise for things I do correct and asks me if I'm getting it correct, so I think she does." She referenced the feedback she obtained from the instructor, which is part of the specifications grading process that informs students' revisions. The instructor's positive feedback seemed to give Katie a sense of the instructor's recognition of her as a math person. Carlos, who was not enrolled in a specifications grading course, reported that his instructor saw him as a math person "no more than anyone else." This interview response aligns with Carlos's low score for recognition from the survey. Overall, Katie and Oscar reported that their instructors recognized them as a math person, but Carlos did not. The use of instructor feedback in specifications grading may allow students to perceive their instructor's recognition of them as a competent doer of mathematics.

Discussion

Implementing innovative pedagogies, such as specifications grading, has the potential to foster students' mathematics identity development. Specifications grading realizes this by adopting a growth-mindset that provides students with multiple opportunities to meet their mathematics learning objectives. However, preliminary evidence suggests that specifications grading may also provide Latin* students with educational spaces that may take advantage of Latin* students' *cultural wealth* (Yosso, 2005), or the idea that Students of Color possess a plethora of capital that, when channeled into their education, can serve as critical factors that contribute to students' success in STEM (Samuelson & Litzler, 2016). Because of this, future studies should also explore to what extend do Latin* students' *aspirational capital* (Yosso, 2005), or their ability to maintain hopes and dreams for the future despite the challenges faced as marginalized populations, may be catapulted when they are provided with low-stakes educational spaces in which they feel more in control of their academic trajectories. Furthermore, their *navigational capital,* or their abilities to traverse White-dominated social constructs, like traditional Calculus curricula, could allow them to traverse their specifications grading Calculus I course with more ease given that the learning space values *persistence*, yet another critical characteristic of many Students of Color. Combined, these two forms of capital could also have the capacity to bolster Latin* students' success in Calculus and STEM in general; however, further studies are necessary. We propose the following discussion question for the audience: What further analyses can help us explore the connection between specifications grading and Latin* students' mathematics identity development? Also, it seems that specifications grading may also take advantage of Latin* students' cultural wealth. If so, what specific characteristics of specifications grading accomplishes this?

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Implementing and Scaling-Up of Research-Based Precalculus Curriculum

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Small-scale teaching experiments focused on how students reason about exponential functions provide an incredibly useful foundation when thinking about the design of classroom tasks. However, scaling-up certain critical features of these experiments such as focused and prolonged instruction on a single topic is unrealistic for those teaching at the university level. Drawing on research on covariational reasoning, exponential functions, and design-based research methodology, we developed and implemented two interactive lessons with the goal of enhancing students' understanding of exponentiation and covariational reasoning. In particular, we sought to identify the quantities and relationships that students focused on during classroom tasks and the ways in which the tasks could be refined to support students' understanding of exponential covariation. From this analysis, revisions made to the curricular materials and other implications for research and teaching are discussed.

Keywords: Precalculus, Covariational Reasoning, Exponential Functions

Researchers in undergraduate mathematics have paved the way for the rise in conceptually oriented undergraduate mathematics courses throughout universities in the United States (e.g., Thompson & Carlson, 2017). For many universities, however, implementation of these courses is constrained by several factors such as large class sizes, lack of research-based curriculum materials, and lack of professional development and support for instructors (e.g., Deshler et al., 2015). While many theories exist about instructional means of supporting conceptually oriented undergraduate mathematics learning, there is still a gap in the knowledge of how to scale-up the research findings from these small-scale teaching experiments to instruction and curriculum design in large university courses. This study seeks to fill this gap in the literature by developing and modifying classroom tasks from small-scale teaching experiments to support students' reasoning about exponential covariation in the scaled-up setting of a large undergraduate precalculus course. In particular, we sought to identify the quantities and relationships that students focused on during the classroom tasks and the ways in which the tasks could be refined to support students' understanding of exponential covariation in this scaled-up context.

Literature Review

Quantitative and Covariational Reasoning

Quantitative and covariational reasoning have been shown to be foundational reasoning abilities important for learning about functions and other related mathematical concepts (e.g., Carlson et al., 2002; Oehrtman et al., 2008; Thompson & Carlson, 2017). When students reason quantitatively, they are able to "conceptualize measurable attributes of some object or situation" (Madison et al., 2015, p. 56) and the relationships between these attributes. This reasoning serves as a basis for covariation which entails coordinating how two quantities change in relationship with each other (Thompson & Carlson, 2017). Instruction aimed at the development of quantitative and covariational reasoning has been shown to improve student learning outcomes in undergraduate precalculus and prepare students for success in future mathematics courses such as calculus I (Carlson et al., 2010; Carlson et al., 2015).

Exponential Functions

Historically, instruction on exponential and logarithmic functions has been based on the ideas of exponentiation as repeated multiplication and as logarithms as either repeated division or simply as the inverse to exponentiation (Ellis et al., 2012; 2015; Kuper & Carlson, 2020). Recent studies (e.g., Kuper & Carlson, 2020) have found that students' understanding of exponential and logarithmic functions is greatly improved when they are given opportunities to reason quantitatively and covariationally.

Ellis et al. (2012; 2015) analyzed three 8th grade students' reasoning while working in the context of a magical growing cactus named the Jactus to explore the conceptual shifts that students made to reason productively about exponential growth. These shifts included moving from a repeated multiplicative view of exponentiation to the coordination of constant multiplicative ratios between consecutive output values with constant additive changes in consecutive input values. An example of this is students' coordination of the ratio between the Jactus' heights at different times and the elapsed growth time. In other words, when given the Jactus' height at weeks 1, 2, and 3, students were able to divide the successive heights and coordinate this multiplicative ratio with the elapsed growth time of 1 week to come to the conclusion that the Jactus' height doubles every week. Ellis and collaborators observed students' levels of correspondence and covariational reasoning for repeated consecutive input values greater than a one-unit change preceded their productive reasoning with non-natural exponents. For example, students had to understand how to reason about the ratio between the Jactus' heights at 2 or 3 week intervals (i.e., the Jactus triples every 2 weeks as opposed to every week) before they were able to productively reason about the ratio of heights at 1 day intervals. For example, the students were told the Jactus doubles in height every week and were asked to find its height at 0.25 weeks. Productive covariational reasoning about these non-natural exponents occurred after students were familiar with natural numbers larger than 1 as the exponent.

Methodology

The analysis presented here is part of a broader study aimed at identifying and analyzing the key features of task design and classroom instruction that are beneficial for student learning when attempting to scale-up teaching experiments to whole classroom settings. We begin this work by developing and modifying classroom tasks used in prior teaching experiments (i.e., Ellis et al., 2012; 2015) to support students' reasoning about exponential and logarithmic covariation in the scaled-up setting of a large undergraduate precalculus course. We used design-based research methodology because its purpose is to simultaneously "develop a class of theories about both the process of learning and the means that are designed to support that learning" (Cobb et al., 2003, p. 10). The strength of this methodology for the purpose of scaling up curriculum and instruction is that we are able to utilize the empirical evidence from studies such as those described in the literature review to analyze and test which features are most crucial for students learning when in the new setting of a large undergraduate precalculus course.

Setting

This study took place at a large research university in the Southeast United States. The participants were 30 undergraduate precalculus students enrolled in a six-week course which included both lecture and lab or recitation sessions. Due to the accelerated nature of a six-week course, students met daily for lectures and twice weekly for lab sessions. During one lab meeting students took a weekly quiz in lieu of unit tests. In the other lab, students worked in small groups of 3–4 on assignments created by the research team which drew on literature such as that described above. The purpose of these labs was to support development of quantitative and covariational reasoning as well as deepen their understanding of key mathematical concepts. For this study, the constraints of this course structure were a crucial part of the design of lab tasks. For example, lab sessions were limited to 60 minutes and the only assistance that students received outside of their small peer group were from their instructor, a graduate teaching assistant, and the first author of this report. Unlike small-scale teaching experiments where one teacher works consistently with 2–3 students, our design needed to account for having two to three instructors working with 30 students during a single lab section.

Data Sources

Students' anonymized written work from the labs were used as the primary source of data analyzed for this study. In total there were six groups of 3–4 students who worked through 6 lab activities. Each lab consisted of a brief pre-lab assignment, a lab activity, and an exit ticket. Additional data that informed our lab revision process and analysis came from field notes taken by the first author during labs and from debrief meetings with the instructor for the course and the research team. Since the focus of this particular study was primarily on students' understanding of exponential functions with an additional emphasis on quantitative and covariational reasoning, we decided to limit our analysis to the lab which introduced these concepts.

The Mathematics. The problem context of the lab was Alice in Wonderland. In the pre-lab activity, students were expected to find the number of sips of potion needed for Alice to fit through the door when given information about her original height and how a potion changes her height as she takes sips. During the lab, students worked through tasks which asked them to calculate the first and second differences in her heights as she sips the potion and to plot the data on a graph. Following this, students had to consider how Alice's height changes when half sips are taken and estimated her height at half-sip values. The second portion of the lab focuses on Alice eating a cake which makes her grow. Students are given a partially-filled table with her height after a number of bites of cake and are asked to fill in the missing height data. Then they are asked to graph their data and to again consider how Alice's height changes when taking half-bites of cake. For this report, we drew on data from two questions (See Table 1) which we felt had the highest potential for supporting students' covariational reasoning.

Table 1. Alice in Wonderland Lab Questions

Problem 2e
Students have the following information: (1) Alice shrinks to ⅔ her current size each time she takes one sip, and one sip is 0.5 oz. (2) Alice was 51 inches tall before drinking any potion, and 34 inches after 1 sip (or ½ oz) of potion. **Students are asked this question:** We know that after her first sip of potion Alice's height decreased by 17 inches. If Alice had instead only taken half a sip of potion (0.25 oz) what is a reasonable expectation for the decrease in her height? Would it have been more than, less than, or equal to 8.5 inches? Explain your reasoning.

Problem 3d

Students have the following information: Alice has eaten 8 bites of cake and we know Alice's height after various bites of cake (e.g., 0 bites, 4.5 inches tall; 2 bites, 8.85 inches; 6 bites, 33.88 inches).
Students have to find the following information: Alice's height after 5 and 8 bites
Students are asked this question: Based on our table of data we know that after two bites of cake, Alice's height increased by 4.32 inches. If Alice had eaten each bite of cake separately would each growth spurt be a 2.16 inch increase in height? Why or why not?

Analysis

Tyburski et al. (2021) argue that covariational reasoning is clearly delineated into levels with associated mental actions involved, but in practice there is little guidance for deciding what moment-to-moment student words or actions are evidence of their level of covariational reasoning. They offer lessons from their work to researchers who want to analyze and make claims about students' covariational reasoning. In the spirit of their work, we began our analysis by drawing on students' written work to identify the quantities that students attended to and evidence of the ways in which they coordinated those quantities. We used this along with the field notes and observations made by the research team to group students' responses into categories we felt were indicative of the types of quantities and covariational reasoning observed.

Findings

The intention of the lab was to support students' reasoning and foundational conceptions about the exponential covariation of quantities. We wanted students to coordinate additive changes in bites/sips with multiplicative changes of Alice's height. Additionally, we wanted students to understand that Alice's height would decrease (or increase) by smaller and smaller amounts (or larger and larger) as she continued to sip potion (or eat cake). In order to get students to think about this, we asked them to consider whether or not Alice's difference in height would be halved when she took a half sip instead of a full sip (i.e., Problem 2e). We anticipated that students would reason about this by noticing that when you multiply by ⅔ (the shrinking rate), that the number shrinks more when you begin with a larger number. In other words, Alice shrinks by larger amounts when she is taller, but her changes in height decreases as she gets shorter. However, we found that the resources embedded in the design of the lab leading up to problem 2e did not provide students with adequate resources to appropriately coordinate Alice's heights and the successive changes in heights with each sip of potion. Indeed, five of the six groups responded incorrected to problem 2e by reasoning in ways such as "In order [for] a full ⅔ decrease, she would have to take a full 0.5 oz sip, but she only took half; therefore the ⅔ decrease would be ½ [as] effective, so she decrease[d] by ⅓, so her height decrease[d] would be 5.66 [inches]," or that "Alice's height [would] decrease by 8.5 inches because 17 divided by 2 equals 8.5." The only group who responded correctly used a previously learned formula (which was not our intention).

In examining student responses to question 3d, however, we found that students had far more ways to reasoning productively about the quantitative relationships involved. Additionally, every group correctly answered that Alice's growth would not be in two equal spurts of 2.16 inches. We grouped the ways of reasoning about this into three (often overlapping) categories as shown in Table 2.

Table 2. Examples of Student Work on Question 3d

Reasoning	Groups	Response
Change is **not** constant	2, 3, 6	"It would not be constant, so she would not grow 2.16 each time, instead the second time would be greater."
Change is multiplicative	2, 5	"No, because we're not adding, we are multiplying."
Change is exponential	1, 4, 6	"No, the rate of change is defined as exponential growth. This means each bite stimulated a greater change in height than the preceding bite."

Discussion

Our findings support those from Ellis et al. (2015) in documenting the conceptual difficulty of reasoning covariationally with exponential functions. We observed a pattern in students' reasoning (and our own when designing and planning the learning trajectory) where one wants to use linear relationships when partitioning both quantities rather than coordinating additive and multiplicative changes. We found that students frequently described changes in height in terms of linear relationships in question 2e (when they had to reason about a partition of a single pair of values). In contrast, students more frequently conceptualized the relationship in question 3d multiplicatively (when they had to reason about successive changes of sips and heights). While it was our intent that both questions supported the coordination of additive changes in one variable with multiplicative changes in the other, students did not reason in the way we intended until they were forced to reason about iterative or successive changes.

In examining other features of the task design, we also noted that the resources available to the students at the time of each problem were starkly different. Prior to question 2e, students were told that Alice's height was 51 inches and that she would shrink to ⅔ of her original height with each sip of potion. Then students were given a table of values filled in with her height at one sip intervals from 0 to 10. In contrast, prior to question 3d students were given Alice's initial height and a partially filled table of values (See Table 1). The students were tasked with figuring out Alice's height given only the information from this table before being asked question 3d. When students had to reason about the quantities and their relationships to fill in the table, it appeared to also support their ability to reason productively in question 3d (where every group answered correctly).

Based on our observations, we refined the content of the lab so that students would have the opportunity to engage in reasoning about the coordination of quantities earlier. We also set up tasks that specifically brought out some of the reasoning from their responses in order to challenge their thinking about constant versus non-constant rates of change. This preliminary report offers one way that researchers might use the findings of small-scale teaching experiments to inform task design in larger classroom settings. Additionally, we can use these findings to further our understanding of when and how conceptual shifts occur for students when learning about exponential functions from the frame of quantitative and covariational reasoning.

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Highlighting Student Voices: Perspectives on Active Learning Advanced Math Courses and Implications on Equity

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The relationship between active learning and equity is a prominent discussion within mathematics education, as it should be. There are conflicting findings on whether active learning promotes or detracts from equitable outcomes. This study aims to highlight student voices in describing the student experience in active learning advanced mathematics classrooms. This preliminary report outlines key themes that surfaced from a cursory analysis of the data; quiet students were seen as less confident or less competent by their peers during whole class discussions, and students were keenly aware of the knowledge distribution within small group discussions. It is important to understand the implications of the sociocultural dynamics that may be emphasized in active learning classrooms, thus being better equipped to attend to equity.

Keywords: active learning, inquiry-based learning, equity, advanced mathematics

It is understandable why advocates of student-centered, active learning instruction (e.g., inquiry-based learning) might purport active learning as means to achieve equity in mathematics education (Tang, 2017). Generally defined, active learning occurs in student-centered classrooms where students are actively engaged in doing mathematics during whole class discussions and collaborate with peers during small group discussions with the goal of building their own mathematical understanding; the instructor merely mediates student learning rather than acting as the sole intermediary of knowledge as is the case in traditional, lecture-based classrooms. As defined, active learning classrooms sound like idyllic classrooms in which equitable student outcomes are bound to occur. Clearly, if the best-case active learning classroom scenario were easy to employ *and* the norm, the answer to how to achieve equity in math classrooms would be more evident –simply implement active learning approaches. However, that is not the case. For the purposes of this preliminary report, my conception of equity focuses on how students position themselves and others in terms of equal access to participate *in* and *with* the mathematics, which is related to the assignment of competence (or power) and one's identity as a capable doer of mathematics (Gutiérrez 2002, 2013). Equity is an ideal that postsecondary mathematics educators strive for, but how we achieve this goal is elusive. There are some that report findings that active learning leads to equitable results, particularly for women (Freeman, 2014; Kogan and Laursen 2014; Laursen, 2014) and marginalized student populations (Fullilove and Treisman, 1990). On the other hand, others are more apprehensive or critical of the claim (Johnson et al., 2020; Ernest, 2019; Evans, 2022). These conflicting findings (and opinions) on the extent to which active learning achieves equitable student outcomes imply that more research is needed. The aim of this study is to highlight those most impacted by dramatic changes in teaching and learning practices, the students who experience learning in active learning classrooms. Much of the existing research on equitable implications of active learning centers around achievement results or learning outcomes, which are important when considering equitable outcomes. However, I believe that undergraduate students are equipped to articulate their perceptions of what works and what does *not* work for them in an active learning classroom. Furthermore, by highlighting student voices, the author answers the call to deepen our field's understanding of the impact of active learning approaches on equity within undergraduate mathematics (Laursen & Rasmussen, 2019; Melhuish et al., 2022). In this preliminary report, I outline the methods used to conduct this qualitative study, describe a couple of significant connections between student responses that have equity implications, and discuss the equity implications.

Methods

The data in this study are comprised of semi-structured interviews (approximately an hour in length) with students who completed an advanced math course taught using an active learning approach. To provide readers with a basic context in interpreting the preliminary results, the self-reported demographics collected from each of the ten participants are presented in Table 1. Six participants were from an abstract algebra (Fall 2021) course and the remaining four participants were from an introduction to proof (Spring 2022) course. Both courses were offered at the same large, public Hispanic-serving university in the southern United States. The selection of the two different advanced math courses was intentional. It was discovered through the first round of interviews with the students from the abstract algebra course, that many of them had previously taken an active learning course. Therefore, the decision to recruit from an advanced math course earlier in the math course sequence seemed appropriate. As hoped, for variety, none of the four students from the introduction to proof course had previously taken an active learning course. Since the interviews took place after the end of the semester, interview questions were provided one to two days prior to the interview to allow adequate time for students to reflect on their experiences. The interview transcripts were generated using an electronic transcription service. Transcripts were verified for accuracy and any repeated words and filler utterances (e.g., uh, um). For preliminary analysis, a single coder open-coded broad interpretations of students' descriptions of firsthand and secondhand experiences in students' respective classes. Codes were then grouped by theme and relationships between the themes were identified. In total, there were 20 questions asked during each interview. For the purposes of this preliminary report, the focus will be on the following subset of questions that are strongly relevant to equity: *Do you believe you had <u>equal access to participate</u> in the course during <u>whole class discussion</u> as your fellow peers? Why or why not?; Do you believe each student (other than yourself) had <u>equal access to participate</u> in the course during <u>whole class discussion</u>? Why or why not?; Do you believe you had <u>equal access to participate</u> in the course during <u>groupwork</u> as your fellow peers? Why or why not?; Do you believe each student (other than yourself) had <u>equal access to participate</u> in the course <u>during groupwork</u>? Why or why not?*

Table 1. Self-reported student demographics

Identifier	**Gender**	**Race/Ethnicity**	**Degree Program**
Rosie	Female	White	Applied Mathematics
Tyler	Male	White	Education; MS Certification
Stephen	Male	White	Mathematics
Nathan	Male	Caucasian	Manufacturing Engineering
Ivan	Male	Hispanic	Applied Mathematics
Jon	Male	Southeast Asian	Computer Science & Mathematics
Nora	Female	White/Hispanic	Mathematics with Teaching Certification
Mary	Female	White/Cuban	Mathematics
Lamar	Male	White	Mathematics
Carlos	Male	Hispanic	Economics

Preliminary Results

The presentation of the preliminary results will be segmented into two broad categories, Whole Class Discussion and Small Group Discussion, which were derived from the foci questions, as listed above. Relationships between codes surfaced from the interview questions related to whole class discussion and small group discussion. I briefly summarize a relationship finding within each broad category and provide support from the data.

Whole Class Discussion: "You Gotta Be Strong"

It is not unusual to have quiet students in any given mathematics classroom. In a lecture-based classroom, they may go unnoticed. However, it is more challenging to hide in an active-learning classroom due to whole class discussions being a focal mode of learning. In analyzing the data, the characteristics of shyness/quietness arose on multiple occasions whether it was someone acknowledging they were the quiet one in class (projected as a deficit), or someone stating that it was easy to identify the shy ones (also considered a deficit). How might this pertain to equity? A student's disposition for shyness was a reason for peers to redraw mathematical competence, or conversely, outspoken students were interpreted as having more competence. The following series of interview excerpts illustrate some social dynamics of vocal and less vocal students within an active learning environment. When Mary was asked whether she believed she had equal access to participate in the course during whole class discussion as fellow peers, she responded as such:

> Yeah, pretty much. I've never been a super talkative person in class. Um, like when it comes to talking to other people in the class one-on-one, I love doing that and I love working in groups where we're all just kind of putting our heads down, thinking about stuff. But then if you were talking about the part where [the instructor] would like ask for sort of ideas on what to do, I would let other people talk and usually I just watched them battle it out. Um, 'cause that was fun for me, but you know, I don't like to be the loudest person in the class or anything, but I do think I had equal access to participate. I just chose not to sometimes.

Participation in whole class discussions may take on different forms. Although instructors ought to challenge and encourage students to vocally contribute to whole class discussions, ultimately, students should have the autonomy to participate in ways that are comfortable to them. However, one trend that surfaced was the tendency to interpret shyness or quietness, a trait often attributed to women, as an indication of mathematical inability or lack of confidence.

> *Jon:* There tended to be those people who were bad at math, and I don't wanna say bad at math like that, but they weren't good at proofs. Like, they said it, it was relatively clear, you know, and those type of people tended to not talk as much.

Jon describes making a connection between math ability and quietness. However, Mary was content engaging, albeit quietly, in whole class discussions while actively participating in small group discussions. Her identity encapsulated a quiet demeanor, which is not a reflection of her mathematical ability as she suggests in the following excerpt.

> *Mary:* Um, I'm just comfortable not saying anything. I actually ended up talking to my really good friend that I made [from class] about this 'cause he said at some point during the semester." It's obvious that you know what's going on and you just prefer not to speak.' Yeah, that's true. It wasn't like, I was feeling like I didn't have equal access. I just kind of was like, I was chilling, you know?

Mary implies that there is a level of assertiveness and strength needed to participate in whole class discussion when she admits, "I didn't talk because I knew someone else was gonna answer, but you know, you gotta be strong. You gotta stand up for yourself if you wanna. If you really want to talk, like you can just do it."

Many of the participants mentioned being able to identify the talkers from the non-talkers, and in fact, stated there were more non-talkers than talkers. In contrast to the negative labels non-talkers were ascribed, the more vocal students were assigned competence. Rosie admits being nervous to talk when there were outspoken people in class, whom she attributed competence due to their perceived confidence. This led her to doubt herself and her potential mathematical contribution.

> *Rosie:* Yeah. I feel like if I wanted to talk, like I definitely had like the opportunity and chance to. It was hard sometimes because we definitely had some more outspoken people in the class and it always made me nervous 'cause sometimes I got different answers than them on stuff, right? And they were very like sure in like what they were saying. So, it kind of made me like a little bit more hesitant to talk sometimes even though I was like, mmm, I don't know. But I'm also just a little bit more shy like that.

When asked a follow-up question related to whether the vocal student referred to earlier in the interview affected Nora's learning, Nora explained that it was helpful to have a different perspective than the professor.

> *Nora:* Not at all. Because I think we all also like learned from him like 'cause he was very vocal about like things that were going through his mind. Like um, if there was like a pattern going on through like a proof that we were learning, he would kind of vocalize it. So, I think it also at points like helped the class learn a little bit more and to see his point of view, like a little bit different than the professor's, because then we're getting kind of like multiple points of view. And so, at times it really helped.

The more vocal students have ascribed competence in contrast to less vocal students who have their mathematical proficiency questioned. Due to the focus on peer-to-peer interactions in active-learning classes, this is important to note. The perceptions of peer competence affect how students work together in small groups, which are prominent in active learning classrooms.

Small Group Discussion: "There was clearly a knowledge disparity"

As a characteristic of active learning classrooms, small group discussions were paramount in both courses within this study. In response to the series of questions asking about equal access to participate in the course during group work, Mario described an obvious "knowledge disparity" between the groups. When the interviewer asked Mario what he meant by "knowledge disparity," he responds:

> The best way I could put it is that you could tell that there's like a team that just really understood everything. And there were other teams that were just kinda like struggling. There were some groups that were kind of like in the middle, they kind of understood some of the stuff, and then there were some teams that were just like… 'I need outside help essentially.' The group assignment remained the same for this course the entire semester. Mario described how frustrating it was to be in a group where everyone was lost, and they had to wait for the professor to make it to their group to offer assistance.

He goes on to explain how difficult it was to get assistance if the entire group was lost due to "knowledge disparity."

> *Mario:* If you had questions or you were stuck, you're just like waiting for the professor to just kind of like come and help you. So, you know, if every team is like, kind of like

needing help on a part, you're kind of just like in this queue trying to solve it, you know, you kind of stall.

By design, active learning is meant to distribute mathematical authority more evenly rather than being solely held by the instructor. However, if an entire group is unable to make progress on a problem or proof, the group may have to rely on the instructor, as was the case described by Mario. Thus, the knowledge disparity persists. If you were fortunate to be with group members who had a stronger understanding of a concept, there were sentiments related to the uneven distribution of knowledge within a group.

Ivan: There would be cases in which some of our other group members, they didn't really do as much because they were confused, or they were lost within the proof itself. So from there, what we would do is we would essentially guide one another and as to the right direction. And just to essentially give them the baby steps of other proofs to show them as a way to kind of help them to learn that proof per se, or topic.

But the consequences of leveraging your group members for instruction may lead to fellow group members feeling as Jon describes.

Jon: There was an effort, but I mean at a certain point you kind of have to stop so that you can actually get the work done. So yeah. I mean, there's a limited time to finish. We had probably, I think at the entire semester, there was like a hundred proofs or something. So, I mean, there was quite a bit to do, and we cannot like hold back because of somebody.

Group members may feel as though they are being held back due to the time it takes to explain mathematics concepts to peers. This shift in authority that occurs in active learning environments is powerful as long as there is an accommodation made for situations as described by the students within this study.

Discussion

This study is a direct response to the call by many (e.g., Adiredja, 2017) for additional research on issues of equity in postsecondary education and, particularly, research that investigates student perceptions of their active learning experiences. The findings from this preliminary analysis indicate that competence may be assigned to students based on their perceived level of participation in whole class discussions. Furthermore, that assigned competence is likely to transfer into small group discussions and affect peer interactions. Additionally, students are perceptive to the unequal distribution of collective knowledge between *and* within small groups. As was stated, it was obvious whether someone was in a group that possessed “less” knowledge. These groups would have to wait around for the professor’s assistance. In the case where there were more knowledgeable students in a group, the more knowledgeable students felt like they were being slowed by having to explain concepts to peers. It is speculated (Brown, 2018; Ernest et al., 2019; Johnson et al., 2020) that since active learning classes increase personal interactions through whole class and small group discussions, student beliefs and peer perceptions may play a more significant role in learning than in traditional lecture-based classrooms. Although instructors cannot eradicate stereotypes and beliefs students possess, instructors can take steps to mitigate the influence of these beliefs as long as we identify common student perceptions. It is imperative to understand the challenges of implementing active learning approaches to make necessary modifications or accommodations to simultaneously and consistently achieve equitable results.

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Student Goals and Motivations in their Undergraduate Abstract Algebra Course

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Mathematics educators assign important learning outcomes to abstract algebra classes. However, it is unclear whether students share these goals or whether they have different motivations entirely. Using the framework of expectancy-value theory, we highlight preliminary results from semi-structured entrance and exit interviews with six undergraduates enrolled in abstract algebra. This report centers around the following research question: What are students' expectations, motivations, and goals for their undergraduate abstract algebra course?

Keywords: abstract algebra, motivation, goals, expectations

Introduction

Abstract algebra is a foundational course in advanced undergraduate mathematics in which students are expected to learn proof technique and abstraction alongside content-specific material (e.g. group theory, field theory) (e.g. Dubinsky et al., 1994). However, it is unclear what students' own goals and motivations are. While there have been many studies on the role of motivation in mathematics courses, there have been comparatively few studies on motivation in advanced proof-based courses like abstract algebra. This distinction is important, as mathematics educators have noted that the content and style of advanced proof-based courses differs from previous courses such as calculus (e.g. Alcock & Simpson, 2002).

We highlight preliminary results from semi-structured entrance and exit interviews with six undergraduates enrolled in abstract algebra. Our central research questions are: *What are students' expectations, motivations, and goals for their undergraduate abstract algebra course? And how do students' motivations contribute to how they approach the course?*

Theoretical Constructs

We adopt the framework of Situated Expectancy Value Theory (SEVT), (Eccles & Wigfield, 2020) to elucidate the relationship between students' expectations, motivations, and perceived task value of abstract algebra. A key assertion of Situated Expectancy Value Theory is that achievement on a task is mediated both by students' expectations for success as well as the task's perceived value (Eccles & Wigfield, 2020; Rosenzweig et al., 2019). Task value is then further separated into the constructs of attainment value, intrinsic value, utility value, and cost.

Eccles et al. (1983) define *attainment value* as the personal or identity-based importance of doing a task. *Intrinsic value* (also referred to as interest value) refers to the students' enjoyment of the task and shares similarities to the constructs of interest (Hidi & Renninger, 2006) and intrinsic motivation (Ryan & Deci, 2016). Within this study, a task is intrinsically valuable when it is enjoyable or helps procure future enjoyment. However, the concepts have different theoretical traditions and focuses. For instance, intrinsic motivation emphasizes the rationale behind an activity as opposed to what makes it valuable (Eccles, 2005). *Utility value* refers to the degree to which a task would be beneficial for a students' present or future plans. For the purpose of this study, we specify that utility value accomplish tasks for reasons other than for one's enjoyment, such as credentialing or seeking employment. Finally, *cost* refers to what an individual must give up in order to accomplish a task. We define success as the student's realization of their goals and is not necessarily based external metrics such as performance on an exam, and is thus a student-centered relative construct.

Literature Review

While motivation has been studied within math education literature (Hannula et al., 2016), and in scholarship on undergraduate STEM education (Cromley et al., 2016), we found no studies attending to motivation in proof-oriented mathematics classrooms. Existing motivation literature has largely consisted of studies drawing correlates between motivation and related constructs like self-efficacy, achievement, and interest (Cromley et al., 2016; Jones et al., 2010). There has been a recent emergence of scholarship on EVT interventions intended to increase student motivation (Rosenzweig & Wigfield, 2016). Much of the interventional scholarship has focused on utility value interventions as it is perceived to be the most easily modifiable task value (Harackiewicz et al., 2014) (i.e., the interventions have focused on increasing students' perceived utility of the tasks on which they work). While there is extensive literature on the statistical relationships between students' self-efficacy, motivation and performance in mathematics courses (for a review, see Wigfield & Eccles, 2020), there is relatively little work on how these factors operate qualitatively.

Eccles and Wigfield adopted SEVT from Eccles and colleague's Expectancy Value Theory (Eccles, 1984; Eccles et al., 1983) as part of a call to attend more closely to the situatedness of motivation. Situatedness, for Eccles and Wigfield, refers not only in the macro sense to the student's sociocultural environment but also to their interpersonal context. We believe our work provides a contribution to SEVT by situating our understanding of our participants motivation within their specific context of undergraduate abstract algebra. Accordingly, previous studies of elementary schoolers point to the domain-specificity of situated task values and academic self-concept.

Methods

Our study consisted of two semi-structured interviews with six participants on their motivation and associated study habits halfway through and at the end of the semester. Our interview protocol contained questions of the form:

1. Why do you think it might be important to study math?
2. What do you expect to get out of studying math in college?
3. What do you expect to get out of Abstract Algebra?
4. In a perfect world what would you want to get out of Abstract Algebra?

The first question asked participants for their perceived task values of studying math while the second and third question ask for participants' expectancies for both math and abstract algebra. The fourth question draws a distinction between participants' expectations and what they perceive they could optimally get out of abstract algebra.

In the second interview, we asked students to reflect on their responses in the first interview and to comment on whether they felt their expectations and/or ideal ("perfect world") goals were achieved. Students were also asked for their perceived task value of abstract algebra. We used thematic coding (Braun & Clarke, 2006) to identify key themes from the interviews.

Results

We present preliminary results from our ongoing analysis. One emergent theme is that some students desired intrinsic value, had utility value, and appreciated but did not require utility value in the form of real world applications. When asked what he expected to get out of abstract algebra, Participant 1 stated:

> It does seem like you know, at least for group theory, that that comes up a lot in things so I'll have an understanding, very foundational conceptual understanding and in a field that does have applications to these, this theoretical work. Um, hopefully, I'll come out of it. Part of the anticipation is also I don't really know where it might be useful, but hopefully, the expectation is, a useful thing will present itself to me. And also, you know, everybody I've talked to has been like, "oh, wow, Galois Theory is really awesome". […] I think it's one of those subjects which is, I'm very good at when I have an intuition of it, but sometimes something will happen I'm like, "why is that? Why does that work?" Or like "what does that work and this not work" Actually, that's the worst thing. It's not "why does this work?" But usually, 'why does this work and that doesn't'

The student remarks that he is unsure in what ways abstract algebra would be useful for him but that he expected that "a useful thing will present itself." We interpret this as the student expecting a utility value even though he does not know what it is yet. Even in the absence of a concrete utility value, the participant still anticipates the class will be intrinsically valuable to him. The expectation of intrinsic value seems sufficient motivation for Participant 1. A different participant, however, anticipated he would struggle with the course in the absence of intrinsic value. When asked what he could get out of abstract algebra under ideal circumstances, he answered:

> Yeah, I'm not too sure. I mean, I think...I don't know. I mean, I think most importantly, *for me even more than like having real life application and would be just to find something I'm interested in*, because like I said, *I have to pass the course*. And I think courses are a lot easier for me when I can find myself getting really interested in them. So for me even more important than having a real life application, just be to have an interest in it because I think it would just kind of be easy to pass the class with an interest in it, but it's kind of going in the direction where it's, it's like I don't have an interest in it. So I'm expecting to be pretty difficult for me to kind of fully understand everything and get myself to study for it. And prepare myself for exams and other things. But I, I guess, like you know, like I said, like, I mean, I don't know that anything off the top of my head in the class so far that would have a realistic you know, real life application for me, but I do think that would be very nice. That would be very cool if I could kind of get something because I think that would motivate me in itself to know that like okay, this is going to be something you have to know and have to understand versus right now I'm kind of in the like, my mind, I'm just like, "Okay, I literally just have to pass this class. And that's it.

Like Participant 1, Participant 2 desired intrinsic value in the course. While "real life application[s]" would be a plus for Participant 2, the participant stressed that, "even more important than having a real-life application [is] to have an interest in" abstract algebra. An interesting topic would make it "easy to pass the class;" without it, he would find it difficult to "get [himself] to study." We interpreted this to mean that even though Participant 2 desired utility value, it was intrinsic value (i.e., personal interest) that was crucial. It is notable that Participant 2 had a high personal utility value for the course—he required it to graduate—but thought that this would not be sufficient motivation for him to do the work to study the material. Later in the interview, Participant 2 described how the effort required to pass would be substantial, requiring frequent personal meetings with the course professor for extra help. The intrinsic motivation seemed necessary for Participant 2 to put in this effort.

Participant 3 noted utility that advanced mathematics had, but found this to be an unattractive reason to pursue the discipline.

> I study it, in part because I like it. Which is a huge motivation factor. But, you know, there's always that question of like, 'what is the real world application of what I'm doing?' and and I did struggle with that for quite a while, especially in that like transition from community college to [university]. Because [in university], a lot of the recruiting programs that you know, try to try to hire students out of the math program, they are government based or like Lockheed Martin, or, you know, all of those other programs and I don't like that very much. You know, and so a lot of times at this higher level, the application of math is based on those kinds of programs. And so, okay, I can name that as an application but I don't like that as an application.

Participant 3 later clarified that while much mathematics is used for a military focus, one can use mathematics for a “community-based focus and to do good things” and learning “flexible thinking and critical reasoning”, so mathematics could potentially have high utility. Nonetheless, Participant 3 stated that this is not why they studied mathematics:

> I think at some point, I just determined that I do math because I like it. And I enjoy it. Just like, I don't have to have a real world application for doing like arts or poetry, or whatever. So personally, I like it.

Discussion and Conclusion

Our analysis is preliminary and speculative at this stage. The data require verification with closer analysis. Our data indicate so far that for some students, attainment value is critical. Participants 1 and 3 were confident they would find attainment value and were optimistic about their ability to succeed. Participant 2, however, anticipated he would struggle in the course due to the absence of intrinsic value. Furthermore, our data suggest a connection between student motivation and the way students approach abstract algebra. Utility value appears to be the least important construct; the students included in this preliminary analysis neither expected it nor needed it. Indeed, Participant 3 thought the utility of mathematics for military use was a negative feature of the discipline. One student, Participant 2, was worried he would be unable to pass the class without finding intrinsic value and stated that he needed to go to office hours. Participant 1 and Participant 3 cited enjoyment as a reason to pursue mathematics.

Much scholarship in undergraduate math education has focused on achieving the instructor’s goals without addressing whether this aligns with student goals. Our work demonstrates that for some students, intrinsic value is the most important factor among intrinsic value, utility value, and attainment value. We suggest more research on how intrinsic value can be fostered in students and, more generally, how students can experience the joy that mathematicians experience in doing mathematics.

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Investigating Student Use of Representation in Quantum Mechanics Change of Basis Problems

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Meta-Representational Competence (MRC), first theorized by Andrea diSessa, has been used widely in Math Education studies. It has been recently used in Physics Education Research in the domain of Quantum Mechanics. Specific problems within Quantum Mechanics have a foundation in Linear Algebra, and may be approached or perceived differently based on the notation used (either Dirac, Matrix, or Spinor notation). Semi-structured interviews were conducted to present physics students with content questions and then ask them about MRC concepts directly, as it related to the content questions or other situations in Physics or Math. Student statements were coded to indicate both previously identified and novel facets of MRC in the context of change of basis problems in Quantum Mechanics. The preliminary analysis of two student interviews demonstrates each student's distinct utility of MRC concepts, and suggests extending the Wawro, Watson, & Christensen (2020) MRC statement codes.

Keywords: Quantum Mechanics, Meta-Representational Competence, Physics, Linear Algebra, Representation.

Literature Review

Notions involving representation and notation in Quantum Mechanics have often been investigated in Physics Education Research (PER). Fredlund, Airey & Linder (2012) noted that "the affordances of different representations determine the role they can play in communication, and thus in the sharing of knowledge." These affordances, or the "inherent potential of that representation to provide access to disciplinary knowledge", are said to "enable certain representations to become legitimate within a discipline such as physics. Physics learning then, involves coming to appreciate the disciplinary affordances of representations." (Fredlund, Airey & Linder, 2012). Additionally, Gire & Price (2015) identified the terms "*individuation, degree of externalization, compactness, and symbolic support*" as "structural features of quantum notations". This structural features framework was adapted by Schermerhorn et al. (2019) into a "Computational Features Framework", in order to "answer questions regarding how and why students use different methods". While PER literature has examined representations in Quantum Mechanics in the past decade, it often did so without leveraging diSessa's Meta-Representational Competence (MRC) Theoretical Framework (2002), (2004).

Investigations using MRC have been recently conducted in the context of Quantum Mechanics, specifically in units involving "spin" which tend to use Eigentheory concepts. In a 2020 study, Wawro et al. enumerated the different MRC concept-related codes that arose in their interviews which were structured around Eigentheory concepts. These codes were in part informed by the "Computational Features Framework" developed by Schermerhorn et al (2019), along with additional codes that were organized and explained through the MRC framework.

The Wawro et al. (2020) study has been cited in a recent paper by Corsiglia et al. (2022), which focused on QM Change of Basis problems. However, the Corsiglia study focused on a phenomenographical approach and did not implement the MRC theoretical framework used in the Wawro study. This current research seeks to incorporate MRC at a boundary between realms of RUME and PER in the specific context of QM Change of Basis problems. This is done through the coding of MRC concept statements that involve codes listed in Wawro et al. (2020)

and novel codes that were not found in this earlier study. We define "MRC concept/statement" here as a concept/statement that could be analyzed by the MRC Theoretical Framework.

Theoretical Framework

The work presented is attempting to make direct measures of Andrea diSessa's theoretical framework, Meta-Representational Competence (diSessa, 2002). Meta-Representational Competence (MRC) "...[describes] the full range of capabilities that students (and others) have concerning the constructions and use of external representations" (diSessa & Sherin, 2000). Iszák Iszák, Çağlayan & Olive (2009) mention regarding a diSessa paper (2002), "[diSessa] observed that students' criteria usually emerge in reaction to particular examples." Iszák also noted, "he reported that different students can make systematically different judgments about external representations."

This framework has seen significant use in the Research in Undergraduate Math Education community (Hillel, 2000; Arcavi, 2003; Iszák, 2003; Iszák, Çağlayan & Olive, 2009) but has seen little use in Physics Education Research. Notions involving representation and notation in Quantum Mechanics have been often investigated in Physics Education Research (PER), but only one paper incorporated Meta-Representational Competence as a framework, Wawro et al. (2020). In Wawro et al. (2020), MRC statement codes were created based on the data observed (see Fig. 1) and proved to be very fruitful in parsing the data observed. However, the theoretical framework was applied post-hoc. This work aims to use an interview protocol designed specifically to parse out student statements regarding MRC in the context of Quantum Spin states.

A. Value-based Preference	B. Problem-based Preference
1. Clarity 2. Speed 3. Familiarity 4. "Likeability" 5. Ease of writing	1. Useful in calculations 2. Makes more direct use of given relationships 3. Needs less information 4. Compactness 5. Individuation 6. Externalization
C. Purpose and Utility Awareness	
1. Has freedom to choose symbols 2. Has an ease with notation, writes symbols to mean what is personally desired 3. Aware of one's own progress in notation use 4. Able to "step back" and weigh options to decide which notation system is best	

Figure 1. Table of MRC categories and codes from Wawro et al. (2020)

Background

In Quantum Mechanics, spin is a property of some particles, often described as "when a particle behaves like it has angular momentum (is 'spinning'), even though the particle is not physically spinning". This often confusing concept is better understood mathematically through the use of Eigentheory. Spin states of a particle or system are represented by Kets (|Psi>), which function similarly to vectors. For Spin-½ particles, the most common type in Intro-Quantum Mechanics situations, applying a directional Spin Operator can only result in one of the two possible eigenvalues and its corresponding eigenvector. The result is either "spin-up" (a "+" eigenvector with a corresponding eigenvalue of $+\frac{\hbar}{2}$) or "spin-down" (with a "-" eigenvector with eigenvalue of $-\frac{\hbar}{2}$). In a certain coordinate direction (namely z, x, or y), a spin-½ particle's spin state can be described as a linear combination of these $|+>$ "spin-up" and "spin-down"$|->$ kets.

These state vector kets, defined as linear combinations of eigenvector kets, can also be written in terms of 2 by 1 matrices. Here, the first entry represents the $|+>$ ket (usually in the z-direction), and the second entry represents the $|->$ ket. Introductory units of instruction may present the same information in both Dirac and Matrix form, so that students can see how one notation is translated to the other. Additionally, "Spinor" notation is sometimes used to write statements without explicitly using a Dirac ket or a column vector for the eigenvectors.

In any notation, the relations of the eigenvector kets, the spin operators, and the eigenvalue kets can be written in eigenequation form: $\hat{S}_z\,|+> = \frac{\hbar}{2}|+>$. Figure 2 (below) includes an example of the 3 notations used in the context of a state vector equation:

Spinor: $\chi = \frac{3}{5}\cdot\chi_+ - \frac{4}{5}\cdot\chi_-$ Dirac: $|\psi\rangle = \frac{3}{5}\cdot|+\rangle - \frac{4}{5}\cdot|-\rangle$ Matrix: $\chi = \frac{3}{5}\cdot\begin{bmatrix}1\\0\end{bmatrix} - \frac{4}{5}\cdot\begin{bmatrix}0\\1\end{bmatrix}$

Figure 2: Equivalent state vectors, or wavefunction kets, written in different notations

Methods

A semi-structured interview was designed to provide participants with content questions and ask them about MRC concepts directly (Browner et al., 1988). Content questions were adapted from questions from the Corsiglia et al. (2022) study, which focused on Change of Basis, including concept-focused and computational-focused questions. The questions were adapted to provide a context for discussing MRC statements, rather than pure tests of conceptual understanding. For some of these content questions, multiple versions were made with different representations in the questions (See Fig. 2). After each set of content questions, a verbal follow-up question was asked. The follow-up questions were intended to target MRC concepts directly. At first, these follow-ups treaded lightly on MRC concepts and focused more on content, but as the interview progressed, the follow-ups were directly aimed at MRC concepts.

This interview structure was selected to organically ease students into talking about MRC concepts without directly explaining the MRC theoretical framework. To avoid dead-ends due to content issues, hints were structured into the content questions, or ready to be provided by the interviewer. The structure of the interview questions was meant to guide students through the most efficient path (Corsiglia et al., 2022) through answering the questions.

Three student interviews were conducted at a midwestern, land-grant research university, each lasting about one hour long. Of these 3 interviews, we elected to focus the preliminary analysis on the interviews of participants PS and CG, as their interviews were the most unlike each other. After the interviews were transcribed, one of the authors reviewed the transcriptions to code mentions of MRC statements, guided by the codes delineated in Wawro et al. (2020).

Preliminary Results

The depth and amount of MRC statements made by student participants with relatively little direct prompting were surprising and encouraging. While physics students may unknowingly make MRC-related statements with their peers (e.g., preferences for notation, etc.) while working on group assignments, there is little evidence that they are instructed or directly graded on MRC concepts within physics courses. However, the interview participants all had some form of vocabulary for expressing some MRC concepts, to varying degrees, without being explicitly introduced to the terminology. Additionally, interview participants seemed to accept and use vocabulary that was explicitly introduced during the interview with good facility.

We focused our preliminary study on Student Participants PS and CG, as they had distinct interview experiences from an MRC concept standpoint. They performed differently on the tougher content questions. They also articulated different preferences in notation. We share some preliminary analyses of each interview separately to best highlight the most prominent features of each participant's interview experience. Future work will aim to synthesize these and other interviews we've performed.

PS Interview: PS is a Physics Undergraduate student, who had taken two semesters of undergraduate Quantum Mechanics from a midwestern land-grant research university. PS identifies as a white, cis-gendered man. PS made fewer MRC comments in total, and it took them longer to 'warm up' to speaking freely about MRC in depth. PS was effective in elaborating on their thought process while working on content questions when prompted. Additionally, most of the MRC statements PS made were "Value-based preferences" (Fig. 1).

Within the first 10 minutes of the interview, PS made statements involving their belief in a "Hierarchy" of notations. PS stated that Dirac (and maybe Spinor as well) notation is just a "shorthand" for the Matrix notation. There was no existing code from Wawro et al (2020), so we defined a new code to encapsulate this. Much later in the interview, PS made a similar remark regarding the Spinor notation in Question 5 ("Q5"), after being given Question 7 ("Q7"), (emphasis added in **bold**):

PS: There's a reason that I already like this notation way better, I got to be honest.

Interviewer: Why do you like it better?

PS: But again, you know, **it's like, hey, spin up -- 1 0, spin down -- 0 1**, and like, like I said before, I just **I like the given** - what exactly **given values for just like shorthand** [points at Q5] like Psi, Psi plus that's the spin-up, and you know, I kinda get, like all screwed up in my mind, like - how do I do this again?

PS: **Where I can actually, like… work out the numbers?** Like, [points at Q5 page] can one over square root of three be multiplied times Chi spin up? **Whereas here [Q7] it's like, I know I can, you know?**

It's possible that PS views this hierarchy of Matrix notation as being the 'real' notation, with Dirac and Spinor notation as 'shorthands' because they prefer doing change of basis computations in Matrix notation. Additionally, PS made many MRC statements about "Clarity", where Matrix notation was often 'more clear', and Dirac notation was 'less clear'. They also mentioned "Likeability" often: liking Matrix notation, and disliking Dirac notation.

PS spent significant time on Q5 but was not able to complete it. They expressed that they did not have enough equations to solve for the constants they wanted. When given Q7, which was a Matrix notation focused instead of Spinor notation focused, PS responded with the reaction in the interview transcript above. Even though the problems were similar, PS was able to complete Q7. Afterwards, PS made conflicting statements regarding their success in Q7 compared to Q5. Initially, they made statements indicating that they understood the question better after they saw it presented in Matrix notation (Q7). However, a couple minutes later, they said that they couldn't be sure that the different notation made it easier. In their opinion, their success may have been due to just seeing the same question a second time. A goal of future work is to investigate the connection between struggling to answer problems and a person's understanding of concepts that fall under Meta-representational Competence.

CG Interview: CG, a current Physics graduate student, has taken Quantum Mechanics at both an Undergraduate level at a small, private college in the midwest, and at the Graduate level at a midwestern land-grant research university. CG identifies as a white, cis-gendered man. CG, who performed extremely well on content questions, spoke fluently about how they represented their work verbally, on the page through equations, and using hand gestures. Relative to PS, CG made

MRC statements earlier and more often over the course of their interview. Many of these statements involved a preference for Dirac notation. "Value-based preferences" MRC statement codes (Wawro et al. 2020) such as "Speed", "Familarity", "Likeability", and "Ease of Writing" were often made by CG in support of Dirac notation and against Matrix notation. CG also made "Problem-based Preference" statements, noting Dirac notation was "useful in calculations" because it had "tons of tricks."

While PS made few statements regarding "Purpose and Utility Awareness" (See Fig. 1), CG made many (at least 10) distinct statements in this category; the first one coming during an early follow-up question. Most frequent were statements that would fall under "Aware of one's own progress in notation use", which encompassed Dirac, Matrix, and Spinor notation. CG also made statements consistent with the "Able to 'step back, and weigh options to decide which notation system is best" code after one of the verbal follow-up questions and while considering how to begin working on Q5.

CG made many statements involving concepts of "Abstractability" and "Visualizability." To CG, Matrix notation was very visible. They used hand gestures to set up a 3-dimensional coordinate system to communicate state vectors. CG then mentioned understanding Matrix notation as applying a linear transform to a vector, to change the direction of the state vector. In contrast, CG viewed Dirac notation as being more "abstract" in nature. They stated that the Dirac Kets could represent anything, or be written as "smiley faces" or "frowny faces" without any loss of meaning. Because of CG's focus on these statements, and that they emphasized the wording, we believe "Abstractability" and "Visualizability" may be identified as new codes for extending the Wawro, Watson & Christensen (2020) MRC codes related to expectation values (See Fig. 1).

Discussion / Conclusion

MRC concepts, which involve the capability to use external representations (diSessa & Sherin 2000), are likely related to skills developed by individuals by the time they are considered an 'expert' in a particular type of problem. Interviewee CG, who performed extremely well on content questions, successfully represented their work verbally, on the page through equations, and using hand gestures. Interviewee PS struggled with several content questions and did not demonstrate many MRC statements outside of the Value-based Preference category. How and when MRC concepts are developed and the role it plays alongside content understanding in Upper-division Physics is still unclear. Future research may shed additional light onto the relationship between students' understanding of their thinking of how to use different representations and their performance on content questions in Upper-division Physics.

Questions for Audience

If students are not directly taught or coached regarding MRC concepts, how do they develop it in a Quantum Mechanics spins context which draws heavily on Eigentheory concepts?

Would directly instructing students on understanding their thinking of how to use different representations help students become adept at this and content understanding?

At what points throughout a mathematics or physics curriculum could instruction be initiated and subsequently revisited to support MRC concept development across a curriculum?

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Exploring the Connection Between Women of Color's Values and Mathematical Identity in Undergraduate Mathematics

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The growing call for more equitable and just mathematics classrooms includes supporting and strengthening students' mathematical identity. Recent scholarship proposes a connection between students' mathematical identities and their values (Kalogeropoulos & Clarkson, 2019). However, limited research acknowledges and explores mathematics identity at the intersection of gender and race (Leyva, 2021; McGee, 2016). Furthermore, undergraduate mathematics in the United States often does not reflect the cultural values of women or students of color (Fong et al., 2019; Leyva, 2016, 2021). Informed by sociopolitical theory and an intersection lens, this study explores women of color's mathematics education values and the connections between those values and their mathematical identity. Through interviews and mathematical autobiographies, participant examples indicate various values, including those related to learning from mistakes and social support. Preliminary results suggest that the presence of these values in mathematics environments may support women of color's mathematical identity.

Keywords: Equity, Mathematical Identity, Values, Gender, Race

In the growing call from research and professional organizations within mathematics education to create more equitable and just mathematics classrooms (e.g., Mathematical Association of Mathematics Instructional Practices Guide, 2018; National Council of Teachers of Mathematics, 2014), one area of interest focuses on supporting and strengthening students' mathematical identities. Mathematical identity encapsulates students' relationship with mathematics and their perceptions of themselves as a doer of mathematics, including their beliefs about ability and participation, sense of belonging, and institutional and interpersonal supports (Voigt, 2020). Research suggests that gendered and racialized mathematical discourses play an important role in students' development of their mathematical identity, however, limited scholarship exists that acknowledges and explores mathematics identity at the intersection of gender and race (Leyva, 2016, 2021; McGee, 2016).

Recent scholarship proposes a connection between identity and values, such as mathematics teachers' professional identities and their values and students' mathematical identities and their values (Kalogeropoulos & Clarkson, 2019; Mandt & Afdal, 2022). There is general consensus that values are sociocultural in nature, embodying cultural norms about what is bad or good, or worth avoiding or doing (Hill et al., 2021; Schwartz et al., 2011). Thus, values in mathematics education includes one's belief about what is important and worth doing in mathematics, as well as what drives one course of action over another (Hill et al., 2021; Seah, 2019). Given that "what we value reflect years of learning and influence from our historical experiences and social interactions as members of the culture we belong," (Seah, 2018, p. 564) I argue that the relationship between identity and values in mathematics must be similarly influenced by gendered and racialized mathematical discourses. To this point, undergraduate mathematics in the United States often does not reflect the cultural values of women or students of color (Abrams et al., 2013; Fong et al., 2019; Leyva, 2016, 2021). Various scholars recognize how mathematical discourses in the US often align with, and thus give power to, dominant masculine and white values such as competition, risk, and individualism (Bullock, 2019; Jaremus et al., 2020; Leyva, 2016, 2021; Martin, 2006). Thus, unique and possibly additional tensions within mathematical spaces emerge at the intersection of race and gender for women of color as they develop their mathematical identity while negotiating their values with dominant mathematics values (Carlone & Johnson, 2007; Leyva, 2017; Rodd & Bartholomew, 2006). Supporting and incorporating women of color's cultural values within undergraduate mathematics may help reduce these tensions and better support women of color's mathematical identity development (Battey & Leyva, 2016; Hunter, 2021; Seah et al., 2016). Thus, this research engages critical frameworks at the intersections of gender and race to challenge exclusionary adherence to dominant values and encourage counternarratives to white, masculine mathematical discourses. In this research I ask, (1) What do women of color STEM majors describe as valuable/important to their undergraduate mathematics education? And (2) In what ways, if any, are these valued/important components connected to their mathematical identity?

Theoretical Perspective

I utilize sociopolitical theory and an intersectional lens to inform this work (Adiredja & Andrews-Larson, 2017; Crenshaw, 1991). This decision stems from the sensitivity of mathematics educational values to both the culture of the learner and societal influences (Hunter, 2021; Lee & Seah, 2015; Zhang, 2019). Given the white, patriarchal discourses within undergraduate mathematics, an intersectional lens helps describe the ways in which interactions between race, gender, class, and other social constructs account for overlapping discrimination imposed on people with multiple marginalized identities within societal systems (Crenshaw, 1991). Within the cultural system of mathematics, we recognize that the values women of color hold are both individually unique as well as shaped by societal influences related to gender and race. In particular, sociopolitical theory emphasizes the ways in which knowledge, power, and identity emerge and are often constrained by the values and norms of the mathematics environment. This theory challenges the currently "accepted" values and norms within the culture of mathematics, which often inequitably affect students' perceptions of themselves as doers of mathematics (Adiredja & Andrews-Larson, 2017). Sociopolitical theory rejects student assimilation into the current dominant culture and instead promotes a system that considers, supports, and affords space to various backgrounds and identities. Thus, I chose to center women of color in this analysis to provide counternarratives to the white, patriarchal mathematical context without comparisons to the dominant group. Instead, I highlight the often submerged and undervalued variety of experiences of women of color related to mathematical identity and values.

Methods

Data Context and Participants

The data for this study include transcripts of follow-up interviews and mathematics autobiographies from participants who completed the *S-PIPS-M* survey instrument administered by the Progress through Calculus (PtC) project in 2017-2018. This survey collected information about undergraduate students' experiences in precalculus and calculus, including mathematical activities, interactions, and affect (Street et al., 2021). To recruit participants for this study, I sent out an interest survey in Spring 2022 to all women of color from the survey who consented to future contact (n = 1121). Women of color within this dataset includes any participant who selected at least woman from the following select-all response options related to gender: Man, Transgender, Woman, Not listed (please specify) and who selected at least one of the following select-all response options related to race and/or ethnicity: Alaskan Native or Native American, Black or African American, Central Asian, Hispanic or Latinx, Middle Eastern or North African, Native Hawaiian or Pacific Islander, Southeast Asian, South Asian. The recruitment email described the study as an exploration of women of color STEM majors' experiences in undergraduate mathematics and thus participants self-selected as a woman of color to participate in this follow-up study. The interest survey included questions about demographics, various experiences and feelings related to undergraduate mathematics, and how those experiences and feelings related to race and/or gender, if at all.

Of those who responded to the interest survey, I selected participants based on results from a previous analysis as well as aiming to diversify the sample in terms of gender, race/ethnicity, university, and major. Previous analysis using this dataset grouped the full *S-PIPS-M* survey sample of women of color into clusters based on components of mathematics identity. Thus, I selected 12 participants to interview that span across these clusters while attending to the characteristics listed above. For this analysis, I focus on two participants, Lyka and Callie. Brief descriptions of Lyka and Callie are presented in Table 1. The participants self-described their gender and racial identities through an open-ended question on the interest survey.

Table 1. Description of participants.

Lyka	Callie
Lyka (she/her) is a Filipino and white cisgender woman. She recently graduated with degrees in Biomedical Engineering and Multidisciplinary Studies from a public university in the Eastern half of the United States. Within the mathematics department at her university, she completed College Algebra, Trigonometry, Calculus I, Calculus II, Multivariable Calculus, and Ordinary Differential Equations.	Callie (she/her) is a Hispanic/Latino cisgender woman. She recently graduated with a degree in Biology from a public university on the East coast. Within the mathematics department at her university, she completed College Algebra, Precalculus, Trigonometry, Statistics, and Calculus I.

Data Collection

In Spring 2022, I conducted 60 - 90 minute, semi-structured virtual interviews with each participant. The interview protocol included questions about participants' experiences and feelings related to undergraduate mathematics, including their perceptions of themselves as doers of math and what they found to be supportive and valuable in mathematics spaces. I divided the interview questions into four categories, three of which were informed by Voigt's definition of mathematical identity and one focused specifically on the potential connections to gender and/or race. This categorization intended to bring forth participants' narratives around their mathematical identity as well as components they felt were important and valuable in their mathematics education. I also asked participants to write a brief mathematics autobiography using an open-ended prompt. The prompt encouraged participants to highlight memorable and influential moments in their life related to mathematics up to and including college, along with any ways in which race and/or gender interacted with these moments. I asked participants to complete the mathematics autobiography prior to their interview.

Data Analysis

To analyze the transcripts, I used Saldaña's (2013) pre-coding, first cycle, and second cycle coding methods. Pre-coding occurred during the interviews and during the transcription process, whereby I noted particularly interesting or powerful statements. During the first cycle I utilized open coding to label any statements related to the research questions, including statements related to cultural values, mathematics education values, and the components of mathematical identity. These codes became the basis of the second cycle coding process which involved categorizing the initial codes into theoretical codes to help describe the overall story connecting the data to the research questions. The coding process is still in progress and thus the results section that follows includes preliminary findings and developing themes.

Results

Initial findings suggest mathematics education values associated with learning from mistakes, connecting to the real-world and future careers, clarity in mathematical explanations, and social relationships, such as with their instructor and peers. There also appears connections between these mathematics education values and their mathematical identity. Intertwined within these mathematics education values and connections to mathematical identity, I also note the role of cultural values, especially related to gender and race. I present here two examples, one from each participant, to exemplify some of these points. Future analysis will provide more detailed and potentially addition results and overarching themes.

Lyka's Example

In her mathematics autobiography, Lyka describes a stigma she perceives in STEM, where "if you mess up that's not only reflective of you as a person, but women as well, the same goes for race." However, she describes how much she valued and enjoyed her mastery-based upper-level calculus courses where she felt that having the opportunity to work through her mistakes made the process "less stressful and made me enjoy doing math. I was able to focus on the meaning of the content." She attributes those courses as the reason she decided to pursue a mathematics minor. This suggests that seeing her mathematics education values reflected in her math courses improved how she felt about math and herself as a doer of math. In Lyka's experience, we see how cultural values around making mistakes in mathematics, and how those interacted with being a woman of color in STEM, were challenged within her mastery-based math courses. When instead the value of making mistakes aligned with her values around learning and improvement, she added a mathematics minor, suggesting a strengthened mathematics identity.

Callie's Example

Callie describes both positive and negative experiences within undergraduate mathematics. She acknowledges how her performance often related to her feelings toward the courses. However, how she describes the importance of social support emerges across these experiences. When detailing her math courses, she recognizes that for most part "you were kinda on your own, and it was up to me… I was really good at forming friends within those classes so we could study together." The role of social support also influenced her perceptions of herself within mathematics and her major. When describing a fellowship she engaged in during college specifically geared toward marginalized students in STEM, she emphasizes how "that fellowship that I had is like a very big reason why I continued to do so well in all these [math] classes… I probably would have changed my major completely if I wasn't in this fellowship." She continues to talk about how the importance of this social support intertwined with her gender and race, "because they did help me a lot with just like the reassurance of like, listen, this isn't a common thing like, you're a woman, you're a minority." Even though Callie describes overall having a negative feeling toward mathematics, having a space that reflected her values around social support and made her feel less alone as a woman of color in STEM provided a positive connection with mathematics and her major.

Discussion

The above preliminary results suggest various values women of color relate to their experiences in mathematics. These include social relationships, such as the importance of peer support and organizational support for Callie, learning from mistakes, such as Lyka described in her mastery-based math courses, connections to the real-world and future careers, and clarity in mathematical explanations. Additional values may emerge as I complete the analysis. There also appears to be a connection between these women of color's values and their mathematical identity. In particular, they both mention the importance of a value-related component in their STEM degree considerations. Future analyses will explore this connection in more detail, including when and in what way values relate to students' mathematical identity. These results also support the use of sociopolitical theory and an intersectional lens given the inclusion of statements regarding the influence of gender and race on their experiences. Once the findings and overall themes are established, I will more robustly consider in what ways these women of color's values connect with previous literature about women and people of color's values and mathematical identity and how these results intertwine with gender, race, and the tensions often present within dominant mathematical systems. The implications of this work aim to consider how women of color's values emerge in mathematical spaces and the role of the university to incorporate those values to better support women of color in undergraduate mathematics.

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The Role of Language in Undergraduate Mathematics: Linear Independence and Limit

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As English is increasingly seen as the language of mathematics instruction in the United States and the world, it becomes important to interrogate the role of the English language in mathematical understanding. This article investigates this question in relation to linear (in)dependence and limit through clinical interviews with undergraduate students. Preliminary results indicated a relationship between students' conceptual understanding of linear (in)dependence and their ability to connect their mathematical and non-mathematical uses of the term (in)dependent. Moreover, students' unproductive conceptions of limit appeared to be influenced by non-mathematical meanings of limit.

Keywords: Linear Algebra, Calculus, Language, Student Thinking, Undergraduate Education

As English is increasingly seen as the language of mathematics instruction in the United States and the world (e.g., Gerber et al., 2005; Planas, 2018), it is crucial to interrogate the role of the English language in mathematical understanding. Prior research suggests that the structure of the English language may be less effective than some other languages, say the Chinese language, for facilitating mathematical performance at primary and secondary school levels (e.g., Han & Ginsburg, 2001). This paper investigates the role of the English language at the undergraduate level in the context of limits and linear independence, as these are two central topics of study in mathematics education (e.g., Larsen et al., 2017; Rasmussen & Wawro, 2017).

Literature Review

Most English mathematics words at the K-12 levels have been created from Greek or Latin (Milligan & Milligan, 1983). For example, the term *quadrilateral* comes from two Latin words: *quadri* which means "four" and *latus* which means "side." Unfortunately, most English-speaking students are not taught Latin or Greek, so many of them may come to see math terms as random combinations of letters with no inherent meaning (Johnston-Wilder et al., 2016). In contrast, Chinese math words appear to have been created as descriptive and conceptually clear compound phrases (Han & Ginsburg, 2001). For example, the Chinese term for *quadrilateral* literally translates into English as *four sides shape*, which clearly depicts the concept of quadrilateral.

Yet, many undergraduate math terms in English such as *linearly independent* already seem conceptually clear to the expert's eye. But are such terms also conceptually clear to the undergraduate learner? Research on linear (in)dependence and limit documents a diversity of student conceptions of these topics. For example, Plaxco and Wawro (2015) analyzed data from clinical interviews with students from a linear algebra course that used a travel-oriented curriculum (Wawro et al., 2012). They identified four student conception categories of linear (in)dependence: travel, geometric, vector algebraic, and matrix algebraic. The travel conception described linear (in)dependence in terms of movement. The geometric conception used language about spatial reasoning or graphical representations. The last two conceptions focused on operations on algebraic representations of vectors and matrices, respectively.

Williams (1991) investigated the following student models of limits: (a) dynamic-practical, (b) acting as a boundary, (c) formal, (d) unreachable, (e) acting as an approximation, and (f) dynamic-practical. He found that it was common for students to exhibit a dynamic view of limit and organized this view into three variants: squeezing a value from both sides, finding approximations, and coordinating changes in x with changes in y. Building on William's (1991) work, Oehrtman (2009) characterized students' spontaneous metaphors for limits and identified five often used student metaphors of limit: as physical limitation, with infinity as number, as proximity, as collapse, and as approximation. He also identified three less often used student metaphors of limit: as motion, as zooming, and as informal version of correct definition. Of all these metaphors, Oehrtman found limit as an approximation to be the most productive.

My study investigates the uses of the terms *limit* and *(in)dependent* in relation to student understanding of limits and linear (in)dependence. Here, student understanding is conceptualized as any or all of the following elements of mathematical knowledge: meanings, images, ideas, connections, ways of comprehending situations, and explanations (Lobato, 2014). With this framework in mind, the research question is: What is the role of language in student understanding of linear (in)dependence and limits?

Methods

To gather background information on student understanding of linear (in)dependence and limits, the research team conducted clinical interviews (Ginsburg, 1997) with two groups of undergraduate students across two universities in the southwest of the United States. One group consisted of 4 students who had recently taken linear algebra and was asked questions about their understanding of linear (in)dependence. The second group had 6 students who were enrolled in Calculus II or III and were asked about their understanding of limits. Student interview responses were transcribed and analyzed. Students' linear (in)dependence conceptions were inferred from the data using open coding from grounded theory (Strauss, 1987). To infer students' conceptions of limits, we used a list of a priori codes from prior research (e.g., Williams, 1991), while allowing room for other codes to emerge from the data within a grounded theory approach (Strauss & Corbin, 1990).

Next, we analyzed the previously identified student understandings of limits and linear (in)dependence, considering the role of language. First, we determined which, if any, student model of limit related to the non-mathematical meaning of *limit.* Then, we analyzed student responses to the interview question regarding why the words *independent* and *dependent* are used for linear (in)dependence. In each response, we determined if the student used meanings of (in)dependence from outside linear algebra to interpret linear (in)dependence. Finally, we analyzed the relationship between each student understanding of linear (in)dependence and their use of outside meanings of (in)dependence.

Results

Background Findings: Student Understanding of Linear (In)dependence and Limits

Three student models of linear (in)dependence emerged: (a) A set of vectors (viewed as a matrix) is linearly independent if and only if it is reducible to the identity matrix; (b) A set of vectors is linearly independent if and only if no two vectors in the set are scalar multiples of each other; and (c) A set of vectors is linearly independent if and only if at least one vector is a scalar multiple of another vector in the set. Student models (a) and (c) were used by one student each (Marco and Arletta, respectively), whereas student model (b) was used by two students (Edward and Alana). While none of these student models was completely correct mathematically, some were more correct and conceptually oriented than others. Student model (a) was more computationally oriented, focused on reducing a matrix through row or column operations. On the other hand, models (b) and (c) highlighted connections between linear (in)dependence and the idea of scalar multiples, with model (b) being more logically correct than (c).

Regarding the limit analysis, each student had multiple conceptions of limit in calculus. We identified eight student models of a function's limit: Coordinating x and y (n=6 students), approaching from both sides (n=5), function value (n=5), continuity (n=3), boundary (n=4), unreachable (n=6), exception (n=2), and stopping point (n=1). The model of limit *coordinating x and y* described a limit in terms of how the function value changes as x approaches a certain value. *Approaching from both sides* described 'limit as x approaches *a*' as the common y-value the graph approaches from the left and from the right of *a*. *Function value* treated the limit value and the function value as the same. *Continuity* viewed the limit's existence as equivalent to the function's continuity. *Boundary* conceptualized a limit as a number or point past which the function cannot go. *Unreachable* conceived of a limit as a number or point that cannot be reached by the function graph. *Exception* viewed the limit as something that does not follow a rule. Finally, *stopping point* conceptualized the function as motion along a graph and viewed the limit as the place where the function is constrained.

The Role of English on Student Understanding of Linear (In)dependence

The results indicated that Edward and Alana, the two students with model (b) of linear (in)dependence, connected their non-mathematical understandings of the terms *independent* and *dependent* to their understanding of linear (in)dependence, whereas the other students (Arletta and Marco) failed to make such a connection.

For example, when I asked Edward, "We've been talking about linearly independent sets of vectors…Why do you think the words 'independent' and 'dependent' are used?," he responded, "Well, dependent means they're *related* to each other. Right? Independent means they're not." Then, when I asked him to elaborate, he said, "Well, dependent means it's related to or reliant upon each other. So, if you have a dependent set of vectors, you can make one vector from another by scaling it. So, they're interconnected, I guess." Based on this excerpt, it appears Edward connected his understanding of the term *dependent* as used outside linear algebra (i.e., "related to or reliant") to his interpretation of that term in linear algebra (i.e., "make a vector from another by scaling it").

Similarly, when Alana was asked the same question, she said:

> Okay, I like that the words are the words they are, because they help me remember the definition. So, when it's dependent– just like, in general the definition of dependence, to me, is when you need something else, or, like, you rely on someone else. So, if you just, like, rely on something or someone– so then, thinking about it mathematically, like– I'll just use this example I've been using. Oops. This [vector] times two gives you this [vector], and so there's that, like, *relationship*. And then independence, to me, in general means completely on your own. So, like, there's no way that these two [vectors] can ever, like, work together and be the same or look the same.

Thus, she used her general knowledge from outside of mathematics that "dependence…is…when you need… or rely on someone else" and that independence means "completely on your own" to help create mathematical interpretations or examples of linear (in)dependence (e.g., "this [vector] times two give you this [vectors]" for dependence and "no way these two [vectors] can ever look the same" for independence).

In contrast, Marco did not leverage his understanding of the words *independent* and *dependent* from outside of mathematics for interpreting linear (in)dependence. When asked the same question above, he responded:

> Yeah, honestly, I've been going through the whole… linear algebra course. I didn't know why it was called independent and dependent. I just thought, well, if it reduces to identity matrix, it's independent. That's all the correlation I got from that.

Hence, Marco claimed that, in interacting with the concept of linear (in)dependence, he had not attended to the meanings of *independent* and *dependent* outside linear algebra. Not connecting his non-mathematical and mathematical understandings of *independent* and *dependent* might have led Marco away from attending to scalar multiples (for interpreting linear (in)dependence) and instead into rote learning.

Like Marco, Arletta initially claimed never connecting her prior linguistic understanding of (in)dependence to linear (in)dependence. However, when I later presented her with a traditional textbook definition of linear (in)dependence, she forcedly made a connection: "The words independent and dependent [are used] because they're depending on whether or not the input values are zero to get a[n] output of zero." Moreover, when I pressed her to exemplify this connection, Arletta responded:

> Um, I guess, just going back to the definition. Like if c_1x_1— Like if c_1, c_2 [the coefficients] were to equal 0, then that means they're [the vectors x_1, x_2] independent… and... in dependent ones [vectors], these values [the coefficients] can be anything, but they don't all have to be zeros.

Although Arletta made an interesting connection between the formal definition of linear (in)dependence and her prior understandings of terms *(in)dependent*, this connection was superficially (not conceptually) aligned with the topic of linear (in)dependence. This is not surprising, given that she admitted not connecting the formal definition to her understanding of linear (in)dependence. This may help explain why, in her model of linear (in)dependence, she labeled a set with at least two vectors which are scalar multiples of each other as linearly independent and a set with no scalar multiples as linearly dependent (which is mathematically reversed).

These results indicate that the students with the strongest conceptual understanding of linear (in)dependence (those with model (b)) were precisely those who connected their mathematical and non-mathematical understandings of the words (*in)dependent* in terms of a *relationship* between vectors.

The Role of English on Student Understanding of Limit

All six students appeared to connect their mathematical and non-mathematical conceptions of limit. All students had at least one mathematical conception of limit that related to the non-mathematical definition of limit as a restraint or limitation – "a measure or condition that keeps something under control or within limits" (2019, n.p.). This non-mathematical definition aligns with the following (mathematically incomplete or incorrect) student models of limit inferred from the data: boundary (n=4), unreachable (n=6), exception (n=2), and stopping point (n=1).

To illustrate the connection between the non-mathematical meaning of limit and student mathematical understanding of limit, consider one student, Joyce, who had a *boundary* model of limit. When Joyce was asked what came to mind when she heard the word *limit*, she said, "When something cannot be something anymore like 'I'm at my limit.'" Then, to elaborate, she sketched a graph and its highest point while saying, "Here's a dot… and there's like a function… and it's like a maximum, but I guess like, that's as high as it goes. Like that's what I'm picturing like it's the limit… like the dot [the highest point on the graph] is the limit." Thus, Joyce related her non-mathematical conception of *limit* as a limitation (e.g., "like I'm at my limit") to her mathematical understanding of limit as a function's "maximum" value (which is mathematically incorrect).

Discussion

Overall, the results indicated that students' non-mathematical language can play an important role in students' mathematical understanding of linear (in)dependence and limits. In particular, it may be that connecting the non-mathematical use of (in)dependence as a *relationship* to the mathematical meaning of linear independence afforded students a deeper understanding of this topic. However, not all students in the study who had taken linear algebra made such a connection, suggesting that there is still room for making the English term *linearly (in)dependent* conceptually clearer to students. On the other hand, the analysis of student understanding of limits suggested that extending non-mathematical understandings of the term *limit* to mathematical interpretations of this term without reflection may also direct students' mathematical understandings astray.

These results highlight the importance for teachers to help students reflect on the terminology used to describe the concepts they are studying. This reflection may encourage students to compare the mathematical and non-mathematical meanings of certain terms. Leading students to see conceptual similarities (when they exist) can allow students to build on their own linguistic resources, whereas encouraging them to see differences can re-direct students away from unproductive mathematical conceptions rooted in limitations of non-mathematical meanings.

For example, a linear algebra instructor could facilitate discussions about the non-mathematical meaning of *dependent* and its connections, not only to the idea of scalar multiples of vectors, but more generally to the concept of linear combinations. This could help students interpret the idea of "a vector being expressible as a linear combination of other vectors" as a relationship in which that vector "depends" or "relies" on the other vectors. Likewise, a calculus instructor could facilitate discussions about the similarities and differences between mathematical and non-mathematical meanings of the term *limit*, so students may confront potentially unproductive conceptions and build productive models of *limit*.

To prompt or enrich such reflections, features of other languages, such as Chinese, could be leveraged. For example, if linear algebra students don't bring up the idea of a "relationship" on their own in class discussion, the instructor could promote such idea by asking students to consider the literal translation of the Chinese term for *linearly dependent* (线性相关: linearly related to each other*)*, which explicitly invokes the idea of a relationship. Similarly, to enhance the discussion about limits in a calculus class, students might be asked to reflect on the constrains of the literal translation of the Chinese name for *limit* (极限: maximum magnitude limit). This literal translation appears to reinforce the mathematically incorrect idea (held by students like Joyce above) that a limit is the maximum value of a function.

Ultimately, whether the multiple meanings embedded in certain mathematical terms support or hinder student mathematical understanding may lie on how students and instructors leverage these meanings. One promising way is for instructors to promote student reflection over the similarities and differences between multiple meanings of important mathematical terms. More research in language and undergraduate mathematics may shed more light on this promise.

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Calculus I Instructors' Use of Representations in Instructional Tasks: Introducing Derivatives with Inquiry

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I present preliminary results from my dissertation research that investigates how eight college Calculus I instructors used representations in an instructional task intended for introducing derivatives with inquiry. Using Zandieh's (2000) framework, the instructors were asked to propose and sequence eight tasks for introducing derivatives to their students. This paper focuses on the instructors' utilization of various representations of derivatives in the first task they proposed in the sequence. The findings show how all instructors found graphical and physical representations appropriate for introducing derivatives, with some of them also reaching out to other representations (symbolic, numerical, and verbal) simultaneously.

Keywords: Instructional tasks, derivatives, representations, inquiry, calculus teaching

Research on the impacts of inquiry teaching suggests positive student outcomes, such as learning and persistence in STEM (Freeman et al., 2014; Kogan & Laursen, 2014; Laursen et al., 2014; Johnson et al., 2019; Rasmussen & Ellis, 2013). It is also known that the implementation of inquiry methods varies considerably among instructors (Laursen & Rasmussen, 2019). Research on the variability of inquiry-oriented teaching in undergraduate mathematics has mostly focused on characterizing and measuring how instructors and students interact in the classroom (e.g., Laursen et al., 2011; Mesa et al., 2020; Shultz, 2020). While showing the range of pedagogical variability in these classrooms is necessary for understanding inquiry teaching, this strand of research overlooks the content instructors and students interact with.

In my dissertation study, I investigate how college calculus I instructors, who use inquiry in their teaching, shape students' mathematical work during instruction of one specific piece of content: derivatives. As one of the basic and core concepts in calculus, derivatives' multiple representations and conceptualizations make its teaching a rich area for investigation. I bring my attention to the content during instruction by looking into the instructional tasks and their sequencing that instructors use to introduce derivatives to their students. For this preliminary report, I focus on instructors' use of mathematical representations, specifically addressing the following research questions:

1. Which representations do college Calculus I instructors, who teach with various inquiry approaches, use within instructional tasks to introduce derivatives with?
2. What patterns exist, if any, in their use of representations in instructional tasks for introducing derivatives?

Research on Representations of the Concept of Derivative

Research on representations of the concept of derivative has mostly attended to student understanding via using various representations, making connections among representations, and extending student understanding to applications of derivatives in other contexts (e.g., Feudel & Biehler, 2021; Jones & Watson, 2018; Maharaj, 2013; Roorda et al., 2007; Siyepu, 2013). Despite this enduring research on students' understandings of the derivative, there is less research on the teaching of derivatives through using representations. The scarce research in this area focuses on the student outcomes of instructional treatments. Kendal and Stacy (2000) compared high school students' competency in derivative representations from two classes. Although both teachers used identical curricular materials, students from the two classes became proficient in different representations (graphical versus symbolic) based on their respective teachers' emphasis on one representation. The authors claimed that "students had different cognitive experiences and acquired different differentiation competencies which related directly to the ways they were taught" (p. 133). Similarly, Hähkiöniemi (2006) carried out a teaching period based on the APOS theory, during which he emphasized visual and symbolic representations. He agreed with Kendal and Stacy's (2000) claim, as he found that students could recognize derivative visually early on and calculate derivative at a point symbolically. Hähkiöniemi (2006) concluded that visual representations are a good fit for introducing and developing understanding of derivatives. Also using the APOS theory, Borji et al. (2018) designed a teaching intervention focused on graphical representations. While both the experiment and the control groups spent equal time on the concept of derivative, the students in the experiment group spent some time using a programming software with graphing facilities, and later performed better on a written tests on derivatives. In another study, instead of emphasizing visual and symbolic representations, Dwirahayu et al. (2017) prioritized physical representations, specifically analogies to the speed of a vehicle, to introduce derivatives to students. Compared to the control group, the authors showed that teaching with analogies resulted in higher student understanding of other representations of the derivative.

These interventions are promising in explicating the relationship between teaching and learning of derivatives using different representations. However, the research has yet to investigate how instructors, outside of interventions, think about their use of representations for teaching derivatives and how they design and sequence their instructional tasks based on their insights about student conceptions of derivatives.

Theoretical Framework

In this study, I use Zandieh's (2000) framework (Table 1) to: 1) ground instructors in both the knowledge at stake and students' conceptions of derivative; and 2) organize instructors' use of representations when proposing tasks for teaching derivatives with inquiry.

Table 1. Zandieh's (2000) adapted framework for the concepts of derivative.

		Representations			
		Graphical (Slope)	Verbal (Rate)	Physical (Velocity)	Symbolic (Difference Quotient)
Process-Object Layer	Ratio	Slope of the secant line	Average rate of change	Average velocity	Difference quotient
	Limit	Slope of the tangent line	Instantaneous rate of change	Instantaneous velocity	Limit of the difference quotient
	Function	Graph of the derivative function	Rate of change of a function	Velocity as a function of time	Derivative as a function

Zandieh (2000) organized students' conceptions by representation (graphical, verbal, physical, symbolic) and process-object layers (ratio, limit, function). The process-object layers are hierarchical, as each layer is found by taking the process of that layer over the previous layer as an object. For example, the limit layer is found by the *process of* finding the limit of the ratio as an *object*. Although not originally captured in the framework, Likwambe and Christiansen (2008) and Roundy et al. (2015) added two new representations: numerical and instrumental.

The numerical representation of the derivative corresponds to the ratio of change ($\frac{y_2-y_1}{x_2-x_1}$) is written with values, often with a table. The limit layer corresponds to when the denominator approaches zero; the function layer is presented as an array of numbers or set of ordered pairs of differences. The instrumental representation of the derivative is the rules of differentiation and lives only at the function layer.

Methods

This qualitative study uses data collected through interviews with eight college calculus instructors who were purposefully selected. To select the interviewees, I administered a survey, including the INQUIRE instrument (The INQUiry-Oriented Instructor REview; Shultz, 2020), to 48 college instructors who teach Calculus I with inquiry. I then used INQUIRE to purposefully select eight interviewees with different patterns of inquiry-oriented practices in the classroom. The final interview sample included eight participants (all pseudonyms): Justin, Adrian, Barry, Matthew, Gopher (he/him pronouns); Monica (she/her pronouns); Max (they/them pronouns); and Alex (all pronouns). All participants were tenured with more than four semesters experience teaching Calculus I with inquiry. I conducted four remote semi-structured interviews (1-2 hour long each; Figure 1) with each instructor, as they proposed up to eight tasks. Instructors could use their teaching materials to choose tasks they had used or to create new ones.

Using Zandieh's (2000) framework, I dedicated an interview to each representation. During each interview, the instructors were given two prompts and were asked to propose a task for each prompt that would transition students' conceptions from one layer (as pre-conception) to the next (as post-conception, see Figure 1). For example, as Prompt 1, participants were asked to propose a task situated within the physical representation of derivative that would help transition students' conceptions from the ratio to the limit layer (without the framework's language):

> Assume that you have already taught students about average velocity and want students to learn about instantaneous velocity at a point. Propose a task with the position of an object varying as a function of time, where students figure out the velocity of the object at a given moment.

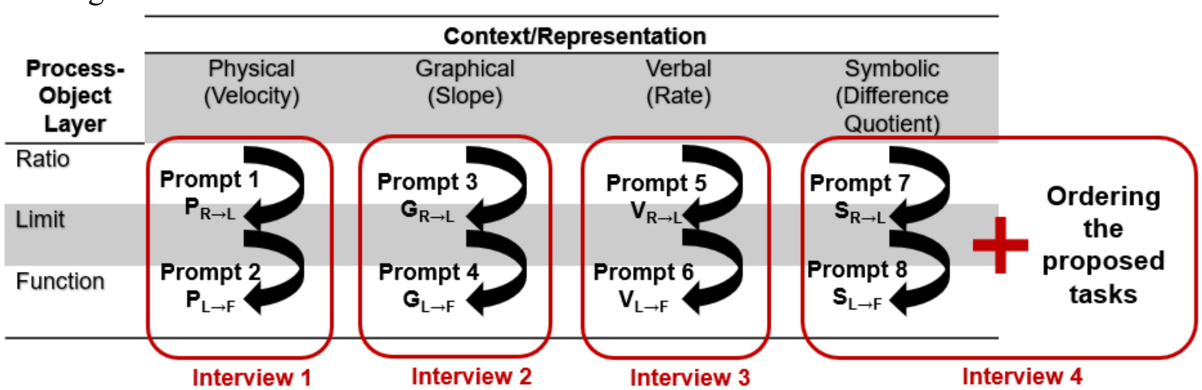

Figure 1. Structure of the interviews using Zandieh's (2000) framework. The code under a prompt, $X_{A\to B}$, gives: the representation X (P = Physical, G = Graphical, V = Verbal, and S = Symbolic), and the process-object layers A and B (R = Ratio, L = Limit, and F = Function). The arrow between the layers indicates the layer's precedence.

At the end of the final interview, I presented each instructor with all their proposed tasks and asked them to order them for introducing derivatives using inquiry. For the analysis reported here, I identified the representations and layers used within the proposed tasks without looking at the prompts' representation/layer code (those in Figure 1). I also coded representations of the derivative not in Zandieh's (2000) original framework. After coding, I listened to the interviews and read the transcripts to identify representations/layers that were mentioned but not written down in the task, as many instructors did not have enough time to write down all task details.

Findings

Due to space limitations, I present the representations used in the instructional task that instructors chose to use *first* when introducing derivatives to their students with inquiry. Because at the end of the final interview and after proposing eight tasks, the instructors re-arranged their proposed tasks in their preferred sequence for teaching derivatives, each instructor's *first task* is not necessarily the product of the *first prompt*. Figure 2 shows the prompts that generated each instructor's first task and the representations and layers identified in each. As their first task, the instructors chose to either use the task proposed for Prompt 1 (physical representation from the ratio to the limit layer) or Prompt 3 (graphical representation from the ratio to the limit layer). As shown in Figure 2's "Other reps/layers row", Gopher, Max, Barry, and Matthew reached out to representations and layers not mentioned in the prompts.

<table>
<tr><td>Prompt</td><td>Prompt(s) with reps/layers given</td><td colspan="4">Prompt 1
P: R → L</td><td colspan="2">Prompt 3
G: R → L</td><td>Prompts 1 and 3
P + G: R → L</td></tr>
<tr><td rowspan="2">First Task</td><td>Task's reps/layers used from prompt</td><td colspan="3">$P_{R/L}$</td><td rowspan="2">P_R</td><td rowspan="2">$G_{R/L}$</td><td>G_L</td><td>$P_{R/L} + G_{R/L}$</td></tr>
<tr><td>Other reps/layers</td><td></td><td>$G_{R/L}$</td><td>$N_{R/L} + S_L$</td><td>$G_F + S_{R/L}$</td><td>$V_{R/L} + S_{R/L}$</td></tr>
<tr><td></td><td>Participants</td><td>Alex
Monica</td><td>Gopher</td><td>Max</td><td>Adrian</td><td>Justin</td><td>Barry</td><td>Matthew</td></tr>
</table>

Figure 2. Derivative's representations and layers in the prompt that generated each instructor's first task, and representations and layers identified in their first tasks. Reps/layer is short for representations and layers. Each task has rep/layer codes similar to those in Figure 1. The task's layers are separated with slash (/) instead of arrows (→) here because I do not claim that the tasks necessarily transfer students' conception from one layer to another.

Alex, Monica, and Gopher provided similar tasks with tables of data for the position of a falling objects at various times and asked students to find the average velocity for various intervals. They then asked their students to find an estimate for the velocity at somewhere arbitrary between the existing data points. Gopher also used the graphical representation by defining secant and tangent lines, asking students to draw them on the curve of best fit for the data in Excel and finding connections between the lines' slopes and the velocities.

While Adrian and Max started their tasks similarly by providing the equation of the position of an object as a function of time, Adrian's task stayed within the ratio layer, as it asked students to only explain the physical meaning of some ratios (e.g., $\frac{R(5)-R(3)}{5-3}$). Max however provided multiple tables with times approaching $t = 5$ (e.g., 4.5, 4.75, 4.875) and asked students to complete the tables with average speed between the given times and $t = 5$. Max then asked students to estimate and write a formula for the object's speed at $t = 5$, thus using the limit layer of multiple representations (numerical and symbolic).

Looking at all representations used by instructors in their first task, everyone, except Justin, used the physical representation for introducing derivatives to their students, either encouraged by the prompt or by choice. Staying within the graphical representation only, Justin used an interactive GeoGebra applet to ask his students to find the slope of a secant line between two points on $f(x) = x^2$, as the distance between the points decreased. He then asked students to calculate the slope if Δx is zero, see what happens, and think about what the calculations reminded them of. Being cognizant of not using the physical representation, Justin explained that he does not use velocity "as the gateway application to move to calculus" because students get trapped in thinking that "a derivative is always a velocity rather than a velocity is always a derivative" (Interview 1).

Like Justin, Barry also chose the task he proposed for prompt 3 as his first task. Using a variety of functions and without mentioning secants, Barry asked students to find slope of tangent lines at various points. He then asked students to consider 2^x and plot $\frac{2^{x+h}-2^x}{h}$ in Mathematica with h getting smaller, ending the task by providing the limit definition of the derivative. As the only instructor to do so, Barry did not include the ratio layer in his task, but used the function layer.

Lastly, Matthew was a special case in the sense that he proposed the same task for both Prompt 1 and Prompt 3, saying that he wants students to "associate graphical representations to various parts of their work in the context of distance, velocity, time" early on (Interview 2). He also was the only instructor that used all four original representations (within ratio and limit layers) in his first task. Starting the task with the equation of the height of a bolt fired from a crossbow as a function of time, Matthew proposed a series of subtasks that asked students to: approximate the speed for a specific *t*, explain their approximation "using words only" (verbal), use the limit notation to write their findings, and identify secant and tangent lines on the graph of the function and find their slopes. Wanting students to become aware of the four main representations of the derivative, Matthew said that the task makes them "repeat the same set of questions over and over in different representations" (Interview 1).

Discussion

The findings show that the instructors in this study found graphical and physical representations, within ratio and limit layers, appropriate for introducing derivatives. These results align with Hähkiöniemi's (2006) and Dwirahayu et al.'s (2017) suggestions of using these representations early on. Nonetheless, half of the instructors also used other representations simultaneously. Moreover, while most instructor's use of the representations reflected the hierarchical order of layers in Zandieh's (2000) framework, one instructor (Barry) chose to not teach in that order. Further inquiry is needed to understand the ways in which calculus instructors utilize representations, separately and in connection to one another, at various conceptual levels to teach derivatives.

The study extends our understanding of inquiry teaching by attending to the content, as opposed to pedagogy, through their use of instructional tasks. As more mathematics faculty implement inquiry-based methods, it is essential that we understand their decision-making at the content level so that their specific needs can be met through curriculum or professional development initiatives. By attending to instructional tasks, we also gain a better understanding of what is actually offered to students during instruction and how it might impact their learning.

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How Do Postsecondary Linear Algebra Instructors Implementing Inquiry-Oriented Approaches Address Goals of Instruction in an Online Work Group?

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Postsecondary instructors interested in inquiry-oriented instruction of Linear Algebra participated in a sequence of eight one-hour online work group meetings with other experienced inquiry-oriented linear algebra facilitators and teachers. Recordings from three meetings were analyzed for how two participants referenced goals of instruction in preparation for teaching a new instructional unit on subspaces. We identified four goals of instruction of teaching subspaces. We discuss the intersections of several goals of instruction and possible implications for those who want to transition to inquiry oriented instructional approaches.

Keywords: goals of instruction, inquiry-oriented instruction, online work group

Linear Algebra is a postsecondary mathematics course that is typically required of all mathematics and many STEM (Science, Technology, Engineering, and Mathematics) majors. The topics learned in this course are foundational for many other math courses. Tucker (1993) and Carlson et al. (1993) point out the importance of linear algebra in addressing not only the theoretical and practical aspects but also its applicability in modern computation in industrial science, technology, computer science, engineering, and economics.

The number of students required to take Linear Algebra provides motivation to search for ways to improve instruction. Active learning in undergraduate STEM education is linked to increased student performance, and some literature suggests it may be linked to more equitable outcomes (Burke et al., 2020; Freeman et al., 2014; Haak et al., 2011; Kogan & Larsen, 2014). Inquiry-oriented Linear Algebra (IOLA) is one approach to active learning in undergraduate mathematics education. Inquiry-oriented instruction derives from Realistic Mathematics Education (RME) with the intent to support students in reinventing key mathematical concepts (Freudenthal, 1991; Kelley & Johnson, 2022). To support the students in these courses with an engaging learning experience, teachers need to be supported (Andrews-Larson et al., 2019) because changing instructional approaches is difficult and long-lasting change requires shifts in instructional practice (Cohen, 1996; Henderson et al., 2011). Therefore, there is "a need for professional development programs that foster the development of undergraduate mathematics instructors' pedagogical reasoning" (Andrews-Larson et al., 2019, p. 129)

Thus, an online work group (OWG) created as part of a larger project to support college mathematics instructors teaching Linear Algebra, gave instructors an opportunity to collaborate with others interested in continuing their pursuit to enact reform-oriented instructional practices. In the OWG of this study, participants with inquiry-oriented instruction (IOI) experience work on implementation of a newly developed set of IOLA tasks and discuss their teaching practices with researchers and facilitators of the OWG.

In this analysis, we focus on the two undergraduate instructors who are not researchers or facilitators of the larger project because this enables us to examine how experienced IOI instructors discuss content and related pedagogical decisions they make before and during instruction. Their discussions can highlight how teachers' knowledge shapes teachers' practices (Borko & Putnam, 1996), in particular practices related to the preparation of a lesson. One of those practices is attending to goals of instruction (Wagner et al., 2007) which is the focus of our study. The practical goal (Maxwell, 2013) of this research is to continue the conversation initiated by Wagner and colleagues, whose focus was on a novice instructor's experience with the inquiry-oriented instructional tools, and extend this to a discussion including experienced IOLA instructors. This research will aid in understanding the reasoning of experienced IOLA instructors, particularly regarding their instructional goals and the challenges of IOLA curriculum implementation. Our research question is "What is the nature of the goals of instruction articulated by experienced Inquiry-Oriented Linear Algebra (IOLA) instructors as they work to implement a new instructional IOLA unit in the context of the Online Work Group (OWG)?"

Theoretical Framework

Wagner et al. (2007) identified four categories related to instructional goals in the context of inquiry-oriented instruction at the undergraduate level: classroom orchestration goals, cognitive/learning goals, assessment goals, and content goals. The four goals emerged as a result of continuous work on studying teacher knowledge, such as pedagogical content knowledge (Shulman, 1986) done by Wagner et al. (2007) that focused on teacher knowledge shaping teacher practices through analyzing data from a novice inquiry-oriented instructor. The goals of instruction are signified through questions that instructors ask in their "transition" to teaching in reform-oriented ways. While instructors are in transition, each goal is evidenced by the, but not limited to, following questions:

1. Classroom orchestration goals: "How does mathematics emerge in the classroom? What is the instructor's role or students' role?" (p. 257)
2. Cognitive/Learning goals: "What supports learning? Is there a correspondence between what students are learning and what the teachers want them to learn?" (p. 259)
3. Assessment goals: "What is evidence of learning? What determines pace?" (p. 261)
4. Content goals: "Specifically what mathematics ought to be covered?" (p. 263)

Using the work from Wagner and colleagues as a priori scheme, this proposal will work to identify the ways in which these kinds of goals were conceptualized by experienced IOLA instructors in the online work group.

Study Context: Inquiry-Oriented Linear Algebra and Online Work Group

Inquiry-Oriented Linear Algebra (IOLA) is a design-based research project in linear algebra education as one approach to active learning (Freeman et al., 2014) and inquiry-based mathematics education (Laursen & Rasmussen, 2019). Laursen and Rasmussen (2019) describe inquiry-based mathematics education as "student engagement in meaningful mathematics, student collaboration for sensemaking, instructor inquiry into student thinking, and equitable instructional practice to include all in rigorous mathematical learning and mathematical identity-building" (p. 140). Specifically, the IOLA-X project focuses on developing student materials composed of challenging and coherent task sequences that facilitate an inquiry-oriented approach to the teaching and learning of linear algebra (Wawro et al., 2013). There are five main phases in the Design Research Spiral: *Design*, *Paired Teaching Experiment* (*PTE*), *Classroom Teaching Experiment* (*CTE*), *Online Work Group* (*OWG*), and *Web* (Wawro et al., in press). The participants of our study come from the OWG in phase 4 of the research project.

The main purpose of the OWG for the IOLA research team is to learn from instructors how IOLA is implemented in various classrooms with various student populations and to gain insights to develop instructor notes and revise tasks (Wawro et al., in press). At the center of the

OWG for this study was the discussion of a unit of tasks about subspaces; the tasks were contextualized in a problem about students walking in one-way hallways past cameras monitoring their traffic (See Figure 1). To draw out the feedback from the instructors, the facilitators manage mathematical discussions about the tasks as well as facilitate discourse about the preparation and implementation of the tasks. Through examining discussion and input from the experienced undergraduate instructors participating in the OWG, questions and thoughts about the goals of instruction and challenges with implementation naturally arise.

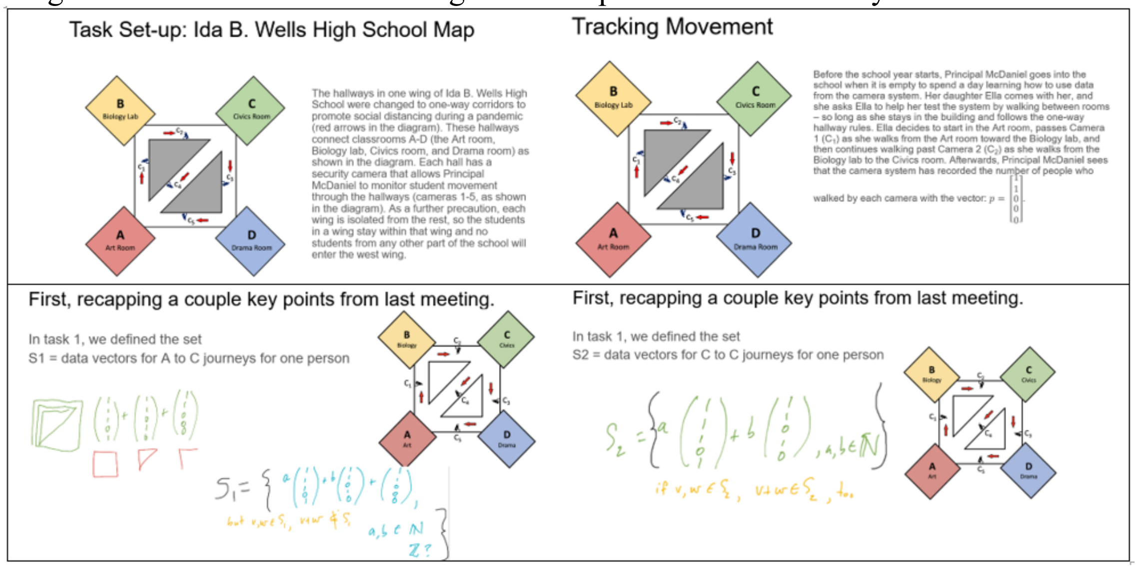

Figure 1. The inquiry-oriented linear algebra tasks in the subspaces unit

Methods

Our primary data source was the recorded videos of the OWG meetings (held and recorded via Zoom); group artifacts such as Google Slides and Jamboards served as secondary data sources. The OWG meetings were held in the Spring 2022 semester. During the first three meetings, participants worked through and discussed the mathematical progression of a new IOLA unit comprised of three tasks. The subspaces unit focused primarily on notions of closure of sets of vectors under vector addition and scalar multiplication, and on null and column spaces. Participants worked on the mathematical problems as a group, then discussed mathematical goals, approaches, and links to other ideas and topics. The remainder of the meetings took place throughout participants' implementation of the sequence, with each participant reporting on how the implementation went, what they liked, how their students reasoned about tasks, what they would change, and what they would do differently.

In the OWG meetings, there were six members. In this proposal, the members are coded as F1 and F2 (facilitators), R1 and R2 (IOLA researchers), and I1 and I2 ("pure" participants who are experienced inquiry-based instructors but not IOLA researchers). This team involved one graduate student (F2), one instructor (I2), three associate professors (F1, R1, and I1), and one full professor (R2). I2 taught linear algebra at a large public university in the Northeastern United States at the time of the OWG; and I1 taught the same course at a small private college in the Northwestern United States. The participation and contributions of I1 and I2 serve as the focus of our study.

Eight hours total of OWG meetings were conducted and recorded over several days. We analyzed transcripts of the first three videos. Narrowing these three videos allowed us to examine the ways in which participants worked through the mathematical tasks and identify the instructional issues that participants anticipated in their implementation. The videos were transcribed by Otter, an online artificially intelligent transcription application. Both authors separately coded all three videos for the four goals of instruction for all the participants of the OWG using Nvivo software. While coding, we assigned four codes, which mean four goals, at the level of a single turn of talk. Then, we compared codes to reach agreements to build inter-rater reliability. We identified common themes within each code in terms of broad questions and agendas that the set of codes seemed to address. Our findings highlight the themes in which each goal of instruction emerged from the two "pure" participants in our data set. Figure 2 summarizes these findings. Each goal of instruction is, not necessarily, but preferably described in question form in order to follow the original form of goals of our theoretical framework.

Findings

Generally, in the OWG meetings, the pure participants discussed classroom orchestration goals involving how to manage discussions of contextualized tasks about "closure". Also, cognitive/learning goals discussed in the OWG meetings are contextualized goals in subspaces and communication in engaging in IOLA tasks. Assessment goals are discussed in terms of pacing, timing, and grading of inquiry-based teaching. Lastly, content goals include curricular trajectories and mathematical content relevant to subspaces reorganized by instructors.

The participants talked about how to orchestrate the discussion of closure under scalar multiplication, considering irrational and complex scalars in the context of the IOLA task.

I1: We had a conversation we were talking about... here's a bunch of examples, what's the span of each of these sets? And, and I thought it was superfluous, but it turned out not to be. The last one I asked was span of the vector, <e, pi squared>. And that really freaked them out. And I was glad that I included [it] because we had already talked about magic carpet ride. Any real scalars are fine. If you want complex scalars, that's a different class, but any real numbers, great. And they seemed okay with that. But no, it was clear, we're going to have that conversation again.

From this excerpt, the instructor is unpacking a class interaction they have had before this OWG meeting where they introduce complex components of vectors in another IOLA task (magic carpet ride). I1 determined that interaction was relevant to the future implementation of the subspaces task because the idea of complex scalars will spark "that conversation again". That demonstrates the instructor's reference to orchestration goals, but also content and cognitive/learning goals. The instructor is connecting mathematical concepts across the IOLA tasks and using the tasks as an opportunity for elaboration beyond the initial exploration.

Furthermore, the participants shared their thoughts on student reasoning that may happen in class discussions and the contextualized goals in subspaces and sets of scalars, coded as cognitive/learning goals.

I2: And, but I think S2 is closed. […] Is it closed under scalars? Scalar multiplication of non-integer or negative scalars? I guess [it] depends on, do we require this to be walkable by number of persons? Right? Or does it just have to fit set description?

As Figure 1 illustrates, S2 is a set of camera vectors that pass through hallways from room C to room C. In this context, scalars mean the number of loops one person creates through one big square loop (through rooms A-B-C-D) and one small triangle loop (through rooms A-B-C). In the contextualized ways, S2 is closed under scalar multiplication but as I2 mentioned, the scalars depend on how much we address that real-life context. The non-integers or negative scalars cannot be defined in the contextualized goals, but this should be clarified to move on to the purely abstract mathematical definition of closure. During this conversation, participants discussed how they manage their trajectories of teaching linear algebra, how to address contextualized mathematical goals with IOLA tasks, and how they orchestrate discussions.

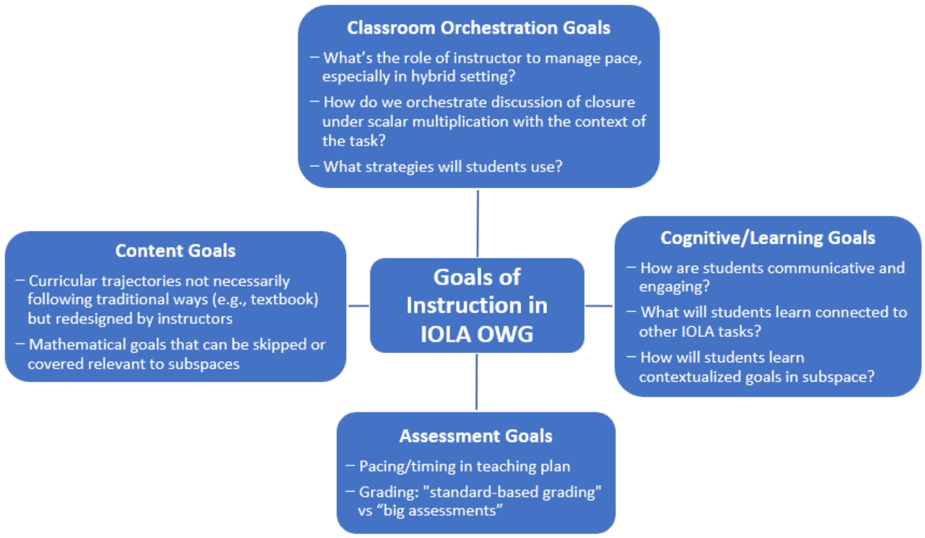

Figure 2. Goals of instruction in inquiry-oriented linear algebra online work group

Discussion

The two participants of our study asked questions and made comments in reference to all four goals of instruction introduced by Wagner et al. (2007). However, some dialogue from the participants were coded as referencing two or more goals. That made clear a separation of assessment goals from the other three goals. The results from the three goals - content, classroom orchestration, and cognitive/learning goals - show an overlap exposed by the instructors. The intersection of those goals may exist due to the contextualized nature of the IOLA tasks that the instructors work through in the OWG. In the task, scalars are constrained due to the context. Therefore, the scalars in the task would not be defined as scalars abstractly because other real numbers (like negative integers) could not be included. The instructors recognized the scope of the tasks and made plans to use the tasks as an opportunity to launch into more (abstract?) concepts chosen by the instructor and guided by student contributions.

On top of referencing goals of instruction as the novice IO instructor did in Wagner and colleagues' (2007) investigation, the experienced IOLA instructors of the OWG were able to expose tensions in instruction and mathematics that goes beyond attending to goals of instruction. The IOLA instructors addressed challenges of teasing through the tension between the aim of Realistic Mathematics Education (RME) and what mathematics is pursued in traditional linear algebra classrooms. This tension can be something to consider for new and transitioning inquiry oriented mathematical instructors.

Acknowledgments

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Successes and Challenges in Supporting Calculus Students to Conceptualize and Express Distance on Graphs

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Jana Turner, Rhodes College

Conceptualizing and expressing distances on graphs of functions are central to understanding why integrals afford measuring areas and volumes in calculus. In this preliminary report, we share the results of implementing an intervention designed to support these skills with 31 Integral Calculus students. Initial results show that the activity did support some students in learning to accurately expressing the horizontal distance between two functions. However, a closer look at students' work revealed that the tasks may not have supported all of the ways of reasoning we intended. We discuss the instance of two students who successfully completed the activity yet displayed evidence of treating symbols as labels or parameters and following a pattern to complete the tasks rather than conceptualizing distance. We close with a discussion of how to improve the next iteration of the intervention.

Keywords: Graphical Representations, Calculus, Distance, Cartesian Coordinate System

One of the key results of Integral Calculus is the ability to find the area of curved regions and volumes of solids of revolution. In order for students to understand conceptually why the integral affords finding the value of such areas and volumes, students must be able to: 1) conceptualize distances within coordinate systems and 2) describe these distances with algebraic expressions from relevant functions. Prior research suggests that students, even at the undergraduate level, often struggle with one or both of these skills (David, 2019; Knuth, 2000; Parr, 2021), and may benefit from targeted interventions in these areas.

Previously, Parr et al. (2021) described the results of a small teaching experiment (Steffe & Thompson, 2000) that utilized a sequence of tasks designed to support students in conceptualizing and expressing distances algebraically, as part of a hypothetical learning trajectory (Simon & Tzur, 2004). Parr et al. (2021) reported that the tasks showed promise at supporting students and also clarified ways of reasoning that students may use when describing distances of segments. One of these interpretations is what Parr et al. (2021) refer to as a *composed magnitude interpretation* of distances on number lines. This interpretation involves conceptualizing a difference of the form $b–a$ (where $b \geq a$) as representing the distance in one-dimension between a and b. The interpretation of the difference $b–a$ is one of a *magnitude*, or amount of distance, within a graphical representation (Parr, 2021). The interpretation is *composed* as it includes the understanding that b and a themselves are magnitudes of distance. Thus, the difference $b–a$ yields the distance between a and b as a result of finding the difference between the distance between b and the origin, and the distance between a and the origin.

The current study is part of an ongoing effort to learn how to support students in conceptualizing and expressing distances for calculus. In this study, we investigate the effectiveness of implementing a revised set of these tasks in a classroom setting. The research questions this study seeks to answer are:
(1) *To what extent do these tasks support students in learning to conceptualize and express distances on graphs of functions in calculus in terms of the input and output variables?*
(2) *What are the affordances and limitations of these tasks in supporting students to develop a composed magnitude interpretation of distances?*

Methods

The data in this study were collected in Spring 2022 from 31 students enrolled in one of two sections of Integral Calculus taught by the lead author. The Integral Calculus course met three times a week and students participated in the study during two consecutive course meetings, ahead of the unit on Areas and Volumes. At this point in the course, students had been introduced to integrals through an accumulation model, without graphical applications. At the end of the first day of the study, students were given ten minutes to complete the pre-test item (Figure 1) individually. The next class meeting, students were instructed to work in groups of two or three to complete the *Interpreting Graphs for Calculus Activity* over the 50-minute meeting time, which included the same pre-test item as the final task. The instructor circulated groups answering clarification questions about the tasks as needed. Students recorded their group's activity via Zoom. Of the 31 students who consented to have their written work included in the study, 26 also consented to include their group's video recording in the data set.

Pre-Test Item & Interpreting Graphs for Calculus Activity

The pre-test item and the tasks comprising the *Interpreting Graphs for Calculus Activity* used in this study were a revised version of the tasks first described in Parr et al. (2021). The pre-test item (Figure 1) asked students to represent the length of the horizontal segment extending from a point labeled (x, y) on the function $y = \sqrt{x-1}$ to the vertical line $x = 2$ in terms of x, in terms of y, and to explain their reasoning for each response.

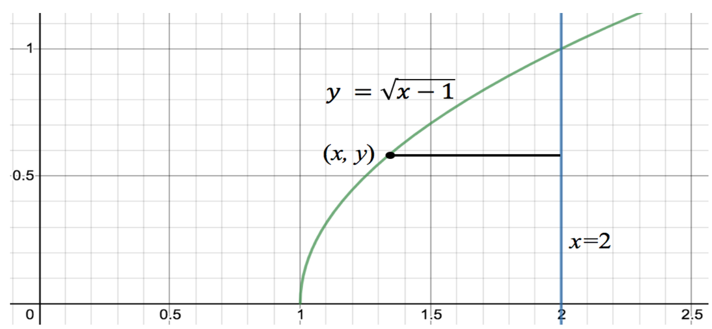
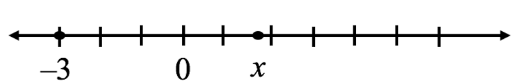


Figure 1. Pre-test item asking students to represent horizontal segment's length (black) in terms of x and y.

The three main goals of the *Interpreting Graphs for Calculus Activity* are to support students in (a) conceptualizing horizontal and vertical distances on number lines and the Cartesian plane, (b) representing these distances with difference expressions, and (c) using algebraic relationships among input and output variables to express distances flexibly in terms of either x or y. In this study, we focus on the first two goals and associated tasks. The first task asked students to use a ruler to mark specified distances from 0 on a provided number line, as well as distances between two values. The next set of tasks asked students to represent various difference expressions as segments, involving whole numbers, integers, rational numbers, and variables as distances on both horizontal (Task 2-Figure 2) and vertical number lines (Task 3).

h. Draw a segment to represent x– (–3) on the x-axis below.



i. Place another value of x on the number line above, and draw another segment representing x– (–3).

Figure 2. Portion of Task 2 from Interpreting Graphs for Calculus Activity.

Task 4 reversed this and provided students with horizontal or vertical segments, asking students to express the length of the given segment as a difference. Finally, Task 8, which we refer to as the post-test item, included the same items as the pre-test item to serve as a comparison for student responses before and after completing the activity.

Preliminary Data Analysis

To analyze our results, we first recorded students' responses to the pre-test and post-test items and developed basic categories for these responses, including correct responses, blank/I don't know, or other expressions. Based upon a student's response to the pre-test and post-test items, each research member selected 3-4 students who did not initially answer the pre-test item correctly and developed initial content logs and analytic memos of students' responses to the Interpreting Graphs for Calculus Activity and group videos of the students completing the activity. The goals of this initial analysis of students' work on the activity was to determine if we could identify shifts in the student's thinking about the tasks that may have supported or hindered them in responding to the post-test item correctly, especially related to their development of a composed magnitude interpretation of distances.

Results

The number of students who correctly represented the segment's length in terms of x and y increased significantly from the pre-test to the post-test item on the graphing activity (Table 1).

Table 1. Number of correct responses on the pre-test and post-test item.

Represent segment's length	Pre-Test	Post-Test
in terms of x: $2–x$	12/31 (38.7%)	24/31 (77.4%)
in terms of y: $2–(y^2+1)$	10/31 (32.3%)	20/31 (64.5%)
Both x and y correct	7/31 (22.5%)	20/31 (64.5%)

While the activity appears to be generally effective in its goal, our ongoing data analysis revealed that some students were limited in their use of variables to express lengths of segments, or may have avoided conceptualizing difference expressions as finding distances, even when they gave correct responses on the post-test items. (We note also that the post-test item was completed in groups and may not reflect each individual's understanding). These findings suggest that the activity may not promote some of the central ways of reasoning that we intended for students to develop and use. We illustrate these two issues using the case of two students who were partners for the in-class activity, Kevin and Amir. Kevin and Amir both incorrectly responded to the pre-test item but correctly expressed the horizontal distance in terms of x and y on the post-test item.

Symbols as Labels or Parameters

Both sets of responses from Kevin and Amir indicate that they tended to use symbols (such as x, y, and z) as either labels or parameters, rather than as variables to represent varying values. We found evidence of this tendency beginning with Amir's work on Task 1 in which he labeled segments with symbols x, y, and z that he introduced himself. This tendency was most apparent when it created some hesitation for Amir on Task 2i. Kevin asked Amir about this task and their discussion reveals the limitations of Amir's use of x. We provide their conversation below and Amir's work in Figure 3.

Amir: (*reading 2i*) Oh, so like place a value for x and just draw it, x equals 2…

Kevin: What is it asking, again?
Amir: Place another value of x, like 1, 2, 0, why did I put 0?... It doesn't make sense to place another value of x. I mean even if you put x over here, so x_0 equals x_1, it's the same thing.
Kevin: Ah (*leaning back*)
Amir: I mean, I like the zero thing.
Kevin: So, wait, what are you doing? (*looking at Amir's work*)
Amir: I'm saying x equals 0, I'm setting
Kevin: Okay
Amir: the segment from 0
Kevin: Yeah
Amir: I don't know.

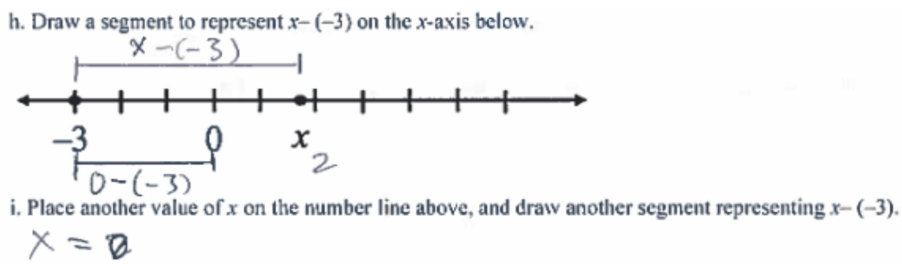


Figure 3. Amir's work, in which he wrote "x= 0," and then labeled "0–(–3)" on a segment below the number line.

Amir explained his interpretation of the directions to "place another value of x" as choosing a value for x, when he listed values "like 1, 2, 0." He goes as far to say that "it doesn't make sense to place another value of x" and continues to then describe different possible x's as "x_0" and "x_1." We infer from these comments that Amir is interpreting x as a parameter (Thompson & Carlson, 2017), in which x may take on different values in different settings, but may not take on different values in the same setting. This interpretation is distinct from interpreting x as a variable in which x may take on multiple varying values in the same context. Amir's interpretation of x as a parameter may help to explain why he thought that it didn't "make sense to place another value of x." Amir's interpretation of x as a parameter is also supported by his written work on this task, in which he sets x to 0, but then he labels the new segment as "0–(–3)" rather than "x–(–3)" as directed, not comfortable with using x to take on two different values in the same number line.

Following a Pattern for Expressing Distances

In addition to treating symbols as labels and parameters, Kevin and Amir also indicate that they may be simply following a pattern in order to express the indicated distances, rather than using a composed magnitude interpretation of distances. Specifically, this pattern involved using a minus sign because previous answers they found were also difference expressions. This issue came to light as Kevin suggested an alternative answer to 4c (Figure 4), a task which asked students to represent the distance between 5 and y on a vertical number line.

Amir: See directions matter. So now like, this above, 5 minus y and this is y minus
Kevin: Yeah. (*pause*) I guess you could say that it's also y plus 5.
Amir: y plus 5 (*slowly, thinking*). Well, yeah, you can say that, too. But because it's differences. Here it doesn't say differences, here it does.
Kevin: I'm just going to use minus for consistency.
Amir: Yeah

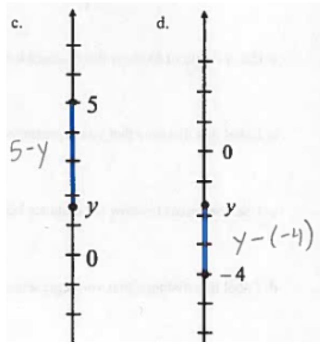


Figure 4. Kevin's work on Task 4c and 4d.

When working on Task 4c, Amir seemed to indicate that the ordering of the symbols mattered. In this case, whether 5 or *y* were on top indicated which was to be first in the expression. At this point, it is unclear whether Amir's distinction about the direction or order of the terms mattering was the result of a composed magnitude interpretation. However, Kevin's suggestion of "y+5" as an alternative expression, and Amir's agreement, indicates that they were not interpreting *y* as a distance from 0 to *y* and 5 as a distance between 0 and 5. Instead, they justify their decision to use the correct answer, 5–*y*, "for consistency," rather than with any reference to distances. This justification suggests that Kevin and Amir may have been learning to use a rule or pattern of "top minus bottom," since they could have just as readily used the expression y+5 to represent the distance between *y* and 5, and may have, if previous expressions had included sums rather than differences. We note that Kevin and Amir provided correct responses to the remaining tasks.

Discussion & Future Plans

At first glance, the *Interpreting Graphs for Calculus Activity* appears promising as an intervention to support students in expressing distances, based on the results of the pre-test and post-test items. However, as designed, they may be limited in the ways of reasoning they can promote. Kevin's and Amir's use of symbols as labels and parameters suggests that the tasks as designed may not effectively support students in conceiving of difference expressions involving symbols as representing varying values of distances. Kevin's and Amir's rule for using a minus sign in their expression to represent distances, to the avoidance of conceptualizing distance measurements given by the expression, may work in many contexts for which a "right minus left" or "top minus bottom" formula yields the correct answer. Yet, we anticipate this rule may not support students in expressing distances in coordinate systems beyond the Cartesian coordinate system. In our future work, we plan to conduct more teaching experiments with students with a next iteration of revised tasks designed to support students in conceptualizing distances more effectively. We hypothesize that it could benefit students to include more reflective prompts throughout the tasks, such as "what does this value represent?" after students calculate the length of each segment and to include some dynamic distances rather than only static distances for students to express with differences. We hope to further discuss with the audience ways to improve the task design to address the two issues we highlight in this study.

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The Role of Ritual and Personalization in Students' Exploration of Subgroup Generation

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We share preliminary results of a commognitive analysis of the participation of twelve undergraduate students in an exploratory subgroup generation activity in an abstract algebra class. We find that while students' collaborative activity contains markers of both ritualized and explorative participation, their appeals to ritual and personalized formulations of inferences and insights about subgroups serve to broaden access to the conceptual substance of the activity. We discuss a possible asset-oriented interpretation of students' ritualized activity as an aid to exploration and reflective of the disciplinary activity of professional mathematicians.

Keywords: Discourse, Inquiry-Oriented Instruction, Abstract Algebra

Advanced mathematical courses such as abstract algebra provide a space where students are apprenticed into the norms and activity of mathematicians. These classroom spaces involve a complex interaction between student reasoning and activity and norms deriving from the professional mathematician community. Frequently, scholars have analyzed student activity in this setting from a deficit point of view, illustrating ways that students may engage in non-normative activity and fail to meet standards of communication associated with norms of formal proving (see Styliandies et al., 2017 for an overview).

To both illustrate this trend and contextualize our study, we use literature about students and subgroups. Subgroups are one of the fundamental topics in group theory (Melhuish, 2019) and have received considerable attention in literature. Some students have been shown to attend to subset properties but not operation. For example, students may suggest that "odds" are a subgroup of the integers (Brown et al., 1997; Titova, 2013). Similarly, several studies have shown students identify $\{0,1,2\}$ as a subgroup of $\mathbf{Z}_6$ (e.g., Dubinsky et al., 1994; Melhuish, 2018) because they unintentionally switch operation from addition mod 6 to addition mod 3. In contrast, Findell (2002) illustrated how a Cayley table could focus a student on the relevant operation, although generating subgroups from that table proved to be a significant challenge. There have also been some studies that pointed to specific issues with applying subgroup test requirements including production of proofs (Ioannou, 2018) or checking closure by only considering combinations of distinct elements (Hazzan, 1994). In general, we can state that the field has characterized a number of "errors" related to subgroups.

In this report, we attempt to re-examine student activity with subgroups through a more asset-based lens. We share analysis from one open-ended task designed to engage students in generating subgroups for a given, known group. We leverage the framework of Lavie, Steiner, and Sfard (2019) to characterize students' activity as exploratory or ritualistic, noting ways that students engaged productively with the task and identifying dilemmas students needed to resolve. We found that our students negotiated many issues reflective of the literature, but productively drew on personalization and ritual as they engaged in open-ended exploration.

Theoretical Framework

In this study we take a commognitive view of mathematics learning, in which thinking is conceptualized in terms of acts and processes of communication (Sfard, 2008). Central to the commognitive perspective is the notion of *routine*, a repetitive pattern of action that retains some invariant features across different performers and implementations (Lavie et al., 2019). Lavie and colleagues distinguish two types of routines: *rituals*, which are process-oriented and require performance of a rigid sequence of actions; and *explorations*, discursive routines guided by the performer's motivation to communicate an idea or pursue a line of inquiry. According to Lavie et al., students initially can only participate in mathematical activity in ritualized ways, because they must first familiarize themselves with the norms and meta-rules of disciplinary discourse. As they gain familiarity, they may begin to participate in "explorative" ways, generating new truths and pursuing inquiries borne of their own motivation and curiosity. Coles and Sinclair (2019) challenge the notion of ritualized activity as meaningless, arguing that participation in ritual can engender meaning for learners and is not mutually exclusive with exploration.

One specific discursive feature that has been associated with ritualized activity in mathematics is *personalization*, in which operations are presented as human actions on mediators rather than in terms of structural relations among mathematical objects (Ben-Yehuda et al., 2005). Personalization in mathematical talk has been associated with lower achievement in empirical studies of children's mathematical activity (Bills, 2002). Our study aims to challenge narratives of personalization as a marker of less sophisticated or less robust mathematical activity by exploring ways in which personalization can help students access algebraic concepts.

Guided by this theoretical framework, we address the research questions:

1. In what ways might collaborative student work on an open-ended task in an abstract algebra course resemble ritualized or explorative mathematical activity?
2. In what ways might ritual and personalization support students' explorative engagement in such a task?

Context and Method of Study

In this report, we analyze an activity on Lagrange's theorem developed by the Orchestrating Discussion Around Proof project which designed and field-tested abstract algebra lessons that engage students in authentic mathematical proof activity (Melhuish et al., 2022). The first activity in the task sequence involved students generating subgroups in order to anticipate the relationship between the order of a group and its subgroups. The participants in our study are twelve undergraduate mathematics majors who worked on the lesson in four teams (Table 1).

Table 1. Teams and group assignments for the subgroup generation activity.

Team	Group	Students	Team	Group	Students
1	$\mathbf{Z}_{10}$	Carlos, Romiax	**3**	D_4	Brad, Fernando, Mona
2	$\mathbf{Z}_{12}$	Diego, Jake, Paolo, Quinn	**4**	S_3	Alan, Eric, Nathan

Each team worked for 20 minutes to create a list of subgroups of their assigned group. In the previous class session, the instructor (Author 1) had demonstrated a possible approach for generating a subgroup of the group $\mathbf{Z}_8$: choose an element to place in a subset, and then use the closure and inverse properties to identify other elements that must be in the subset if it is a subgroup. Although he modeled how to identify elements that must be included in a subgroup, he did not demonstrate any systematic approach for generating all subgroups of a group.

We audio recorded each team's discussion and photographed all board work, then generated a transcript. To investigate the interplay between ritual and exploration in students' discussions, we analyzed all four team transcripts, noting any talk turns that contained instances of personalization or possible indicators of ritualized or explorative activity (see Table 2). For each instance we further described the problem-solving context, developing analytic descriptions of how personalization, exploration, and ritual supported the group's problem-solving process.

Table 2. Discursive indicators of personalization and ritualized and explorative activity.

	Indicator	Example
Personalization	Talk about human involvement with objects or symbols	"Yeah, we would need 1 and then we would need 11."
Ritualized activity	Description of subgroup generation routine as rigid algorithm with fixed, prescribed steps	"So you do have to add [an element to] itself?"
	Settling conceptual/procedural questions by appeal to authority or analogy with instructor-led example	"Last time over here, [Instructor] adds them, he adds the same element by themselves"
Explorative activity	Exploration of hypotheticals, as evidenced by "if-then" language	"If you compose [a transposition] by itself it just goes back to the identity?"
	Settling conceptual/procedural questions by definition or deduction	"Does 0 always have to be in here?" "Yes. Because it always has to contain the identity."

Preliminary Results

We found that each team's discussion contained some instances of personalization, though different teams used personalization in different ways. Each team's discussion contained some indicators of ritualized mathematical activity, but was largely reflective of explorative activity, with multiple members engaging together in exploration of hypotheticals. For this report, we present analyses of two cases chosen based on the contrast in their discursive features.

Team 1: A Progression from Ritual to Exploration

Team 1 first identified the singleton {0} and the entire group as subgroups of $\mathbf{Z}_{10}$. They then attempted to resolve the question of which operation (addition or multiplication) was the group operation by referring to the instructor's previously worked example. When this failed to clarify the issue, they asked the instructor, who specified that the operation was addition. We interpret this as pointing toward a ritualized form of participation; while it is possible to determine that $\mathbf{Z}_{10}$ is a group under addition but not under multiplication, the group considered the operation to be indeterminate without external specification ("Yeah, well, it could be any binary operation").

As they proceeded with subgroup generation, Team 1 used personalization to generate informal descriptions of subgroup properties, interpreting closure as follows: "So closed means whenever you're adding, whenever you get another element that's also in that set." Throughout the activity, the team frequently referred to a generic "you" operating on elements of subgroups to generate additional required elements.

At one point, Team 1 grappled with whether one should add an element of a subgroup to itself in order to generate additional subgroup elements. Rather than consult the definition of subgroup, Romiax attempted to resolve the issue by referring to the $\mathbf{Z}_8$ example: "Last time over here, he adds them, he adds the same element by themselves." Carlos appears to accept this as a warrant, then recalls an instance of this himself: "And then he did eight plus eight or something." It is possible that for Team 1 the ritual of subgroup generation had not yet fully fused with the team's understanding of the definition of subgroup. This is not to say that the team was inattentive to this definition; on the contrary, the team refers repeatedly to the properties that define a subgroup. We hypothesize that for this team, fluency in the application of the definition of subgroup was a work in progress, and the sense of a subgroup generation ritual or "game" (a descriptor used by the instructor and subsequently by some students) supported their exploratory efforts. After settling these procedural issues, Team 1 went on to fluently explore other subgroup possibilities, determining that the set of even elements of $\mathbf{Z}_{10}$ would form a subgroup, but deducing that the set of odd elements would not satisfy the closure property.

Team 4: Ritual Embedded in Explorative Activity

Team 4 began by identifying the trivial subgroup of the symmetric group S_3. Alan then asked, "Where do we start with the–?" Eric responded, "Yeah, just pick any couple of them and see what they generate." Eric thus explicitly frames his envisioned approach to the activity as an exploration, where they "pick" group elements and "see" what subgroup they generate. This explorative framing is consistent throughout Eric's portion of the transcript: "So if you include these ones, you need the whole group, is what we *discovered*." (emphasis ours).

However, Team 4's discussion is not without indicators of ritualized activity. At one point, Alan asks, "Dude, just a question, when we're composing, do we have to do it the other way too?" This question implicitly frames composition as a part of a prescribed routine that must be performed in a specific way. Assuming Alan's question is about whether closure requires one to be able to compose subgroup elements in different orders, this is a novel dilemma for the team: the instructor's prior example involved an abelian group. The team avoids the dilemma by discovering that the two permutations they used as starter elements generate the entire group, at which point the question is dropped; this suggests that the team's successful exploration supersedes the need to clarify procedural aspects of the subgroup generation "game."

In terms of personalization, we observe that while both Team 1 and Team 4 talk about the subgroup generation routine in terms of human actors performing operations on elements, Team 4 uses personalized language more often to refer to elements they "need": for example, "if we do this with all of them are we going to end up needing (1 3)? What if we did?" We hypothesize that the language of "needing" elements supports the team's communication about elements that must be in a subgroup without the use of conditional statements.

Discussion and Implications

In all four discussions, we see moments when team members are uncertain or disagree about the rules of the subgroup "game." We note two features in how students engage in the game.

Teams often attempted to resolve dilemmas by consulting a prior instantiation of the game, as one might when participating in a ritual whose rules are determined by an external authority. One might consider this antithetical to authentic mathematical activity: one powerful disciplinary norm in mathematics is that we resolve dilemmas by consulting foundational definitions and using deductive reasoning (see Dawkins & Weber, 2017). Second, we observed students engaging in personalization: rather than using abstract mathematical language, they introduce themselves or objects as acting agents. Again, this in some ways defies norms of valued activity in advanced mathematics. Further, literature on how students engage in abstract algebra content often suggests personalization as a deficit or lower level of reasoning (Hazzan, 1999). Others have suggested that using an "I" statement in reference to a quantified statement (in this case related to isomorphism) reflects the lowest level of development related to a quantified statement (Leron et al., 1995). If we were to use the standards of proof communication and disciplinary norms, we might view the student activity as lacking valued disciplinary qualities.

We wish to suggest an alternative, asset-based framing (cf., Adiredja, 2019) that interprets students' personalization and ritualized activity during an exploratory task not as evidence of a deficit with respect to acquisition of formal mathematics, but as resources that facilitate students' fluent participation in the task. We find that the students' personalization of the necessity of an element belonging to a subgroup due to the closure or inverse property (*e.g.*, "*You would need* at least the two [3-cycles] because they are inverses of each other"; emphasis ours) allows students to follow implications of hypotheticals without getting bogged down in the more demanding structure of conditional statements. Similarly, ritualization of the rules of the subgroup generation "game," and the opportunity to discern these rules from a worked example, offers students an alternative pathway to access the task even if, due to uncertainty about a definition or convention, they are unsure whether a certain "move" is permitted or required. This advances one of the goals of [*blinded project*]: to broaden access to mathematical activity beyond those students who quickly master the much more complex "game" of formal mathematics, whose rules are intricate and often difficult to decode. Further, we suggest that personalization and ritualized activity *are* reflective of disciplinary activity in meaningful ways. Weber and Melhuish (2022) argued that personalization is a key part of how mathematicians understand ideas highlighting metaphors and connection to their lives. Additionally, mathematicians also defer to outside sources of authority and have been shown to find arguments more convincing if written by respected authors (Inglis & Mejia-Ramos, 2009) and will defer to authoritative sources to assume validity (Weber & Mejia-Ramos, 2011). One of the main reasons mathematicians read proofs is to find ideas to transfer to their own research (Mejia-Ramos et al., 2012; Weber & Mejia-Ramos, 2011), which is also reflective of much of the students' ritualized activity.

Finally, we note that the instructor's choices of words in introducing the activity (*e.g.*, "game", action verbs such as "generate") likely had some influence on students' participation and discourse. We note that these word choices were reflected in student talk, and envision future work exploring ways in which careful orchestration of a task launch can invite students into exploration of open-ended problems with ritual playing a strategic and productive role.

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Connecting Instructor Mathematical Knowledge for Teaching, Intercultural Competence, and Undergraduate Learning

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We report on work to explore instructor intercultural competence and its relationship with the effectiveness of mathematics teaching in post-secondary settings. Results are from quantitative and qualitative inquiry into the mathematical knowledge for teaching and intercultural orientations of 15 instructors and 476 of their undergraduate students. Instructors participated to varying degrees in a professional short-course for faculty learning to teach mathematics for future grade school teachers. The greatest learning gains among undergraduates were in the classes of those instructor-participants who completed short-course activities about interculturally responsive mathematical knowledge for teaching future teachers.

Keywords: teacher preparation, intercultural competence, professional development, equity

Lee Shulman (1998), in reviewing the education of professionals (e.g., doctors, lawyers, teachers, clergy), noted six defining characteristics of a profession:

1. the obligations of *service* to others, as in a “calling”;
2. *understanding* of a scholarly or theoretical kind [e.g., mathematics, pedagogy];
3. a domain of skilled performance or *practice*;
4. the exercise of *judgment* under conditions of unavoidable uncertainty [e.g., instructional decision-making in classroom contexts];
5. the need for *learning from experience* as theory and practice interact; and
6. a professional *community* to monitor quality and aggregate knowledge. (p. 516)

These characteristics play out in a variety of ways. Item #1 includes the fact that:

> The professional educator’s challenge is to help future professionals develop and shape a robust moral vision that will guide their practice and provide a prism of justice, responsibility, and virtue through which to reflect on their actions. Medicine’s Hippocratic Oath, therefore, is a central manifestation of the moral foundations of a profession. (Shulman, 1998, p. 516)

Now, consider the current state of the art in the professional education of college mathematics instructors. What is clearly and explicitly offered to instructors for how to make morally and contextually complex judgment calls? Not much, yet (Braley & Bookman, 2022). Certainly characteristic #2 is accepted as essential (Herman, 2011) while #3, and how to measure it, has had increasing attention in the recent past (e.g., Castro Superfine & Li, 2014). If characteristic #4 is not well addressed early on, then the alternative is #5: Learn it from experience. Learning from experience requires a multifaceted mirror of reflective practice, one that instructors can use to see encounters with others as opportunities for professional learning about mathematics teaching. Both #5 and #6 are emerging as areas for research and development (e.g., among graduate students and novice faculty in their early teaching experiences).

Theoretical Perspectives

Professional growth is shaped by professional and personal cultures: dynamic social systems of values, beliefs, behaviors, and norms for a group, organization, or other collectivity in which the shared values, beliefs, behaviors, and norms are learned, internalized, and changeable by members of the group (McGrath et al., 1995). Many reports indicate that culture is a significant factor in the inequities of persistence and disparities in achievement in mathematics (e.g., Greer et al., 2009; Laursen & Rasmussen, 2019). University programs are responding (e.g., see Equity Alliance, equityalliance.stanford.edu). From anti-racism training to culturally responsive pedagogies, post-secondary instructor professional development efforts have emerged largely from the same arena as K-12 teacher education: psychology. Yet there is another area of the academy from which professional educators can draw insight: anthropology (Ladson-Billings, 2001). Psychology tackles professional education through an approach based in a teacher's *disposition,* attempting to change it through focused reflection on *behavior*; social anthropology offers the idea of movement along a developmental continuum of *orientation to differences* through focus on *relationships* and *communication*. In particular, healthcare and international relations professions offer theories of intercultural development and conflict resolution that are now being applied in education (e.g., Bennett, 2004; Hammer, 2009; Hauk et al., 2021; Hauk & Speer, 2022; Leininger, 2002; Wolfel, 2008). The core of intercultural orientation development is building skill at establishing and maintaining relationships in culturally complex situations.

Though some instructors have largely monocultural classrooms – in the sense that most students share experience of a particular set of cultural-general norms and practices – the nature of diversity in the U.S. is shifting from such segregated monocultural circumstances to cultural heterogeneity (Aud et al., 2010). We argue that intercultural competence is an essential component in the shift from teacher-centered to student-centered approaches to instruction. Knowing one's orientation, or the normative orientation of a group, can inform instruction. In particular, the authors are researchers in a project that has created professional learning for faculty new to teaching future K-8 teachers. Here we report on a small exploration into the results of infusing intercultural competence development into professional development.

Research Question: What evidence is there that intercultural orientation is part of the mathematical knowledge for teaching needed by post-secondary instructors to be effective in teaching future teachers?

Mathematical Knowledge for Teaching (MKT) and teaching future K-8 teachers

There are particular understandings and skills associated with effectively teaching children, a sociological synergy of mathematics and mathematics education called *mathematical knowledge for teaching* that encompasses types of *subject matter knowledge* and *pedagogical content knowledge* (MKT; Ball et al., 2008). Subareas of these include knowledge of the intended curriculum (Herbel-Eisenmann, 2007), "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content" (Hill et al., 2008, p. 375); and knowledge about teaching actions and decision-making (e.g., productive ways to respond in-the-moment to students to support learning). All of the components of MKT are situated in a seventh aspect, *knowledge of discourses,* about communication in/with/through mathematics among people in and outside of the classroom (Hauk et al., 2014; Scheiner, 2019).

A related idea at the post-secondary level is *mathematical knowledge for teaching future teachers* (MKT-FT; Castro Superfine et al., 2013, 2014). A deep MKT-FT is vital in the student-centered and inquiry-oriented approaches to teaching shown to improve learning, increase persistence, and reduce inequities (Bressoud et al., 2015; Laursen et al., 2014). Generally, instructors build MKT-FT by learning from experience and professional community: grading, examining their own learning, observing and interacting with students or colleagues, reflecting on and discussing their own practice *and* the practices known to be effective in teaching (Kung, 2010; Speer & Wagner, 2009). Those with advanced degrees in mathematics who teach future teachers are faced with a need to connect mathematical learning by their future-teacher-students to the special mathematical knowledge needed for working with children as well (Castro Superfine & Li, 2014).

Model of Intercultural Competence Development

Our effort in applying the ideas of intercultural competence development to mathematics teaching and learning is based in the developmental model of intercultural sensitivity (Bennett & Bennett, 2004). The model ranges from a monocultural, ethno-centric view to an ethno-relative, intercultural view. Figure 1 is a schematic of the five benchmark orientations and developmental processes that support intermediate transitions.

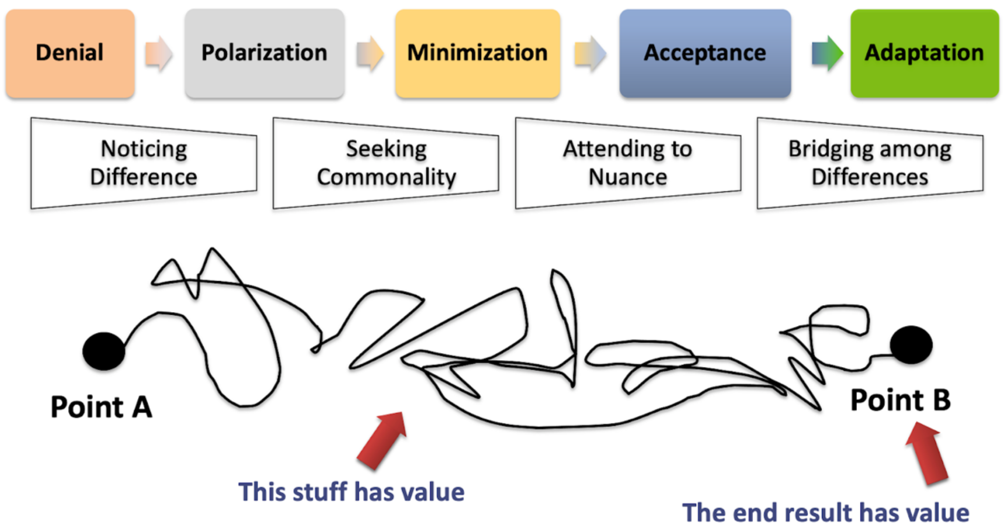

Figure 1. Milestones and processes for the intercultural development continuum.

The starting point of the continuum of orientations towards cultural difference, *denial*, is a lens for perceiving the world based on denying difference and assuming "Everybody is like me." A person with this orientation will avoid or express disinterest in cultural difference. The transition to the next benchmark, *polarization*, comes with the recognition of self as distinct from "other" through a noticing of difference. A polarization orientation can be driven by the assimilative assumption "Everybody *should* be like me/my group" where cultural differences is viewed in terms of "us" and "them." Transitioning *minimization* involves noticing commonalities beneath the surface differences, in particular a growing awareness of norms. This middle orientation is a minimization of difference, a lens for experience based on the idea, "Despite some differences, we really are all the same, deep down" and focuses on similarity and universals (e.g., biological similarities – we all have to eat and sleep; and presumed universal values). A minimization orientation will, however, ignore recognition and appreciation of subtle difference. Through increased attention to nuance in the differences that exist within noticed commonalities, one begins the transition to an *acceptance* orientation. Here, "accept" is used in its socio-cultural sense – the action or process of consenting to receive (rather than its psychological one – believe or come to recognize as valid or correct). Someone with an acceptance orientation has both some mindfulness of self as having a culture and awareness of moving among multiple cultures (plural). However, the importance of relative context, how to respond and what to respond in-the-moment of interaction with others is underdeveloped. The transition to *adaptation* involves developing culture-general frameworks for perception and behavioral shifts that are responsive to a full spectrum of detail in an intercultural interaction along with a concomitant awareness that one's own perceptions are limited and the whole picture is bigger than what is perceived. Adaptation is an orientation wherein one bridges cultural perspectives, without losing or violating one's authentic self, and adjusts communication and behavior in ways viewed as appropriate by each cultural group involved.

In the particular context of post-secondary mathematics teaching and learning, the continuum from ethno-centric to ethno-relative orientation may begin in a denial of difference (e.g., mathematics is acultural). Further development will go through a phase characterized by a tendency to polarize (e.g., there is one best or right way to solve every problem). From there, development can progress to a minimizing search for universals (e.g., there are "objective" or "mastery based" ways that are universally applicable to assess *all* students, and grades become the essential element of interaction for instructor and student, disconnected from students' mathematical funds of knowledge). With time and intentional development, one can learn more about mathematical ideas, contexts, and human interactions and reflect on teaching with greater attention to relational details (e.g., learning about implicit bias and suddenly noticing it in every word problem in the text), but how to use this knowledge to improve opportunities to learn, classroom climate, and interactions with others remains elusive. At its most developed, adaptation in mathematics teaching and learning involves networks of people and ideas; interactions can be anticipated (enough) that teaching responds to the needs of those involved (Hauk et al., 2014). Development along the continuum is not direct or linear. Folding back to previous orientations (particularly in times of stress) is common. Also, the time spent in learning about self and others during transitions and folding back hold value in developing more variety in the ways in which one perceives, makes sense of, and enacts culturally-informed relationships.

Methods

The *Professional Resources & Inquiry in Mathematics Education* (PRIMED) *for K-8 Teacher Education* project field-tested a 15-hour, 10-week online short-course for college instructors new to teaching future K-8 teachers. This online professional development (PD) experience covered topics such as mathematical knowledge for teaching, inquiry- and task-based learning, and tools for equitable instruction. The PRIMED experience was hybrid: Modules 2 and 5 included both asynchronous and live online activity-based work; the other three modules were sets of asynchronous web-based activities completed by instructors individually and in teams (see Figure 2). Participant teams rarely met in real-time and tended to check-in with each other by email or through online discussion boards.

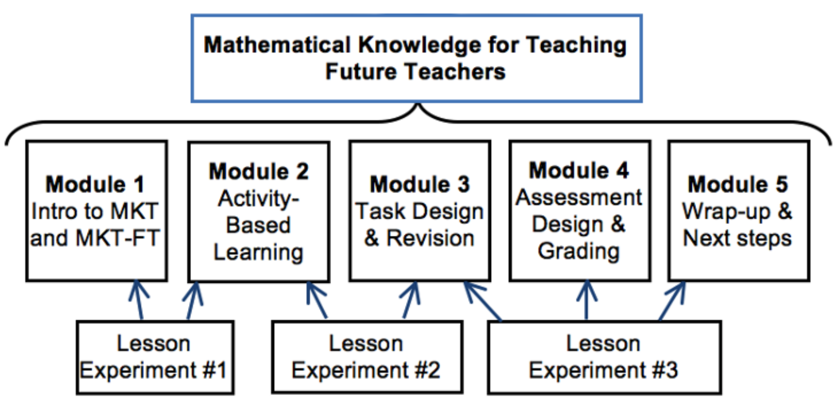

Figure 2. Overview of the PRIMED short-course for instructors new to teaching future teachers.

The short-course included consideration of MKT and MKT-FT in interculturally rich discourse contexts (e.g., examining a scenario where students are learning mathematics while also learning English as an additional language). Instructors had opportunities to learn about, witness (though video-clip-based activities) and practice (through lesson experiments) how to deal with the realities of negotiating the multiple experiences and relationships in a math-for-teachers classroom (see Jackson et al., 2020 for more on short-course development).

Each participating instructor agreed to complete research tasks and to include assignments in their course through which undergraduate students completed assessments of MKT. Instructors (n=15) and undergraduates (n=476) providing data for the study were from 12 different institutions of higher education (2 public community colleges, 7 master's granting universities [6 public, 1 private], and 3 doctorate-granting universities [2 public, 1 private]).

One of the project's main measures was the valid and reliable elementary grades number and operations *Learning Mathematics for Teaching* (LMT) assessment (Hill et al., 2008). Items on the LMT capture whether respondents correctly solve primary school mathematics problems as well as how they address teaching tasks (e.g., evaluating unusual solution methods, representing content to students, identifying adequate mathematical explanation). Instructors and their future-teacher-students completed the LMT at both the beginning and end of the semester in which instructors participated in PRIMED. Matched pre-post data were available for 12 instructors and 226 undergraduates. All also completed the Intercultural Development Inventory (IDI), a validated and reliable measure of intercultural development (instructor-participants did the IDI pre- and post-short-course, the undergraduates once, at the beginning). Also, each instructor had a PD involvement index (low, substantial, high) based on three characteristics: count of contributions to course online discussions, number of course tasks completed, and self-evaluation of effort. Though limited, these data allowed initial investigation of the relationships among PD, instructor and undergraduate MKT, and intercultural competence development.

Results

According to the LMT, faculty and their students gained MKT. The overall mean gain for instructors on the LMT was 0.5 standard deviations (sd) and for undergraduates was 0.29sd (see Figure 3, next page). Other research on prospective teacher learning has indicated that one semester of instruction in a course carefully designed and taught by an experienced teacher educator can lead to gains of up to 1sd (Castro Superfine et al., 2013). However, far smaller changes are usual (Phelps et al., 2016). When faculty have PD to strengthen their teaching, LMT gains among their students have been reported at about 0.25sd (Laursen et al., 2016). In the PRIMED study, when involvement-index is considered, average undergraduate learning gains were highest in the classes taught by instructors who participated most consistently in the short-course (student gain of 0.35sd), lower for those who skipped lesson experiments or modules (gain of 0.28sd for substantial participation group), and lowest for those who did only a few activities (0.13sd for this group).

The distribution of intercultural orientations for students was similar to that in previous research among pre- and in-service K-8 teachers (the majority at or before polarization). Figure 4 illustrates this along with the fact that the distribution of orientations for instructors was different from that of students (the majority at or after minimization) – which also is similar to results in other studies, (see Hauk et al., 2017 and references therein). That is, the distribution for students is rooted in polarization (seeing difference) while that for instructors is heaviest in minimization (minimizing difference and focusing on similarity). We have reported elsewhere on the challenges posed by the distinction (e.g., Hauk et al., 2015).

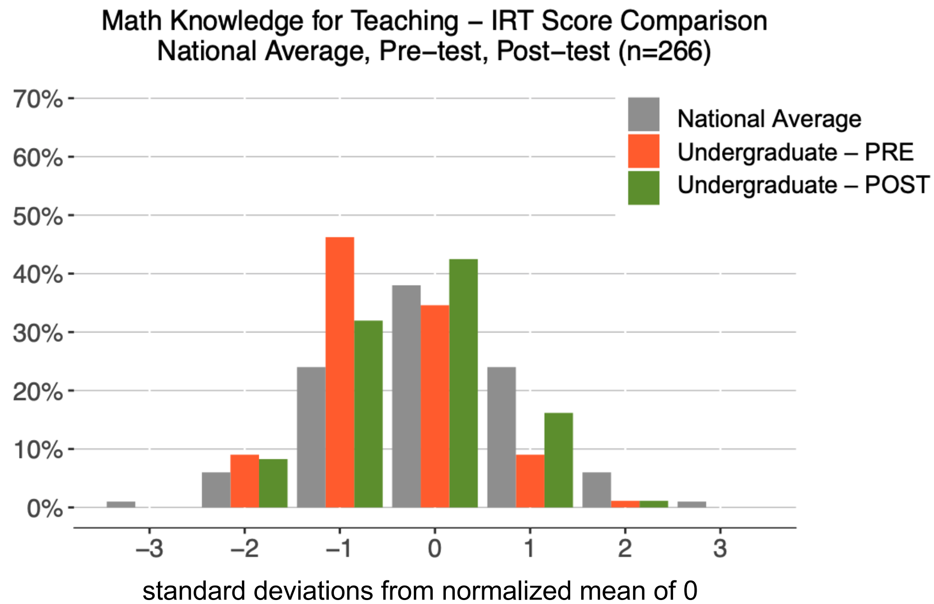

Figure 3. Comparison of LMT scores among future teachers

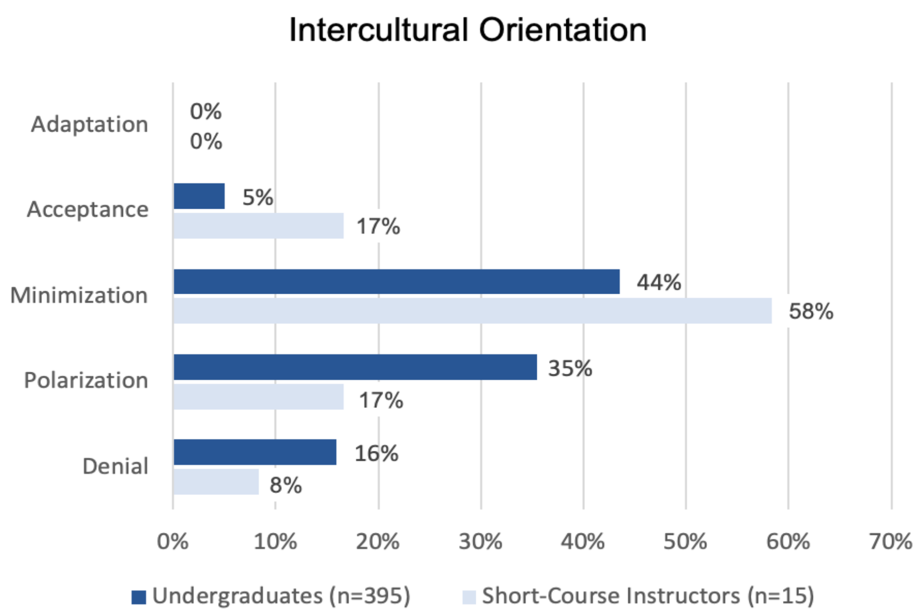

Figure 4. Pre-course IDI results as % of group for undergraduate students and instructor-participants.

Juxtaposing the two data sets, Figure 5 (next page) is a visualization of undergraduate student LMT average gains (horizontal) plotted according to the difference between the instructor and class average IDI orientations (vertical). The central boxed values indicate where the instructor and class (on average) had statistically insignificant differences in IDI score. Away from the central box the differences in IDI were statistically significant (p<0.5). For example, the dot furthest to the lower right is for a high-participation instructor (blue dot) whose class' polarization orientation was more interculturally developed than his denial orientation.

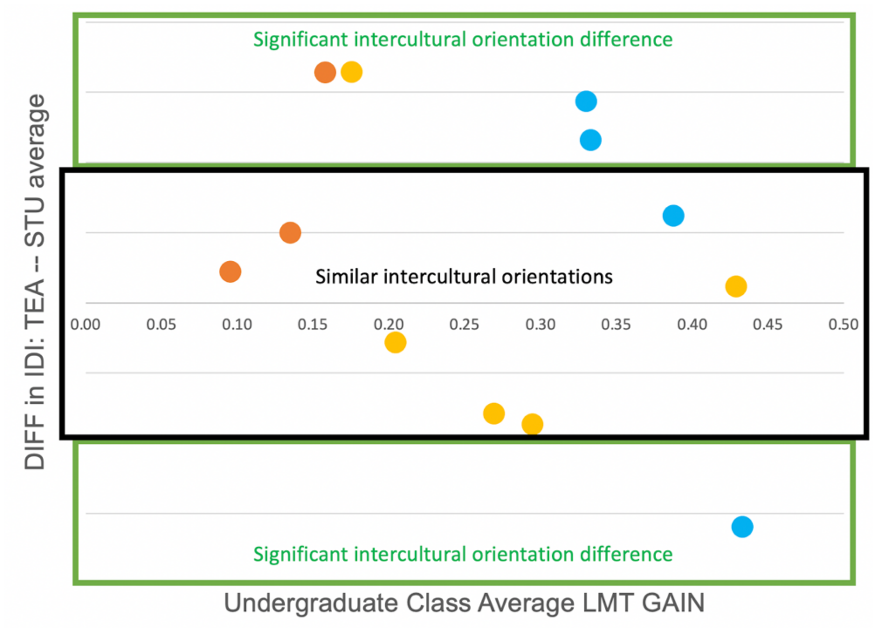

Figure 5. Scatterplot of student LMT gain by difference in IDI score. Dot colors indicate instructor level of participation in PRIMED (blue=high, yellow=substantial, orange=partial).

The research question driving the analysis sought insight into whether and how faculty intercultural competence development and MKT-FT development might be related to changes in undergraduate MKT. As indicated in Figure 5, instructors with high participation in the interculturally explicit PRIMED project had classes with the largest measured learning gains, even when there were significant differences between their own intercultural orientations and those of students. Figure 5 also suggests that PRIMED supported at least one instructor with a highly ethno-centric view to be effective with a diverse class group (rightmost dot in Figure 5).

Discussion and Conclusion

This study echoes, on a small scale, recent work on PD for inquiry-based learning in college settings that demonstrated an instructors' perceptions of their ability to enact behavior which they can control was positively associated with instructional practice aligned with PD aims (Archie et al., 2022). Our group continues to explore the relationships among intercultural competence, professional learning, and classroom practice (Hauk et al., 2017; Jackson et al., 2020). Finally, on a practical note, at the end of the short-course semester, three common goals for future professional learning emerged from participants, each related to interconnecting MKT and orientation:

(1) build *awareness of self as having a cultural lens for viewing the world* in a post-secondary mathematics/department culture that is distinct from the culture of K-12 teaching;
(2) find guidance in the transitions from minimization and into acceptance, particularly how to *be mindful of one's cultural filter(s) for interacting with the world*;
(3) engage in *building a knowledge base about equity*, including knowledge about culturally normative values and distinguishing these from essentializing or stereotyping approaches.

Acknowledgments

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Visualizing Conceptual Change: Using Lakatosian Conflict Maps to Analyze Problem Solvers' Structures of Calculus Definite Integral Tasks

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The work of Imre Lakatos has been influential in the philosophy of science and mathematics. His writings on how science and mathematics progress as fields have been utilized in mathematics and science education. In this study, we apply Lakatosian theory in analyzing task-based interviews. We use conflict diagrams to center the discrepant events participants experience while solving calculus tasks. In so doing, we illustrate how problem context plays a vital role in the evolution of problem-solver's structures in problem situations.

Keywords: Lakatos, Conceptual Change, Data Analysis

Qualitative data is rich. Researchers can present complex, nuanced perspectives on mathematical reasoning with interview data. However, rich data is often also dense and difficult to understand and/or overwhelming in scope. We present conflict maps as a tool for analyzing task-based interview data rooted in the Lakatosian theory of scientific and mathematical progress that illustrates the rich problem-solving that evolves over the course of a task-based interview. As an analytic tool applied to interview data, conflict maps present problem-solving strategies as they emerge and how they change. This highlights and makes evident moments of cognitive conflict.

Lakatosian conflict maps exist independently as pedagogical tools in science education (Tsai, 2000). In utilizing such maps, lessons are designed so that learners engage with discrepant events, moments where they perceive and must respond to a discrepancy between expectations and results. Specific discrepant events are regarded as critical events when alternative conceptions of science are engaged and evolve into traditional scientific conceptions. These critical events are identified by the instructor ahead of the lesson and result from the experiences of instructor and/or research on student thinking. Our research goal is to explore the usefulness of Lakatosian conflict maps as analytic tools for task-based interviews. In the current study, we illustrate their use in analyzing how undergraduate students solved calculus accumulation tasks. Our research question is what discrepant events emerge in the solving processes of individuals while solving calculus definite integral tasks?

Literature Review

Lakatosian Theory

In this study, we draw from the perspectives of Lakatos (1970). Lakatos claims that scientific theories do not progress linearly matching objective reality better as time goes on. Instead, Lakatos states that theories either progress or degenerate over time. Theory falsification does not happen immediately upon identifying a contradiction for Lakatos. Lakatos describes scientific theories as having both a "hard core" of central assumptions and a "protective belt" of auxiliary hypotheses (p. 133). Seemingly contradictory evidence can be redirected towards the protective belt instead of instantly disproving the theory outright. The hard core can be altered, but only after accumulating significant contradictory evidence that leads to the development of a new hard core and protective belt that explains the observations and makes new predictions. Within the domain of mathematics education, there have been several works to emerge utilizing Lakatosian principles. Larsen and Zandieh (2008) explore guided reinvention in abstract algebra using processes rooted in Lakatos' Proofs and Refutations methodology. Nunokawa applied Lakatos' theory to mathematical problem solving in many of their studies, specifically examining the "solver's structures of a problem situation" (Nunokawa, 1996, p. 274).

Enhanced Conflict Maps

Several researchers in science education have made explicit use of Lakatosian principles in their work. Specifically, they have connected these ideas with students' conceptual change and developed what are called (enhanced) conflict maps (Tsai, 2000; Oh, 2011, 2014). These maps serve as pedagogical tools for building effective science lessons. See Figure 1 for the template for an enhanced conflict map.

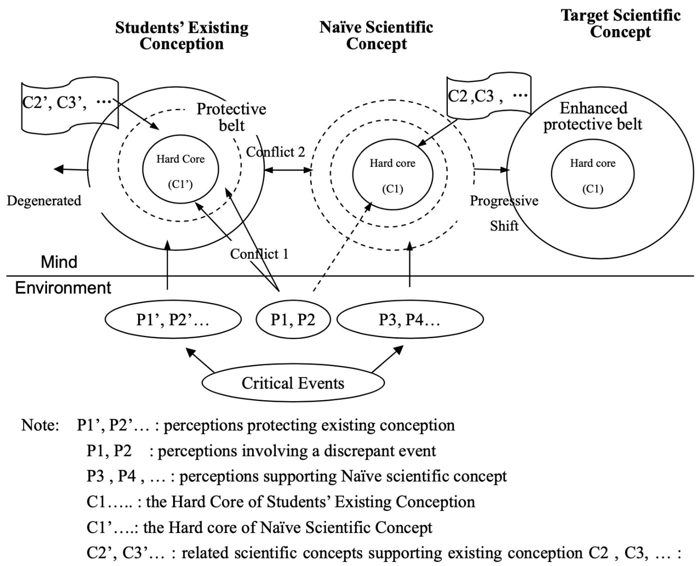

Figure 1. Enhanced Lakatosian conflict map Oh (2014)

Conflict maps have been utilized as pedagogical tools rather than analytical ones. These approaches map out how students are expected to move from one naïve or alternate conceptions to a scientific or mathematical one and include the planned events that (hypothetically) move students from one conception to the other. Ogbonnaya & Dimitriou-Hadjichristou (2022) illustrate such as approach in mathematics education, specifically for a student learning about surface area of a cone. We claim these diagrams can be used for the purpose of data analysis in research. Whereas discrepant and critical events have been presented intentionally by the instructor to move students through the various conceptions, we map these events as they occurred naturally during a task-based interview. This approach offers an alternative way to analyze qualitative data and provides an opportunity to reflect on analytic techniques in qualitative research based in visualization of data at several levels simultaneously, which may serve a researcher with neurodivergence better than traditional text-based techniques.

Theoretical Perspective

Problem Solving

Nunokawa (1994) defines problem solvers' structures as "structures given by the solver to the problem situation" and that "a solving process is described as a sequence of changes in the problem solver's representation structures" (p. 276). Nunokawa (1992) illustrated that the "sense" a problem-solver generates for a problem situation impacts the problem-solver's thinking during problem-solving. We use problem situation as Nunokawa does, to describe the in situ problem-solving environment as perceived by the participant.

Knowledge as an Adaptive Function

A view of knowledge as an adaptive function played a key role in the data analysis in this study. von Glasersfeld (1982) claims "knowledge for Piaget is never (and can never be) a 'representation' of the real world. Instead it is the collection of conceptual structures that turn out to be adapted, or as I would say, viable within the knowing subject's range of experiences" (p. 4). Viability is the crucial idea. Just as with the evolution of an organism in an ecosystem, what students learn is not driven by matching some objectively true reality, but what the student finds viable. For this study, the personal ecosystem is what Nunokawa calls the problem situation**,** and the problem-solver's structures exist (and thrive or perish) within that ecosystem. von Glasersfeld states that "in the sphere of cognition, though indirectly linked to survival, equilibrium refers to a state in which an epistemic agent's cognitive structures have yielded and continue to yield expected results, without bringing to the surface conceptual conflicts or contradictions" (p. 5). This is the heart of the concept of viability in constructivism, that learning is the development and application of stable cognitive structures.

Methods

This study concerns data from a broader study of the ways students reason in calculus accumulation tasks. In the broader study, twelve (12) undergraduate students participated in task-based interviews lasting approximately one hour. Participants in the study were undergraduate students from a large research university in the United States. All had majors or intended majors in the biological or life sciences and were solicited from across various upper and lower division undergraduate courses. Participants solved five calculus tasks concerning accumulation and rate of change set within contexts relevant for biological and life science students. In this study, we focus on one of the tasks and how students processed moments of conflict in their problem-solving work.

> Interview Task: The greenhouse effect is the rise in temperature that the Earth experiences because certain gases prevent heat from escaping the atmosphere. According to one study, the temperature is rising at the rate of $R(t)$=0.014$t^{0.4}$ degrees Fahrenheit per year, where t is the number of years since 2000. Given that the average surface temperature of the Earth was 57.8 degrees Fahrenheit in the year 2000, predict the temperature in 2200.

Diagramming the evolution of problem structures

Diagrams were created to visualize the evolution of problem solvers' structures during their work. Each diagram represents a student's full solution strategy for one interview task. Each diagram illustrates the central assumptions problem solvers made about the given rate of change function, this is referred to as the hard core of their problem structure. The hard core is surrounded by a belt of supporting concepts, assumptions, and calculations that the participant accepts as viable given their current assumptions. When contradictions arise, these are visualized externally to the current structure in the diagram. These are discrepant events for the participant. Each idea within the belt is presented chronologically how it appeared within the interview, read from top to bottom. The diagrams were first sketched by hand and then they were transferred to PowerPoint. The diagrams were validated using external checks, other expert researchers reviewing and providing feedback comparing the diagrams to the interview transcripts.

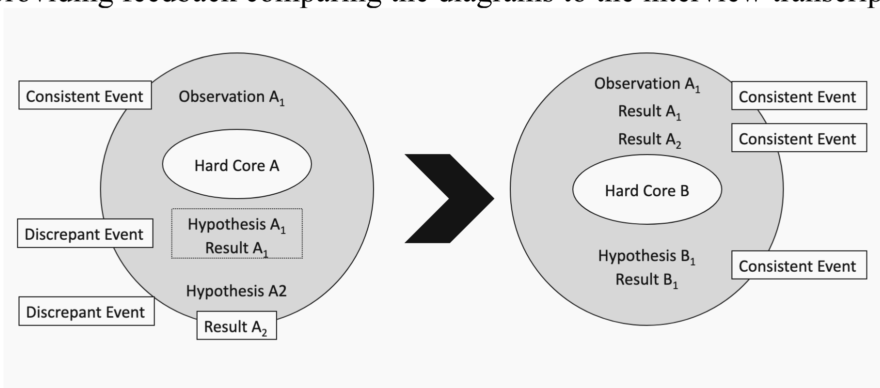

Figure 2. Theoretical problem structure diagram

In this report, we describe the discrepant events for eight of the participants. We focus on only those diagrams with discrepant events and identify the nature of those events and whether/how they were resolved. Figure 3 shows a theoretical problem structure diagram.

Results

We present the problem structure diagrams for two participants, Jake and Brenda. These visualizations capture the participants' full work on the task and what they recognized as viable and what discrepant events they experienced.

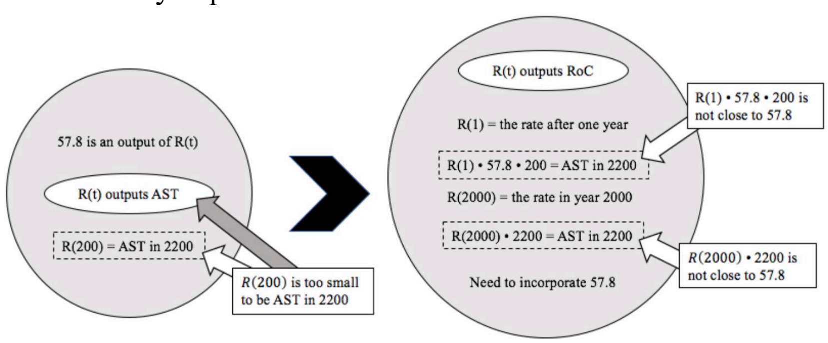

Figure 3. Problem structures diagram for Jake

In Jake's first problem structure, he claims the output of the function for t=200 will be the average surface temperature for the Earth in the year 2200. Jake's observation that R(200) is too small to the be average surface temperature is a discrepant event, similar to the case of Brenda. Jake formulates a new assumption about the values of the function R(t), namely that the outputs are rates of change. While this is accurate, his calculations lead him astray and new discrepant events result.

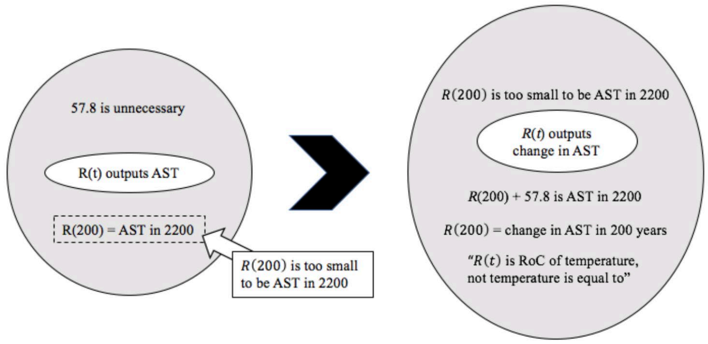

Figure 4. Problem structures diagram for Brenda

Table 1 contains a summary of the discrepant events observed in the eight participants under consideration. Each of the participants reasoned about the size of the value they calculated in relation to their expectations for climate change and the initial given temperature. Several others utilized the setting of the interview in reflecting on discrepant events or their work on other task.

Table 1. Summary of discrepant events

Participant	**Discrepant event description**	
Jane	Value is too low for AST(2200)	
	R(0) not 57.3	R(2000) very small
Anne	R(200) too small for AST(2200)	
	Interviewer direction	Work on Task 5
Shauna	R(200) too small for AST(2200)	
Andy	R(200) too small for change in temperature	
	Computed value too large for AST(2200)	Computed value too small for AST(2200)
Ron	R(200) too small for AST(2200)	
	R(200) too small for change in temperature	Work on Task 5
Jake	R(200) too small for AST(2200)	
	Computed value not close to 57.3	
Brenda	R(200) too small to be AST(2200)	
Gina	R(200) too small to be AST(2200)	

Discussion and Questions

Our research question is: what discrepant events emerge in the solving processes of individuals while solving calculus definite integral tasks? Most of the discrepant events observed involved the context of the task. Participants assessed the viability of their assumptions by using their expectations and knowledge of the context. The calculated values being either unreasonably low or high for the context of average surface temperature of the Earth was often a key indicator that assumptions were not viable. Questions for discussion: What are the benefits of adopting this analytical approach in mathematics education research? How can Lakatosian conflict maps benefit us in analyzing student thinking as they work through a task or broader concept? We feel visualizing interview data on problem solving math tasks leads to greater accessibility. We also believe the diagrams preserve the participants' voice, their authentic ideas as opposed to solely the researchers' perspective of the work.

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A Genre Analysis of Mathematical Remarks

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This report describes the preliminary results of a genre analysis of mathematical remarks, which was conducted using Bhatia's (1993) seven-step genre analysis model. As part of the analysis, discussion threads on Mathematics Stack Exchange about mathematical remarks were analyzed as well as a corpus of remarks from first-year mathematics lecture notes of a British university. An overarching communicative purpose and seven non-mutually exclusive "mini communicative purposes" (Hyon, 2018, p. 30) of the genre of mathematical remarks are identified.

Keywords: Discourse, Genre Analysis, Mathematical Remarks

The American Mathematical Society's (AMS) Publications Technical Group (2017) observed:

> In mathematical research articles and books, theorems and proofs are among the most common elements, but authors also use many others that fall in the same general class: lemmas, propositions, axioms, corollaries, conjectures, definitions, remarks, cases, steps, and so forth […] (p. 2)

To account for differences between elements of this list, the amsthm package supports three theorem styles: plain (e.g., for theorems and lemmas), definition (e.g., for definitions and axioms), and remark (e.g., for remarks and notes). By being named alongside theorems and proofs as well as receiving their own theorem style by the AMS, "remarks" hold a privileged place in the writing of research mathematicians. The purpose of the study outlined in this report is to describe the genre of mathematical remarks using genre analysis, in particular by using Bhatia's (1993) seven-step genre analysis model.

As my work is of preliminary nature, I will only elaborate on a subset of the seven steps (i.e., steps 1, 2, 4, and 6). Further, in the interest of space, I will only report on findings regarding the communicative purposes of the genre of mathematical remarks, that is, I omit my findings regarding the form of the genre. This choice is motivated by the definition of genre I draw on (i.e., Swales's [1990] definition), which places communicative purpose at the center of a genre. Thus, this report's overarching question, which also guides much of the genre analysis, is: What are the communicative purposes of mathematical remarks?

Background Literature

As discussed by Hyon (1996) as well as Bhatia and Salmani Nodoushan (2015), researchers from three different (Anglophone) research traditions have sought to study genre: (a) New Rhetoric studies, (b) systemic functional linguistics (SFL), and (c) English for Specific Purposes (ESP). But, as Bhatia (2012) remarked, "[g]enre theory, in spite of these seemingly different orientations, covers a lot of common ground" (p. 241). For an in-depth comparison of the three traditions, see Hyon (1996) and Hyon (2018). In short, differences between the traditions that were more pronounced in the 1980s and 1990s have begun to fade in recent times with particularly the New Rhetoric and ESP traditions having ever-greater overlap (Hyon, 2018). I draw on the ESP tradition and provide a few brief remarks on this tradition below.

English for Specific Purposes

The ESP tradition of genre theory draws on Swales's (1990) definition of genre, which—in abbreviated form—reads:

> A genre comprises a class of communicative events, the members of which share some set of communicative purposes. […] In addition to purpose, exemplars of a genre exhibit various patterns of similarity in terms of structure, style, content and intended audience. (p. 58)

Building on Swales's (1990) definition of genre, Bhatia (1993) developed a seven-step genre analysis model attending to purpose and form: (a) placing the given genre-text in a situational context, (b) surveying existing literature, (c) refining the situational/contextual analysis, (d) selecting corpus, (e) studying the institutional context, (f) levels of linguistic analysis, and (g) specialist information in genre analysis. As Bhatia (1993) explained about these seven steps:

> These steps have been artificially separated for the sake of convenient formalization and systematic discussion. Moreover, it is not the intention to suggest that in all such investigations, the analyst must go through all the stages listed above and certainly not in that order. (p. 40)

For an application of Bhatia's (1993) model, see Singh et al. (2012). For four reviews of Bhatia (1993)—as well as his response to these reviews—see Nielsen et al. (2017).

Method

Data Collection

I collected two sets of data, each for a different step of Bhatia's (1993) seven-step genre analysis. For step 2 (i.e., surveying existing literature), I compiled all discussion threads related to mathematical remarks I could find on Mathematics Stack Exchange (MSE). The search was conducted on August 29, 2022 and consisted of searching for "remark" using the MSE search function. The top 500 results (sorted by "relevance") were then examined to determine whether the question asked pertained to conventions about writing mathematical remarks. I then inspected each discussion thread's "related" section to find any additional remark-related questions but found none. All ten relevant discussion threads were downloaded.

For step 4 (i.e., selecting a corpus), I compiled all remarks from the PDF lecture notes of all mandatory courses for first-year mathematics majors at my British undergraduate institution. These eight courses are: Foundations, Differential Equations, Analysis 1 & 2, Introduction to Abstract Algebra, Linear Algebra, Mathematics by Computer, and Geometry & Motion. These PDF files were written by the lecturers of these classes and uploaded or sold (in printed form) by the mathematics department (before or after the semester) for a small fee (£1–3). During the compilation process, three situations occurred: (a) remarks were the notes' only labeled exposition (in which case the remarks were included), (b) remarks were not the notes' only labeled exposition (in which case only the remarks were included), or (c) the notes contained no exposition labeled as remarks, only exposition labeled as "note" (in which case the "note"s were included). "Note"s were excluded (case b) or included (case c) to account for some authors possibly using the term synonymously with remarks, whereas others may have chosen to make a distinction. I ended up with a corpus of 31 remarks.

Given the preliminary nature of this work, rather than trying to create a large representative corpus of remarks (e.g., by compiling remarks from talks, lecture notes, textbooks, and research papers), I chose to focus on remarks from first-year lecture notes because this corpus was of a manageable size. Further, I suspected that this corpus might emphasize different communicative purposes than the discussion threads on MSE—a difference of pedagogical interest to me. Thus, for step 6, I am studying the communicative purpose(s) of a sub-genre of mathematical remarks: the remarks made to first-year British mathematics majors in lecture notes.

Data Analysis

To analyze the genre of mathematical remarks, I am using Bhatia's (1993) seven-step genre analysis model. The four steps I detail in this report are steps 1, 2, 4, and 6. For step 1 (i.e., placing the given genre-text in a situational context), I drew on my knowledge and experience. Step 4 (i.e., selecting corpus) has already been described in the Data Collection.

For step 2 (i.e., surveying existing literature), I analyzed the MSE discussion threads by using structural coding (Saldaña, 2009) to identify segments of the data that: (a) asked a question (or made an assertion) about the form of mathematical remarks (e.g., "Can I have a 'remark' after a 'lemma' before its 'proof'?"), or (b) asked a question (or made an assertion) about the purpose of mathematical remarks (e.g., "Remarks may also be used to provide motivation"). As described in the introduction, I will only report on the latter analysis in this report.

For step 6 (i.e., levels of linguistic analysis), Bhatia (1993) outlined three levels of linguistic analysis for analyzing one's corpus. I report on my level 3 analysis (i.e., the largest grain size), which Bhatia (1993) referred to as structural interpretation of the text-genre. Such an analysis may break down a genre into different moves, each of which has its own "mini communicative purposes" (Hyon, 2018, p. 30). Given that remarks are typically short texts of only a few sentences, I considered it unlikely to find a series of moves the genre follows. Nevertheless, I used the idea of "mini communicative purposes" to identify different sub-purposes that remarks can have. To identify these purposes, I process coded (Saldaña, 2009) the remarks in my corpus.

Preliminary Results

As Bhatia (1993) specified, researchers need neither follow all his steps nor follow them linearly. With this in mind, I share my preliminary results for steps 1, 2, and 6 below.

Step 1: Placing the Given Genre-Text in a Situational Context

To place the given genre-text in a situational context, Bhatia (1993) encouraged analysts who come from the relevant speech community to draw on their own experience and background knowledge. As a mathematics education researcher with a background in mathematics, I am familiar with the communication of proof-based mathematics and consider myself a member of the relevant speech community. Thus, in this section, I share experience-based observations I made before beginning my analysis. For the purposes of this report, I include only my observations about the communicative purpose(s) of mathematical remarks.

Context. Most of my relevant experience comes from studying mathematics at a British university. Lectures at the university consisted of a chain of labeled "blocks" (e.g., definition, theorem, proof, remark). These blocks—including remarks—were both spoken and written. Remarks were announced in spoken discourse (e.g., "Let me make a remark.") and written discourse (e.g., "<u>Remark:</u> The function …"). The only spoken discourse not recorded on the boards was unlabeled exposition, whose purpose it typically was to motivate material.

Communicative purpose. A remark's purpose is to highlight a piece of information that does not fit into the mathematical quartet of definition, statement-to-be-proved, proof, and example but is too important to be left as unlabeled exposition. Remarks can, for instance, relate one theorem/proposition/lemma to another (e.g., one being a special case of another), warn the reader about a potential confusion, indicate an alternative proof, highlight an idea or nuance that may have been lost amid mathematical notation, or share an alternative notation.

I suspect that the distribution of communicative purposes of remarks in a text depends on the intended audience. For instance, "highlighting an idea or nuance that may have been lost amid mathematical notation" might be a more prevalent purpose for remarks aimed at newcomers to proof-based mathematics, whereas remarks "relating one theorem/proposition/lemma to another" might be more prevalent when the audience is research mathematicians.

Step 2: Surveying Existing Literature

In his call to survey existing literature, Bhatia (1993) took an expansive view of possibly pertinent literature. Particularly relevant to this study is the inclusion of practitioner advice. Thus, as described under "Data Collection," I searched MSE for "remark" and found ten relevant discussion threads. After analyzing the discussion threads, I identified the following purposes of mathematical remarks:

1. to be able to refer to some piece of exposition (this requires that the remarks are numbered);
2. to draw attention to a piece of exposition;
3. to isolate (important or noteworthy) exposition tangential to the main exposition;
4. to give the reader a break (e.g., with an historical anecdote);
5. to provide motivation;
6. to state why some hypotheses are necessary or natural;
7. to state why one cannot hope to get stronger results;
8. to justify a seemingly unnecessarily circuitous route;
9. to address a potential distraction in the reader's mind;
10. to point out pitfalls; and
11. to point out (and possibly explain) the following of a certain convention.

Step 6: Levels of Linguistic Analysis

Analyzing my corpus of mathematical remarks, I identified the following non-mutually exclusive (mini) communicative purposes of remarks:

1. justify the author's choices on a meta-level (e.g., why they made a particular choice);
2. respond to a possible reader assumption or reaction (e.g., pointing out a pitfall);
3. point out something the reader may have overlooked (e.g., pointing out a border case, pointing out an additional implication);
4. introduce new or different notation and terminology;
5. point out an alternative (e.g., to a definition, an equation);
6. give advice; and
7. share background or additional context (e.g., historical anecdote, practical application, related concept).

As aforementioned, these "mini-purposes" of remarks are not mutually exclusive and, while studying their intersections, one combination of moves stood out due its frequency and structure: purpose 2 followed by purpose 7. That is, around half the time, in addition to responding to a possible reader assumption or reaction, authors chose to share background or additional context.

Discussion

Synthesizing across steps 1, 2, and 6, I observe that there are different grain sizes of communicative purposes of mathematical remarks. The overarching communicative purpose appears to be to separate some exposition from the main exposition—either for elevation or isolation. For example, whereas some remarks warn of a pitfall and are numbered to be able to refer to them more easily (i.e., elevated exposition), other remarks isolate exposition tangential to the main exposition to the extent that they can be skipped—one of the analyzed remarks even included the phrase “feel free to ignore this remark.” Discounting the possibility of reverse psychology, mathematical remarks are thus sometimes (almost) integral (elevated exposition) and at other times skippable (isolated exposition). I suspect this duality may leave novice readers of proof-based mathematical texts wondering whether remarks can be skipped or not.

Moving to the mini purposes (or sub-purposes) of mathematical remarks, I offer the following synthesized list of mini purposes from steps 1, 2, and 6 of my genre analysis:

1. justify the author’s choices on a meta-level (e.g., why some hypotheses are necessary or natural, why one cannot hope to get a stronger result, why one took a seemingly circuitous route, why one made a particular choice);
2. respond to a possible reader assumption or reaction (e.g., pointing out a pitfall, addressing a possible distraction in the reader’s mind);
3. point out something the reader may have overlooked (e.g., pointing out a border case, pointing out an additional implication, pointing out that B is a special case of A);
4. introduce new or different notation and terminology;
5. point out an alternative (e.g., to a definition, an equation, a proof);
6. give advice; and
7. share background or additional context (e.g., historical anecdote, practical application, related concept).

As the step 6 analysis showed, these purposes are not mutually exclusive and there may be some that combine more frequently (e.g., purpose 2 followed by purpose 7). As aforementioned, I also hypothesize that some purposes (e.g., sharing background or additional context) may lead to remarks which authors consider more ignorable than remarks with other purposes (e.g., responding to a possible reader assumption or reaction).

During the analysis, I also observed three differences between the mini communicative purposes identified in steps 2 and 6. First, the discussion threads seemed to emphasize purpose 1—a purpose only seen once in the lecture notes. Second, purpose 6 appeared in the lecture notes but did not explicitly appear in the discussion threads. Third, purpose 3 appeared in the notes but not in the discussion threads. (This last observation is partially in line with my suspicion voiced in pre-analysis step 1.) I am left wondering to what extent these differences are due to a difference in intended audience.

Compiling the corpus, I encountered a large variety of labeled exposition: “advice,” “aside,” “comments,” “digression,” “notation,” “note,” “observation,” and “warning.” Comparing these to the mini purposes, some similarities appear: warning (purpose 2), observation (purpose 3), notation (purpose 4), advice (purpose 6), and digression and aside (purpose 7).

Finally, I wish to share with the reader two of my wonderings: (a) What are the pedagogical ramifications of the mini purposes and the duality of the central purpose(s) of remarks? and (b) What possibilities would applying genre analysis to other mathematical genres offer?

Acknowledgments

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Utilizing Cognitive Interviews to Improve Items That Measure Mathematical Knowledge for Teaching Community College Algebra

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The Algebra Instruction at Community Colleges: Validating Measures of Quality Instruction project (VMQI) developed an instrument to measure mathematical knowledge for teaching community college algebra (MKT-CCA) and conducted cognitive interviews with 12 community college instructors of College Algebra. Thirty-six drafted MKT-CCA test items were reviewed with each item being reviewed by two instructors. This paper presents the lessons learned from this preliminary analysis of the cognitive interview transcripts.

Keywords: Cognitive interviews, mathematical knowledge for teaching, community college, college algebra

Introduction

The Algebra Instruction at Community Colleges: Validating Measures of Quality Instruction (VMQI) is a National Science Foundation-funded project that is developing an instrument to (1) measure mathematical knowledge for teaching community college algebra (MKT-CCA) and (2) identify the dimensionality of this knowledge. We began by situating our conceptualization of mathematical knowledge for teaching vis-a-vis one of the most common conceptualizations produced by Deborah Ball and colleagues (Ball et al., 2008; Hill et al., 2008). Ko and Herbst (2020) further proposed organizing the subject matter knowledge in Ball et al.'s framework by tasks of teaching. To move this work further in community colleges (CC), we have hypothesized to measure two main constructors, Tasks of Teaching (Choosing Problems and Understanding Student Work) and Function Types (linear, exponential, and rational functions that are foundational for advanced work in calculus) (see Figure 1). This conceptualization of algebra knowledge for teaching framed the instrument's development. Cohen and Swerdlik (2009) outline the development of an instrument in five main stages: conceptualization, construction, tryout, analysis, and revision. In the construction stage, cognitive interviews (CIs) play a significant role.

Cognitive interviews are an important qualitative method used by researchers to discern interviewees' understanding of survey items (Meadows, 2021; Willis, 2005). As part of the development of our MKT-CCA instrument, we conducted CIs with CC college algebra instructors to understand whether the items we developed to assess the dimensions (see Figure 1) are interpreted by respondents as intended and whether the participants used the intended knowledge (Mesa et al., 2020-2023). We used the following research questions to guide the analysis.

1. How do CC college algebra instructors interpret our developed items in relation to the dimensions of our blueprint?
2. What knowledge do CC college algebra instructors use to respond to the MKT-CCA instrument?

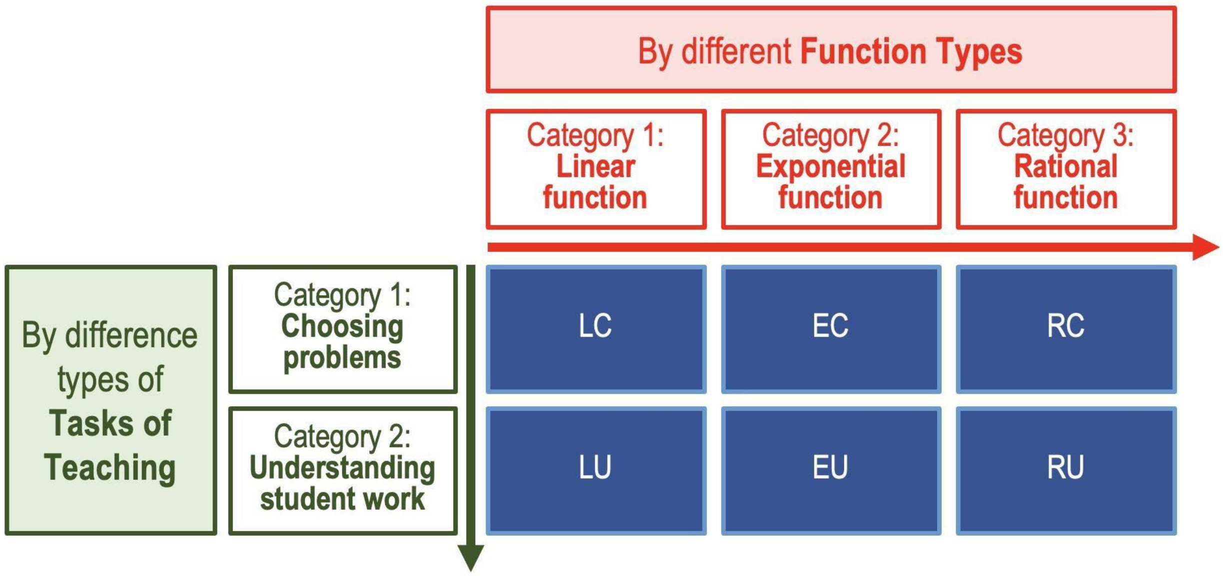

Figure 1. MKT-CCA blueprint showing its six dimensions.

Method

Materials and Procedures

Research on CIs suggest conducting "rounds" that consist of 5 to 15 interviews, which are ideally repeated following efforts to revise questions and eliminate problems (McColl, 2001; Willis, 1994, 2005). After the initial instrument construction and before piloting the instrument, CIs provide information to evaluate whether the items proposed uses the knowledge that is intended. Typically, CIs are planned to take a 3:1 ratio (Willis, 2015) of time (three times more than it would take to answer the item without think-aloud or probing). As a team, we decided to limit interviews to at most 90 minutes to eliminate fatigue (D'Ardenne, 2015). For this reason, we sent two items out of six planned for each respondent, ahead of time so that respondents could think about them and take notes beforehand. By doing this, we anticipated that we could average 12 minutes per item during the CIs. At the time of our CIs, our item pool consisted of 60 draft items that we had deemed ready for review. The 36 items selected (six items per dimension) would provide us with the information to understand whether the draft items were interpreted by respondents as intended, used the knowledge we thought was needed, and targeted the dimensions of the MKT-CCA blueprint.

We developed a protocol that we agreed to use for all interviews. Three teams of two researchers each performed the interviews (one person would be asking the questions, whereas the other was in charge of taking field notes and managing technical aspects). Each team interviewed four participants. We selected 36 items with the goal of having each item be reviewed by two participants. All interviews were conducted online in the last week of July 2021 using Zoom with screen sharing, and audio and video recording with transcription. We used Jamboard to present the items one at a time since it allows for real time documenting the participants' item solving process. We gave each participant two items to work on ahead of the interview and asked them to send us the documentation of their work before the scheduled interview. We asked them to keep track of how long it took them to complete each item. The interview protocol asked for their impressions of the content of the item, whether the content was relevant to their work, the answer they chose and their reasoning, and the reason(s) for eliminating other options.

Analysis

In this paper, we use the interview response from a respondent on an item as our unit of analysis. This gave us a total of 72 (one missing data point due to time restraint) interviews on 36 items. All the interviews were fully transcribed and summarized. The field notes were also coded. Analysis was done in two phases. The first phase focused broadly on answering whether respondents (a) choose the correct answer, (b) use correct reasoning in choosing the answer, (c) choose an incorrect option with correct reasoning, and (d) refer to the dimensions of the blueprint. This phase of coding was done by five members of the research team.

The second phase involved adapting Conrad and Blair's (1996) cognitive coding scheme and was done by the three authors. The coding scheme proposed attending to three categories: *Understanding the Intent*, *Performing the Primary Task,* and *Formatting the Response*. Conrad and Blair used *Understanding the Intent* to code the respondents' comprehension and appreciation of the researcher's objectives; *Performing the Primary Task* was used to code how well the respondents were able to retrieve information from memory, do mental arithmetic and evaluate a response. *Formatting the Response* was used to map how respondents responded to prespecified answer options. We adapted this coding scheme for our purposes as follows:

Understanding the Intent. This category examined whether the respondent was able to place the item in a dimension of our blueprint. Did the respondent recognize both the task of teaching and the function type? We coded a yes if the respondent recognized both the function type and the task of teaching, a partial if only one was identified and a no if neither was present.

Using Knowledge. This category examined whether the respondent understood the instructional goal embedded in the stem. Did the respondent reason about the stem in ways that conformed with our reasoning? Did the respondent use mathematical knowledge to make sense of the stem? Did the respondent's response indicate whether the stem was clear? Did the respondent give any indication of confusion of the stem? Did the respondent understand the context presented in the stem? This included instances where the wording was not clear but did not include situations in which the respondent did not understand the mathematical ideas or terminology in the stem. We coded the response as yes if it indicated a clear understanding of the stem, partial if the understanding was not clear and no if the response indicated that the respondent did not understand the stem.

The stem is the part of the item that defines the mathematical task. It is followed by possible solutions (optional answers).

Formatting the Response. This category examined whether the respondent's reasoning on the options was mathematically correct or aligned with our template. Was the respondent's correct or incorrect selection of options and elimination of distractors expected? We coded the formatting as expected if the justification matched our template intent or was found to be mathematically reasonable. It was coded as problematic when the respondent gave a correct answer or eliminated an option for the wrong reasons, when the reasons could not be mathematically justified, or when the justification given did not conform to our intent. It was coded as partial if the response was a mix of expected and problematic.

Findings

We present the preliminary findings of our analysis in this section. Table 1 shows the summary of the analysis using the three categories. Overall, 88.73% of participants' responses showed a recognition of the function type and task of teaching, 85.92% of responses showed an understanding of the stem and 61.97% of participants responses to the items had mathematically appropriate justifications.

Table 1: Percentage of Responses to the Individual Items (N=72).*

Dimension of Blueprint	Understanding the Intent			Using Knowledge			Formatting the Response		
	Yes	Partial	No	Yes	Partial	No	Expected	Partial	Problematic
EC (n=12)	100.0	0.0	0.0	91.7	0.0	8.3	66.7	8.3	25.0
EU (n=12)	91.7	8.3	0.0	91.7	8.3	0.0	66.7	0.0	33.3
LC (n=11*)	100.0	0.0	0.0	90.9	9.1	0.0	45.5	0.0	54.5
LU (n=12)	83.3	16.7	0.0	91.7	0.0	8.3	100.0	0.0	0.0
RC (n=12)	66.7	25.0	8.3	83.3	8.3	8.3	41.7	0.0	58.3
RU (n=12)	91.7	8.3	0.0	91.7	0.0	8.3	50.0	8.3	41.7

*One data point is missing. One participant only responded to five items.

Research Question One

Research question one sought to answer how CC college algebra instructors interpret our developed items in relation to the dimensions of our blueprint. This will be addressed using the *Understanding the Intent* code.

Understanding the Intent. As expected, preliminary analysis shows that participants mostly indicated a recognition of the function type of the items with only one exception. However, it was noted that the function type was mentioned in the stem of the item and is likely to be the reason the participant did not mention it in the interview. One main observation noted is that participants mostly were able to identify the task of teaching except in the case of choosing problems for rational functions where we observe a 33.3% of responses not indicating the task of teaching.

Research Question Two

We wanted to know what knowledge CC college algebra instructors will use to respond to the MKT-CCA instrument for research question two. We will apply the *Using Knowledg*e and *Formatting the Response* codes for this research question.

Using Knowledge. Participants demonstrated an understanding of the instructional goal in the stem for most of the items. With the exception of choosing an exponential function problem, we recorded high percentages for this category. We observed that the participants were distracted from the objectives of the items by the context of the item. For example, on an item to introduce students to interpreting the meaning of vertical asymptotes in a real world applied context, Justina (pseudo) stated, “I did not get where those numbers are coming from, 0.5 and 6380. I did not get . . . those assumptions being made about the surface or something”. Justina was clearly focused on the context rather than the mathematics in the item and was distracted by the source of numbers and assumptions.

Formatting the Response. Only a little over half the reasons given for the choice of options were expected or mathematically justified. This was concerning since we had recorded high percentages for the *Understanding the Intent* and *Using Knowledge* codes. Further analysis revealed that out of the 25 problematic codes recorded, 12 (about half) of them involved rational functions. About 39% of the reasons (involving 17 items) given were deemed problematic for varied reasons. The top reasons were (1) lack of knowledge needed to correctly answer items (especially with rational functions), (2) selecting familiar options over mathematically sound options, and (3) some also were distracted by multiple contexts in one item and tend not to reason through such items mathematically. Additionally, a few items were noted to be complicated.

Emmanuel (pseudo) gave the following reason for selecting an option: “I go with [option] A only because I already do it with my students. And it's one that they enjoy and it's one that I have expanded on over time.” Emmanuel does not offer a mathematical justification for his choice.

Conclusion

We learned some lessons from conducting these CIs. Respondents comprehend an item better when the item contains fewer applied contexts and confusions associated with parts of an item distract respondents from the objective. Additionally, items need to have more precise language when describing the stem and options.

Based on the feedback received from the CIs, four complicated items were discarded, the stems of seven items were revised, and the options of 14 items were revised to make them more focused, independent from each other, reasonable for the context, and possible to assess mathematically. Ideas collected from the respondents were also used to create new items and revise items that were not part of the cognitive interviews.

Questions for Discussion

1. Based on our review of the literature, isolating factors of mathematical knowledge for teaching has been elusive? What affordances or limitations do you see in our design to isolate two tasks of teaching? What else should be considered?
2. Is there another cognitive interview coding scheme that you have used to extract useful, reliable, objective, and quantifiable data from Cognitive Interviews? We may be doing cognitive interviews after our next version of the instrument. Any advice?

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Co-Construction of Course-Taking Narratives in Undergraduate Mathematics Advising

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This study explores a mathematics department's advising process for incoming, first-year undergraduate students. Through the analysis of video-recorded advising sessions, the interactional dynamics of an undergraduate mathematics advising program are explored. In particular, we describe how student-advisor interactions co-construct students' course-taking narratives in ways that shape their mathematical opportunities.

Keywords: Advising, Discourse Analysis, Gender

Introduction

Mathematics coursework plays a crucial role for undergraduate students as a prerequisite for many programs of study, particularly in STEM fields. The continued underrepresentation of women in mathematically demanding STEM fields (Watt et al., 2017) has been attributed to attrition of women at various points between participation in early education and persistence in STEM careers (Blickenstaff, 2005). While research has linked women's attrition from the STEM pipeline to a variety of causes such as discrimination, societal gender inequality, gendered differences in academic interests, and differential high school academic preparation (Watt et al., 2017, p. 255), it has rarely demonstrated how these factors operate to impact academic and career choices for individual women at the timescale of moments (i.e., seconds to minutes). In this paper, we explore mathematics advising for incoming undergraduates as an undertheorized site of attrition along the STEM pipeline and illustrate how interactions at a small timescale can shape students' mathematical trajectories. Although we do not extend our analysis to examine the influence of gendered discourses or gender performances, we see this work as a necessary first step to learning more about the experience of women moving through the STEM pipeline. Our preliminary analysis of the impact of gender performance can be found in the poster sessions for this conference.

The Context of Mathematics Advising

Academic advising serves a critical purpose in the process of welcoming new undergraduates to the complex system of course offerings and institutional requirements at any university. While academic advising programs serve similar purposes across universities, they are uniquely structured to fit the needs of a particular institutional context. We use the term *mathematics advising* to describe academic advising programs for incoming undergraduates and offered by mathematics departments for the sole purpose of guiding students toward a decision about which mathematics course(s) to register for.

The mathematics advising sessions we examine take place over a very short timescale (five to ten minutes) and occur between an advisor and student who have no knowledge of each other. In this brief interaction, the advisor needs to learn something about the student and provide relevant information about available courses. At the same time, the student must share something about themselves and gather the information they deem relevant to the decision-making process. Course registration typically occurs within days of the advising session, so students have limited opportunities to seek out additional information or support before making a decision.

Mathematics Advising as Genre

We take a discursive approach in which written representations of advising sessions (transcripts) are thought of as *texts* that fit within a particular *genre*. The genre of a text is defined as “a staged, goal-oriented social process” (Martin & Rose, 2003, p. 7), meaning that instantiations of a particular genre generally follow the same steps to accomplish a specific goal and are enacted in social contexts. The genre of a text, and the stages that comprise that genre, constrain the range of discursive acts that are possible (or likely) to occur at various points within the text. Genre analysis provides a way of justifying the reader’s context-specific interpretations of discursive acts within a text. For example, the question *What do you think?* can be interpreted as referring to what the student is considering for an academic major or for a math course depending on when in the advising session it is spoken.

The genre of *mathematics advising* follows a particular logic in the pursuit of a clear goal. Namely, the purpose of an advising session in this study’s context is for the student to decide which course they will register for. To accomplish this goal, the advisor and student engage in a conversation that consists of three stages: Opening, Deliberation, and Closing. In the Opening, the advisor solicits information from the student about their prior mathematics experiences as well as academic and career goals. During the Deliberation, the advisor and student evaluate available course offerings in relation to the information shared during the Opening. Finally, the student and advisor arrive at a consensus during the Closing. The outcome of a mathematics advising session is an informal agreement between the advisor and student about which math course(s) the student will register for.

Mathematics Advising by Co-Constructing Narratives

We conceptualize mathematics advising sessions as sites in which the student and advisor co-construct a narrative about the set of courses that are appropriate for the student to take. The goal of a mathematics advising session is for the student to decide which math course to take, given the set of options made available by this co-constructed narrative. The act of registering for courses can then function as a collapse of students’ potential mathematical futures, particularly when they elect not to take a math course or to take a course that will not support their entry into a mathematically demanding field of study.

Stories told about the self by oneself are likely to have the most immediate impact on an individual’s actions (Sfard & Prusak, 2005). The stories that students tell about themselves during the math advising sessions that we observed are particularly likely to have an impact on their actions because students registered for classes within a day or two of their math advising sessions. Given that the explicit goal of advising sessions is to influence students’ action with respect to course registration, the stories that students tell about themselves in this context are very likely to influence their mathematical trajectories. Although students are narrating stories about themselves, they are doing so through interaction with an advisor. Advisors may ask questions that surface aspects of a student’s narrative that wouldn’t have otherwise entered the discursive space. Even seemingly innocuous utterances have the potential to shift the way a student’s narrative unfolds. In this study, we are trying to learn how the advisor-student interactions shape the narratives about students’ mathematical potentialities by influencing their actions with respect to course registration.

Theoretical Framework

Following Sfard and Prusak (2005), we take a narrative theory of identity that equates “identities with stories about persons” (p. 14), told by the self or others, each of which can exist simultaneously and change dynamically in response to contextual factors. In this view, the discursive construction of narratives comprise the identity of an individual rather than functioning as a window into some "true" sense of self that is represented by the narratives. Not all narratives about an individual count as identities, however. Sfard and Prusak (2005) define identities as collections of stories about a person that are *reifying* (they describe properties of the individual), *endorsable* (the narrator believes the story to be a faithful reflection of how things are in the world), and *significant* (changing some aspect of the story is likely to impact the narrator's feelings about the identified person). Identity-narratives are important in the context of mathematics advising because the goal of each session is to determine which course, out of a large number of options, is most appropriate for the students given who they are. In our conceptualization, *who the student is* is the collection of stories about the student that have been co-constructed within the advising session. In this study, we focus on the mechanisms through which students and advisors co-construct reifying narratives about the student during mathematics advising sessions.

Narratives have reifying qualities when utterances are about *being* or *having* (e.g., *I am a mathematician* or *I have a math brain*) rather than *doing* (e.g., *I do math*). Sfard and Prusak (2005) identify use of the *is*-sentence (e.g., *she is an able student*) as a particularly effective mechanism for reifying identity because it turns "properties of actions into properties of actors" (p. 16). In addition, they note that identity narratives can be analyzed for reifying qualities by looking for "verbs such as *be*, *have*, or *can* rather than *do*, and with the adverbs *always*, *never*, *usually*, and so forth, that stress repetitiveness of action" (p. 16). Additionally, given that the context of the interaction is an advising meeting where the information provided by the student is used to set an academic trajectory, sentences utilizing sensing processes like *love, hate,* and *want* also result in the reification of identities. For example, the statement *I hate math* functions like *I am not a person who likes math* which is a *being* process.

Given the potential impact of the reifying narratives co-constructed in mathematics advising sessions, this study seeks to address the following questions:

1. How are reifying narratives co-constructed through student-advisor interactions in mathematics advising sessions?
2. How do reifying narratives shape students' mathematical opportunities?

Method

Context & Participants

This study is part of a larger project exploring the mathematics advising program situated across three sites at Midwestern University (a pseudonym): the general student body in the liberal arts college, the honors program in the liberal arts college, and the engineering school. The participants included mathematics advisors consisting of faculty, lecturers, postdoctoral fellows, and doctoral students in the mathematics department, as well as incoming undergraduates. The data was collected over several weeks at these sites during the summer of 2019. In this study, we narrow our focus to examine 32 advising sessions from the honors program conducted by one man and one woman advisor.

Data Sources

The data corpus for the larger project consists of several sources: video-recordings of 100 advising sessions across six advisors at three sites, 100 web-based surveys from undergraduate students after their advising session, 1 video-recording and fieldnotes of the training session, and

4 audio-recordings of interviews with mathematics advisors after the advising sessions. For this study, we draw on video-recordings of the advising sessions as well as transcriptions constructed by our research team (Hammersley, 2010).

Analysis of the Co-Construction of Reifying Narratives

After applying stage and timeline codes to each transcript, we looked at the student's turns of talk and marked instances where reifying narratives were being constructed. In particular, we looked for markers of reification such as the first person equivalents of *is*-sentences, *I am* and *I was*; use of verbs such as *be*, *have*, or *can*; adverbs that stress repetitiveness of action such as *always*, *never*, and *usually*; and sentences utilizing sensing processes like *love, hate,* and *want*. After marking the reifying narratives constructed through the student's talk, we looked at the advisor's talk preceding each instance of reification to determine what the narrative was in response to. Looking ahead from each instance of reification, we created memos about how the advisor's talk took up (or did not take up) aspects of the student's narratives. For each session, we generated a diagram depicting the co-construction of narratives and indicated whether each utterance was a bid to move the student toward or away from mathematics coursework. Lastly, each session was placed in a two-dimensional matrix based on whether the student and advisor co-construction was cooperative (both moving toward or both moving away from math) or competing (moving in opposite directions), and whether the student was ultimately moved toward or away from mathematics.

Researcher Positionality

This analysis was carried out by a multi-ethnic, multi-racial (Black, White, and Latinx) research team, who identify as women. Each team member has an academic background in mathematics and is now focused on mathematics education. We use Black feminist perspectives to acknowledge the relationships between gender, race, and class within this analysis and draw on our own experiences as women working in and through mathematics spaces to inform our questions and interpretations of the data. Therefore, gender and gendered dynamics are taken as ubiquitous within social interactions and, thus, ground our analytical focus for the present study.

Preliminary Analysis & Discussion

Given constraints on space, we present an excerpt from one transcript and the accompanying diagram in Table 1. In the diagram, circles represent reifying narratives spoken by the student and squares represent responses to, and elicitations of, reifying narratives by the advisor. Arrows pointing left indicate narratives that move the student away from math, and arrows to the right represent narratives that move the student toward math. Shading is used to track particular narratives; white indicates *I plan to be a psychology major*, light gray indicates *I plan to take a math course*, and dark gray indicates *I plan to take a statistics course*.

In this session, the student repeatedly states that she enjoys math and would like to take calculus. The advisor, taking up the narrative of the student as a psychology major, repeatedly points out that math is not required and points the student toward statistics courses as more relevant. The student ultimately concedes that she may not take a calculus course, so we characterized this session as Away From Math. The co-construction in this session was classified as Competing to reflect the push and pull between the student's narrative of mathematical interest and the advisor's narrative of mathematics not being necessary.

Text	Narrative	Diagram
Advisor: Well you don't need any math for psychology. Did you want to take a math class or do you want to avoid math?	Psych majors don't need math Do you want to avoid math?	
Student: I did. I– **I do enjoy math** even though it's not really a part of psychology, but **I still really enjoy math**. Like **I am thinking of taking statistics**, but I wanted to. **I'm thinking also of including calculus** if I have enough room for it.	I enjoy math. I really enjoy math. I'm thinking of taking statistics. I'm thinking about taking calc.	
Advisor: So stats and math are two different departments here and I agree that if you're going into psych you're probably gonna have to take some stats and that's probably more relevant. But if you want to take math that certainly. (I encourage you to do that).	Stats isn't math. Stats is more relevant for psych. If you want to take math, I encourage that.	
Student: Yeah, **I really enjoy it**. […]	I really enjoy math.	
Advisor: [your AP score] means that you don't have to take calc one. So if you're going to take calculus, it would make sense to start with calc two. But you don't need that for psych. So, th- the only reason you should take calculus is if you really want to take more calculus.	You placed into Calc 2 You don't need Calc 2 for psych The only reason to take Calc 2 is if you really want to take more calc.	
Student: Okay. Yeah. I- **I'm pretty interested in taking it**, but **I think I'm actually going to talk to my advisor first**	I'm pretty interested in taking calc I may not take calc	
Advisor: I think that makes sense.	Not taking calc might make sense	

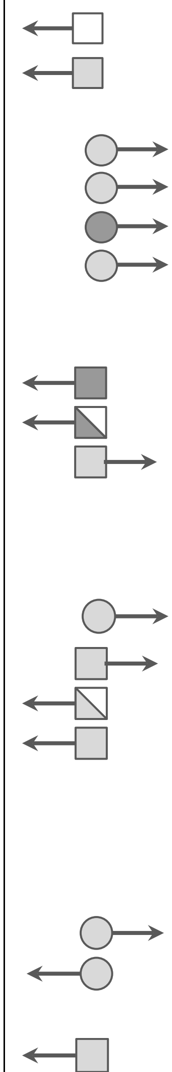

Figure 1. A transcript excerpt from a math advising session, reifying narratives, and diagram of co-constructed narratives.

This preliminary analysis included eight (of 32) randomly selected sessions from the honors college. Of those sessions, three were characterized as Cooperative/Toward Math and one was characterized as a Competing/Toward Math. All four of those sessions were with students who identify as men. Of the four sessions characterized as Away From Math, one was Cooperative (with a Woman student) and the other three were Competing (two women and one man). Although we expect to find variability as we analyze more sessions, it is notable that all three randomly selected sessions that have women students featured co-constructed narratives that moved these women away from further mathematics study. While there are many reasons for women to not pursue the study of advanced mathematics, that decision has long-term implications for the range of academic majors and careers that remain viable options. The results of this study illustrate how interactions at the timescale of seconds and minutes can impact the mathematical trajectories of students and act as a site of attrition for women in STEM.

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Examining A Network of Communities Focused on Transforming Mathematics Instruction

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Communities of practice (CoPs) offer a structure for individuals with common goals to engage in collective learning (Wenger-Trayner & Wenger-Trayner, 2015). The CoMMIT Network was developed to support regional Math CoPs of undergraduate mathematics faculty interested in using active learning and inquiry teaching approaches in their courses. This study utilized the value framework (Wenger et al., (2011) to interrogate the value participating in the CoPs and COMMIT Network provides participants.

Key Words: communities of practice, active learning, inquiry teaching

Mathematics courses at institutions of higher education often act as "gatekeeper courses," leaving many students with minimal access to or interest in pursuing STEM careers. One strategy for increasing success is to use evidence-based teaching methods, like active learning (Freeman et al, 2014; Theobald et al, 2020). Despite research that supports this approach, traditional lecture-style teaching is still the predominant method of instruction in many college mathematics courses (Jaworski & Gellert, 2011; Laursen et al., 2019). Using networks and communities of practices (CoPs) to initiate, promote, and sustain change is one method shown to overcome these barriers to instructional change (Austin, 2011; Gehrke & Kezar, 2017). The COMMIT Network provides a flexible structure to support regional CoPs of undergraduate mathematics faculty in their mission to influence instructional change and improve student learning outcomes.

Methods

This study focused on two research questions: 1) To what extent does the COMMIT Network create value for the regional CoPs? 2) What aspects of the COMMIT Network model are most valuable to individual faculty participants, CoP leaders, and the Network? We utilized the value creation framework (Wenger et al., 2011) and descriptive analysis to qualitatively examine the layers of value identified by both faculty participants and CoP leaders to understand how the COMMIT Network can best support the sustained, systemic implementation of evidence-based teaching practices in institutions of higher education.

Results and Implications

Faculty participants shared finding immediate value in the ability to network with like-minded peers and share resources and ideas across their CoPs. CoP leaders expressed the importance of the COMMIT Network in providing a structure that supports: 1) regularly scheduled activities, 2) opportunities for networking, 3) structured channels of communication, 4) distribution of leadership, and 5) mentoring, which helps the CoPs within the Network to develop a sense of collective identity and legitimacy.

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The Value of COMMIT-ing to Teaching with Inquiry: Examining a Network of Communities Focused on Transforming Mathematics Education

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Introduction

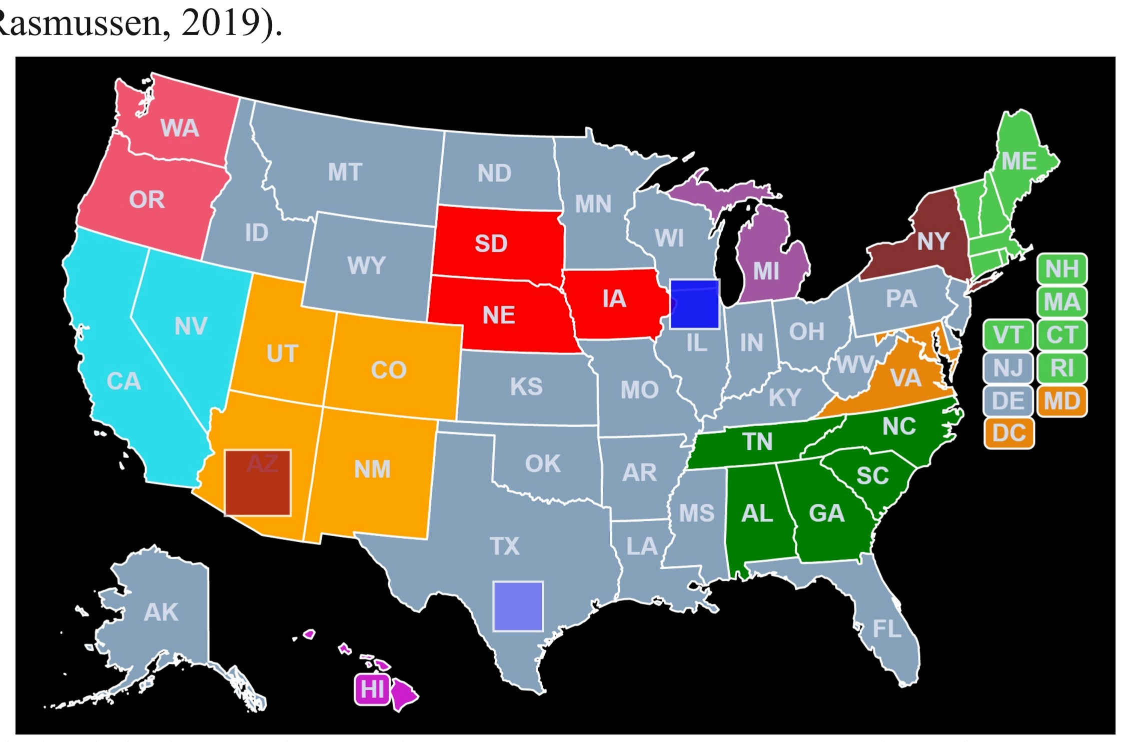
The 21st century workforce needs highly qualified STEM professionals. It is projected that STEM jobs will grow 8% over the next ten years (Seitz, 2021). Mathematics courses at institutions of higher education (IHEs) often act as "gatekeepers," leaving many students with minimal access to or interest in pursuing STEM careers. One strategy for increasing success, including for students from historically under-represented groups, is to use evidence-based teaching methods, such as active learning (Freeman et al, 2014; Theobald et al, 2020). Despite research supporting this approach, traditional lecture-style teaching is still the predominant method of instruction in many college mathematics courses (Jaworski & Gellert, 2011; Laursen et al., 2019; Nolan, 2006, 2010). Using networks and communities of practices to initiate, promote, and sustain change is one method shown to overcome these barriers to instructional change (Austin, 2011; Gehrke & Kezar, 2017) and is the focus and exploration of this project. Communities of practice (CoP) are groups of people with common interests who engage in shared learning via ongoing, regular interactions and collaborate toward a common goal(Lave & Wenger, 1991; Wenger-Trayner & Wenger-Trayner, 2015).

Context

The **COMMIT (COMmunities for Mathematics Inquiry in Teaching) Network** provides a common infrastructure and flexible structure to support regional communities of practice in their mission to influence instructional change, address faculty isolation, and ultimately, improve student learning outcomes. The COMMIT Network collectively spans over half of the United States, with over 800 educators actively involved. Each regional COMMIT brings high quality professional development to their local members, focused on infusing active learning, equity, and inquiry into college mathematics classrooms.

A regional **COMmunity for Mathematics Inquiry in Teaching (COMMIT)** is a local group of college mathematics educators interested in practicing and disseminating teaching and learning techniques centered on student inquiry.

COMMIT Members center their classrooms on the Four Pillars of inquiry that include: **(1)** ability for instructors to foster equity in their design and facilitation choices , **(2)** deep student engagement with coherent and meaningful mathematical tasks, **(3)** opportunities for students to collaboratively process mathematical ideas with peers, and **(4)** opportunities for instructors to inquire into student thinking (Laursen & Rasmussen, 2019).



The highlighted areas below are where current COMMIT communities are located.

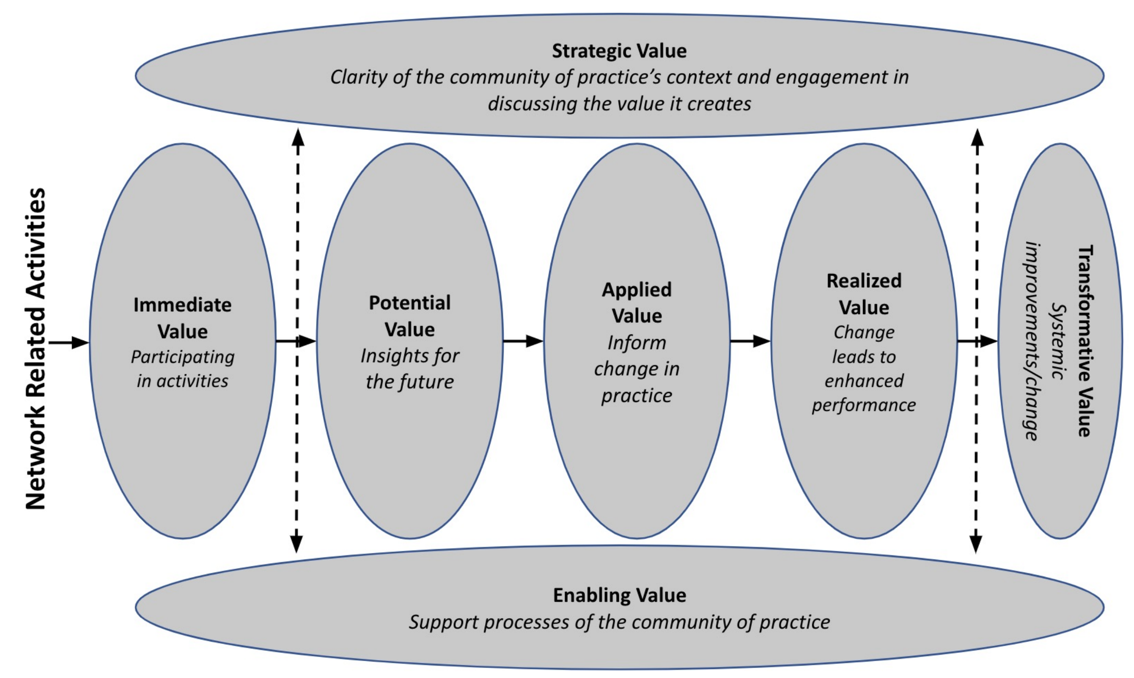
Research Questions

1) To what extent does the COMMIT Network create value for the regional COMMITS?

2) What aspects of the COMMIT model are most valuable to individual faculty participants, COMMIT leaders, and the broader network?

Theoretical Framing

Utilizing Wenger et al.'s value framework (Wenger et al., 2011), we examine the multiple layers of value identified by both faculty members and leaders as we seek to understand how the COMMIT Network can best support the sustained and systemic implementation of evidence-based teaching practices in IHEs. The framework incorporates five cycles of value creation - immediate (in the moment resources, information, connections), potential (for the future), applied (tested implementation), realized (actualized implementation), and transformative (broader dissemination to others). In addition to the cycles, value at the CoP level is also supported by strategic value (clarity of the context and vision, ability to engage in strategic conversations) and enabling value (support processes that make network life possible).



Findings

The Value the COMMIT Network Creates for Regional COMMITs

Interviews of COMMIT regional leaders in Year 2 of the project indicate that the ability to share resources and ideas and to develop interpersonal connections as the most prominent "in the moment" Immediate Value aspects of engaging in COMMITs. One CoP leader explained,

> *"We have been trying to learn from others and what they are doing…it has been inspiring for us." The opportunities to connect with like minded peer groups also surfaced as important to CoPs."*

Leaders shared that it was important to "...have access to the leadership teams and a shared voice" as they developed processes and events for their region and having access to other faculty and CoPs allowed them to build strong interpersonal relationships with colleagues, not only within their own COMMIT but also from across the network.

Findings (Continued)

Regional COMMIT leaders identified the role that the Network plays in fostering a sense of collective identity around utilizing teaching with inquiry practices. Leaders noted the importance of this in developing a sense of purpose and support to "change the way we think about effective teaching" in their CoPs and at their individual institutions. "Having the general CoP Network in the background helps us generate some legitimacy," explained a leader. "It is not something we are making up because it is fashionable...that support from the back is helpful. We have a more defined structure to what we were trying to do more organically."

Finding Value in the COMMIT Network Model

CoPs are often ephemeral—they develop organically out of shared need, but often dissolve, particularly when they do not exist within a formal organization. One of the goals of the COMMIT Network project has been to examine the structures and supports that individual CoPs need to sustain over time in order to facilitate long term, systemic change in mathematics education. As such, it is important to understand the aspects of the Network model itself that leaders and participants view as being integral to the health of their COMMIT.

- **Regularly scheduled, ongoing events**
- **Formal and informal opportunities for faculty to network and share ideas/resources**
- **Structured channels of communication within CoPs and across the Network**
- **Distributing leadership by recruiting and training new and upcoming leaders in each CoP**
- **Mentoring support for individual participants, workshop and activity facilitators, and COMMIT leaders provided by COMMIT members and the Network**

These forward-looking insights aligned with our consistent evidence of the value both individuals and regions find in the strong community connections within the network. When looking at the future goals of each region, it is often about engaging within the network in addition to continuing to expand the reach of the network and its regions.

> *"It's nice… our community has been around for awhile. We have a history of things we have done and tried, we could give that experience and share that history with new CoPs to help out."*

> *"There is something powerful about the regional anchoring of things but there is enough connection that regions can inform each other. There are many similarities to having a network where we can learn together what works well."*

Quick Facts About the Network

13 current regional COMMITs (8 included in the study), which expanded from the original 4 regions funded in 2019.

438 documented participants engaged in one or more COMMIT Network activities over the past 3 years (e.g. book clubs, teas, workshops, classroom observations).

213 mini-grants awarded to faculty to support their participation in activities such as peer classroom observations, course collaborations, and workshop planning/facilitation.

78.5% of surveyed participants indicated expanding their COMMIT Network with math faculty peers by engaging in at least one inquiry teaching activity

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The Story of Sunny: Consequences of Faculty Mentoring for Perception of Mathematics Career Pathways

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Keywords: pure mathematics majors, faculty mentorship, women in mathematics, perception of career pathways, mode of belonging

Numerous studies have underlined the persistent gender gap in mathematics related fields as a potential focus to be explored further with respect to learner identity, self-concept, stereotype threat, and sense of belonging in the discipline (Blackburn, 2017; Lahdenperä & Nieminen, 2020). The notion of belonging was cited as an important construct to investigate and understand factors in women's decision to pursue a career in mathematics (Good et al., 2012). Therefore, we investigated the "mode of belonging" of women with regard to engagement, alignment, and imagination components using Wenger's (1998) framework. For the purposes of this poster, we emphasize the imagination component of mode of belonging to address women's perception of career pathways in pure mathematics and its link to faculty mentorship.

We adopted narrative inquiry to examine women and underrepresented students' experiences in various mathematics environments (Clandinin & Rosiek, 2007). For the purposes of the study, we selected Sunny as the focus case among five participants. The study has longitudinal characteristics since Sunny participated in one-hour interview and weekly seminar discussions in Spring 2020. She also participated in three interviews and eleven weekly reflections in Spring 2022. Sunny graduated in Spring 2022. We focus on the case of Sunny, who began as a pure mathematics major, and then later pursued a career path in actuarial science. We concentrated on the aspects that led Sunny to switch from pure mathematics to actuarial science. Sunny's case was intriguing because she expressed a passion in pure mathematics and stated that up until her third year, she was interested in pursuing a research career in pure mathematics. Sunny changed her mind, though, after being discouraged by her Topology professor in her pursuit of research opportunities. Instead, she decided to pursue a career in actuarial science, which does not involve the pure mathematics and proofs that she most appreciated in her studies. Sunny remarked that this professor made it seem as though she lacked the ability to involve in research as an undergraduate student and was unhelpful in helping her find a suitable research topic for her directed independent study. Although Sunny had taken for granted a career in academia up until her third year, she considered this particular moment as a tipping point in which she realized she needed to shift her attention away from pursuing graduate-level study in pure mathematics.

It is worthwhile to examine the shift in Sunny's intentions: her initial plan to attend graduate school to pursue research, which morphed into a career path in actuarial science after an undesirable outcome from consulting with her Topology professor. We focused on Sunny's story because, despite her outstanding success in her major, Sunny's experiences revealed potential aspects of women's experiences in the mathematics major that could impact women's decisions not to pursue a career in pure mathematics. Despite earning a degree in pure mathematics, Sunny claimed that she did not feel valued or accepted in her program since she did not feel as a contributor to pure mathematics research. Our research indicates that faculty mentoring impacted women's decisions to pursue careers that are (or are not) in line with their academic background because women in STEM are more prone to social and institutional influences (Xu, 2017).

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Gamification and a Leaderboard-Based Mathematics Game

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Bowling Green State University	BGSU	BGSU

One of the most used game design elements in gamification design is leaderboard, a scoreboard showing participants' current scores and rankings. Though many studies suggest the positive effects of leaderboard on participants' learning and motivation (Kalogiannakis, Papadakis, & Zourmpakis, 2021), research also shows that not all students benefit from the use of leaderboard (Andrade, Mizoguchi, & Isotani, 2016; Nicholson, 2013). Based on self-determination theory (SDT) (Ryan & Deci, 2017), we designed a study where undergraduate math students completed a leaderboard-based review, and addressed two questions: (**RQ1)** How are students' perceived autonomy and competence associated with their enjoyment and intention of continued participation? (**RQ2)** How is student actual competence associated with their enjoyment and intention of continued participation?

Method

In an undergrad math course, *Real World Math Skills,* we created a gamified quiz (Quizalize, n.d.) to review content that focused broadly on *Contending with Change* (e.g., linear & exponential modeling, & rates of change). During class, 47 students answered 24 Quizalize problems, while a leaderboard was projected in real time. Finally, students completed a survey with 19 Likert questions (1: strongly disagree to 7: strongly agree). These survey items also aligned with a specific sub-category (Competence, Autonomy, Enjoyment, & Intention of continued participation) and we considered the mean of students' ratings on survey items in each sub-category. We used student performance data (i.e., total number correct on quiz) to represent student actual competence.

Findings

Considering the correlations among variables, student actual competence was not correlated with either enjoyment or intention of future participation. Instead, student perceived competence was significantly correlated with both. We also found strong correlations ($r >= +.70$) (Creswell & Guetterman, 2019) between perceived autonomy and both enjoyment and intention of future use. Based on these correlations, we ran linear regressions to understand how well perceived competence and perceived autonomy predict student enjoyment and intention of future participation, respectively. The findings showed that perceived competence and perceived autonomy significantly predict student enjoyment and intention of future participation.

Significance of the Study

Our findings suggest that effects of leaderboard-based math practice on student enjoyment and intention for future participation are varied. These variations were not related to student actual competence. Instead, they were related to students' perceived competence and autonomy as predicted by SDT. Regarding Math Education and gamification implications, findings helped researchers and educators understand why and how students responded to gamified activities differently. Thereby offering a way to explain inconsistent findings in leaderboard research and providing insights about how to tailor leaderboard-based activities to optimize effects of intervention on all students.

Acknowledgement

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SYLLABUS 2.0: USING VIDEO TO MAKE THE MATH SYLLABUS ACTIVE

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INTRODUCTION

There is a widely held view that a college course syllabus serves as a contract for students rather than a teaching and learning opportunity (Fornaciari & Lund, 2014). Earlier studies on syllabus development have focused on necessary components to include in the syllabus, such as professor information, course information, disability information, student support services, grading, and technology requirements (Doolittle & Siudzinski, 2010; Garavalia et al., 1999; Jenkins et al., 2014; Perrine et al., 1995). A second wave of studies focused on how the syllabus is presented in terms of professor tone (Harnish & Bridges, 2011) and length and use of images (Harrington & Gabert-Quillen, 2015; Nilson, 2002). More recent research on syllabus development has expanded on previous design considerations and has focused on accessibility and universal design for learning (Womack, 2017), the infusion of diversity, equity, and inclusion across the various components of the syllabus (Fuentes et al., 2021), and the inclusion of social justice elements across sections (Taylor et al., 2019). Given the wealth of research available on what components to include in a syllabus and possibly how that affects instructor perception, there is a lack of research on how to present the syllabus in a way that engages students in learning about the course. Our study examined the implementation of two multimedia syllabi developed for large lecture math courses: a graphic syllabus and a video version.

RESEARCH QUESTIONS

1) Which components of a video syllabus do students access the most frequently?

2) What are students' attitudes toward an undergraduate math course after reviewing its syllabi?

3) How does the syllabus modality (graphic or video) impact students' posttest performance on syllabus content?

CONTEXT/METHOD

Two sections of students in Precalculus College Mathematics (N=172) participated in the study. All students were first-semester School of Engineering students in a one-semester Precalculus course.

One section was assigned a graphic syllabus, whereas the other was assigned a video syllabus. After engaging with the syllabus, 107 students consented to take the syllabus survey, which represents 62% of the 172 students enrolled in the class. Sixty-one (57% of the sample) of these students had been assigned the graphic syllabus and 46 (43% of the sample) the video syllabus.

ANALYSIS

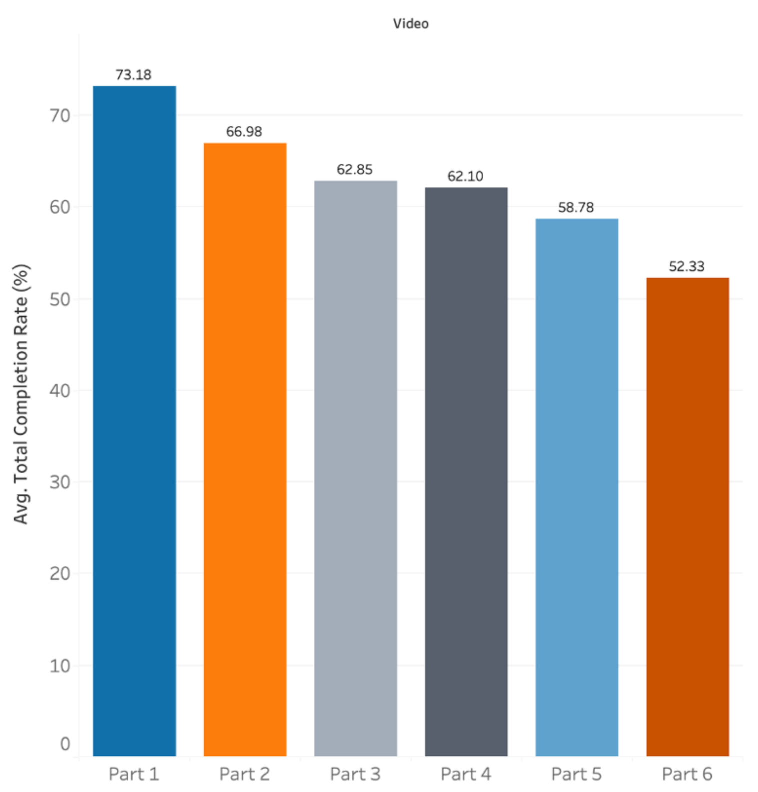
As most of our data was quantitative, we ran descriptive statistics for each of our survey and quiz questions to describe the data set. We also ran a t-test between groups to see if having either syllabus experience resulted in a significant difference in overall syllabus quiz score or a difference between each quiz or survey item. For the two survey questions that had qualitative responses, we used open coding to determine whether any themes emerged from the students' responses. For example, we classified both positive and negative reactions to the syllabi based on the one-word descriptions used to describe the syllabus. Video-watching data was extracted from Kaltura and organized into charts for ease of interpretation.

RESULTS

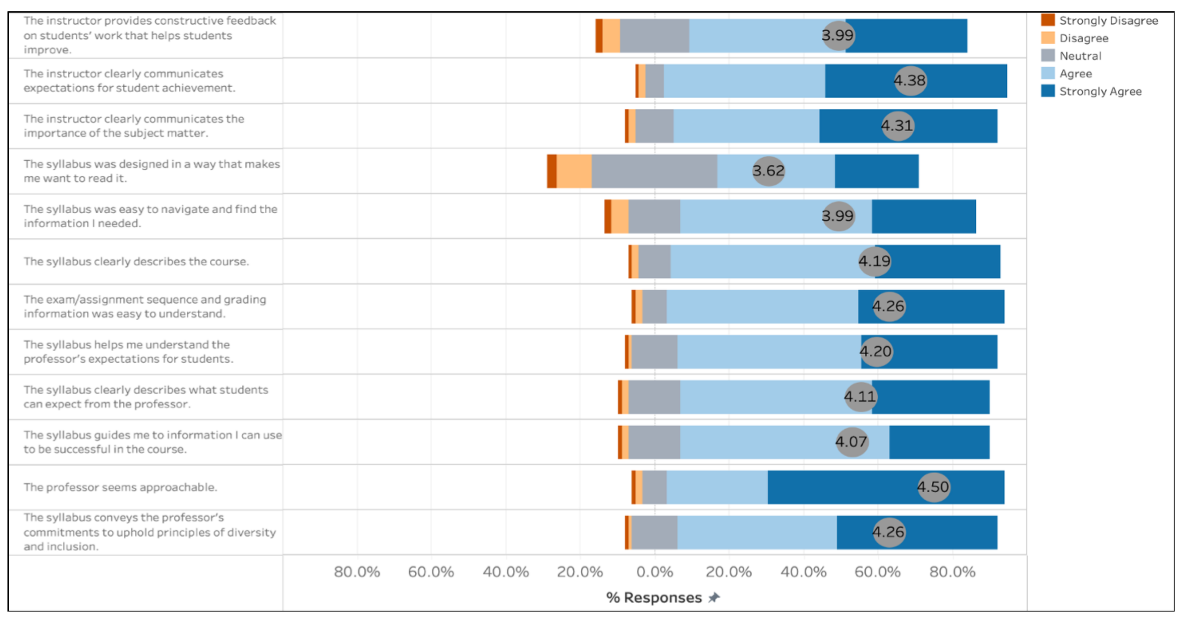
Findings indicate no significant difference between students' syllabus quiz performance and overall survey responses between groups.

Students responded favorably to the survey regardless of syllabus type (100% positive on graphic syllabus, 91% positive on video).

Last, there was no observed relationship between the length of each syllabus segment video and the frequency at which students watched each segment.



Distributions and Means of Responses to Syllabus Perception Questions



Note. The length of the bar denotes the percentage of instructors who selected that option. The center of the middle item (Neutral) is anchored at 0%. The gray circle on the bar shows the mean.

IMPLICATIONS

The results of our study indicate that first-semester college students may need additional explicit instruction on what a syllabus is, how to use it, and why it is important. Through our data analysis, we uncovered that there could be more effective ways of engaging students with the syllabus. When looking at this data coupled with some students' desire for a paper-based syllabus, we suggest offering students the option of viewing a graphic syllabus or video syllabus to accommodate learning preferences. To increase implementation fidelity and ensure students engage with the syllabus, we offer two suggestions for implementing a video syllabus and a graphic syllabus: a video-based syllabus quiz with embedded prompts and socially annotating the syllabus. Our next study will investigate this implementation.

Related literature

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Meritocrats, Wallflowers, and More: Characterizing Obstacles to DEI Engagement

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Keywords: diversity, equity, inclusion, post-secondary STEM education

There is a pressing and continuing need to bolster equity and inclusion for marginalized people in all aspects of American society, including undergraduate STEM education. This paper draws on data from a broader study (Johnson et al., 2022) which collected survey data from postsecondary STEM instructors regarding their attitudes toward diversity, equity, and inclusion (DEI) efforts and how those views changed as a result of the events of 2020 (e.g., the onset of the COVID-19 pandemic, nationwide protests in support of racial justice). While many participants reported changes in their views, this report focuses on those who stated that their views were unchanged. In particular, we present profiles of the attitudes of 21 people who were not previously invested in social justice, and did not change their views in 2020. From these data, we identified archetypes of those who hinder DEI efforts in hopes of elucidating their motivations.

Theoretical Perspective

Both the original study (Johnson et al., 2022) and this one are informed by critical race theory (CRT), specifically where this framework intersects with education (Ladson-Billings, 1998). CRT posits that American educational structures are inherently beholden to a "White supremacist master script" (p. 18) under which minoritized students are seen as deficient. In particular, our study reports on the perceived differences in talent, opportunities, and interest in STEM fields among racial groups.

Data Reduction, Collection, and Analysis

Approximately 1300 post-secondary mathematics instructors were originally surveyed (Apkarian et al, 2022), 305 of whom responded to a follow-up survey. Of these respondents, 21 indicated that they had no DEI engagement prior to 2020 or thereafter. For these individuals, we compiled their race, gender, and beliefs on racial differences in STEM. We also captured their free responses which elaborated on their lack of engagement. In the spirit of grounded theory (Corbin & Strauss, 1998), these responses were coded to find themes to their motivations.

Results and Discussion

Of the 21 respondents in our data set, 18 were white men, perhaps suggesting that a privileged position obscures the need for DEI advancements. Five archetypes emerged from analyses of their responses – *opponents* who actively resist DEI efforts, *meritocrats* who believe that STEM is inherently neutral (e.g., Alvarado, 2010; Au, 2013; Liu, 2011), *good apples* who suggest that respect for all is sufficient (Martin, 2003), *wallflowers* who cite higher priorities elsewhere, and *exceptionalists* who posit that DEI engagement is only needed elsewhere. Understanding the motivations behind DEI hesitance is essential in order to overcome them.

Acknowledgements

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Meritocrats, Wallflowers, and More: Characterizing Obstacles to DEI Engagement

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Background and Original Study

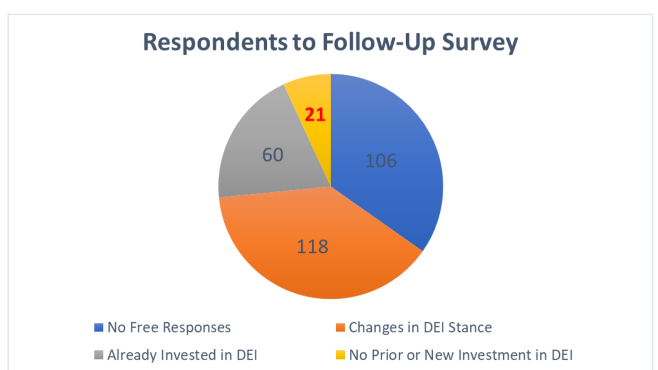
- Data taken from **Evaluating the Uptake of Research-Based Instructional Strategies in Undergraduate Chemistry, Mathematics, & Physics**, an NSF-funded study.
- Approximately 1300 post-secondary math instructors were surveyed
- Respondents asked how events of 2020 affected views and actions with respect to diversity, equity, and inclusion.
- Apkarian et al., 2022 focused on changes reported by those surveyed.

Framing of Current Study

- Now focused on those who were not previously invested in DEI efforts and indicated no change following 2020.
- Goal was to characterize those who might present resistance to DEI efforts and progress.

Data Reduction

- 305 respondents completed follow-up survey.
- 199 of these provided free responses
- 21 self-reported as having no prior investment in DEI efforts and no changes following 2020



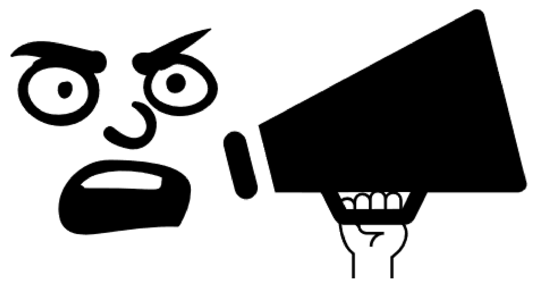
- Demographic information and free responses gathered for this subset of 21 respondents.

Data Analysis

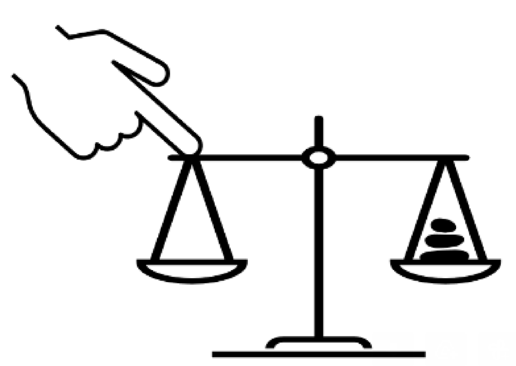
- Open coding (Corbin & Strauss, 1990) performed on free responses to identify reasons given for lack of DEI engagement
- Axial coding revealed five categories with common themes or motivations.
- Categories reframed as character archetypes.

Archetype Detail and Example Responses

Opposition (n = 2)



- Both respondents are white men.
- Neither believes they are personally responsible for DEI.
- Example Responses
  - *"It is actually annoying to see how this agenda gets pushed down people's throats."*
  - *"Affirmative action has gone too far and should be eliminated."*



Meritocrats (n = 6)

- All men (5 white, 1 Hispanic)
- 4/6 believe that DEI is beneficial to STEM
- Example Responses
  - *"Mathematics is inclusive for every race…This has nothing to do with equity issues."*
  - *"Personal equity issues are a bit like…disabilities such as autism or dyslexia or depression…Success is the best cure and so empowering."*

Good Apples (n = 6)

- All white (5 men, 1 woman)
- All agree that students from different groups have equal aptitudes
- Example Responses
  - *"I would just like to teach, treating each person as infinitely precious in the eyes of God."*
  - *"Both before and during 2020, I have tried to maintain a principled, fair, conservative position on diversity."*
  - *"I support diversity and opportunities for all."*

Wallflowers (n = 5)

- 3/5 believe they have a personal responsibility to DEI engagement
- Example Responses
  - *"We are teaching online right now. That is the focus of my teaching."*
  - *"I am not on campus and do not have opportunities to participate in diversity events."*
  - *"My views continue to be the same: less focused on equity and more on the overall unfair treatment of adjunct teachers."*

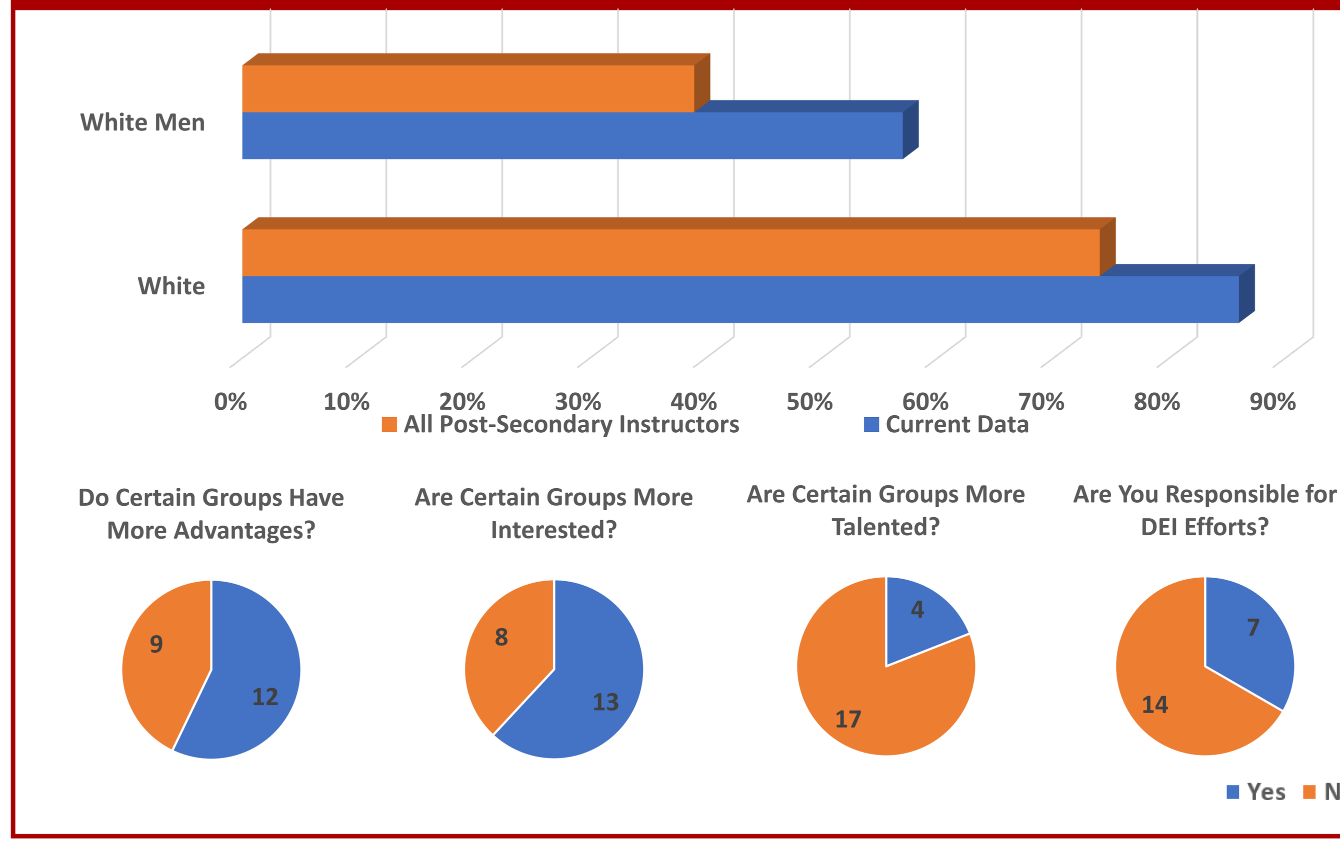
Exceptionalists (n = 7)

- 6 men, 1 woman
- 6 white, 1 declined to identify their race
- Example Responses
  - *"While the return of overt personal racism in recent years is dismaying, I just don't see it in math and science."*
  - *"There isn't an issue with [DEI] in any of the department I've worked in. I've never witnessed such issues in the workplace."*

Overview of Archetypes

- Opposition
  - *DEI efforts are inherently wrong.*
- Meritocrats
  - *STEM fields are inherently neutral.*
- Good Apples
  - *I'm not a part of the problem.*
- Wallflowers
  - *Higher priorities exist elsewhere*
- Exceptionalists
  - *The problem may exist, but it doesn't exist **here**.*
- **Note**: Some respondents classified as multiple archetypes.

Summary of Reduced Data Set



Discussion and Key Takeaways

- Majority of data set is white, perhaps suggesting that privilege conceals the need for DEI advancements.
- In this study, most obstacles to DEI are moderate, as opposed to vehemently opposed to DEI efforts.
- This study corroborates prior research on belief in meritocracy (e.g., Alvarado, 2010; Au, 2013; Liu, 2011).
- Good Ones archetype are consistent with "for all" rhetoric (Martin, 2003).
- Understanding the motivations behind people's reticence to embrace DEI efforts may facilitate engagement with them.

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Initial data coding performed by Jason Guglielmo and Matthew Park
Archetype illustrations provided by Jessica Ruiz.

Research on Mathematical Proof: A Review of RUME Conference Proceedings 2020-2022

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Keywords: Proof, literature review, methodology

Mathematical proof is one of the major topics of research in the RUME community. In recent years, we have anecdotally noticed a trend in RUME that proof research is often conducted in laboratory settings rather than within authentic classrooms. Further, we, along with our collaborators, have special interest in researching proof as a collaborative activity (Bleiler-Baxter et al., 2021, 2023; Heath, 2022; Heath & Bleiler-Baxter, under review; Heath et al., 2022) because of our conviction that proof is a "communal, negotiated, and sense-making process" (Ko et al., 2016, p. 618). Yet, research on collaborative proving is still considered a new direction for proof research (Heath & Bleiler-Baxter, under review). In this project, we seek to review recent literature produced in the RUME conference proceedings to identify trends in up-and-coming research on proof and proving at the undergraduate level. We seek to answer the question: What kinds of research are being conducted on proof and proving in the RUME community?

Methods

In a methodical search of RUME conference proceedings (Karunakaran & Higgins, 2021, 2022; Karunakaran et al., 2020) during the period 2020-2022, we included contributed, preliminary, and theoretical reports containing the words "proof" or "proving" in their title. Two researchers developed and revised categories of interest for the reports, separately coded all reports according to these categories, and resolved any disagreements. The coding categories included: research conducted in real classrooms, survey research, interview research (non-task-based), task-based interview research, artifact analysis, assessment/inventory measures, text/document analysis, collaborative proof research, and early-career scholar first authorship.

Findings

A total of 30 reports qualified for inclusion in the review: 11 out of 148 reports (7.4%) in 2020, 4 out of 49 reports (8.1%) in 2021, and 15 out of 145 reports (10.3%) in 2022. Notable findings include the strong presence of interview research on proof (17/30 reports, 56.7%). Moreover, an overwhelming majority of interview-based research involved task-based interviews (13/17 reports, 76.5%). Our suspicions of a lack of research in authentic classroom contexts were confirmed, as only 6 reports (20%) used data collected from a real classroom. The number of reports concerning collaborative proving has slowly grown, with 2 reports in 2020, 3 reports in 2021, and 4 reports in 2022 concerning collaborative proving. Although this is still a minority of proof research at RUME in the last three years (9/30 reports, 30%), we noticed an overwhelming majority of RUME reports on collaborative proving had early-career first authorship (8/9 reports, 88.9%) compared to the presence of early-career scholarship in the reports included (22/30 reports, 73.3%). We will include a complete summary of the findings in our poster presentation, including trends in proof research in RUME over the past three years.

This review of literature can help the RUME community identify trends in proof research and identify ways in which the research base can grow. During the poster session, we will solicit feedback for how to continue this investigation and questions about trends in proof research the RUME community would like to see answered.

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Research on Mathematical Proof: A Review of RUME Conference Proceedings 2020-2022

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Middle Tennessee State University

Mathematics and Science Education Doctor of Philosophy Program

Focus of Review

Mathematical proof is one of the major topics of research in the community of Research in Undergraduate Mathematics Education (RUME). We aimed to review literature produced in the RUME conference proceedings from the past three years to identify trends in up-and-coming research on proof and proving at the undergraduate level.

Research Question: What kinds of research are being conducted on proof and proving in the RUME community?

Method and Procedure

In a methodical search of RUME conference proceedings during the period 2020-2022, we included contributed, preliminary, and theoretical reports related to proof research containing the keywords "**proof**" or "**proving**" in their title. A total of 30 reports qualified for inclusion in the review: 11 out of 148 reports (7.4%) in 2020, 4 out of 49 reports (8.1%) in 2021*, and 15 out of 145 reports (10.3%) in 2022.

Two researchers developed and revised categories of interest for the proceedings papers and separately coded all reports according to these categories.

*There was no RUME Conference in 2021 due to COVID-19. In lieu of a face-to-face conference, the organization published a set of research reports open only to early-career scholars as lead authors.

Authorship of RUME Proof Reports 2020-2022 (All Authors)

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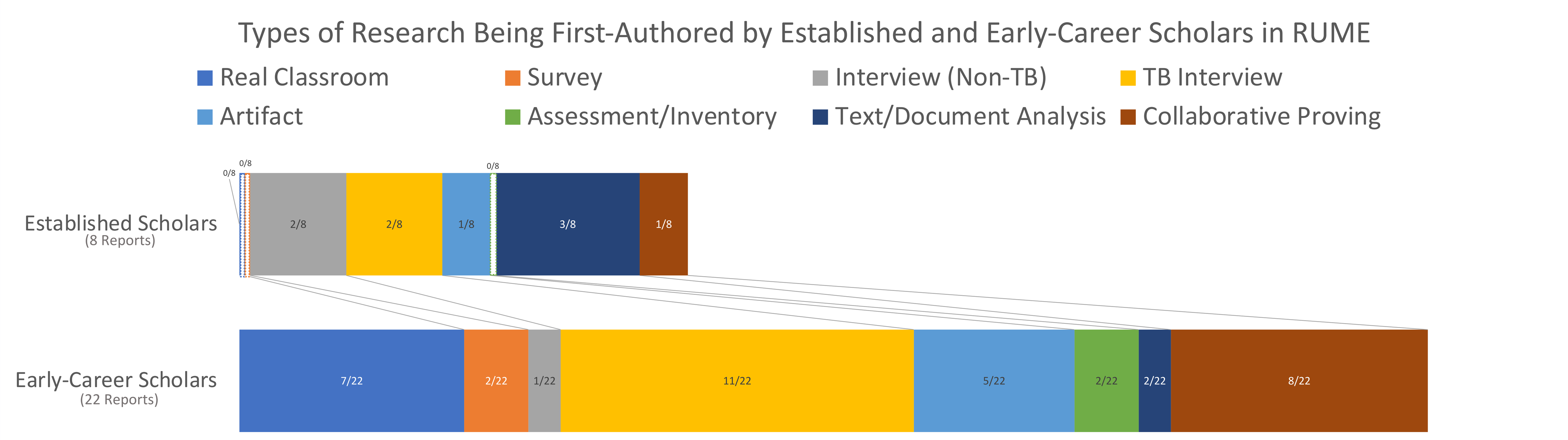
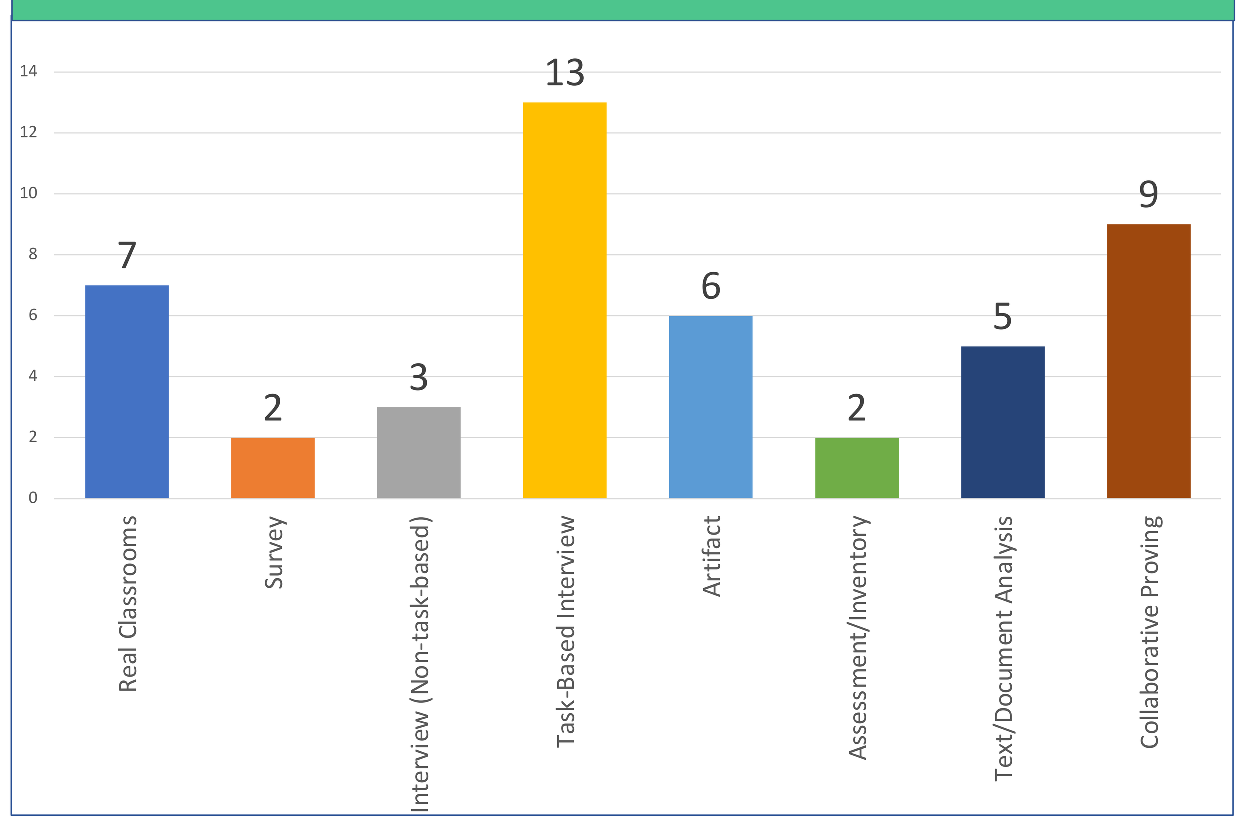
Trends Among Early-Career Scholars in Proof Research



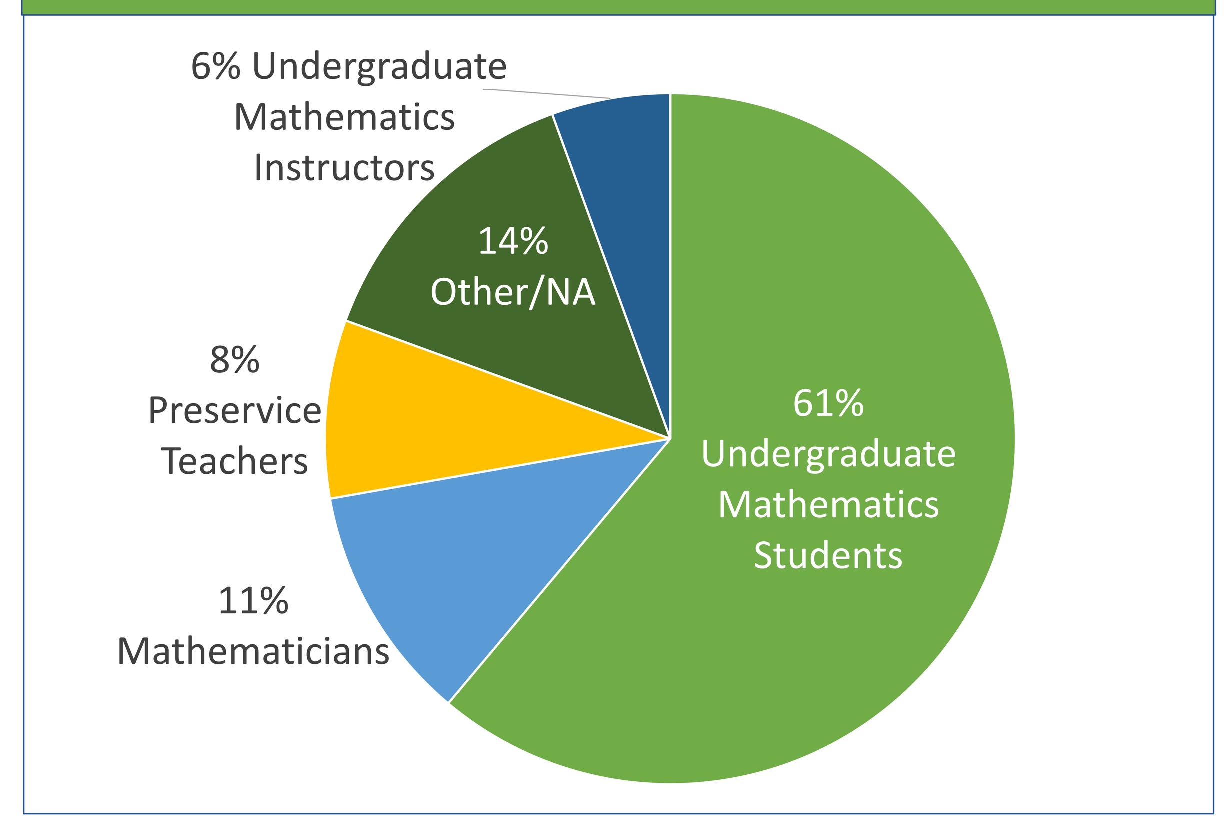
Adopting the definition of "early-career scholar" from the 2021 RUME Reports request for proposals:

- Current graduate or undergraduate student
- Post-doctoral scholar
- Pre-tenure faculty members
- Faculty who are not in a tenure-track position, but are at most 6 years removed from their graduate degree

Data in Current Proof Research



Populations of Interest



Code Definitions

Code	Description
Real Classroom Research	The study employed data collected from an instructional space, either physical or virtual, shared by instructor(s) and students, which mimics as closely as possible the naturalistic setting of their class meetings on any given day. This does not include classroom lessons of pre-service teachers delivering lessons as a course assignment.
Survey Research	The study employed data collected through a survey of opinions or experiences of participants.
Assessment/Inventory Measure(s) Research	The study employed data collected through an assessment or inventory, which assigned a score of some kind (e.g., pre-test/post-test, self-efficacy inventory).
Interview Research (Non-Task-Based)	The study employed data collected through a general interview not rooted in a think-aloud mathematical task.
Task-Based Interview Research	The study employed data collected through an interview rooted in having participants narrate their thoughts and actions while completing a mathematical task.
Artifact Analysis	The study employed data from classroom or course artifacts such as homework, classwork, or written reflections.
Text/Document Analysis	The study analyzed written resources (textbooks, exams, research publications, etc.) in order to answer a research question about the state of available resources.
Collaborative Proving Research	The study employed data from the context of collaborative proving (whether in a classroom or in an interview setting) or the study aimed to investigate collaborative proving as a primary goal. It is not enough for the course context to involve collaborative proving or collaboration without it being an element of the research.
Research Subject	A categorical variable describing the population of interest in the report. Pre-Service Teacher, In-Service Teacher, Undergraduate Mathematics Instructor, Mathematician, Multiple of these, or Other/NA.

Results

This review discovered three trends in undergraduate proof research in recent RUME conference proceedings:

01
The RUME Proceedings Reports concerning proof are dominated by early-career scholars, but there may be a new wave of interest in research on collaborative proving from and conducting research in real classrooms among early-career scholars in RUME.

02
Research was most often conducted in laboratory settings, with frequent use of task-based interviews

03
Research was most often focused on undergraduate math students.

Discussion

Why Conference Proceedings?

We chose to first look at the RUME conference proceedings because these best reflect the current research happening in undergraduate mathematics education research, as the process of publishing in a peer-reviewed journal may take an extended period of time and not best reflect current endeavors.

Future Directions

As we recognize some proof research may have been overlooked due to our search methods, we intend to expand this search in the future to more systematically identify research related to proof. This may occur either within the RUME proceedings by attending to the primary and secondary categories listed in accepted RUME submissions, or by expanding our keyword search in reputable research journals.

Questions

- When we expand the search to journal publications, what journals should we consider in order to get a broad representation of the field?
- What kinds of questions do you have about the proof research being conducted in RUME?

References, Contact, and Proceedings Paper

How one GTA used Data to Promote Student Engagement

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Derek Williams
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Keywords: student engagement; social engagement; GTA development; group work

Student engagement is essential for rich learning (Middleton et al., 2017). Research shows that student engagement consists of emotional, social, cognitive, and behavioral components (Middleton et al., 2017); tends to decline in mathematics classrooms over any time interval (e.g., Martin et al., 2015), in-the-moment states fluctuate (e.g., Williams, 2022); and that classroom environments can be designed to promote engagement (Fredricks et al., 2004). Given research on college instructors attending to engagement is rare, the purpose of this study is to investigate how one GTA reflected on student self-reports of engagement during a precalculus course to promote engagement. We also discuss whether and how students responded to the GTA's strategies.

Theoretical Framework

Classrooms are dynamic environments influenced by a myriad of factors. We draw on dynamic systems theory (Vauras et al., 2013) to attend to the interpersonal back-and-forth between instructors' pedagogical strategies and students' reactions.

Methods

We address the questions: (1) How did a GTA use student engagement data to make pedagogical decisions?, and (2) How did students respond to the GTA's implementation?.

We collected self-reported information about student engagement using the experience sampling method (Nakamura & Csikszentmihalyi, 2009). The survey instrument was informed by flow theory, and has been used in college mathematics classrooms (e.g., Williams, 2022). These data were then aggregated and visualizations were shared with the GTA during virtually conducted and recorded meetings. We also recorded classroom observations. We used qualitative analysis methods and state space grids (e.g., Turner et al., 2014) to produce results.

Results and Conclusions

The GTA's primary strategy to promote engagement was group work, and the students' self-reported engagement, on aggregate, improved when group work was implemented. However, the GTA could rationalize when and why group work would either not be appropriate or would potentially not influence student engagement with hypotheses that included activity length and problem difficulty. Classroom observations also revealed patterns in student-teacher conversations and group work activity. The GTA's location in the classroom was related to whether students asked the GTA questions versus asking their peers. The amount of visible interaction between group members was based on the details and instructions provided by the GTA's when introducing activities. Our results show a link between the GTA's expected student involvement in groups with the self-reported levels of engagement and data to offer potential explanations for these relationships. Thus, we propose GTAs can attend to student engagement in ways that support students' growth and instructors' pedagogical goals during a single semester in a mathematics course.

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How one GTA used Data to Promote Student Engagement

Katie Taylor, The University of Alabama
Derek Williams, Montana State University

PURPOSE & MOTIVATION

- Research shows that student engagement
  - is a complex meta-construct consisting of behavioral, social, emotional/affective, and cognitive components (Fredricks, Blumenfeld, & Paris, 2004; Middleton, Jansen, & Goldin, 2017)
  - tends to decline in mathematics classrooms over any time interval (e.g., Martin et al., 2015)
  - in-the-moment states fluctuate (e.g., Williams, 2022)
  - and that classroom environments can be designed to promote engagement (Fredricks et al., 2004).
- Given research on college instructors attending to engagement is rare, the purpose of this study is to investigate how one GTA reflected on student self-reports of engagement during a precalculus course to promote engagement.
- We also discuss whether and how students responded to the GTA's strategies.

RESEARCH QUESTIONS

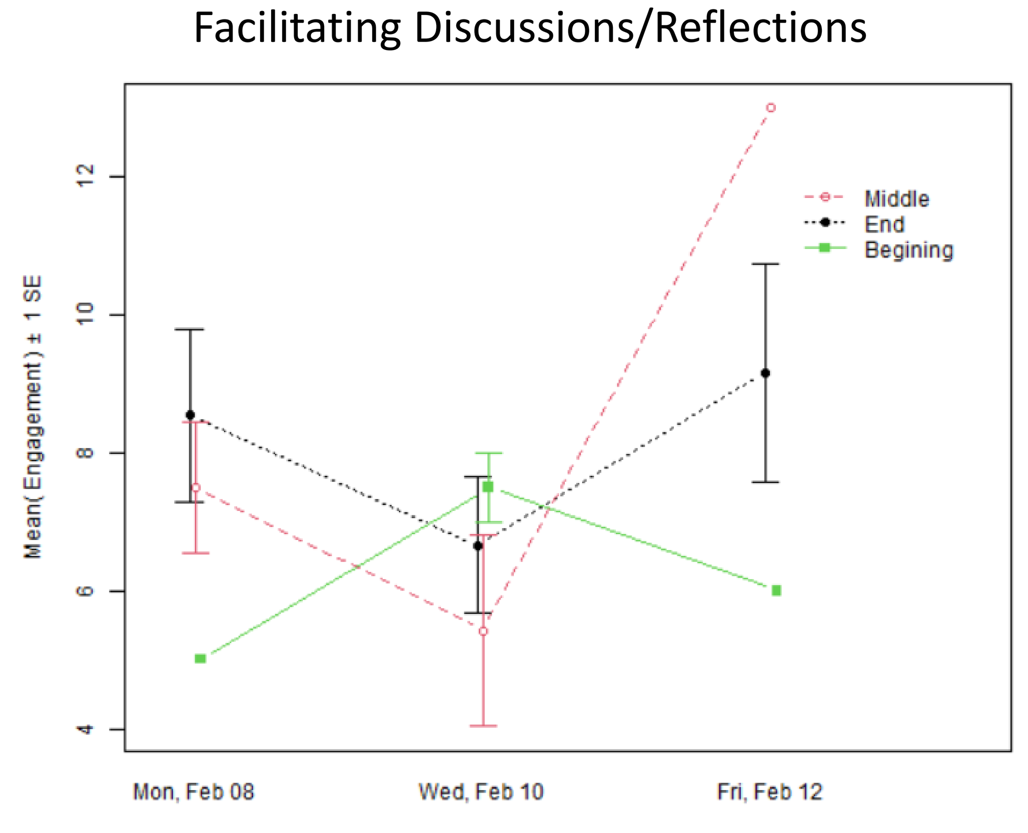
1. How did a GTA use student engagement data to make pedagogical decisions?
2. How did students respond to the GTA's implementation?

THEORETICAL FRAMEWORKS

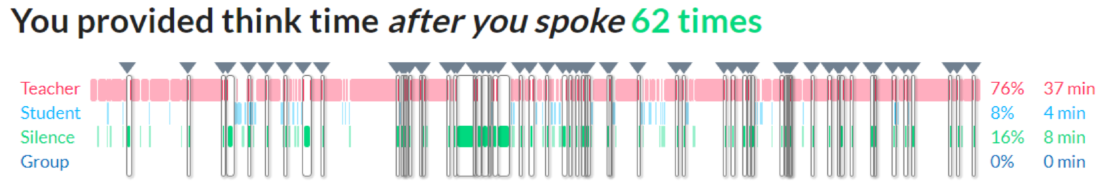
Classrooms are dynamic environments influenced by a myriad of factors. We draw on dynamic systems theory (Vauras et al., 2013) to attend to the interpersonal back-and-forth between instructors' pedagogical strategies and students' reactions.

METHODS

Facilitating Discussions/Reflections



You provided think time *after you spoke* 62 times



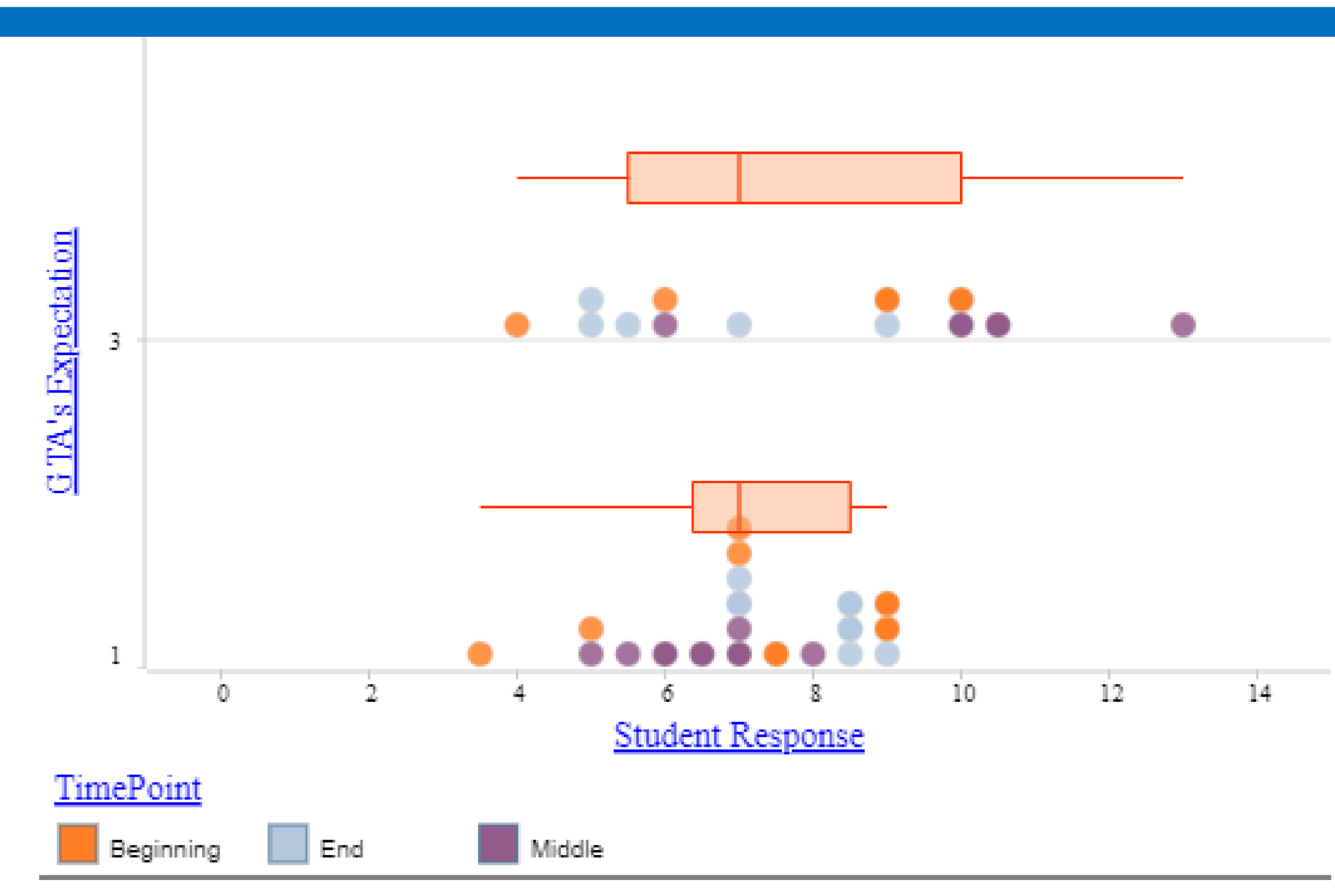
METHODS

Analyzing Video Data

subInterval	GTA Expectations	Student Report	Date	TimePoint
4	0	7.5	10-Feb	Beginning
8	1	7.5	10-Feb	Beginning
12	0	7.5	10-Feb	Beginning
16	1	7.5	10-Feb	Beginning
20	2	5.5	10-Feb	Middle
24	1	5.5	10-Feb	Middle
28	1	5.5	10-Feb	Middle
32	0	5.5	10-Feb	Middle
36	1	7	10-Feb	End
40	0	7	10-Feb	End
44	3	7	10-Feb	End
48	3	7	10-Feb	End

GTA Expectation	Student Response	Date	TimePoint
1	7.5	10-Feb	Beginning
1	5.5	10-Feb	Middle
3	7	10-Feb	End

RESULTS



RESULTS

GTA Statements During Discussions/Reflections

"On Friday, it is supposed to be all group work…I think that will be good. Maybe even more so than on Monday…or maybe group work goes too far sometimes."

"Admittedly, today was more of a lecture format because I wanted to get a lot of stuff out of the way, so I expected those on the right to be way lower."

"I had them working together on the end but through the whole like first part it was like I had some questions where it was supposed to just be like alright them themselves think about it."

CONCLUSION

- The GTA's primary strategy to promote engagement was group work
  - The students' self-reported engagement, on aggregate, improved when group work was implemented
  - However, the GTA could rationalize when and why group work would either not be appropriate or would potentially not influence student engagement with hypotheses that included activity length and problem difficulty.
- Classroom observations also revealed patterns in student-teacher conversations and group work activity.
  - The GTA's location in the classroom was related to whether students asked the GTA questions versus asking their peers.
  - The amount of visible interaction between group members was based on the details and instructions provided by the GTA when introducing activities.
- Our results show a link between the GTA's expected student involvement in groups with the self-reported levels of engagement and data to offer potential explanations for these relationships.
- Thus, we propose GTAs can attend to student engagement in ways that support students' growth and instructors' pedagogical goals during a single semester in a mathematics course.

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Network-Based Trajectory Analysis and Framework Development of Precalculus Curricula

Cristina Flores Vlad Nita John O'Meara Ashwin Vaidya

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Keywords: network theory, curriculum, identity, complexity science, connectivity, precalculus

This synthesis of previous work and ongoing research efforts calls for a network theory mediated analysis to investigate the affordances of curriculum and, in particular, its alignment with student learning processes. The three factors that are identified and considered in this study are the student, curricular materials, and the respective environments from which these members arise and how these act as determinants of curriculum development and design. There are three primary research questions addressed here: **(1)** How does the development of connections between topics within a specific precalculus course's curricular materials (textbook and online learning platform) over the course of a semester influence and/or align with the development of students' connections? **(2)** How does the development of connections between precalculus topics as perceived by students with respect to average path length (APL), clustering coefficient (CC), and the distribution of highly connected topics (Hubs) correlate with their performance in the course? **(3)** To what extent do equitable and identity-based curricular supplements (having students provide specific personal connections to the material, as well as draft problem sessions based on student interests) have an impact on student performance and/or perceived connectivity of precalculus topics?

A network theoretic perspective of various precalculus curricula (O'Meara & Vaidya, 2021) proves helpful in identifying and motivating key features of the subject as they appear in course texts. Based on this framework, an undirected network representation of a specific precalculus course's materials is constructed in which nodes represent topics introduced throughout the materials, and edges represent explicit connections formed between topics. Edges are weighted based on frequency of relation between two given topics, and nodes are weighted based on overall degree distribution. Students are administered an IRB-approved survey on a biweekly basis in which they are provided with sets of topics covered in the course up to a given point. They are then asked to form any connections they see fit, and these self-reported connections are stored as undirected networks whose properties and relevant metrics are analyzed upon each submission period through the end of the semester to gain insight into student learning trajectories. The corresponding poster will display the dynamic nature of the evolution of students' networks, as well as insights into how course performance can be correlated with both features of the students' networks and the course's 'ideal' network structure. This project aims to lay the foundation for meaningful intervention and predictive analytical tools that instructors can make use of in order to identify which students are likely to struggle in the course, and with what particular topics they are failing to fully comprehend. Such an extension is mitigated by a qualitative analysis of identity-based surveys and collaboratively constructed problem sets where the presence of personal connections in this medium are compared and correlated to the students' connected network structures and their overall performance in the course.

The implications of this work seek to assist in the optimization of the complex system synthesizing various feedback-based designs within educator roles, student response, and the interactions between the two as they pertain to various stages of the curriculum development process. The results of this study aim to provide a deeper understanding of the extent to which the components of an andragogical system can be refined via regulated and personalized learning methods.

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O'Meara, J., & Vaidya, A. (2021). A Network Theory Approach to Curriculum Design. *Entropy*, *23*(10), 1346. https://doi.org/10.3390/e23101346

Mathematical Modeling of Student Learning Network

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Introduction

This synthesis employs a network theory mediated analysis to investigate the affordances of curriculum and, in particular, its alignment with student learning processes. Via the utilization of various lenses of network theory, 'connected curriculum' design (Fung, 2017), and modern learning curve theory (Thurstone, 1918; MacLellan et al, 2015), we collectively postulate that education is inherently a complex system at various scales and stages of the learning and teaching process. A network perspective of precalculus curricula (O'Meara and Vaidya, 2021) proves helpful in identifying and motivating key features of the subject as they appear in course texts. Of these, hubs and time-series developments of relevant computed metrics have been particularly useful in mapping the alignment between the preset goals of the precalculus course, as identified in previous literature, and its execution. At the same time, this analysis has also been valuable in identifying the trajectory of the textbook curricula in adequately preparing precalculus students for success in calculus and beyond.

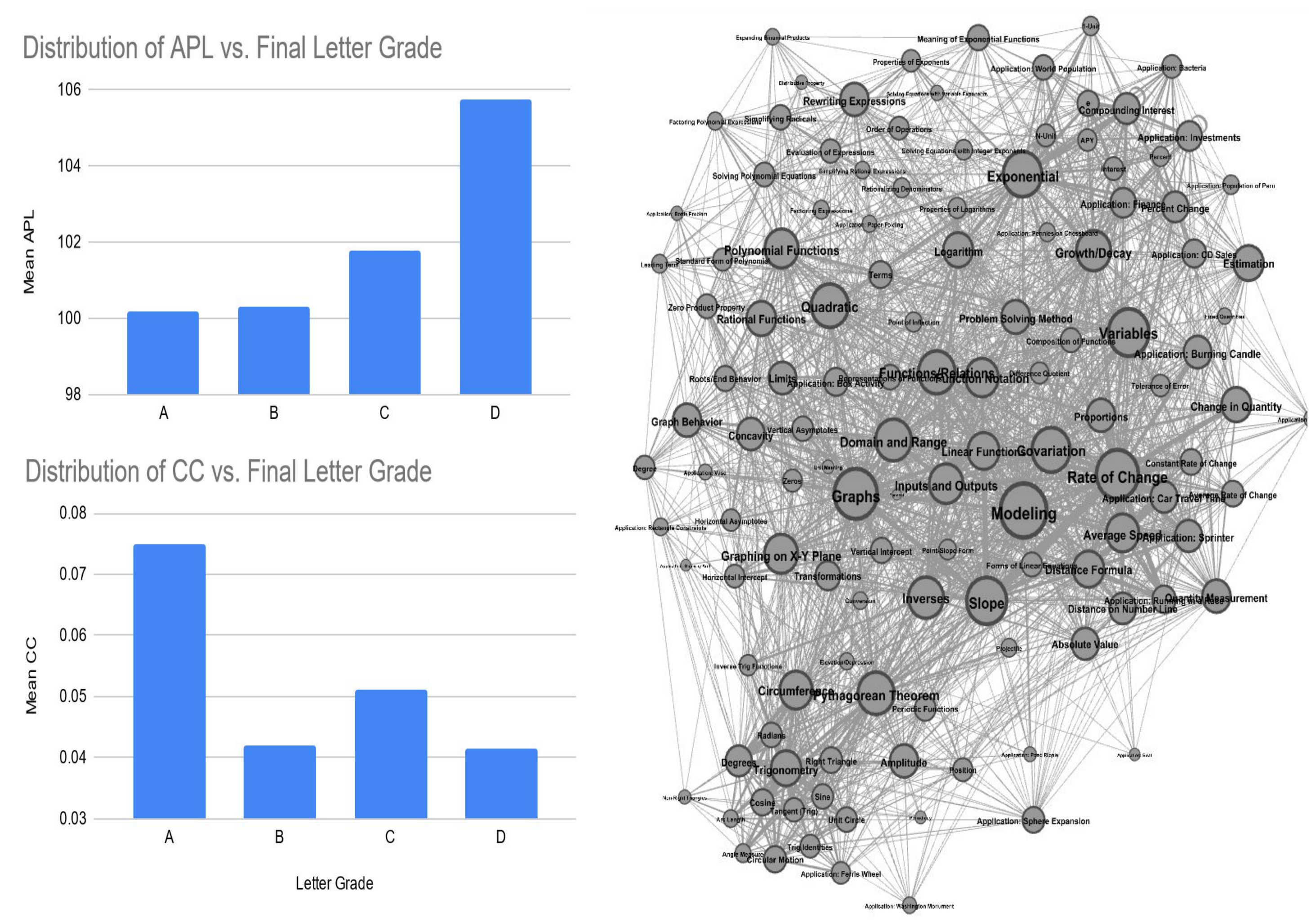

Figure 1: Aggregate results of student concept maps / self-reported networks in which individual perceived connections are formed amongst course topics. Pictured above are histograms displaying relevant network metrics with respect to final grade distribution (left), and the union of all student networks with node size adjusted for degree/span of connectivity (right).

Research Questions

(1) How does the development of connections between topics within a specific precalculus course's curricular materials (textbook and online learning platform) over the course of a semester influence and/or align with the development of students' connections?

(2) How does the development of connections between precalculus topics as perceived by students with respect to average path length (APL), clustering coefficient (CC), and the distribution of highly connected topics (Hubs) correlate with their performance in the course?

(3) What affordances do the alignment between research-based curricular taxonomy and course materials provide? How do observations in precalculus influence the transition for students into calculus and beyond?

(4) To what extent do equitable and identity-based curricular supplements (having students provide specific personal connections to the material, as well as draft problem sessions based on student interests) have an impact on student performance and/or perceived connectivity of precalculus topics?

Research Design

Students are administered an IRB-approved survey (FY21-22-2488) on a biweekly basis in which they are provided with sets of topics covered in the course up to a given point. They are then asked to form any connections they see fit, and these self-reported connections are stored as undirected networks whose properties and relevant metrics are analyzed upon each submission period through the end of the semester to gain insight into student learning trajectories. This acts as an intervention and predictive analytical tools that instructors can make use of in order to identify which students are likely to struggle in the course, and with what particular topics they are failing to fully comprehend. Such an extension is mitigated by a qualitative analysis of identity-based surveys and collaboratively constructed problem sets where the presence of personal connections in this medium are compared and correlated to the students' connected network structures and their overall performance in the course.

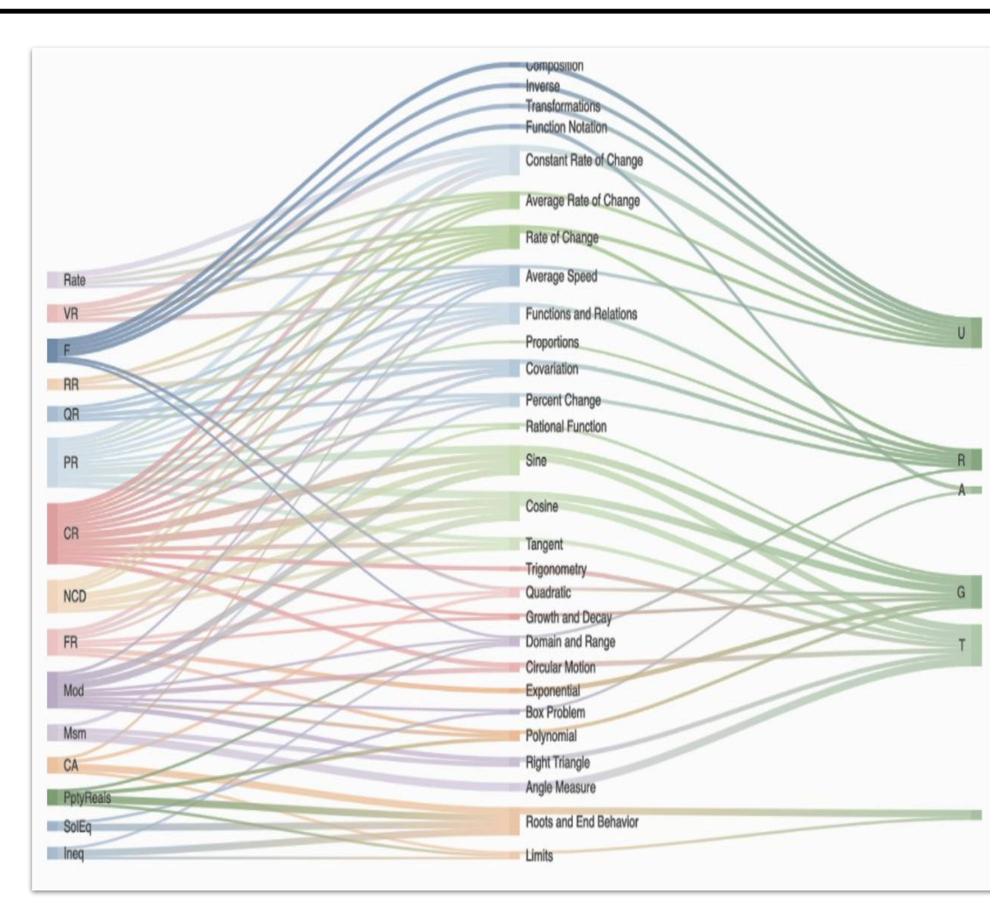

Figure 2: Representation of prominent precalculus topics as an informational exchange between precalculus taxonomy, course textbook, and calculus taxonomy principles.

Findings

Highly successful texts have been found to display a power law with respect to frequency distribution of connected topic inclusion ($\alpha = 0.05$), among consideration of other theoretical distributions, to assess whether or not meaningful learning patterns retain consistent underlying patterns and/or distributions. The most well-connected topics (Hubs) are regarded as necessary markers of classroom discourse, and the extent to which they are considered in taxonomical goals was measured with respect to the empirical distributions of both the intended curriculum and enacted curriculum.

The results of this study aim to not only provide a deeper understanding of how intended and enacted curriculums interact with each other, but also considers to what extent the components of an andragogical system can be refined via the magnitude of their presence in both course materials and feedback provided directly by active participants in the learning environment.

Keywords

network theory, curriculum, identity, complexity science, connectivity, precalculus

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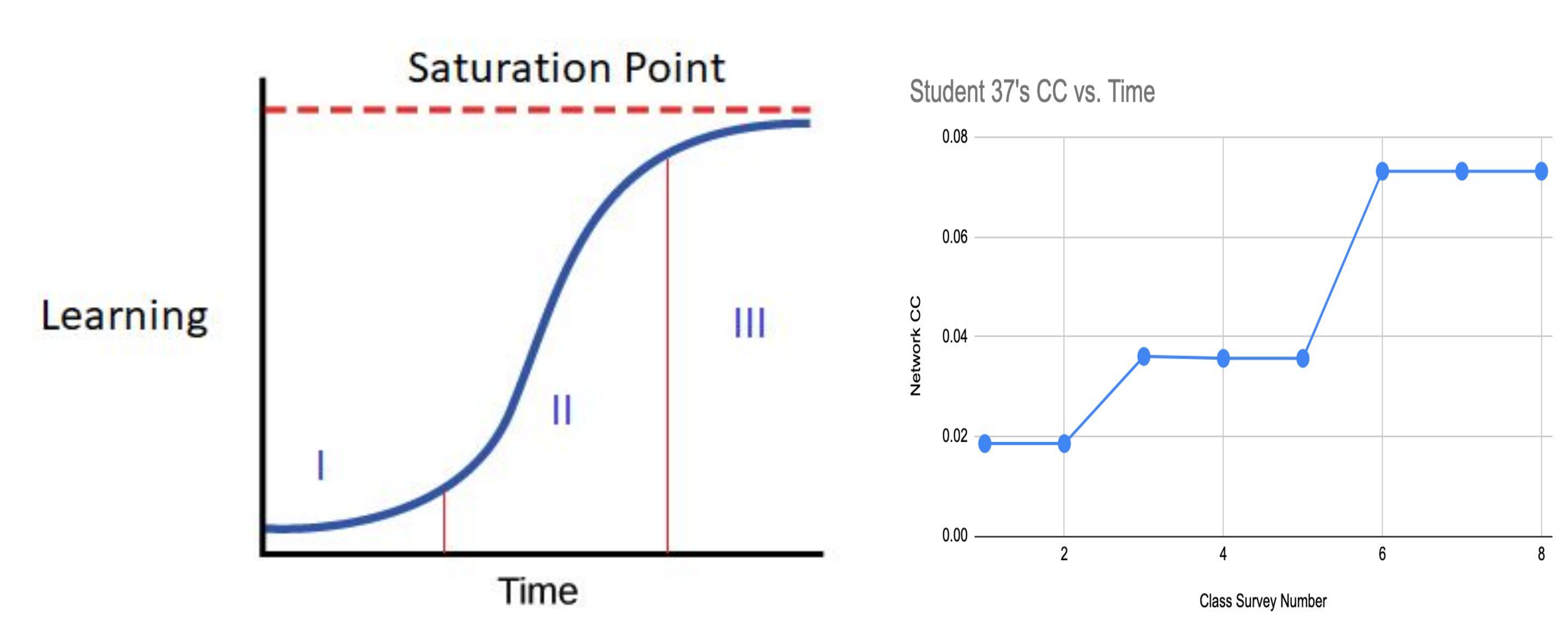

Figure 3: A schematic of the logistic learning curve (left), in which I indicates the phase of slow growth until critical knowledge is reached, II is the phase of rapid learning phase, and III is the phase at which saturation of knowledge is approached, with critical transition points between phases. This is accompanied by a model of an anonymous student's clustering of precalculus concepts over time (right).

PEER and Faculty Development Lessons from the Pandemic

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Keywords: Professional Development, Practitioners, Emerging Researchers

Faculty involved in education research have highly varied backgrounds. Some have received formal training through graduate programs in discipline-based education research (DBER),others may begin as a postdoctoral researcher, still others may not have begun until they were established faculty at an institution. Some are hired because of their education specialities (Bush et al., 2017), while others grow into this role as part of their engagement with the pedagogical mission of their unit.

Researchers who complete graduate programs or post-doctoral programs in education research are prepared for research. Those who become involved with education research as faculty have limited options for formal learning and guidance. The Professional development for Emerging Education Researchers (PEER) program was established to support these emerging DBER practitioners (Franklin et al., 2018). PEER offers workshops and mentoring to support emerging DBER practitioners in developing their education research skills.

PEER is strongly grounded in two theoretical perspectives: responsive teaching, and communities of practice. Responsive teaching is a model of teaching in STEM education that places high importance on centering and responding to the ideas of students and the connections they are making (Robertson et al., 2015). PEER workshops also seek to develop a local community of practice among participants (Wenger, 1999). Many emerging practitioners are still developing their identity as education researchers, and a community of fellow practitioners is critical to supporting that development.

The challenges posed by the COVID-19 pandemic led to changes to PEER's structure as we pivoted to an online format. PEER has returned to offering in person workshops, but lessons learned during the pandemic have had lasting effects on how we assess and respond to the needs and interests of participants. Notably the existence of new hybrid and online formats, and the addition of an online "kickoff" meeting which is now held before most PEER workshops.

In this poster we present lessons learned during the pandemic, and the impacts on PEER. We discuss implications for professional development initiatives, focusing on integrating online sessions for community building and pre-workshop preparation. Pre-workshop online meetings can serve as tools for informing both participants and facilitators about the structure of the workshop, and identifying adjustments that may need to be made to the program.

Acknowledgements

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PEER and Faculty Development Lessons from the Pandemic

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Pandemic Changes

Online Field Schools

- Keep sessions short: reduce burnout
- Lots of small group discussions
- Facilitator panels: divide facilitators across different online rooms

Virtual Kickoffs

- Pre-fieldschool introductions!
- We are doing these even for in person field schools now
- Establish expectations for program ahead of time
- Get early participant feedback

Gateway Workshops

- Lightning fast, often online, 1.5-3 hours
- Quick introduction to fundamental topics
- Inexpensive, can be run at professional meetings/conferences
- Provide exposure

PEER

- Professional development for Emerging Education Researchers
- Est. 2014
- 22 in person field schools
- 4 online workshops
- Hundreds of participants
- Across 3 continents

Responsive

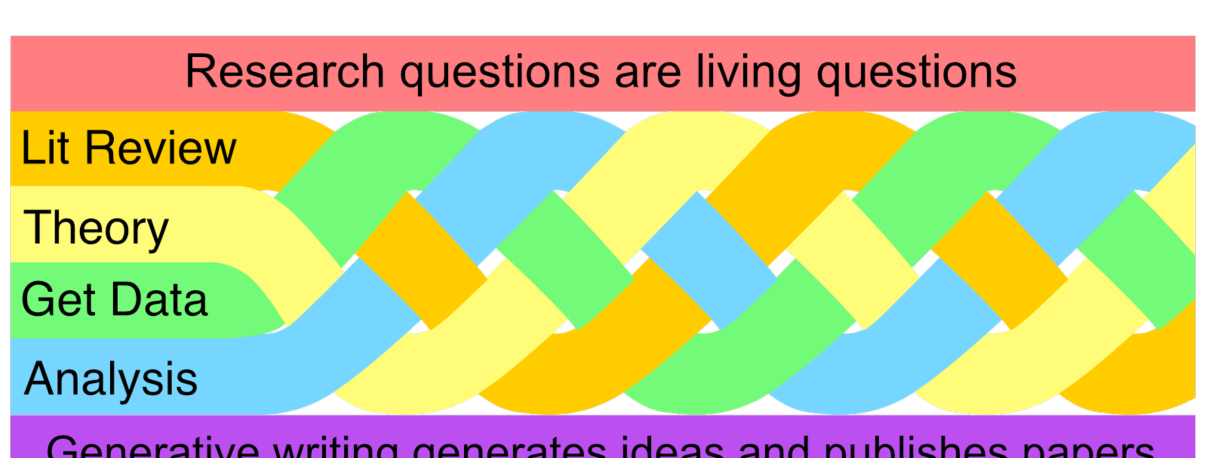
Collaborative

Research is...

Communicative

Playful

Our Research Model



Get Involved!

- Participate in the program
- Be a local host, and we'll come to you
- Join us as a facilitator

- Online/in-person gateways (15.3 hours)
- Introductory field schools (~5 days)
- Advanced field schools (~5 days)

Modules

Research Life	Teaching	Ethics	Analysis	Communication	Research Design
Personal Paths	Listening to Students	Collaboration	Video Coding	Literature Reviews	Research Questions
PEER Principles	Fostering Equity	Human Subjects	Emergent Coding	Paper Structure	Data & Access
		Authorship	Doing Interviews	Choosing a Journal	Case vs Recurrence
			Quantitative with R	Skeptical Audiences	Finding a Group
				Beautiful Posters	Project Planning

Acknowledgements

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Using Multiple Disciplinary Lenses to Explore Students' Graphical Interpretation: A Case Study

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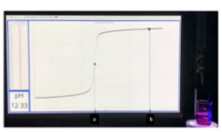
Key Words: Graphical Reasoning, Knowledge in Pieces, Shape Thinking, Multidisciplinary

We present a case study of how one student, Aubrey, utilized ideas from biology, mathematics, and chemistry to make sense of a graph common to introductory chemistry: a titration curve. Via multidisciplinary analysis, we found that Aubrey exhibited shape thinking activated by biology resources. However, while she also activated math resources, her interpretation was bound by treating the graph as a static object. Furthermore, Aubrey activated chemistry resources to 'read beyond the data,' but her inferences appeared to be influenced by biology resources. With the current findings, we argue that students may memorize graph features without understanding what is happening conceptually or mathematically and then transfer these learned facts to other disciplinary contexts. Therefore, instructors should be aware of the knowledge and reasoning skills a student may bring from other introductory STEM courses when interpreting graphs.

While previous literature has explored graphical interpretation generally and from specific disciplinary perspectives (e.g., Roth et al., 1999; Rodriguez et al., 2019), there is limited multidisciplinary work. This poster fills that gap by presenting a unique case study of a graphical interpretation task from chemical, biological, and mathematical points of view. Participants (n=15) were recruited from introductory biology and chemistry courses. Data include semi-structured, hour-long interviews involving three tasks. We present only the results of one task: Aubrey's graphical interpretation of a titration curve (Figure 1). A multidisciplinary analysis was conducted using Corbin and Strauss's (2008) constant comparison method.



Task 1: In a titration experiment, a base is added to an acidic solution at a constant rate. The graph (Figure 1) shows how the pH of the solution changes with the amount of base added. Using the graph, explain the relationship between the pH of the solution and the amount of base added to the solution.

Figure 1. The Titration Curve Task.

Aubrey employed resources from each discipline to interpret the titration curve, but her sense-making was bound to her understanding of a biology graph with a similar S-shape. This suggests students may memorize and transfer learned facts about graph features to other disciplinary contexts. However, to engage in authentic interpretation across disciplines, students need to be able to make sense of all curvature and critical points on a graph and understand how they emerge from simultaneous changes in quantities. Therefore, it may be valuable to prompt science students to think through mathematically meaningful critical points or graph features in the context of the physical phenomenon they represent. In conducting a multidisciplinary analysis, we discovered tensions between each discipline's interpretation and expectations. Thus, while this poster presents data and conclusions from a unique case study, it also provides pedagogical insights from adopting a multidisciplinary approach.

Acknowledgments

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Using Multiple Disciplinary Lenses to Explore Students' Graphical Interpretation: A Case Study

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National Science Foundation
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Background

Problem Statement

How do students reason about quantities and quantitative relationships as part of interpreting and making inferences about graphs commonly discussed in introductory biology and chemistry courses?

Theoretical Orientation

Quantitative Reasoning ⟷ Covariational Reasoning

Representations ⟷ Graphical Reasoning

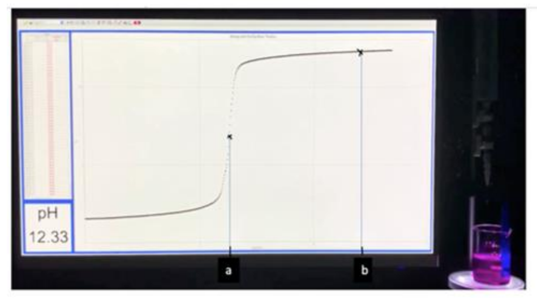
Data Collection

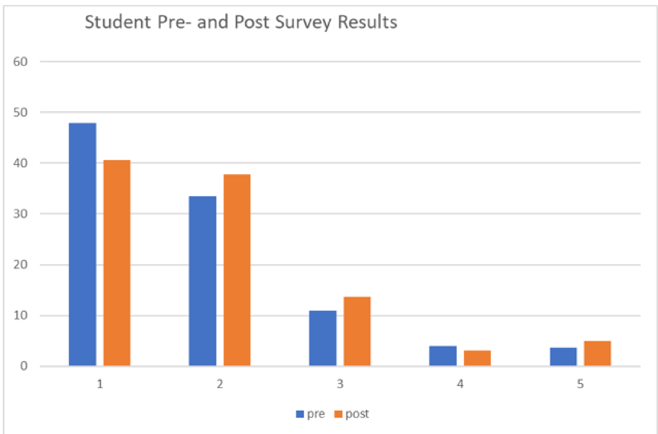
- Method
  - Surveys
    - Pre-survey (n=660)
    - Post-survey (n=492)
  - Interviews
    - Pre-interview (n=16)
    - Post-interview (n=15)

Titration Task

In a titration experiment, a base is added to an acidic solution at a constant rate. The graph below shows how pH of the solution changes with amount of base added. Using simulation video and the graph below, explain the behavior of function f that represents pH on the interval from $x = a\ , and\ x = b$.

1. pH is increasing at an increasing rate.
2. **pH is increasing at a decreasing rate.**
3. pH is increasing at a constant rate.
4. pH is decreasing at a decreasing rate.
5. pH is decreasing at an increasing rate.





Analytical Method

- Three disciplinary-based education experts (mathematics, biology, and chemistry) analyzed Aubrey's response from their disciplinary perspective
- Knowledge-in-pieces
- Open Coding
- Constant comparison

Findings

Biology

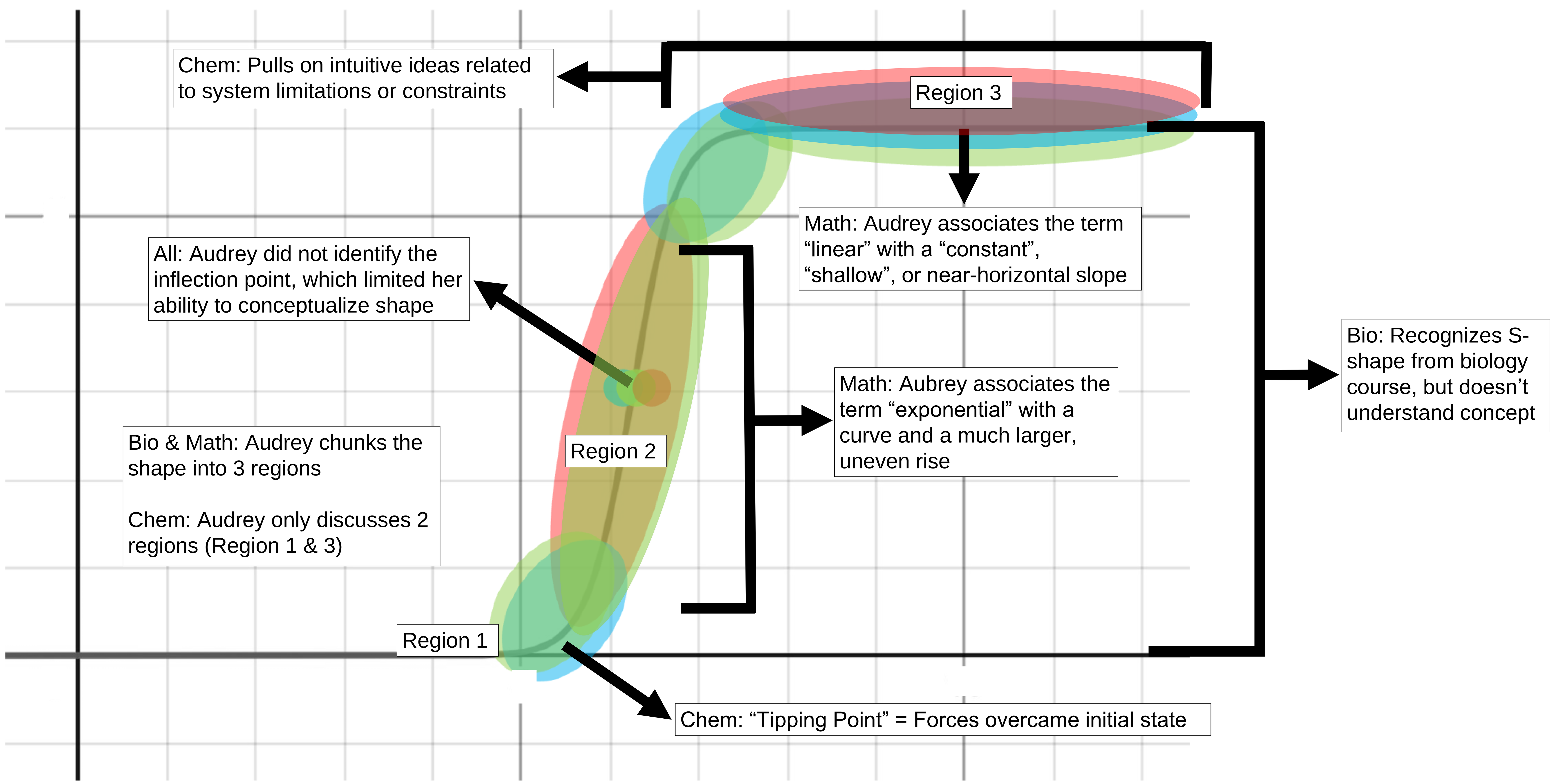
Overall: Aubrey activates biology resources that are rooted in viewing the graph as a static object

Chemistry

Overall: Aubrey activates chemistry resources to 'read beyond the data,' but her inferences appear influenced by biology resources

Mathematics

Overall: Aubrey activates math resources, but her interpretation is bound by treating the graph as a static object



Key Takeaways

- Instructors should be aware of the knowledge and reasoning skills a student may bring from other introductory STEM courses when interpreting graphs
  - Students may be viewing graphs as disconnected chunks, instead of a continuous set of data
  - Students may also be missing important mathematical features of the graph, such as the inflection point
- Graph features matter differently across disciplines, which may impact the type and sophistication of reasoning used
  - In science, it may be sufficient to associate concepts with chunks or shapes as part of focusing on interpretation and "reading beyond the data"
  - However, this may promote students to think of graphs as objects composed of shapes, rather than as emerging through covariation
  - Therefore, it may be valuable for science instructors to prompt students to engage with graphs from a more mathematical perspective

Preliminary Findings Regarding Student Learning from Lecture and Homework

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Keywords: homework, lecture, relearning

Lectures and homework both provide opportunities for students to learn mathematics. Researchers have focused on learning from lectures (e.g.,Krupnik et al., 2018; Lew et al., 2016; Lew & Zazkis, 2019) and learning from homework (e.g., Dorko, 2021; 2020a, 2020b; 2019; Ellis et al., 2015; Glass & Sue, 2008; Kanwal, 2020; Krause & Putnam, 2016; Lithner, 2003; Rupnow et al., 2021). Dorko and Cook (under review; 2022) propose attending to learning across the two settings is crucial to improving student learning in both settings. To that end, our study addresses the questions (1) *What did two developmental mathematics students learn in lecture and what did they learn from homework?* and (2) *How can we explain why they learn particular things in a particular context?*

The data collected spanned eight 75-minute lectures, three written homework assignments, six online homework assignments, and one exam from a developmental math class at a large US university. This constituted the material from the class period following the first exam to the class period directly prior to the second exam, and the exam itself. The content included creating graphs on a graphing calculator, using the graphs to solve inequality and optimization problems, solving linear equations by hand, and using slope to find horizontal or vertical changes in right triangle problems. The data corpus includes videos of the lectures, videos of each student doing the homework, and an interview with each student about their graded exam. In the homework sessions, the interviewer asked students to “think aloud”, asked where they had learned particular ideas, and showed students clips from the lecture video and asked what sense they had made of that portion of the lecture. In the post-exam interview, the interviewer and student discussed where the student felt they had learned the material tested in each question.

Because the course is a developmental math class, our theoretical perspective is that of *relearning*, which Amman and Mejia Ramos (2022) define as “learning about content that one has tried to learn before in a previous math course” (p. 752). Relearning is a suitable perspective for our work because it allows us to look at student learning in the context of homework and lecture in terms of the outcomes that are occurring and how the circumstances of the contexts support or limit those outcomes (Amman & Mejia Ramos, 2022). Data analysis is ongoing, following a thematic analysis method (Braun & Clark, 2020). Preliminary findings indicate students learned (a) how to identify independent and dependent quantities; how to use the calculator to make graphs, (b) how to interpret graphs, (c) how solving an inequality is different from solving an equation and how to do both on the calculator; (d) how to optimize functions on the calculator. Students commented that getting answers wrong in the online homework helped them understand how to solve inequalities and optimization problems.

For most of the items above, the students said they learned the ideas both in lecture and homework. Their comments indicated lecture exposed them to ideas and homework afforded an opportunity to “have to like remember and go back over how to do it” (Student 3) and have ideas “cemented in my brain” (Student 3).

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Preliminary Findings Regarding Student Learning from Lecture and Homework

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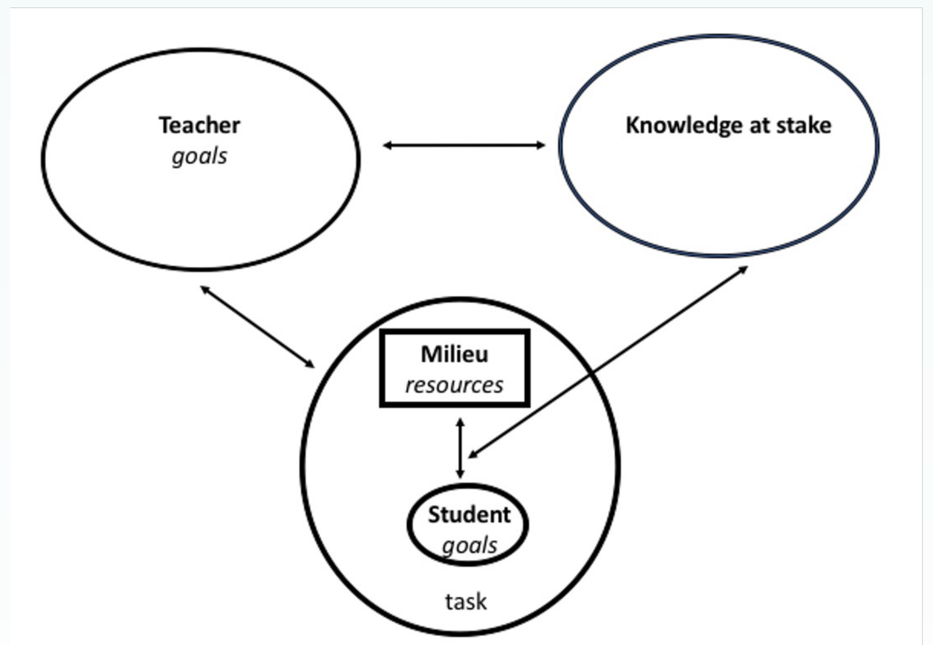
Abstract

Lectures and homework both provide opportunities for students to learn mathematics. Researchers have focused on learning from lectures (e.g., Krupnik et al., 2018; Lew et al., 2016; Lew & Zazkis, 2019) and from homework (e.g., Dorko, 2021; 2020a, 2020b; 2019; Ellis et al., 2015; Glass & Sue, 2008; Kanwal, 2020; Krause & Putnam, 2016; Lithner, 2003; Rupnow et al., 2021). However, these studies have focused on each milieu individually. Dorko and Cook (R&R, 2022) propose attending to learning across the two settings is critical for improving student learning in both settings. To that end, our study addresses the questions (1) *What did two developmental mathematics students learn in lecture and what did they learn from homework?* and (2) *How can we explain why they learn particular things in a particular context?*

Theoretical Perspective

Our work is broadly framed with the instructional triangle, and we make sense of an individual's learning through Piagetian learning theory.

- Instructional triangle (Cohen et al., 2005; Dorko, 2021; Herbst & Chazan, 2012): instruction is the interactions among teachers and students around content, in environments
  - Milieu: a space in which interactions around mathematical tasks occur
    - Lecture is one milieu; HW is another



- Piagetian learning theory (von Glasersfeld, 1995)
  - Students' schemes entail (among other things) an expected result of applying the scheme.
  - Perturbation occurs when an outcome differs from the expected result.
  - Perturbation may or may not result in accommodation to achieve equilibration.
    - We define learning as accommodation to a scheme.

Literature

Though student learning from online homework is a relatively new focus for undergraduate mathematics education researchers, studies have shown:

- Online homework is as effective as paper-and-pencil homework (Dorko, 2020).
- Most successful US college calculus programs have an online homework component (Ellis et al., 2015).
- Students commonly rely on example problems to complete homework.
  - In one study, 84.7% of students reported using worked examples while doing homework.

When completing online homework students used reasoning that included (Dorko 2021, 2020)

- Mathematical thinking
- Guessing
- Copying answers from various sources
- Mimicking the steps of a similar problem
- Reasoning based on expectations from experience

Students learn the procedures from online homework problems, but rarely develop the conceptual insights professors hope some problems might promote (Dorko, 2020, 2021, 2019)

Issues that professors face

- Students worry more about the procedure than the concept itself (Lew et al., 2016; Thompson, 2013; Thompson et al., 1994)
- Students only write what the professor writes (Lew et al., 2016)
- Students may miss verbal-only comments (Lew et al., 2016)

Data Collection Methods

Course context

We collected data from a developmental math class at a large US university. The content was from the class period following the first exam to the class period directly prior to the second exam, and the exam itself. The content included creating graphs on a graphing calculator, using the graphs to solve inequality and optimization problems, solving linear equations by hand, and using slope to find horizontal or vertical changes in right triangle problems.

The data collected include

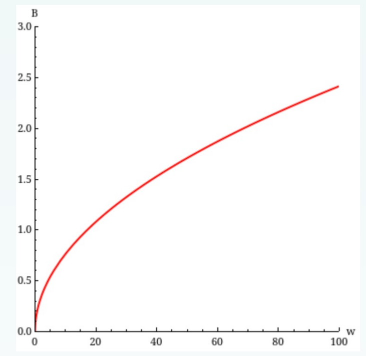
- Videos of eight 75-minute lectures,
- Videos of interviews with students as they completed
  - three written homework assignments
  - six online homework assignments
- The students' exam responses
- An interview with the students about their exams

Data Analysis Methods

Data analysis is ongoing, following a thematic analysis method (Braun & Clark, 2020).

1. Read lecture transcript, make list of the knowledge at stake (e.g. when/how/why to use CALC, INTERSECT on graphing calculator)
2. .Read transcript of HW that covered that content
   - Looked for influences of lecture on HW activity (e.g. doing something that was done in lecture, like using the TI-83 to find an intersection on a graph; using a word/phrase the instructor used, like "isolate" the variable)
3. In HW, identified instances student had to attempt problem multiple times, and sought to unpack how they worked their way to a correct answer
4. Repeat 1-3 for all lecture/HW pairs
5. For exam work and interview, looked at HW problems that covered same content as exam problem and what the student's activity looked like on that problem (e.g. multiple attempts, influences from lecture - see (1))

Example: Student 4 learned...



Online homework task: $B(w) = \frac{\sqrt{210w}}{60}$ gives the body surface area, B, in m^2, for a 210 cm tall man who weighs w pounds. Would weight gain have a greater effect on surface area for a lighter man or for a heavier man? (multiple choice)

Theoretical tenet	Data excerpt	Notes / explanation
Scheme	"I think that it's as someone gets heavier, their surface area increases, so would the weight gain have a greater effect on surface area for a lighter man or a heavier man, I think that means it's going to be heavier."	In S4's scheme for interpreting graphs, the only thing that matters is whether the graph is increasing or decreasing.
Expectation of scheme's outcome	That 'heavier' is correct	
Outcome	'Lighter' is correct	
Evidence of perturbation	[expresses surprise that actual outcome did not match expected outcome	
Evidence of equilibration via accommodation (evidence of learning)	"From 0 to 20 there's a sharper increase than up here…. Because it'll kind of level off somewhere… I think that it's asking more about like the slope and not, not just about the surface area. So I think that's why I got that wrong."	Getting the answer wrong resulted in S4 attending to the graph's rate of change. He modified his scheme to include rate of change as relevant to interpreting graphs.

Results

Preliminary findings indicate students learned (a) how to identify independent and dependent quantities; how to use the calculator to make graphs, (b) how to interpret graphs, (c) how solving an inequality is different from solving an equation, and how to do both on the calculator; (d) how to optimize functions on the calculator. Students commented that getting answers wrong in the online homework helped them understand how to solve inequalities and optimization problems.

- Influence of lecture on HW activity
  - **Taking time to make sense of given information:** Like the instructor, S1 took time to make sense of and employ given information
    - Identifying independent/dependent variables
    - Units
    - Labeling graph
  - **Using words the instructor used** (e.g. "isolate" a variable, "assign a variable")
  - **Problem contexts**
    - Remembered similar problem contexts from lecture and used this similarity to solve HW problems
    - Connected different problem contexts that both had concave up, decreasing graphs
  - **Use of skills:** table to set graphing window, table to select correct graph, CALC menu features, using graph instead of table to solve problems
  - **Ways of looking at graphs** (steepness for concavity, identifying increasing/decreasing as ways to justify use of procedures)
  - **Applying definitions** (recognized an equation as linear based on class definition)
- Learning from HW activity
  - **"Trial and error", "mastery," "putting stuff into practice":** student's description of what and how she learned from homework
    - She primarily went through CALC menu features via trial and error, taking several problems to sort out what to use when
  - **Solution to one problem influencing solution method to next problem** (e.g., generalizing the practical meaning of an increasing, concave up function)
  - **Quantitative meaning for concavity:** computing AROC over different intervals helped her see the changing rate of change and how it influences the shape of the graph

Discussion & Conclusion

- We think students' learning from mistakes can be explained by Piagetian learning theory: seeing an "incorrect" mark perturbs them
- Lecture appears to give students tools, (e.g. features of the calc menu) but they often go through a trial-and-error process in homework figuring out when to use each tool
- Students notice similar problem contexts, as well as abstracting underlying similarities in mathematical structure (e.g. concavity) .

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Analyzing Student Understanding of Derivatives Through Tasks Presented Online

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Introduction

Calculus plays a crucial role in STEM (Bressoud 2021) and studies have indicated that **derivate** is a complex concept to learn (Park, 2015; Zandieh, 2000). Researchers have unpacked student cognitive processes and existing mental structures when doing mathematical tasks involving derivatives (Baker, 2000; Asiala, et al., 1997). The COVID-19 pandemic caused disruptions in mathematics courses and more instructors are using online tools like Desmos or WebAssign. We investigate research questions:

(1) To what extent are derivative tasks facilitated with prompting in an online environment helpful for student learning?

(2) What mental structures and connections exist in the student process of doing online derivative tasks?

Framework & Research Questions

APOS theory (Arnon et al., 2014; Dubinsky & McDonald, 2001) is utilized to study student understanding of derivatives. Examinations through APOS theory generally begins with the development of a **genetic decomposition** (GD) and following the GD for derivatives (Asiala et al., 1997), we explored student interpretation of derivatives. To unpack students' mental structures and processes involved, we focused on a Baker et al. (2000)'s **graphing schema** for derivatives which includes the **interval schema** and the **property schema**. Aligned with that, **the triad** (intra-, inter-, and trans) for schema development is used in the context of APOS theory.

Methodology

We conducted semi-structured clinical interviews with **five students** who had taken Calculus I at a college level. Sergio (white male) and Maxine (female of color) are incoming freshmen at state midwestern universities who had completed AP Calculus AB. Amber (white female) and Trisha (white female) completed while Vilma (female of color) was enrolled in Calculus I at a liberal arts midwestern university in the United States. All the interviews were **conducted via Zoom** while the students interacted with **Desmos**. Students described their thought processes as they worked on the two tasks related to derivatives. **The first task** pushes students to think critically about the conceptual understanding of the derivative as the instantaneous slope or slope of the tangent line while **the second task** requires creating a graph of f(x) solely based on given conditions and no equations. During the interviews, the researchers used prompts including "what is a derivative?" or asking students to evaluate f'(x) where x is an arbitrary point.

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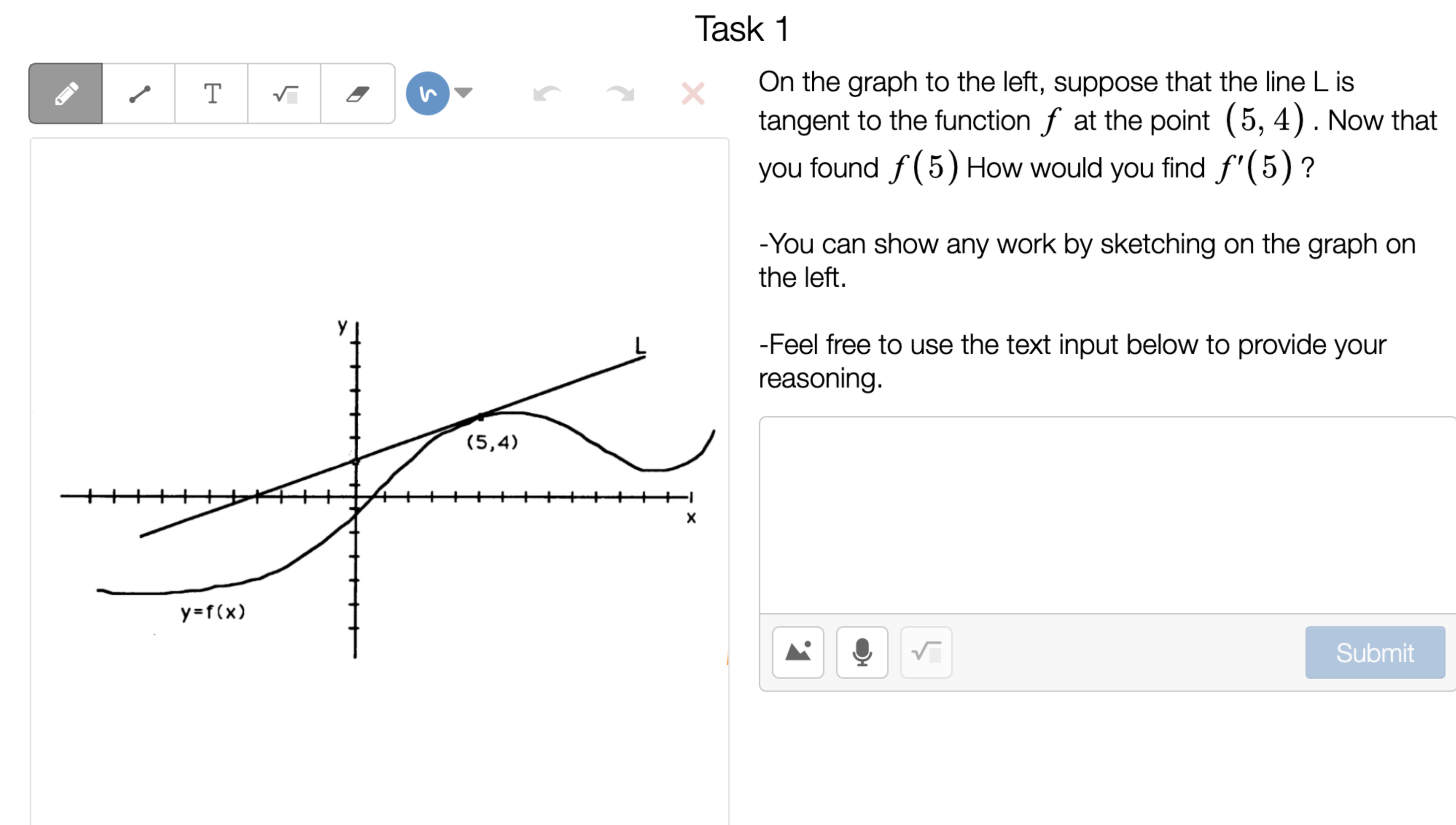
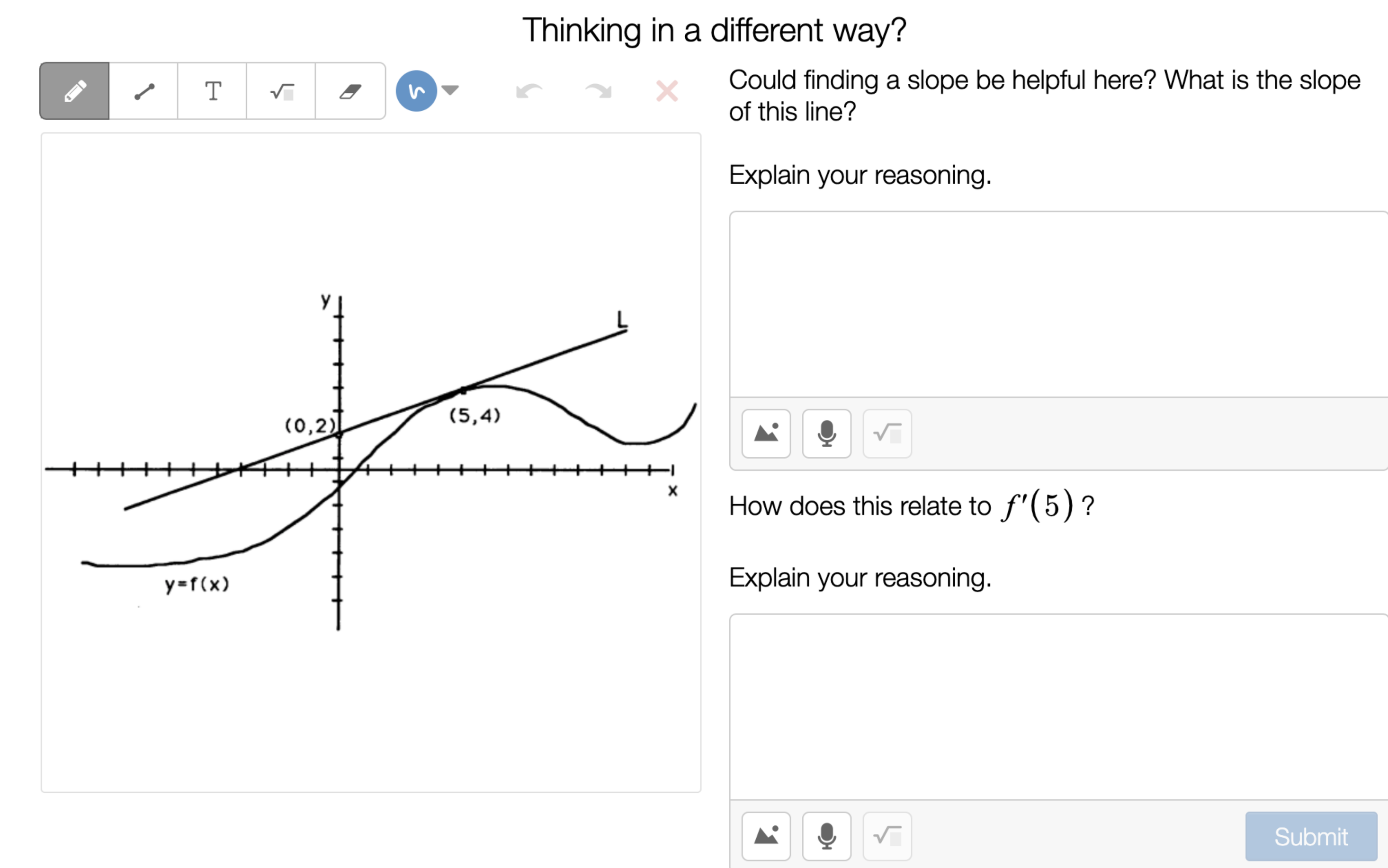
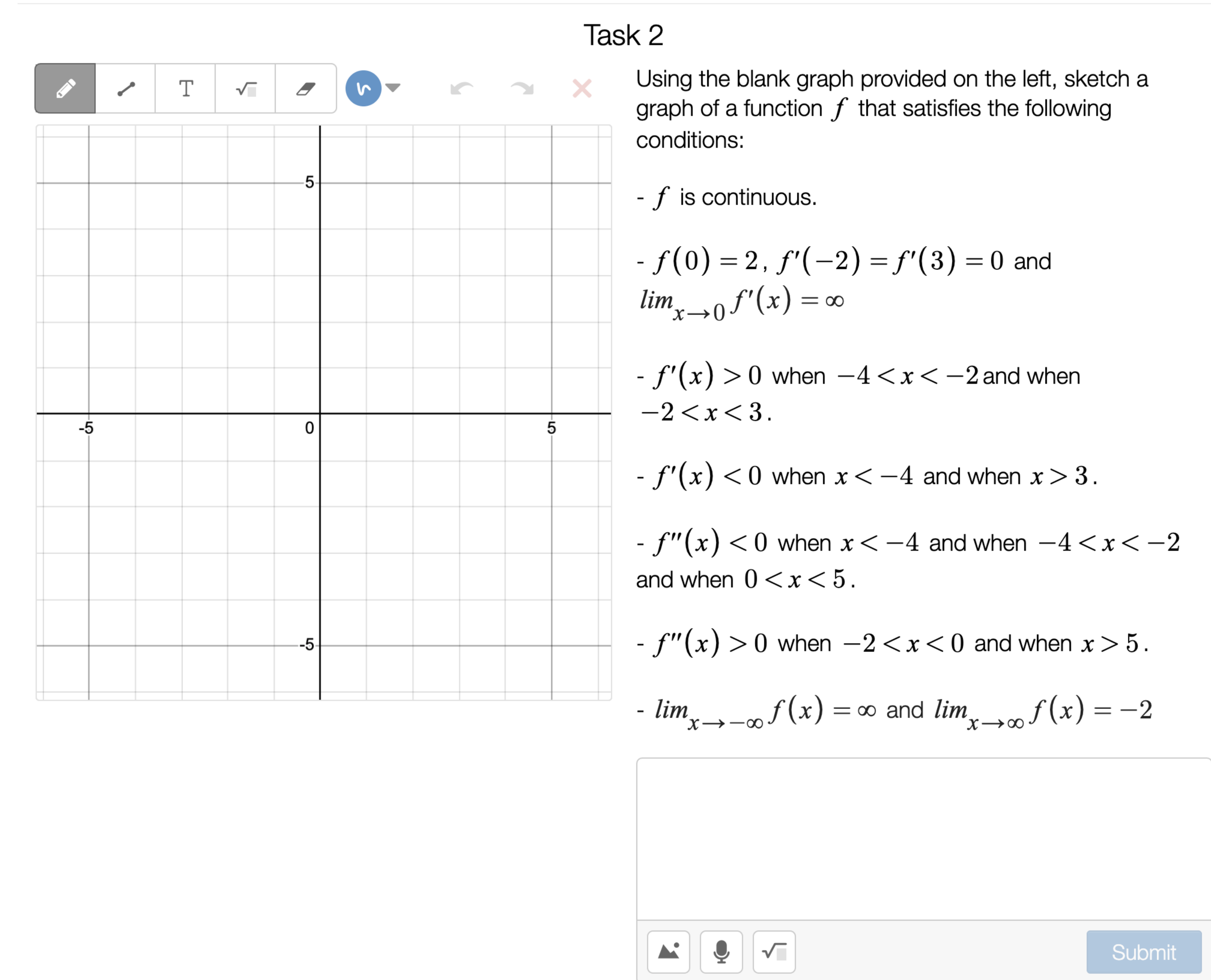




C. Graphical interpretation of the derivative at a point
   i. Overcoming the need to differentiate some formula
   ii. Coordinate with A to see f'(a) as the slope of a tangent line
   iii. Coordinate several interpretations of f'(a)
D. Graphical interpretation of the derivative as a function
   i. Seeing the derivative as the function, x → slope at (x, f(x))
   ii. Identifying f' with the line tangent at a point
E. Several coordinations to get the graph of f
   i. Graphical interpretation of f(x), for a single x
   ii. Interpretation of f'(x) for a single x as the slope
   iii. Process of x moving through an interval
      a. monotonicity of the function and sign of the derivative
      b. infinite slope (vertical tangent) and infinite derivative
      c. concavity of the function and sign of the second derivative
   iv. Drawing a complete or fully representative graph



The nine dimensions of the Derivative Schema.		Intervals ((-4, -2, (0, 3), etc.)		
		Intra-	*Inter-*	*Trans-*
Properties (monotonicity, concavity, differentiability, etc.)	*Intra-*	Intra-intra	Intra-inter	Intra-trans
	Inter-	Inter-intra	Inter-inter	Inter-trans
	Trans-	Trans-intra	Trans-inter	Trans-trans

Above we present the first task and GD used to analyze (Asiala et. al., (1997) on the left side and second above and the triad model used to analyze that data on the table above.

Results & Discussion

All participants remarked that the tasks looked very different from the type of problems that they had done in-class or for homework in Calculus I. None of the participants had difficulty with using the online tools and Maxine appreciated constructing graphs online. Sergio notes that "...on the [usual] homework you can just find a pattern and knock'em out pretty quick…" but explains that for the novel tasks " … you can't cheat it if it's a conceptual problem, you have to actually think about it." Results suggest that students continue to have difficulties in **going from just "taking derivatives" to conceptual understanding.**

We identified components of the genetic decomposition and their critical role in student understanding of derivatives. We also found evidence of Baker's (2000) two dimensions of student understanding of derivatives (property & interval). Vilma was the only student at trans-trans dimension. We found that using certain prompts like "What is a derivative" or "what's slope elsewhere?" via online environments could also be helpful, but there is a need to investigate this further.

Connecting Modern Algebra with Secondary Math Content

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Keywords: modern algebra, in-service teachers, connections with secondary curricula.

The 2010 report of the Conference Board of the Mathematical Sciences states that among four-year institutions with secondary pre-service teaching certification programs, 89% of the math departments require students to take abstract/modern algebra (Blair, Kirkman, & Maxwell, 2013). In addition, this requirement is advocated by many mathematical societies. However, there is a continued interest in and need for research on how knowledge of mathematical content is employed in teaching practice, as well as how secondary teachers view the relevance of their post-secondary courses in teaching (Zazkis & Leikin, 2010). In May 2019, the presenter attended *Connecting Advanced and Secondary Mathematics* conference, that brought together researchers, mathematicians, and teacher educators to explore these connections and develop plans for further research and/or professional development on these topics (Murray et al., 2017).

In fall 2019, the presenter taught a modern algebra class (covering topics of ring and field theory) for the Graduate Certificate in Mathematics for secondary level math in-service teachers. They implemented ideas from the conference by making intentional connections between the material taught and the secondary math curriculum. The presenter was interested in researching how these intentional teaching interjections changed the in-service teacher's opinion about the relevance of the advanced material covered with respect to the secondary math content the in-service teacher taught as well as the connections they see between the two curricula.

The students in the class (10 in-service teachers) were given pre- and post-surveys, that contained quantitative opinion questions about the relevance and connections between the advanced and secondary content in the curricula. The feedback obtained confirmed the advanced algebra material could be meaningful to teachers by connecting the material with the secondary mathematics and could also support teachers to re-conceptualize, re-structure, and re-understand their knowledge of secondary mathematics (Wasserman, 2016). The pre- and post-survey results showed the students' opinions changed significantly about the relevance and connections of the advanced algebra to secondary math content and about recognizing connections in their own classes.

In addition, students were asked to provide reflective feedback on the exam and the class projects about advanced topics they learned in the class that were the most useful/meaningful for them as well as sharing their opinions and suggestions on these topics. Almost all the advanced topics chosen by students as their favorites/most meaningful/useful, were the ones that were the most applicable in their high school classroom settings. In their comments, students expressed confidence that the advanced class material and connections made in class helped them understand the majority of topics that were covered in the course. As a result, they were now familiar to them on a deeper level. The in-service teachers expressed confidence in better understanding and being able to explain many seemingly true statements (taken for granted) they had been using in their own classes and making their students see relevance in mathematics.

Moreover, the presenter will discuss the questions of the pre- and post-survey results, which show a small increase in change of the students' opinions and brainstorm further (follow up) research questions.

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UTAH VALLEY UVU UNIVERSITY Department of MATHEMATICS

Connecting Modern Algebra with Secondary Math Content

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Motivation / Literature Review

- Every five years, the Conference Board of the Mathematical Sciences(CBMS) executes a national survey of undergraduate mathematical and statistical sciences in two- and four-year colleges and universities.
- The 2010 report states that among four-year institutions with secondary pre-service teaching certification programs, 89% of mathematics departments require students to take abstract/modern algebra (Blair, Kirkman, & Maxwell, 2013).
  - ❖ Blair, R., Kirkman, E. E., & Maxwell, J. W. (2013). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States: Fall 2010 CBMS survey*. American Mathematical Society.
- The Mathematics Association of America along with other professional mathematics societies also advocate for advanced courses such as these to be taken by future secondary mathematics teachers.
- Despite this requirement and advocacy, there continues to be a need for research on how knowledge of mathematical content is employed in teaching practice, as well as how secondary teachers view the relevance of their post-secondary courses to teaching (Zazkis & Leikin, 2010).
  - ❖ Zazkis, R., & Leikin, R. (2010). *Advanced mathematical knowledge in teaching practice: Perceptions of secondary mathematics teachers*. Mathematical Thinking and Learning, 12(4), 263–281.

Connecting Advanced and Secondary Mathematics (CASM) Conference (http://casmconference.org/)

Held in Minneapolis, MN, May 20-22, 2019
NSF Funded (53 participants)

- "The conference will bring together researchers, mathematicians, and teacher educators interested in exploring connections between advanced and secondary mathematics. The CASM Conference has three main objectives:
  - I. To develop plans for research and/or professional development to expand current knowledge on connections between advanced and school mathematics, so that attendees can take concrete steps to move projects forward.
  - II. To provide a space for mathematicians, mathematics educators, and other stakeholders to discuss their work and develop a common vision for future directions.
  - III. To share current research on mathematical connections between advanced and secondary mathematics, focusing specifically on abstract algebra."

❖ Murray, Eileen; Baldinger, Erin; Wasserman, Nicholas; Broderick, Shawn; White, Diana. *Connecting Advanced and Secondary Mathematics*. Issues in the Undergraduate Mathematics Preparation of School Teachers, v1 Aug 2017 (https://files.eric.ed.gov/fulltext/EJ1151024.pdf)

CASM Conference

- Five theme groups formed
  - ❖ Curriculum
  - ❖ Educator Knowledge
  - ❖ Item Development
  - ❖ **Professional Development for Teachers** (I was part of this group)
  - ❖ Project Design

Implementation

- Fall 2019:
  - ❖ Class: Modern Algebra (Ring and Field Theory)
  - ❖ Program: Graduate Certificate in Mathematics
  - ❖ 10 Students: Secondary-level math in-service teachers
- Proof-based class - hence very demanding and hard for students
  To address:
  - ❖ Why taking this class?
  - ❖ How to make the class relevant for students?
  - ❖ What is the connection to the material they teach at the high schools?

Research Project

- Collecting data from the in-service teachers (pre- and post-surveys) about the material/topics covered;
  - ❖ **What is the relevance of this material with respect to the secondary math content they teach**?
  - ❖ How to better equip them in teaching secondary math topics?
  - ❖ Which topics will prepare math education students better for understanding secondary mathematics?
  - ❖ What are in-service teachers' opinions and suggestions on the topics?

Survey Questions IRB Protocol #s 349; 350	Pre Survey	Post Survey
# of surveys	10	10
The *Abstract Algebra* material is relevant to secondary math content.	2.6	3.9
The *Abstract Algebra* provide connections to secondary math content.	2.5	4.3
The *Abstract Algebra* material equips the pre-service teachers to better teach secondary math content.	2.9	3.4
I have used *Abstract Algebra* material/concepts when teaching my secondary math classes.	2.3	3.9
I believe that *Abstract Algebra* material/concepts should be taught to pre-service teachers.	3.1	3.6

5 – strongly agree 4- agree 3 – neither agree nor disagree
2 – disagree 1 – strongly disagree

Student Feedback – Reflections

I. *Exam question – reflection*:
Describe your favorite part (or topic) of *Modern Algebra* class this semester. Explain why that part (or topic) was your favorite.

II. *Project question* – **Reflections on the course material**
- ❑ **address the topics covered;**
- ❑ **which topic(s)/concept(s) was/were most meaningful and useful for you;**
- ❑ **which topic(s)/concept(s) was/were the most exciting and new to you;**
- ❑ **which connections you see with this material and the high school curriculum;**
- ❑ **what would you change or leave the same in terms of the course content; etc.**

- Almost all the topics chosen by students as their favorites/most meaningful/useful, were the ones that
  - ✓ were most applicable in their high school classroom settings
  - ✓ provided connections to the secondary math content
  - ✓ were familiar to the students
- In their comments, teachers expressed confidence that the class material and connections made helped:
  - ✓ them understand the majority of topics that were covered in the course and were familiar to them on a deeper level;
  - ✓ them to understand and explain many seemingly true statements (taken for granted) they use in their classes;
  - ✓ them communicate mathematics to their students by connecting concepts;
  - ✓ their students see relevance in mathematics.

Conclusion

- The feedback above confirms that the *Abstract Algebra* material
  - ❖ can be meaningful to teachers by connecting it with the secondary mathematics;
  - ❖ can support teachers to re-conceptualize, re-structure, and re-understand their knowledge of secondary mathematics.

  (Wasserman, N. (2016). Abstract algebra for algebra teaching: Influencing school mathematics instruction. *Canadian Journal of Science Mathematics and Technology Education, 16*(1), 28-47).
- The pre- and post-survey results show that students' opinions changed significantly about
  - ✓ the relevance of the *Abstract Algebra* to the secondary math content;
  - ✓ connections of the *Abstract Algebra* to the secondary math content;
  - ✓ recognizing these connections in their own classes.
- Why the small increase in change of the opinion of the students about the last question on the survey? Few conjectures:
  - ✓ the proof-based classes are difficult, demanding, and time consuming;
  - ✓ most of the in-service teachers have taken the needed prerequisite (*Abstract Algebra* course (Graph Theory)) long time ago;
  - ✓ lucking of good proof writing skills.
- **Further research questions:**
  - ❖ What connections of *Abstract Algebra* can impact teacher instructions?
  - ❖ How can Variation Theory be implemented in *Abstract Algebra* classes with goal to impact teacher instructions?

MAA SIGMAA-RUME, Omaha, NE, February 23-25, 2023
25th RUME Conference, February 23rd, 2023

Prospective Secondary Mathematics Teachers Integrating Reasoning and Proving in their Teaching

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Keywords: Reasoning and Proof, Secondary Teachers, Beginning Teachers, Teacher Education

Transitioning from teacher preparation programs to teaching in school is fraught with challenges (Smagorinsky et al., 2004; Thompson et al., 2013). Specifically, beginning teachers struggle to implement reasoning and proving in real classrooms and recontextualize what they learned in their teacher preparation programs (Gomez Marchant et al., 2021; Stylianides et al., 2013). In this study, we investigate how prospective secondary mathematics teachers (PSTs) who participated in a capstone course *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020), envision and enact what they learned in the course in their student teaching experience and first year of teaching their own class.

We followed two PSTs, Olive and Diane, through the course, student teaching, and first year of teaching. Data included, for each teacher, six video-recorded lessons and six post-lesson interviews, which invited reflection on integrating reasoning and proof in the lesson. Data were analyzed through open coding (Patton, 2002) inspired by activity theory (Ellis et al., 2019; Engeström, 1987) with attention toward aspects that enable or inhibit enactment of reasoning and proof, such as their cooperating teacher, the school and classroom norms, and set curriculum.

As PSTs, Olive and Diane developed knowledge and skills for teaching reasoning and proof in secondary classrooms. As student teachers, they expressed a deep desire to integrate reasoning and proof into their teaching but encountered challenges which hindered their enactment of this vision. They felt obliged to conform to the norms of teaching established by their cooperating teachers; mostly notetaking and computational practice. Nevertheless, as student teachers working with a cooperating teacher, they still found ways to engage students with reasoning and proof by pushing for student explanations and designing short proof-related activities. As time progressed, they stepped into teaching their own classrooms. They embraced their independence and carved out unique paths to engage students with reasoning and proof. They devoted class time to group work, exploration-based activities, and invited reasoning and justifying.

This inquiry aims to provide an increased understanding of prospective teacher education with respect to mathematical reasoning and proof and how PSTs' knowledge and practices continue to develop during their cooperative and independent early-career teaching experiences. Notably, the proof-related practices learned in the course reemerged as the PSTs taught their own classes, despite facing pressures to teach in more traditional ways during student teaching. Our study contributes to a better understanding of the PSTs' development as mathematics teachers in their university programs and beyond.

Acknowledgments

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Students' Voice in an Active Calculus Learning Environment: How Do Students Perceive Purposeful Design Learning Environments

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Keywords: Active Learning, Calculus, Collaborative Small Groups

Previous studies emphasized the advantages of student-centered over instructor-centered teaching and learning of Calculus (Ely, 2021). In addition, collaborative group work in calculus classrooms has been shown beneficial to meaningful student engagement with calculus problems (Hitt & Dufour, 2021) and greater students' success compared to traditional classrooms (Code et al., 2014). Yet, some studies observed students' resistance to active learning (Tharayil et al., (2018). Thus, it is important to continue investigating students' perceptions of active learning environments in Calculus courses.

This study was conducted at a four-year R1 university in the northeastern United States. Participants were students in a Calculus course, which runs in a large lecture (3 times a week) and recitation (twice a week) format. In Spring 2022, we introduced active learning into recitations and activities that focus on Scientific, Mathematical, and Metacognitive Processes (SMMPs). The students worked on SMMP activities in groups and presented their solutions in class. The recitation sections were facilitated by graduate teaching assistants and undergraduate learning assistants. At the end of the semester, we collected survey data on students' perceptions of their learning experiences in Calculus recitations. Data include surveys (n=150) with 7 Likert-scale questions and 3 open questions. We also collected students' classroom work, related home assignments and test items on the topics of SMMPs. The current analysis focuses on Students' perceptions of the active learning environment structure of the recitations and how it affected their learning experiences.

We found that 73% of the students indicated that communicating with their peers in the recitations helped them to clarify their mathematical thinking, 71% of the students stated that Calculus concepts became clearer after they heard explanations from other students in their group, 80% of the students stated that they would recommend group work for Calculus recitations to other students, and 72% of the students stated that they would like other courses to offer opportunities of group work. Student-written comments supported these results. One typical comment was: "I think that being in a group where we all have different levels of understanding or ways of understanding the class concept is helpful because it allows me to think through my thought process and explain it to my other group members."

Our results show that most students perceive small collaborative group works to provide meaningful learning experiences for them. These results echo previous studies (e.g., Hitt & Dufour, 2021) and extend them by identifying students' justifications underlying their views of the benefits of group work in Calculus recitations to their learning. These results, along with the analysis of students' achievement contribute to the growing body of knowledge on reforming the teaching of Calculus.

Acknowledgments

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Students' Voice in an Active Calculus Learning Environment: How Do Students Perceive Purposeful Design Learning Environments

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NSF Transforming Undergraduate STEM Gateway Teaching and Learning by Emphasizing Core, Interdisciplinary STEM Practices (DUE # 2013427)

Background

Previous studies emphasized the advantages of student-centered over instructor-centered teaching and learning of Calculus (Hitt & Dufour, 2021). In addition, collaborative group work in calculus classrooms has been shown to be beneficial to meaningful student engagement with calculus and lead to greater students' success compared to traditional classrooms (Code et al., 2014). Yet, some studies observed students' resistance to active learning (Tharayil et al., 2018). Thus, it is important to continue investigating students' perceptions of active learning environments in Calculus courses.

Aim of the Project

We aim to support students meaningful learning experiences in Calculus via designing active learning environments. This study is focused on exploring students' perceptions of active learning environments in Calculus courses.

Research Question

How do students perceive purposefully designed active learning environments?

Method

Research Site: This research is conducted at a four-year R1 university in the northeastern United States. Participants were students in a Calculus course, which runs in a large lecture (3 times a week) and recitation (twice a week) format.

Data Collection and Participants: Data include surveys (n=150) with 7 Likert-scale questions and 3 open questions. Participants were undergraduate students enrolled first Calculus course at the university.

Design of the Study: We introduced active learning into recitations and activities that focus on Scientific, Mathematical, and Metacognitive Processes (SMMPs). The students worked on SMMP activities in groups and presented their solutions in class. The recitation sections were facilitated by graduate teaching assistants and undergraduate learning assistants. At the end of the semester, we collected survey data on students' perceptions of their learning experiences in Calculus recitations.

Design of Active Learning Environment

SMMP Activities Topics
Continuity
Implicit Differentiation
Derivative of Logarithmic Functions
Optimization Problems
Graphing
Antiderivatives

Instructional Routines
Launch the Activity
Small Group Exploration
Whole Class Discussion
Wrap-up/Closure

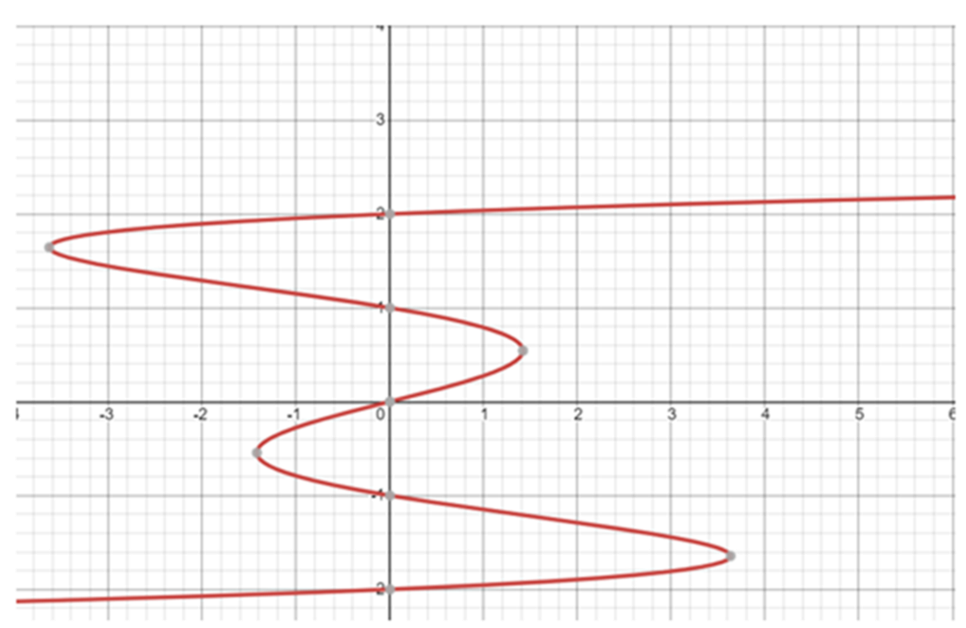
Small Group Exploration

Whole Class Discussion

An Example of SMMP Activity

Directions: Working with the peers in your group, solve the following problems. Make sure to show and justify all your work. Next, make sure everyone in the group understands the solution and participates. Finally, be prepared to report your answers to the whole class.

Task 1: In your small group, consider the curve on the graph and the equation of the curve: $x = y^5 - 5y^3 + 4y$. With your small group, explain how you would answer the following questions using both the symbolic equation and the curve graph.

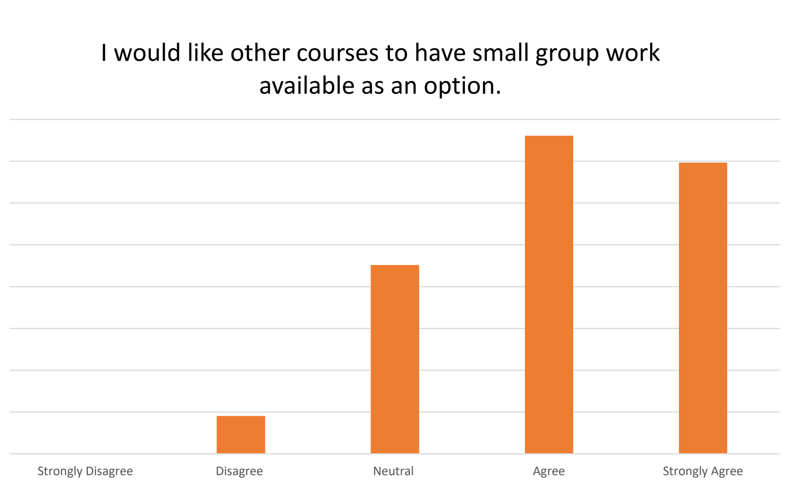


1) Is it possible to express y as an explicit function of x?

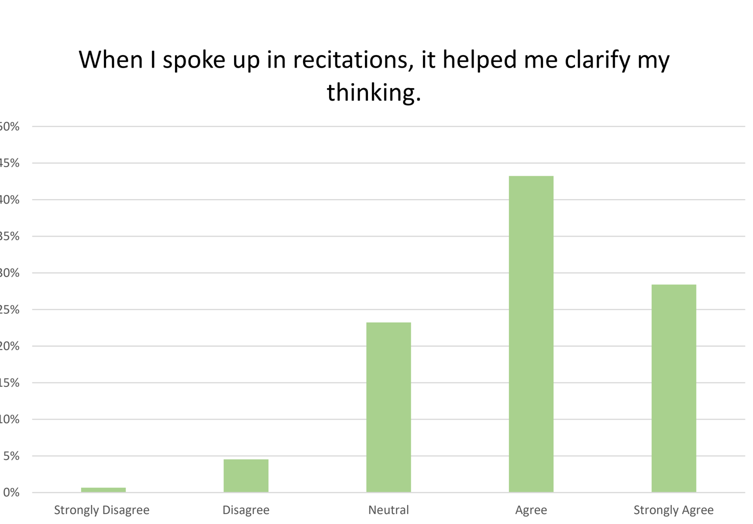
a) Explain your response using the graph.

b) Explain using the symbolic equation for the curve: $x = y^5 - 5y^3 + 4y$.

Survey Results

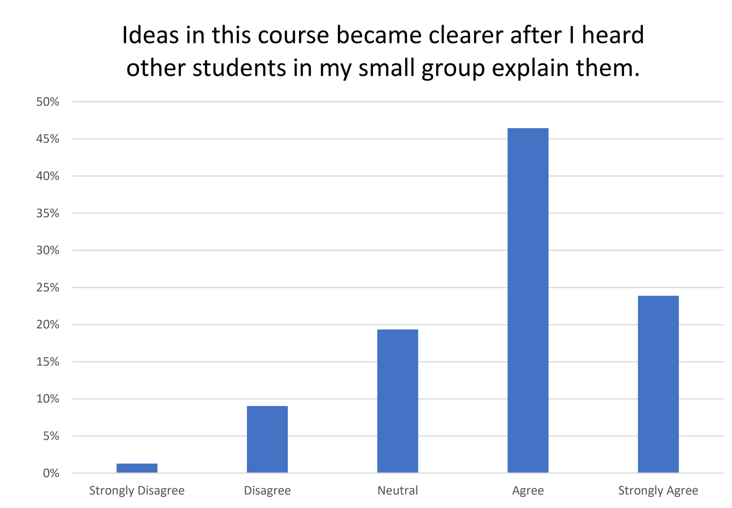


72% of students stated that they would agree or strongly agree with the statement: "I would recommend group work at Calculus recitation to other students."

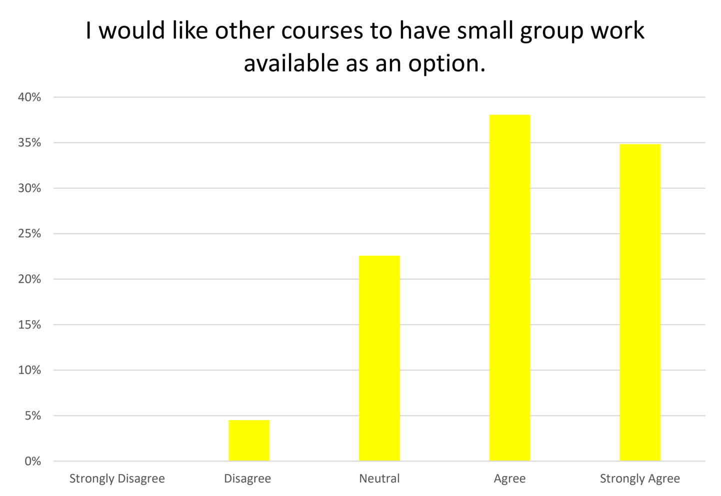


71% of students stated that they would agree or strongly agree with the statement: " When I spoke up in the recitation, it helped me to clarify my thinking."

Survey Results



70% of students stated that they would agree or strongly agree with the statement: "Ideas in this course become clear after I heard other students in my small group explain them."



73% of students stated that they would agree or strongly agree with the statement: "I would like other courses to have small group work available as an option."

Open-ended Questions and Samples from Students' Answers

Please comment on the aspects of the recitations that you think were most helpful for your learning

- Direct help from the TA and LA was super helpful. I learn better by working through things with other people. **Also teaching what I knew to my classmates helped me understand the content on a deeper level**
- I think that being in a group where we all have different levels of understanding or ways of understanding the class concept is helpful because **it allows to me to think through my thought process and explain it my other group members.**

Please comment on the aspects of the recitations that you think were least helpful for your learning

- I think sometimes the **worksheets are very long**
- I learn significantly more from Recitation than lecture, and don't have many complaints. Only thing I can think of would maybe **letting students design their groups at the beginning of the class**, but this also could be a downfall.
- Group work. Sometimes it can be helpful but **most of the time group members would rather work through their own work quickly versus helping other members.** Most of the help I receive in this class is from the TA or the LA.

If you were a TA designing recitation: What are at least two things, you would keep? Explain why these are strengths

- I would keep **group work** because it helps students collaborate and work through problems together.
- I would also keep **the end of class where we put problems on the board to help other groups that are still struggling with an explanation on how we did it.**
- I would **keep the LA** because having two people to help explain things is much better than one. Especially if they have different ways of explaining things.

Summary of Justifications

- Groupwork improved students' articulation of their thoughts.
- Groupwork helped students understanding of the content on a deeper level.

Discussion

- Our results show that most students perceive that small group work provides them with meaningful learning experiences.
- These results echo previous studies (e.g., Hitt & Dufour, 2021) and extend them by identifying students' justifications underlying their views of the benefits of group work to their learning in Calculus recitations.
- These results, along with the analysis of students' achievement, contribute to the growing body of knowledge on reforming the teaching of Calculus.

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Novice Calculus Learning Assistants' Learning to Notice

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Keywords: Learning Assistants, Teacher Education, Near-Peer Tutoring, Calculus

Learning Assistants (LAs) are undergraduate near-peer tutors who aid in the instruction of large lecture STEM courses (Otero et al., 2010). While many LAs do not pursue a career in teaching, they assume a para-professional role in classrooms. This experience has been shown to be transformative for students in these courses and for LAs themselves (Schick, 2018); however, the precise mode of LAs' development is not well understood. This study extends prior research on LAs as potential future teachers (Fineus & Fernandez, 2012; Gray et al., 2012; Vandegrift et al., 2020) by asking how the professional noticing of novice LAs develops over their first semester in a calculus classroom. Noticing refers to what teachers attend to and interpret in their instruction. Noticing is widely recognized as an important element of mathematics teacher development (van Ess & Sherin, 2008).

This study is a part of a multi-disciplinary project for reforming introductory STEM courses at a large, public, northeastern university. The four LAs participating in this study were in their first semester of teaching in a calculus course. The course follows a format of a large lecture (approx. 160 students) and recitations (approx. 20 students) led by a graduate teaching assistant. LAs were only present in recitations, helping small groups work through conceptually rich activities. The LAs also attended a weekly meeting on pedagogical topics, e.g., metacognition, types of questions, student ideas and content preparation. Data includes beginning and end of semester interviews, four written reflections throughout the semester, and video of recitations. In the interviews, LAs were asked to analyze samples of student work, reflect on clips of their teaching, and share their thoughts on the social dynamics in class. Interviews were coded on Stockero's (2008) five dimensions of noticing: actor, topic, stance, specificity, and artifact focus.

In their first interviews, LAs' comments were fairly general – discussing mathematics without relation to teaching or learning, and often evaluative in nature – LAs judged their pedagogical ability to provide students with clear, correct mathematical explanations. Comments about student thinking the classroom environment lacked detail, using the labels "good" or "not as good". This changed in their final interviews as LAs discussed specifics of student thinking and offered conjectures and explanations on classroom events. The focus of their comments shifted toward how the mathematics of a task created space for student learning. LAs also talked at length about specific students' ideas in relation to social dynamics of their classroom. LAs largely dropped their evaluative stance for making more assertions about teaching and learning.

Undergraduate peer tutoring is a growing phenomenon in introductory STEM courses, which provides unique teaching and learning experience. By characterizing how LAs professional noticing evolves, our study will advance understanding of LAs' growth as mathematics teachers and of their interactions with students. This will also provide greater insight into how LAs function within and contribute to undergraduate mathematics classrooms.

Acknowledgments

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Prematriculation Interventions using ALEKS: Common Lessons-learned in two Programs

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Keywords: Math Intervention, Precalculus through Calc II (P2C2), Student Success.

Students who enter university level calculus without a strong mathematical foundation in algebra and functions often struggle to succeed: repeating courses, leaving STEM, or dropping out entirely (Chen, 2013). Students from traditionally underrepresented groups are at particularly high risk of STEM attrition (Alexander et al., 2009). Here we describe student behavior in two interventions designed to provide a prematriculation experience for first-year university students. Each summer intervention paired content preparation with evidence-based learning strategies in an effort to reduce STEM attrition and increase success in gateway mathematics courses.

In Canada and the United States, the COVID-19 pandemic has had a large impact on K-12 students, with a major concern being mathematical preparation. Evidence indicates that current and future incoming university students are less prepared than their pre-pandemic counterparts (NEAP, 2022; EQAO, 2022). We know that high school teachers were forced to make curricular decisions that did not have in mind calculus preparation; for example, omitting logarithmic functions from the curriculum (Sacka, 2021). To shore up the mathematical skills of incoming students in an efficient way, the programs described here make use of a popular online intelligent tutoring system called Assessment and Learning in Knowledge Spaces (ALEKS). Within ALEKS, the curriculum for each program was shaped to be closely aligned with the local context and curricula, ensuring that student engagement with the material will be served in their upcoming courses. This alignment was directly communicated to students, instructors, advisors and administrators in each program.

Despite differences in local context and format of the "non-mathematical" components of each program, we found common experiences with respect to student engagement with ALEKS:

1. ALEKS can be offered as a stand-alone product, but ALEKS alone was not sufficient to motivate student engagement or open lines of communication with faculty or peers.
2. Synchronous support for ALEKS (e.g. office hours) were underutilized, with the majority of participants in one program reporting that they were able to find answers to their questions within ALEKS. Students in both programs instead used office hours to ask questions related to placement and course selection.
3. While students in both programs retained access to ALEKS into the start of the Fall semester, very few students remained active in the platform once classes started. This result could be attributed to lack of time during this busy transitional period, and/or decrease in motivation as seen in previous applications of ALEKS (Fang et al., 2019)

In addition to these shared insights, we will also report on differences between ALEKS use across the two programs, including learning gains, time spent on ALEKS and the distribution of topics mastered. Through this poster presentation, we aim to:

- learn more about experiences with ALEKS at other institutions,
- hone in on strategies for better student engagement with the tool, and
- gauge community interest for support in analyzing the local data needed to establish and evaluate similar interventions.

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Prematriculation interventions using ALEKS: Common lessons-learned in two programs

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Emily Braley, Johns Hopkins University

Keywords: Math Intervention, Precalculus through Calc II (P2C2), Student success

Why provide early support for STEM students?

Students are much more likely to succeed in university-level calculus when equipped with a strong foundation in algebra and pre-calculus. Success in Calculus is crucial for students who intend to pursue a degree in STEM. Literature suggests that without early success in college STEM courses, students may struggle to succeed: repeating courses, leaving STEM, or dropping out entirely (Chen, 2013). Students from traditionally underrepresented groups are at particularly high risk of STEM attrition (Alexander et al., 2009).

Here we describe student behavior in two summer interventions designed to provide a prematriculation experience for first-year university students.

Each summer intervention paired mathematical content with evidence-based learning strategies in an effort increase success in calculus.

Has the pandemic affected STEM-preparedness?

In Canada and the United States, attempts to measure impact of the COVID-19 pandemic indicate that current and future incoming university students may be less prepared than their pre-pandemic counterparts, with a major concern being mathematical preparation. (NEAP, 2022; EQAO, 2022).

Not all students were able to interact with the full high school curriculum; for example, some classes omitted exponential and/or logarithmic functions from the pre-calculus curriculum (Sacka, 2021).

As a result, we may see a higher degree of heterogeneity in terms of preparedness of incoming students.

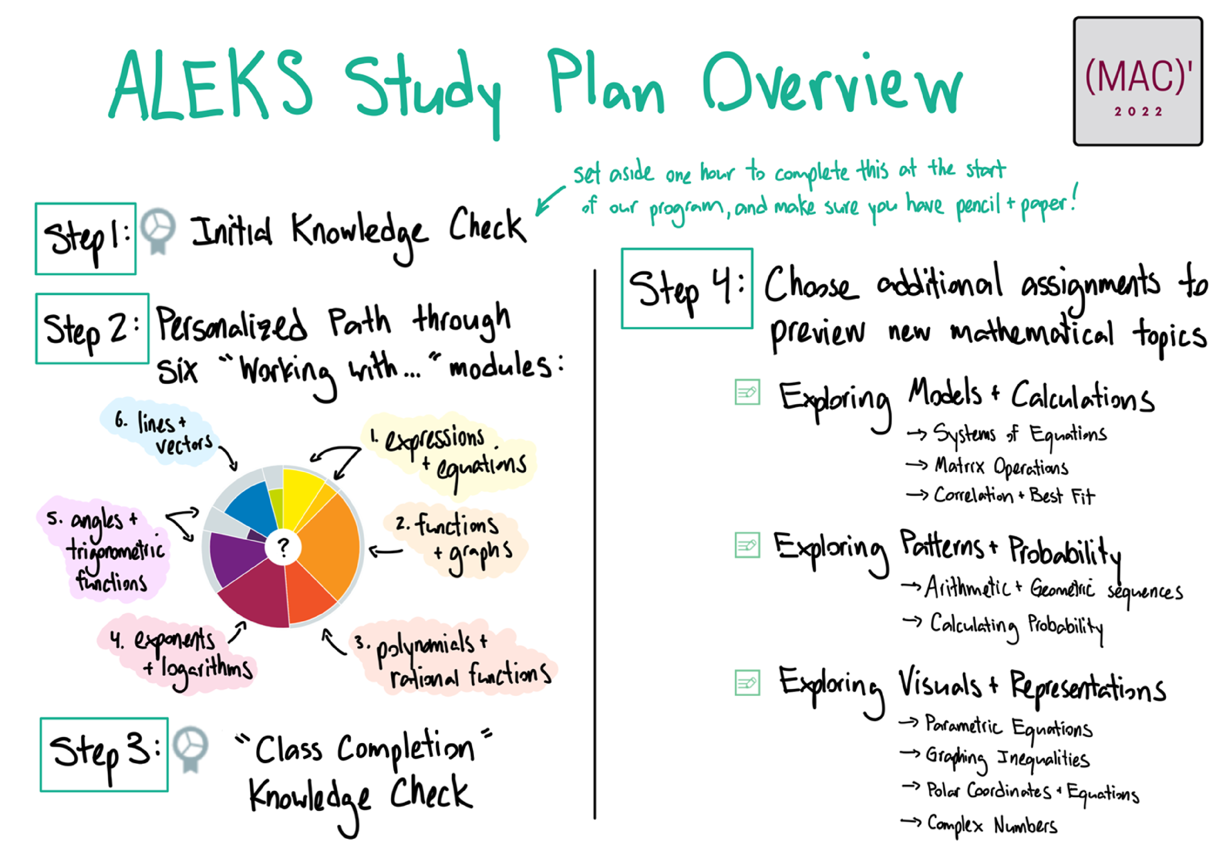
How can the intervention be customized?

To target relevant mathematical skills in an efficient way, the programs described here make use of an online intelligent tutoring system called Assessment and Learning in Knowledge Spaces (ALEKS).

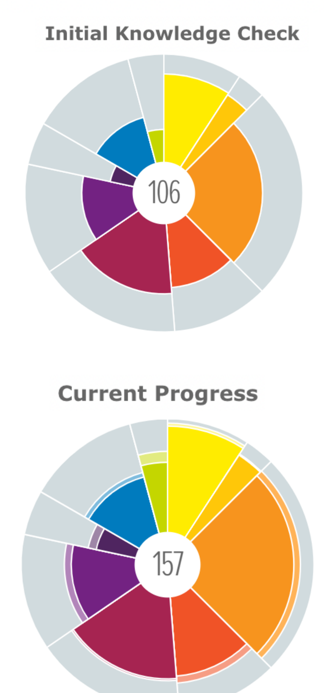
Within ALEKS, the curriculum for each program was shaped to be closely aligned with the local context and curricula, ensuring that student engagement with the material will be served in their upcoming courses.

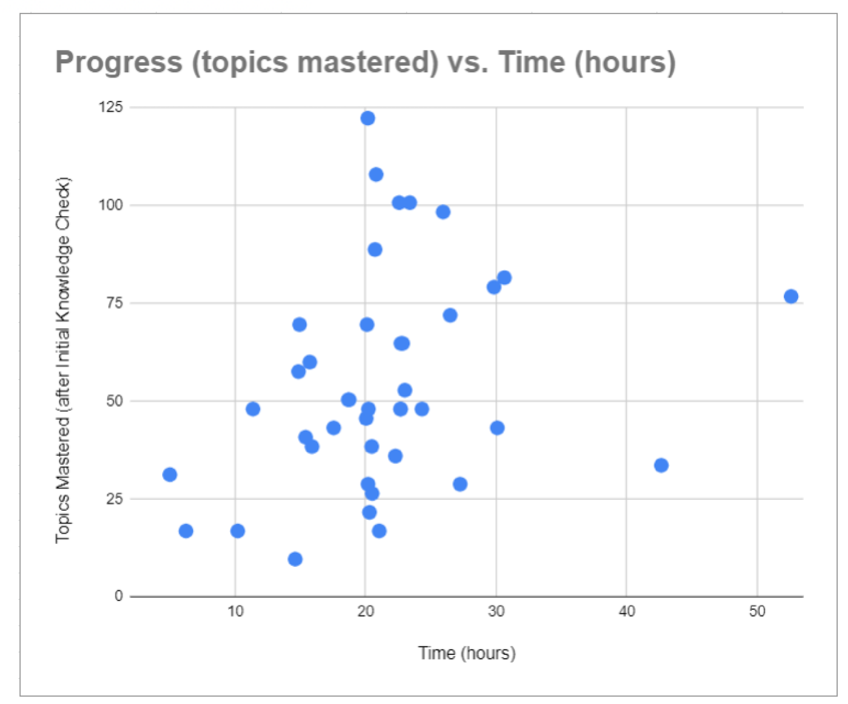
This alignment was directly communicated to students, instructors, advisors and administrators in each program.

McMaster's Peer-Run Inclusive Math Experience (MacPrime)

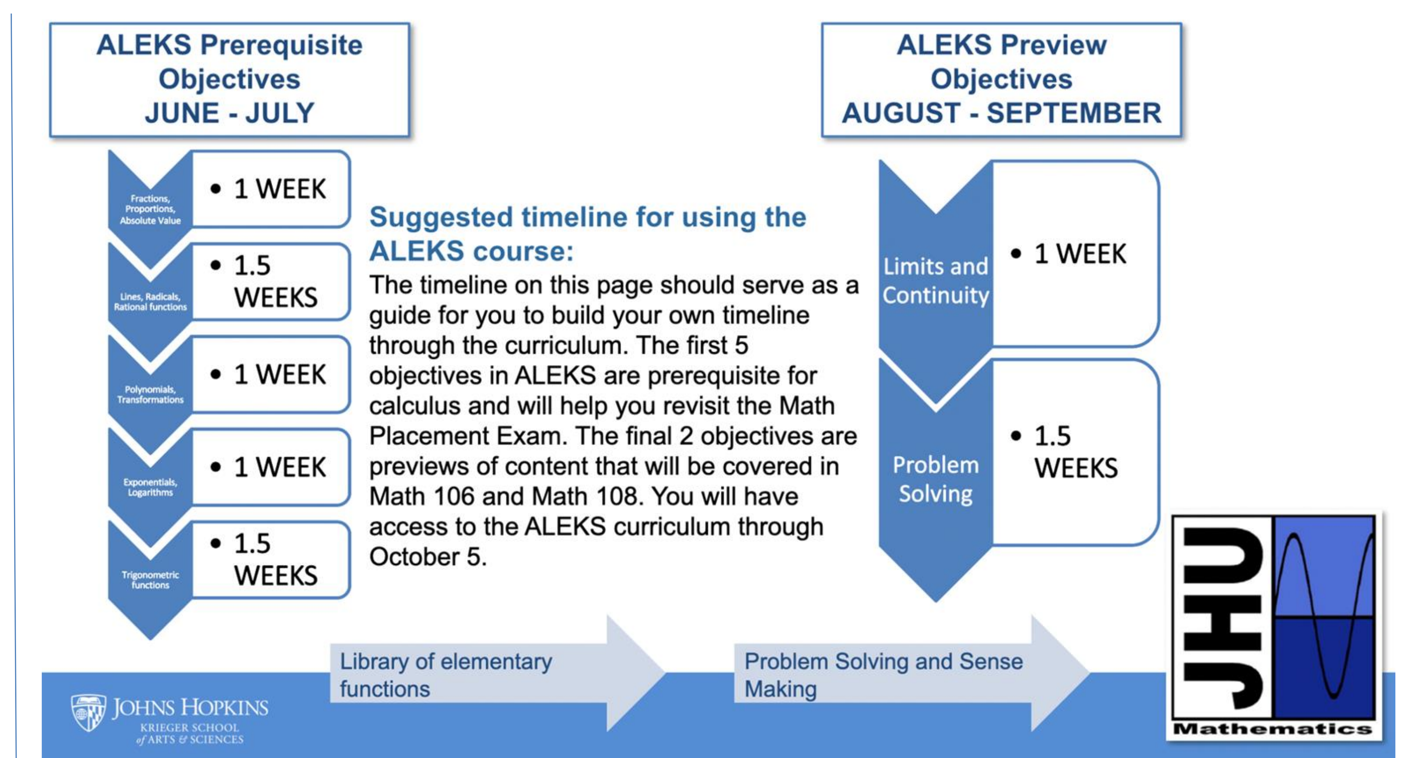


Summer 2022 - ALEKS Pie

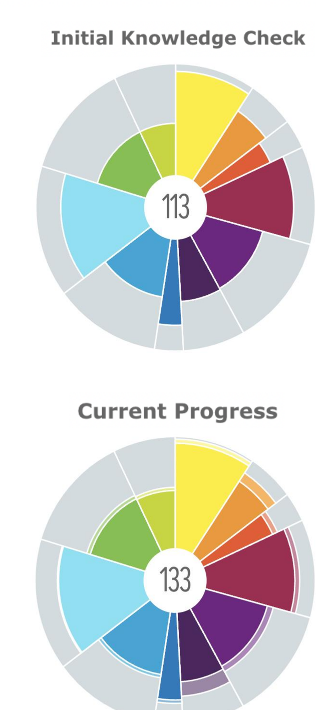


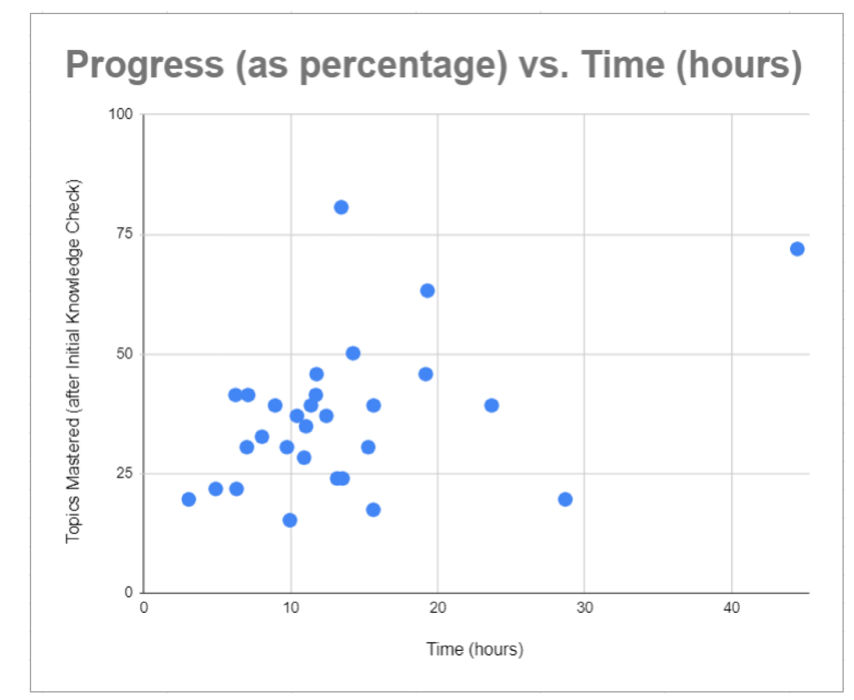


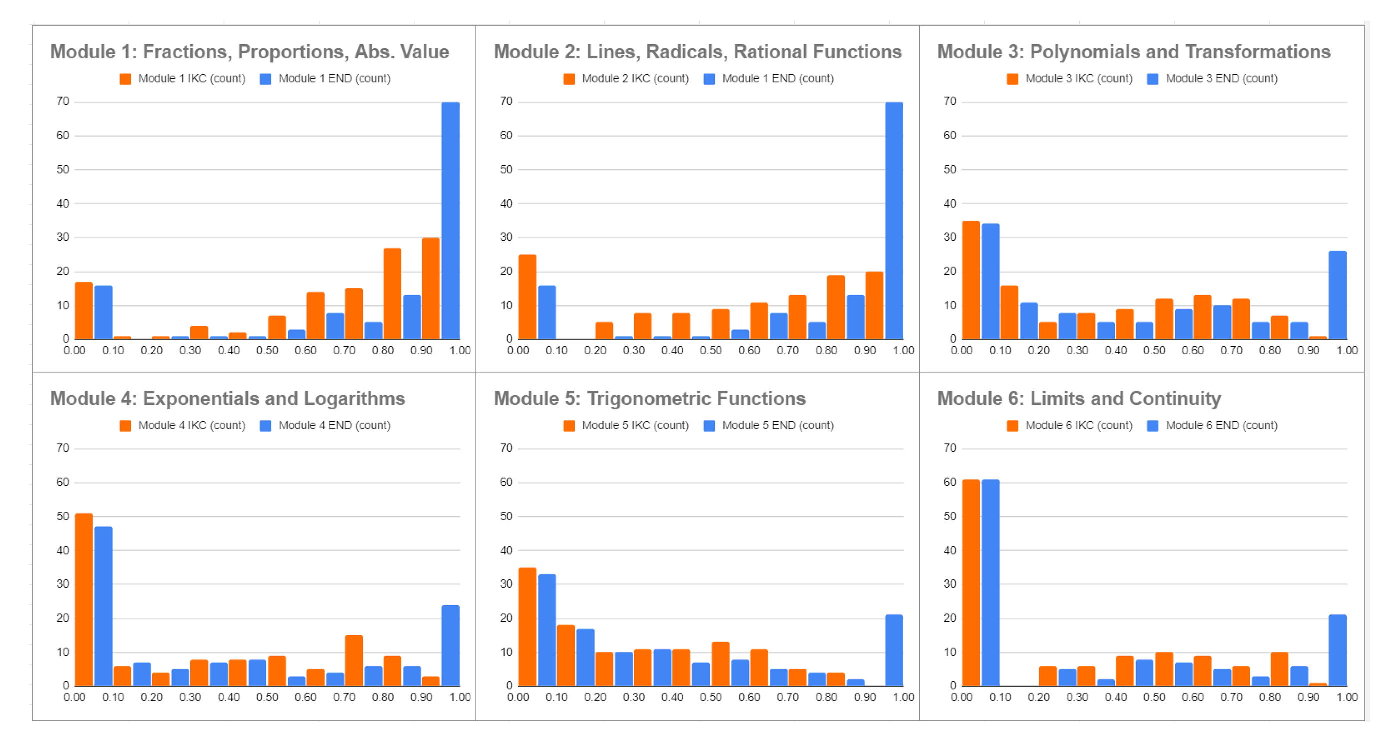
Johns Hopkins' Student Success Programs



Summer 2022 - ALEKS Pie







Student Engagement with ALEKS

- ALEKS alone was not sufficient to motivate student engagement or open lines of communication with faculty or peers.
- Synchronous support for ALEKS (e.g., office hours) were underutilized, participants at McMaster reporting that they were able to find answers to their questions within ALEKS. Students in both programs instead used office hours to ask questions related to placement and course selection.
- Time spent in ALEKS (time active in the system) was weakly to moderately correlated with progress through either program's curriculum (number of topics mastered after Initial Knowledge Check): Pearson $r = 0.30$ (McMaster) and $r = 0.43$ (Johns Hopkins),
- Very few students remained active in the platform once classes started. This result could be attributed to lack of time during this busy transitional period, and/or decrease in motivation as seen in previous applications of ALEKS (Fang et al., 2019)

Can we use program data to better "Mind the Gaps"?

Kornelson, Moore-Russo and Reeder (2020) identified six categories of algebraic errors commonly exhibited by calculus students, found in Modules 1-3 of each program:

- Simplifying
- Distributive property
- Variable isolation
- Operations with rational expressions
- Operations with radicals
- Composition of functions

The authors suggest that even when mastery of these skills is demonstrated in ALEKS, instructors should consider a just-in-time approach, as well as mitigating factors that could be causing mistakes. Data collected from these programs could provide guidance for aligning additional interventions for a given cohort of students.

We hope to learn more about:

- **Experiences with ALEKS products at other institutions (courses/programs, placement, etc.)**
- **Strategies for better student engagement with ALEKS**
- **Community interest for support in analyzing data needed to evaluate programs using ALEKS**

Take our survey:

bit.ly/3ujlyMQ

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Longitudinal Study of Women's Sense of Belonging in Undergraduate Calculus

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Keywords: sense of belonging, gender, undergraduate calculus

Undergraduate women leave STEM majors at a higher rate than men – especially after taking Calculus I (Chen et al., 2013; Seymour & Hunter, 2019). Research identifies low sense of belonging as a major reason why women leave STEM majors (Seymour & Hunter, 2019; Shapiro & Sax, 2011). One feels a *sense of belonging* (SB) when they feel valued or accepted in an academic community, and that their contributions are valued (Good et al., 2012). Studies have shown that students' SB can be attributed to their social connectedness (SC) with the people in their class and their perceived competence (PC) in the course material (Rainey et al., 2018). Further, Rainey et al. (2019) found that students who experienced active learning reported a stronger SB. However, few quantitative studies have investigated students' SB in relation to its influencing factors. This study examines undergraduate women's SB over the course of one semester in an active learning Calculus course. I address the research questions: For undergraduate women enrolled in an active learning Calculus course, (1) How do SC and PC contribute to their SB?, (2) In what ways do their SB, SC, and PC change over the course of the semester?

This study takes place at a Mid-Atlantic research university during the Fall 2022 semester. Two sections of the active learning Calculus course were offered, with a total enrollment of 127 students. These students are typically freshmen intending to major in STEM and do not know upon enrolling that the course provides substantial active learning opportunities (primarily small-group work and whole-class discussions). To measure students' SB, SC, and PC, a survey was designed and administered to all students during Weeks 1, 7, and 14 of the semester. The survey consisted of Good et al.'s (2012) Mathematical Sense of Belonging scale, items from several scales measuring SC (Frisby & Martin, 2010; Hoffman et al., 2003; Maloney & Matthews, 2000; Wanders et al., 2019), and items from Wigfield and Eccles' (2000) Expectancy-Value scale and Cribbs et al.'s (2015) FICSMath scale to measure PC. The data set was reduced to only include participants who self-identified as women (e.g., women and trans women).

Findings indicate that both SC and PC have a significant effect on women's SB. A regression of SB on SC and PC accounted for 47% of the variance in SB at Week 1 ($F[2,46]=20.684$, MSE=.374, $p< .001$), 67% at Week 7 ($F[2,43]=44.145$, MSE=.235, $p<.001$), and 69% at Week 14 ($F[2,40]=44.522$, MSE=.259, $p<.001$). Interestingly, PC held greater influence at Week 1 ($\beta_{SC}=.379$, $p_{SC}=.001$; $\beta_{PC}=.492$, $p_{PC}<.001$), while SC held greater influence at Week 7 ($\beta_{SC}=.641$, $p_{SC}<.001$; $\beta_{PC}=.305$, $p_{PC}=.003$) and Week 14 ($\beta_{SC}=.655$, $p_{SC}<.001$; $\beta_{PC}=.328$, $p_{PC}=.001$). Additionally, students' SB, SC, and PC each increased significantly between Week 1 and Week 7 and plateaued from Week 7 to Week 14 (Table 1). Results from this study confirm that SC and PC hold a significant influence over women's SB. Further, women in an active learning Calculus class can experience positive changes in their SB that align with changes in their SC and PC.

Table 1. Women's Sense of Belonging, Social Connectedness, and Perceived Competence Means.

	Week 1	Week 7	Week 14	p_1	p_2
Sense of Belonging	3.9312	4.5488	4.6159	<.001	.517
Social Connectedness	4.3676	4.7735	4.8537	.001	.444
Perceived Competence	3.7591	4.0335	4.0518	.008	.855

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Longitudinal Study of Women's Sense of Belonging in Undergraduate Calculus

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Research Questions & Definitions

For undergraduate women enrolled in a year-long active learning Calculus course,

1. In what ways, if any, do their **sense of belonging**, **social connectedness**, and **perceived competence** change over the course of the first semester?
2. How do **social connectedness** and **perceived competence** contribute to their **sense of belonging** at the beginning, middle, and end of the first semester?

Sense of Belonging: feeling accepted in an academic community, and that your presence and contributions are valued

Social Connectedness: feeling socially connected or similar to those around them

Perceived Competence: feeling like they understand the material or received good grades

Context

- **Setting:** Undergraduate Calculus course with frequent opportunities for students to engage in active learning via group work and whole-class discussion; 2 sections offered with 127 students total
- **Participants:** Women enrolled in active learning Calculus; mostly first-year students intending to major in STEM

Data Collection & Analysis

Data were collected during Fall 2022 from women enrolled in active learning Calculus using online surveys.

Regression was used to analyze data at each time point.

Repeated Measures ANOVA was used to analyze data across time.

Major Findings

1. Sense of belonging, social connectedness, and perceived competence…

a) All **increased** between Week 1 and Week 7

b) All **plateaued** between Week 7 and Week 14.

Table 1.
Women's Sense of Belonging, Social Connectedness, and Perceived Competence Means.

	Week 1	Week 7	Week 14
Sense of Belonging	3.9312	4.5488***	4.6159
Social Connectedness	4.3676	4.7735***	4.8537
Perceived Competence	3.7591	4.0335***	4.0518

Note. These results are based on students who completed all three surveys, so N=41. Asterisks are used to denote the *p*-values (* for $p<.05$, ** for $p<.01$, *** for $p<.001$ significance levels) corresponding to the significance of the difference between that time-point and the previous time-point.

2. Social connectedness and perceived competence **significantly contributed to** students' sense of belonging throughout the semester.

a) **Perceived competence** held greater influence at Week 1

b) **Social connectedness** held greater influence at Weeks 7 & 14

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Related Tables

Table 2.
Regression of Sense of Belonging on Social Connectedness and Perceived Competence.

		Sum of Squares	df	Mean Square	F	Sig.
Week 1 N=49	Regression	15.472	2	7.736	20.684	<.001
	Residual	17.204	46	.374		
	Total	32.676	48			
Week 7 N=46	Regression	20.788	2	10.394	44.145	<.001
	Residual	10.125	43	.235		
	Total	30.913	45			
Week 14 N=43	Regression	23.037	2	11.518	44.522	<.001
	Residual	10.349	40	.259		
	Total	33.385	42			

Note. Sample sizes at each week differ here because each regression was run without regard for how many participants completed multiple surveys.

Table 3.
Regression coefficients.

		Unstandardized coefficients		Standardized coefficients		
		B	Std. error	Beta	t	Sig.
Week 1 N=49	Constant	-.214	.685		-.312	.756
	Social Connectedness	.510	.148	.379	3.438	.001
	Perceived Competence	.514	.115	.492	4.467	<.001
Week 7 N=46	Constant	-.086	.500		-.172	.864
	Social Connectedness	.702	.106	.641	6.632	<.001
	Perceived Competence	.326	.104	.305	.3152	.003
Week 14 N=43	Constant	-.304	.525		-.580	.565
	Social Connectedness	.759	.109	.655	6.955	<.001
	Perceived Competence	.302	.087	.328	3.480	.001

Conclusions

- **Social connectedness** and **perceived competence** do in fact significantly influence women's **sense of belonging** in mathematics.
- The amount of influence each contributor had on **sense of belonging** was different at different time points.
- Women in an active learning Calculus class can experience positive changes in their **sense of belonging** over time.

Instructor Perceptions of Student Example-Use

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Keywords: Proof, Example-Use, Mathematical Knowledge for Teaching

Researchers have commonly found students from K-16 struggle to understand the purpose of a proof and how to construct proofs (e.g., Harel & Sowder, 2007; Healy & Hoyles, 2000; Stylianides, 2007). One component leading to this struggle is the difficulty in understanding the purpose of examples in a proof context (Aricha-Metzer & Zaslavsky, 2019; Ellis et al., 2019; Epp, 2003). Few studies in the field of example-use in proof investigate the mathematical knowledge for teaching (MKT) that would support instructors of these proof classes to effectively build upon and communicate strategies of using examples to their students (Zaslavsky & Knuth, 2019). In particular, mathematical knowledge for teaching proof (MKT-P) is a promising framework to investigate instructors' knowledge of the example-use subdomain of proof. I will use MKT-P to further describe the sub-domains of knowledge for content and students (KCS) and knowledge for content and teaching (KCT). This study applies the MKT-P framework to answer the following research question:

1. How do instructors of introductory proof classes perceive students' understanding and use of examples?

This study is part of the author's dissertation. Instructors of introductory proof courses across the southeastern United States were interviewed from September 2022 to November 2022. During the interviews, participants were asked to respond to six sample student responses to two questions adapted from Balacheff (1987) regarding example-use in proofs. Later in the interview, the participants sorted and ranked the six responses. Interviews were coded according to the MKT framework outlined by Ball and colleagues (2008) and further coded with a lens of MKT-P outlined by Stylianides and Ball (2008), Steele and Rogers (2012), and Lesseig (2016).

One of the main findings of this study includes the variety in ways instructors value the role of example use in their students. Some instructors held a strong stance that any example used to aid understanding by the student should be fully excluded in the written product. Other instructors instead said they wanted to see more reasoning in the written product by their students and to include more examples. For instance, Steven noted when reflecting on a student submission from Gina, "I mean, it's reasoning by example, which is always rough." This starkly contrasts with Mark who responded to the same instance from Gina with, "She is arguing from the picture, which I appreciate." The field of example use in proof would benefit from more explanation as to the roles of examples for practicing instructors of introduction to proof classes. Aricha-Metzer and Zaslavsky (2019) argued productive instances of example-use aided student understanding and proof production. There is a potential disconnect between the state of the literature of example-use in proof and the practice of practitioners of proof courses. Presenting this in a poster format will allow me to share quotes from my interviews and detailed information about my participants. I hope to gather insight from attendees for future directions with this work and how to disseminate information to practitioners hesitant to include examples in their proving.

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Instructor Perceptions of Student Example-Use

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Introduction

- Understanding the purpose of examples in a proof context can help students develop proving skills (e.g., Harel & Sowder, 2007; Healy & Hoyles, 2000).
- Although work has been done understanding the student side of examples, far less work has been done on understanding how to aid instructors to help their students use examples more efficiently (Zaslavsky & Knuth, 2019).

Research Question

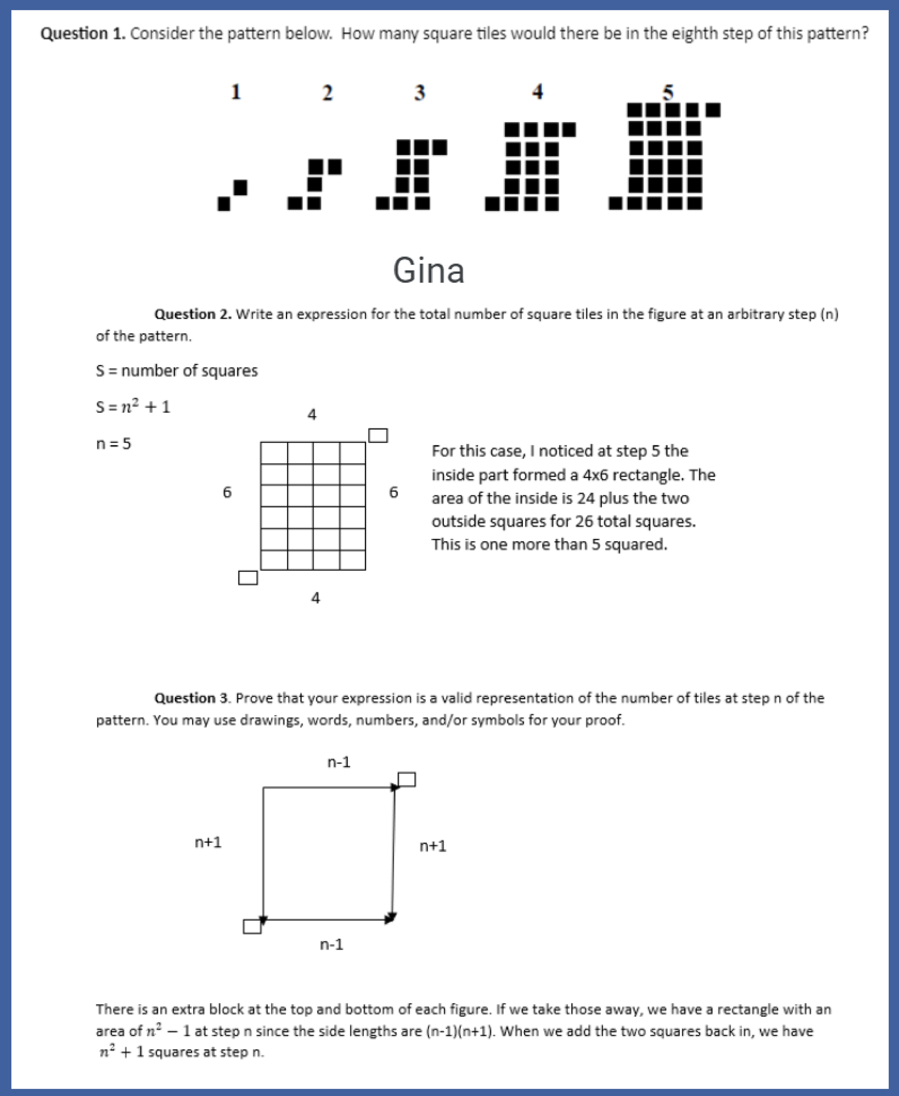
- How do instructors of Introduction to Proof courses perceive students' understanding and use of examples?

Methods

- 1-hour semi-structured interviews with 11 professors of Introduction to proof classes across the SE US.
- Instructors were presented with sample student answers to basic proof tasks.
- Instructors responded to these tasks and gave feedback about their understanding of the student and how they would respond to the student in their class.
- Then, instructors were asked to organize the responses in whatever manner they deemed necessary.
- Finally, the 6 student work samples were ranked from "best progress toward a proof to least progress toward a proof."

	R1/R2	Masters	Baccalaureate
Math	2	2	2
Math Ed	2	1	2

Do it, but don't show me



Question 1. Consider the pattern below. How many square tiles would there be in the eighth step of this pattern?

Gina

Question 2. Write an expression for the total number of square tiles in the figure at an arbitrary step (n) of the pattern.

S = number of squares

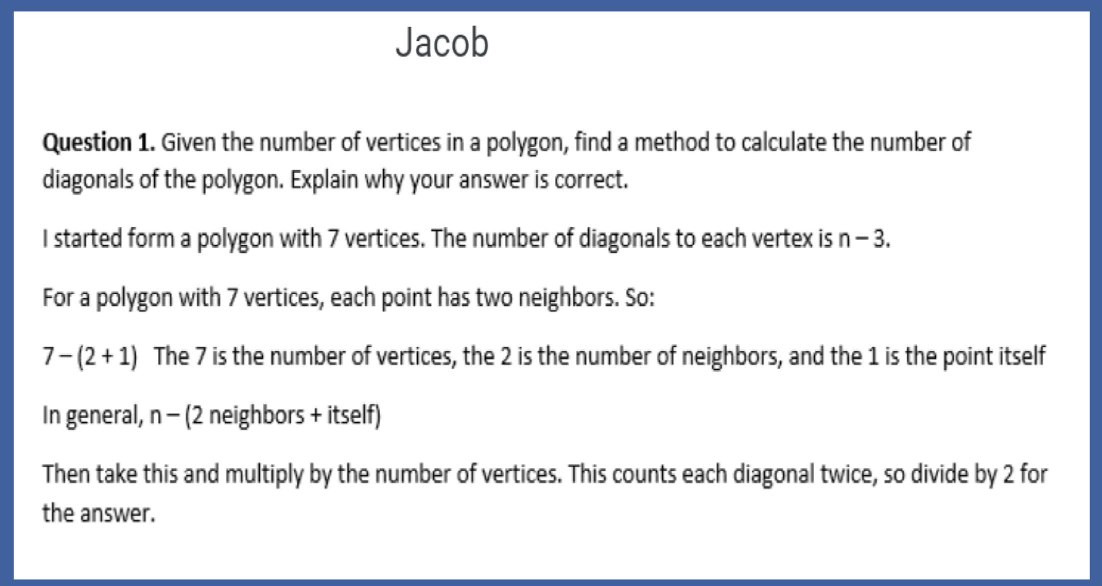
S = $n^2 + 1$

n = 5

For this case, I noticed at step 5 the inside part formed a 4x6 rectangle. The area of the inside is 24 plus the two outside squares for 26 total squares. This is one more than 5 squared.

Question 3. Prove that your expression is a valid representation of the number of tiles at step n of the pattern. You may use drawings, words, numbers, and/or symbols for your proof.

There is an extra block at the top and bottom of each figure. If we take those away, we have a rectangle with an area of $n^2 - 1$ at step n since the side lengths are (n-1)(n+1). When we add the two squares back in, we have $n^2 + 1$ squares at step n.



Jacob

Question 1. Given the number of vertices in a polygon, find a method to calculate the number of diagonals of the polygon. Explain why your answer is correct.

I started form a polygon with 7 vertices. The number of diagonals to each vertex is n – 3.

For a polygon with 7 vertices, each point has two neighbors. So:

7 – (2 + 1) The 7 is the number of vertices, the 2 is the number of neighbors, and the 1 is the point itself

In general, n – (2 neighbors + itself)

Then take this and multiply by the number of vertices. This counts each diagonal twice, so divide by 2 for the answer.

Evan (Assistant Professor at Private Baccalaureate/ Math Ed/ 4 years experience/Lecture):
"So she has a nice little example here, of when if n is equal to five, so for this case, you notice the step five, the inside part formed a four by six rectangle, the area inside is 24, plus the two outside squares for 26 Total squares. This is one more than five squared. Okay, that's great for that particular example. So it kind of shows me how she came up with that formula."

Steven (Professor at Public R1/ Math/ 22 years experience/Lecture):
"I mean, it's, it's reasoning by example, which is, which is always rough."
"I would say that the n minus one n plus one could have helped me early on."

Barry (Associate Professor at Public M1, Math, 7 years experience/ Lecture):
"So, I do like the, I guess, kind of started out with, you know, thinking of a fixed number, just to kind of get an idea of what to do with it."

Lisa (Assistant Professor at Public R1, Math Ed, 4 years experience/ IBL):
"Okay, so it's interesting, because like, I think Jacob is thinking about seven-sided figure, right? And he's taking it as like a generic example. But then the seven is actually never important. Like in his work. Like, he could have, you know, thought about seven and then gone through and replaced all the seven with an M, and now think would have been enough."

Howard (Professor at Public M1, Math Ed, 17 years experience, lecture)
"But as an educator in a first course, I want to see the example, right? And if you're leaving the example out, you're actually leaving out a step in the learning."

Data Analysis

- Data was first coded by episode looking for instances of knowledge of content and teaching (KCT) or knowledge of content and students (KCS) using the mathematical knowledge for teaching framework (Ball et al. 2008).
- Then, a thematic analysis was conducted across episodes looking for themes within KCS and KCT.

Results

- Almost even split of instructors who wanted to see the students written example in their final product (HW).
- No difference based on PhD, time at university, self-identified teaching style.
- Slight difference on if the instructor was at an R1/R2 institution versus anything else.
- Almost all participants asked students to show their thinking through examples when asked how to respond to students.

Discussion

- As a field, researchers have argued that examples, when used productively, aid student understanding and production of proofs (Aricha-Metzer & Zaslavsky, 2019).
- There seems to be a disconnect between what researchers believe, and what we believe in practice.
- Should students be expected to write textbook proofs in introduction to proof classes for all assignments?

Mathematics and Science Education Doctor of Philosophy Program

MIDDLE TENNESSEE STATE UNIVERSITY

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Online Undergraduate Mathematics Classes Impact on Student Perceptions

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Keywords: online learning, student perceptions, undergraduate mathematics, challenges

Background

Online learning has been present in universities and other post-secondary institutions since the introduction of internet access in the 1990s (Johnson & Seaman, 2022). The COVID-19 pandemic has since caused higher demand for online learning and remote instruction to assist with mitigation efforts of the global pandemic (Barlow et al., 2020; Ramadhani et al., 2021). Specifically in mathematics courses, online platforms such as Desmos, GeoGebra, MyMathLab, and WeBWorK are used to supplement learning and teaching (Hauk & Segalla, 2005; Oates et al., 2014). A research study by Serhan and Almeqdadi (2020) on student perceptions of MyMathLab and WebAssign found advantages of using online platforms to be immediate feedback and flexibility of access. Another study by Valdez and Maderal (2021) indicated students preferred online homework over pen-and-paper traditional methods due to the accessibility of lecture notes and math software. Despite the possibilities that online learning offers, there are possible challenges for students in undergraduate math classes. Issues with technology, internet difficulties, and ease of forgetting assignments were challenges students faced in undergraduate online math classes (Serhan & Almeqdadi, 2020). Furthermore, if students were not trained on given technology, they faced difficulty with using online tools and technology effectively. Other challenges are how time-consuming and non-user friendly various online tools can be (Hauk & Segalla, 2005; Naidoo, 2020). With these challenges in mind, the aim of this research is to explore student perceptions in online undergraduate math classes.

Methodology and Results

This study was conducted at a public university in the northeast region of the United States. In spring 2021, 19 online sections of Probability and Statistics were included in the study with 39 out of 464 students consenting to participate. Face-to-face courses were not an option due to the worldwide pandemic. The study participants completed a 17-item survey measuring their perceptions of learning mathematics online. Data was analyzed quantitatively using descriptive statistics.

Despite the small response rate, this study yielded some interesting preliminary findings. Some results included 41% of the students stated they were motivated to complete assignments, 39% were not confident in succeeding, and 59% believed they would perform better had the course been offered in person. These findings mirror those from Serhan and Almeqdadi (2020). When students were asked about the challenges they were facing, motivation was selected with the highest frequency (64%), and 51% of students stated they struggled with understanding course material. The pandemic forced universities to move courses online, and students had no choice in their course delivery. Many participants shared common suggestions and concerns to make online learning better. They requested live meetings (face-to-face or video conference), more accessible content, and the inclusion of clear explanations. Further results and areas for future research will be presented in the poster.

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Impact of Online Undergraduate Mathematics Courses on Student Perceptions

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Abstract

The mode of instruction, online versus face-to-face classes, may be a positive or a negative experience. During the spring 2021 semester, data was collected from three online statistics courses to measure student perceptions. This poster presentation will discuss the impact of online undergraduate mathematics classes on student perceptions.

Background

While online learning has played an important part of curriculum and teaching in universities, it has become more prominent due to the need for distance education to help mitigate the spread of COVID-19 (Ramadhani, 2021). Online platforms like Zoom, OER (open educational resources), and Blackboard allow for learning to be flexible, adaptable, and accessible to students. However, with online learning comes unforeseen challenges. Researchers have found difficulties in communication (Ramadhani, 2020), issues with technology, and ease of forgetting assignments (Serhan & Almeqdadi, 2020). Other researchers found that online platforms such as WeBWorK were not user-friendly (Hauk & Segalla, 2005). Research is needed to understand the influence of online classes on student perceptions.

Research Question

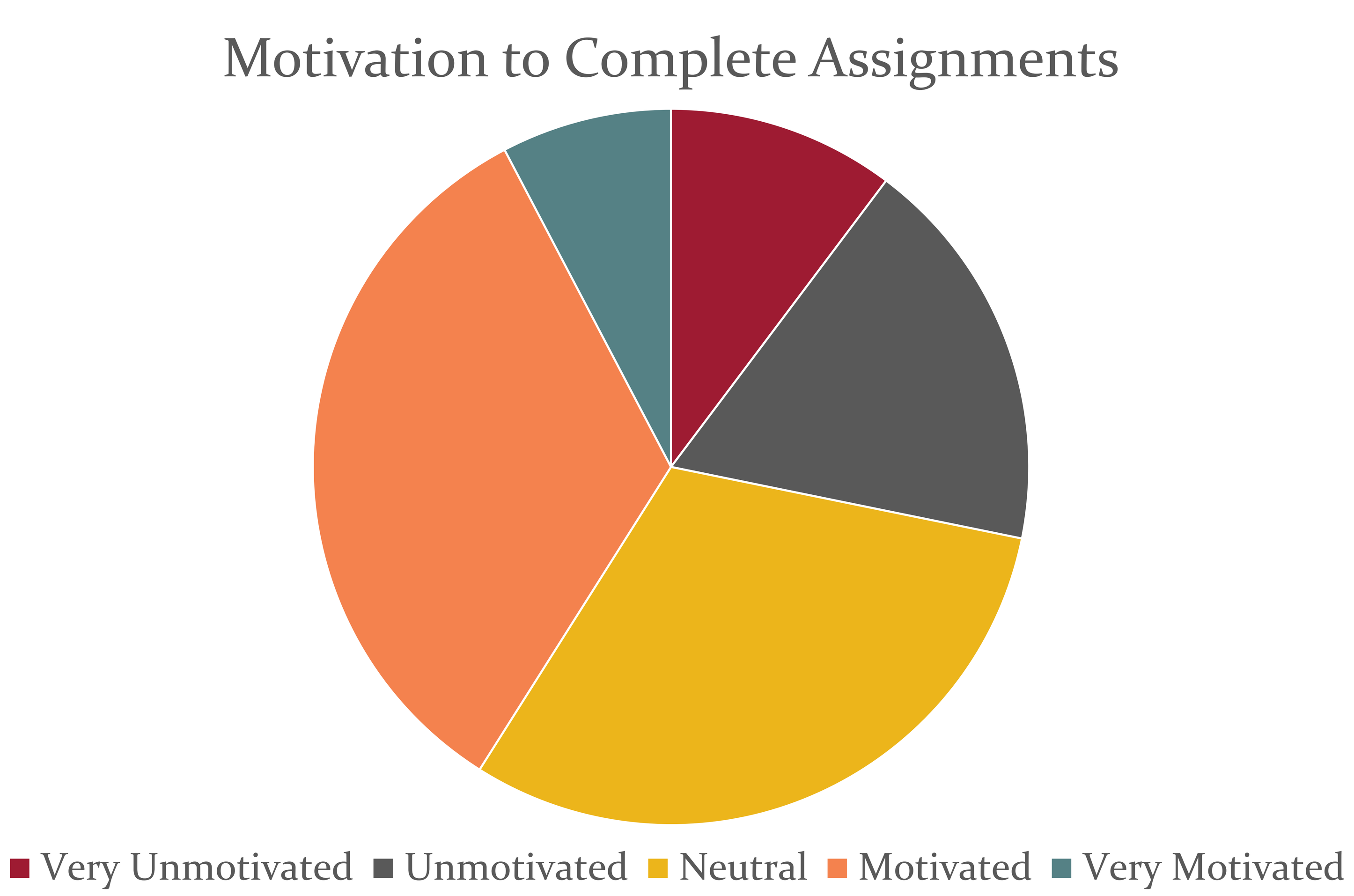
What are the impacts of an online undergraduate mathematics class on student perceptions?

Methodology and Procedures

Due to the variety of majors represented amongst statistics classes, the perceptions of students from three undergraduate statistics courses were examined in this study; Probability and Statistics, Probability and Statistics for Business Majors, and Probability and Statistics for Natural Science.

A 17-question online survey was distributed among 464 students across 19 sections of these courses, of which 39 responded yielding a response rate of 8.41%. Data was analyzed using descriptive statistics.

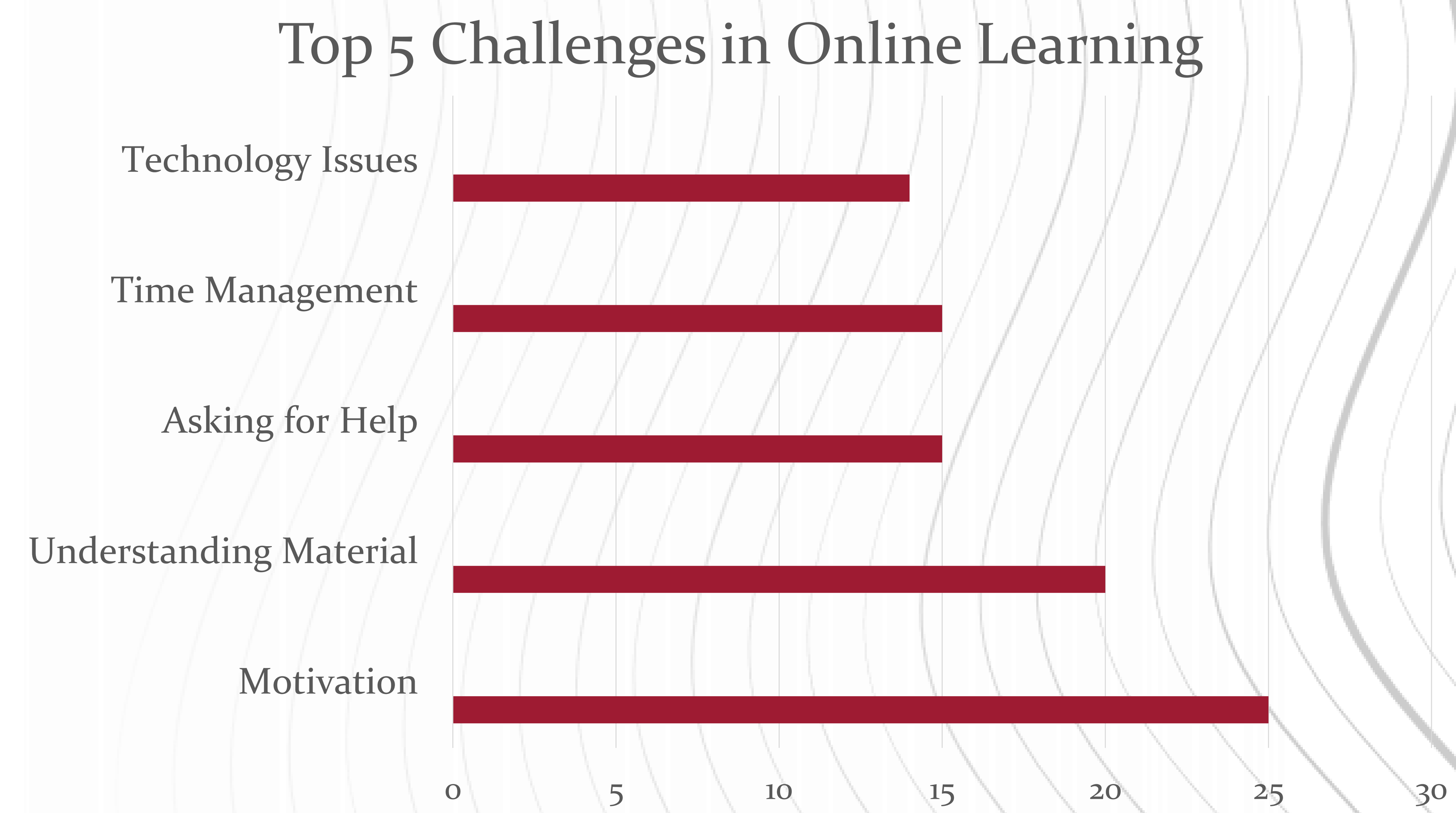
Results and Discussion



41% of the students from this sample stated they were motivated to complete assignments. Approximately 39% of students were not confident in succeeding in their statistics course. However, about 59% stated they would perform better had the course been offered in person.

Results and Discussion Cont'd

Students shared their opinions about challenges they faced while learning statistics online. The top 5 challenges were staying motivated, understanding the material, asking for help when struggling, time management, and technology difficulties.



Some general themes emerged when students were asked "what could make this course better?". The themes were

- the request to have live meetings (face-to-face or video conference)
- need more access to the content, and
- include more clear explanations/instructions.

Student statements specifically addressed technology, such as WeBWorK not being user-friendly, asking for lectures to be posted online, and having specific instructions and training for how to use course software (R Studio).

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The Teaching and Learning of Geometric Proof: Role of the Teacher

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Keywords: geometry; proof; teacher content knowledge; beliefs; attitudes; pre-service; novice

The purpose of this study was to explore preservice and novice secondary mathematics teachers' proof-writing abilities and beliefs about geometry (particularly proofs), in order to propose ideas about how to improve the teaching and learning of geometric proofs. This study contributed to the pre-existing research on Shulman's (1986) categories of teachers' knowledge. Although many studies distinctly examined subject matter content knowledge, pedagogical content knowledge, or curricular knowledge, the present study began to examine these factors in conjunction. Moreover, the researcher did not find pre-existing studies focusing solely on the beliefs, attitudes, and content knowledge of pre-service and novice secondary mathematics teachers with regard to geometric proof.

I created a five-item content assessment (eMATHinstruction, 2017, 2018a, 2018b; The University of the State of New York, 2015, 2016) to administer to preservice and novice secondary mathematics teachers. Each content assessment item was taken from end-of-year assessments for high school students. I met with 29 participants to administer the assessment while asking questions regarding their feelings, beliefs, and confidence in regard to teaching proof in high school geometry. The administrations of these content assessments and simultaneous interviews were audio-recorded and transcribed, and subsequently analyzed for trends in the participants' responses using five qualitative codes: Beliefs and Attitudes (BAA), Issues of Correspondence Between Substance and Notation (ICSN), Expressing Understanding and Self-Doubt (EUSD), Mathematical Language (ML), and Pure Mathematical Issues (PMI).

The results of the study showed that the majority of the preservice and novice secondary mathematics teacher participants do not currently feel prepared to teach proof in high school geometry, from standpoints of both mathematical content knowledge and confidence. The participants with no geometry teaching experience mainly drew on their own experiences from high school, and tried to remember specific rules and standards that their particular teachers tried to instill. This finding supports the ideas of Schoenfeld (1985) and Battista (2009), who also found that most secondary school students see proof-writing as nothing more than following a set of prescribed rules. This study, however, also illuminated issues within participants' Mathematical Knowledge for Teaching (MKT). Specifically, participants did not connect the ideas of rigid motions and equality of measures to congruence, used circular reasoning, and made incorrect visual assumptions when considering diagrams. In response to the interview questions that addressed participants' beliefs and feelings about proof and proving, the participants overall voiced preferences to teach other courses over geometry. Many participants gave up without attempting one or more content assessment items, displaying the need for stronger beliefs in their own abilities. Although participants were not told that the assessment items were high-school level problems, many of them expressed that they knew that from previous exposure via tutoring or student teaching.

Since teachers' attitudes are so easily transferrable to the students in their charge, it is important to continue to conduct research in this area so that the state of the teaching and learning of geometric proof can be improved.

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On the Evolution of Graduate Student Mentees' and Mentors' Conceptualizations of Peer Teaching Mentorship

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Keywords: Graduate Teaching Assistant (GTA), Professional Development, Peer Mentoring

Peer mentoring programs can provide instructional support for graduate teaching assistants (GTAs) (Rogers & Yee, 2018; Yee & Rogers, 2017) through more specialized and detailed discussions than just working with faculty (Speer et al., 2015; Yee & Rogers, 2016). Lanius et al. (2022) explored how mentees and mentors participating in a comprehensive multi-component GTA pedagogical training program, *Promoting Success in Undergraduate Mathematics Through Graduate Teacher Training* (Harrell-Williams et al., 2020), at three universities at the start of an academic year conceptualized the role of an effective mentor. In this poster, we explore whether this conceptualization of the mentor role changed over the course of the academic year after participation in components of the training program: a GTA Teaching Seminar, Critical Issues Seminar, and peer mentoring (including mentor training).

Forty GTAs across three universities completed online surveys in August 2020 and May 2021, answering the same items both times. Two Likert-type items asked about their view of the mentor role in a mentor-mentee relationship. For the first item, a score of "1" indicated viewing the role of mentor as an authority figure whereas a score of "5" indicated more of a collaborative relationship. For the second item, a score of "1" indicated the mentor is a problem-solver and "5" indicated being an empowerer. Mentors also answered two items rating themselves on these continuums. All GTAs were asked to describe the characteristics of an effective mentor.

The related samples Wilcoxon signed-rank tests indicated that there were no statistically significant changes for mentees or mentors for the Likert-type items from August to May. Specifically, six mentees of 23 mentees and five of 17 mentors did not "change" on either of the role conceptualization items. Five of 11 mentors did not "change" on the self-as-mentor items. However, Spearman correlations across timepoints only ranged from .20 to .56, indicating some individual change which was also supported by the qualitative data. The poster will also report results from a psycholinguistic analysis of the responses to the open-ended items. Future work will explore changes in role conceptualization as mentees become mentors.

Acknowledgements

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On the Evolution of Graduate Student Mentees' and Mentors' Conceptualizations of Peer Teaching Mentorship

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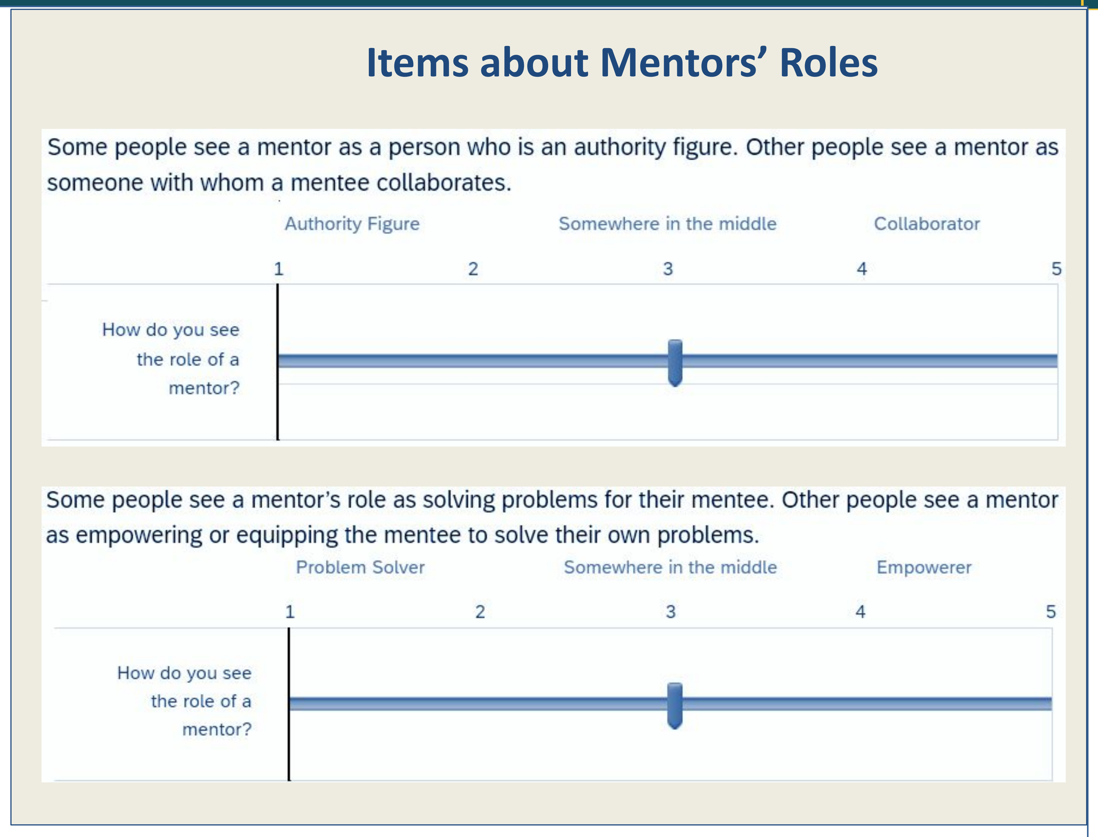
Background

There is ample evidence that peer mentorship among graduate students benefits their academic, professional, psychological, and social learning (Lorenzetti et al., 2019). Specifically, peer teaching mentorship supports graduate teaching assistants' (GTAs) instructional effectiveness, leadership, and confidence (Belnap, 2005; Lorenzetti et al., 2019; Funkhouser et al.,2021; Yee & Rogers, 2017). Views on what makes an "effective" mentor vary throughout literature (NASEM, 2019). Based on the social theory of learning, mentors and mentees tend to conceptualize the mentor role as an authority figure or collaborator, and a problem solver or empowerer (Lanius et al., 2022). In this poster, we examine how GTAs' conceptualization of a peer-mentor's role changes over time as they engage in the peer-mentoring component of the PSUM-GTT program.

Methods

- Research Question: How did mentees and mentors' conceptualizations of peer teaching mentorship change?
- Sample: Start-of-year survey completed in August 2020 and May 2021 by 23 mentees and 17 mentors.
- Survey items:
  - 2 Likert-type items about mentors' roles (see box to the right)
  - 1 open-ended question:
    "What do you believe are the characteristics of an effective mentor?"
- Analysis:
  - Related samples Wilcoxon signed-rank tests & Spearman's correlation to assess change over time
  - Qualitative coding analysis to organize and group descriptive terms for characterizing an effective mentor obtained from the open-ended survey question.



Take-aways

- Neither the repeated samples tests nor the "problem solver/empowerer" correlations were significant.
- The four Transition Diagrams provide insight into interesting cases for further qualitative analysis.
- Both mentees and mentors increased their mentioning of "good listener" as important for effective mentors.
- Mentors increased their mentions of effective mentors as "knowledgeable" while mentees were unchanged.
- Mentees prioritized the "availability" of mentors more in May whereas mentors were unchanged.
- "Constructive feedback" mentions increased over time for mentees but decreased significantly for mentors.
- "Relevant experience" of mentors decreased in mentions for both mentees and mentors over time.

Data Context: About PSUM-GTT

The Promoting Success in Undergraduate Mathematics through Graduate Teaching Assistant Training (PSUM-GTT) program employs a multi-component approach to training.

- Seminar on teaching with a focus on equity and inclusion
- Critical Issues in STEM Education seminar series
- One-to-one peer mentoring
- Support from a peer TA coach
- Visits to K-12 mathematics classrooms and enrichment experiences

2016

Program launched at University of Colorado – Denver (NSF# 1539602)

2019

Program introduced at Auburn University and University of Memphis

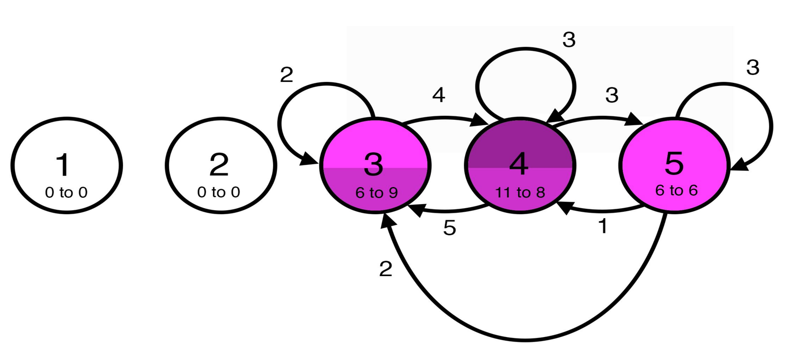
(NSF# 1821454, 1821460, 1821619)

Now

Program leaders from all three universities work together to coordinate program implementation and refinement as well as research on program experiences and outcomes

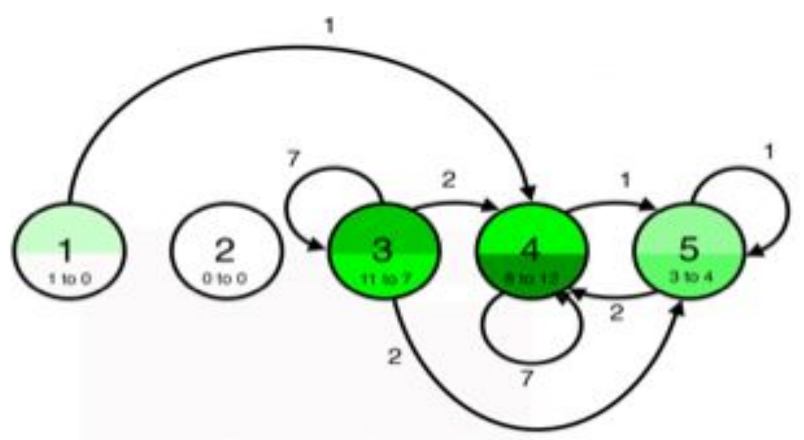
Results

QUANTITATIVE RESULTS



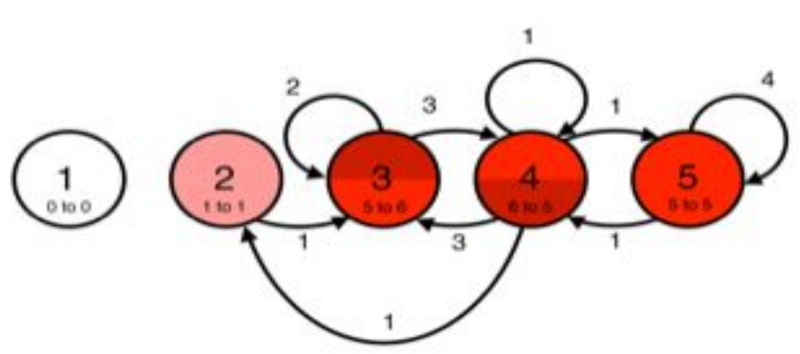
Evolution of Mentee Thought:
Mentor as More Problem Solver (1) or More Empowerer (5)

repeated samples Wilcoxon signed rank test: z = -0.68, p= .50
Spearman's rho: r = .20



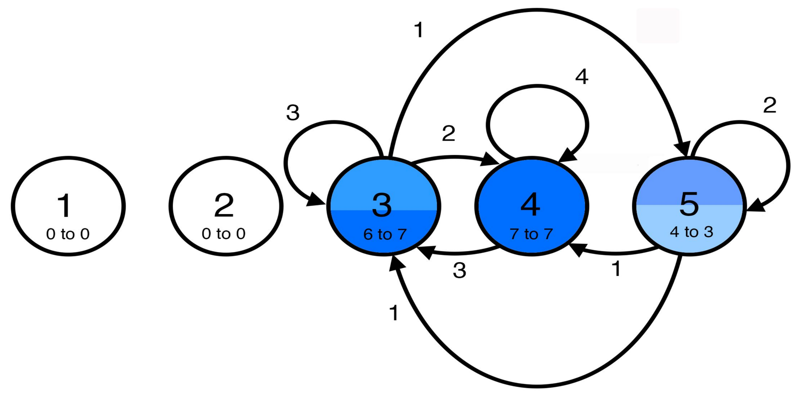
Evolution of Mentee Thought:
Mentor as More Authority(1) or More Collaborator (5)

repeated samples Wilcoxon signed rank test: z = 1.73, p=.08
Spearman's rho: r = .44*



Evolution of Mentor Thought:
Mentor as More Authority (1) or More Collaborator (5)

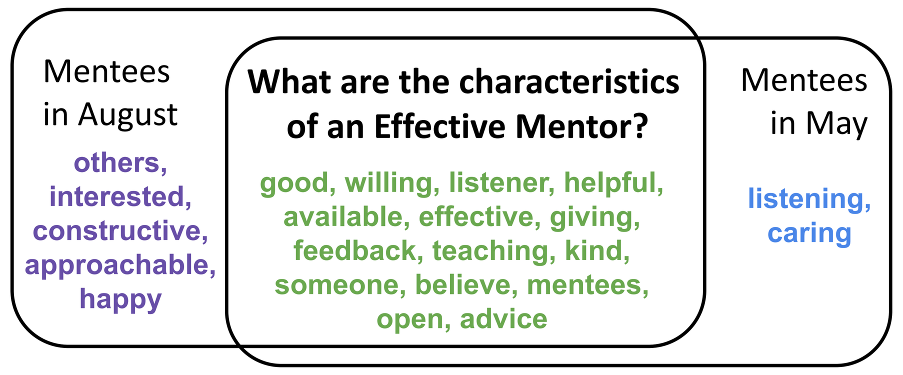
repeated samples Wilcoxon signed rank test: z = -0.28, p=.78
Spearman's rho: r = .56*



Evolution of Mentor Thought:
Mentor as More Problem Solver (1) or More Empowerer (5)

repeated samples Wilcoxon signed rank test: z = -0.51, p=.61
Spearman's rho: r = .25

QUALITATIVE RESULTS



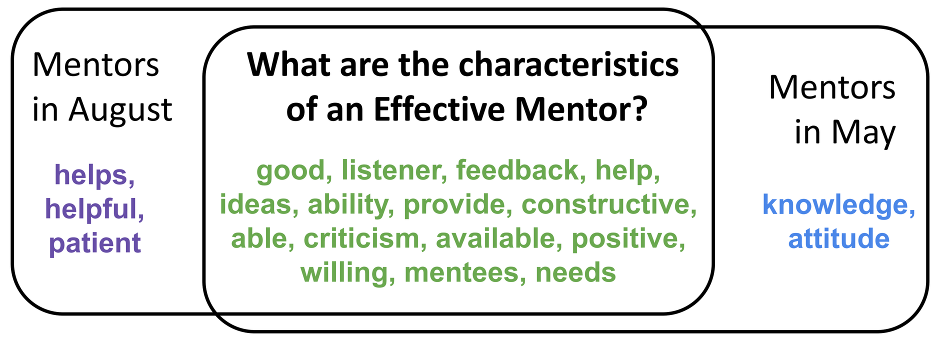


Table summarizing frequency counts for qualitative codes of open-ended question

Mentees	Relevant Experience	Knowledgeable/ Know what to do	Good Communicator	Good Listener	Approachable	Constructive Feedback	Helpful	Available	Uplifting
August	10	4	10	5	5	8	7	1	3
May	6	4	8	11	8	12	8	7	5
Mentors									
August	13	2	4	10	6	14	9	9	5
May	8	9	2	15	8	4	3	9	5

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More information is available on the project ResearchGate page, accessible by using the QR code.

Students' Justifications and Their Proofs by Mathematical Induction

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Keywords: mathematical induction, justification, proof

Mathematical induction has proven to be difficult for students to understand (Ernest, 1984) and, more broadly, the purpose of proof; often remaining unfulfilled as to the certainty of $P(n)$ for values of n for which the statement is proven (Avital & Libeskind, 1978; Harel, 2002). The research perspective in this paper is the proving-as-convincing research paradigm (Stylianides et al., 2017).

Some studies have focused on students' convictions with how they develop or shift (Brown, 2014) and how students convictions may be understood (Weber & Mejia-Ramos, 2015). Others have studied how mathematical arguments may be convincing (Stylianides & Stylianides, 2009). Generally, students may show evidence of this with their justifications regarding proofs.

As part of a larger study, we had students complete a task asking to produce a proof by mathematical induction and provide justifications for the truth value of $P(n)$ for certain values of n. The purpose of this study is to examine the relationship between students' abilities to construct a proof by mathematical induction and their types of justifications based on their written responses to the task. In this study, 59 undergraduate STEM majors across two semesters were given the following task:

a) Prove the statement $P(n)$ by mathematical induction for $n \in \mathbb{N}$, $n \geq 6$,
$P(n)$: $17 + 20 + 23 + \ldots + (3n - 1) = n(3n + 1)/2 - 40$.
b) Is the statement true for $n = 10$? Justify your answer.
c) Is the statement true for $n = 5$? Justify your answer.
d) Comment on your difficulties, if any.

For a preliminary analysis, a framework has been developed loosely based on work by Harel and Sowder (1998) who described the act of justifying as "how students ascertain for themselves or persuade others of the truth of a mathematical observation" (p. 243). We have incorporated the domain of discourse (Stylianides et al., 2007) and domain of validity (Healy & Hoyles, 2000) into a framework to help define these terms.

For proof justification, a student provides justification using his or her proof of the statement. For $P(10)$, it is inferred that the student understands the associated domain of validity for his or her proof. For $P(5)$, it is inferred that the student is convinced that for any value not within the domain of validity (but in the domain of discourse), the statement is false. For empirical justification, the student does not use his or her proof for justification. However, a student may feel the need to provide an additional verification, such as a calculation, that is different from his or her proof after having already proven the statement.

We found that students often relied on empirical justifications (53%), as reported in the literature. However, we also found interesting results. Some students (17%) relied on the proof for both cases, and a few (7%) used proof justification for $P(5)$ and empirical justification for $P(10)$, a result found only to be reported once in the literature (Stylianides et al., 2007). These findings contribute to the field by showing that students may use different sources for justifying statements depending on where the value in question lies. Educators may use these results to attend to how students may operate in the domain of validity and domain of discourse when understanding the purpose of proof.

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Students’ Justifications and Their Proofs by Mathematical Induction

Mario A. Gonzalez and Valentina Postelnicu

Department of Mathematics in the College of Science and Engineering at Texas State University

Rationale

Mathematical induction has proven to be difficult for students to understand (Ernest, 1984) and, more broadly, the purpose of proof; often remaining unfulfilled as to the certainty of $P(n)$ for values of n for which the statement is proven (Avital & Libeskind, 1978; Harel, 2002).

The research perspective in this paper is the proving as convincing research paradigm (Stylianides et al., 2017). Stylianides et al. (2017) stated that this research paradigm on proof revolves around studying students’ convictions with evidence (i.e., mathematical arguments and proofs) and how it is related to the mathematics community’s accepted forms of proof.

In this proving as convincing research paradigm, deductive arguments, such as proofs by mathematical induction, convince mathematicians while arguments relying on specific calculations or having the appearance of proof remain unconvincing (Stylianides et al., 2017).

Theoretical Background

Ernest (1984) described the necessary steps to prove a statement by mathematical induction which grounds itself in *modus ponens*, a deductive argument, and the Principle of Mathematical Induction (PMI).

For mathematical statements, the domain for finding whether a statement is true is called the domain of discourse (Stylianides et al., 2007). The domain of discourse for proofs by mathematical induction is any natural number, n, greater than the basis step, n_0 ($n \geq n_0$). In Healy and Hoyles (2000), they called the domain for which a statement is known to be true the domain of validity. The domain of discourse for $P(n)$ in the task for this study is $n \geq 5$ where n is a natural number, however, the domain of validity is $n \geq 6$.

For this study, the term justification describes how a student determines the validity of a mathematical statement for a value (Harel & Sowder, 1998). Students provide these justifications with written responses to the task. Note that the proofs with which students interact are their own proofs by mathematical induction.

We have incorporated the domain of discourse (Stylianides et al., 2007) and domain of validity (Healy & Hoyles, 2000) into a framework to help define these terms.

Categories

For proof justification, a student provides justification using his or her proof of the statement. For $P(10)$, it is inferred that the student understands the associated domain of validity for his or her proof. For $P(5)$, it is inferred that the student is convinced that for any value not within the domain of validity (but in the domain of discourse), the statement is false.

For empirical justification, the student does not use his or her proof for justification. However, a student may feel the need to provide an additional verification, such as a calculation, that is different from his or her proof after having already proven the statement.

Purpose

As part of a larger study, we had students complete a task asking to produce a proof by mathematical induction and provide justifications for the truth value of $P(n)$ for certain values of n. For these values, we investigated students’ responses for which source, either their proofs or non-proof related reason(s), was used to determine the truth of the statement.

The purpose of this study is to examine the relationship between students’ abilities to construct a proof by mathematical induction and their types of justifications based on their responses to the task.

The Task

In this study, 59 undergraduate STEM majors were given the task below. With the students’ responses, we performed preliminary analyses of the justifications, and we discuss these students’ justifications.

a) Prove the statement $P(n)$ by mathematical induction for $n \in N, n \geq 6$,

$$P(n)\text{: } 17 + 20 + 23 + \ldots + (3n - 1) = \frac{n(3n+1)}{2} - 40.$$

b) Is the statement true for $n = 10$? Justify your answer.

c) Is the statement true for $n = 5$? Justify your answer.

d) Comment on your difficulties, if any.

Table 1. The interactions between the responses of whether P(10) and P(5) are true.

		$P(10)$ result	
	Evidence of Source	Proof Justification	Empirical Justification
$P(5)$ result	Proof Justification	17 (29%)	4 (7%)
	Empirical Justification	7 (12%)	31 (53%)

(Top Left) Proof Justification for P(10) and Proof Justification for P(5).

b) Is the statement true for n=10? Justify your answer.

Yes, the statement is true for n values greater than or equal to 6. Since 10 is greater than 6, the statement holds true.

c) Is the statement true for n=5? Justify your answer.

No, the statement is not true for n = 5. The statement is only true when n = 6 or greater than 6. Since 5 is a lower number than 6, the statement is not true for when n = 5.

(Top Right) Empirical Justification for P(10) and Proof Justification for P(5).

b) Is the statement true for n=10? Justify your answer.

$\frac{10(3(10)+1)}{2} - 40$

b) is false

$3(10)-1=29$ $\quad$ $155 - 40 = 115$ $\quad$ $29 \neq 115$

n=10 is false

c) Is the statement true for n=5? Justify your answer.

no because $n \geq 6$

$5 \leq 6$ which makes this false

(Bottom Left) Proof Justification for P(10) and Empirical Justification for P(5).

b) Is the statement true for n=10? Justify your answer.

Yes because we checked ~~it~~ and the statement is true for any $n \geq 6$ by the PMI, and $10 \geq 6$

c) Is the statement true for n=5? Justify your answer.

It is not true for n=5.

$P(5)\text{: } 3(5)-1 = \frac{5(3(5)+1)}{2} - 40$

$14 \neq 0$

(Bottom Right) Empirical Justification for both P(5) and P(10).

b) Is the statement true for n=10? Justify your answer.

$\frac{10(31)}{2} - 40 =$ $\quad$ $17+20+23+26+29$

$155 - 40 = 115$ $\quad$ $37 + 49 + 29$

$86 + 29 = 115$

$n = 10$ ✓

I was thinking too fast and I forgot the -40

c) Is the statement true for n=5? Justify your answer.

$(3(5)-1) = 14$ $\quad$ $\frac{5(16)}{2} - 40$

$\frac{80}{2} = 40 - 40 = 0$

$14 \neq 0$ $\quad$ n=5 not true

Findings

Note that students’ written comments on part d) of the task are included in the figures. Some students declined to respond to part d).

We found that students often relied on empirical justifications (53%), as reported in the literature. However, we also found interesting results. Some students (17%) relied on the proof for both cases, and a few (7%) used proof justification for $P(5)$ and empirical justification for $P(10)$, a result found only to be reported once in the literature (Stylianides et al., 2007).

Some students responded to part d) by writing that they did not experience difficulties. This is interpreted that students may not perceive the relationship between a proof and the domains of discourse and validity.

These findings contribute to the field by showing that students may use different sources for justifying statements depending on where the value in question lies. Educators may use these results to attend to how students may operate in the domains of discourse and validity when understanding the purpose(s) of proof.

Next Steps

We plan to use these findings to help develop tasks to better understand how students’ justifications change based on the domains of discourse and validity.

Future research investigations may include these constructs and the relationship with students’ convictions with mathematical statements and arguments.

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Exploration of Students' Experiences Transitioning from Secondary to Tertiary Mathematics

Samuel Waters
University of Northern Colorado

Keywords: Transition, University Mathematics, College Algebra, Phenomenology

Early experiences in undergraduate mathematics contribute to student persistence STEM majors (Courturier & Cullinane, 2015). The transition from high school to undergraduate mathematics remains problematic for both students and instructors despite increased attention on the issue (O'Shea & Breen, 2021). Motivated by the need to understand STEM-specific issues related to student transitions to college, this study investigated two research questions: (a) What is the essence of the lived-experiences of first-year undergraduate students transitioning to their first post-secondary mathematics course? and (b) How do students view their transition after their first undergraduate mathematics course?

Previous research on transition to college found when students do not feel like they belong at their institution or in a classroom, there is less academic engagement and lower achievement (Bowman et al., 2019). While many researchers have investigated the transition to collegiate mathematics, most are focused on specific aspects of the mathematics courses such as content, instruction, or classroom environment (Blackmore, 2021; Chuene, 2011; O'Shea & Breen, 2021). This study enhanced existing research by taking a more holistic view of transitions including factors in and out of the mathematics classroom.

Through a constructivist lens, this study focused on the meaning that students construct while experiencing their transition to undergraduate mathematics and their reflections on these experiences. As the qualitative approach of the study, an interpretative phenomenological analysis (IPA) centered the study on individual experiences and the meaning-making of both participant and the researcher (Smith & Shinebourne, 2012). Goodman et al.'s (2006) transition theory served as an analytical framework. Participants were three first-year students who were enrolled in College Algebra during their first semester at a mid-sized university. The first interview occurred at week ten of the first mathematics course. One participant was selected for a follow-up interview at the start of their second semester. Each interview was transcribed and coded using In Vivo coding followed by pattern coding to identify themes.

Three overarching themes were: (a) adjusting to new course expectations, (b) navigating challenges outside the classroom, and (c) changes in mindset and motivation. All participants shared experiences adjusting to new expectations for participation and working with others, which two participants noted helped them learn and made them more comfortable in the classroom. Participants described an emerging independence which contributed to changing mindsets in the mathematics classroom. One participant shared that since he was in college now, "I had to think differently… I had to change the mindset of what I needed to do." Another spoke about how her experiences in College Algebra provided an outlet to explore a new sense of self, stating "I felt more in my own skin." Findings from this study point to several implications for instructors including setting and communicating clear expectations to assist students in understanding their new environment. Encouraging group work early in courses can help students build connections and develop a sense of belonging both socially and academically within the classroom. Further research can explore how students' mathematical identities change during the transition to college mathematics.

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Exploration of Students' Experiences Transitioning from Secondary to Tertiary Mathematics

UNIVERSITY OF NORTHERN COLORADO
Mathematical Sciences

Samuel Waters, Graduate Student, Educational Mathematics Ph.D. and Advisor Dr. Gulden Karakok

Motivation, Purpose, Research Questions

Motivation

- High dropout rates at universities linked to failure in mathematics point to a remarkable **subject-specific challenges** for students transitioning from secondary to tertiary educational environments (Rach & Heinze, 2017).
- Understanding student experiences provides institutions and faculty with needed information to set up supports and interventions to promote student success in mathematics (Blackmore et al., 2021).

Purpose

The purpose of this longitudinal phenomenological study was to describe first-year undergraduate students' experiences transitioning from high school to undergraduate mathematics classrooms.

Research Questions

- Q1 What is the essence of the lived-experiences of first-year undergraduate students transitioning to their first post-secondary mathematics course?
- Q2 How do students view their transition after their first undergraduate mathematics course?

Research Design and Methods

Research Paradigm: Constructivism

The central tenant of constructivism is that meaning is constructed by human beings as they interact with the world; reality cannot exist separate from the knower's experiences (Crotty, 1998).

Methodology: Interpretive Phenomenological Analysis (IPA)

Phenomenological studies attempt to describe a phenomenon by focusing on the common meanings that individuals have towards a shared experience (Creswell & Poth, 2018).

Setting: mid-sized, regional university in the Rocky Mountain Region

Participants: Participants were three undergraduate student volunteers enrolled in College Algebra during Fall 2021, their first semester of college after graduating high school in Spring 2022.

Dan (he/him)	Caucasian; majoring in English Education; enjoys fantasy and board games
Sela (she/her)	Hispanic; majoring in Psychology; oldest of five; first-generation college student; "quiet and chill"
Solus (he/him)	Hispanic; majoring in Business Marketing; out-of-state student; "shy"

Note: Pseudonyms, pronouns, and all personal descriptors were selected and provided by the participants

Data Collection: Each participant partook in an interview ten to twelve weeks into their first semester. One participant, Sela, was selected for follow-up and was interviewed two additional times during Spring 2022.

Data Analysis: Interview transcripts were reviewed and coded using in vivo and descriptive codes. Codes were also initially categorized based on Schlossberg's 4S components. Pattern coding was then used to identify main themes.

Students' Voices

"when I walked into college my first day, **I didn't really see myself [how I do] now**." - Sela

"I feel like it [College Algebra] just made the semester a bit more better, I guess. **I felt more in my own skin** than I did…a class that I actually felt motivated doing, felt pretty good in." - Sela

"I guess the thing that changed was **having a new mindset since it was college now.** I had to think differently… I had to change the mindset of what I needed to do and what I needed to change about myself." – Solus

"During the year I'd been really good about sticking to a sleep schedule and just overall I was doing a lot better with mental health and physical health and a lot of that stuff. **So a lot of the reasons I really loved that math class weren't exactly about the math class. There was other contributing things.**" - Dan

Results

Adjusting to New Course Expectations

Description	Example
Differences in perceived instructor support	"A lot of professors aren't really there to teach." – Dan
Greater focus on process and conceptual understanding.	In high school math class, "I didn't have to explain a lot of it and you just had to get the answer." - Solus
Access to peer supports and multiple solution strategies	"I got to learn more about how the other peers did math, like how they solved their problems. They would explain it and it taught me a new way than the original way I learned and so it's a new strategy to put in the books." –Sela

Navigating Challenges Outside the Classroom

Description	Example
Other responsibilities	"I started working this semester my first job because I never really worked before, so it's something new that I've been trying to get used to and then also [getting] used to making time around it. I get less time to complete it [homework] and so I think I'm rushed to do it [homework]." - Sela
Difficulty in making friends	"I've been really slow to meet new people and make friends." – Dan
Outside stress interacting with mathematics	"it's getting to the point where I'm just like, I need to ask for an extension time because there's too many things to do. And so, when I do have time, it's usually for more work." - Solus

Changes in Mindset and Motivation

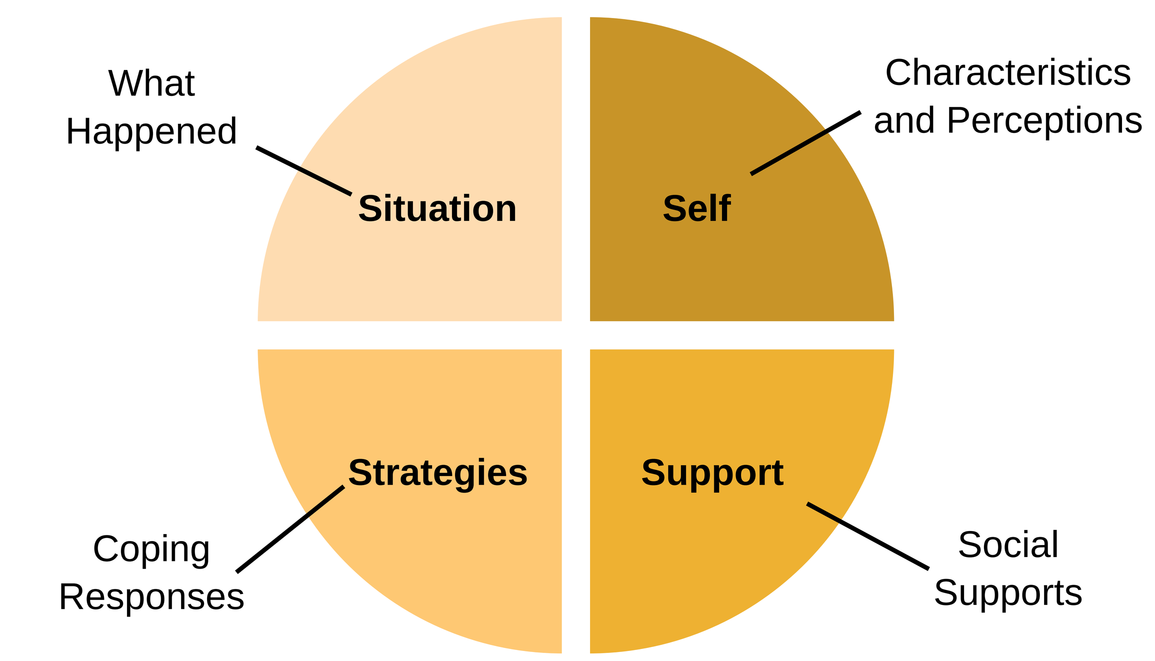
Description	Example
Growing sense of independence	"It's just adjusting to living on my own." – Dan
Independence related to mindset	Solus made a conscious decision to adopt a new mindset when entering college (quoted above). "Not having to rely too much on others." – Solus
Independence related to motivation	Dan took ownership of his own learning and did not rely on others for motivation in mathematics.

Reflections on the Transition

Description	Example
Assessing support and strategies	"teaching them was teaching myself" – Sela
Developing independent identity	Sela was able to see herself as an individual instead of just as a member of her family. The interactions Sela had in her mathematics course provided an outlet to explore this new sense of self (quoted above).
Applying lessons learned moving forward	"Because of that previous group work I had in College Algebra… I didn't feel pressure…I felt easily able to really speak with them and share my opinion as well…I know that if it were younger me, I wouldn't have been able to do that." - Sela

Analytical Framework

Schlossberg et al.'s (1995) Transition Theory



Trustworthiness and Limitations

Trustworthiness

- Research journal, reflexivity, and bracketing
- External checks from peers and other researchers
- Member checks during and after interviews

Limitations

- Sample: size; same course at same institution; enrolled in mathematics in first semester right after high school
- Dan and Solus were only interviewed once

Implications and Future Research

Implications for Practice

- Undergraduate instructors can reflect on how they communicate course expectations to help students assess their new situations.
- Consider how to incorporate group work early in the course to assist students with their academic and social transitions.
- Be understanding of factors outside of class that might be contributing to academic challenges.
- Build structured peer supports to take advantage of lessons learned by students who have gone through the transition.

Recommendations for Future Research

- Examine secondary-to-tertiary mathematics transitions in other settings (e.g., institutions, courses, etc.)
- Investigate how transitions are impacted by recent and ongoing educational experiences related to COVID-19
- Further explore the potential of group work to facilitate student transitions in mathematics.
- Research how students' mathematical identities change during transition.

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Embodiment as Evidence for Student Engagement in an Inquiry-Based Mathematics Classroom

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Keywords: Embodiment, inquiry, engagement

The multi-dimensional construct of student engagement is an important aspect of students' experiences with mathematics, and increased levels of engagement support students' academic achievement (Carini et. al, 2006; Pilotti et al., 2017) and positive mathematics identity (Voigt et al., 2022). We propose that embodiment is a useful lens for exploring evidence of such engagement. Embodiment is the notion that "thinking does not occur solely in the head but also in and through a sophisticated semiotic coordination of speech, body, gestures, symbols and tools" (Radford, 2009, p. 111). While scholars have studied student engagement in the context of specific embodied tasks through self-reporting instruments (Georgiou & Iannou, 2020; Lindgren et. al, 2016), there is a gap in the literature regarding how embodiment might provide evidence of students' engagement. As student engagement with meaningful mathematical tasks is one of four pillars of Inquiry-Based Mathematics Education (IBME) (Laursen & Rasmussen, 2019), the IBME classroom is an ideal setting to study observable, embodied student engagement. In this poster, we present a microanalysis of three episodes of embodied engagement between two students (Rebecca and Lauren) in an IBME Discrete Mathematics classroom.

We draw on Fredericks et al.'s (2004) summaries of *behavioral* (on-task, participatory classroom actions), *affective* (emotional response or investment to the task at hand and its associated discoveries), and *cognitive* (students' self-regulation of learning) engagement to assess whether embodiment captures classroom engagement. These categories of engagement are not disparate, and thus we consider them as "dynamically interrelated" (Fredericks et al., 2004, p. 61). We also leverage definitions of embodiment that draw from Nemirovsky et. al (2012) in which mathematics is framed as a perception-action cycle and is evidenced by bodily activities including gesture, gaze, and body position to address our research question: *In what ways can embodiment provide evidence for students' engagement with meaningful mathematical tasks?*

Our three selected episodes support the notion that all three categories of student engagement are observable, embodied phenomena in the IBME mathematics classroom. For example, in one episode Rebecca's body positioning, gesture, and use of materials, without verbiage, illustrates her behavioral and cognitive engagement as the professor and Lauren converse. An embodied lens enabled attention to Rebecca's rich engagement that otherwise may have gone unnoticed. Thus, we assert that leveraging embodiment to study student engagement provides new insights into how students engage with meaningful tasks in an IBME classroom. This perspective can be utilized by both researchers and instructors in constructing a more nuanced understanding of how students engage with mathematics tasks.

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Embodiment as Evidence for Student Engagement in an Inquiry-Oriented Mathematics Classroom

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Motivation

Engagement in the math classroom is an important component of student experience and achievement. (Carini et al., 2006; Pilotti et al., 2017; Voigt et al., 2022)

"Developing valid and reliable measures [for student engagement] is especially important in math and science because **engagement** in these subjects **is so critical to academic achievement and career choices** related to STEM." (Fredericks et al., 2016)

Measuring student engagement is often limited to self-reporting or observation scales which privilege verbal participation. (Fredericks & McColskey, 2012; Hodgson et al., 2010)

We propose embodiment as a lens through which to observe student engagement in the classroom,

Research Question: In what ways can embodiment provide evidence for students' engagement with meaningful mathematical tasks?

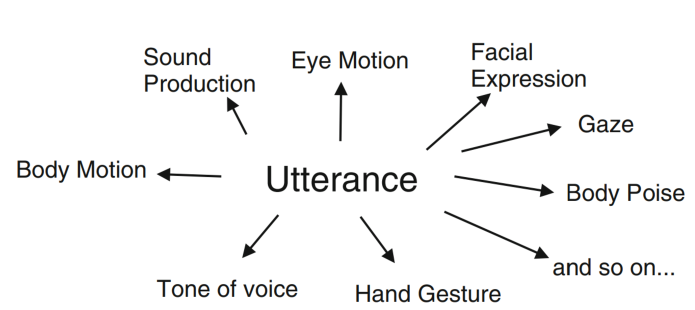
Embodied Cognition

Embodied cognition asserts that "…thinking does not occur solely in the head but also in and through a sophisticated semiotic coordination of speech, body, gestures, symbols and tools." (Radford, 2009, p. 111)

Embodiment can *enhance* student engagement with tasks. (Georgiou & Iannou, 2020; Lindgren et al., 2016)

We categorized the embodiment of our participants using Nemirovsky and Ferrara's (2009) notion of **multimodal utterances.**

Nemirovsky & Ferrara (2009)
Figure 2: The multimodality of an utterance



Student Engagement

Fredericks et al. (2004) provide three categories of student engagement:

- **Behavioral Engagement:** on-task, participatory classroom actions
- **Affective Engagement:** emotional response and investment in the task
- **Cognitive Engagement:** students' self-regulation of learning, thoughts, and perseverance

Student engagement with meaningful mathematics tasks is one of four pillars of Inquiry-Based Mathematics Education (IBME). (Laursen & Rasmussen, 2019)

Methodology

Participants:

- Students of an undergraduate Discrete Mathematics course
- An experienced instructor (Dr. A) who follows an Inquiry-Based Mathematics Education (IBME) teaching paradigm

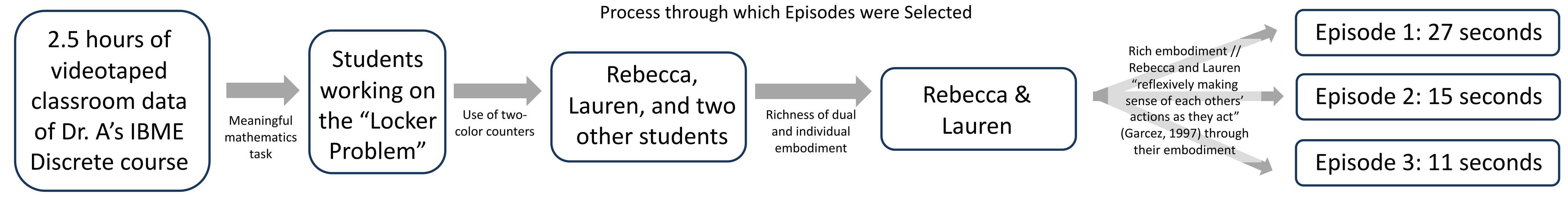
Data Collection and Selection:

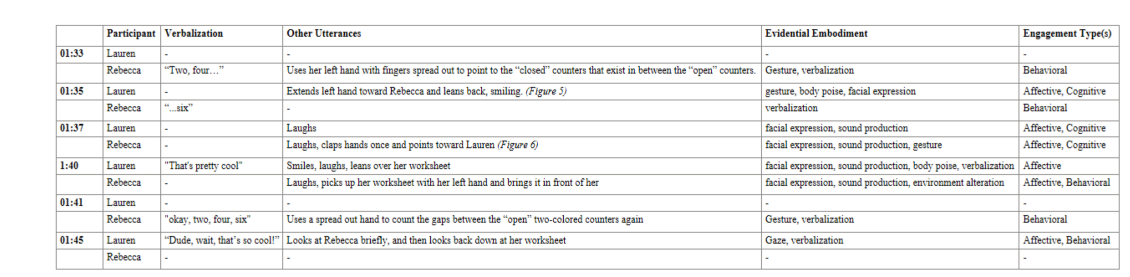
- Videotaped a total of 2.5 hours of classroom footage
- Narrowed our focus to a *meaningful mathematical task* that the students worked on the "Locker Problem" *(physical copy of worksheet available for viewing)*
- Analyzed the engagement of two students, Rebecca and Lauren, who utilized two-colored counters and worked both individually and together to solve the Locker Problem

Analysis:

- Microanalysis (Garcez, 1997): second-by-second transcription and description
- 2x Thematic Analysis (Braun & Clarke, 2006) – *a priori* coding according to Nemirovsky & Ferrara's (2009) multimodality of utterances and Fredericks et al.'s (2004) three domains of engagement.

Process through which Episodes were Selected





Screenshot of Full Microanalysis and Coding for Episode 2

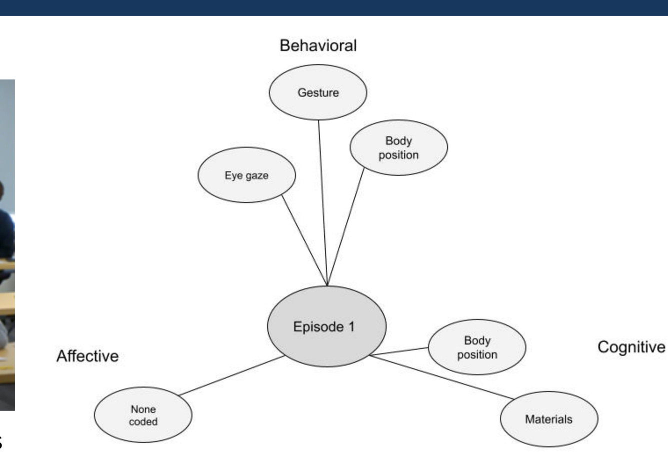
Results

Episode 1: Responsive Resetting

Rebecca interacted with her two-colored counters in direct response to the conversation she heard Lauren and Dr. A having.

Rebecca (right) leaned towards her counters and began resetting them to their original "closed" position when Dr. A (center) prompted Lauren to reconsider the problem.

Rebecca pushed additional counters towards her original line of counters, while Dr. A and Lauren continued discussing Lauren's worksheet.

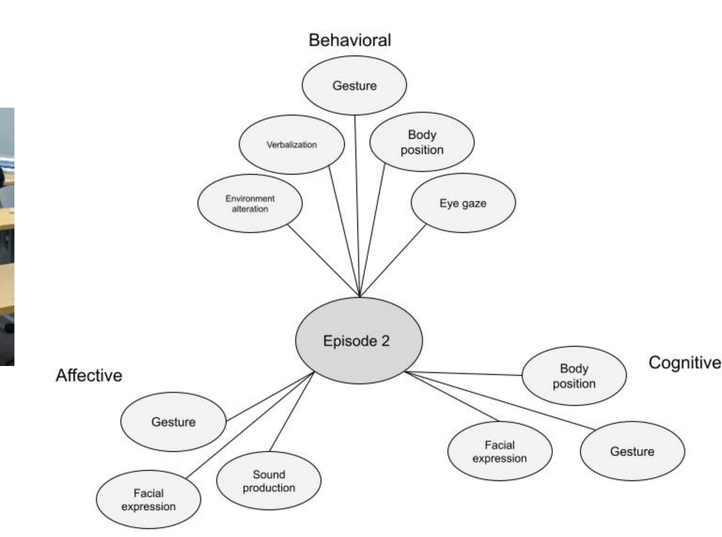


Episode 2: Collaborative Conjecturing

Lauren and Rebecca discovered a pattern in their counters, and expressed excitement when the pattern continued.

Lauren and Rebecca nonverbally and excitedly acknowledged each other, demonstrating a shared understanding of the pattern they both noticed, without ever verbalizing the pattern itself.

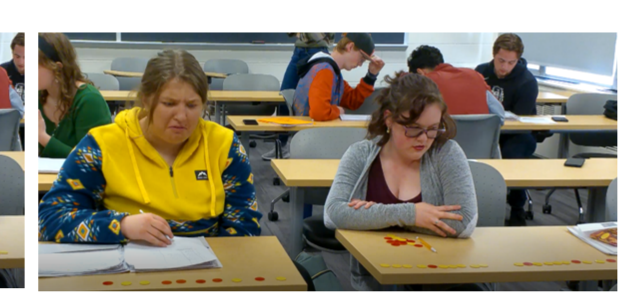
Rebecca laughed, clapped, and pointed to Lauren in a continued display of excitement. This exchange was demonstrative of affect, cognitive, and behavioral engagement by both students.



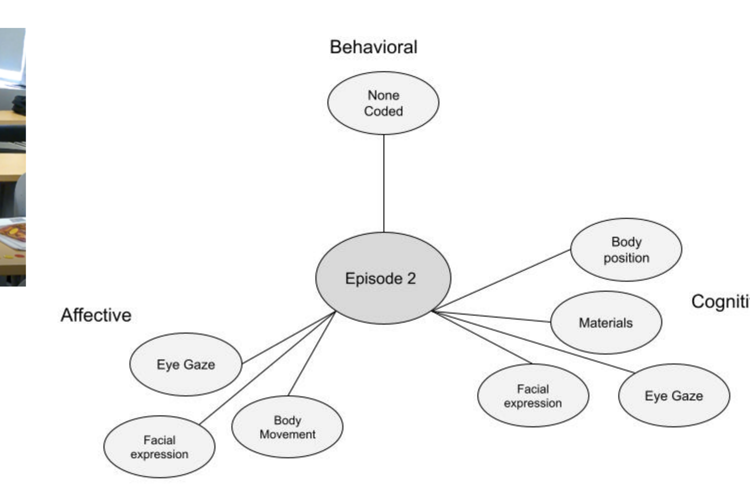
Episode 3: Second-guessing Squares

Lauren and Rebecca both expressed deep confusion when they overheard Dr. A validate a classmate's conjecture, which did not match their materials.

Both students looked towards Dr. A and a fellow classmate, while listening to their discussion. Lauren adopted a confused expression in response to a statement made by Dr. A that seemed contradictory to the pattern discovered in Episode 2.



Lauren shook her head in a continued state of confusion. Rebecca stared intently at the counters in front of her. Both students remained silent, seemingly attempting to reconcile Dr. A's statements with their own discovered pattern.



Findings & Discussion

Embodiment **can** and **does** provide evidence of student engagement in all three domains!

Embodiment provides evidence for students' engagement in ways which…

1. …illuminate engagement which may have gone unnoticed under observational scales which privilege verbalization
   - Episode 1 - Rebecca responds non-verbally to Dr. A's guidance
2. …account for the multimodal nature of both utterances and engagement as constructs (Fredericks et al., 2004; Nemirovsky & Ferrara, 2009)
   - Episode 2 - Rebecca laughs, claps her hands, and points to Lauren, indicating emotional and cognitive engagement
3. …consider the broader classroom context in which the students are engaging with the task (Hodgson et al., 2017; Kahn, 2014; Keith, 2016)
   - Episode 3 - Rebecca and Lauren's cognitive engagement is largely impacted by Anthony's conjecture in the classroom environment

Using embodiment to learn about students' classroom engagement **broadens the definition** of what it means to be a **classroom participator**, and captures engagement that might otherwise go unnoticed. Embodiment has strong potential to help educators notice, understand, and empower non-verbal participation. Further, embodiment offers an additional lens through which researchers may assess engagement.

Future Work

- Triangulating observations with self-assessment data (particularly with relevance to the students' cognitive engagement)
- Examining what a microanalysis of engagement would look like under different teaching paradigms

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How Calculus Students Describe and Use Volume in Calculus and Non-Calculus Tasks

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Existing research indicates that some learning challenges Calculus students face may not be related to the Calculus-specific content, but rather from issues related to prerequisite knowledge. Several Calculus topics such as volume by slicing and volumes of revolution have a prerequisite topic of volume. These Calculus topics tend to be challenging for students in various ways. Better understanding how these students describe and use volume may help inform instructors as well as potential reforms to practices to better support student learning of these Calculus topics.

Keywords: calculus, volume, integration, student description

Existing literature documents the influence of prerequisite knowledge on the learning of limits and differentiation (e.g., Dorko, 2012; Ferrini-Mundy & Graham, 1994; Orton, 1983). It is possible that these issues could also extend to other Calculus topics. In particular, could student understanding of volume influence thinking on volume by slicing or volumes of revolution problems? A 2012 study indicated that, on tasks prompting Calculus students to find the volume of prisms, issues arose with conceptions and calculations, including some students making calculations more suggestive of surface area than volume (Dorko, 2012). Building on findings about understanding of integration (e.g., Jones, 2015; Sealey, 2014), we replicated Dorko's study's design with a focus on volume-related topics found in second-semester Calculus.

Approached from a cognitive perspective, data were collected via written response to surveys distributed to second-semester Calculus students from a large public university. Three questions were prompts to find the volume of a rectangular, a circular, and a triangular prism. Data was also collected in the form of follow up interviews to the survey task. During the interviews, students were asked to solve two volume of revolution tasks, similar to those typically presented in their Calculus class. Students were asked to describe their solutions, as well as why they think their solutions are or are not accurate. Data included what was recorded on paper, as well as video and audio interview recordings. We collected 97 responses to the survey task and have conducted 7 interviews. Of the students who completed the survey, 20% of students used an incorrect formula to calculate the volume of the cylinder and 11% for the triangular prism. These findings are similar to those of Dorko (2012). Analysis of interview data indicates some issues with volume in Calculus tasks as well as uncertainty about the volume ideas. One student stated, "it's the basic geometry that's gonna get me." While this student ultimately correctly completed each task, the student was consistently unsure of the volume formula they were using for both the disk and shell method when setting up each problem. The student was also unable to sufficiently justify the volume formula.

These findings suggest that volume-related ideas are influencing thinking about calculations done in second-semester Calculus. The poster will display student-generated volume sketches and will expand on these findings as well as others from the larger study. This will include findings from additional interviews and discussion of implications.

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How Calculus Students Describe and Use Volume in Calculus and Non-Calculus Tasks

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Background and Purpose

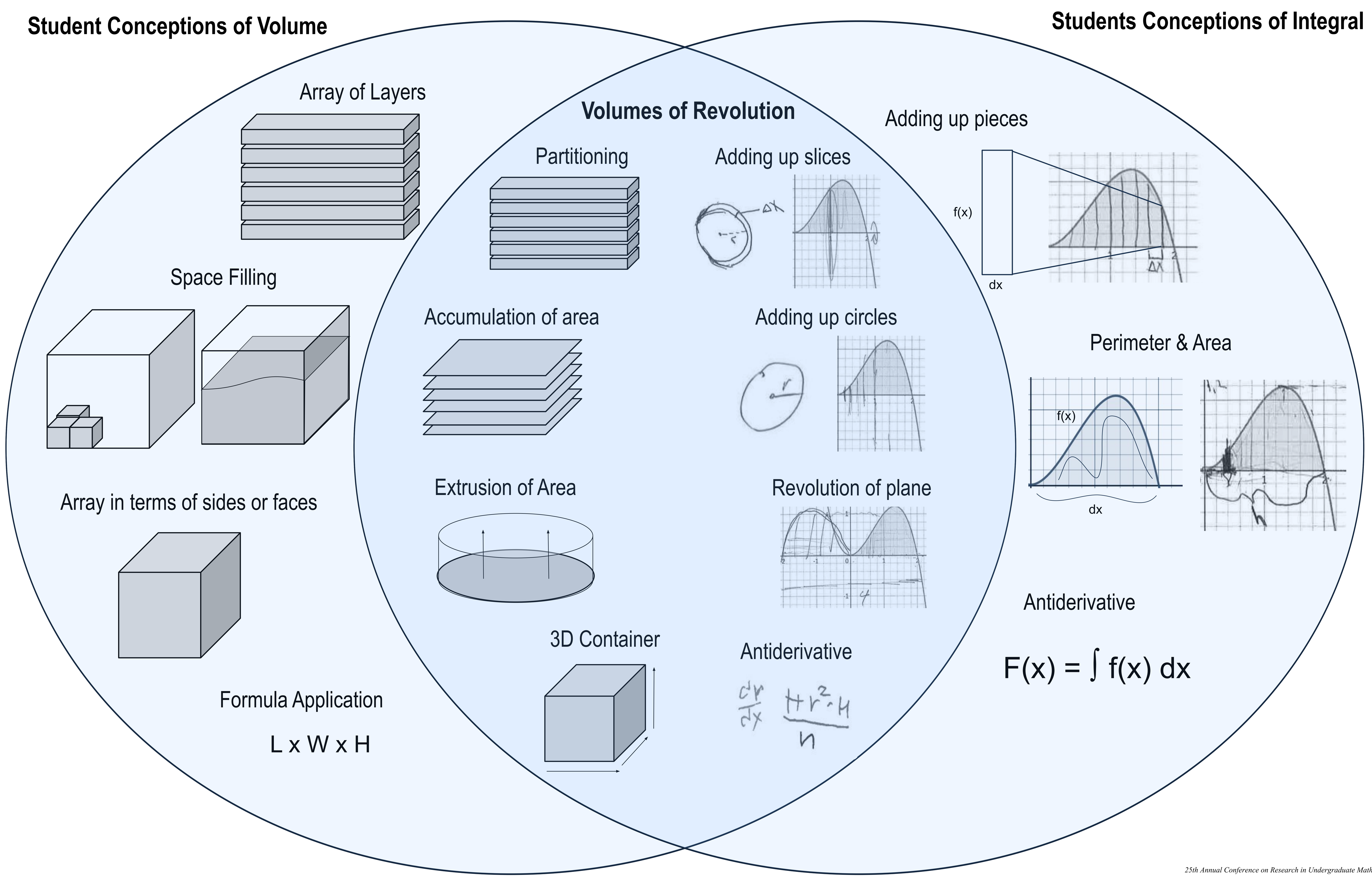
Existing literature documents the influence of prerequisite knowledge on the learning of limits and differentiation (e.g., Dorko, 2012; Ferrini-Mundy & Graham, 1994; Orton, 1983; Stewart & Reeder, 2019). It is possible that this is the case for other Calculus topics. In particular, could students' conceptions of volume influence their thinking on volume by slicing or volumes of revolution problems? Based on the interview data, we contend that Calculus students draw on both their conceptions of the integral and their conceptions of volume when solving these types of problems.

Jones (2013, 2015) investigated how Calculus students conceptualize the integral. Three of the student conceptions of the integral are highlighted in this study.

- **Adding up pieces**
The integral as the area under a curve partitioned into infinitely many rectangular pieces that are summed up. A representative rectangle indicates the multiplication of the function and the differential and applies to all other rectangular pieces. The bounds of integration indicate where the pieces start and stop.
- **Perimeter and Area**
The integral is the area under a curve, but as a singular unit rather than partitioned into pieces. The multiplication inside the integrand is given by the perimeter of the region bounded, with the function being the top curve, the differential being the bottom base, and the bounds of integration being the left and right sides.
- **Antiderivative**
The integral is the inverse process of differentiation. Students use "function matching," where the function inside of the integral must have some original function for which it is the derivative of.

Battista and Clements (1998) investigated how Elementary students conceptualized volume through three-dimensional cube arrays. The categories presented in this study are supported by Dorko (2012) and survey data gathered as part of the larger study of this research.

- **Array of Layers**
Volume is the summation of finitely many three-dimensional unit slices.
- **Space Filling**
Volume is the amount something can hold. This conception can take the form of three-dimensional unit cubes or as the filling of a whole unit.
- **Array in terms of sides or faces**
Volume is bounded by the perimeter or surface area. In this conception, the student measures volume only using the exterior of the shape.
- **Formula Application**
Volume as simply a formula, such as length times width times height, to be plugged in and evaluated.



Integral Conceptions

- **Adding up slices**
The integral represents the volume of the solid formed by revolving the region around the axis of rotation, with the solid partitioned into infinitely many slices, where the function represents the radius of the disk and the differential represents the thickness of the slice.

"Alright, so you could imagine that, if you wanted to revolve it around the axis, you could imagine a bunch of little circles, going around the x-axis. You could like slice it up into a bunch of little circles . . . and if you sum them up using an integral, that will give you a volume. . . . but it's technically gonna be little disks."

- **Adding up circles**
The integral represents the volume of the solid formed by revolving the region around the axis of rotation, with the solid partitioned into infinitely many two-dimensional slices, where the function represents the radius of the circle and the differential represents the infinitesimal difference between each slice.

"We're finding the areas between zero to two, so that's the integral you have. You are forming a bunch of circles with rotating it around the x-axis. . . . they're like slices. . . . it wouldn't be any width."

- **Revolution of plane**
The integral represents the volume of the solid formed by revolving the region around the axis of rotation. The area under the curve is a singular unit of area and the integral "does the work" of revolving it around the axis.

"So we can find the circumference of the circle, which would be the value four, cause you have negative two to two . . . so you have four pi and then we multiply that by, um, the area under a curve."

- **Antiderivative**
The integral is the inverse process of differentiation.

"It would just be, you know, looking at the problem and then applying the differentiation rules, you know, in reverse."

Volume Conceptions

- **Partitioning**
Volume is the summation of infinitely many infinitesimally thin three-dimensional slices.

"You can think about volume of something as first smaller volumes. . . . So if you have a big rectangle, you can first think about it as maybe not like a rectangle, but maybe like a few boxes on top of each other."

- **Accumulation of area**
Volume is the summation of infinitely many two-dimensional regions of area.

"What you could imagine it's volume as turns into it's area as you keep making it's height much much thinner, and if you stack a ton of these essential areas- what are essentially areas- on top of each other, you can effectively get a volume."

- **Extrusion of area**
Volume is a region of area multiplied by some depth.

"A disk being a circle that's kind of raised up."

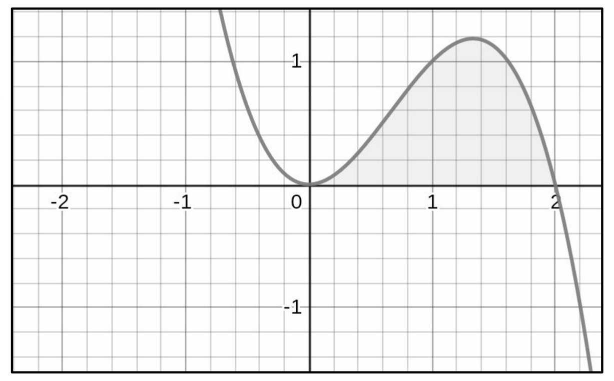
- **3D Container**
Volume is the space inside of a container, bounded by planes.

"to describe volume, I'd say it's like the amount of water a closed container could hold."

Methods

- **Survey tasks (n=97)**
  - distributed to second and third semester Calculus students
  - prompted students to find the volume of three prisms
  - supports literature on student conceptions of volume
- **Follow-up interviews (n=7)**
  - conducted after students completed summative assessment on volumes of revolution
  - included two volumes of revolution tasks
    - revolving a region around the x-axis (disks)
    - revolving a region around the y-axis (shells)
  - students were asked to explain their process

Using a grounded theory inspired approach, segments of the transcribed interviews were coded into emerging categories for both conceptions of the integral and conceptions of volume.



Findings and Implications

- **For instruction**

Our data suggests that students recall their prior conceptions related to volume when solving volumes of revolution or volume by slicing tasks. We contend that **students use both their conceptions of the integral and of volume when solving these tasks.** We suggest that instructors of Calculus should care about their students' understanding of volume. Presenting Calculus students with integration by slicing of simple prism tasks may help reveal students' conceptions of volume and how their conceptions do or do not align with their conceptions of integration.

- **For further research**

The research presented in this poster is part of a larger study investigating student conceptions of the integral and volume. Jones (2015) quantitatively analyzed the productivity of the integral concepts they identified in both the mathematics and applied contexts. It is beyond the scope of this research to quantitatively analyze the conceptions identified in this poster. We suggest further research **to investigate these conceptions of integral and volume to help determine if the productivity of some concepts over others remains the case in the context of applied Calculus problems.**

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Student Engagement in Online Calculus 1: Cognitive and Social Dimensions

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Keywords: affect and cognition; argumentation model; online calculus; student engagement

At the university level and in STEM fields students' drop out in part due to not being able to pass mathematics courses, particularly Calculus I. According to the Mathematical Association of America (MAA), Calculus I occupies a unique position as a gateway course to STEM degrees. Almost all STEM majors need to take at least the first course in Calculus. Hence, there is the need to understand how to create successful Calculus courses, particularly in online settings. This study examines the ways students engage with the mathematical content, each other and the course while learning about the concept of derivative in an asynchronous online Calculus I course. The purpose of this research is to investigate the nature of students' cognitive and social engagement in an asynchronous online Calculus I course.

Methods

In this qualitative study, we investigate the question, What is the nature of students' (a) cognitive, and (b) social engagement as they learn about limits and derivatives in an interactive asynchronous online calculus 1 course?

We analyzed weekly discussion posts from 23 participants for evidence of argumentation (Toulmin, 1958) and social presence (Swan & Shih, 2005) to learn about students' cognitive investment with mathematical content and ways of interacting with each other, respectively. We also drew on features of the learning management system used to facilitate the course to understand students' behaviors. Additionally, transcripts from interviews with six students were analyzed to further understand student engagement during the course.

Results & Conclusion

In general, students' initial posts to discussion boards almost always contained evidence of cognitive engagement in the form of components of mathematical arguments; however, dramatic differences in whether replies to initial posts contained components of mathematical arguments occurred. We found that the cognitive demand (Smith & Stein, 1998) of the prompt influenced whether reply posts contained evidence of cognitive engagement. Specifically, high-demand tasks tended to yield replies with components of mathematical argumentation more often than low-demand tasks. Regarding social engagement, students tended to participate in more back-and-forth interactions when they were assigned to small discussion groups. We also noted that the composition of initial posts could influence whether other students replied. Specifically, when initial posts involved a level of uncertainty from the original author, classmates tended to reply more often. Results from this study demonstrate the importance of conceptualizing student engagement as a multidimensional construct, and indicate that pedagogical practices such as using high-cognitive demand tasks, promoting rough draft thinking (Jansen, 2020), and grouping students for online discussions will promote student engagement and learning in asynchronous online courses.

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Comparing Student and Instructor Perspectives of Teaching Actions to Foster Creativity

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Keywords: Creativity, Student-Teacher Perception, Teacher Actions

Research Question/Background Literature/Methods

Do students and instructors perceive teacher actions to foster creativity in the same way? Zepke et al. (2013) found that tertiary students and teachers similarly valued particular teacher classroom behaviors such as providing feedback, enthusiasm, and encouraging critical thinking, while Fitzgerald et al. (2020) noted that teachers' perceptions of teaching tend to be more positive than students' perceptions of teaching or teaching actions. We define teacher actions to foster creativity to include acknowledged actions (in or out of class) from the teacher that made a student feel creative (Satyam et al., 2022; Tang et al., 2022). These actions may be intended by the instructor, but may not be perceived as such - this is the impetus of our study.

Structured interviews were conducted via Zoom with two groups of Calculus 1 instructors: those who participated in a professional development series focusing on fostering creativity in the classroom (PD) and a group who received no such direction (Non-PD). Instructors were asked questions regarding their approach to teaching through the lens of fostering creativity, reflections on specific classroom interactions, and the effectiveness of their approaches. Interviews were also conducted with students from the instructors' classes. Students were asked about creativity in the classroom and the instructor's actions in fostering creativity. Transcripts were coded for instances of teacher actions reported to foster creativity identified in Satyam et. al (2022). Transcripts received multiple coding passes in order to reach agreement. Once coding was complete, counts were aggregated and organized by individual instructor for analysis.

Results/Discussion/Acknowledgments

Instructors from PD and Non-PD cohorts reported a similar number of creativity-fostering teaching actions (an average of 20.29 for PD instructors and 19 for Non-PD instructors), but students of PD instructors reported higher actions (8.83) than their Non-PD counterparts (5.92). Standard deviation in responses from PD students was higher (7.08) than the standard deviation from the Non-PD students (5.31).

We believe the findings suggest two pertinent things: that instructors and students do perceive creativity-fostering teacher actions differently; and that when instructors are intentional about creativity practices, students acknowledge and recognize the attempts through the higher total actions. However, the difference in standard deviations suggests that students who experience intentionally creative practices from the PD instructors perceive a larger range of creativity-fostering actions than those students who see similar but unintentionally creativity-fostering practices from Non-PD instructors. This is consistent with the findings of Lew et al. (2016) - without cues, students have trouble identifying the key elements that teachers intend to convey. Future research might focus on persistence of students' perceptions of creativity in future math classes.

This material is based upon work supported by the National Science Foundation under DUE Grant #1836369, 1836371. Any opinions, findings, conclusions, or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the NSF. We also thank the Creativity Research Group for their input and support.

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Comparing Student and Instructor Perspectives of Teaching Actions to Foster Creativity

Research Question

Do students and instructors perceive teacher actions to foster creativity in the same way?

Introduction

Zepke et al. (2013) found that tertiary students and teachers similarly valued specific teacher classroom behaviors such as providing feedback, enthusiasm, and encouraging critical thinking, while Fitzgerald et al. (2020) noted that teachers' perceptions of teaching tend to be more positive than students' perceptions of teaching or teaching actions.

We define **teacher actions to foster creativity** to include acknowledged actions (in or out of class) from the teacher that made a student feel creative (Satyam et al., 2022; Tang et al., 2022). These actions may be intended by the instructor, but may not be perceived as such.

What is Mathematical Creativity?

We situate creativity following Satyam et al. (2022): "if any item (process or product) is new to the student then that is an act of creativity." We believe creativity to be relative to the student rather than oriented toward the mathematical community and that creativity can be developed and fostered.

Data Collection/Methods

Structured interviews were conducted via Zoom with two groups of Calculus 1 instructors: those who participated in a professional development series focusing on fostering creativity in the classroom (PD) and a group who received no such direction (Non-PD). Instructors were asked questions regarding their approach to teaching through the lens of fostering creativity, reflections on specific classroom interactions, and the effectiveness of their approaches.

Interviews were also conducted with students from the instructors' classes. Students were asked about creativity in the classroom and the instructor's actions in fostering creativity. Transcripts were coded for instances of teacher actions reported to foster creativity identified in Satyam et. al (2022). Transcripts received multiple coding passes in order to reach agreement. Once coding was complete, counts were aggregated and organized by individual instructor for analysis.

Teacher Actions

Satyam et al. (2022) grouped teaching actions which foster creativity into four themes. An action is coded as:

Task-Related
Properties of a mathematical content task that were (re)designed, evaluated, or assessed by the instructor

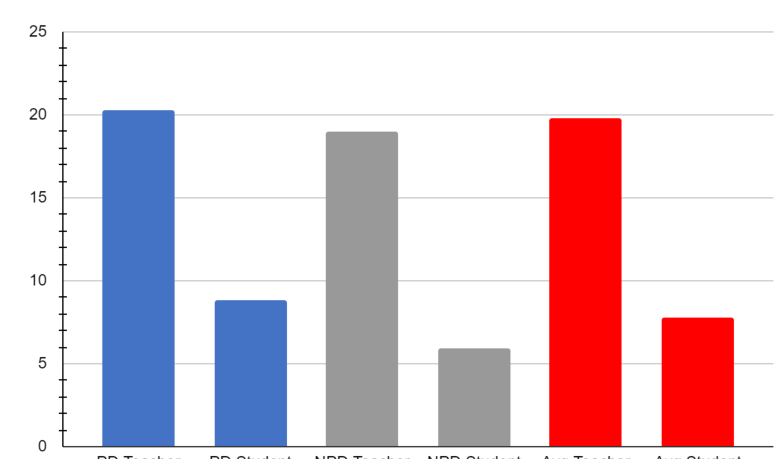
Active Learning
action that is student-centered, including inquiry-oriented (or -based) instruction (Kuster et al., 2018; Shultz & Herbst, 2020)

Holistic Teaching
actions that do not require a response from students, yet psychologically build an environment for fostering creativity

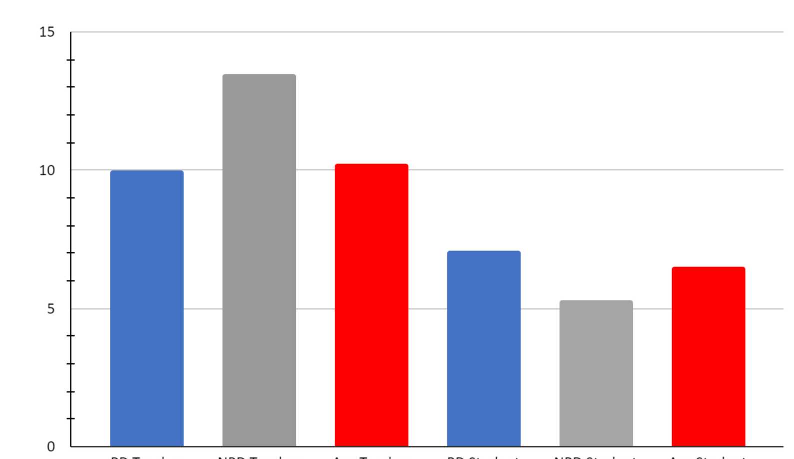
Teacher-Centered
action that involves instructor input with little to no student feedback, whether it be verifying correctness or connecting topics

Results

Average Code Counts for Teachers and Students



Standard Deviation of Code Counts for Teachers and Students



PD Teacher Codes (by Action Type)				
	Task-Related (n=52)	Teacher-Centered (n=12)	Inquiry Teaching (n=35)	Holistic Teaching (n=43)
Average	7.43	1.71	5	6.14

PD Student Codes (by Action Type)				
	Task-Related (n=74)	Teacher-Centered (n=12)	Inquiry Teaching (n=59)	Holistic Teaching (n=65)
Average	3.08	0.5	2.46	2.71

Non-PD Teacher Codes (by Action Type)				
	Task-Related (n=20)	Teacher-Centered (n=9)	Inquiry Teaching (n=22)	Holistic Teaching (n=25)
Average	5	2.25	5.5	6.25

Non-PD Student Codes (by Action Type)				
	Task-Related (n=17)	Teacher-Centered (n=9)	Inquiry Teaching (n=21)	Holistic Teaching (n=31)
Average	1.31	0.69	1.62	2.38

Discussion and Future Work

We conclude that **students and instructors do perceive teacher actions to foster creativity differently**, and the difference is independent of participation in PD.

- When instructors are intentional about creativity practices, students acknowledge and recognize the instructor's attempts.
- PD students perceive a larger range of creativity-fostering actions than Non-PD students. This is consistent with the findings of Lew et al. (2016) - without cues, students have trouble identifying the key elements that teachers intend to convey.

Future work might investigate:

- Persistence of student perceptions of creativity in future classes
- Differences in perceptions of teaching actions between instructors at different stages of their careers or between contingent/non-contingent faculty
- Longitudinal data on math attitude beliefs for students whose mathematical creativity is encouraged and fostered

Acknowledgements

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Opportunities to Learn Provided by a Commonly Used Business Calculus Textbook

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Contributing to the growing body of research on opportunities to learn provided by undergraduate mathematics textbooks, this study reports on the analysis of optimization examples given in a commonly used business calculus textbook in the United States. Findings of this analysis indicate that the textbook consistently encourages students to interpret critical numbers and extrema, in addition to encouraging students to verify whether or not critical numbers produced minimum/maximum values of the objective functions from which they were determined. The analysis further revealed that the textbook rarely encourages students to include units of extrema, provides limited opportunities for students to reason about objective functions that have more than one critical number, and provides minimal opportunities for students to reason about absolute extrema problems, respectively. Recommendations for different stakeholders, including business calculus textbook authors and business calculus instructors, are included.

Key words: Opportunity to learn, optimization problems, business calculus, textbook analysis

Textbooks play an important role in students' learning of mathematics at all levels. In fact, according to Reys et al. (2004), "… the choice of textbooks often determines what teachers will teach, how they will teach it, and how their students will learn" (p. 61), a sentiment that is shared by several other researchers (cf. Alajmi, 2012; Begle, 1973; Kolovou et al., 2009). Although much research as reported on opportunities to learn about various mathematical topics at the K-12 level (cf. Alajmi, 2012; Charalambous et al., 2010; Dole & Shield, 2008; Jones & Tarr, 2007; Pickle, 2012; Stacey & Vincent, 2009; Stylianides, 2009; Thompson et al., 2012), there is a paucity of similar research at the undergraduate level, which is the motivation for the present study.

The study reported in this paper is part of a larger study (Mkhatshwa, 2016b) that examined: (1) students' reasoning about optimization problems, (2) opportunities to learn about optimization problems via two course lectures each of which was attended by approximately 300 students, and (3) opportunities to learn about optimization problems provided by a course textbook. The current study reports on opportunities to learn about optimization problems via examples provided by the course textbook (Haeussler et al., 2011), a widely used textbook in the teaching of business calculus in the United States (Mkhatshwa & Doerr, 2016a).

Consistent with findings reported by other researchers (cf. Reys et al., 2004), one major finding of the larger study is that the presentation of optimization problems during course lectures closely followed the presentation of optimization problems in the textbook. Findings of the present study suggests that the textbook encourages students to interpret critical numbers and extrema, and to verify extrema using various methods such as the first or second derivative test, respectively. In addition, the textbook rarely encourages students to include units of extrema, provides limited opportunities for students to reason about objective functions that have more than one critical number, and provides minimal opportunities for students to reason about absolute extrema problems. In light of the findings of the present study, we recommend, among other things, that business calculus textbook authors provide more opportunities for students to reason about units of extrema, objective functions that have at least two positive critical numbers, and absolute extrema problems (Mkhatshwa, 2019) in economic contexts.

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Foregrounding Inversely Proportional Relationships in Integration by Substitution

Andrew Izsák, Tufts University

Background

Foci of Past Calculus Research[1]

- Co-variation
- Functions
- Limits
- Derivatives
- Integrals

Significant Omission

- Students' understandings of multiplication and division *with quantities* in the context of calculus topics

Why it Matters

- College students often have limited understandings of multiplication and division *with quantities*
- Multiplication and division *with quantities* foundational for derivatives, integrals, families of functions central to calculus
- Studying formulae alone unlikely to help students understand quantitative foundations for calculus

The Study

Developed Novel, Semester-Long Course

- Introduced an explicit meaning for multiplication based in measurement

N	$\bullet$	M	$=$	P
# of units in 1 group		*# of groups in 1 product amount*		*# of units in 1 product amount*

- Emphasized drawings of lengths and areas
- Worked only with piecewise linear functions and step functions. (No discussion of limits)
- **Strand 1**:Division, prop. relationships, slopes of lines, function composition, chain rule
- **Strand 2**: Multiplication, areas, inversely prop. relationships, integration by substitution

Data Collection

- 18 undergrad. students, Fall 2021
  - 15 completed at least 1 calculus course
  - No STEM majors
- In-class group work on sets of word problems
- Whole-class discussion focused on how the $N \bullet M = P$ structure fit problem situations
- Collected written homework and exams

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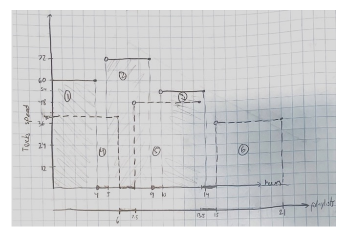
Results

Strand 2

- Students debated whether rectangular areas could represent products other than areas. For instance, can areas represent distances (e.g., rate • time = distance) or volumes (e.g., area • height = volume)? This is NOT obvious to all students.
- Students were able to reason appropriately about situations in which P was fixed.
- One final exam 13 students gave appropriate solutions to the following problem:
  - Three friends on a road trip. **Steve** drives 60 mph for 4 hrs. They rest 1 hr. **Chara** drives 72 mph for 4 hrs. They rest 1 hr. **Zawadi** drives 54 mph for 4 hrs.
  - Graph speed vs. time. Explain how the graph shows distance each friend drives.
  - Consider same situation but time is measured in playlists; 1 playlist is 40-min.

One Student's Solution



"Area 1 correlates to the distance driven by Steve…. N • M = P ….

60 miles in a column • 4 column = 240 miles

….The new function is drawn with a dotted line….the distance traveled is still represented by the area underneath the function….Area 1 = Area 4….

40 miles in a column • 6 column = 240 miles"

Comments

- A special case of integration by substitution, stated as

$$\int_{g(a)}^{g(b)} f'(u)du = \int_{a}^{b} f'(g(x)) \bullet g'(x)dx$$

- This formulation treats integrand as rate-of-change, consistent with the FTC.
- Justifies equation by explaining that each rectangle in a Riemann sum for the left-hand side has the same area as the corresponding rectangle in a Riemann sum for the right-hand side

Math Placement and Academic Achievement: An Internal Analysis

Sean Gruber, Johns Hopkins University
Emily Braley, Johns Hopkins University
Richard Brown, Johns Hopkins University

The results presented here offer preliminary insights into the course patterns and academic achievement of students taking entry-level math courses at Johns Hopkins University (JHU) from Fall 2017 through Fall 2021. First, we used students' math placement exam results and end-of-semester grades in their first math course to predict academic achievement when considering whether students followed their placement recommendation. Second, we compared trends in placement data and academic achievement of students taking math courses at JHU before and during the COVID-19 pandemic.

Keywords: Math Placement, Contradictory Recommendations, Entry-Level Courses

Empirical evidence suggests undergraduates' persistence and interest in majoring in STEM fields are associated with their achievement (Chen, 2013) and experiences in entry-level and first-year STEM courses (Bressoud et al., 2013; Crisp et al., 2009; Gainen, 1995, Seymour & Hewitt, 1997). Considering many students must take the recommendations of a math department (Apkarian et al., 2017), it is important that departments review their math placement processes and recommendations to students on a regular basis, a common practice of successful calculus programs (Bressoud & Rasmussen, 2015).

The Math Placement Exam (MPE) at Johns Hopkins University (JHU) is broken into two parts, MPE1 and MPE2, which provide unique course recommendations. Aggregating data from Fall 2017 through Fall 2021, we conducted a binary logistic regression using students' first-semester course grades as the outcome variable. The placement recommendation and the course a student took served as the independent, nominal variables. Our study focused on students who were recommended to take precalculus, calculus I, or calculus II.

Many students at JHU taking calculus courses are on a pre-medical track and, thus, aim to earn a relatively high GPA in preparation for applying to medical school. As a result, it was important for us to choose a grade threshold for our binary outcome variable that reflected our Department's efforts to "prepare students to be numerate citizens and productive employees" while conducting this internal review of the MPE (American Mathematical Association of Two-Year Colleges, p. 14). In turn, we chose a grade of B- or above as the grade threshold.

Preliminary results suggested that students who were recommended to take precalculus or calculus I on MPE2 were statistically significantly less likely to earn a grade of a B- or better if they took the course they were recommended for when compared to students who 'jumped' one course above their recommendation (e.g., were recommended for precalculus and took calculus I). Students recommended to take calculus I by MPE1 or MPE2 who took precalculus were less likely to earn a B- or better when compared to those recommended for calculus I who took calculus I. A similar pattern held for students recommended to take calculus II on MPE2 but who took calculus I. Finally, only for the 2020 and 2021 cohorts did students receive contradictory recommendations to take calculus I (on MPE1) and precalculus (on MPE2). The preliminary findings presented here may be used to inform other institutions in their review of how students are placed into calculus-sequence courses while considering how to measure the mathematical preparedness of students affected by the pandemic who are now entering college.

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Math Placement and Academic Achievement: An Internal Analysis

Sean Gruber, Emily Braley, and Richard Brown - Johns Hopkins University (JHU)

Background and Motivation

Undergraduates' persistence and interest in majoring in STEM fields are associated with their achievement (Chen, 2013) and experiences in entry-level and first-year STEM courses (Bressoud et al., 2013; Crisp et al., 2009; Gainen, 1995, Seymour & Hewitt, 1997).

Considering many students must take the recommendations of a math department (Apkarian et al., 2017), departments should review their math placement processes and recommendations to grow successful calculus programs (Bressoud & Rasmussen, 2015).

Research Questions

1. Aggregating data between Fall 2017 and Fall 2022, where are students ending up after their placement recommendation results?
2. Are placement recommendations indicative of performance in the math class they first take?

Context

Many students at JHU taking calculus courses are on a pre-medical track.

- Only 9 (2%) did not make the grade satisfaction of a C- or better.
- 1,746 of 5,534 (31.6%) students between 2017 and 2021 did not meet the grade satisfaction cutoff of a B- or better.

Math Placement Exam

The Math Placement Exam (MPE) at Johns Hopkins University (JHU) is broken into two parts, MPE1 and MPE2, which provide unique course recommendations:

- Students with no prior calculus experience take MPE1 (makes recommendations for precalculus or calculus I).
- Students with some calculus experience, then take MPE2 (makes recommendations for precalculus, calculus I, calculus II, or one of calculus III, linear algebra or differential equations).
- If MPE2 recommends precalculus, then students go back and take MPE1.

Model

Binary Logistic Regression:

- Outcome variable: First-semester course grades categorized into two groups using a threshold of B- or better.
- Independent variables: Placement recommendation and the course a student took.

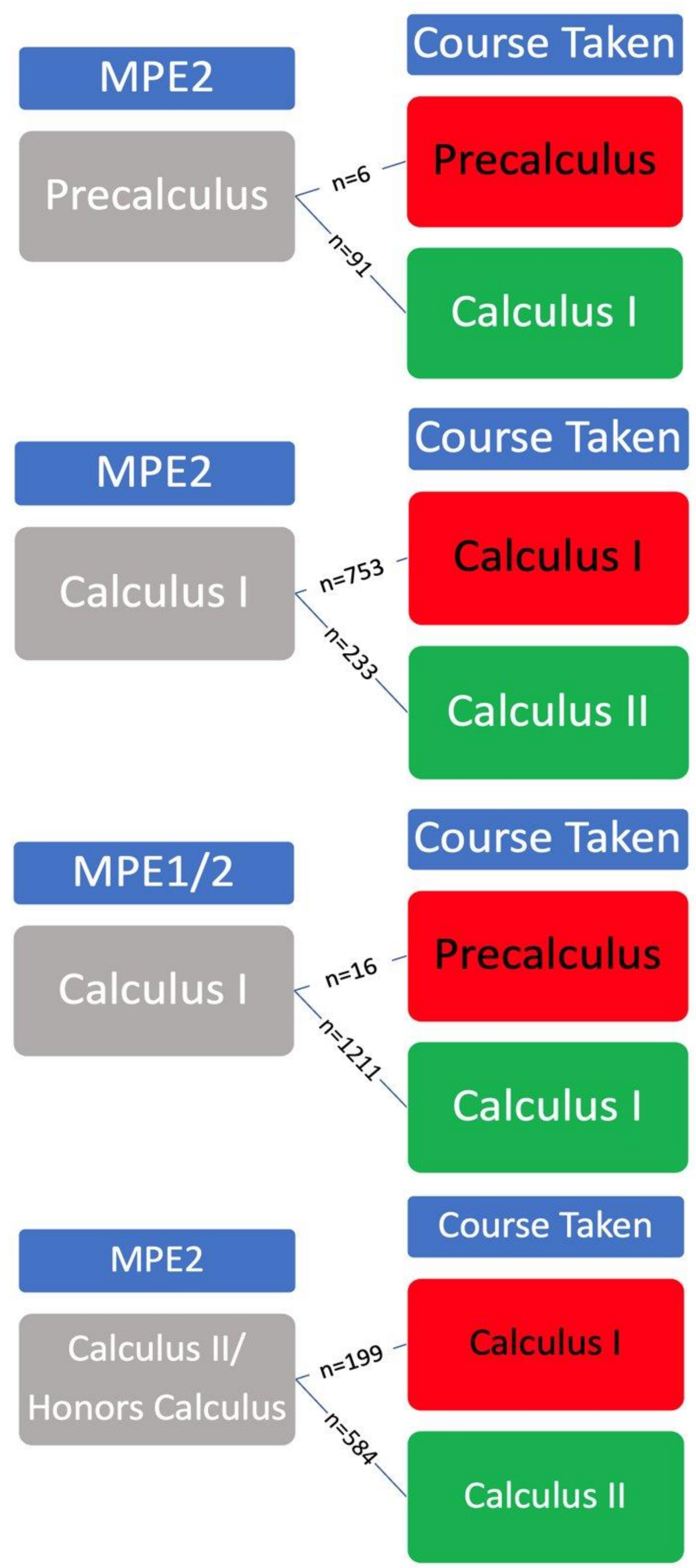
$$P(s_i) = \frac{1}{1 + e^{-(\beta_0 + \beta_1 Rec_i + \beta_2 Take_i)}}$$

where $P(s_i)$ is the probability student i earns a B- or better considering they are recommended for course Rec_i and take course $Take_i$ (could be different from recommended course).

Results

We report on students who were recommended to take precalculus, calculus I, or calculus II.

Red - less likely to earn at least a B-
Green - more likely to earn at least a B-



Contradicting recommendations (MPE1 recommends calculus I, MPE2 recommends precalculus):

This impacts a small (n=69) group of students and only occurred in the 2020 (n=39) and 2021 (n=30) cohorts.

Conclusions and Next Steps

Strengths	Weaknesses
MPE2 is doing a reasonable job of recommending students for calculus II or above.	Neither MPE1 nor MPE2 are effectively sorting students into precalculus or calculus I.

1. Update the placement tests based on what we have learned and STEM-preparedness literature from ALEKS and "Minding the Gaps" (Kornelson et al., 2020).
2. Connect with advising (broadly) and admissions to inform changes and share outcomes.
3. Discuss a path for those who receive contradicting recommendations.
4. Continue to build peer-to-peer networks, mentoring relationships with faculty, and other outside-of-class supports found to help female and students of color remain and succeed in STEM fields of study (Ellington & Frederick, 2010; Griffin et al., 2010; Palmer et al., 2011; Seymour & Hewitt, 1997).
5. Track changes and continue tracking efficacy of recommendations from the tests.

References: tinyurl.com/jhuReferences

1.

Provider Motivation for Working with Novice College Mathematics Instructors

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Keywords: novice college instructor professional learning, Provider, TA

A *Provider* is a person who offers opportunities for professional learning to novice college instructors (e.g., new faculty, graduate student teaching assistants and associates [GTAs]) Motivation theory has been studied in a variety of work contexts, including higher education (Daumiller et al., 2020). However, there is no existing research specific to the motivation of Providers of professional development much less about those who work in college mathematics contexts. Knowledge of Provider motivation would be valuable for at least two reasons: (1) knowing the motivations of people who become providers has implications for employers to attract and retain Providers and (2) understanding Provider motivation can inform the identification and development of new Providers (i.e., moving the field to purposeful development and away from the accidental Provider, who "fell into" it).

The research question is the following: What do Providers report as motivating them to choose to be a Provider of professional learning for novice college mathematics instructors? The framework for answering this question relies on motivation theories, change theory about people, power, structures, symbols in departments (Reinholz & Apkarian, 2018), and research on Provider professional orientations (Martinez et al., 2021). These interact to shape Provider work. In particular, structures include the interrelationships of people and roles like Course Coordinator, instructor, GTA, seminar leader.

Data were from surveys and interviews among people who had participated in a Provider workshop. Of the ten people who did the survey five agreed to interviews. For data analysis, I transcribed the interviews and did open coding for each interview, then cross-interview thematic coding, then case story development that used existing frameworks for motivation, professional orientation, and change theory to structure each story. I wrote case stories for each interviewee and contacted them for feedback. The poster will present a cross-case analysis of similarities and differences among interviewees, including charts documenting sources and enactments of motivations.

Implications for research include the need for investigation of the relationship between Provider motivation and Provider effectiveness in supporting professional learning by novice instructors. Participants' reflections and stories also connect to the emerging need for research on what motivates Providers to do scholarly work related to being a Provider.

My goals in sharing the poster and having conversations with RUME attendees are for both feedback and thinking about future reporting. In particular, I wonder how attendees at the conference would talk about their own motivations for doing academic research, including RUME. Conversations about this would be useful for me to refine how I report on the motivation of Providers who do academic research about professional development of novice college instructors.

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Reasoning Students Employed When Mathematizing During a Disease Transmission Modeling Task

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Keywords: mathematical modeling, differential equations, student reasoning

Few studies explicitly focus on students' understanding of differential equations for representing real-world phenomena (Lozada et al., 2021). Rowland and Jovanoski (2004) studied the meanings students have for the terms of an already constructed differential equation and found that across contexts, students tend to conflate amount with rates of change of amounts. Our goal is to document students' reasoning while constructing differential equations that represent real-world phenomena (called mathematization) in a way that helps instructors connect their students' emergent models with normatively correct models. We frame students' reasoning while mathematizing using theories of quantitative, co-variational, and multi-variational reasoning (Carlson et al., 2002; Jones, 2018; Thompson, 2011). Using this lens builds on previous findings that the operations students use on quantities reflect the quantitative relationships perceived by the student, (Larson, 2013), and that the models that students could potentially make depend on (and are constrained by), the quantities the student imposes onto the situation (Czocher et al., 2022).

The purpose of this study was to (1) catalog the quantities students impute while modeling a disease transmission scenario from first principles and (2) document students' rationales when combining those quantities. We wanted to know: What are participants' existing ways of reasoning when writing equations to model of the spread of a disease throughout a community? To answer this question, we present work from STEM undergraduates who participated in a set of individual cognitive task-based interviews via Zoom. Data were analyzed by first identifying the quantities participants imposed onto the task scenario and then documenting the participants' (inferred) reason for using specific arithmetic operations($\mathrm{e.g.}, +, -, \div, \times$) on quantities.

Our results reveal the rationales for using arithmetic operations to depict specific aspects of the dynamics of spread of disease throughout a community. All three participants constructed models for the instantaneous rate of change of the sick and healthy populations, though at times using unconventional notation. All three participants constructed a model for the change in the healthy population by covarying healthy people and sick people. They did this by noting that the decrease in the healthy population is equal in magnitude to the increase in sick population. Two participants depicted this relationship using the arithmetic operation $-$, while one participant, Rua, chose not to use any notation to depict this relationship. We infer that Rua was covarying the number of sick people and the number of healthy people, but not multi-varying sick, healthy and time. Our contribution extends what is known about students understanding of differential equations by reporting on how covariational and multivariational reasoning played a role in students' construction of a differential equation that models disease transmission. We infer that covariational and multivariational reasoning were observable because our participants were working on differential equations modeling task focused on disease transmission.

Acknowledgments

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Reasoning Students Employed When Mathematizing During a Disease Transmission Modeling Task

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Introduction

- Few studies exist on students' understanding of differential equations for representing real-world phenomena (Lozada et al., 2021).
- Students tend to conflate amounts of change and rates of change of amounts in many different contexts that require differential equations (Rasmussen & King, 2000; Rasmussen & Marrongelle, 2006; Rowland & Jovanoski, 2004; Mkhatshwa, 2018).
- **Goal:** document students' reasoning while constructing differential equations that represent real-world phenomena (called mathematization) in a way that can inform instructors about how to connect their students' emergent models with normatively correct models.

Theoretical Perspective

Mathematical model - a conceptual system consisting of **elements, rules governing interactions**, the **relationships between elements**, and **operations**. Expressed into the world through different representations (Lesh & Doerr, 2003)

Elements are:

- **Quantitates** - triple of an object, attribute, quantification (Thompson, 2011).
  - Ex attributes: length, volume, cardinality, speed, and density (Ellis, 2007).
  - A quantity is cognitively distinct from (mathematical) variables.

Relationships between elements and operations are described by

- **Covariational reasoning** - cognitive activities for coordinating two varying quantities (Carlson et al., 2002)
- **Multivariational reasoning** - the cognitive activities for coordinating more than two varying quantities (Jones, 2018)

RQ: What are participants' existing ways of reasoning when writing equations to model the spread of a disease throughout a community?

Results

- 3 participants constructed a model for the change in the healthy population by covarying healthy people and sick people.
  - All noted that the decrease in the healthy population is equal in magnitude, and inversely related to, the increase in sick population.
  - 2 depicted this relationship as $\frac{dS}{dt} = -\frac{dh}{dt}$
  - 1 depicted this relationship as $\frac{dS}{dt} = \frac{dh}{dt}$
- We infer that the decision to use/not use the symbol– is rooted in the participants multivariation of sick, healthy and time.

Methods

Data collection

- Cognitive task-based interviews with 20 STEM majors who completed or were enrolled in differential equations
- Report 3 computer Science majors on a Disease Transmission (SI compartment model) task.

Analysis

- Identify quantities participant imposes onto task
- Note arithmetic operations used on quantities
- Document participants' reason for using that arithmetic operation
  - quantitative, covariational, and multivariational reasoning.

Rua

$$\Delta S(t) = (H(t) \cdot S(t) \cdot r) \qquad \Delta H(t) = (H(t) \cdot S(t) \cdot r)$$

Rua: It's (referring to $\frac{dW}{dt}$) the same (as $\frac{dI}{dt}$). Um, I think there was one of the equations that we were working through, where I believe the aspect of negativity or positivity was something I was kind of trying to internalize or view, probably, where I think this is a similar outlook just depending on how you're looking at it. Uh, it would either-- you know, if that rate is something that's representative of decay or growth with regards to sick, then it should be the opposite for healthy, and vice versa. If it was growth or decay for healthy, then it should be the opposite-- I guess that's because they're inversely related-- for the sick.

Winnow & Roion

Winnow:

$$\frac{ds}{dt} = \frac{m^{5}}{s(t) \cdot h(t)} \cdot h(t) \cdot r \qquad \frac{dh}{dt} = -\left(\frac{m}{s(t) \cdot h(t)} \cdot h(t) \cdot r\right)$$

Winnow: That would show the rate of change of healthy people, so I guess it would be a negative number because the healthy people would be-- the number of healthy people would be decreasing. Later says: We got was minus 5 healthy people because we had 5 new sick people

Roion:

$$\frac{\partial I}{\partial t} = 1.\left(a \cdot \left(c\, \underbrace{w(t)}_{\text{well}}\right) \underbrace{I(t)}_{\text{infected}}\right) \qquad \frac{\partial W}{\partial t} = -\left(a \cdot c\, I(t)\right) \cdot w(t)$$

Roion: OK. So for this one (referring to $\frac{dW}{dt}$), it'd pretty much be the same (as $\frac{dI}{dt}$) but in the negative direction, as the amount of well people would be decreasing as more infected…

ER: OK. So are you saying that $\frac{dW}{dt}$ is the same as $\frac{dI}{dt}$ but negative?

Roion: Yes.

ER: OK. And what tells you that that's the case?

Roion: It's because as more people are infected, that means less people in the community are actually well. So it's decreasing as time goes on, as more of the infected people interact with the well people.

Discussion

- We extend what is known about students' understanding of differential equations by reporting on how covariational and multivariational reasoning played a role in students' construction of a differential equation that models disease transmission.
- We infer that covariational and multivariational reasoning were observable because our participants were working on modeling disease transmission using differential equations.

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Data Scientists' Use of Embodied Ideational Mathematics in Conveying Concepts Rooted in Linear Algebra

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Keywords: Embodiment, data science, ideational mathematics, linear algebra

Embodied cognition is the theory that learning is influenced by the body through a variety of modalities. Nathan (2022) summarizes literature indicating that forms of embodiment such as body form, gesture, simulation, and materialist epistemology can support student learning. Other studies in embodied cognition focus on ways in which people's perceptions and ways of thinking about a topic can impact teaching and learning and place a high level of emphasis on ideational mathematics. Ideational mathematics is defined as an individual's perception of formal ideas (Schiralli & Sinclair, 2003; Sinclair & Gol Tabaghi, 2010). While previous studies centered on mathematicians, our work centers specifically on the emerging field of data science. Despite the significant amount of literature on embodiment in mathematics (Alibali & Nathan, 2012; Hall & Nemirovsky, 2012; Sinclair & Gol Tabaghi, 2010), there exists a gap in the literature on the impact of embodiment as it relates to data science. Thus, we wish to better understand how instructors use ideational mathematics regarding concepts they present in their classrooms. In conducting this investigation, we leverage Nathan's (2022) summary of various forms of embodiment to address our research question: In what ways do mathematicians use embodiment to convey ideational mathematics when discussing data science concepts rooted in linear algebra?

We conducted one-hour audio and videotaped interviews with two data scientists (Hunter and Elizabeth) with varying backgrounds. Hunter's area of expertise is topology, and Elizabeth's expertise is in harmonic analysis. In this poster, we present an analysis of two episodes from these interviews. The selected episodes provide evidence of Nathan's (2022) summary of the four forms of embodiment. The two data scientists primarily discussed concepts ideationally, and used representational gestures, materials, and simulation that accompanied personified verbiage. For example, Elizabeth explained an activity that she uses with students in her capstone course to teach cosine similarity within natural language processing. Her activity, which she referenced as a "Bag of Ingredients" model, uses grounded, real world examples of recipes to clarify the broader, more complex "Bag of Words" model. She related comparing ingredients in recipes to comparing words in documents of various length to discuss the comparison of common words and synonyms. Hunter integrated various tools and materials such as drawings on a whiteboard and a model of a molecule to explain principal component analysis, a key concept in the field of data science. The concepts discussed in both episodes are rooted in linear algebra, a mathematics subject which both participants identified as an area of struggle for students. The implications of the insights offered through analysis of these episodes is particularly relevant as data science programs become increasingly predominant and institutions search for the best ways to convey material.

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Data Scientists’ Use of Embodied Ideational Mathematics in Conveying Concepts Rooted in Linear Algebra

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Motivation

- Existence of outlines of introductory data science courses (Baumer, 2015; Hicks & Irizarry, 2018)
- Studies exploring mathematicians’ ideational mathematics (Sinclair & Gol Tabaghi, 2010; Soto-Johnson et al., 2016) and embodiment (Marghetis et al., 2014).
- Linear algebra in the formal, symbolic, and embodied worlds of thinking (Stewart, 2018).
- Minimal research on the teaching and learning of data science.

Research Question: In what ways do data scientists use embodiment to convey their ideational mathematics when discussing concepts rooted in linear algebra?

Principles of Teaching Data Science

- Organize around Case Studies
- Integrate Computing
- Minimize Notation
- Mimic Experience
- Demonstrate Critical Thinking

Hicks and Irizarry (2018)

Theoretical Framing

Embodied cognition: The theory that learning is influenced by the body through a variety of modalities.

Type of Embodiment	Definition	Examples
Body Form	Use of body and the environment	Tracing a circle on hand
Gesture	Spontaneous Movements of hands, arms, and other body parts	Indexical (pointing), Representational (iconic and metaphoric), Beat
Simulation	Using imagination to extend ideas	Use of MATLAB or other technologies
Materialist Epistemology	Use of materials and manipulatives	Annulus, water wheel, blocks

(Nathan, 2022)

Conceptual Mathematics

- Formal, public representations
- Examples
  - Textbook definition of Principal Component Analysis (PCA) : dimensionality reduction method which transforms a large data set into a smaller one that still contains most of the information

Ideational Mathematics

- An individual’s perception of formal ideas
- Examples
  - PCA produces the best view in 2-D or 3-D space for physical, geometric intuition
  - Matrix multiplication as gobbling up a vector, throwing information away, and spitting out a vector

(Schiralli & Sinclair, 2003; Sinclair and Gol Tabaghi, 2010)

Methodology

Participants: Two **data scientists** working in a mathematics department at a research university in the Rocky Mountain area.

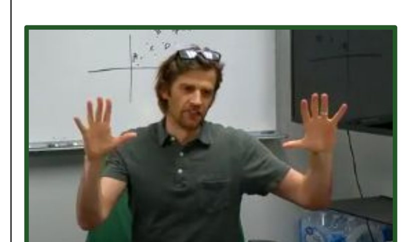
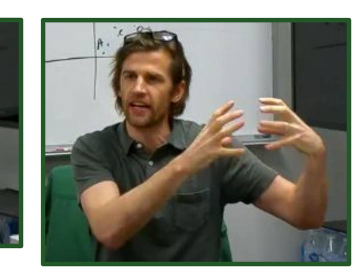
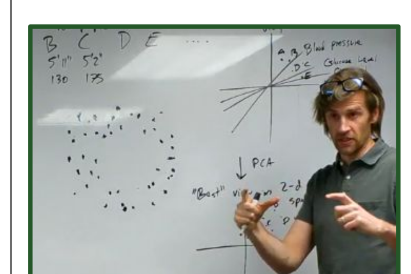
- Hunter - Topologist
- Elizabeth - Harmonic Analyst

Data Collection and Analysis:

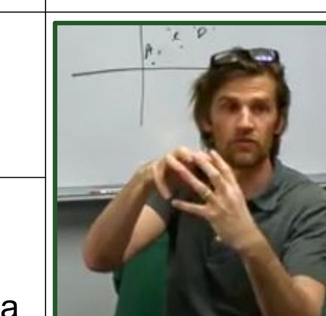
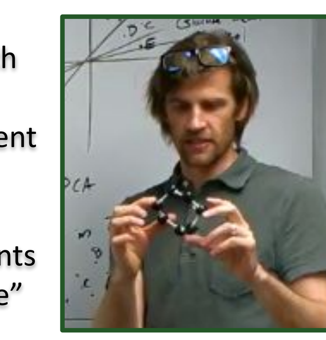
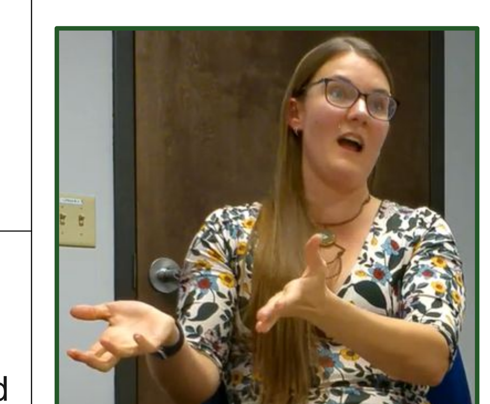
- Videotaped approximately hour-long interview with each participant.
- Open coded entire interview for embodiment (Nathan, 2022).
- One episode was selected from each interview, approximately 10 to 15 minutes in length, after the first round of coding.

Results

(1) Mathematical areas of expertise influence ideational mathematics and forms of embodiment

Participant	Embodiment Type	Transcliptions
Hunter Focused on visual & geometric representations through **body form, gesture, & materials**	**Movement, Body Form, Perception** • Used body to represent data science and mathematics as vast fields, and topology’s role in each	➢ “Inside of data science, topology is a very small part, on the order of 1% of all data science” 
	Gesture - metaphoric • Topology a “pretty big area” (left) of the “vast landscape” of mathematics (right) • Described how to formalize geometric representation of data	 
	Materialist Epistemology • Whiteboard inscriptions visually represented a high-dimensional data set as: ○ Chart of numbers ○ 5D and Cartesian planes ○ Points arranged in circular formation	 ➢ “We formalize blurring one’s vision by maybe growing a ball around each data point”
Elizabeth Represented physical & abstract objects through **body form, gesture & simulation**	**Movement, Body Form, Perception** • Utilized environment to fix “locations” of abstract mathematical objects	➢ Matrix multiplication “gobbles up vector” and “throws away” information” ➢ Measured by nullity 
	Gesture - metaphoric • Established “locations” of null space and column space on either side of her	
	Simulation • Described matrix multiplication as “gobbling up” and “spitting out” vectors, and “throwing away” information • Nullspace fixed to her right • Column space fixed to her left	 ➢ A vector is “spit out”, measured by rank

(2) Data science can be explained using relatable materials and simulations, reducing need to rely heavily on symbolism

Participant	Embodiment Type	Transcliptions
Hunter Use of relatable PCA example through **gesture, simulation, & materials**	**Gesture** - metaphoric • Uses hands to represent familiar shapes to describe data	 ➢ “Maybe the data set is arranged among a sphere, and maybe there’s a good reason why the data is well-modeled by a sphere
	Simulation • Relatable PCA example: data represents medical characteristics of a doctor’s patients	➢ “It can stretch and bend in fold in different ways. . .you have many different points in 24-D space” 
	Materialist Epistemology • Molecule represents 24-dimensional data set • Annulus represents the 24D data “living” in two dimensions	 ➢ “Turns out, this molecule lives on a very lower-dimensional, 2D space inside of 24 dimensions”
Elizabeth Use of relatable and immersive **simulations & gesture**	**Gesture** - metaphoric • Returns to established “location” of nullspace for song decomposition • Collided hands together representing orthogonality	 ➢ In the outputs of the decomposed songs, the “stuff that was thrown away was all the low frequency things”
	Simulations *Rank-Nullity Theorem Demo* • Decomposition of various songs based on frequencies • Used songs familiar to students *Bag-of-Ingredients Model* • Modified Bag-of-Words model using recipes and ingredients instead of random words	➢ Bag-of-ingredients didn’t recognize that two different cookies were both sweet because they used different sugars, so “those two parts of the vectors were orthogonal” 

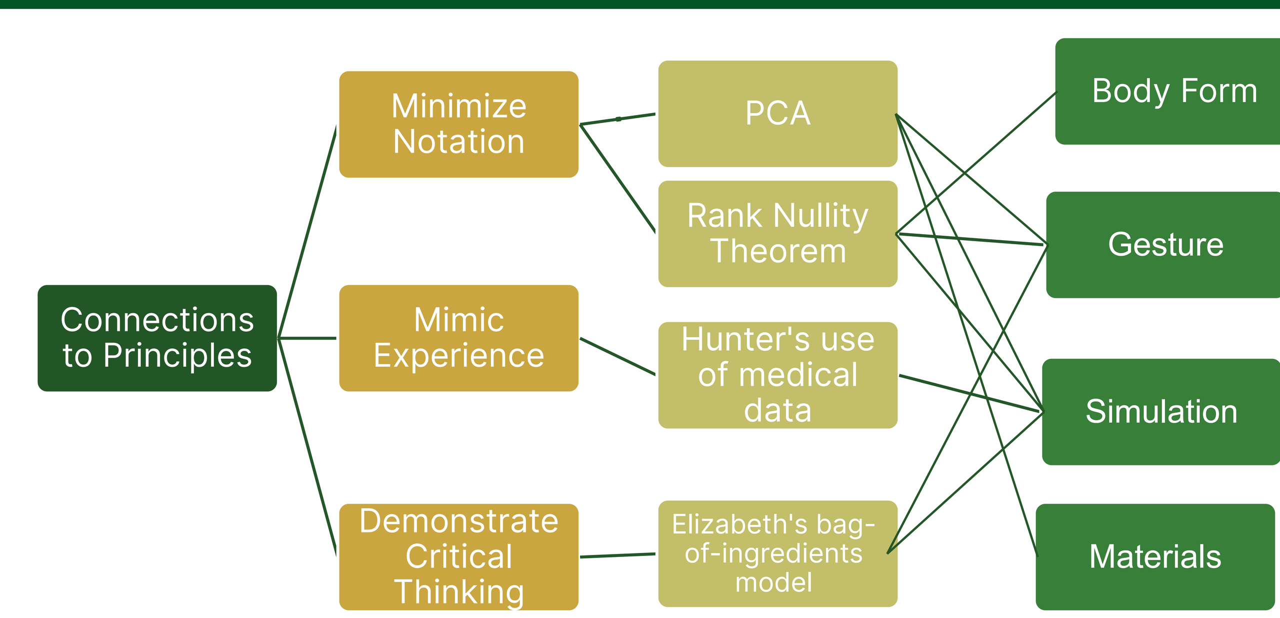
Discussion and Implications

Expertise Influence in Embodiment:

- Both participants utilized gesture,
- Hunter used **physical** materials and Elizabeth used **imagined** materials
- Need faculty with **varied expertise** to effectively **teach all students** while keeping the plethora of approaches to data science in mind (National Academies of Science, Engineering, and Medicine, 2018).

Relatable Explanations:

- Opens up “multiple pathways for students of different backgrounds to engage at levels ranging from basic to expert” (National Academies of Science, Engineering, and Medicine, 2018, p. 64)



While discussing the linear algebra concepts they thought were important in data science, the participants explained their ideational mathematics with varied embodiment reflective of their backgrounds.

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ELITE PD

Engaged Learning, Inclusive Teaching, & Equity

professional development

Adapting the TRU Framework: Tracking Changes in MGTAs' Instructional Practices

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Background

- Lecture-based teaching continues to dominate undergraduate mathematics, despite its negative impact on student success in STEM.
- Mathematics graduate teaching assistants (MGTAs) make up significant portion of the undergraduate teaching force.
- However, MGTAs are often not adequately prepared to teach in engaging, inclusive, and equitable ways.
- Without robust professional development (PD) for teaching that focuses on active, engaged teaching practices, MGTAs often replicate the lecture-based teaching practices that they have experienced.
- Consequently, the lecture-based teaching culture is perpetuated, and undergraduate students do not receive the benefits of evidence-based teaching.

Research Questions

***RQ1*. How does the ELITE PD program support MGTAs to transform their teaching practices to be more engaged, inclusive, and equitable?**

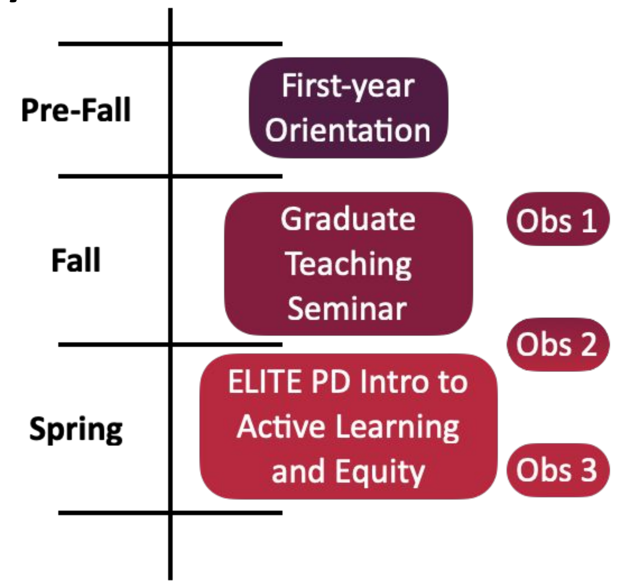
RQ2. What are undergraduate students' experiences and perceptions of MGTAs' teaching practices and how do these change over time?

RQ3. How do the teaching practices of MGTAs in the ELITE PD program compare to MGTAs who do not participate in the program, both within the same graduate program as well at different institutions?

RQ4. What identifiable aspects of individual, institutional, and departmental cultures inhibit or support sustainable change?

Data & Methods

Classroom observations of MGTAs' teaching in their first year:



Theoretical Framework: TRU

The Mathematics	Cognitive Demand	Access to Mathematical Content	Agency, Authority, and Identity	Uses of Assessment
The extent to which the mathematics discussed in the observed lesson is **focused** and **coherent**, and to which **connections between procedures, concepts and contexts** (when appropriate) are addressed and explained.	The extent to which classroom interactions **create and maintain an environment of productive intellectual challenge** that is conducive to students' mathematical development.	The extent to which classroom activity structures **invite and support the active engagement of all of the students** in the classroom with the core mathematics being addressed by the class.	The extent to which students have opportunities to **conjecture, explain, make mathematical arguments, and build on one another's ideas**, in ways that contribute to students' development of **agency, authority, and their identities** as doers of mathematics	The extent to which the teacher **solicits student thinking** and subsequent instruction response to those ideas, by **building on** productive beginnings or **addressing emerging misunderstandings**.

The Mathematics

Small Group Work

TRU	ELITE PD
The focus is on the grade level mathematics with the role of the teacher focused on supporting students exploring and justifying	The focus is on a particular mathematical idea or concept where the role of the GTA is to support students exploring and justifying, with a focus on conceptual understanding

Individual Work

TRU	ELITE PD
The focus is on the designed lesson and materials.	The focus is on the delivery of content rather than the structure of the designed lesson.

Cognitive Demand

TRU	ELITE PD
The focus is on the designed lesson and materials as well as the teacher interactions.	The focus is on the delivery of the lesson and materials as well as the GTA interactions.

TRU	ELITE PD
The focus is on the grade level mathematics and its development.	The focus is on helping students understand concepts introduced.

Original TRU Framework

Discussion

Why TRU?

- Captures aspects of equity and inclusivity not captured by other instruments, as well as identity and agency
- Other instruments did not capture the mathematics and the content element, nor did it capture the interactions between students.

What is different?

- TRU
  - Created for use in K-12 classrooms under Common Core
  - Focus is on the materials themselves in addition to the implementation of them
  - Teachers have more control over the materials they use in the classroom
  - Teachers are the ones introducing the topics and reinforcing them
- ELITE PD
  - Courses are focused on building on past material
  - Focus is on the facilitation of engagement with the prepared materials rather than the materials themselves
  - MGTAs often do not have control over the materials/activities used in their classrooms
  - Often, MGTAs are not introducing the topic to the students for the first time; rather they are reinforcing it

Observation 1

- The first observation of each MGTA indicates where they are in their teaching.
- Initial results indicate that there is room for growth for the MGTAs
  - One particular area for growth is in their understanding of group work versus individual work

Future Work

- How do undergraduate student perceptions align with what the framework helps to capture?
  - If they do align, this is an opportunity to use these reports as a way to gauge what is happening in the classroom, which is more sustainable

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Effects of Representations in Complex Analysis on Student Understanding in Real Analysis

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Research suggests that providing students opportunities to learn about concepts in a different context can deepen their understanding of these concepts (Kemp & Vidakovic, 2022; Troup et al, 2022; Baker et al. 2000, Dreiling, 2012; Hollebrands et al., 2010). Additionally, recent research has suggested that students reasoning about mappings from $\mathbb{R} \to \mathbb{R}$ rather than graphs in $\mathbb{R}^2$ may open new affordances for students in their reasoning about mappings from $\mathbb{C} \to \mathbb{C}$, as they attempt to generalize from graphs in $\mathbb{R}^2$ rather than mappings from $\mathbb{R} \to \mathbb{R}$ (e.g., Soto & Hancock, 2019; Troup et al., 2017). The proposed intervention investigated in this study represents a potential converse to Troup et al.'s observation in that learning about concepts in Complex Analysis may offer opportunities for student understanding of concepts in Real Analysis to deepen. This study was conceived due to students in one of the researcher's Complex Analysis class expressing that learning about sets and neighborhoods in the Complex plane made aspects of real analysis, such as the $\epsilon - \delta$ definition of a limit, more understandable. The researchers' intent for this pilot study was to determine more precisely the possible ways in which this realization can occur. We hoped to identify, (1) In what ways is student understanding of concepts (such as sets and neighborhoods) in Real Analysis affected by being introduced to these concepts in the Complex space?, and (2) What pedagogical implications emerge from obstacles students may face seeing different representations of these concepts?

In this poster, we outline an ongoing study intended to characterize, through the lens of APOS (Action, Process, Object, and Schema) Theory, ways in which learning concepts in Complex Analysis might render concepts in Real Analysis more approachable for a student. Baseline data was collected via a Real Analysis class assignment in Spring 2022 at a Western Hispanic Serving Institution. We then introduced the students to an 18-minute video mini-lecture covering geometric representations of ϵ and δ neighborhoods in the complex plane and a corresponding definition of limit. We collected data afterward with another class assignment and compared this to our baseline data to determine what, if anything, changed in students' reasoning and responses after watching the video. Our findings from this pilot study revealed some easily correctable limitations of the current methodology, including the format and timing of data collection. Despite these limitations, initial results suggest a common perspective the researchers named *range-axis entanglement* where the student had a strong tendency to believe the inputs and outputs of a function must be graphically represented on an xy-plane. We believe this may be a consequence of an early coordination of Processes within the Limit schema.

We hoped that exposure to these concepts in a different context could offer affordances for students to overcome obstacles and better reason abstractly. In future data collection, we plan to adjust the presentation of the relevant concepts in the context of Complex Analysis as an activity in class instead of a video. We also plan to collect data through interviews to target student understanding in both Complex and Real Analysis. Particularly, that as students become proficient with mappings $\mathbb{C} \to \mathbb{C}$, and in so doing disentangle the range and axis concepts, they might become more proficient with mappings $\mathbb{R} \to \mathbb{R}$ as a resultant side effect. For this poster, we present these preliminary findings, including examples of the range-axis entanglement, from the first stage of our study. We also provide future research plans, including ways we intend to refine the methodology for future stages of our larger project.

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Effects of Representations in Complex Analysis on Student Understanding in Real Analysis

CALIFORNIA STATE UNIVERSITY BAKERSFIELD

Aubrey Kemp Brian Ryals Jonathan Troup

Abstract: Research suggests that providing students opportunities to learn about concepts in a in a different context can deepen their understanding of these concepts. Additionally, recent research has suggested that students reasoning about mappings from $\mathbb{R} \to \mathbb{R}$ rather than graphs in $\mathbb{R}^2$ may open new affordances for students in their reasoning about mappings from $\mathbb{C} \to \mathbb{C}$, as they attempt to generalize from graphs in $\mathbb{R}^2$ rather than mappings from $\mathbb{R} \to \mathbb{R}$. The proposed intervention investigated in this study represents a potential converse to this observation that the lack of opportunities for students to engage with $\mathbb{R} \to \mathbb{R}$ in comparison with $\mathbb{R}^2$ leads to difficulty with mappings $\mathbb{C} \to \mathbb{C}$. In this poster, we outline an ongoing study intended to characterize, through the lens of APOS (Action, Process, Object, and Schema) Theory, ways in which learning concepts in Complex Analysis might render concepts in Real Analysis more approachable for a student. We present preliminary findings from the first stage of our study. We also provide future research plans, including ways we intend to refine the methodology for future stages of our larger project.

Key words: Limits, $\varepsilon - \delta$ proofs, APOS Theory, Visualization, Complex Numbers

Research Questions

1. In what ways is student understanding of concepts (such as sets and neighborhoods) in Real Analysis affected by being introduced to these concepts in the Complex space?
2. What pedagogical implications emerge from obstacles students may face seeing different representations of these concepts?

Literature Review

Providing students opportunities to learn about concepts in a different context can deepen their understanding of these concepts (Kemp & Vidakovic, 2022; Troup et al., 2022; Baker et al. 2000, Dreiling, 2012; Hollebrands et al., 2010)

Limits:

- Students rely heavily upon graphs and formulas in understanding the limit concept (Williams, 1991)
- *Obstacle of formalism*: the tendency for students to view the formal representation of a mathematical object as if the representation was itself the object (Sierpinska, 2000)
- In addition to relying on a formal definition of limit, students can benefit from dynamic/geometric ideas and representations of limits (Tall & Vinner, 1981)
- Reasoning about mappings from $\mathbb{R} \to \mathbb{R}$ rather than graphs in $\mathbb{R}^2$ may open new affordances for students in their reasoning about mappings from $\mathbb{C} \to \mathbb{C}$, as they attempt to generalize from graphs in $\mathbb{R}^2$ rather than mappings from $\mathbb{R} \to \mathbb{R}$ (e.g., Soto & Hancock, 2019; Troup et al., 2017)

APOS (Action-Process-Object-Schema) Theory (Dubinsky, 2002; Arnon et al., 2014):

- *Action* – can transform objects with external stimuli or by performing steps

Ex: For $\lim_{x \to a} f(x) = L$ - may evaluate f at a single point x that is close to or equal to a.

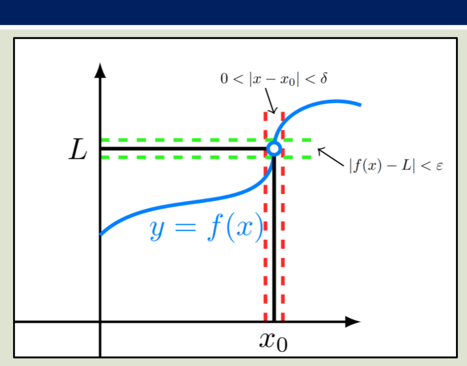
- *Process* – can imagine performing these Actions without necessarily doing so; can *coordinate* Processes with others to form connections between concepts

Ex: May evaluate the function at more than one point, each point getting nearer to a, interiorize this action, and construct a domain process in which x approaches a. They also may construct a range process in which $f(x)$ approaches L, and coordinate these processes by applying the function f to both processes

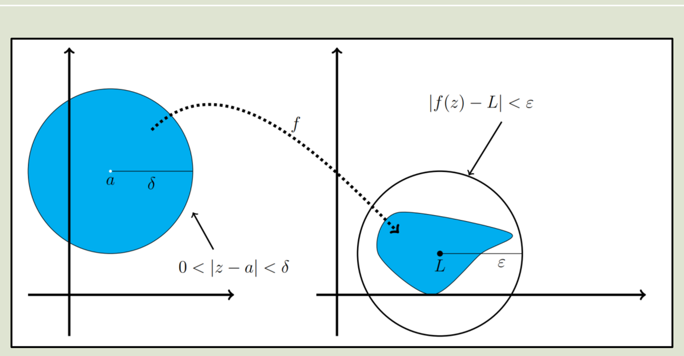
- *Object* – can think of a Process as a totality to which Actions or Processes may be applied

Ex: Considering limits at a point of combinations or compositions of functions

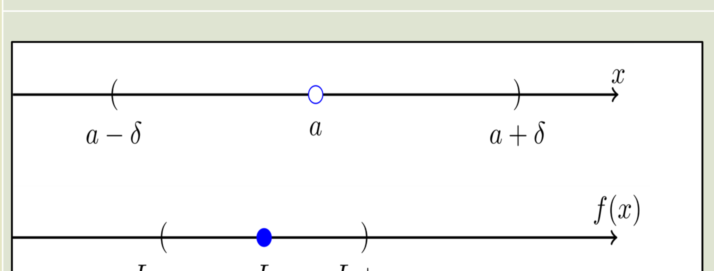
Visualizations



Typical geometric picture in $\mathbb{R}$ (focus on pre-image and image as one graph)



Typical geometric picture in $\mathbb{C}$ (focus on pre-image and image as two graphs)



Desired geometric picture in $\mathbb{R}$ (focus on pre-image and image as two graphs)

Methodology

- 13 volunteers from a Real Analysis course at a western university (out of 20 students enrolled)
- Coursework (homework, assessments) collected for analysis
- Participants:
  - Saw traditional classroom algebraic and geometric treatment of one dimensional $\varepsilon - \delta$ limits in $\mathbb{R}$
  - Watched short lecture on the algebraic and geometric treatment of one dimensional $\varepsilon - \delta$ limits in $\mathbb{C}$.
  - Were asked to draw "before" and "after" pictures that correspond to the pre-image and image and to elaborate on their thinking
- Using APOS Theory, analyzed whether there were then any changes in their geometric understanding in $\mathbb{R}$.

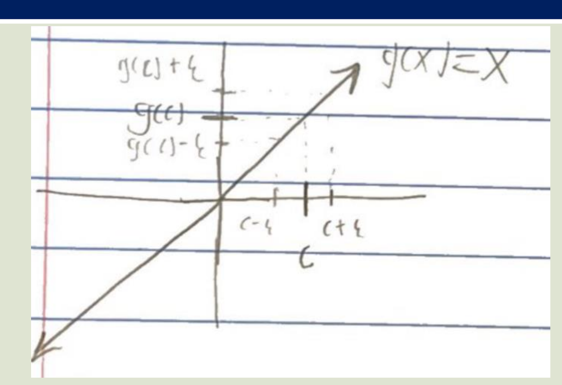
Definition Given in Class

4.1.4 Definition Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of** f **at** c if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Before Exposure to $\mathbb{C}$ | After Exposure to $\mathbb{C}$ | Analysis Highlights

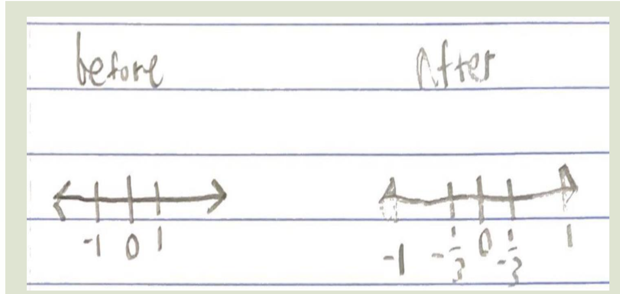
Student 1

Before Exposure to $\mathbb{C}$



"The limit of the polynomial $g(x) = x$ is precisely $g(c)$ for c in $\mathbb{R}$, since if $0 < |x - c| < \varepsilon \to |x - c| < \varepsilon$. In other words, **as we approach the c value, we also approach the $g(c)$ value.**"

After Exposure to $\mathbb{C}$



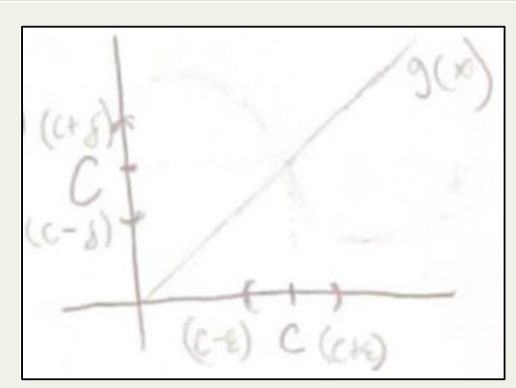
"The function $3x$ **stretches the real number line** by a factor of 3. Since $\dim(\mathbb{R}) = 1$, **our pictures only have one dimension**, whereas $\dim(\mathbb{C}) = 2$ as seen as $\mathbb{C} = \mathbb{R} \times \mathbb{R}$."

Analysis Highlights

- Before: Process conception ("approach")
- After: Process conception (if prompted may show evidence of Object)
- *Limit schema* – Evidence of successful coordination between image and pre-image processes
- No evidence of Range-axis entanglement

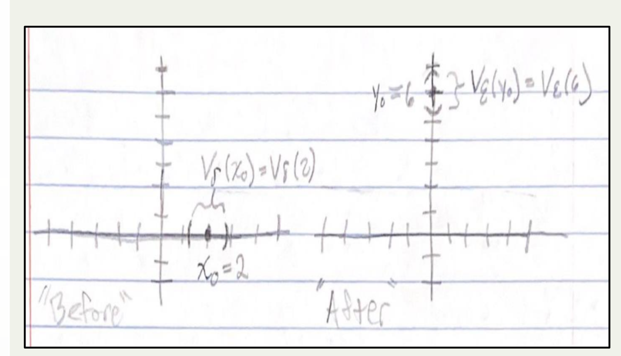
Student 2

Before Exposure to $\mathbb{C}$



"If an **infinite amount of points** is in the ε –neighborhood of c, then there also [is] **an infinite number** of points in the δ –neighborhood of c. So the **number $g(x)$ approaches as x approaches c is also c.**"

After Exposure to $\mathbb{C}$



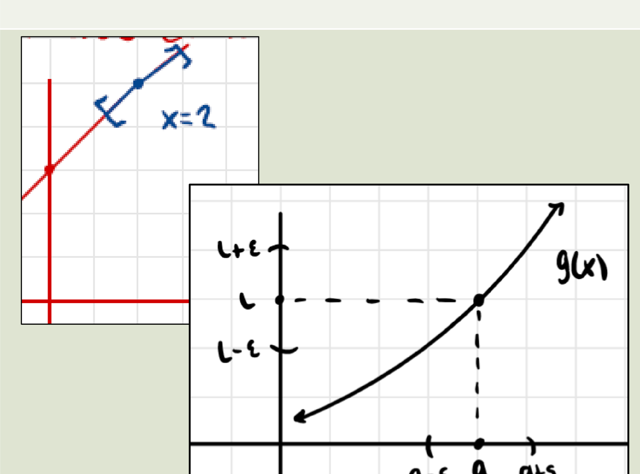
"In this function **$f(x)$ is our output and x is our input**. So we can have $f(x)$ be our after image and x be our before image. So if we have an ε-neighborhood around $y_0 = 6$ then we know **there must be a corresponding** δ-neighborhood around $x_0 = 2$."

Analysis Highlights

- Before: Process conception (relating neighborhoods as processes)
- After: Process conception (relating the neighborhoods as objects)
- *Limit schema* – Object conception of **Function**; May need to de-encapsulate **Function** object to strengthen the coordination of **Range** and **Graphing** processes
- Range-Axis Entanglement

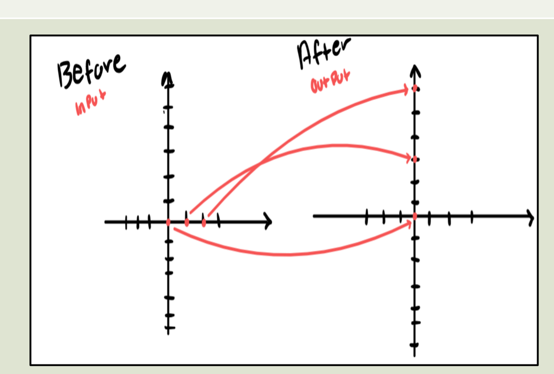
Student 3

Before Exposure to $\mathbb{C}$



"Could be that c is the same distance as epsilon, not sure. Cannot write a full explanation as I don't know where c is.

After Exposure to $\mathbb{C}$



"For the before graphs, I **choose the x values** with $x \in \mathbb{Z}$. I stopped at $x = 2$ because $c = 2$. All of these points remain on the x-axis. For the after graphs, **I used the points from the x-axis and plugged into $y = 3x$.** My **y values only stuck to the y-axis.**"

Analysis Highlights

- Before: Pre-Action conception (could refine understanding of cluster point)
- After: Action conception (picking specific points to relate)
- *Limit schema* – Action conception of **Function**; May need to work on coordination of **Pre-Image** and **Image** processes to interiorize

"Why Complex?"

- Symbolically, the definitions of a limit in the Real and Complex space are nearly identical, though the visual interpretation is different
- Promotes thinking of neighborhoods and their images as sets
- Detangles the function f from the y-axis
- Visual generalizes to more abstract setting (traditional visual doesn't!)
- Complex still uses same notation as $\mathbb{R}$ (unlike $\mathbb{R}^2$)
- Offer affordances for a better comprehension in $\mathbb{R}$?

Preliminary Results

- Did not clearly affect understanding
- Possibly allowed for coordination reversal for some students
- Illuminated schema development and obstacles within
- *Range-Axis Entanglement*: Strong tendency to believe the inputs and outputs of a function must be graphically represented on an xy-plane (face an obstacle in "separating" pre-image and image)
  - Consequence of early coordination of Processes within the Limit schema?
  - How can we effectively guide students?

Limitations of Present Study

- Format of data collection
  - No interviews to probe for understanding
  - Questions in prompts did not target understanding in both $\mathbb{R}$ and $\mathbb{C}$ as well as they could
  - Video was 18 minutes → In class activity next time
- Timing of data collection
  - Collect after semester or at least not during finals
  - Introduce concepts $\mathbb{C}$ in class before this video

Future Directions

- As students become proficient with mappings $\mathbb{C} \to \mathbb{C}$, and in so doing disentangle the range and axis concepts, do they become more proficient with mappings $\mathbb{R} \to \mathbb{R}$?
- Continuing investigating this idea and *range-axis entanglement*
- Student understanding of absolute value in relation to the limit definition
- How does replacing the notation $|x - y|$ with $d(x, y)$, where d stands for distance, affects student conceptions.

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Designing Tasks to Teach Mathematical Induction

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We conducted a study on the teaching and learning of mathematical induction, based on the principles of didactical engineering, with undergraduate students enrolled in a Discrete Mathematics course at a four-year university. Students' performance on induction tasks improved. We focus here on designing tasks with the potential to foster inductive reasoning and give meaning to the Principle of Mathematical Induction.

Keywords: Mathematical Induction Tasks, Didactical Engineering, Undergraduate Students

The recent research on mathematical proofs is rich, but not many studies specifically address mathematical induction. A proof by mathematical induction may be used whenever we have a statement $P(n)$, where n is a natural number. According to Ernest's (1984) framework, the steps of a mathematical induction proof are:

- basis step, checking that $P(n_0)$ is true (n_0 is the smallest value for which $P(n)$ is true),
- inductive step (checking that $P(k) \rightarrow P(k+1)$, $k \geq n_0$), and
- invocation of the Principle of Mathematical Induction (PMI).

We report on a study on the teaching and learning of proofs by mathematical induction, with undergraduate students enrolled in a Discrete Mathematics course taught by the first author at a four-year university. We focus on the application of the principles of didactical engineering (Artigue, 1994) to design tasks with the potential to foster inductive reasoning and give meaning to the PMI. Figure 1 illustrates the phases of the didactical engineering process.

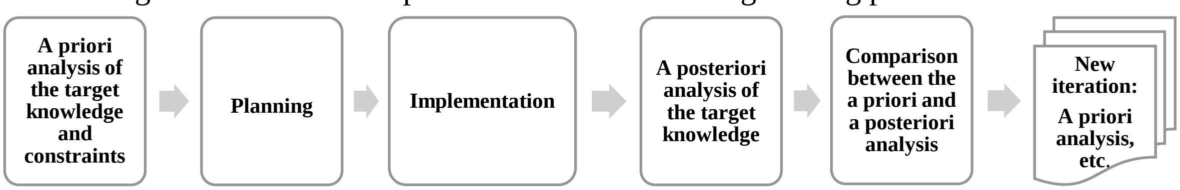

Figure 1. The didactical engineering process adapted after Artigue (1994)

At the beginning of the fourth iteration, the a priori analysis revealed that our students had difficulty giving meaning to mathematical induction as a method of proof. Although our students' performance improved with each iteration, the induction step remained procedural in nature. Ernest (1984) drew our attention to the usefulness of analogies to the PMI. Harel (2001) used tasks that defined sequences recursively and fostered inductive reasoning, thus giving meaning to the PMI. Other studies highlighted the importance of visual representations (Stylianou and Silver, 2004) and gestures used during the teaching and learning process (Kokushkin, 2020). Based on research and our own experience with the teaching and learning of mathematical induction, we designed, implemented, and refined tasks with the potential to foster inductive reasoning and give meaning to the PMI. One such task asked the students to create their own analogy to the PMI (Postelnicu and Gonzalez, 2020), while another task used visual representations. We discuss the use of visual representations, intended to help the students define a sequence recursively and find the general term of the sequence and the formula for the sum.

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“What is Equitable Teaching?”: Graduate Teaching Assistants’ Perceptions of Equity

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Keywords: Equity, Graduate Teaching Assistants, Professional Development

In response to national movements for social justice in education, there has been an increase in efforts to develop institutional structures to support diversity, equity, and inclusion (DEI) in the classroom. Teachers are tasked with supporting diverse groups of students from varied backgrounds (Ameny-Dixon, 2004). However, research indicates that there is a lack of support to empower practitioners to create equitable learning environments (Perez et al., 2020).

One significant portion of the teaching force in undergraduate mathematics is Mathematics Graduate Teaching Assistants (MGTAs). Iverson (2012) noted that current graduate education is insufficient for developing MGTAs’ skills and knowledge for fostering social change. While there is scant literature on teachers’ conceptions of DEI - the little that exists indicates that many practitioners have varied understandings of these concepts (Bell et al., 2007; Fifty et al., 2022; Garmon, 2005). Therefore, there is a need to explore how to support MGTAs in developing equitable and inclusive teaching practices.

In this study, we investigate MGTAs’ initial conceptions of equitable teaching practices. Additionally, we report our initial observations on how the MGTAs interacted with specific activities throughout a teaching seminar focused on active learning and DEI topics. This work is part of a larger project that aims to design and implement a multi-stage, MGTA professional development program centered on equity and inclusion in a range of academic institutions.

In our initial findings through semi-structured interviews and post-class surveys, we noticed that MGTAs’ conceptions of equity generally referred to a deficit in students’ content knowledge. For example, a significant portion of our subjects thought that equitable teaching involved providing extra office hours or spending more time with students in the classroom based on how well they understood the material. Few MGTAs demonstrated an awareness of differences in student backgrounds and identities that may have impacted their schooling.

We observed that directly discussing the diverse populations that take undergraduate mathematics courses supported the MGTAs in expanding their conception of equitable teaching practices. In particular, providing these MGTAs with descriptions and illustrations of students’ access needs, mathematical backgrounds, and classroom experiences aided the MGTAs in discussing ways in which they could create a more equitable teaching environment.

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ELITE PD

Engaged Learning, Inclusive Teaching, & Equity
professional development

"What is Equitable Teaching?": Graduate Teaching Assistants' Perceptions of Equity

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Context, Research Questions, & Methodology

Context:

- There has been an increase in efforts to develop institutional structures to support diversity, equity, and inclusion (DEI) in the classroom
- Current graduate education is insufficient for developing Mathematics Graduate Teaching Assistants' (MGTAs) skills and knowledge for fostering social change (Iverson, 2012)

Research Questions

- What are MGTAs' conceptions of *equitable teaching*?

Data Sources

- 31 Clinical Interviews with MGTAs at 3 institutions
- Clinical Interview consisted of questions asking about MGTAs teaching experience and ideas about equitable and inclusive teaching

Working Definition of "Equitable Teaching"

Equitable Teaching refers to focusing on the needs of every individual and ensuring the right conditions are in place for each person to achieve their full potential. An equitable classroom values the unique contributions that students of all backgrounds bring to the classroom and allows diverse groups to grow side by side, to the benefit of all.

Teaching practices include attending to:

- student identity, addressing stereotypes and bias
- intentional language use (e.g., Su, 2015)
- small-group structure and roles (e.g., Brown, 2018; Dasgupta et al., 2015; MAA, 2018; Reinholz, 2018)
- addressing needs of current-day students, including access and resources

What does "Equitable Teaching" mean to you?

MGTA Conceptions of Equitable Teaching

"How would you describe equitable teaching at the post-secondary level or college level?"

- Overall, the MGTAs had varying ideas on what "equitable teaching" entailed. Coding revealed overlapping themes.
- Besides "helping students", there was no one theme that appeared in more than 50% of MGTAs' responses.

Equitable Teaching as…

Using multiple/diverse teaching styles

"Equitable teaching is a way of designing **teaching methodologies** that allow for **multiple methods of learning** from students"

"I think having a **diverse set of ways that you present material.** Some students may be **more visual**. Some students really like to just **hear what you're saying**. So yeah, providing all the students with the best change to do well in your class I guess."

"So like **explaining things a different way**, playing a video, and doing something that's more quiet individual work"

"Split it up so there's a little of everything for **each type of student**… like for example if I'm speaking a lecture to you, I'm also trying to write down what I'm saying on the whiteboard so that students can get **both audio and visual** there"

Helping students who are behind

"Trying to figure out if there are students who are **falling behind** and figure out how to get them **back on pace**"

"To be equitable you want to make sure, like focus on students that **might not have the material down** and come back to them more often and engage with them more often"

"So students with different backgrounds like there's a lot of **students with gaps in their mathematical content** knowledge from prior courses and **so being able to help fill those**"

"So equity for me it's more like I give everybody what they need. So, for example, if I have office hours, and as they come in, if one student, I notice they have **more gaps**, they might need more time to spend with me vs. somebody who has just one or two questions that might be more of like a clarifying question."

Being accessible and available to students

"Equitable teaching would be providing all of the students with **equal access to a teacher or TA**. So **office hours that are accessible to everyone.** Maybe someone has a job on the side or is on a team or, you know, in the choir, something like that. And making sure you have office hours or **a way to reach you** even when everyone has all these different schedules."

"I want everyone to be **able to have that opportunity to come and study or ask questions,** or whatever they need to do. So I've definitely pushed myself to, **if someone can't make it to office hours, then I schedule other times"**

Equal access/provision to resources and learning

"Equity is every student has **equal access** to a laptop and reliable internet access or something like that"

"Equity, to me, probably implies a lot of **free available resources**. Things like that are kind of open door policies to allow students to try and **seek out as much support as they can**."

Involving Active Learning

"Of all of the class, like most of the time where it's **the most equitable is the active learning process**, because those are designed to be smaller **group work** activities, **working together**, and getting **questions answered**"

"Like **interactive with group work**, or you know, they're like creative and exploratory ways to engage with the material and those things are helpful for equity and inclusion. Having **students work together and talk to each other and collaborate**."

"It's most equitable when we're kind of in a **discussion mode**. When it's not me standing up there giving them information. More when it's **students asking questions or working on problems in smaller groups."**

Discussion

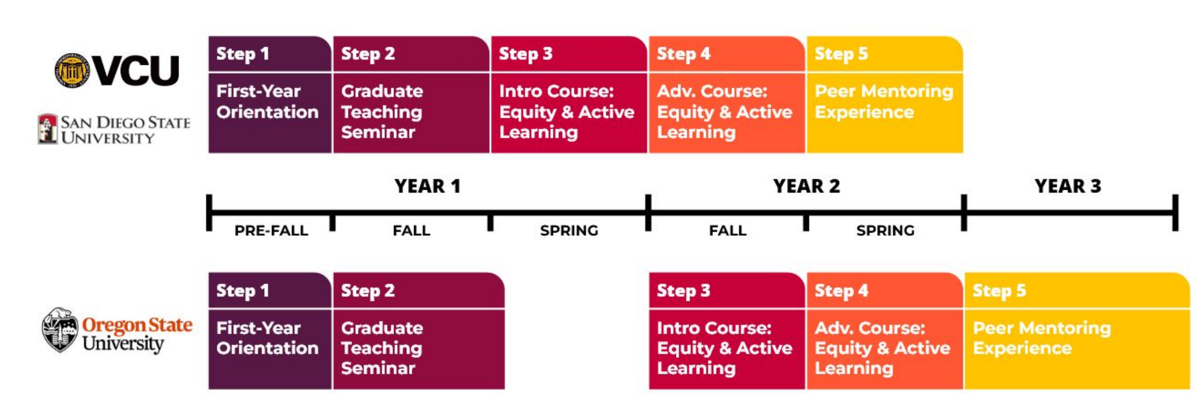
- Few MGTAs explicitly mentioned an awareness of how students' identities impact their classroom experience:

  *"Each student has **individual experiences** that they're bringing to the classroom and that, if we can **acknowledge those different identities** and those different backgrounds, that **we can help students have access** to the material in a better way."*

- While many MGTAs explained equitable teaching as "giving according to needs", most of these MGTAs associated this with different learning styles (Audio/Visual/Kinesthetic) or helping students who are behind on content knowledge.

The Elite PD Program

- We designed a professional development (PD) program for MGTAs focused on engaged learning, inclusive teaching, and equity (ELITE PD).
- The ELITE PD program spans 2-3 years and will be implemented at three collaborating institutions with differing contexts.
- MGTAs who participate in the program will be supported in making incremental changes to transform their teaching and broaden their understanding of equity and inclusion.
- Cohorts will be recruited from each institution and progress through a sequence of PD workshops, classes, and peer mentoring



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Mathematics professors' goals through an integrated lens of guided reinvention and creation

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Keywords: Realistic mathematics education, Rehumanizing mathematics, alternative conceptions, conjecture, undergraduate mathematics, creation, guided reinvention

Guided reinvention or the guidance principle (Realistic Mathematics Education) examines how teachers can provide opportunities for students through "realistic" contexts to facilitate them to reinvent "higher" level formal concepts. (Gravemeijer, 2008; Van den Heuvel-Panhuizen, & Drijvers, 2020). Guided reinvention acknowledges collaborative students' work like whole group discussions. We assume that students can show or develop multiple understandings and even non-traditional knowledge in such discussions. Therefore, we consider the creation dimension (Rehumanizing Mathematics) which acknowledges students' conceptions and diverse forms of expressing mathematics (Gutiérrez, 2018) as a way of expanding guided reinvention. Being able to stray from traditional knowledge forms/representations can feel humanizing for students and help transform their relationship with mathematics as authors/creators of mathematics. By way of illustration of diverse forms, there can be sense-making alternative axiomatic models of a concept, which are different from the traditional ones. Moreover, students might show "proofs" that are non-deductive, non-symbolic, or visual. Creation privileges Nepantla moments of tensions within multiple perspectives (Gutiérrez, 2012); however, Gravemeijer's mathematization in guided reinvention process focuses on developing standard procedures, "limiting interpretations," and validations (Gravemeijer, 2000). This encourages us to identify how guided reinvention and creation can interact with and expand each other's use.

To obtain empirical evidence, we recruited five mathematics professors in India for our ongoing qualitative case study to explore how their views of their teaching roles might influence their interest/propensity to use guided reinvention and creation. One out of three interviews for each participant centers guided reinvention, another interview centers creation, and the final interview centers both concepts. "Member checking" is conducted in the last interviews by summarizing interpretations of the first two interviews to the participants. With a lens of guided reinvention and creation, we seek responses to our research questions on mathematics professors' perceptions of their role in facilitating students for abstract concepts, students' alternative conceptions (Fujii, 2020), students making conjectures, and students' participation. The findings can potentially exemplify and guide us on how guided reinvention and creation can be developed as an integrated framework, which can be used in later studies and support a more humanizing experience for students studying mathematics.

The poster presents the study's preliminary results, participants' quotes, future steps, and a Venn diagram depicting how guided reinvention and creation potentially interact. Among other results, we focused on one professor and noticed he finds it essential to provide realistic scenarios but thinks it becomes difficult to find such scenarios at "higher" abstraction levels. We identified that, for this professor, he finds it important to alert his students about informal writing so that they can avoid "errors," which might prevent guided reinvention or creation. We look forward to discussions with conference attendees about more results mentioned in the poster.

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Enriching Assessment of Physics Quantitative Literacy: IRT with Multiple-Response Items

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Keywords: Quantitative Literacy, Item Response Theory, Physics

Students in introductory physics courses are often asked to use familiar mathematics in novel ways. Improving students' quantitative literacy within physics contexts (i.e. physics quantitative literacy, or PQL) is a goal for many physics instructors. A challenge in assessing PQL is the blended relationship between students' general mathematical reasoning and conceptual understanding of physics. A recently developed multiple-choice instrument for measuring PQL includes several multiple-choice-multiple-response (MCMR) items for which students may select as many responses as they think are correct (White Brahmia, et al., 2021). The MCMR item format provides an opportunity to see the interaction between students' quantitative literacy and conceptual physics understanding.

Previous approaches to analyzing data from MCMR items fall into three categories: scoring response combinations as either completely correct or incorrect (Kubinger & Gottschall, 2007; Smith, et al., 2020; White Brahmia, et al., 2021), defining an *a priori* partial credit score for each response combination based on expert opinions (Wilcox & Pollock, 2014), or avoiding item scores by treating responses as categorical variables (Smith, et al., 2019; Vlach & Plašil, 2006).

In this poster we present a novel approach to defining a data-driven partial credit score for each combination of responses to MCMR items based on item response theory (IRT) analyses. We seek to address three research questions: 1) What score is assigned to each combination of responses based on IRT parameters? 2) How do correct and incorrect responses impact these scores? 3) How do scores based on IRT parameters relate to traditional dichotomous scores?

Data come from 3424 introductory physics students at a large research university. We apply IRT methods to MCMR data by treating each item as a combination of individual dichotomous response-items and grouped nominal response-items (Smith, et al., 2022). IRT parameters are used to define a score for each response combination (Smith & Bendjilali, 2022).

Our results show that the correct response combination for each item always has the highest IRT-defined score, even though the correct responses are not specified in the model. Correct individual responses have positive scores, and incorrect responses have negative scores. Total IRT-defined test scores strongly correlate with dichotomous test scores ($r = 0.8$). This supports the interpretation of IRT parameters as indicating levels of correctness and understanding.

Some analyses of grouped responses yielded counterintuitive results: we identified one item for which certain incorrect responses corresponded to positive parameters, and another item for which selecting two (out of three) correct responses was scored lower than selecting only one.

Our overall results indicate a strong potential for using IRT analyses to inform partial credit scores for MCMR items used to measure students' PQL; however, more consideration is needed to define the transformation from IRT parameters to partial credit scores.

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Enriching Assessment of Physics Quantitative Literacy: IRT with Multiple-Response Items

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Goal: Define partial credit for response combinations using both dichotomous and nominal IRT models to analyze data from MCMR items

Introduction

- Improving students' physics quantitative literacy (PQL) is a goal for many physics instructors.
- PQL is the blended relationship between students' general mathematical reasoning and conceptual understanding of physics.
- The **P**hysics **I**nventory of **Q**uantitative **L**iteracy (PIQL), includes multiple-choice-multiple-response (MCMR) items; students may select as many responses as they think are correct (White Brahmia et al., 2021).
- The MCMR items can show the interaction between students' quantitative literacy and conceptual physics understanding.
- Previous approaches to analyzing data from MCMR items fall into three categories:
  - scoring response combinations as either completely correct or incorrect (Kubinger and Gottschall, 2007; Smith et al., 2020; White Brahmia et al., 2021),
  - defining partial credit score based on expert opinions (Wilcox and Pollock, 2014),
  - treating responses as categorical variables without scores (Smith et al., 2019; Vlach and Plašil, 2006).

Methods

- PIQL response data analyzed using item response theory (IRT).
- Data come from 3424 introductory physics students.
- Each item is a combination of individual dichotomous response-items and grouped nominal response-items (Smith et al., 2022). See Table 1
- IRT parameters are used to define a score for each response combination (Smith and Bendjilali, 2022).

Item Response Theory

The probability of choosing a particular response k, given the student's overall physics quantitative literacy θ.

Dichotomous Two-parameter Logistic Model (2PL): A response option is either chosen or not chosen, e.g., response "A" to item 17.

$$P_{2pl}(k|\theta) = \frac{e^{a_k\theta+d_k}}{1+e^{a_k\theta+d_k}}$$

Nominal Response Model (NRM): A response option is a set of related responses or combinations of responses, e.g., responses "D", "E", "both D& E", or "neither D nor E" to item 18

$$P_{nrm}(k|\theta) = \frac{e^{a_k\theta+d_k}}{\sum_i e^{a_i\theta+d_i}}$$

Joint probability of selecting a particular combination of responses is the product of individual functions.

$$P_{16}(A,C,F|\theta) = P_{2pl}(C|\theta)P_{2pl}(D'|\theta)P_{nrm}(A\cap F|\theta)P_{nrm}(B'\cap E'|\theta)$$

$$= \left(\frac{e^{a_C\theta+d_C}}{1+e^{a_C\theta+d_C}}\right)\left(\frac{1}{1+e^{a_D\theta+d_D}}\right)\left(\frac{e^{a_{AF}\theta+d_{AF}}}{\sum_i e^{a_i\theta+d_i}}\right)\left(\frac{1}{\sum_j e^{a_j\theta+d_j}}\right)$$

$$= \frac{\exp\left[(a_C+a_{AF})\theta+d_C+d_{AF}\right]}{Z_{16}(\theta)}$$

where $i = \{0, A, F, AF\}$ and $j = \{0, B, E, BE\}$. $Z_{16}(\theta)$ is a summation including one term for every possible combination of responses.

The "a_k" parameters indicate level of PQL (Smith and Bendjilali, 2022), and a_0 = 0. We use these to define partial credit scores.

For the dichotomous 2PL model, a negative a_k value indicates that the probability of selecting response k is a monotonically decreasing function of θ. But for the nominal model, positive or negative a_k values indicate higher or lower PQL than not selecting any of the responses, respectively.

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Wilcox, B. R. and Pollock, S. J. (2014). Coupled multiple-response versus free-response conceptual assessment: An example from upper-division physics. *Phys. Rev. ST Phys. Educ. Res.*, 10(2):020124.

Important: The remainder of the questions on this survey may have more than one correct answer. Select ALL ANSWERS you believe are correct.

15. The internal energy of a system can be increased by doing positive work on the system or by heating it, and it can be decreased by cooling the system or if the system does work. Which if the following equations represent(s) this relationship (U is the internal energy of the system, Q is positive when energy flows into the system, and W is positive when work is done on the system)? ***Choose all that apply.***

   a. $\Delta U = Q - W$
   b. $\Delta U = -Q + W$
   c. $\Delta U = Q + W$
   d. $-\Delta U = Q + W$
   e. $-\Delta U = Q - W$
   f. $-\Delta U = -Q + W$

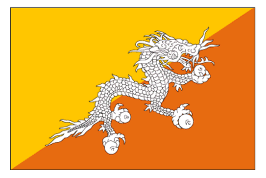
16. Consider the following statement about subatomic particles:

   "There are three times as many quarks as nucleons."

   Some students were asked to write an equation to represent this statement, using Q for the number of quarks and N for the number of nucleons. Which of the following are correct?
   Choose all that apply.

   a. $3Q/N$ b. $3Q = N$ c. $3Q + N$ d. $Q = 3N$ e. None of these are correct.

17. Shown at right is a picture of a small flag of the Kingdom of Bhutan, which has a dragon along its diagonal. A second version of the flag, not shown, is 1.5 times taller than the smaller flag and is also 1.5 time wider.
   Which of the quantities below are larger by a factor of 1.5 for the larger flag compared to the smaller flag? ***Choose all that apply.***



   a. The distance around the edge of the flag.
   b. The amount of cloth needed to make the flag.
   c. The length of the curve forming the dragon's backbone.
   d. The diagonal of the flag.
   e. None of these quantities are larger by a factor of 1.5.

18. The following problem is constrained to one dimension. A hand exerts a constant, horizontal force on a block as the block moves along a frictionless, horizontal surface. No other objects do work on the block. For a particular interval of the motion, the hand does $W = -2.7$ J of work on the block. Recall that for a constant force $\boldsymbol{F}$, and displacement $\Delta\boldsymbol{s}$, $W = \vec{F}\cdot\Delta\vec{s}$.

   Consider the following statements about this situation. Select the statement(s) that **must be true.** ***Choose all that apply.***

   a. The work done by the hand is in the negative direction.
   b. The force exerted by the hand is in the negative direction.
   c. The displacement of the block is in the negative direction.
   d. The force exerted by the hand is in the opposite direction as the block's displacement.
   e. The force exerted by the hand is in the same direction as the block's displacement.
   f. Energy was added to the block system.
   g. Energy was taken away from the block system.

19. Recall that the electric field is a vector quantity. At a location on the x-axis, the x-component of the electric field is measured to be $E_x = -10$ N/C, and the y and z-components of the field are measured to be zero.

   Consider the following statements about this situation. **Select the statement(s) that must be true.** ***Choose all that apply.***

   a. The source of the field is a negative charge.
   b. At that location of the x-axis, the field is in the negative x-direction.
   c. In this field, the motion of any charged particle is opposite to the direction of the field.
   d. The magnitude of the electric field is decreasing.
   e. The magnitude of the electric field at that location is 10 N/C less than it is at the origin.

20. Two objects can move in either direction along the x-axis. The two objects collide. As a result of the collision, object 1's velocity vector changes, $\Delta v_x = -10$ m/s.

   Consider the following statements about this situation. Select all statement(s) that **must be true.** ***Choose all that apply.***

   a. Object 1 is moving faster after the collision than it was before the collision.
   b. Object 1 is moving slower after the collision than it was before the collision.
   c. Whether object 1 is moving faster or slower after the collision than it was before the collision cannot be determined using only the information given.
   d. Object 1 moved in the positive direction before the collision, and in the negative direction after the collision.
   e. Object 1 moved in the negative direction before the collision and in the positive direction after the collision.
   f. Object 1 moved in the negative direction both before and after the collision.

Figure 1. MCMR items on the PIQL

Dichotomous and Nominal Response-items

Item	Independent Dichotomous Responses	Grouped Nominal Responses	Item	Independent Dichotomous Responses	Grouped Nominal Responses
15	C, D	A+F, B+E	18	A, B, C	D+E, F+D
16	A, C	B+D	19	A, B, C, D, E	–
17	A, B, C, D	–	20	D, E, F	A+B+C

Table 1. Strongly correlated response options are grouped and analyzed using the nominal response model (Smith et al., 2022).

IRT Parameter Results

	Expected Ranking	IRT Score		Expected Ranking	IRT Score
Item	C*	+0.67	Item	DG*	+1.39
15	AF	-0.20	18	D†	+0.04
	BE	-0.53		G†	+1.35
	D	-0.74		EF	+0.07
	A	-1.08		E; F	-0.46; +0.53
	F	-0.50		A; B; C	-0.20; -0.36; -0.57
	AC‡	-0.40		DE‡; FG‡	-0.48; -0.20
Item	D*	+2.27	Item	B*	+0.49
16	B	+1.54	19	A	0.00
	A	-0.41		C	-0.51
	AB	+1.13		D	-0.84
	AD‡	+1.86		E	-0.38
Item	ACD*	+1.48	Item	C	+1.76
17	A†D†	+1.11	20	C†D	+0.90
	A†C†	+1.17		D	-0.86
	A†	+0.77		A	+0.23
	C†; D†	+0.37; +0.34		B	+0.99
	B	-0.89		AB‡	-0.23
				ABC‡	-0.10
				E	-1.22

Table 2. Expected rankings of responses to MCMR items (higher on top), and associated partial credit "scores" defined from IRT parameters. Asterisks (*) indicate completely correct responses. Daggers (†) indicate partially correct responses. Double daggers (‡) indicate contradictory response combinations.

Takeaways

- Choosing mathematically equivalent responses indicates higher understanding than just choosing one of a pair. Example from item 15: $a_{AF} > a_A + a_F$ and $a_{BE} > a_B + a_E$.
- Incorrect independent (dichotomous) responses always have negative parameter values. Incorrect responses that are part of a nominal group may have positive or negative IRT parameter values. This suggests that any response from some groups indicates higher physics quantitative literacy (PQL) than not choosing any from the group.
- The order of the IRT parameter "scores" matches the expected ranking for most items, but there are some discrepancies. For example: choosing the two incorrect responses "E" and "F" on item 18 has a higher score than choosing the single correct response "D". More work is needed to reconcile these results.

Research Questions and Results

RQ1: What score is assigned to each combination of responses based on IRT parameters?

- See Table 2

RQ2: How do correct and incorrect responses impact these scores?

- All correct response scores are positive.
- All incorrect dichotomous response scores are negative.
- Incorrect nominal responses may be positive or negative, but all are lower than correct.

RQ3: How do scores based on IRT parameters relate to traditional dichotomous scores?

- IRT parameters used as scores give a range of total test scores from about 52 to 82 (compared with [0,20] for dichotomous scoring)
- When partial credit scores are scaled to have a range of [0,20], most (but not all) students have higher total scores when including partial credit.

Partial Credit vs. Dichotomous Scores

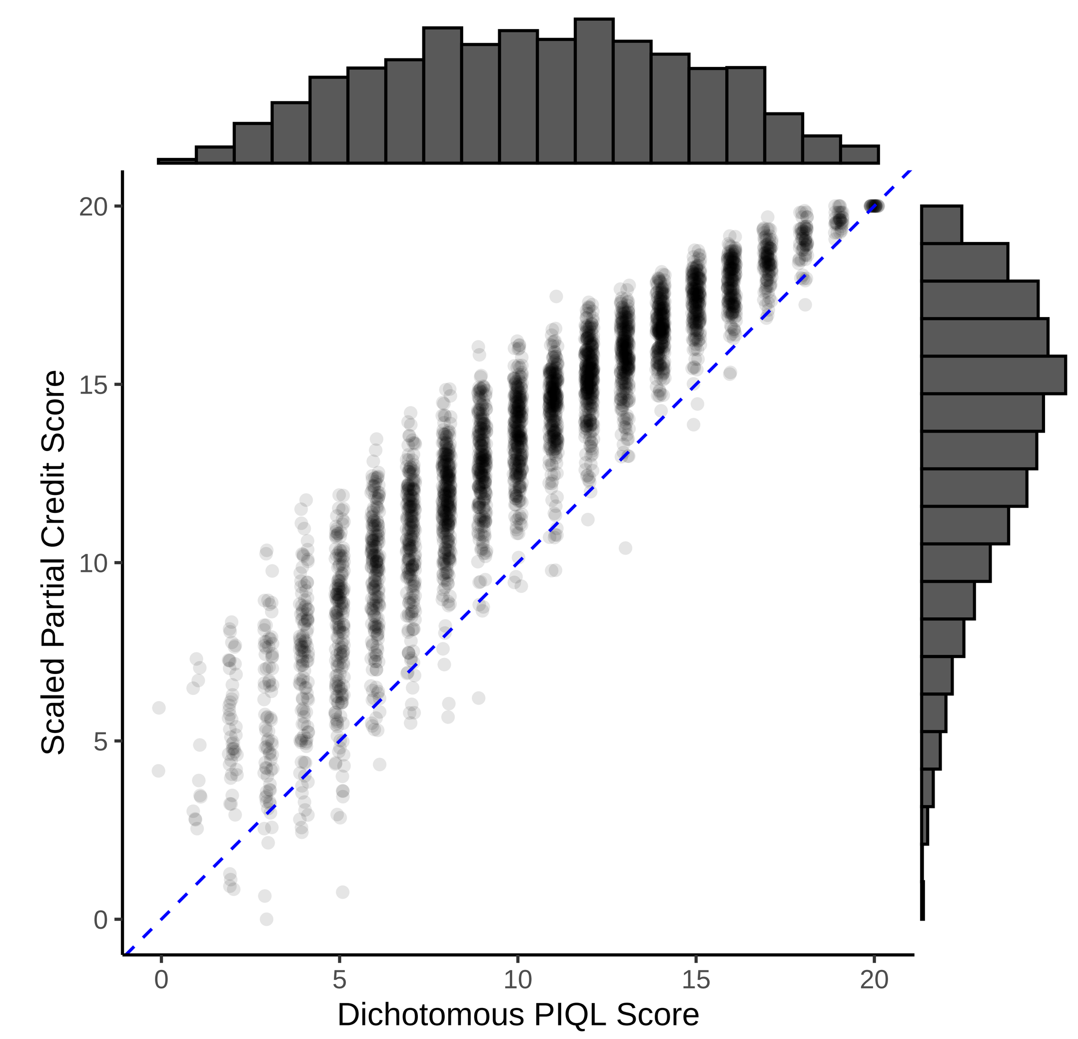


Figure 2. Partial credit scores have been scaled to have the same range as dichotomous scores: [0,20]. Points above the dashed blue line indicate higher partial credit scores, and vice versa. Histograms show the overall distribution for each score.

Conclusion

- The correct response combination for each item always has the highest IRT-defined score, even though the correct responses are not specified in the model.
- Correct individual responses have positive scores, and incorrect responses have negative scores.
- Total IRT-defined test scores strongly correlate with dichotomous test scores ($r \approx 0.8$). This supports the interpretation of IRT parameters as indicating levels of correctness and understanding.
- Some analyses of grouped responses yielded counterintuitive results: we identified one item for which certain incorrect responses corresponded to positive parameters.
- Our overall results indicate a strong potential for using IRT analyses to inform partial credit scores for MCMR items used to measure students' PQL; however, more consideration is needed to define the transformation from IRT parameters to partial credit scores.
- IRT parameters indicate how much knowledge or understanding is indicated by each response. Translating this into a partial credit score is not entirely straightforward.

Pre-Calculus Students' Mathematics and Art Cognitions in Making and Testing Conjectures About the Fibonacci Spiral and the Great Wave of Kanagawa

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Keywords: Cognition, Pirie-Kieren Model, Pre-Calculus

There are many forms of thinking and sense-making through artistic expression (e.g., painting, music, dance, and emotion). Similarly, there are many forms of thinking and sensemaking through mathematical expression (e.g., quantities, symbols, diagrams, logic). In mathematics, the Pirie-Kieren (P-K) framework has been useful for exploring the development of mathematical understanding (Pirie & Kieren, 1994). According to the P-K framework, a learner goes back and forth among eight cognitive layers when working on a mathematical task (primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, inventizing). No similar framework exists for the development of artistic understanding. Research has suggested that bringing arts into a mathematics classroom can be viewed as a rich way of engaging learners (Baird, 2015). A growing interest in cognitive development at the intersection of mathematics and arts offers an opportunity to explore how it might be modeled as a non-linear dynamical recursive process as in the P-K framework.

Thinking about art can inform/shape thinking about mathematics and vice versa. The poster presents research exploring the nature of pre-calculus students' mathematics and art cognitions in making and testing conjectures about recursive relationships. The work examines the potential of using the P-K model for documenting students' mathematics and art understanding and how this understanding develops while including attention to responses such as gesturing and body language. I developed and conducted interviews with four pairs of college precalculus students. The interview task used the Fibonacci spiral and an Edo-period woodblock print, the Great Wave by Hokusai, as spurs for student thinking and interaction about mathematics and art. Qualitative inductive analysis examined the audio- and video-recorded interviews using the P-K framework to identify and describe the mathematical and art-centered/visual thinking by students as they made sense of and responded to aspects of the task.

This poster will illustrate the findings on how the Pirie-Kieren framework is useful in unpacking the math-rich and diagram/visual-rich aspects of cognition by pre-calculus students. This poster aims to provide a space for RUME participants to share and discuss how art (song, painting, poem, etc) can support thinking about an idea/concept in mathematics. Additionally, it is a goal of this poster's graduate student author to talk with visitors about existing research about or near to mathematics and arts cognition, both to learn what they suggest as related readings and what more or different they would hope to see in a long report about this research.

The implications of the research include the value of (1) revisiting existing research at the intersection of art and mathematics with multiple P-K diagrams in hand (one for art, one for mathematics, and one for both) to make better/different sense of cognition involving both mathematics and art and (2) designing new curriculum and related research on how to combine art and mathematics that is better informed by the ways the combining may influence mathematical understanding.

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Pre-Calculus Students' Mathematics and Art Cognitions in Making and Testing Conjectures About the Fibonacci Spiral and the Great Wave of Kanagawa

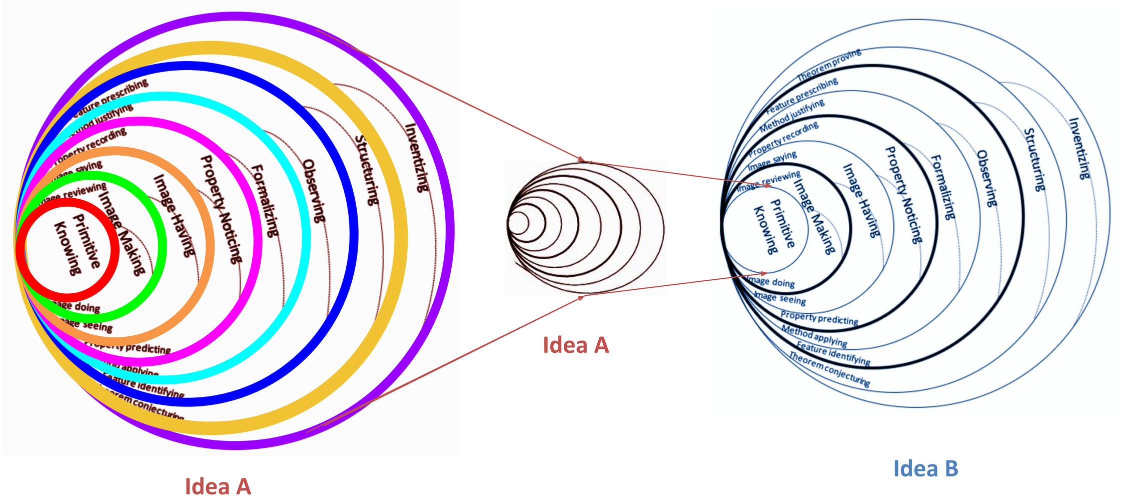
Tuto Lopez Gonzalez, San Francisco State University

Project Overview

The Pirie-Kieren (P&K) framework

The Pirie and Kieren (P&K) framework (Pirie & Kieren, 1989; 1994) provides a way to explore the development of mathematical understanding by suggesting that a learner move through eight cognitive levels of sophistication as they go through a learning situation.

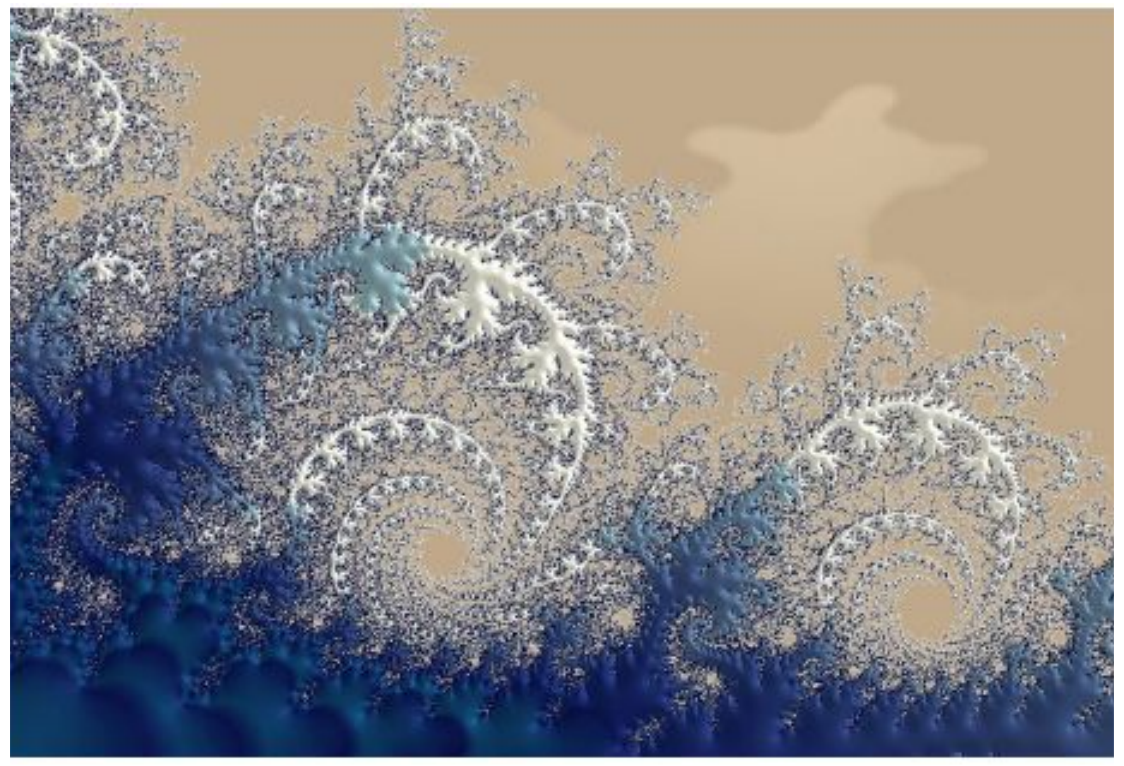
1. **Primitive knowing:** Learner's information brought to the learning situation.
2. **Image making:** The learner acts (mentally or physically) to gain an idea about a concept.
3. **Image having:** Single-activity associated images are replaced by a mental picture.
4. **Property noticing:** Notice properties and connections between mental pictures.
5. **Formalizing:** Description of class-like mental objects to produce full mathematical definitions.
6. **Observing:** Question formal statements verbalize cognitions about the formalized concept.
7. **Structuring:** Determine if formal observations are true by an axiomatic system (proof)
8. **Inventizing:** Complete understanding of a concept and pose questions that lead to new concepts



Self-similar nature - Encapsulated Idea A becomes Primitive Knowing for Idea B

Developing a framework for the development of artistic fulfillment/understanding

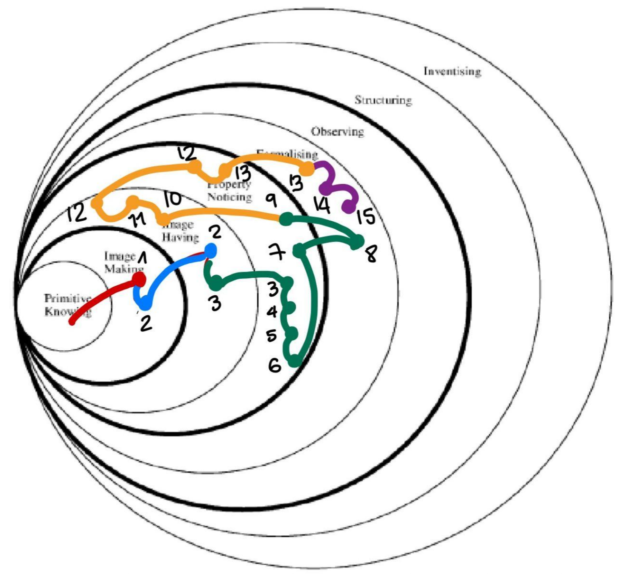
Research has suggested that bringing arts into a mathematics classroom can be viewed as a rich way of engaging learners (Baird, 2015). A growing interest in cognitive development at the intersection of mathematics and arts offers an opportunity to explore how it might be modeled as a non-linear dynamical recursive process as in the P&K framework.



I created an experience completed by 3 pairs of pre-calculus students. Each pair interacted with ***The Great Wave off Kanagawa (GWK)*** (1831) by artist Katsushika Hokusai. Participants were asked to analyze the GWK. Then, pairs completed a mathematical task with the **Fibonacci Spiral (FS)**. Finally they interacted with the GWK using the FS.

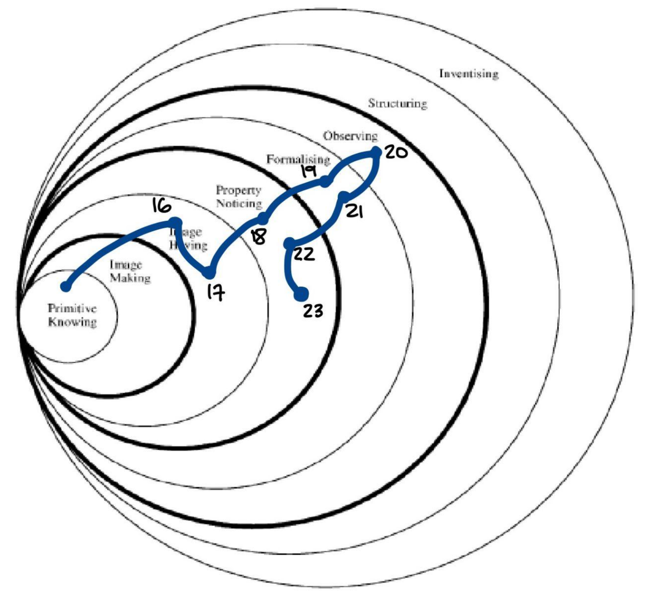
Maps of Development of Understanding

Part 1, Pair 1: Analyzing the GWK.



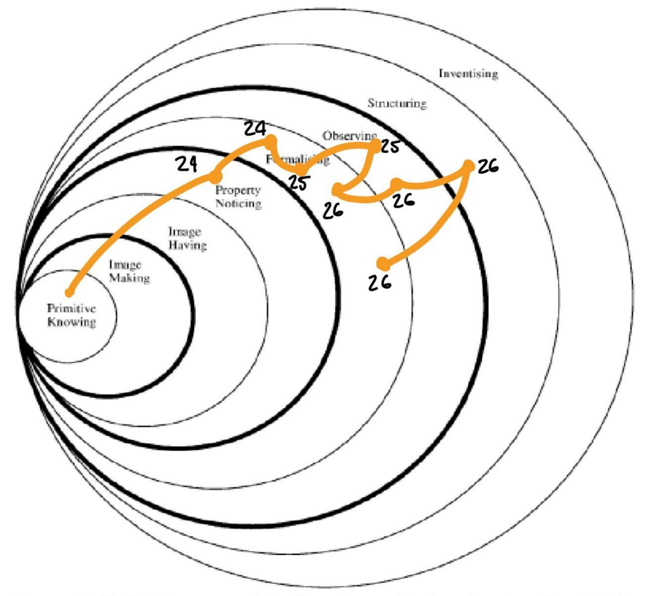
The P&K framework was extended to track the learner's development of fulfillment/understanding of the GWK.
(# indicates prompt of 27 questions total)

Part 2, Pair 1: Building the Fibonacci



The P&K framework for the development of mathematical understanding was used to draft the map on the second part of the experience
(# indicates prompt)

Part 3, Pair 1: Fitting the spiral



The P&K framework was extended to track the learner's development of fulfillment/understanding of both the GWK and the FS
(# indicates prompt)

Findings

In studying the nature of pre-calculus students' mathematics and art cognitions based on the experience, the following observations were done:

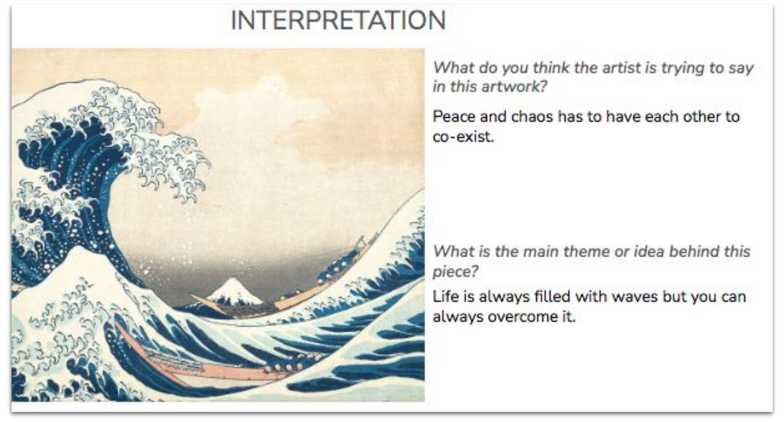
- P&K is useful for understanding mathematical cognition about the FS and documenting student's sense making.
- P&K is useful for exploring student's sense making and fulfillment of arts.
- A multi dimensional version of the P&K framework is proving useful in documenting how students use visual and mathematical information together to make sense of mathematical ideas related to the FS.
  - In analyzing art (Part 1) learners come up to the boundary (Meel, 2003) of formalizing.
  - When interacting with **both** the GWK and the FS (Part 3), learners crossed the boundary and entered mathematical formalizing.
- Preliminary analysis indicates gesturing when interacting with artwork is a form of embodied cognition that supports formalizing concepts beyond noticing.

Future Work

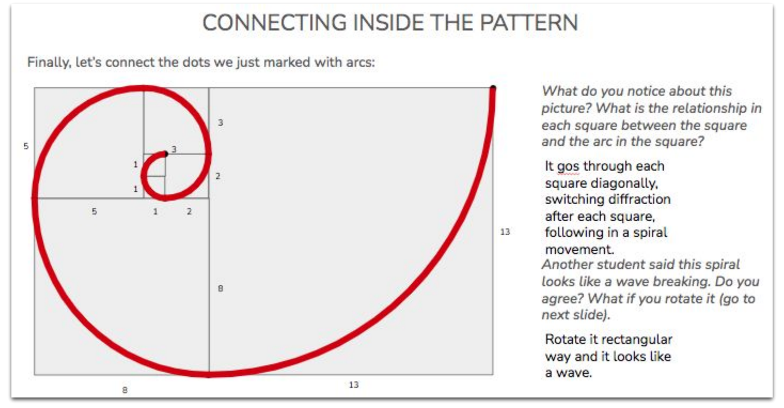
- Testing different Machine Learning algorithms such as decision trees, regressions and neural networks to help identify the cognitive level of sophistication a learner is in for a given mathematical task. Such algorithms can be useful for assessment, personalized learning, curriculum development, and future research.
- Extending the P&K map to a 3-dimensional map where the axis represent art fulfillment growing, mathematics understanding growing and time.

The Experience: Interacting with the Great Wave off Kanagawa

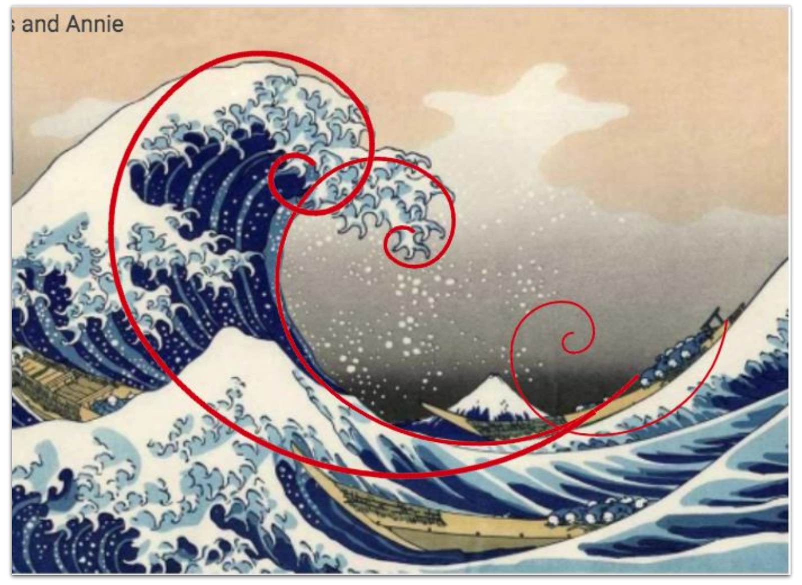
Part 1: Analyzing The GWK using the the 4-step art analysis framework i.e., **Description**, **Analysis**, **Interpretation**, and **Evaluation**.



Part 2: Building the Fibonacci Spiral by making and testing conjectures about its patterns.

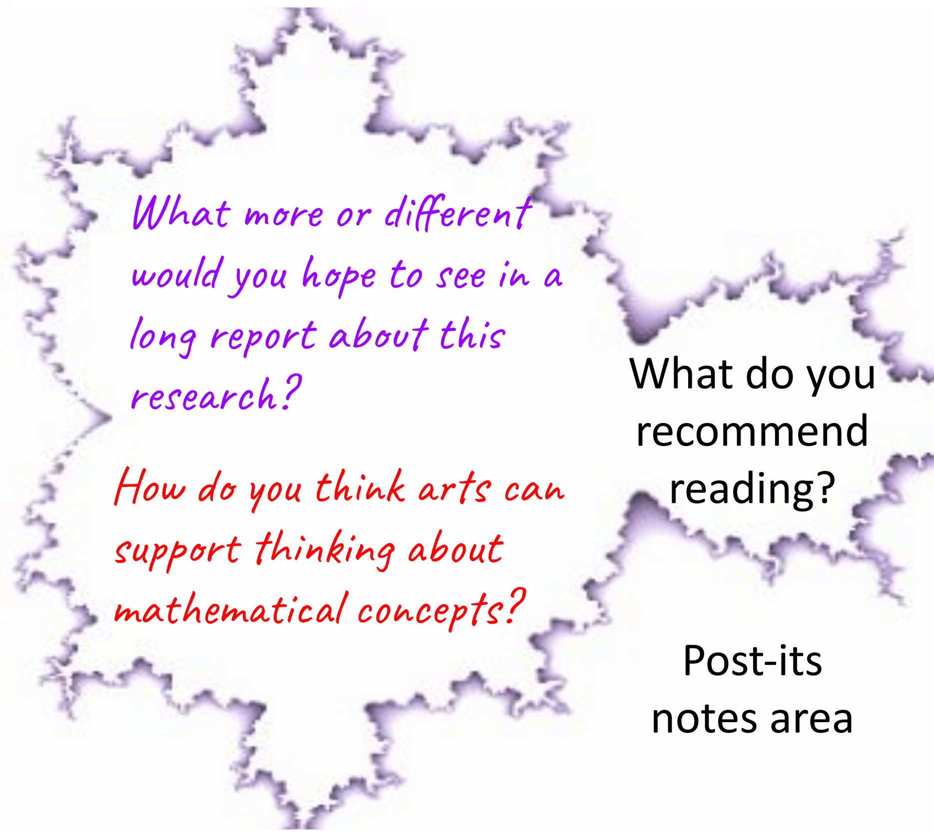


Part 3: Fitting the FS within the GWK and communicating art and math with math key words



Sharing

Let's talk arts and math in research :)



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Engaging an Industrial Advisory Board for Curriculum Development

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Rikki Wagstrom, Metropolitan State University
Wei Wei, Metropolitan State University

Keywords: Data Science, Industrial and Applied Mathematics, Curriculum Development, Industrial Advisory Board, Technical and Non-technical Skills

With rapid changes in STEM careers, gaps still exist between skills required by industry and learning objectives created by colleges and universities. Even well-designed curriculum must be constantly reevaluated and adjusted. Data Science and Industrial & Applied Mathematics are fast-evolving programs whose graduates are highly sought by industry. Prior research has provided valuable general insight into skills valued by industry and pedagogy effective for teaching these skills (Ferns et al., 2021; Jang, 2016; Lamancusa et al., 2008; Schwab-McCoy et al., 2021; SIAM and COMAP, 2016).

In this poster, we will describe how we engaged our Industrial Advisory Board to provide external input for our newly developed Data Science and Industrial & Applied Mathematics programs. We will then show how we have applied that feedback and made changes to our curriculum and pedagogies. Our Industrial Advisory Board consists of nine volunteer members representing multi-national companies with headquarters or having a large presence in the local metropolitan area. This includes several Fortune 500 companies. Specifically, Industrial Advisory Board members were queried for their open-ended responses on topics such as: What do you look for when you hire? What is the appropriate balance between developing technical and nontechnical skills, in terms of preparing students for careers in industry? What computer applications and programming languages are beneficial to integrate into an undergraduate mathematics and statistics curriculum? For which career paths?

There was strong agreement among our board members on the importance of the following technical skills: Multiple linear regression, logistic regression, machine learning techniques, model formulation and model building (choosing the appropriate model and communicating why a specific model is chosen) and being able to interpret and explain results. Familiarity with an object-oriented programming language is typically required for all industries represented by the board members. Most importantly, students need to have the ability and drive to learn different applications as they will likely have to learn multiple new programming languages, modeling techniques and business practices throughout their careers.

The primary focus of non-technical skills is on communication. Specifically, our board members stressed the importance of students learning to communicate technical details to both technical and non-technical audiences. Another essential non-technical skill is the ability to work in teams and have project experience.

We describe how the recommendations have been incorporated into the Data Science and Industrial & Applied Mathematics programs. Initial feedback from students and the Industrial Advisory Board members to both programs has been positive. In particular, the students like working on the open-ended problems and enjoy working on projects in teams. We identify areas we would like to further emphasize or add to our curricula. They include additional model formulation activities, more PowerPoint presentations (emphasis on summarization of results), code documentation, asking questions of clients and importance of continuous education and learning of new skills. The Advisory Board has also provided guidance on several career-readiness items such as resumes, networking and interviewing which we are expanding on.

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Engaging an Industrial Advisory Board for Curriculum Development

David Jacobson, Rikki Wagstrom, Wei Wei

Metropolitan State University, St. Paul, MN

Introduction

- **About Metro State University:**
  - Mathematics and Statistics Department historically offered an Applied Mathematics BS degree
  - Students are frequently location-bound adult learners
  - Financial factors frequently require students to seek employment in the corporate sector after graduation
- **Key theme of 2017 survey of Applied Mathematics alumni working in the corporate sector:**
  - Mathematical skills are valuable, but
  - Increased background in statistics and computing would have improved marketability and preparation
- **Actions taken by Mathematics and Statistics Department:**
  - Creation of a Data Science BS and Industrial & Applied Mathematics BS
  - Creation of Industrial Advisory Board to provide curricular guidance and learning experiences for the two new programs

Industrial Advisory Board Members

- **Role of Industry Advisory Boards**
  - Assist university departments in ensuring that academic programs prepare students with the skills and capacities currently valued and expected by industry.
  - Provide external input on additional ways to enhance the student learning experience, such as providing data sets, guest lectures for courses, and real time experiences such as internships, capstone design projects, and mentoring opportunities
  - Serve as external reviewers to evaluate and adjust specific curriculum

- **Our industrial Advisory Board consists of nine volunteer members representing multi-national companies with headquarters or having a large presence in the Twin Cities metropolitan area.**

  - Best Buy
  - Target (2 members)
  - Traveler's Insurance
  - General Mills
  - US Bank
  - SPS Commerce
  - Cargill
  - CH Robinson

- **Board members were queried for their open-ended responses on several topics.**

Industrial Advisory Board Survey Questions and Summary of Responses

Question	Responses
What do you look for when you hire?	• Soft skills: Strong communication skills, effectively work in teams, time/project management skills • Programming experience: Python, R, Pearl, Java, SQL • Ability to learn; interest in learning; ambition • Experience working on projects: Want them to be able to effectively communicate their train of thought as they discuss prior work that they've done • Problem solving: Perseverance, willingness and ability to ask questions • Work with vendors effectively • Understanding of models: Being able to choose appropriate models and communicate rationale, understand advantages and disadvantages of different models
What is the appropriate balance between developing technical and nontechnical skills, in terms of preparing students for careers in industry?	• Individuals need both • Communicating technical details to technical and non-technical audiences is difficult but essential • Being able to make a clean, concise PowerPoint presentation is essential • Need to be interested in taking ownership for projects and have the drive to do it as well as possible • It's important to be able to ask (the right) questions • Problem solving strategies are important: (One board member asks the following of her employees: If a problem is not clearly stated, what could you do? What are some other strategies you can try?)
What computer applications and programming languages are beneficial to integrate into an undergraduate mathematics and statistics curriculum?	• Python and R: Most widely used; May or may not be a deal-breaker for hiring • SQL: Not a deal-breaker for hiring, but definitely helpful for database queries • Java: Helpful, but not essential • SAS and Excel VBA: Used at one company • Excel: Familiarity with spreadsheets is essential; VBA is helpful, but not essential • MATLAB: Not essential, used primarily in engineering applications • In general, familiarity with an object-oriented programming language is important • Data visualization software (Tableau, Power BI): Used at several of the companies represented, but not essential • Amazon Web Service (AWS): Helpful • If employees have the ability and drive to learn different applications, they can generally pick up what they need to know

Industrial Advisory Board Recommendations

- **Technical Skills**
  - Academic background must include multiple linear regression – what it is and when it works well
  - Knowledge of logistic regression and machine learning techniques are plusses
  - Model formulation is critical
  - Training versus validation versus test data
  - Be able to understand and translate business problems into a math / stat problem
  - Interpret and explain results
- **Non-technical Skills**
  - Communicating technical details to technical and non-technical audiences
  - Ability to work in teams and have project experience
  - Time/project management skills
  - Ability to lean new things (programming languages, modeling techniques, and business practices)
- **Overall**
  - Include project-based or inquiry-based classes
  - Open-ended problems which require practical skills, especially formulating and selecting models
  - Present results of projects in non-technical industry terms
  - Embed mini-projects into the curriculum
  - Emphasize code documentation

Implemented Recommendations

- **Industrial & Applied Mathematics**
  - Four-course sequence in modeling focused on formulating models and problem-solving
  - Incorporation of a team-based investigation of a large open-ended real-world problem
  - Instruction in technical communication
  - Communication of analyses, results, and recommendations to both technical and non-technical audiences
  - Completion of a statistics or mathematics internship, a Statistical Consulting course or a Mathematics Capstone
- **Data Science**
  - Project-based interdisciplinary focus with courses in Math/Stat, Computer Science, and Management Information Systems
  - Programming courses in R, Python, Java, SQL, and Tableau
  - Integrative experience through a statistics or data science internship, Statistical Consulting or a Data Science Capstone
- **Both Industrial & Applied Math and Data Science**
  - Regression Analysis course with a focus on model formulation
  - Career Center presentation on career opportunities
  - Courses with team- based projects with data analysis in R and presentations to both technical and non-technical audiences
  - Creation of student portfolios

Future Work

Initial feedback has been positive. Students like working on open-ended problems and team-based projects.

Areas of further curriculum emphasis:

- Model formulation activities
- More PowerPoint presentations with an emphasis on summarization of results
- Code documentation
- Asking pertinent questions of clients
- Importance of continuous education and learning of new skills

Work with the Industrial Advisory Board on expanding career-readiness guidance:

- Resume construction
- Importance of networking
- Job interviewing skills

Regular surveys of recent alumni for ongoing evaluation

Novice College Mathematics Instructor Communication About Cognitive Demand

Helena Almassy
San Francisco State University

Keywords: cognitive demand, scaffolding, novice instructors

Many incoming graduate students become novice instructors with little to no experience teaching at the college level. Such novices may take part in professional learning opportunities from general workshops on teaching to specific work to prepare for teaching with a particular curriculum (e.g., course coordination). One aspect of preparing to teach is developing skill in supporting students to "successfully engage with a mathematical task" (Stein et al., 2009). To do so, novice instructors must consider the context of the student learning experience as well as the level of cognitive demand for the mathematical task (Smith & Stein, 1998). A curriculum may have a wealth of mathematical tasks available to assign to students, requiring varying levels of cognitive demand (e.g., recalling and applying memorized facts, using procedures with and without connection to other knowledge, and the most cognitively demanding work of doing mathematics in a novel situation). Though research exists on cognitive demand and the preparation of teachers to attend to it for K-12 mathematics, little research has focused on the needs of novice college mathematics instructors. In this poster, the focus is on addressing the research question: How are novice instructors' instructional decisions informed and shaped by consideration of cognitive demand?

Previous research on cognitive demand in the preparation of novice instructors focused on how they interpreted and transformed the definitions of "cognitive demand" and "scaffolding" of student learning (Milbourne, 2018). The poster builds on this, focusing on how novice instructors make sense of and communicate with colleagues about cognitive demand in the context of a weekly teaching workshop. The data collected included nine novice instructors' responses within asynchronous online activities and live audio and video of three consecutive workshop discussions about cognitive demand in various college mathematics classroom examples (e.g., video case materials, Hauk et al., 2013). The inductive qualitative analysis uses van de Pol's (2010) framework for scaffolding, involving contingency, fading, and transfer of responsibility. Initial analysis indicates novice instructors described various "benefits" from lowering cognitive demand that expand on the collection reported by Milbourne. As in previous work, novices talked about "getting through more material". New "benefits" voiced by novices in this study included improving student confidence and self-efficacy. The new findings are related to consideration of the student's perspective.

The goal at the poster is to engage RUME conference participants in discussing novice instructor decisions on how to set cognitive demand goals and why they might set those goals. Poster-side conversation can inform the ultimate reporting of study findings. Results of the research can be used in (re)development of professional learning opportunities offered to novice college mathematics instructors and can shape future research on how novices improve their instructional skills in task-level decision-making.

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Novice College Mathematics Instructor Communication About Cognitive Demand

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Poster Purpose: To collect ideas while discussing novice instructor decisions on how to set cognitive demand goals and why they might set those goals.

Context

Project Objective:

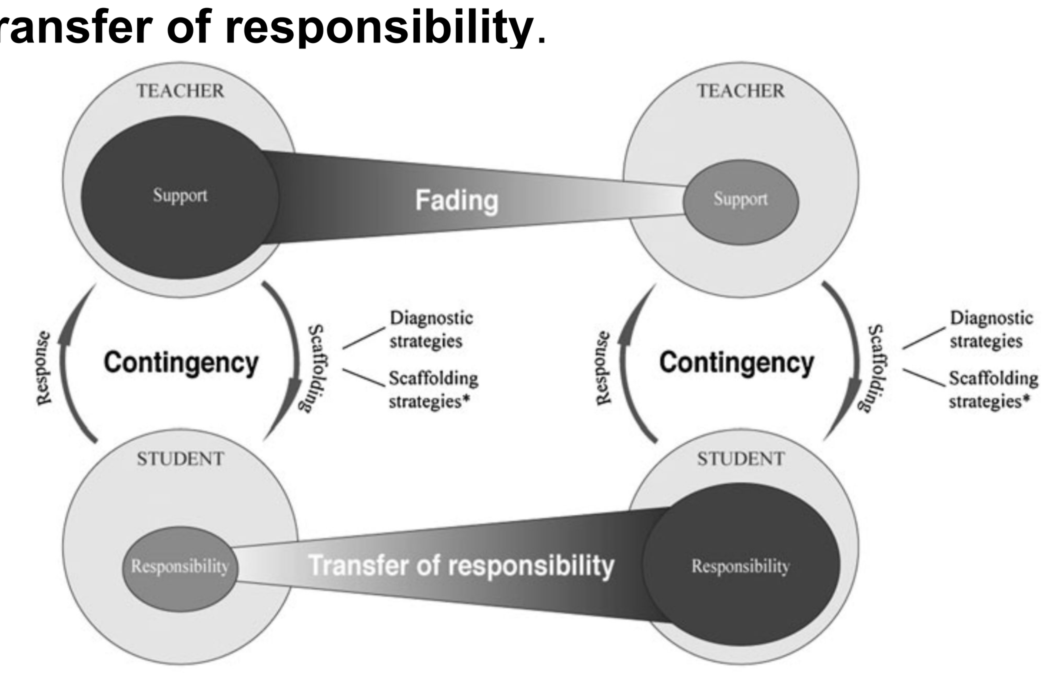
Understanding how are novice instructors' instructional decisions informed and shaped by consideration of cognitive demand.

Background:

Many incoming graduate students become novice instructors with little to no experience teaching at the college level. Such novices may take part in professional learning opportunities from general workshops on teaching to specific work to prepare for teaching with a particular curriculum (e.g., course coordination). One aspect of preparing to teach is developing skill in supporting students in successfully engaging with mathematical tasks. To do so, novice instructors must consider the context of the student learning experience as well as the level of cognitive demand for the mathematical task (Smith & Stein, 1998). Though research exists on cognitive demand and the preparation of teachers to attend to it for K-12 mathematics, little research has focused on the needs of novice college mathematics instructors.

Framework:

This project uses van de Pol's (2010) framework for scaffolding student learning. The characteristics of scaffolding included **contingency**, **fading** and **transfer of responsibility**.



Conceptual Model of Scaffolding from van de Pol (2010)

Research Questions

1) In the context of a particular classroom activity and associated assessment, what is the nature of GTA communication about and use of cognitive demand?

2) How are curriculum and assessment decisions for scaffolding student learning informed/shaped by GTA consideration of cognitive demand?

Mini-Course Embedded within a Teaching Workshop

Setting: Nine graduate students pursuing masters' in mathematics enrolled in a 3-credit Graduate Teaching Workshop. Each was a new graduate teaching associate (GTA) – instructor of record for undergraduate math courses for the first time. A 2-week section of the workshop was dedicated to noticing and responding to cognitive demand.

Asynchronous Activities:
GTAs did a mini-lesson that was organized as exploration, discussion boards, and self-quiz in classifying cognitive demand.

Synchronous Class:
Included small and whole group discussion of portions of the mini-course and readings related to student group work and classroom management techniques.

Levels of Cognitive Demand

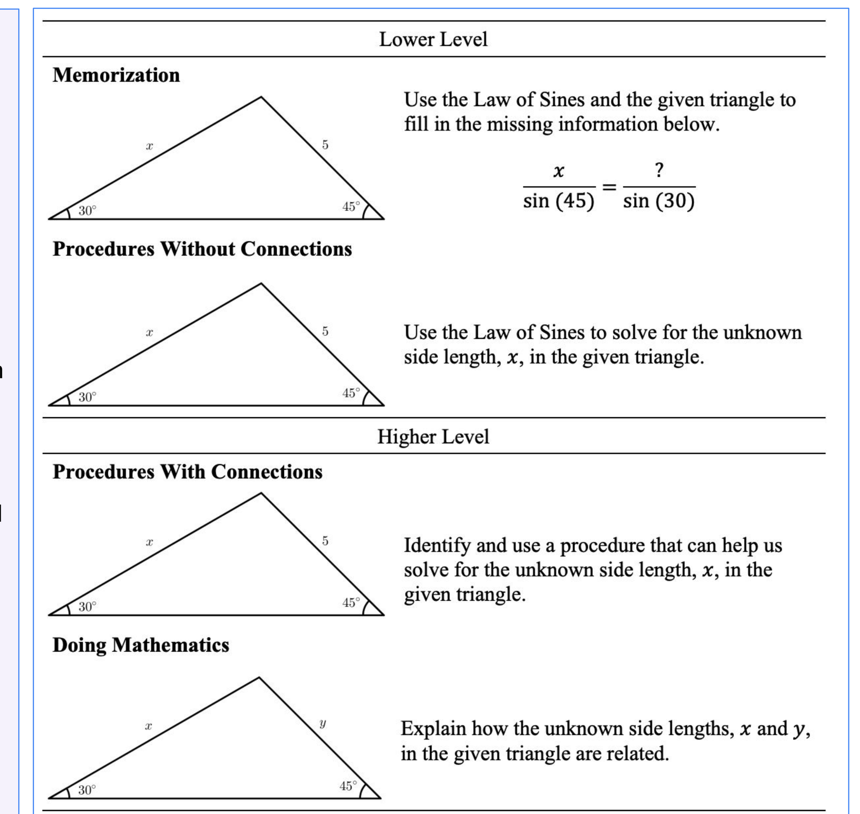
Lower-level demands *(memorization)*
- Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules formulas or definitions to memory
- Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure
- Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated

Lower-level demands *(procedures without connections)*
- Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task.
- Little ambiguity exists about what needs to be done and how to do it.
- Have no connection to the concepts or meaning that underlie procedures being used
- Are focused on producing correct answers instead of developing understanding
- Require no explanations or explanations only describing the procedure that was used

Higher-level demands *(procedures with connections)*
- Focus student's attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas
- Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts
- Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations.
- Although general procedures may be followed, they cannot be followed mindlessly

Higher-level demands *(doing mathematics)*
- Require complex and nonalgorithmic thinking – no predictable, well-rehearsed approach or pathway is explicitly suggested by the task, task instructions, or a worked-out example.
- Require students to explore and understand the nature of mathematical concepts, processes, or relationships
- Demand self-monitoring or self-regulation of one's own cognitive processes
- Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task
- Required students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions
- Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required

Based on Smith and Stein (1998)

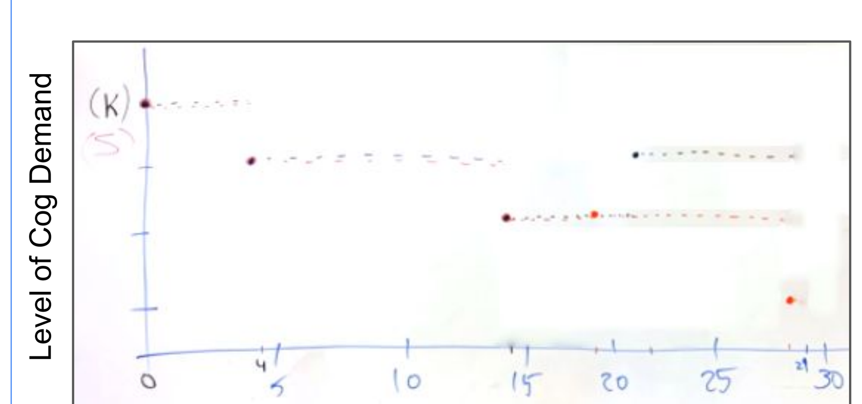


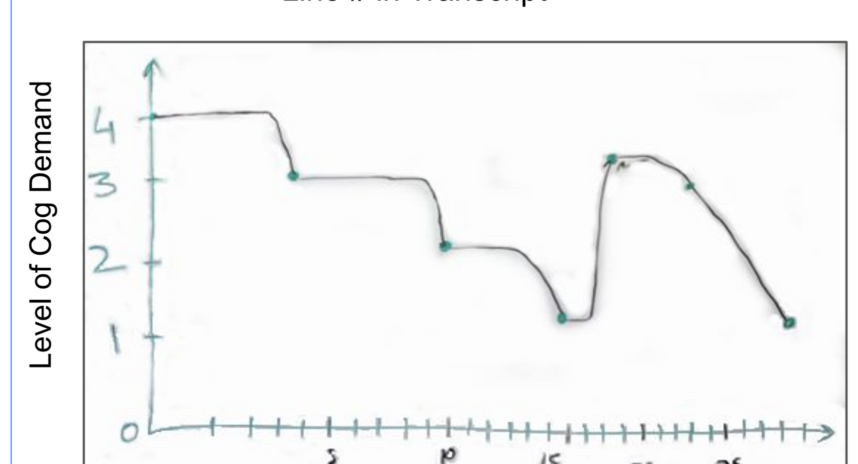
Data Collection:
Audio and video was recorded over three consecutive days during the synchronous workshop (~30 min recording each period). Participants completed portions of the asynchronous work in the previous week and some concurrently.

Results

Tracking Cognitive Demand

Given a conversation among Kristen, Brad, and Ted, (video & 29 lines of transcript), plot the cognitive demand across time/lines of transcript. GTAs agreed the interaction started at 4 (Doing Math) and ended at 1 (Memorization). But plots differed.





Later, individual GTAs described their decision-making. That communication highlighted their sense-making in trying to understand "level" of cognitive demand.

Changes in Cognitive Demand

I also think in the parts when the **cognitive demand __keeps constant__** it is because the question the instructor was proposing **gave the student the opportunity to continue thinking** and using their knowledge to try to solve the problem.

She does say "cancel" and it is **telling a student exactly what to do.** That is the exact point where the **__cognitive demand drops__**.

Brad answered by memory (he knows the rule). Then … the **__[cognitive demand] level increased__** because Ted found that it doesn't match up, they found a problem and **now they need to make some connections** to solve it.

Emerging Awareness of Cognitive Demand

"I was thinking that each of those is sort of mapped to one assignment like, one assignment is the level of doing math, or one assignment is the level of memorization. But then … **two students could look at the same problem** and be at **__different levels__** [of cognitive demand]. "

Personally, I believe that the learning shift should be continuous and that **there exist __sub-levels in between__ the 4 main cognitive demand levels** that were indicated because even in those 4 basic levels, there are already some **overlapping characteristics.**

Discussion

GTAs described various "benefits" from lowering cognitive demand that expand on the collection reported by Milbourne. New "benefits" voiced by novices in this study included **improving student confidence and self-efficacy**. The new findings are related to **consideration of the student's perspective**.

GTAs described practices that they thought would maintain cognitive demand of *Procedures With Connections.* Some would, some might not:

GTA suggestion	*# of GTAs*	*Likely Cognitive Demand Result*
__Ask the student:__ *What do we know about sides and angles?*	4	**Maintain** cognitive demand
__Direct the student:__ *Think about concepts related to trigonometry.*	2	**Reduce** cognitive demand
__Redirect the student:__ *Look at your notes and see what might be a useful procedure.*	1	**Maintain** cognitive demand

Posterside Conversation

- How might novice instructors set cognitive demand goals?
- Why would the particular cognitive demand goal be set by a novice instructor?
- How and why might the goals set by novice instructors about cognitive demand differ from experienced instructors?
- Other questions? Comments?

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A Calculus Student's Relative Size Reasoning

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Keywords: relative size reasoning, relative size, precalculus, rational functions

The ideas of measurement and measurement comparisons (e.g., fractions, ratios, quotient) are introduced to students in elementary school and understanding these ideas as relative size comparisons (Faulkner, 2013, Joshua et al., 2022) are a part of the Common Core State Standards for Mathematics. Researchers have also reported how conceptual (Thompson et al.,1994) meanings for ideas in calculus such as the difference quotient (Byerley & Thompson, 2014), derivative (Byerley et al., 2012; Zandieh, 2000), and the fundamental theorem of calculus (Thompson,1994) involve a multiplicative comparison of two quantities.

Someone who engages in relative size reasoning would conceptualize comparing two quantities' values in a static or dynamic context as measuring one quantity's magnitude in units of the other quantity's magnitude (Lock, 2021). For instance, suppose a student engaged in relative size reasoning to compare two girls' heights. In this context, both quantities are of the same unit of measurement. Thus, the student, when comparing Lyndsey's height to Darien's height, would imagine Darien's height as the unit of measure used to determine how many times as tall Lyndsey's height is in relation to Darien's height. Therefore, she might say Lyndsey is 5/4 times as tall as Darien.

When the two quantities' values are not measured using the same unit, engaging in relative size reasoning will involve imagining the measure of one quantity's value in multiples of the other quantity's value. For example, speed is a multiplicative relationship between an amount of distance and an amount of time. If someone is traveling at a constant speed and travels 8 feet per 3 seconds, one would imagine that the number of feet one travels will always be 8/3rd times as large as the number of seconds it takes to travel that distance. Relative size reasoning is a way of thinking that a person has developed when conceptualizing the comparison of two quantities' values multiplicatively in static and dynamic contexts (Lock, 2021). The purpose of this poster investigates the questions:

What mental actions do beginning calculus students engage in when responding to tasks designed to assess students' relative size reasoning?

Radical constructivism (Glasersfeld, 1995) is the lens used in this study which takes on the perspective that every individual has their own reality and knowledge is constructed based on previous personal experiences. For this study I conducted 4 clinical interviews (Clements, 2000) where 5 calculus students from a large southwestern university worked on 22 tasks associated with different topics (fractions, rational functions, concavity, trigonometry, etc.) that were designed to elicit students' relative size reasoning. In this poster, I characterize a student's thinking by determining what meanings the student had in the moment while working through two tasks. Future research will aim refine my current framework that describes the mental operations involved in relative size reasoning as well as refine my current multiple-choice assessment based on my analysis of my data.

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A Calculus Student’s Relative Size Reasoning

Kayla Lock

Under the advisement of Dr. Marilyn Carlson

Motivation and Research Question

- Common Core State Standards for Mathematics asks that students understand fractions as comparisons of relative size (Faulker, 2013, Joshua et al., 2022).
- This way of thinking supports students in understanding proportional relationships, rates, growth patterns, and derivatives (Byerley 2019; Joshua et al., 2022; Thompson and Thompson 1996, Zandieh 2000).
- Researchers have documented that students in university mathematics courses have difficulties comparing two *quantities* in terms of their *relative size* (e.g. Byerley, 2019; Tallman, 2015).

Research Questions:

- *What mental actions do beginning calculus students engage in when responding to tasks designed to assess students’ relative size reasoning?*

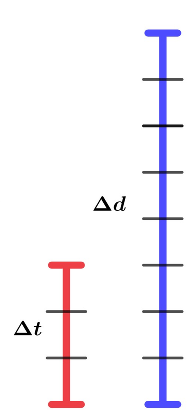
Relative Size Reasoning

- Someone who engages in ***relative size reasoning*** would conceptualize comparing two quantities’ values in a static or dynamic context as measuring one quantity’s magnitude in units of the other quantity’s magnitude.
- When the two quantities’ values are not of the same type of unit of measure (e.g., speed-relationship between an amount of distance and an amount of time), engaging in relative size reasoning will involve imagining the measure of one quantity’s value in multiples of the other quantity’s value (Lock, 2021).
- Relative size reasoning is a way of thinking (Thompson et al. 2014) that a person has Developed when conceptualizing the comparison of two quantities’ values multiplicatively.

Example: Constant speed of 8 feet per 3 seconds

the change in the number of feet one travels will *always* be $\frac{8}{3}$rd times as large as the number of seconds it takes to travel that distance.

$\frac{\Delta d}{\Delta t} = \frac{8}{3}$



constant speed of $\frac{8}{3}$ feet per second

Theoretical Perspective & Methodology

- **Radical constructivism** (Glasersfeld, 1995) is the lens used in this study which takes on the perspective that knowledge is constructed based on one’s previous personal experiences, and knowledge lies in the mind of the individual.
- ***Quantitative reasoning*** is the analysis of a situation into a network of quantities (measurable attributes of objects) and quantitative relationships (Thompson, 1994, 2012).
- ***Covariational reasoning*** refers to the mental actions involved in coordinating the value of two varying quantities and how the values vary in relation to each other (Carlson Jacobs, Coe, Larson, & Hsu, 2002; Saldanha & Thompson, 1998; Thompson & Carlson, 2017)
- This work draws on theories of students’ understanding in the moment and communicating mathematics:
  - Students explained their meanings in the moment for each response they had to the questions. Each question posed was used as a point to characterize the students’ meanings (Thompson, 2013).
- I report on 2 tasks from conducting five 1.5hr long clinical interviews (Clements, 2000) with 1 calculus student from a large southwestern university worked on 22 tasks from different topics (measurement, exponential growth, rational functions, and concavity) designed to elicit students’ relative size reasoning.

Results

Task 1: Paul is Skiing

Suppose Paul is skiing on a circular trail and he skied 27 miles. If one blip is 14 miles and the entire trail is 9 blips, what angle measure (in degrees) did Paul sweep out as he skied 27 miles on the circular trail?

After reading the problem Gabriel stated, **“we know there are 9 blips so that would be split into 9 pieces and each curvature would be 14 miles”** and then explained that since Paul skied 27 miles, **“that would be like almost two full blips.”** In this excerpt below, Gabriel’s statements and drawings suggest that he coordinated the 3 units of measure: blips, miles, and the circumference of the circular trail.

Excerpt 1

Gabriel: Since the, since the trail is circular and he is going through the, the trail, like in that circle, when you draw out the circle, you could kind of see that the, he is going to like traveling through or by the circumference of the circle.

Interviewer: So would you relate your idea of circumference in this question with what you said earlier about how he skied almost “2 full” <blips>.

Gabriel: Well the almost two full parts of like the split I did of the circle. Since we know **that the trail is made up of 9, equal blips, and each blip is 14 miles, we could kind of split the whole trail into 9 different equal pieces. And since we know that each blip is 14 miles, we know that each piece is also going to equal 14 miles. So when it comes to like something like 27, It's not quite two full pieces since two full pieces is 28**, but it's pretty close.

Interviewer: So how does that relate to what you talked about with circumference?

Gabriel: Since you can also split the circumference into 9 equal pieces and **since he is traveling, on the circumference, then the measure of the circumference is also equal to the 9 blips in total.**

Interviewer: So how would you compare the amount that Paul swept out in comparison to what you just talked about with circumference?

Gabriel: Oh, it was about it was about two ninths of the whole circumference.

Interviewer: How did you think about two and 9 and like, what is that two divided by 9 represent?

Gabriel: **The 9 represents the entirety of the circumference since it is split into 9 equal pieces. And the two is how much of those 9 pieces he was able to actually like get through.**

Gabriel then explains how Paul skiing 2/9 of the circumference is related to the angle measure in degrees that Paul swept out while skiing 27 miles (Excerpt 2)

Excerpt 2

Interviewer: What if we change this question to degrees (instead of measured in radians) how would you imagine going about that?

Gabriel: **We divide the 360 by 9 in order to get what the measurement of one of those slices is, and then we would be able to kind of find what the angle is of like the, the 27 or at least close to that.**

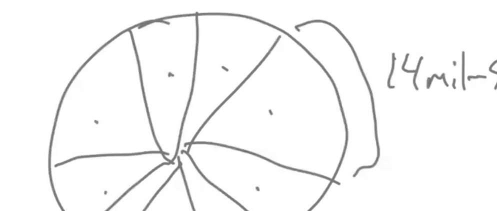
Interviewer: So would you be able to go through that process and kind of walk me through that?

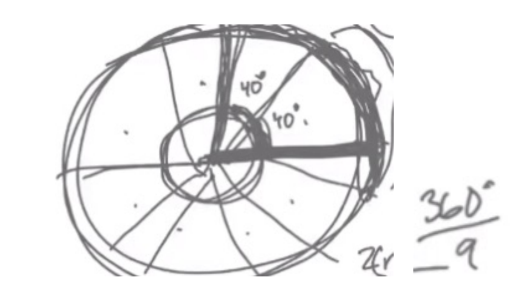
Gabriel: **So 360 divided by 9 is 40, which means that each angle is about 40 degrees. And since it takes up two of those, this angle right here, or 27 takes up about 80 degrees.**

Interviewer: How did you know it was about 80 degrees?

Gabriel: Oh, because of dividing the whole, uh, the whole circle. **Since the circle is going to be 360 degrees, no matter what if I divided that by 9, I got 40 degrees, which meant that every section of this would be also 40. Since 27 takes up about two of those sections, then I'd just add another 40 to that and it would be 80.**

This statement indicates that Gabriel leveraged his understanding of the length of the trail, which is 9 blips, and a full rotation in degree, which he stated is 360 degrees, in order to partition a full rotation into 9 parts and determine the angle measure swept out for each blip. Then he leveraged his thinking about skiing 2 chunks of the circumference to determine Paul swept out 80degrees as he skied 27 miles





Task 2: Gatorade Problem

5gallons of liquid flavoring are poured into a vat. Water will be added to the vat and mixed with the flavoring to produce a drink that will be bottled and sold. The ratio of flavoring to water in the mixture to the total volume of the mixture, x, can be defined as $f(x) = \frac{5}{x-5}$. As the total number of gallons of liquid increases, explain how the value of f varies.

Initially Gabriel read the problem aloud and substituted different values for x as he attempted to make sense of what the rational function conveyed about the context. As an example, Gabriel followed by explaining what the rational function represents when 5 is substituted for 3 (Excerpt 3).

Excerpt 3

Gabriel: So **the number of gallons has to be over 5 in order to produce anything because if it’s 5 or lower, it would just make either zero or a negative number for the denominator, which would mean that there is no like mixture at all.**

Interviewer: What do you mean by that?

Gabriel: There will be less, right? **Because the denominator is the thing that is increasing, and the numerator seems to be constant at a constant state of five.** Let's say we have 6 minus 5 down here. This is the variable. We would have 5 over 1, so that would be equal to 5. If we have like, let's say like 100 down here, minus 5, that would be like 95, which is a number that's a lot smaller.

Interviewer: So, in the first calculation you did where you did 6 minus 5, …what does that 5 represent and what does the 6 represent?

Gabriel: Oh, **The five represents five gallons of liquid flavoring are poured into a vat.** It doesn't really say how many units of the mixture there is. It just kinda says total volume of mixture. **I'm assuming it's gallons so I would assume that this (6) is like how many gallons of the mixture in total there are.** How many gallons of well, like ratio of mixture there is when six gallons of the liquid is put into the mixture.

This suggests Gabriel understood that the function represents the relationship between the total volume of the mixture and the liquid flavoring. In the next excerpt, Gabriel discussed the variation of the function f in the problem context.

Excerpt 4

Interviewer: So here as the number of gallons of liquid varies from 10 to 15, how does the comparison of flavoring to water vary?

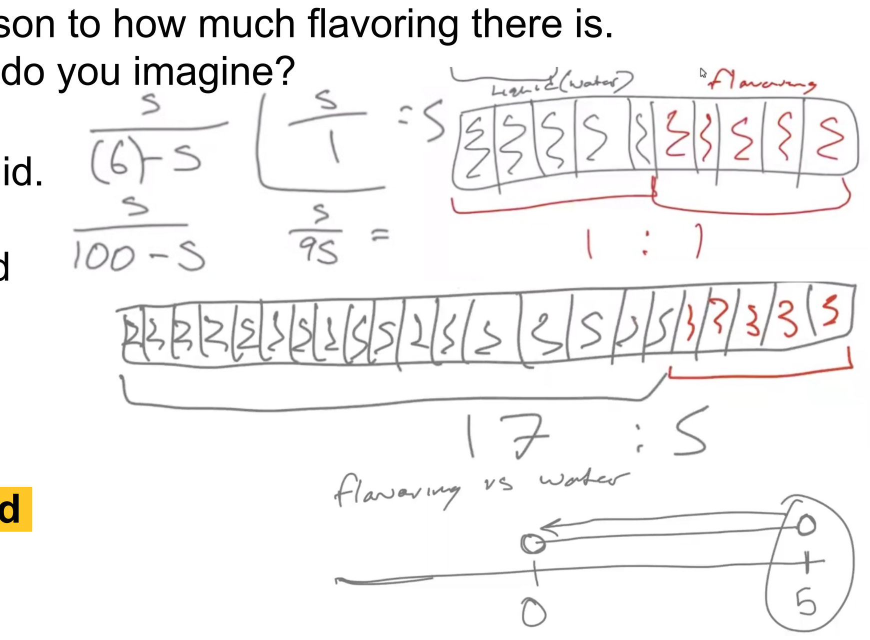
Gabriel: So it's kinda like the same thing like from over here where **the flavoring is overpowered by the amount of liquid that there is. Since the amount of flavoring isn't really changed, it remains at a constant five. If you keep on adding liquid without adding any more of the, the flavoring, then eventually there will be more liquid than the flavoring instead of having like a good, like equal ratio or a constant ratio between the two.**

Interviewer: So, what do you mean by a ratio?

Gabriel: So how much liquid(water) there is in comparison to how much flavoring there is.

Interviewer: When you say comparison and ratio what do you imagine?

Gabriel: So if there's five mixture or, or five parts of the mixture or of the flavoring, then there's five of the liquid. **So as the amount of liquid increases,** the flavoring doesn't also increase meaning that eventually instead of it being like a equal one to one, **the ratio isn't a one-to-one ratio. It's this, 17 to 5, which is what I meant earlier by the liquid overpowering the flavoring. The amount or the proper ratio of the liquid to flavoring decreases… like the more liquid that's added, the more of a smaller fraction the whole equation becomes.**



Conclusion

- This interview suggests that Gabriel’s fraction schemes include to partition a quantity into equal parts, and iterating it repeatedly based on the reference length he chooses.
  - This way of thinking appeared to support him in coordinating several units at the same time in task 1 in order to determine the angle measure Paul swept out as he skied 27miles.
- Gabriel’s language and drawings suggest he reasoned similarly while considering the ratio of flavoring to water as the number of gallons of water increased without bound. Gabriel coordinated the varying number of gallons of total volume, water, and a constant amount of liquid flavoring, while determining the relative size of flavoring to water.
- I argue that Gabriel was engaging in relative size reasoning in task 1 (static context) as well as a in task 2 (dynamic context).

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NOTE: More citations available upon request

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Future Research

- If he continues to reason similarly in other tasks and if other students reason similarly.
- What is the role of relative size reasoning in understanding key ideas of precalculus and calculus.

How Groupwork Influences Students' Sense of Belonging in The Mathematics Classroom

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Motivation and Background

Current STEM education research has shown that a low sense of belonging in STEM leads to lower academic achievement (Stachl & Baranger, 2020). There have been several studies that have explored how shifting to an inquiry-based learning model helps increase student understanding (Connors, 1990; Coopers et al., 2015; Studies et al., 2015), but few have gone into whether this change also leads to an increased sense of student belonging (De Corte et al., 2008).

> Belonging
> *the extent to which the undergraduate precalculus student believes that they are accepted and included as a valuable part of the of the learning community*
> adapted from (Stachl & Baranger, 2020)

The word to stress here is **value.** The idea of value is what belonging is framed as in this study; if someone can express that they add value, it gives insight into the fact that they feel they belong. Prior studies (Stachl & Baranger, 2020) have characterized belonging through perceived peer and faculty support, class comfort in sharing ideas, social connectedness, and empathy from instructors.

The goal of this study is to begin to explore the link between groupwork and a sense of belonging.

In what ways do students report an increased sense of belonging in a precalculus course with a groupwork-based learning norm?

Setting

The University of Maine Precalculus course utilizes the *Pathways to Calculus* curriculum from Arizona State University. This curriculum focuses on introducing function types through the lens of covariation: how one quantity varies with another quantity. Forgoing a traditional textbook, students complete a workbook of problems to learn key concepts in the course.

Precalculus meets five days a week for the whole semester. In Fall 2022, the three, 75-student student sections met three days per week in active learning classrooms on "lecture" days. These rooms (example at right) help to facilitate groupwork and establish it as a class norm. The other two days of class meet in smaller breakout groups (recitations) lead by teaching assistants. All class meetings also were aided by learning assistants, undergraduates who had previously taken the course.

On "lecture" days, the instructor does very little lecturing in the traditional sense. Students are tasked with working in their groups to solve the problems assigned that often allow them to produce the mathematical idea themselves. The whiteboards around the room provide a space for students to write out their work and solutions. When in the room, you can often see other groups checking their work with that produced by other groups. Most "lecture" days often have at least one cycle of the instructor introducing a problem context, students working in groups and displaying their solutions, and a subsequent whole-class discussion of the results.

Methods

Survey

A pre- and post-survey was administered to all three lectures. The survey was given to collect initial and final student opinions concerning belongingness and mathematics. It was also utilized to source students for further interviews (below). Because of low response rates in the post-survey, only pre-survey data is shown in the results (right).

Interview

Interviews (between 20 and 45 minutes) were conducted with four students to gain more insight concerning their experience in the class.

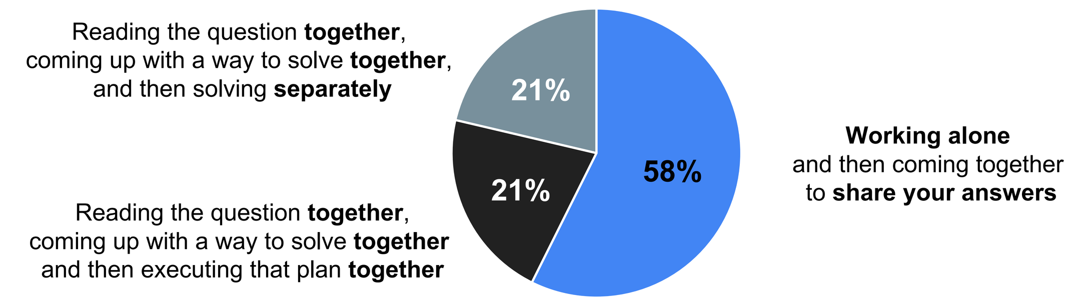
Example questions:
- What has your learning experience been in this class so far?
- What value do you feel you get out of groupwork?
- How comfortable do you feel sharing with your groupmates?
- How valuable is your input to the groupwork taking place?
- How do you feel about math and groupwork?

Demographics:
- 3 students solicited from survey responses, 1 volunteered
- 2 male, 2 female
- 1 student retaking the course
- 1 non-traditional student (outside 18–22-year-old demographic)

Results

Pre-survey results showed most students' conceptions of groupwork entailed only sharing answers. Less than a quarter of students believed groupwork meant working together during the entire problem solving process. (n = 136)

Which of the following best describes how you work on math in groups?



In the pre-survey, interviewed Students 2, 3, and 4 all expressed the view of working alone and only sharing answers. Student 1 expressed the view of preparing together and working separately. Thus the interviews showed growth in their belongingness, because all students reported working together to execute plans of action.

- All four students valued their participation in groupwork, increasing belonging.
- Students 1, 2, and 3 expressed evidence of emotional support (Student 1 example) as well as content support (Student 2 example) in increasing belonging.
- Only Student 3 expressed evidence of reinterpreting the social norms of groupwork through their own social norms as a sign of belonging.

Examples of Types of Belonging Evidence — Student Quotes and Analysis

"I like the like group dynamic and all. And it makes it a little bit **less scary** to like ask questions or give like an opinion, because you're not like sitting there in like a quiet room, with like all eyes on you. You just have the like five or six other people at your table"

"Just if someone gets it, they can explain it [to the group] in a way that the teacher hadn't. Um, so that's always **really valuable**. and then just like **emotional support in general**"

Student 1 – Evidence of Emotional Support

- Lessens the pressure of sharing out
  The norm of sharing with their individual group allows the student to feel less intimidated about discussing their own thinking.
- General emotional support
  The student can lean on their group members to get a new explanation and the moral support they need to get through the problem.

The presence of emotional support implies a sense of belonging.
Being able to emotionally lean on groupmates is evidence that they feel like they belong.

"I don't have to strain myself so hard on, what honestly is, a like simple aspect I am getting hung up on…. Uh, when I can discuss that with my group, and their like 'oh this.' I'm like 'ohhhh, okay' and then we continue on and if they hit a roadblock maybe I'm farther along then them and its that. So, its kinda like **distributed thinking**, but we are all still getting something out of it."

"Um, and there have been plenty of times where we have come to the wrong consensus and our group's…reasoning was wrong. **But we were wrong confidently together.**"

Student 2 – Evidence of Content Support, Distributed Thinking

- Remembering prior concepts and fixing errors
  Completing problems together allows the student to rely on their groupmates to help them remember the 'basic' math concepts they may have forgotten. They can utilize their group members to help them catch algebraic errors so the group can focus on new concepts.
- Distributed thinking
  Mixing of ideas allows everyone to get something out of group discussion, with each member brining something to the work.
  Talking through solutions is beneficial for everyone, no matter the math background they possess.
- Group confidence
  Groupwork gives them a sense of confidence when sharing, even if it is false confidence.

The presence of content support implies a sense of belonging.
Being able to be confident together about the mathematical content means one must feel a sense of belonging within the group.

"For like chem, where [teacher] has like the whole structured lecture set up, I think it is still slightly important, but its more important in a like, in like that detail moves the story along kind of."

"We made…a little **Snapchat group chat**. And if you are struggling with one of the questions on the homework… you can like… send a picture and its like, um, someone will usually get back to you and be like 'oh this is how I did it' [and] kinda explain it a little bit, or show what they did to do it or get it correct and stuff….. I have tried, I made group chats in like **other classes that haven't really gone as well**. Like everyone usually forgets about them and all. Like I tried making like a chem chat, like a chem group chat with some people and I think like no one has ever really used it"

"As a person with like, [mental heath issues], speaking in front of the whole class…. can make me pretty anxious… But like I'm just sitting with like 4, like really nice girls… like asking questions isn't as anxious, it takes some of the pressure and anxiety off."

Student 3 – Evidence of Reinterpreting Social Norms of Groupwork

- Comparison of groupwork norms to other courses
  This student contrasted their experience in precalculus with another course they were taking concurrently (chemistry). What they shared and how they shared impacted their learning of the material between the two courses.
- Transcending classroom social norms
  Continuing the groupwork-based norms outside of class time through the Snapchat group chat allowed the student to get or provide help anytime. This shows a reinterpretation of the classroom norms of groupwork through the students' own social norms. The student unsuccessfully attempted to carry this over to another class where there is not a groupwork-based learning norm.
- Reinterpreting social norms as emotional and content support
  The group chat provided both emotional support through the relief of anxiety, and content support through students helping each other with content.

The presence of reinterpreting classroom social norms through everyday social norms (establishing an active group chat) implies a sense of belonging.
Being able to take the class-based social norms around groupwork and reinterpret them through the students' own social norms is evidence that they feel like they belong.

"[I am] **still not very [comfortable sharing]**, I'm not the most open person most of the time, so I think like I said earlier the **layout of the room really helps** me to be a little bit more open about it"

"When I'm right, its really valuable, but I kinda said it again earlier, [only] **if I have something of value** I will share it"

Student 4 – Evidence of Value of Participation

- Importance of setting for groupwork
  Being placed at tables where students are forced to look at seatmates enabled this person to share when they felt comfortable. They embraced groupwork partly because of the room they were in. This student did not enjoy the course but was still able to get something out of it

This student did share, but only when they had something they felt was valuable.
If this student had an increased sense of belonging, they would feel comfortable sharing even when they weren't sure it was the correct answer.

Conclusions

Preliminary Results

The initial finding for the survey and four unique case studies show that, by the end of the course, they felt a sense of belonging. The students interviewed were able to talk about multiple ways they **added value to the class** or to the group they worked with. Majority of the interviewees also stressed how much groupwork was utilized to further their understanding of the course material.

Further Research

More data is currently being collected (Spring 2023), as the class is offered every semester. The initial results provide a strong case for continued data collection, as the preliminary data shows evidence of increased belongingness on the class and individual levels.
- Are there other impacts on belongingness from groupwork that have yet to be observed?
- What are the mechanisms that cause in-class groupwork norms to be reinterpreted by students into their own social norms of belongingness?
- What differentiates students for whom groupwork creates an increased sense of belongingness, and for those where groupwork doesn't increase belongingness?

Belongingness in mathematics has not been widely explored in undergraduate mathematics education, but its potential effects on success and retention deserve further study.

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Challenges and Resources for Linguistically Diverse Students in Inquiry-Based Undergraduate Mathematics Curricula: The Case of Inquiry-Oriented Linear Algebra (IOLA) Tasks

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Keywords: Inquiry-Based Mathematics Education; Linear Algebra; Equity and Diversity

While there is consensus that inquiry-oriented (IO) instruction is a better alternative overall to lecturing (e.g., Laursen & Rasmussen, 2019), IO approaches may not be equitable for students from certain demographics (e.g., gender inequities in Reinholz et al., 2022). This analysis explores a question related to equity considering linguistic diversity: What linguistic challenges and resources do linguistically diverse undergraduate students experience in IOLA tasks?

Grounded in a sociocultural theory (e.g., Moschkovich, 2015), I constructed a framework that captures linguistic challenges and resources along three interrelated dimensions: (a) lexical – words and phrases in a task (e.g., Spanos et al., 1988), (b) situational – material, activity, semiotic, and sociocultural aspects of a task (e.g., Gee & Green, 1998), and (c) normative – social and sociomathematical norms engendered across tasks (e.g., Yackel et al., 2000).

The data analyzed were one-on-one semi-structured interviews (each lasting about 30-60 minutes) with 5 male linguistically diverse students from one IOLA course. The 5 students spanned a diverse range of cultural backgrounds (Korean, Vietnamese, Malaysian, Hispanic, and White) and comfort levels with English. In the interviews, each student was asked to arrange the first 12 mathematical tasks from their class from most to least challenging linguistically and to explain their ranking of each task. The tasks came from an expanded version of Wawro et al.'s (2013) IOLA curriculum. Each interview was analyzed task by task using the constant comparative method (Miles et al., 2020), allowing for both a priori codes and emergent codes.

Below are some of the main linguistic challenges and resources that students reported:

1. *Lexical:* Challenges included unfamiliar/unintended meanings of words in problem contexts, unfamiliar/unintended meanings of vector notation, and new mathematical terms; Resources included familiar words as well as translations and informal definitions of unfamiliar words and notation.
2. *Situational*: Challenges included high numbers of words and sub-questions, the exclusive use of English, the formality of definitions, symbolizing problem contexts, and algorithmatizing; Resources included low numbers of words and sub-questions, students' primary language(s), translation tools, visuals, and symbolic mathematical language that prompted specific procedures.
3. *Normative*: Challenges included violated expectations about the mathematical complexity level, a single "right" way to solve a task, and the prescription of generalized procedures.

In the presentation, further details will be provided about these findings, which suggest the need for pedagogical linguistic supports in task design and implementation that address and leverage the identified challenges and resources.

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Linear Algebra Proof Constructions & Tall's Worlds of Mathematics

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Keywords: proofs, linear algebra, Tall's worlds of mathematics

Proof is central to mathematics, but proofs are notoriously difficult (Alcock & Weber, 2010; Britton & Henderson, 2009; Caglayan, 2018; Dorier et al., 2000; Epp, 2003; Moore, 1994; Stavrou, 2014; Weber, 2001). Generic example proofs are thought to be effective ways of developing and understanding proofs because they reveal the structure of a proof while providing the prover something concrete to focus on (Leron & Zaslavsky, 2013; Rowland, 2002). When linear algebra students need help writing a proof, they perform better on subsequent tasks if given a generic example proof rather than a formal proof (Malek & Movshovitz-Hadar, 2011). Researchers also recommend emphasizing geometry to develop students' understandings (Dogan, 2018; Ertekin et al., 2010; Hannah et al., 2013; Stewart & Thomas, 2010).

Through this study, I aim to add to the knowledge of linear algebra students' ways of thinking and working with proof constructions by examining the tools and strategies (e.g., generic examples) students use when encouraged to think about linear algebra concepts in different ways (e.g., geometrically). Tall (2013) describes mathematical thinking as occurring in three worlds. The embodied world consists of objects, graphs, diagrams, and their properties. The symbolic world contains symbols, operations, formulas, and calculations. The formal world consists of formal definitions, axioms, and proofs made up of logical deductions from axioms and definitions. My research question was: *What tools and strategies do students choose to use when working on proofs in the embodied, symbolic, or formal world of mathematics?*

To answer this, I conducted a multi-case study with four students enrolled in linear algebra: Rap, Advik, Jun, and Min (pseudonyms). In task-based interviews, I encouraged participants to use a given world of mathematics as they worked on proving or disproving a given statement. I used open coding (Corbin & Strauss, 1990; Strauss & Corbin, 1990) to detail the mathematical tools and strategies students used, and I coded the worlds of mathematics they used. To ensure that my coding was systematic and rigorous, I continually updated a master list of codes, keeping notes on the individual codes and how they were applied. I also performed member checks (Creswell & Miller, 2000; Merriam & Tisdell, 2016).

Comparing tools and strategies across constructions, some were used regardless of the world I encouraged participants to use. For example, Jun used cases and examples. Other strategies and tools were only used when I prompted participants to use a specific world. For example, Rap attempted to work directly from the hypotheses to the conclusion when creating symbolic proofs, but he worked backwards from the conclusion when using formal logic.

All participants used generic examples when asked to use the embodied world and chose other strategies when asked to use the symbolic or formal worlds (with a few exceptions). One interesting task was to prove or disprove the statement: *Let* $S = \{U_1, U_2, \ldots, U_p\}$ *be a set of two or more vectors. If* S *is linearly dependent, then there is a subset of* S *that is linearly independent and has the same span as* S*.* In their embodied generic examples, Rap and Advik began with linearly independent vectors and considered other vectors afterwards. However, their symbolic proofs began with a set of dependent vectors, and they removed vectors until they reached an independent set. This shows that one potential strength of the embodied world is to prompt students to create generic examples and to find different ways of structuring an argument.

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Linear Algebra Proof Constructions & Tall's Worlds of Mathematics

Kelsey Walters

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Introduction

Proof is central to mathematics, but proofs are notoriously difficult.[1-8] Generic example proofs are thought to be effective ways of developing and understanding proofs because they reveal the structure of a proof while providing the prover something concrete to focus on.[9-10] When linear algebra students need help writing a proof, they perform better on subsequent tasks if given a generic example proof rather than a formal proof.[11] Researchers also recommend emphasizing geometry to develop students' understandings.[12-15] Through this study, I aim to add to the knowledge of linear algebra students' ways of thinking and working with proof constructions by examining the tools and strategies (e.g., generic examples) students use when encouraged to think about linear algebra concepts in different ways (e.g., geometrically).

My research question was: **What tools and strategies do students choose to use when working on proofs in the embodied, symbolic, or formal world of mathematics?**

Methods

I conducted a multi-case study with four students enrolled in linear algebra: Rap, Advik, Jun, and Min. In task-based interviews, I asked participants to use a given world of mathematics as they worked on proving or disproving a given statement.

I used open coding[16-17] to detail the mathematical tools and strategies students used, and I coded the worlds of mathematics they used. To ensure that my coding was systematic and rigorous, I continually updated a master list of codes, keeping notes on the individual codes and how they were applied. I regularly discussed codes with a senior researcher. I also performed member checks.[18-19]

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Tall's Worlds

Tall describes mathematical thinking as occurring in three worlds.[20] The embodied world consists of objects, graphs, diagrams, and their properties. The symbolic world contains symbols, operations, formulas, and calculations. The formal world consists of formal definitions, axioms, and proofs made up of logical deductions from axioms and definitions. Table 1 gives examples of linear algebra concepts in the three worlds.

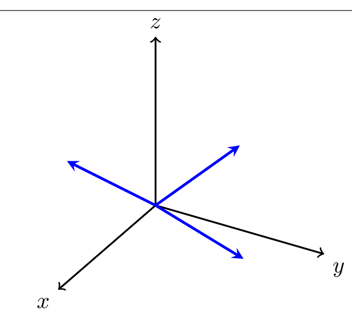
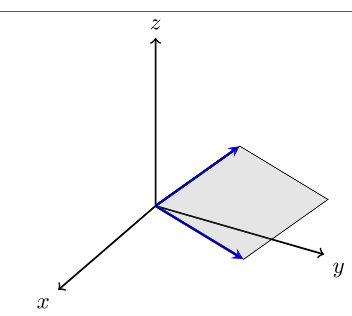
	Embodied World	Symbolic World	Formal World
Linear Independence		$a\begin{bmatrix}1\\0\\0\end{bmatrix}+b\begin{bmatrix}1\\1\\0\end{bmatrix}=0$ $a+b=0,\ b=0$ $a=0,\ b=0$	$c_1v_1+c_2v_2+\ldots+c_nv_n=0$ implies $c_1=c_2=\ldots=c_n=0$
Span		$c_1v_1+c_2v_2+\ldots+c_nv_n$ is in span$\{v_1,v_2,\ldots,v_n\}$	span$\{v_1,v_2,\ldots,v_n\}$ is the set of all linear combinations of $v_1,v_2,\ldots,v_n$

Table 1: Linear Algebra Concepts in the Three Worlds of Mathematics

Findings

Comparing tools and strategies across constructions, some were used regardless of the world I asked participants to use. For example, Jun used cases and examples. Other strategies and tools were only used when I prompted participants to use a specific world. For example, Rap attempted to work directly from the hypotheses to the conclusion when creating symbolic proofs, but he worked backwards from the conclusion when using formal logic.

All participants used generic examples when asked to use the embodied world and chose other strategies when asked to use the symbolic or formal worlds (with a few exceptions). In their work on the task in Table 2, Rap and Advik began with linearly independent vectors and considered other vectors afterwards when creating embodied generic examples. However, their symbolic proofs began with a set of dependent vectors, and they removed vectors until they reached an independent set. Table 2 shows parts of Rap's symbolic and embodied work.

Prove or disprove the following statement.
Let $S=\{U_1,\ U_2,\ldots,U_p\}$ be a set of two or more vectors. If S is linearly dependent, then there is a subset of S that is linearly independent and has the same span as S.

Rap's Symbolic Proof

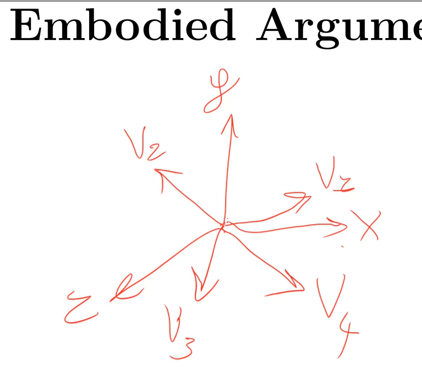
$$x_1U_1+x_2U_2\ldots x_pU_p=0$$

$$x_k\neq 0$$

$$U_k=\frac{x_1U_1\ldots+x_pU_p}{-x_k}$$

"which means that the other $n-1$ vectors is a subset of S, which may or may not be linearly independent. But it has the same span"

Rap's Embodied Argument



"you have V_1 and V_2. V_3 is, in this case, for now, it's linearly independent. Then V_1, V_2, V_3 span the entire space. $\ldots V_4$ is a part of that space."

Table 2: Portions of Rap's Symbolic and Embodied Work

Discussion & Further Research

Participants used generic examples when asked to use the embodied world, whereas other tools and strategies were used regardless of the world I asked participants to use. Further research could determine why students choose particular strategies and how educators can best support their use of various strategies across the mathematical worlds.

One potential strength of using the worlds of mathematics in the classroom is to aid students in thinking about different ways of structuring an argument. Participants also said the three-world framework was helpful for organizing their thoughts.

Through this study, I became interested in how students structure arguments when working in different worlds of mathematics. At times, Rap and Advik structured their arguments differently when reasoning in different worlds. I am interested in studying this further and, if this is a common phenomenon, finding ways to support students' use of various argument structures to develop student understanding of concepts and proofs.

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Using Refutation to Address a Graphical Misconception in Calculus

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Background

Refutation

- Refutation texts have been shown to repair misconceptions held by students in science contexts.[6]
- Students have shown greater gains in multimedia learning environments when misconceptions are addressed.[5]
- Students prefer expository refutation over expository text and view refutation as more efficient.[3]
- There is less research reflecting the use of refutation in mathematics contexts and there is little, if any, research exploring refutation with the use of graphical representations.

Connecting *f* and *f'*

- Students have difficulty coordinating multiple graph features at once (e.g., how can a graph be decreasing and concave up?)[1]
- Students also struggle to interpret the derivative of a function when it is presented graphically.[4]
- Sketching the graph of *f* based on a graph of *f'* involves coordinating the properties of each and representing that graphically.

Research Questions

1. What is the effect of refutation videos and graphs on student performance?
2. Is there potential value in using refutation in a mathematics education setting?

Design & Procedure

Pilot Study

Pre-test → Expository Video n=96 → Post-test

Experiment

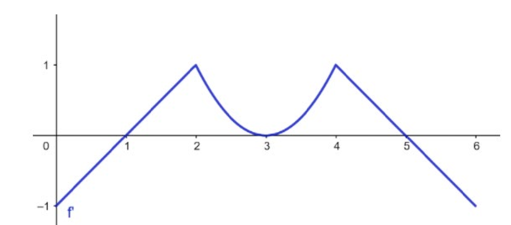
Pre-test → Refutation (Target n=50) / Expository (Target n=50) → Post-test → Spatial Reasoning Test

Materials

Pretest: 5 items
- 2 true/false
- 3 multiple choice

Example item:

The graph of f', the derivative of f, is shown to the right. Decide if the following statement is true or false:
The graph of f is increasing for 4 < x <5
a) True
b) False

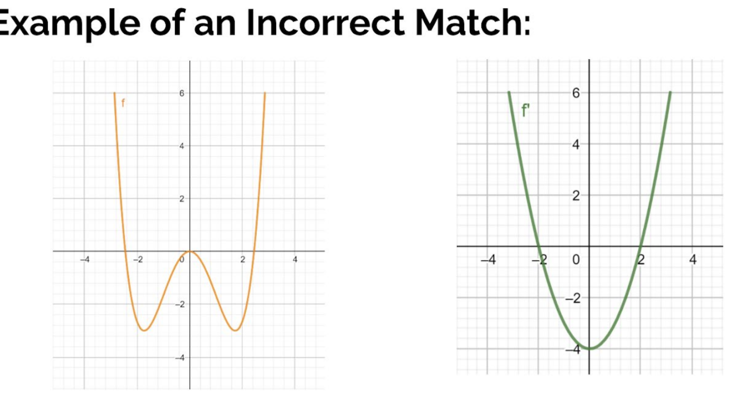


Refutation

Some students think that the behavior of ' matches the behavior of f. That is
- When f is positive, f' is positive
- When f is negative, f' is negative
- When f is 0, f' is 0

For example, students might match the following graphs of a function f and its derivative f'…

Example of an Incorrect Match:



Notice that the graph on the right matches the behavior of the graph on the left. However, the idea that the behavior f matches the behavior of f' is **not true!**

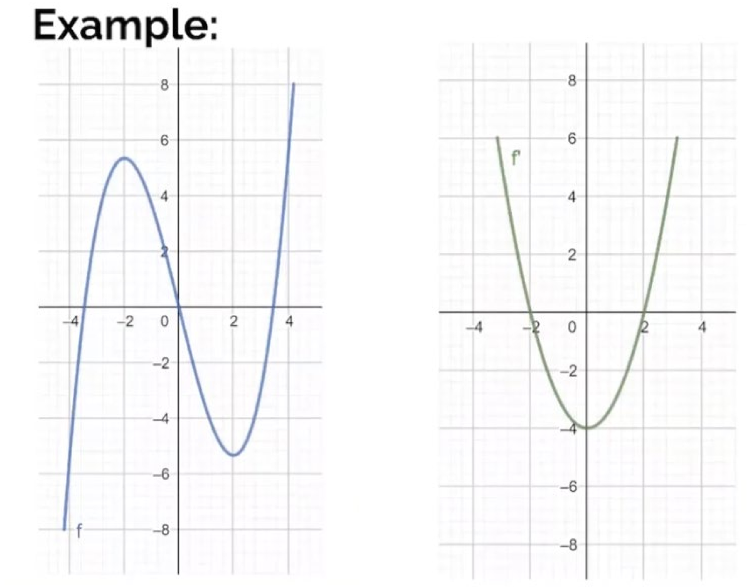
Exposition
(see right)

Exposition

Since the derivative value represents the slop of the tangent line the relationship is characterized as follows:
- When f is **increasing** the slope is positive, so f' is **positive**
- When f is **decreasing** the slope is negative, so f' is **negative**
- When f attains a maxima or minima, the slope is flat so f' is **0**

Behavior of f		Behavior of f'
$f(x)$	Is increasing	$f'(x) > 0$
$f(x)$	Is decreasing	$f'(x) < 0$
$f(x)$	Attains a local Maxima/minima	$f'(x) = 0$

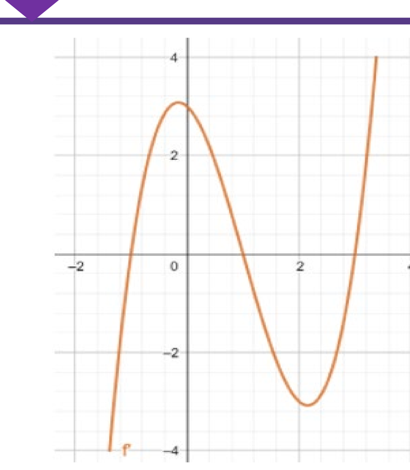
Example:



Post-test: 11 items
- 4 true/false
- 6 multiple choice
- 1 free response

Example item:

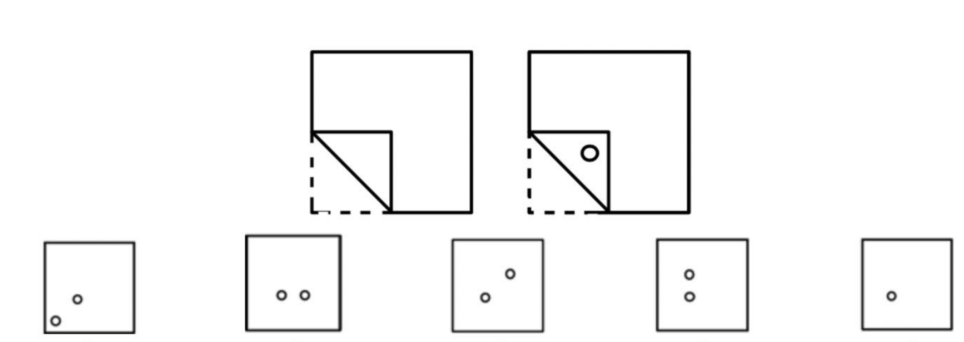
The graph of f', the derivative of f, is shown to the right. For -2<x<4 what are all the intervals in which the graph of f is increasing?
The graph of f is increasing for 4 < x <5
a) -1 < x < 1 and 3 < x ≤ 4
b) -2 < x < 0 only
c) -1 < x < 0 and 3 < x < 4
d) -2 < x < 0 and 2 < x ≤ 4
e) 3 < x ≤ 4 only



Spatial Reasoning Test:

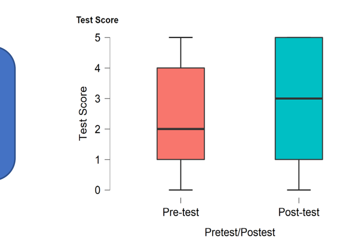
Spatial Reasoning has been shown to influence Calculus learning (Cromley et al., 2017).

Example item:



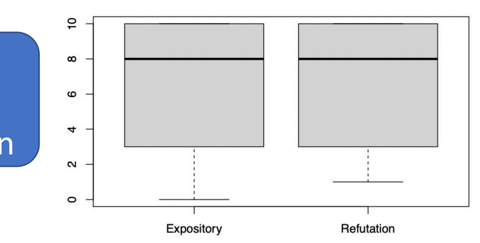
Results

Pre/Post Test Overall



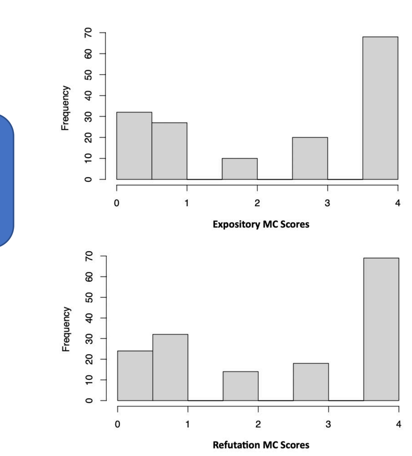
- Saw increases in learning overall

Post Test Score Comparison



- No significant differences in conditions were observed

Question Type Score Comparison



- Grouping by question type (T/F, Multiple Choice (MC)) yielded similar results

Analysis of FR Question

Expository	Refutation
39.3%	40.3%

- Same prevalence of misconception
- Differed in nature

Condition	Example
Expository	"I decided based on where the graph was decreasing"
Refutation	"From the video it stated that f and f' mirror each other"

Discussion

- Results suggest that our lessons improved learning overall
- Refutation failed to improve learning when compared to expository, and some cases reinforced the misconception
- The misconception was prevalent (40% of incorrect responses)

Implications

We interpret the results in several ways:
1) Acknowledging limitations of refutation (see Zengilowski et al., 2021)
2) Additional instructional supports are needed for building robust conceptions related to this topic
3) Using refutation in this way may reinforce existing prior conceptions about f and f'

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Students' Solution Routines in a Nonstandard Problem on the Existence and Uniqueness Theorem for First Order Linear Differential Equations

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Keywords: differential equation, existence and uniqueness theorems, conceptual understanding, commognition theory

University training of modern engineers should prepare them for the solution of applied problems, many of which are described by differential equations (DEs). Recent shift towards computer-led solution of engineering problems undermines the role of advanced mathematical thinking in the education of future engineers negatively affecting their conceptual understanding of mathematical subjects, including DEs. Teaching and learning of DEs at the university level is a relatively new area of education research. Recent empirical studies indicate that students tend to concentrate attention on specific solution techniques and "often fail to relate them to other concepts or ideas" (Camacho-Machín et al., 2012, p. 76).

We continue the research in (Rogovchenko & Rogovchenko, 2022; Rogovchenko et al., 2020; Treffert-Thomas et al., 2018) analyzing the work of students in mechatronics in their fourth year of studies on a set of non-routine problems on Existence and Uniqueness Theorems for linear DEs. For a graded assignment, we designed "tasks and projects that stimulate to ask questions, pose problems, and set goals" whereas students "must learn to inquire systematically" and "must actively construct their own knowledge" (Richards, 1991, p. 38). The data include students' written work (three scripts), answers to pre- and post-questionnaires, and transcribed audio recordings of small group discussions and class presentations.

We analyze the development of students' discourse in one of the assessment problems using the constructs of narrative and routine from the commognitive theory (Sfard, 2008). Mathematical narratives are endorsed with the help of routines. The routine is performed by a person P in a task-situation TS and consists of a pair task-procedure. The same procedure may become a basis for different types of routines, depending on the performer's vision of the task. Expert participants of mathematical discourse interpret most task-situations as requiring a (re)formulation and endorsement of a particular type of mathematical narrative. Such outcome-oriented routines can be called explorations (Lavie et al., 2018). Solution routines were coded, and students' scripts were classified. Based on the data analysis we conclude that the students were able to participate in an explorative routine and produce solution procedures depending on their vision of the task.

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Students' Difficulties With the Understanding of the Concept of the General Solution to a Differential Equation: Can We Teach It Better?

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Keywords: differential equation, general solution, conceptual understanding, textbook analysis

Differential equations find numerous real-life applications in natural and social sciences, economics. Teaching of differential equations to undergraduate students in mathematics, physics, engineering has a long history, and the content of the courses did not change much, paying attention primarily to the use of analytic techniques for finding closed form solutions. Recent advances in technology prompted changes to the syllabus of the courses in differential equations at many universities with a shift towards graphical and numerical approaches for the analysis of differential equations and understanding of the behavior of solutions, "the move away from a "cookbook" course to one that emphasizes modeling, qualitative, graphical and numerical methods of analysis" (Rasmussen & Wawro, 2017, p. 555). However, the research on teaching and learning of differential equations is scarce, "researchers are just beginning to form models of students' learning of differential equations" (Stephan & Rasmussen, 2002, p. 460).

Recent empirical research indicates that students face difficulties with the understanding of the concept of a differential equation itself and to the concepts of the general and particular solutions (Arslan, 2010; Camacho-Machín et al., 2012; Raychaudhuri, 2014). The authors analyzed tasks suggested in the literature for assessing conceptual understanding of solutions to differential equations, concluding that only one task out of five was stimulating students' reflections about the concept (Rogovchenko & Rogovchenko, 2022). Our analysis of students' work on non-routine problems that required understanding of several important theorems and notions including that of the general solution suggested that the explanation of the concept and its use in the course textbook could be one of the reasons for student difficulties.

The research question we address in the paper is: *How was the concept of the general solution introduced in the textbooks on differential equations during the last hundred and fifty years?*

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What's OPDOC? A Rubric for Characterizing Online Professional Development

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Keywords: professional development, online workshops, rubric, checklist

Professional development (PD) has the potential to improve teaching in STEM by helping educators gain knowledge and skills in research-based instructional strategies (RBIS). Instructor use of RBIS has been shown to improve student performance and support equity among students (Freeman et al., 2014; Laursen et al., 2014), and effective PD can help educators implement more RBIS in their classrooms. There has been a lot of research about what practices contribute to effective PD (e.g., Kennedy, 2016; Archie et al., 2022), but as more PD opportunities move online, new tools are needed to describe its characteristics. For instance, spontaneous group work may be quick to coordinate in person, but the same task would require a different set of tools and skills to be executed online. As a way to identify and describe some of these online characteristics, we have developed a rubric called the Online Professional Development Observation Checklist (OPDOC).

The OPDOC was created as an evaluative tool for a series of online PD workshops meant to improve the teaching of undergraduate mathematics instructors. The rubric is designed to be broad enough that it could be applied to a wide variety of online PD programs. It includes 19 items that are based on literature about effective PD practices, including strong logistics (Gaumer Erickson et al., 2017), interactivity among participants (Elliot, 2017; Gaumer Erickson et al., 2017), and support for implementation (Elliot, 2017; Ritzmann et al., 2013). The rubric also includes items that measure equity and inclusiveness, which were drawn from anti-racist rubrics (Blonder et al., 2022) to promote a supportive and collaborative PD environment.

The rubric was used in summer 2022 for both synchronous and asynchronous observation for over 156 hours of total observation of 8 different workshops on varied topics relevant to teaching college mathematics, each 24-30 hours long. There were two workshop models: an intensive model spanning 6 hours a day for 4 days, and a minicourse model spanning 3 meetings a week for 3 weeks. The content of the workshops, the facilitation teams, and the participants were also variable. The OPDOC was used to collect observations of the workshops based on four general categories: logistics, educative content, interactivity, and community-building. For each item within the categories, a numerical 3-point rating scale was used to indicate the strength of evidence for each item.

These numerical scores were then compared to survey responses from participants who attended the workshops. Results showed that the numerical distinctions among workshops were minimal in the logistics and interactivity categories, which could suggest a need for item revision or differential weighting of items. It is also possible that some of the lack of deviation in the OPDOC scores can be attributed to the training that workshop leaders attended beforehand, designed to support leaders in creating organized and interactive workshops. Additionally, as every workshop leader already had some PD experience, it is not unexpected that the workshops were highly rated on average. In future research, we hope to link workshop characteristics with instructor outcomes, so participant survey responses will be used to further identify central components of effective PD and the rubric will be refined to identify variations in greater detail for future workshop iterations.

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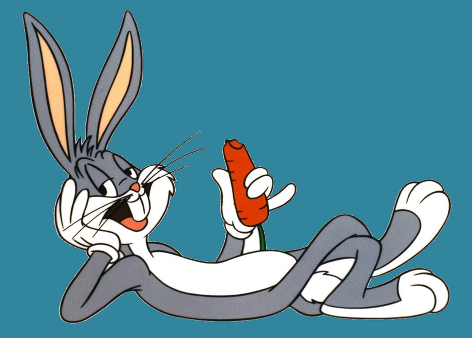

What's OPDOC? A Rubric for Characterizing Online Professional Development

Kyra Gallion, Tim Archie, & Sandra Laursen

Need

Professional development (PD) can help STEM educators gain knowledge and skills in research-based instructional strategies. A few dimensions of effective PD include:

- Strong logistics[1]
- Interactivity among participants[2]
- Support for implementation[3]

As more PD moves online, however, new tools are needed to describe its characteristics. This rubric was developed to evaluate online PD workshops meant to improve the teaching of undergraduate mathematics instructors.

Instrument Development

The OPDOC rubric was first used in the summer of 2022 to evaluate online workshops for a PD program. The observations included:

- 8 workshops on various topics about teaching college mathematics
- Over 156 hours of synchronous and asynchronous observation
- Variable workshop content, models, facilitations teams, and participants

Observations were then compared to post-survey feedback from participants who attended the workshops. Based on these comparisons, we made the following modifications:

- Revised broad categories to emphasize implementation of new practices learned in workshops
- Included new items based on frequently stated participant feedback
- Eliminated repetitive items
- Changed Strength of Evidence from a 3-point scale to 5-point to allow for greater nuance in coding

Acknowledgements

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University of Colorado Boulder

The Rubric

The Professional Development Provider:	Strength of Evidence
Logistics	
1. Follows a schedule that is comfortably paced and honors breaks	None (1)
2. Uses technology that is clear and accessible to participants	None (1)
3. Organizes information so that participants can easily find agendas, links, resources, etc.	None (1)
Content	
4. Provides activities that are relevant to the workshop content, well-paced, and worthy of participant time and conversation	None (1)
5. Describes learning activities clearly and their function within the workshop	None (1)
6. Includes the empirical research foundation of the content (e.g., citations, verbal references to research literature, key researchers)	None (1)
7. Provides information about effective teaching practices within the context of the topic	None (1)
8. Provides information about inclusive and equitable teaching practices within the context of the topic	None (1)
Interactions	
9. Incorporates opportunities for participants to interact with each other related to training content	None (1)
10. Provides guidance during activities to ensure all participants understand the task and are being treated equitably	None (1)
11. Invites participants to express personal perspectives (e.g., experiences, thoughts on concept)	None (1)
12. Includes opportunities for participants to safely ask questions and communicate their needs to facilitators	None (1)
13. Fosters an environment of community, rapport, and openness among the participants	None (1)
Implementation	
14. Provides examples of the content/practice in use (e.g., case study, vignette)	None (1)
15. Connects the topic to participants' context (e.g., course, department, institution) and addresses potential barriers to implementation	None (1)
16.Conveys an expectation that participants are to implement workshop lessons or material into their own contexts	None (1)
17.Includes opportunities for participants to consider and/or practice implementation within their own contexts	None (1)
18.Offers continued support after the workshop (e.g. resources, check-ins, communities of practice)	None (1)

Qualitative Survey Responses

"Frequent breakout rooms and breaks helped to make the long days work well."

"I was *slaughtered* mentally and physically most days... could use longer breaks [...]"

"The workshop descriptions doesn't seem to match the actual workshop."

"The organizers did an excellent job of teaching how to do active learning by explicitly using active learning techniques."

"I loved the interactivity and random group work to meet colleagues and hear fresh ideas."

"As a group we unfortunately fell into the habit of not responding to open prompts for volunteering comments. We needed to be prodded a little more [...]"

"Giving us practical examples we can take directly to the classroom was wonderful."

"I think it would be helpful to maintain an online space for those of us in this workshop so we could share our experiences with implementing our tasks in the future."

"The workshop gave me a wealth of resources I can consult if I need help and a network of colleagues that I can reach out to!"

Quantitative Survey Responses

A comparison of post-survey responses from participants in two example workshops that ranked high and low within the initial sample of eight. The contrasts between the scores suggest participants are perceptive about differences in workshop design and facilitation that reflect elements we seek to document in OPDOC.

Conclusions and Future Work

Based on the data collected at summer 2022 workshops, feedback has already been given to the upcoming summer 2023 workshop teams to emphasize participants' needs for:

- Longer breaks
- More implementation strategies
- Follow-up support

The revised OPDOC will be used for workshops planned for summer 2023 and will be further revised as new observations and survey data are collected. Our intention is to develop a rubric that captures qualitative information to describe workshop features and that also provides quantitative scores, which together can predict or explain short-term outcomes (satisfaction, learning) and long-term outcomes (implementation of new practices).

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Developing and Supporting Underrepresented Prospective Teachers' Pedagogical Content Knowledge for Teaching Diverse Students in High Needs Schools

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Keywords: Teacher Education, Professional Development, Pedagogical Content Knowledge

Having an experienced, well-prepared teacher makes a difference to student outcomes (Cardichon, et al., 2020) and retaining teachers in high-needs schools is even more challenging. In particular, teachers of students in high needs schools need to transform content knowledge into a form appropriate for the diverse interests and abilities of students by drawing on culturally and linguistically appropriate instructional strategies and by building supportive and productive classroom practices. We operationalize components of Pedagogical Content Knowledge (PCK) (Shulman, 1987,1986) and Pedagogical Ways of Knowing (Cochran, et al, 1993) using the coding framework proposed by Meir et. al (2020) which draws from the mathematical practices (National Governors Association, 2010) and sets of instructional strategies for supporting students' mathematical literacy. The purpose is to understand in particular, how participants develop and utilize PCK that speaks to teaching diverse students in high needs schools.

A mixed methods sequential case study design (Creswell & Plano Clarke, 2018) was used, and participants are five Noyce scholars who have graduated from the program and transitioned to teaching in high-needs schools. We use longitudinal evaluation data collected from these participants during their sophomore and senior years in the integrated mathematics credential program as sources, in particular beginning teacher assessment data (TAD), open-ended interview transcripts and responses to closed and open-ended questionnaires.

Our result shows that participant average scores on the Beginning Teacher Assessment were higher than campus averages and that the Noyce mentoring program increases retention of integrated mathematics credential students in the major. In addition, thematic analysis of their TAD lessons and interviews evidenced the ways in which Noyce activities supported them as they integrated understanding of student characteristics and learning context, of mathematics specific pedagogy with their strong mathematical knowledge acquired in the Integrated Mathematics Credential Program. While an integrated Math Credential program is well positioned to foster in prospective teachers' connections between subject matter expertise and pedagogical competence our results suggest that the additional opportunities offered in the Noyce mentoring program are critical in order to develop the types of pedagogical content knowledge necessary to teach in high-needs schools.

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Developing and Supporting Underrepresented Prospective Teachers' Pedagogical Content Knowledge for Teaching Diverse Students in High Needs Schools

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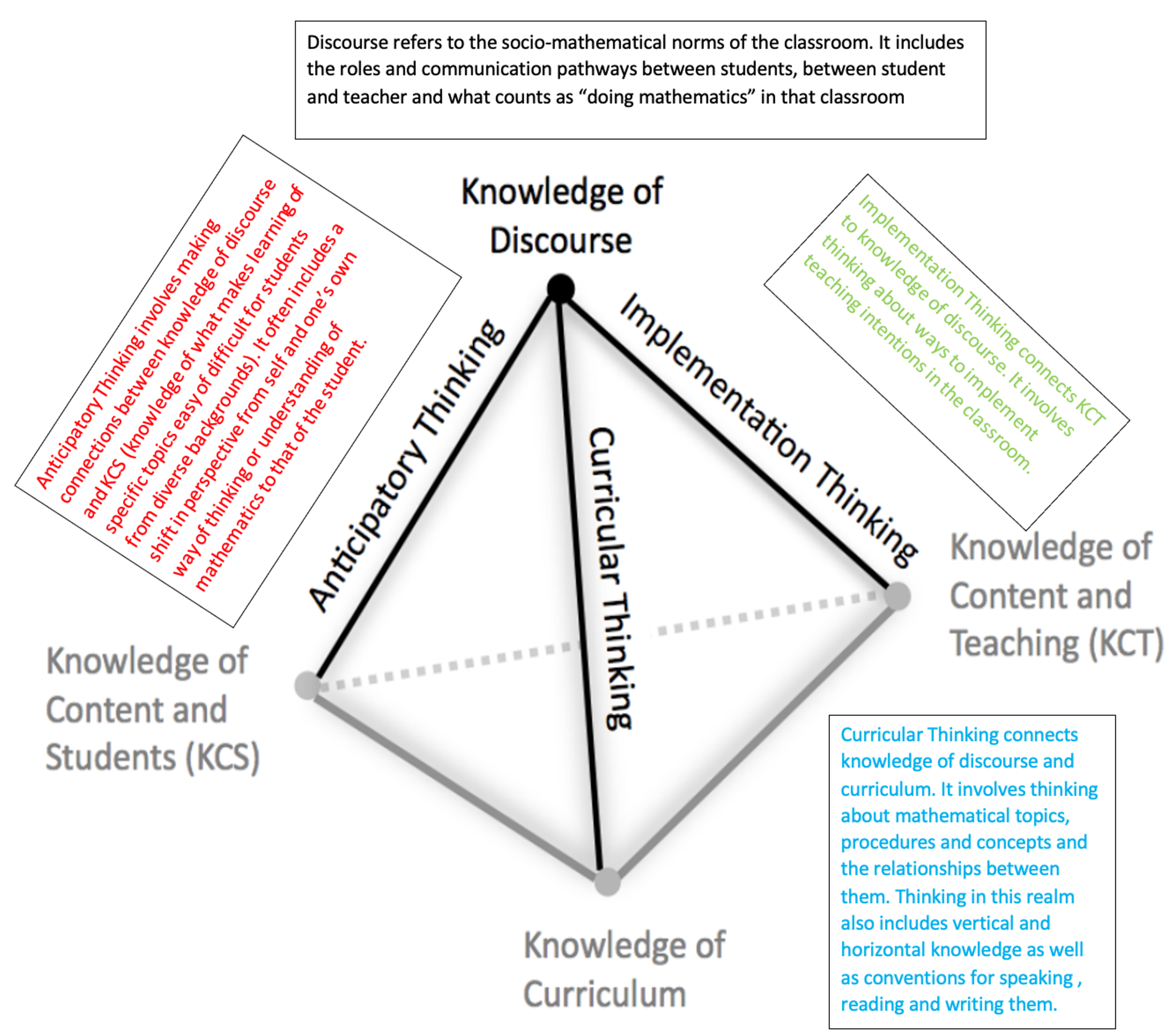
Introduction

Having an experienced, well-prepared teacher makes a difference to student outcomes (Cardichon, et al., 2020) and retaining teachers in high-needs schools is even more challenging. In particular, teachers of students in high needs schools need to transform content knowledge into a form appropriate for the diverse interests and abilities of students by drawing on culturally and linguistically appropriate instructional strategies and by building supportive and productive classroom practices. This study, set in the context of a Noyce-funded mentoring program where participants are underrepresented and talented integrated mathematics credential majors, addresses how prospective teachers can be concurrently supported in a rigorous mathematics program as they develop the deep pedagogical knowing so necessary for teaching students in high needs settings.

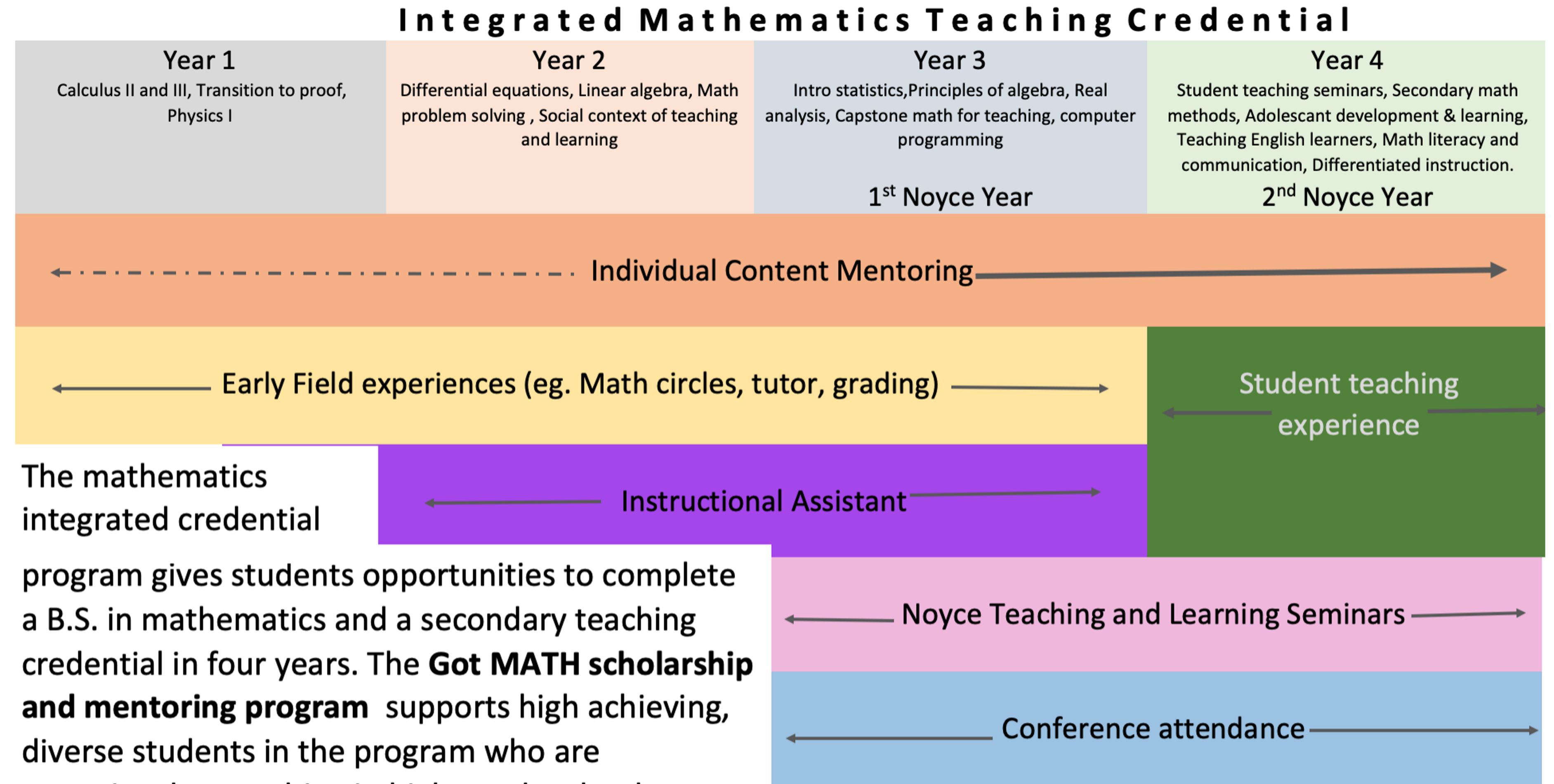
Framework for Pedagogical Content knowledge



Methods

A mixed methods sequential case study design (Creswell & Plano Clarke, 2018) was used. Participants are five Noyce scholars who have graduated from the program and transitioned to teaching in high-needs schools. We use: Longitudinal evaluation data collected during sophomore and senior years in the GotMATH program sources:

- Beginning teacher assessment data (TAD)
- Open-ended interview transcripts
- Responses to closed and open-ended questionnaires
- Reflections from teaching and learning seminars.



The mathematics integrated credential program gives students opportunities to complete a B.S. in mathematics and a secondary teaching credential in four years. The **Got MATH scholarship and mentoring program** supports high achieving, diverse students in the program who are committed to teaching in high needs schools.

GotMATH provides individual and group content mentoring. Early field experiences allow mathematical and pedagogical knowledge to develop simulataneously and Noyce Teaching and Learning seminars given by university and school based mathematics educators model math specific PCK appropriate for underrepresented students in high needs schools.

Category	Code	Example of instance	Lorry	Jacob	Nathan	Bella	Nadine
Creating a meaningful context for instruction	KCS	Using a jeopardy game context for math exam review.	x	x			
	KCT	Group work with small white boards/game format.	x	x	x	x	x
Helping students produce discourse	Anticipatory Thinking	Asking questions to promote focus on math problem solving, strategies and argument.	x	x	x	x	x
	Implementation thinking Using sentence frames and/or sentence starters to support mathematical oral and written discourse.	Student answers to strategic questions were scaffolded with sentence frames to help them justify their reasoning about a problem.	x	x	x	x	x
	Implementation thinking Using word walls	Teacher creates written strategy bank as they are discussed in groups (for later use)			x	x	
	Anticipatory Thinking Using multiple modes for language (speaking, listening, reading, writing)	White boards/notebooks/ posters/exit tickets verbally expressing solutions to near peers and whole group/using visual aids	x		x	x	x
Making input comprehensible	Implementation thinking	Vocabulary and/or syntax support		x	x	x	x
		Implementation thinking Using visuals, drawings, gestures, or realia		x	x		x
Organizing instruction	Anticipatory thinking Strategically placing students in small groups based on needs	Group members are given responsibilities for ensuring all students understand how to solve the equations, so students with stronger understanding are instructed how to give helpful hints and explain their thinking.	x	x		x	x
	Anticipatory thinking	Monitoring student work during small group work to select groups to present.	x			x	
	Socio mathematical norm	Students self-select level of difficulty of problem type and progress upward.		x	x	x	x
	KCS	Working in teacher led small groups for extra instruction			x	x	x

Table 1. Mathematical Literacy Strategies to Engage in Mathematics Discourse Used by Noyce Beginning Teachers.

Results and Discussion

Participant average scores on the Beginning Teacher Assessment were higher than campus averages and our analysis of their TAD evidenced the ways that participants used anticipatory and implementation thinking as they drew from their strong mathematical backgrounds with and understanding of students and teachings to provide opportunities for their students to participate in mathematical discourse. In particular students in scholar classrooms were given opportunities to construct viable arguments and critique the arguments of others (MP3), use appropriate tools strategically (MP5), make sense of problems and persevere in solving them (MP1), look for and make use of structure (MP7) and/or make use of repeated reasoning (MP8) while learning important content standards.

Themes from our stage II analyses show that students in the mentoring program:

1. Valued multiple opportunities during Noyce seminars to experience and reflect on practices and strategies learned during credential courses within the specific context of teaching mathematics.
2. Learned about pedagogical issues specific to students living in poverty from experienced teachers working in local high-needs schools through the Noyce seminars.
3. Gained knowledge of students and content as well as knowledge of content and pedagogy from concurrent opportunities to work as instructional assistants on campus and tutors in elementary settings while themselves learning higher level mathematics content in major coursework (offered through Noyce)

Conclusions

Our result shows that participant average scores on the Beginning Teacher Assessment were higher than campus averages and that the Noyce mentoring program increases retention of integrated mathematics credential students in the major.

- In addition, thematic analysis of their TAD lessons and interviews evidenced the ways in which Noyce activities supported them as they integrated understanding of student characteristics and learning context, of mathematics specific pedagogy with their strong mathematical knowledge acquired in the Integrated Mathematics Credential Program.

While an integrated Math Credential program is well positioned to foster in prospective teachers' connections between subject matter expertise and pedagogical competence our results suggest that the additional opportunities offered in the Noyce mentoring program are critical in order to develop the types of pedagogical content knowledge necessary to teach in high-needs schools.

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Designing Activities to Support Preservice Teachers' in Promoting Generalizations

Kayla Heacock
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Keywords: algebra and algebraic thinking

The practice of generalizing, identifying a relationship to describe multiple examples or instances of a phenomenon, engages students in algebraic thinking and allows them to identify, investigate, and represent relationships (Blanton et al., 2011; Kieran et al., 2016). Generalizing is fundamental to learning mathematics (Carraher & Schliemann, 2002; Kaput, 1999), yet students often have trouble developing (Lannin, 2005) and justifying generalizations (Breiteig & Grevholm, 2006). To first build their understanding of generalizing as a mathematical practice, we introduced a cohort of preservice secondary teachers (PSTs) to two frameworks for generalizing and asked them to use these frameworks to analyze others' teaching. Then we designed a teaching activity embedded in their methods course to help them promote generalizing among their grades 7-12 students. Ellis's (2011) generalizing promoting actions (GPAs) are teacher actions that foster the development or refinement of a generalization. Strachota's (2020) priming actions (PAs) usually precede GPAs and set the stage for more explicit attention to generalizing and prepare students to build on an idea.

We provided an outline for a lesson focused on generalizing and asked three PSTs to customize the lesson for a high school audience and to plan questions to elicit student generalizations. All PSTs used a pattern task, a visual representation of a series of objects where the number of objects changes based on a rule, as the focus of their lesson. The goal of a pattern task is to use the structure of the presented images (e.g., objects organized in arrays) to develop multiple ways of representing the rule. We chose pattern task as the focus for the potential of the tasks to promote generalizing because we believed it served as an accessible entry point for our novice PSTs.

We video-recorded each PST's teaching and asked them to watch their video and analyze the effectiveness of their practice. We coded video data for GPAs (Ellis, 2011), PAs (Strachota, 2020), and student generalizations to identify teacher actions that promoted and hindered students in generalizing. Through our analysis, we sought to understand the patterns of generalizing actions that supported students in articulating the generalizations and determine how teacher educators can support PSTs in promoting student-created generalizations.

Patterns of interactions that promoted generalizing included using multiple GPAs in a short span of time and following student-created generalizations with a new GPA to prompt them to expand or deepen their contribution. Additionally, using PAs and GPAs together helped support students in generalizing more so than using GPAs alone. The PA of naming a critical tool proved especially useful when students invented and described their own strategies. Patterns of interaction that hindered students in generalizing included using GPAs when the actions were centered on a teacher strategy instead of a student strategy or when the teacher was not able to use repeated GPAs in a series. There was not a one-to-one link between GPAs or PAs and students' generalizations, which demonstrates the complexity of generalizing. Enacting the pattern task revealed much about the shortcomings and strengths of teachers' understandings of generalizations; future research will explore how to better support preservice teachers in promoting generalizing.

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Designing Activities to Support Preservice Teachers in Promoting Generalizations

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BACKGROUND

- *Generalizing*: identifying a relationship to describe multiple examples or instances of a phenomenon
- Engages students in algebraic thinking and allows them to identify, investigate, and represent relationships (Kieran et al., 2016)
- Generalizing is fundamental to mathematics (Carraher & Schliemann, 2002)
- Students have trouble developing (Lannin, 2005) and justifying generalizations (Breiteig & Grevholm, 2006)

METHODS

Participants: 3 teaching fellows in one-year master's program

Learning about Generalizing: Fellows completed pattern tasks as learners and analyzed teaching videos for promoting generalizing

Data Collection: Implement pattern task activity in field placement

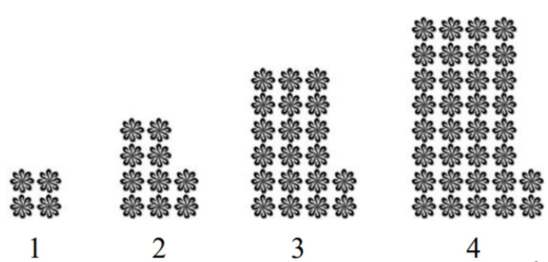
- Choose pattern task & solve in many ways
- Plan for questioning to elicit generalizations using five practices approach (Stein and Smith, 2011)

Data Analysis: Video data analyzed in 15-second segments for generalizing promoting actions and student contributions

RESEARCH QUESTIONS

1. What patterns of generalizing actions best supported students in articulating generalizations?
2. How can teacher educators support preservice teachers in promoting student-created generalizations?

SAMPLE PATTERN TASK



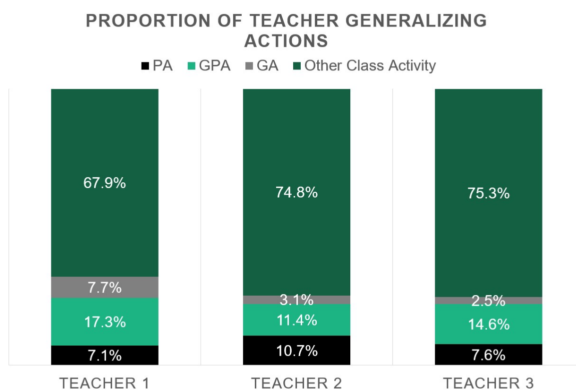
Nguyen, (2020)

FRAMEWORK

Priming Actions (PAs)[1]	
Naming a phenomenon, clarifying critical terms and tools	"Offering a common way to reference a phenomenon or emphasizing the meaning of a critical term or tool."
Constructing or encouraging constructing searchable and relatable situations	"Creating or identifying situations or objects that can be used for searching or relating. Situations that can be used for searching or relating involve particular instances or objects that students can identify as similar."
Constructing extendable situations	"Identifying situations or objects that can be used for extending. Extending involves applying a phenomenon to a larger range of cases than from which it originated."
Generalizing Promoting Actions (GPAs)[2]	
Encouraging Relating and Searching	"Prompting the formation of an association between two or more entities; prompting the search for a pattern or stable relationship."
Encouraging Extending	"Prompting the expansion beyond the case at hand."
Encouraging Reflection	"Prompting the creation of a verbal or algebraic description of a pattern, rule, or phenomenon."
Encouraging Justification	"Encouraging a student to reflect more deeply on a generalization or an idea by requesting an explanation or a justification. Includes asking students to clarify a generalization, describe its origins, or explain why it makes sense."

[1]Categories and descriptions from Strachota (2020, p. 7).
[2]Categories and descriptions from Ellis (2011, p. 316). Adapted encouraging relating and searching, following Strachota (2020), by combining these into one category.

RESULTS & DISCUSSION



Conditions that Supported Students' Generalizing:

- Teacher used multiple GPAs in short amount of time
- Teacher followed student generalizations with GPA
- Teacher used PA and GPA together to elicit and make public student thinking and connect it to multiple representations

Conditions that Hindered Students' Generalizing:

- GPA centered around a teacher strategy, not a student strategy
- Pattern task not at appropriate level for students

Other Findings

- Generalizing is complex
- There is not a one-to-one link between GPA, PA, and student-created generalizations

FUTURE RESEARCH

- Examine influence of teacher questioning on GPA and student-created generalizations
- Examine quality of PSTs' reflections on pattern tasks
- Track use of generalizing promoting actions across student teaching year

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The Implications of Perspectives on MKT for the Examination of Teacher Noticing

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Keywords: Mathematical Knowledge for Teaching, Teacher Education, Decentering

Mathematical knowledge for teaching (MKT) has been widely used in mathematics education research, with many researchers developing varying definitions. The categorization of MKT developed by Ball et al. (2008) has been influential in the study of teachers' mathematical knowledge. Their framework differentiates between pedagogical content knowledge and subject matter knowledge, and breaks these categories into subcategories that describe what the knowledge is of. While this categorization is helpful for characterizing the existence of knowledge it does not provide a mechanism for its development. The framework for MKT introduced by Silverman and Thompson (2008) does not categorize knowledge, but it does propose a potential mechanism for the development of MKT. The framework posits that MKT first begins with a personal understanding of a concept and then can be transformed into knowledge with pedagogical power via decentering. To transform their knowledge, a teacher must imagine how students might understand the content, and how students might be supported in developing an understanding consistent with that of the teacher.

While the use of student thinking in teaching has been examined in multiple ways, one particularly influential construct is professional noticing as defined by Jacobs and colleagues' (2010). They defined noticing to consist of attending to strategies, interpreting student understanding, and deciding how to respond on the basis of these understandings. A related idea is using decentering as a lens to analyze classroom practice (Bas-Ader & Carlson, 2021). When a teacher decenters they intentionally put aside their mathematical perspective and attempt to adopt the perspective of a student. They develop a mental model of their student's mathematics which informs their response to their student's thinking (Bas-Ader & Carlson, 2021).

Supporting graduate student instructors and new faculty in developing MKT and using student thinking is an important aspect of designing professional development programs. This poster will explore the affordances and constraints of using combinations of MKT frameworks and frameworks for the use of student thinking to study the relationship between MKT and teachers' use of student thinking. Using Ball, Hill, and colleagues' characterization of MKT could allow one to study relationships between teachers' knowledge in particular domains and their use of student thinking. One could map knowledge used when interpreting and responding to student thinking to domains of knowledge within the framework to understand how different domains support teachers in using student thinking. A teacher might draw on Knowledge of Content and Students to recognize a student understanding, and Knowledge of Content and Teaching to decide how to move that understanding forward. The use of Silverman and Thompson's framework aligns well with the use of decentering as a lens to examine teachers' use of student thinking. A teacher decenters to develop MKT, which supports them in decentering in the classroom. If a teacher has constructed ways that a student might understand an idea, they might assimilate a student's thinking to one of these ways. If a teacher encounters student thinking for which they do not have a viable pre-constructed model, they develop MKT as they decenter to build a model of their student's understanding. Thus, development of MKT and attention to student thinking can be seen as a reciprocal process, in which MKT supports teachers in making use of student thinking, and attending to student thinking can develop MKT.

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Affordances and Challenges Associated with Euler Diagrams as Representations of Logical Implications

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Keywords: Euler diagrams, logical implication, proof, spatial representations

Research in mathematics education has begun to capture students' epistemic experiences with logic and proof in general, and with logical implications in particular (e.g., Antonini, 2004; Dawkins, 2017; Durand-Guerrier, 2003; Durand-Guerrier et al., 2012; Epp, 2003; Roh & Lee, 2018; Stylianides et al., 2004; Yopp, 2017). However, much is still unknown about how instruction can support student reasoning and learning about logical implication. Some researchers have explored the use of Euler diagrams for representing relations between sets and, in particular, truth sets of open statements (e.g., Bronkorst et al., 2021; Dawkins & Roh, 2022; Hub & Dawkins, 2018). In our ongoing research (the Proofs Project), we use Euler diagrams in introduction-to-proofs courses as one way to represent logical implications and their transformations. We have been conducting clinical interviews with students in which we investigate their conceptual mappings between logical implications and Euler diagrams. We share preliminary findings from our initial analyses of interview data to address the following question: *What affordances and challenges do Euler diagrams present as visual-spatial representations of logical statements?*

One affordance of Euler diagrams comes from their spatial nature: they provide a means to *visualize* relationships between two truth sets. One student (Carmen) explained, "I think with the if-then statements, it helps me to think of the Euler diagrams… and thinking of a circle within a circle." Another student (Kai) commented that, while he preferred to use symbolic logic over Euler diagrams, especially for representing more complex statements, he added, "Otherwise, I think [an Euler diagram] shows relationships very clearly if you have the initial setup correct."

The spatial nature of Euler diagrams also presents particular challenges for students. For instance, ideally, the statement $P \rightarrow Q$ is mapped to an Euler diagram in which the truth set of P is a subset of the truth set of Q. Some students in our study represented this implication with P as a *superset* of Q. As Carmen explained, "The P engulfs the entire space of Q, so that if I have the statement P, then I'm definitely going to have the statement Q." Additionally, for contrapositive statements (e.g., $\sim Q \rightarrow \sim P$), visualizing the truth set of $\sim Q$ as a subset of $\sim P$ in a 2D Euler diagram in which the truth sets of P and Q are depicted first may be difficult as it requires first visualizing the complements of the truth sets for P and Q as closed regions in 2D Euclidean space. Furthermore, the containment of $\sim Q$ within $\sim P$ may be obscured for students because the resulting depiction of $\sim Q \rightarrow \sim P$ is not homeomorphic to the original. We note that, ideally, Euler diagrams for two logically equivalent statements would be homeomorphic to each other. Because these spaces *are* homeomorphic when visualized on a sphere (rather than 2D Euclidean space), we discuss potential affordances and challenges of such a representation, and how Peircian diagrams on a sphere may provide alternative spatial representations (cf. Pietarinen, 2016).

Acknowledgments

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Affordances and Challenges Associated with Euler Diagrams as Representations of Logical Implications

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Purpose & Motivation

Goals of the Proofs Project

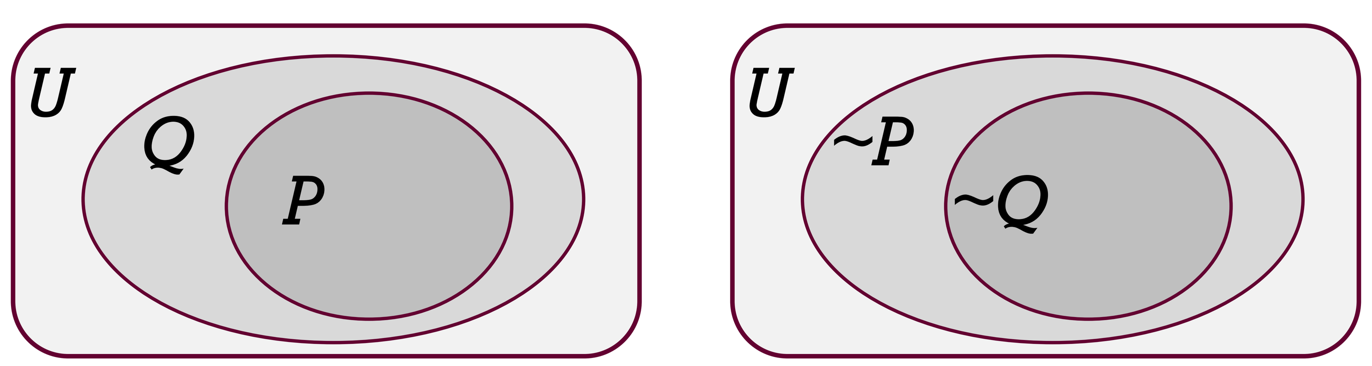
- Identify and address students' epistemological obstacles in introductory proofs courses
- Support teachers by suggesting instructional interactions that may provide students with opportunities to identify, reflect on, and overcome these obstacles

Research Questions Regarding Euler Diagrams

- How do students interpret, construct, and make sense of Euler diagrams as representations of logical implications?
- What epistemological affordances and challenges may be associated with this mode of representation?

What are Euler Diagrams?

- Visual-spatial representations showing relationships between sets
- Statement $p \to q$ represented as $P \subseteq Q$, where P and Q are the truth sets of statements p and q, respectively
- U represents the universal set, or the domain of objects under consideration



(a) (b)

Figure 1: Euler diagram representations of (a) $p \to q$ and (b) $\sim q \to \sim p$

Why Euler Diagrams?

- Hub and Dawkins (2018) suggest students can abstract the logic of conditionals by populating sets with familiar mathematical objects.
- Conditionals describe a relationship between properties. If students populate sets with those properties, they can represent the relationship spatially, as a relationship between sets.
- Dawkins and Roh (2021) have employed this instructional approach, relying heavily on the use of Euler diagrams, within their own research project.

Methods

- Classroom data from an introductory proofs course, taught in the Fall 2022 semester
- Three clinical interviews with each of four students in the course: (1) Logical Implication, (2) Quantification, and (3) Mathematical Induction and Stimulated Recall

Affordances

Visual Representation of an Abstract Relationship

- **Carmen:** "I think with the if-then statements, it helps me to think of the Euler diagram... and thinking of a circle within a circle."
- **Kai:** Stated that he prefers symbolic representations, but also said, "I think [an Euler diagram] shows relationships very clearly if you have the initial setup correct."
- Allows students to offload cognitive demands of maintaining relationships between logical structures by establishing a spatial representation

Visual Facilitation of Transformations

- Provide students with a visual aid in interpreting transformations of logical implications (e.g., negation)

Challenges

Different Interpretation of Subset Relation

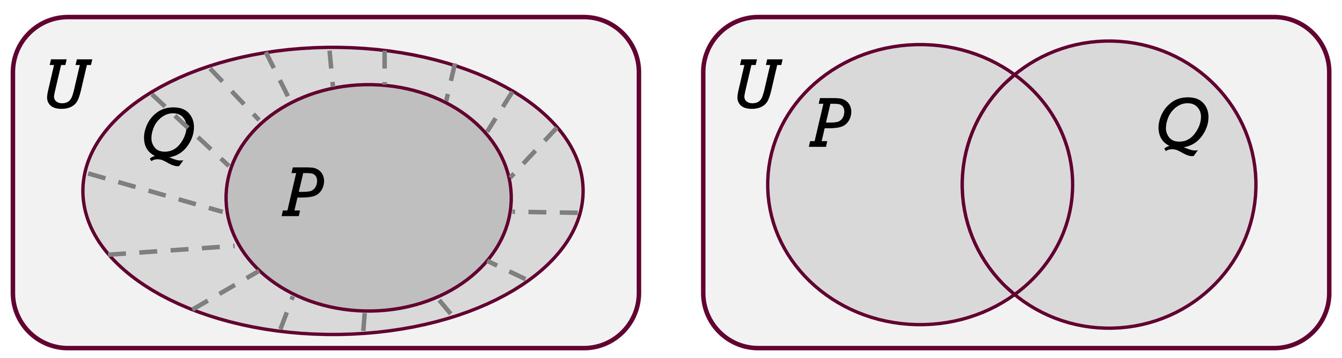
- **Carmen:** "So, if p, then q. The P engulfs the entire space of Q so that if I have the statement p, then I'm definitely going to have the statement q."
- Some students interpreted the containment of P in Q as suggesting that, if you have the space P (or statement p), then you have the space Q (or statement q)
- As **Kai** noted in excerpt above, Euler diagrams can be useful, but they require
  - structuring spatial representations of logical information in a meaningful way, along with
  - shared understanding of what set containment within an Euler diagram represents

Size Matters Not

- Qualitative nature of Euler diagrams may hide quantification from students
- Canonically, the size of a space within an Euler diagram is not an indicator of the cardinality of the set that it represents
- A circle may be used to represent a singleton set, whereas another circle (perhaps smaller than the original) could be used to represent an infinite collection

Empty or Not Empty?

- Generally, the existence of a closed space in an Euler diagram implies the space represents a non-empty set
- Not always the case: in the Euler diagram for $p \to q$, the space $Q - P$ may be empty if $q \to p$ also holds (see Fig. 2a), or the space P may be empty (vacuous case)
- By contrast, in the Euler diagram shown in Fig. 2b, $P \cap Q$ is presumed to be non-empty



(a) (b)

Figure 2: (a) $Q - P$ may be empty, but (b) $P \cap Q$ is assumed to be non-empty

Representing the Contrapositive

- Euler diagram for $p \to q$ not topologically equivalent to Euler diagram for $\sim q \to \sim p$ (compare Fig. 1a and Fig. 1b)
- One potential remedy: construct and transform Euler diagrams on a sphere (see Fig. 3)
- Starting with $P \subseteq Q$, dilate P and Q, reaching to the back side of the sphere, until only a small region not contained in Q is remaining, then invert relationship to show $\sim Q \subseteq \sim P$
- Unclear what additional obstacles this may introduce

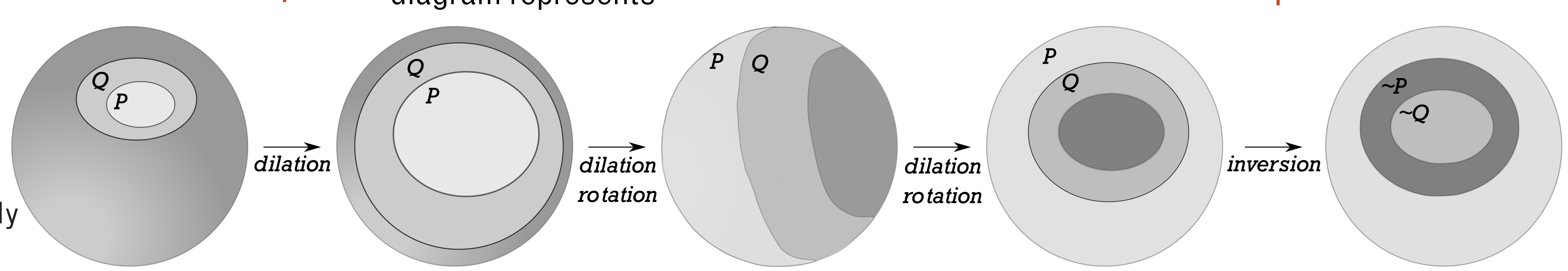


Figure 3: Transforming Euler diagram for $p \to q$ on a sphere to show equivalence to its contrapositive

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Proofs Project

Metacognition in Proof Construction: Connecting and Extending a Problem-Solving Framework

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Keywords: Metacognition, Problem Solving, Proof

Mathematical proofs are an integral component of undergraduate mathematics education. Providing students opportunities to develop the necessary writing technique, and deductive reasoning skills to communicate mathematical ideas via proof is a central goal of undergraduate mathematics programs (Committee on the Undergraduate Programs in Mathematics [CUPM], 2015). For undergraduate students, developing these skills and utilizing them appropriately can be an arduous feat and has implications for their success in a mathematics course or program (Alcock & Simpson, 2002). Due to the difficulty and importance of proof, the mathematics education community has contributed to a varied body of research related to the teaching, learning, and cognitive skills relevant to mathematical proof.

Research in the cognitive processes related to proof construction have provided insight into how undergraduate students, graduate students, and mathematicians interact with the proof construction process. Inglis and Alcock (2012) demonstrated that the processes with which each of these three groups construct proofs varies and that proof construction is not uniform even within a group. Their participants also showed inconsistency between how they said they approach proofs and the actual processes they implemented when given a proof construction task. To address these different approaches and their implications for the teaching and learning of proofs, we need to know more about what beliefs and judgements individuals have about their own cognitive processes during proof construction, and how these impact the choices they make during this process. Metacognition, defined to be "any knowledge or cognitive activity that takes as its object, or monitors, or regulates any aspect of cognitive activity; that is, knowledge about, and thinking about one's own thinking" (Lerman, 2020, pp. 608-609) captures exactly this aspect of an individual's thinking.

Existing metacognition research in mathematics education has focused on the relevant metacognitive processes that occur during mathematical problem solving more broadly. Proof related tasks can be a subset of mathematical problem solving, and therefore many of the findings from problem-solving based metacognition research may be applicable to proof. However, research shows that there are aspects of the proof construction process that may not be addressed in other problem-solving research (e.g., Savic, 2015). The language, structure, and reasoning required in proof constructions can be viewed to be unique to mathematical proofs and form a special writing genre (Selden & Selden, 2013). Savic (2015) also found that when the multi-dimensional, problem-solving framework developed by Carlson and Bloom (2005) is utilized in context of a proof task, adjustments of the framework are needed for the proof processes; in particular, there is a need to study metacognitive processes in proof more directly.

In this poster, we discuss the research that aims to explore the phenomenon of individuals' metacognitive processes during the construction of mathematical proofs. We identify the areas of the Carlson and Bloom framework in which metacognitive processes could potentially be important for proof construction, and we will demonstrate these with active engagement during poster presentation with the audience. During this interactive poster presentation, participants will be asked to share their thinking processes on proof tasks while we highlight the aspects of their metacognitive processes.

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Active Learning Professional Development Series for Both Experienced and Inexperienced Instructors

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The poster describes a pilot study following the outcome of a three-part professional development workshop to promote active learning in STEM classrooms for instructors with a wide range of teaching experience. The series introduced strategies and benefits of active learning and had participants workshop their own strategies. We share the results of an exit survey given to participants and additional observations. The survey seeks to find what kinds of faculty participate and whether the professional development series motivated instructors to use more active learning in their classroom.

Keywords: Active Learning, Professional Development, Post-Secondary Education, STEM

How do we design professional development opportunities, including those that are discipline specific and promote active learning in STEM for an audience of both experienced and inexperienced instructors? We share observations and survey results from a pilot study following a three-part professional development workshop to promote active learning in STEM classes at a university in the Northeastern United States.

The three-session workshop took place over the course of 6 weeks. In order to be flexible, each session included information and activities that built on the previous meeting but were presented in an independent way so that instructors weren't required to attend each session. The College Mathematics Instructor Development Source (CoMInDS) Library on MAA Connect was used to generate activities, including the classroom practices found in (Yee, 2021). An article about shifting the focus from the teacher to student (Webel, 2010) was used for a discussion.

Information for this study was collected though post workshop observations and participant survey questions. There were 16 participants in the three sessions, with no participant attending all three. The six-question survey asked participants about their teaching experience, their professional development experience, and their experience using active learning.

Six faculty members responded to the survey. A majority of the respondents had 6 or more years of teaching experience. While most respondents attend professional development sessions more than once per year, the majority of professional development attended was not designed specifically for teaching in STEM fields. After the workshop, most instructors said they would likely use active learning in their classrooms. Teaching responsibilities was an obstacle to participating, with many instructors arriving late or leaving early due to class schedules. While the sessions were offered in a hybrid format, most instructors preferred the online option. We found that many instructors had trouble working on open ended projects to develop their own active learning examples during the workshop.

From these results, we consider the following future questions. What scheduling incentives, and structure could be implemented to improve attendance? What improvements in the workshop structure could be implemented to improve engagement and better promote active learning? In addition, we would like to investigate ways to better integrate the different experiences, backgrounds, and variety of fields represented by the participants to improve professional development for a wide audience.

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Active Learning Professional Development Series for Both Experienced and Inexperienced Instructors

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Purpose

Share observations and survey results from a pilot three part professional development workshop to promote active learning in STEM classes. The workshop was designed to be flexible, for both novice instructors and veterans, as well as instructors with varying levels of experience using active learning.

Resources and Literature

- College Mathematics Instructor Development Source (CoMInDS) Library on MAA Connect.
- Active Learning Instructional Resource: Classroom Practices from the CoMInDS Library (Yee, 2021)
- Group Reading: “Shifting Mathematical Authority from Teacher to Community.” (Webel, 2010)

Research Question

How do we design professional development opportunities, including those that promote active learning in STEM classes, for an audience of both experienced and inexperienced instructors?

Methods

- Survey Questions
- Post Workshop Observations

Workshop

The workshop had three sessions which took place over the course of 6 weeks. Instructors were encouraged to attend as many sessions as possible. While each session included information and activities that built on the previous meeting, they where presented in an independent way so that instructors weren’t required to attend each session.

Survey Questions

Survey questions asked about the following:

- Teaching Experience
- Professional Development Experience
- Experience using active learning.

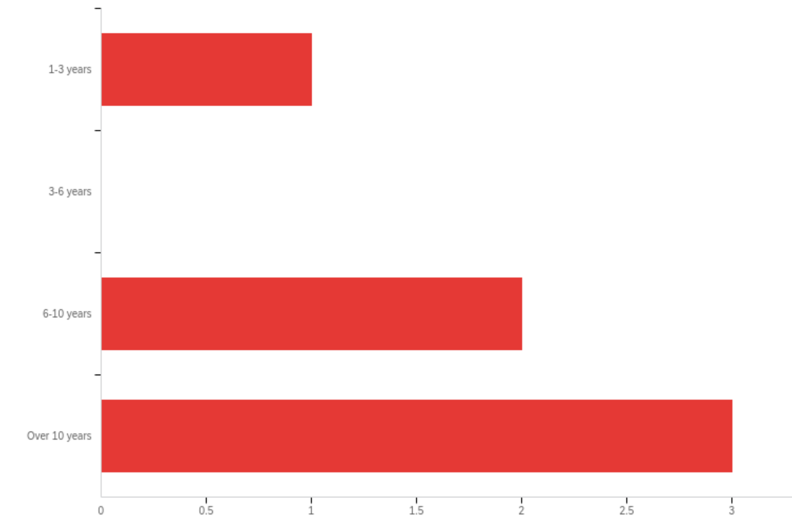
Particpants

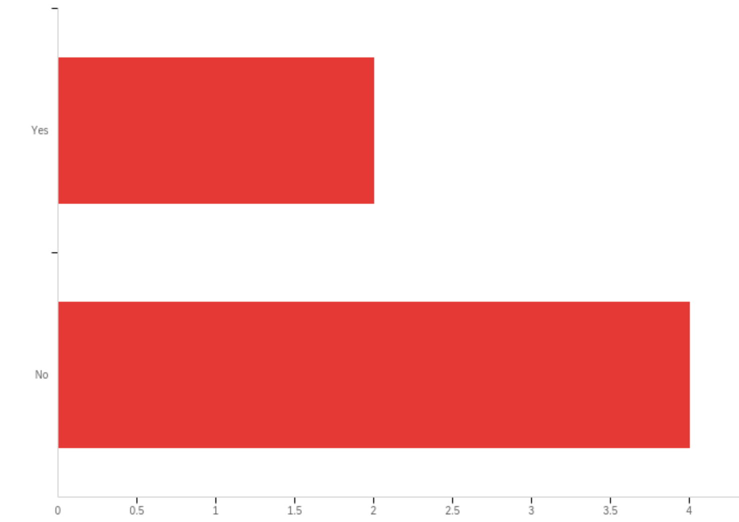
There were 16 participants in the three professional development sessions. No participants attended all three sessions, with several attending two sessions. Many participants only participated for parts of sessions due to teaching schedules.

Survey Questions and Results

Question 1: How many years have you taught?

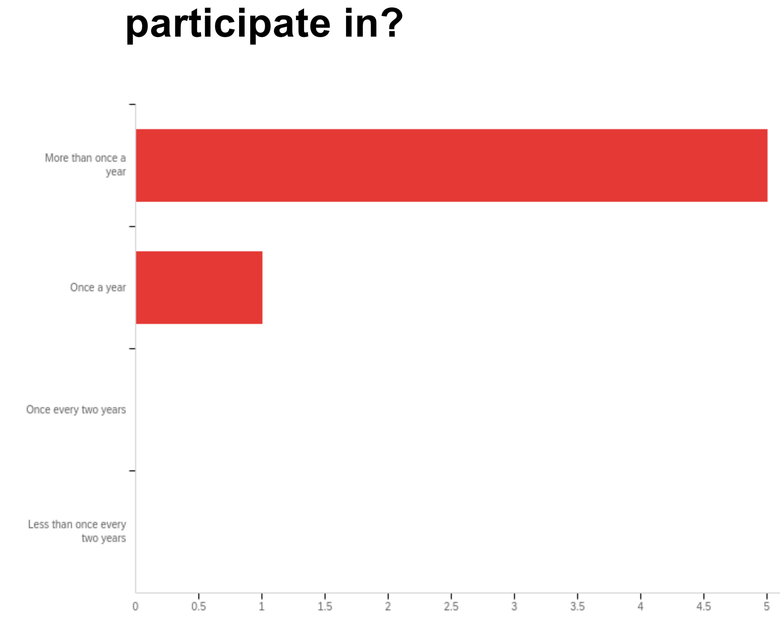
Question 2: Before becoming the instructor of record of a course, did you have any experience being a graduate or teaching assistant?

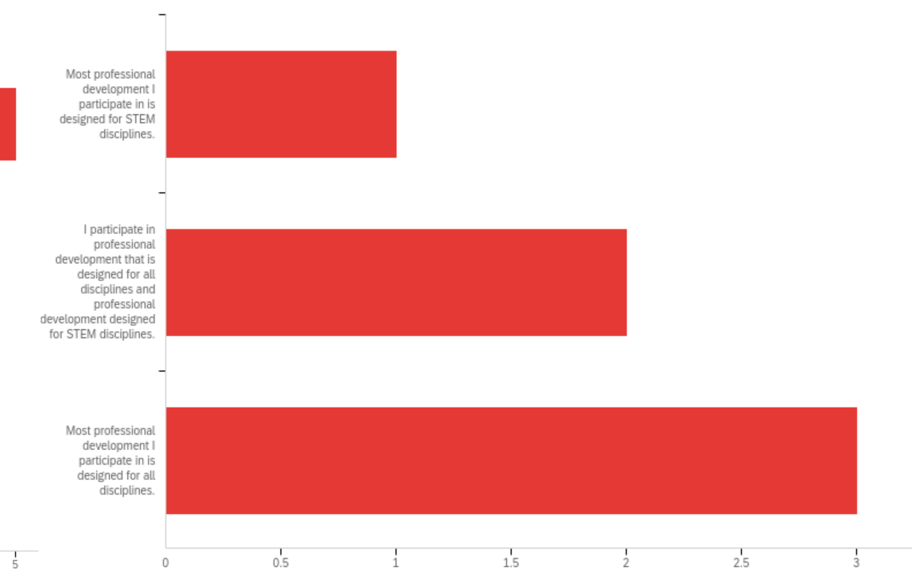




Question 3: Approximately how much professional development in teaching do you participate in?

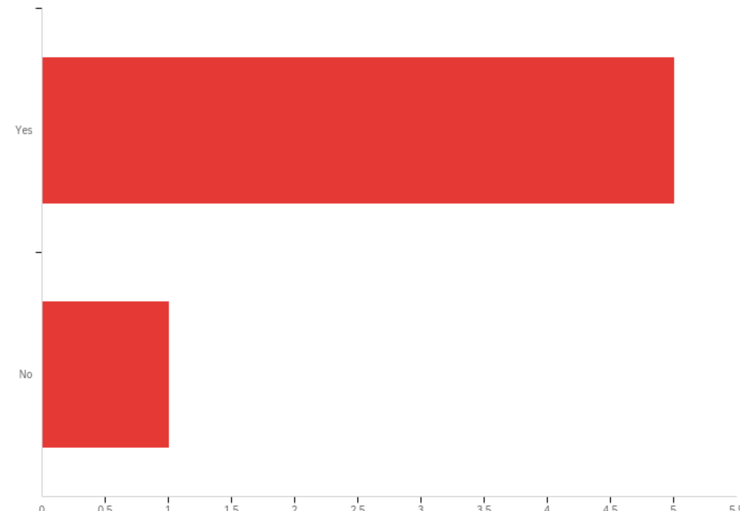
Question 4: Which of the following describes the professional development you participate in?

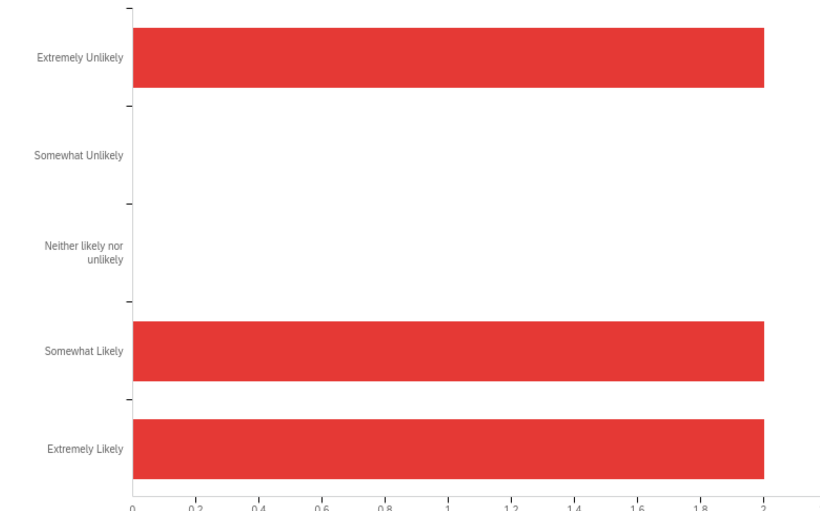




Question 5: Have you used active learning in your classes before?

Question 6: How likely are you use active learning in your classes in the future?





Findings & Discussions

Survey

- A majority of instructors attending these development sessions had 6 or more years of teaching experience.
- A majority of participating instructors did not have experience as a graduate or teaching assistant.
- Most instructors participate in professional development more than once per year.
- The majority of the professional development that the instructors attend is not designed primarily for teaching in STEM fields.
- Most instructors believe they will likely use active learning in the future.

Observations

- Teaching schedule was an obstruction to participating. Multiple instructors either could not participate or could only participate for part of the sessions due to this obstacle.
- While the sessions were available in a hybrid format, both in-person and online, most instructors participated online.
- Instructors had trouble working on open ended projects to develop their own active learning classroom activities during the workshop.

Future Research

- What scheduling, incentives, and structure could be implemented to improve attendance?
- What are improvements in the workshop structure that could be implemented to improve participation and better promote active learning?
- Investigate ways to better integrate the different experiences, backgrounds, and variety of fields represented by the participants to improve this type of professional development in STEM fields.

Acknowledgements

We would like to thank the instructors for their willingness to participate in the pilot study. We would also like to thank the administration for supporting us through the college’s Faculty Fellows program.

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PennState
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Confidence-to-Certainty Metacognitive Shifts for Students in Undergraduate Linear Algebra

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Assessing students' metacognition can help educators better understand student learning. In the present study, the authors evaluated students' pre-assessment confidence, response correctness, and post-response certainty during routine examinations in an undergraduate linear algebra course to investigate the magnitude and direction of students' metacognitive shifts (Δ meta) from mean confidence to mean certainty as compared to response correctness. Results of this study provide insights to student perceptions of their knowledge during examination performances in linear algebra and can inform efforts to improve mathematics teaching and learning.

Keywords: confidence, certainty, correctness, metacognition, metacognitive shift, correlation

Motivation and Study Objectives

Scholars have studied the alignment of confidence and certainty to response correctness (Gardner-Medwin & Curtin, 2007; Hunt, 2003; Preheim, Dorfmeister, & Snow, 2022), but Δ meta has not been empirically compared to overall correctness or performances on different types of questions (*e.g.*, multiple choice vs. process-based questions). The objective of this study is to investigate this gap in knowledge within an undergraduate linear algebra student population.

Methods and Preliminary Results

Students' Δ meta was ordered and evaluated for correlation with performance. Interestingly, five students demonstrated no overall Δ meta (boxed), and one student stood out on either end of the ordered Δ meta (Fig. 1A). Preliminary analyses show a correlation ($r = 0.43$, $p < 0.001$) of Δ meta to overall score (Fig. 1B). Confidence and certainty were separately examined with respect to ordered Δ meta (Fig. 1C). Only certainty showed positive correlation to Δ meta ($r = 0.57$, $p < 0.001$). The presentation will further describe these correlations, the five students who demonstrated no Δ meta, and the comparison of Δ meta between question types.

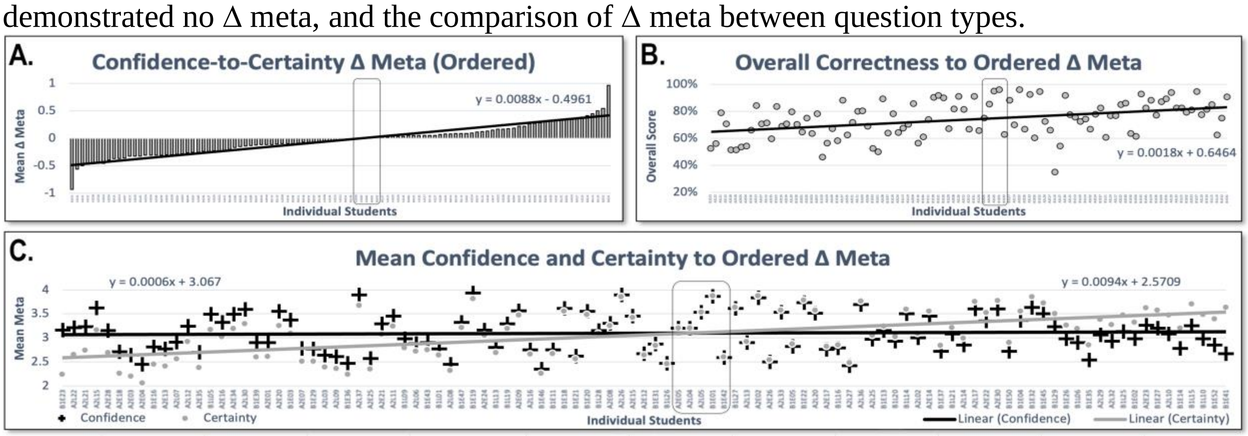

Figure 1. Correctness (B), confidence (C), and certainty (C) correlations to Δ meta (A).

Discussion and Conclusions

This study demonstrates how examining Δ meta can provide mathematics educators with a deeper understanding of student metacognition, learning, and performance, especially regarding undergraduate linear algebra and multiple choice vs. process-based examination questions.

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STEM Students' Views on the Meta-Roles that Function Plays Across the Mathematics Curriculum

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Keywords: Function, Curricular Coherence, Narrative Learning Theory, Multivariable Calculus

One of the purported goals of undergraduate mathematics is to encourage students to draw connections between the various mathematical subfields (Zorn, 2015, p. 11). The function concept—which is crucial in almost every area of mathematics—could serve a primary role in forming these connections, yet this role often goes unnoticed by students (e.g., Melhuish et al., 2020). Indeed, function is a critical recurring theme throughout undergraduate mathematics (Zandieh et al., 2017) that could be leveraged by students to increase their sense of curricular coherence across courses. This theme recurs implicitly and even explicitly in most mathematics courses, but research reminds us that students do not always take up course ideas and themes in the ways instructors intend (Clift & Brady, 2005; Lew et al., 2016). Despite this, there are few studies that speak to the ways that students leverage function as a theme to organize and make sense of their curricular experiences both within and across courses. In particular, there is much research on the metaphors students use for the *concept* of function across contexts (e.g., Lakoff & Núñez, 2000; Melhuish et al., 2020; Zandieh et al., 2017), but much less research on the *meta-roles* that students view function as serving in organizing the mathematics curriculum.

In this poster, I address two questions: (1) What do students' stories about functions reveal about the meta-roles that the function concept plays for them in organizing the mathematics curriculum? (2) In what ways do these general meta-roles appear (or not) in students' stories for functions in a specific course context? I answer these questions by analyzing the stories of three STEM students pursuing different majors who were enrolled in multivariable calculus (MVC). These stories were told during a sequence of four, 1.5-hour, task-based interviews spread throughout a semester—the first concerned the role of function across mathematics and STEM courses and the remaining interviews focused on single-variable; multivariable, real-valued; and multivariable, vector-valued functions, respectively. Using narrative learning theory and Dietiker's (2015) analytic framework for viewing the mathematics curriculum as a story, I juxtapose students' stories about the meta-role of function across the undergraduate curriculum with the stories they tell about different types of multivariable functions they encountered throughout MVC. I attend to students' stories because, as Clark and Rossiter (2008) put it, "Every day we are bombarded with a dizzying variety of experiences and we make sense of those by storying them, by constructing narratives that make things cohere. Coherence creates sense out of chaos by establishing connections between and among these experiences" (p. 62). In other words, stories are a primary representation of students' sense of curricular coherence.

Based on this preliminary, exploratory analysis, I find that students' general stories reveal developed, idiosyncratic views of the meta-role that function plays as an organizing, cross-curricular theme (e.g., "functions are like the ABCs of mathematics", "functions are kind of like canvases, and we're being trained as painters"). On the other hand, students' specific stories about functions in MVC did not explicitly leverage the same meta-roles or thematic structures that were present in their general stories. This suggests that if we want students to use function as a productive, organizing curricular theme, we must provide further reflective opportunities for students to coherently organize their course experiences with respect to this theme.

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A Coordinate System Framework

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Coordinate systems underlie many important topics in undergraduate mathematics, such as functions, equations, and graphs. By providing a framework that describes the aspects of coordinate system thinking, it offers a foundational framing to examine a wide range of characterizations of student reasoning in undergraduate mathematics. This proposal introduces a conceptual framework that emphasizes how one engages with an object viewed in a single coordinate system and multiple coordinate systems. The framework consists of four sub-parts: Naming, Locating, Re-Naming, and Re-Locating.

Keywords: linear algebra, calculus, coordinate system, change of coordinate system, framework

A coordinate system promotes student spatial thinking by imposing a visual aspect to undergraduate mathematics that may be abstract and computational. Many researchers have investigated how undergraduate students reason about coordinate systems when transitioning from one system to another in the context of precalculus, calculus, and linear algebra (Hillel & Sierpinska, 1993; Arcavi, 2003; Montiel, Vidakovic, & Tangul, 2008; Sayre & Wittmann 2008; Montiel et al., 2009, 2012; Lee et al., 2020; Moore et al., 2014; Hillel, 2000; Selby, 2016; Wawro et al., 2013; Zandieh et al., 2017). Despite these studies, there is no overarching framework for interpreting student reasoning across a variety of mathematical contexts. Therefore, this paper proposes a framework to provide a foundation for interpreting a wide range of undergraduate students' thinking with coordinate systems.

The coordinate system framework encompasses two fundamental processes for representing objects within a single coordinate system: Naming and Locating. (1) Naming involves assigning numbers and symbols to spatial and geometric objects, such as points, vectors, lines, and curves in accordance with conventions established by a coordinate system. For example, (2,3), <7,5>, and $y=-3x^2+5x-1$ are all examples of objects represented through the naming process. On the other hand, (2) Locating is the process of plotting symbolic expressions into visual representations, such as dots or graphs in space, using a coordinate system. An example of this process is graphing the function $y = x^3 - 10x$. These two processes can be extended to include reasoning about an object in the context of multiple coordinate systems: Re-Naming and Re-Locating. (3) Re-Naming involves describing the coordinates of a spatial object using a new coordinate system that is laid atop the object without moving it. In other words, the object does not receive a new name until a new coordinate system is imposed over its location. An example of Re-Naming would be taking a point (1,1) in the Cartesian coordinates renaming it as $(\sqrt{2}, \frac{\pi}{4})$ in the Polar coordinates. (4) Re-Locating involves plotting a symbolic expression that does not change into different spaces and transforming the shape of an object into a different shape by mapping points in the first coordinate system to points in the second coordinate system. Although the transformed location may appear different from the original, the two shapes are images of each other.

By using an overarching lens, researchers can better analyze student reasoning by focusing on how students think about each part of a problem, and how they leverage their understanding of coordinate systems to reason about multiple coordinate systems. This approach can also serve as a useful framework for teachers and researchers when designing tasks that involve coordinate systems.

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EDUCATION

Narratives of math affect and math utility are co-constructed during advising sessions through talk & embodiment. Advisors take up narratives inconsistently, resulting in students being "cooled out" from pursuing mathematics.

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Orienting Details

Research Question:
What aspects of talk and embodiment influence the co-construction of narratives about women students in advising sessions with a man advisor?

What is Math Advising?
A 5-10 minute optional conversation between a member of the math department and an incoming undergraduate who is unsure of which math class to take. The session results in a non-binding agreement that guides the student's registration decision. The advisor and student have had no prior interaction.

Our Positionality:
We are math education researchers and trained advisors. We use Black feminist perspectives and our own experiences to inform our questions & interpretations of data.

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Narratives of math affect are taken up inconsistently across sessions.

We illustrate with two sessions with women students and the same man advisor.

Session 1: The advisor ignores the student's bids to construct a narrative of positive math affect. Narratives of math utility are used to cool out the student.

Okay good. <u>Well you don't need any math for psychology.</u> Did you want to take a math class **or do you want to avoid math?**

I did. I-I- do enjoy math even though it's not really a part of psychology, but **I still really enjoy math**. Like I am thinking of taking statistics, but I wanted to. I'm thinking also of including calculus if I have enough room for it.

<u>So stats and math are two different depts here</u> and I agree that if you're going into psych you're probably gonna have to take some stats and <u>that's probably more relevant</u>. **But if you want to take math .. I encourage you to do that.**

Yeah, **I really enjoy it**. So I do want to try like, uh [more challen–

//How far did you go in high school?

I just took AP calc AB. Um, I want to know what [classes–

//Did you take the exam?

Yes. Do I have to show you the scores? {pulls out paper} Uh. I got a 5.

Session 2: The advisor repeatedly takes up the narrative of negative math affect to cool out the student. The student uses math utility to resist those efforts.

Okay. Any idea what you want to study?

Um, yeah, actually that's what I was gonna talk to you about because of the math requirement. Um, (..) **I hate math** {laugh}. [And–

//Ok. Honesty.

I went up to calc BC in high school. Got A's in the classes, did poorly on the exam **cause I hate math** and I just did it for the A w/o studying {laugh}

What did you get on the BC test?

A one, but I didn't try [Oh, ok.] Yeah, but I got an A in the class {both laugh}. So, um, uh, I want to go either premed or go to law school myself. I'm [leaning–

//<u>Well good news is that doesn't require so much math.</u>

Yeah. So I want to get like, **absolute minimum of math**.

Well you do have to satisfy a quantitative reasoning requirement, <u>but you don't have to use math for that</u>.

So the requirement.. is that different than how a grad school will look upon your transcript? Because–

Oh that's an entirely different question.

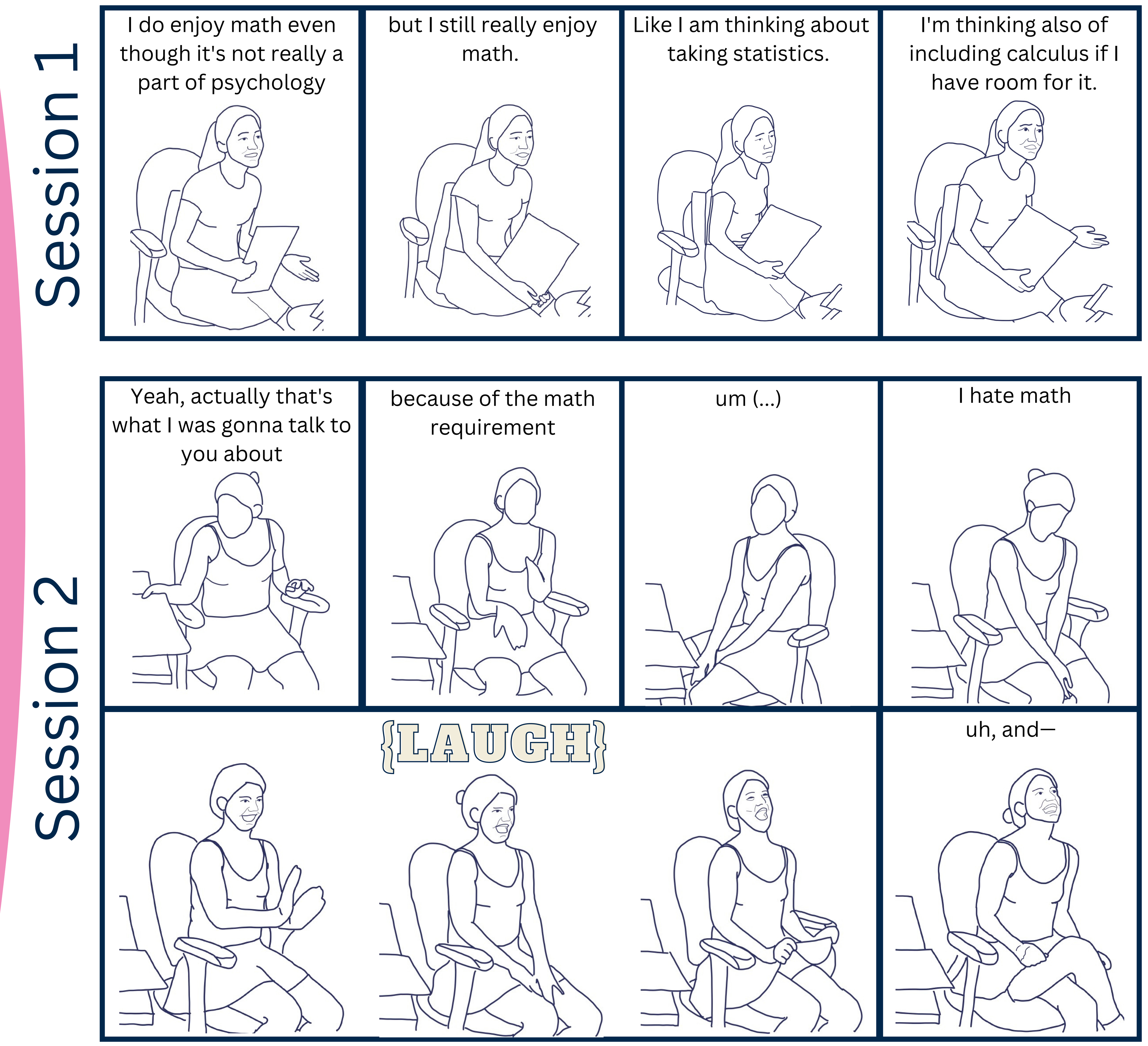
Narratives & Math Advising: We conceptualize math advising as a genre of interaction in which students and advisors co-construct narratives about the set of courses that are appropriate for the student to take. Narratives are co-constructed by the advisor and student through verbal and non-verbal interactions.

Narratives of Math Affect & Math Utility: A math affect narrative is how the student feels about math. A math utility narrative is how well math fits into a student's possible educational career.

Theory of Self-Authorship: Higher Education practitioners should encourage students to explore and design their own educational journey instead of assigning them pre-determined paths.

Why is this important? Students often change paths from orientation to graduation. Decisions that are made during summer orientation can severely limit their options in future semesters, particularly if they opt out of math.

Next Steps: Incorporating Embodiment into our Analysis



The narrative of math affect is not taken up by the advisor in Session 1, but is taken up in Session 2. Why might the advisor be taking up one narrative of math affect and not the other?